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Second-order eikonal cross sections for the excitation of atomic hydrogen

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Abstract. The scheme of Wallace for obtaining higher-order eikonal approximations than the Glauber approximation, is used to calculate the cross sections for the excitation of the 2s and 2p states of atomic hydrogen by electron or positron impact, correct to the second order. As expected, inclusion of the second-order term introduces differences between electron and positron scattering. For electron scattering, an estimate of the eikonal exchange amplitude has also been included. Differential cross sections are presented and discussed in the angular range 0–30°, where the Wallace formula may be expected to be valid.

1. Introduction

The eikonal expansion for potential scattering, proposed by Wallace (1973), has recently been applied to elastic scattering of electrons and positrons from atoms represented by static potentials (Roy and Sil 1978, 1980), to obtain cross sections correct to the second order, the first order being identical with the Glauber approximation. These results show that in addition to distinguishing between electrons and positrons (unlike the Glauber approximation), the second-order calculation also produces better agreement with exact partial-wave calculations, at small angles. It is therefore of interest to explore the possibility of performing such a calculation for inelastic scattering also. As has been indicated by Wallace (1973), his formalism can be extended to many-body problems just as the conventional Glauber theory (Glauber 1959, Gerjuoy and Thomas 1974). As pointed out by Yates (1973), the Glauber formula for excitations, which satisfies the requirements of time reversal invariance, can be derived by assuming that the magnitude of the average momentum K of the incident particle in the centre-of-mass system is equal to

$$(k_i^2+k_f^2)^{1/2}/\sqrt{2},$$

where $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are, respectively, the initial and final momenta. (We use atomic units throughout.) Since Wallace (1973) has shown that at small angles the most significant effect of the second-order term can be reduced to a correction to the usual Glauber phase χ, the second-order scattering amplitude for a transition from state i to state j of the target and projectile system may be written as

$$F_{if}(\boldsymbol{q})=\frac{\mathrm{i}K}{2\pi}\int\langle f|1-\exp[\mathrm{i}(\chi+\tau_1)]|i\rangle\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{b})\,\mathrm{d}\boldsymbol{b} \tag{1}$$

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where $\boldsymbol{q} = \boldsymbol{k}_i - \boldsymbol{k}_f$ and τ_1 is the phase correction, to be defined explictly below. In the usual cylindrical coordinate system in which the z direction is chosen perpendicular to $\boldsymbol{q}$, $\boldsymbol{b}$ is given by $\boldsymbol{r} = \boldsymbol{b} + \boldsymbol{z}$, where $\boldsymbol{r}$ denotes the coordinates of the incident particle with respect to the atomic proton. Decomposing the position vector $\boldsymbol{r}_j$ of each atomic electron also as $\boldsymbol{r}_j = \boldsymbol{b}_j + z_j\hat{\boldsymbol{z}}$, χ may be expressed as

$$\chi(\boldsymbol{b}, \{\boldsymbol{r}_j\}) = \chi_-(\boldsymbol{r}, \{\boldsymbol{r}_j\}) + \chi_+(\boldsymbol{r}, \{\boldsymbol{r}_j\}) \tag{2}$$

$$\chi_+(\boldsymbol{r}, \{\boldsymbol{r}_j\}) = -\frac{1}{v}\int_{-\infty}^{z} V(\boldsymbol{b}, z', \{\boldsymbol{r}_j\})\, \mathrm{d}z' \tag{3}$$

$$\chi_-(\boldsymbol{r}, \{\boldsymbol{r}_j\}) = -\frac{1}{v}\int_{z}^{\infty} V(\boldsymbol{b}, z', \{\boldsymbol{r}_j\})\, \mathrm{d}z'. \tag{4}$$

In these expressions, $v = K/\mu$, μ being the reduced mass, V is the sum of the two-body Coulomb potentials between the incident particle and each frozen-target particle, and $\{\boldsymbol{r}_j\}$ denotes the collection of atomic coordinates. τ_1 is given by (Wallace 1973)

$$\tau_1(\boldsymbol{b}, \{\boldsymbol{r}_j\}) = \frac{1}{2K}\int_{-\infty}^{\infty} \nabla\chi_- \cdot \nabla\chi_+\, \mathrm{d}z. \tag{5}$$

In view of the fact that the first-order amplitude is available in closed form for collisions between electrons or positrons with hydrogen atoms (Thomas and Gerjuoy 1971), it is advantageous for numerical work to write equation (1) as

$$F_{if}(\boldsymbol{q}) = G_{if}(\boldsymbol{q}) + W_{if}(\boldsymbol{q}) \tag{6}$$

where G_{if} is the usual Glauber amplitude given by

$$G_{if}(\boldsymbol{q}) = \frac{\mathrm{i}K}{2\pi}\int \langle f|1 - \exp(\mathrm{i}\chi)|i\rangle \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{b})\, \mathrm{d}\boldsymbol{b} \tag{7}$$

and W_{if}, the second-order term, is

$$W_{if}(\boldsymbol{q}) = \frac{\mathrm{i}K}{2\pi}\int \langle f|\exp(\mathrm{i}\chi)[1 - \exp(\mathrm{i}\tau_1)]|i\rangle \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{b})\, \mathrm{d}\boldsymbol{b}. \tag{8}$$

It is clear from the foregoing that χ changes sign with V while τ_1 does not, so that the amplitudes would differ in magnitude for the scattering of electrons and positrons. The evaluation of τ_1 for collisions of electrons and positrons with atomic hydrogen is dealt with in the next section. The reduction of the expressions for W_{if} in the case of the 1s–2s transition as well as the 1s–2p transition is described in the following sections and the final section presents numerical results for differential cross sections.

2. Phase correction for electron- or positron-scattering from hydrogen

The appropriate potential in this case is

$$V = \frac{e}{r} - \frac{e}{|\boldsymbol{r} - \boldsymbol{r}_1|} \tag{9}$$

where $e = 1$ for positrons and $e = -1$ for electrons. Using this in equation (3), we obtain

$$\nabla\chi_+ = \frac{e}{v}\int_{-\infty}^{z} dz'\left(\frac{\boldsymbol{b}}{r'^3}+\frac{z'}{r'^3}-\frac{\boldsymbol{b}-\boldsymbol{b}_1}{|\boldsymbol{r}'-\boldsymbol{r}_1|^3}-\frac{z'-z_1}{|\boldsymbol{r}'-\boldsymbol{r}_1|^2}\right)$$

where $\boldsymbol{r}' = \boldsymbol{b} + \boldsymbol{z}'$. This reduces to

$$\nabla\chi_+ = -\frac{e}{v}\left[\frac{\hat{\boldsymbol{z}}}{r}-\frac{\boldsymbol{b}}{b^2}\left(1+\frac{z}{r}\right)-\frac{\hat{\boldsymbol{z}}}{|\boldsymbol{r}-\boldsymbol{r}_1|}+\frac{(\boldsymbol{b}-\boldsymbol{b}_1)}{|\boldsymbol{b}-\boldsymbol{b}_1|^2}\left(1+\frac{z-z_1}{|\boldsymbol{r}-\boldsymbol{r}_1|}\right)\right]. \tag{10}$$

Similarly, we find for χ_-,

$$\nabla\chi_- = -\frac{e}{v}\left[-\frac{\hat{\boldsymbol{z}}}{r}-\frac{\boldsymbol{b}}{b^2}\left(1-\frac{z}{r}\right)+\frac{\hat{\boldsymbol{z}}}{|\boldsymbol{r}-\boldsymbol{r}_1|}+\frac{(\boldsymbol{b}-\boldsymbol{b}_1)}{|\boldsymbol{b}-\boldsymbol{b}_1|^2}\left(1-\frac{z-z_1}{|\boldsymbol{r}-\boldsymbol{r}_1|}\right)\right]. \tag{11}$$

The use of equations (10) and (11) in (5) gives

$$\tau_1(\boldsymbol{b}, \boldsymbol{r}_1) = \frac{1}{Kv^2}\int_{-\infty}^{\infty} dz\left[\frac{1}{r|\boldsymbol{r}-\boldsymbol{r}_1|}+\frac{\boldsymbol{b}\cdot(\boldsymbol{b}_1-\boldsymbol{b})}{b^2|\boldsymbol{b}-\boldsymbol{b}_1|^2}\left(1-\frac{z(z-z_1)}{r|\boldsymbol{r}-\boldsymbol{r}_1|}\right)\right]. \tag{12}$$

To evaluate the above integral, we may write $z = \alpha b$, $\boldsymbol{b}_1 = \boldsymbol{\beta} b$, $z_1 = \gamma b$ and $\boldsymbol{b}_1 - \boldsymbol{b} = b\boldsymbol{a}$, so that τ_1 can be expressed as†

$$\tau_1 = \frac{1}{Kv^2 b}\left(I_1 + \frac{\boldsymbol{b}\cdot\boldsymbol{a}}{ba^2}I_2\right) \tag{13}$$

where

$$I_1 = \int_{-\infty}^{\infty} (1+\alpha^2)^{-1/2}[a^2+(\alpha-\gamma)^2]^{-1/2}\, d\alpha \tag{14}$$

and

$$I_2 = \int_{-\infty}^{\infty}\left(1-\frac{\alpha(\alpha-\gamma)}{(1+\alpha^2)^{1/2}[a^2+(\alpha-\gamma)^2]^{1/2}}\right) d\alpha. \tag{15}$$

Reduction of these integrals to closed forms is outlined in the appendix. The final result is

$$I_1 = 2gK(k) \tag{16}$$

and

$$I_2 = 2g^{-1}E(k) \tag{17}$$

where K and E are the complete elliptic integrals of the first and second kinds respectively, and g and k are functions of a and γ, as defined in the appendix.

3. Excitation to the 2s state

Inserting the expression for χ, namely

$$\chi(\boldsymbol{b}, \boldsymbol{b}_1) = 2\eta \ln(|\boldsymbol{b}_1 - \boldsymbol{b}|/b) \qquad \eta = -e/v \tag{18}$$

† After completing the revision of this paper, another calculation of τ_1 has come to our notice (Byron *et al* 1981). The expression for τ_1 derived by these authors contains four terms, of which the last two can easily be shown to cancel each other. The first two terms are exactly equivalent to the corresponding terms given here, as may be shown by means of the hypergeometric function repesentations of the various special functions.

(Franco 1968), in equation (8), we have for the 1s–2s transition

$$W_{1s,2s}=\frac{\mathrm{i}K}{8\sqrt{2}\pi^2}\int \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{b})a^{2\mathrm{i}\eta}\left\{1-\exp\left[\frac{\mathrm{i}}{Kv^2b}\left(I_1+\frac{\boldsymbol{b}\cdot\boldsymbol{a}}{ba^2}I_2\right)\right]\right\}$$
$$\times\exp[-\lambda b(\beta^2+\gamma^2)^{1/2}][2-b(\beta^2+\gamma^2)^{1/2}]b^4\,\mathrm{d}b\,\mathrm{d}\phi_b\,\beta\,\mathrm{d}\beta\,\mathrm{d}\phi_\beta\,\mathrm{d}\gamma \tag{19}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and γ are as defined in the previous section, and ϕ_b and ϕ_β are, respectively, the azimuthal angles of $\boldsymbol{b}$ and $\boldsymbol{\beta}$ in the plane perpendicular to the z axis. The constant λ is equal to $\frac{3}{2}$. On changing the area element $\beta\,\mathrm{d}\beta\,\mathrm{d}\phi_\beta$ to $a\,\mathrm{d}a\,\mathrm{d}\phi_a$, and writing $\phi_{ab}=\phi_a-\phi_b$, the integration with respect to ϕ_b may be performed to give

$$W_{1s,2s}=\frac{\mathrm{i}K}{4\sqrt{2}\pi}\int_0^\infty a^{1+2\mathrm{i}\eta}\,\mathrm{d}a\int_{-\infty}^{\infty}\mathrm{d}\gamma\int_0^{2\pi}\mathrm{d}\phi_{ab}\int_0^\infty J_0(qb)\exp[-\lambda b(t+2a\cos\phi_{ab})^{1/2}]$$
$$\times[2-b(t+2a\cos\phi_{ab})^{1/2}]\left\{1-\exp\left[\frac{\mathrm{i}}{Kv^2b}\left(I_1+\frac{\cos\phi_{ab}}{a}I_2\right)\right]\right\}b^4\,\mathrm{d}b \tag{20}$$
$$t=1+a^2+\gamma^2.$$

The integration over b in the above equation can be performed, using standard formulae (Gradshteyn and Ryzhik 1965), to yield finally

$$W_{1s,2s}=\frac{\mathrm{i}K}{\sqrt{2}\pi}\int_0^\infty a^{1+2\mathrm{i}\eta}\,\mathrm{d}a\int_0^\infty\mathrm{d}\gamma\int_0^\pi\mathrm{d}\phi_{ab}[2(A-C_5)-(t+2a\cos\phi_{ab})^{1/2}(B-C_6)] \tag{21}$$

where

$$A=\frac{24}{(\delta^2+q^2)^{5/2}}\left[1-\frac{5q^2}{\delta^2+q^2}+\frac{35}{8}\left(\frac{q^2}{\delta^2+q^2}\right)^2\right] \tag{21a}$$

$$B=\frac{120\delta}{(\delta^2+q^2)^{7/2}}\left[1-\frac{7q^2}{\delta^2+q^2}+\frac{63}{8}\left(\frac{q^2}{\delta^2+q^2}\right)^2\right] \tag{21b}$$

$$C_i=\left(-\frac{\partial}{\partial\delta}\right)^i[2J_0(\alpha[(\delta^2+q^2)^{1/2}-\delta]^{1/2})K_0(\alpha[\delta^2+q^2)^{1/2}+\delta]^{1/2})] \tag{21c}$$

where

$$\delta=\lambda(t+2a\cos\phi_{ab})^{1/2} \tag{21d}$$

$$\alpha=\left[-2\mathrm{i}\left(I_1+\frac{\cos\phi_{ab}}{a}I_2\right)(Kv^2)^{-1}\right]^{1/2}. \tag{21e}$$

4. Excitation to the 2p state

Quantising the 2p state wavefunction along the z axis, the second-order term for the excitation of the magnetic sublevel m is seen to be given by

$$W^m_{1s,2p}=\frac{\mathrm{i}K}{(96\pi^3)^{1/2}}\int\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{b})a^{2\mathrm{i}\eta}\left\{1-\exp\left[\frac{\mathrm{i}}{Kv^2b}\left(I_1+\frac{\boldsymbol{b}\cdot\boldsymbol{a}}{ba^2}I_2\right)\right]\right\}r_1$$
$$\times\exp(-\lambda r_1)Y^*_{1,m}(\hat{\boldsymbol{r}}_1\cdot\hat{\boldsymbol{z}})b^3\,\mathrm{d}\boldsymbol{b}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\gamma. \tag{22}$$

Since, for $m=0$, the spherical harmonic is an odd function of γ, whereas the rest of the integrand is even, $W^{0}_{1s,2p}$ vanishes, just as the Glauber amplitude does itself (Tai *et al* 1970, Thomas and Gerjuoy 1971). To evaluate $W^{\pm1}_{1s,2p}$, we again change the area element $d\boldsymbol{\beta}$ to $d\boldsymbol{a}$ and apply the procedure of Thomas and Gerjuoy (1971) to obtain

$$W^{\pm1}_{1s,2p}=\frac{-K\exp(\mp i\phi_q)}{8\pi}\int_0^\infty a^{1+2i\eta}\,da\int_{-\infty}^{\infty}d\gamma\int_0^{2\pi}d\phi_{ab}\int_0^\infty J_1(qb)\times\exp(\mp i\phi_{\beta_b})\beta[1-\exp(-\alpha^2/2b)]\exp(-\delta b)b^5\,db \qquad (23)$$

where α and δ are as defined in equation (21). With the help of the auxiliary relations

$$a\sin\phi_{ab}=\beta\sin\phi_{\beta_b}$$

and

$$1+a\cos\phi_{ab}=\beta\cos\phi_{\beta_b}$$

equation (23) reduces, as before, to

$$W^{\pm1}_{1s,2s}=\frac{-K\exp(\mp i\phi_q)}{2\pi}\int_0^\infty a^{1+2i\eta}\,da\int_0^\infty d\gamma\int_0^\pi(1+a\cos\phi_{ab})(E-C_6)\,d\phi_{ab} \qquad (24)$$

where

$$E=\frac{360q}{(\delta^2+q^2)^{7/2}}\left[1-\frac{7}{2}\frac{q^2}{\delta^2+q^2}+\frac{21}{\delta}\left(\frac{q^2}{\delta^2+q^2}\right)^2\right].$$

When τ_1 is small, which is expected to be the case at sufficiently high energies, the complexity of equations (21) and (24) can be considerably reduced by expanding $\exp(i\tau_1)$ and dropping powers of τ_1 higher than the first. In this case the angular integration can also be performed analytically.

5. Exchange effects

In order to afford a better comparison with other calculations which make allowance for electron exchange, and also with experiment at low energies, an estimate of the eikonal exchange amplitude may be provided in the Ochkur approximation, using the formulation of Franco and Halpern (1980). With the present choice of the z axis, the relevant amplitudes are readily shown to be given by

$$g_{1s,2s}=C\lambda^{-i\eta-2}(\lambda^2+q^2)^{i\eta-3}(2-i\eta)\{(q^2-\lambda^2)[3\lambda-i\eta(q^2-\lambda^2)]-3\lambda(\lambda^2+q^2)\} \qquad (25)$$

$$g^{\pm1}_{1s,2p}=C[\exp(\mp i\phi_q)/4\pi]q(\eta+i)\lambda^{-i\eta-1}(\lambda^2+q^2)^{i\eta-3}[9+i\eta(q^2-\lambda^2)] \qquad (26)$$

$$g^{0}_{1s,2p}=C(-\eta/8\pi)\lambda^{-i\eta-2}(\lambda^2+q^2)^{i\eta-2}[3\lambda^2+q^2+i\eta(q^2-\lambda^2)] \qquad (27)$$

$$C=\frac{8\pi^2\eta\exp(\pi\eta/2)}{K^{2+i\eta}\sinh\pi\eta}.$$

6. Numerical procedures

Much of the effort that a numerical solution of equations (21) and (24) entails arises from the evaluation of the functions C_i and the need to continue the integrations in

a and γ up to sufficiently large values of these variables to obtain convergence. In the present work, the first problem was handled by writing a subroutine which computes the derivatives of products of functions of functions, in terms of the derivatives of these functions with respect to their respective arguments. As regards the latter, it was found that a series representation of the integral over b in equations (20) and (23), obtained by expanding the Bessel function present therein, converged very rapidly for large values of a and γ.

7. Results and discussion

The differential cross sections for the excitation of the 2s and 2p states for incident energies 54.4, 100 and 200 eV are presented in tables 1 and 2, repectively. Since the validity of the Wallace formula has not been established for large-angle scattering, these data are confined to the angular range 0–30°. For potential scattering, Wallace (1973) had conjectured that his approximate formula might hold for large angles also. In the present case, however, it was found that at sufficiently large angles (e.g. after about 70° at 100 eV), the correction term became so large as to cause a steady increase in the cross section, which one would hardly expect to happen, on physical grounds. It may be mentioned in this connection that the extension of Wallace's second-order formula to inelastic processes, as represented by equation (1), implies a further approximation in that the intermediate atomic states are assigned an average value of energy.

The electron scattering cross sections given in tables 1 and 2 include the exchange contributions, estimated for an unpolarised incident beam. At 200 eV comparative computations showed that the small-τ_1 approximation yielded essentially the same

Table 1. Differential cross sections ($a_0^2\ \mathrm{sr}^{-1}$) for the excitation of the H(2s) state by electron or positron impact.

	54.4 eV			100 eV			200 eV		
		Second-order eikonal			Second-order eikonal			Second-order eikonal	
Angle (deg)	Glauber $e^\pm$	e^-	e^+	Glauber $e^\pm$	e^-	e^+	Glauber $e^\pm$	e^-	e^+
0	1.83+0[a]	1.96+0	1.54+0	2.00+0	2.12+0	1.83+0	1.88+0	1.94+0	1.80+0
2	1.70+0	1.83+0	1.43+0	1.71+0	1.82+0	1.54+0	1.36+0	1.42+0	1.29+0
4	1.42+0	1.54+0	1.17+0	1.24+0	1.34+0	1.10+0	9.05−1	9.53−1	8.54−1
6	1.10+0	1.21+0	8.99−1	8.72−1	9.55−1	7.70−1	6.11−1	6.46−1	5.80−1
8	8.27−1	9.13−1	6.69−1	6.15−1	6.78−1	5.43−1	4.04−1	4.25−1	3.89−1
10	6.11−1	6.78−1	4.92−1	4.31−1	4.76−1	3.85−1	2.57−1	2.69−1	2.54−1
12	4.46−1	4.96−1	3.59−1	3.00−1	3.28−1	2.74−1	1.58−1	1.64−1	1.61−1
14	3.22−1	3.57−1	2.62−1	2.06−1	2.23−1	1.93−1	9.46−2	9.65−2	9.98−1
16	2.32−1	2.55−1	1.90−1	1.40−1	1.48−1	1.36−1	5.57−2	5.60−2	6.09−2
18	1.66−1	1.80−1	1.38−1	9.40−2	9.78−2	9.45−2	3.28−2	3.24−2	3.71−2
20	1.19−1	1.27−1	1.00−1	6.34−2	6.44−2	6.55−2	1.97−2	1.92−2	2.28−2
25	5.55−2	5.58−2	4.44−2	2.53−2	2.48−2	2.63−2	6.68−3	6.60−3	7.64−3
30	3.19−2	3.16−2	1.98−2	1.27−2	1.31−2	1.13−2	3.25−3	3.40−3	3.30−3

[a] The last two characters indicate powers of ten.

Table 2. Differential cross sections ($a_0^2\ \mathrm{sr}^{-1}$) for the excitation of the H(2p) state by electron or positron impact.

	54.4 eV			100 eV			200 eV		
		Second-order eikonal			Second-order eikonal			Second-order eikonal	
Angle (deg)	Glauber $e^{\pm}$	e^-	e^+	Glauber $e^{\pm}$	e^-	e^+	Glauber $e^{\pm}$	e^-	e^+
0	4.19+1	4.10+1	4.24+1	9.56+1	9.15+1	9.59+1	2.13+2	2.12+2	2.13+2
2	3.69+1	3.61+1	3.74+1	6.52+1	6.47+1	6.54+1	7.23+1	7.20+1	7.25+1
4	2.67+1	2.59+1	2.72+1	3.15+1	3.11+1	3.18+1	2.11+1	2.09+1	2.13+1
6	1.76+1	1.69+1	1.81+1	1.54+1	1.49+1	1.56+1	7.92+0	7.70+0	8.06+0
8	1.12+1	1.06+1	1.17+1	7.94+0	7.57+0	8.13+0	3.34+0	3.17+0	3.45+0
10	7.17+0	6.61+0	7.53+0	4.29+0	3.98+0	4.46+0	1.49+0	1.37+0	1.57+0
12	4.61+0	4.13+0	4.92+0	2.40+0	2.14+0	2.54+0	6.87−1	6.05−1	7.47−1
14	3.00+0	2.59+0	3.26+0	1.37+0	1.17+0	1.49+0	3.23−1	2.68−1	3.66−1
16	1.97+0	1.63+0	2.19+0	7.91−1	6.36+0	8.91+0	1.55−1	1.19−1	1.86−1
18	1.31+0	1.03+0	1.49+0	4.64−1	3.47+0	5.46+0	7.60−2	5.31−2	9.80−2
20	8.78−1	6.49−1	1.03+0	2.75−1	1.89−1	3.42−1	3.84−2	2.37−2	5.41−2
25	3.38−1	2.11−1	4.34−1	7.93−2	3.97−2	1.19−1	8.18−3	3.31−3	1.53−2
30	1.40−1	7.56−2	2.01−1	2.55−2	8.22−3	4.97−2	2.32−3	7.02−4	5.89−3

results as the exact results and hence the former has been employed for calculations at 200 and 300 eV. Differential cross sections for the excitation of the $n=2$ level (2s+2p) by electron scattering at 100 eV are shown in figure 1, along with the corresponding data according to the recent distorted-wave second Born approximation (DWSBA) of Kingston and Walters (1980), as well as the available experimental data (Williams and Willis 1975). Though there are some differences between the present results and the DWSBA results for the excitation of the 2s and 2p states considered separately, the summed up cross sections, which are what have so far been actually measured, display quite good agreement.

Appendix

Evaluation of the integral I_1

Defining

$$J(y, y_1)=\int_{y_1}^{y}\frac{\mathrm{d}\alpha}{(1+\alpha^2)^{1/2}[a^2+(\alpha-\gamma)^2]^{1/2}}. \tag{A.1}$$

we may write (cf equation (14))

$$I_1=J(\infty, y_1)-J(-\infty, y_1). \tag{A.2}$$

Noting that the zeros of the denominator of the integrand in (A.1) are $\pm\mathrm{i}$; $\gamma\pm\mathrm{i}a$, we may use the formula (267.00) of Byrd and Friedman (1971) to obtain

$$J(y, y_1)=gF(\phi, k) \tag{A.3}$$

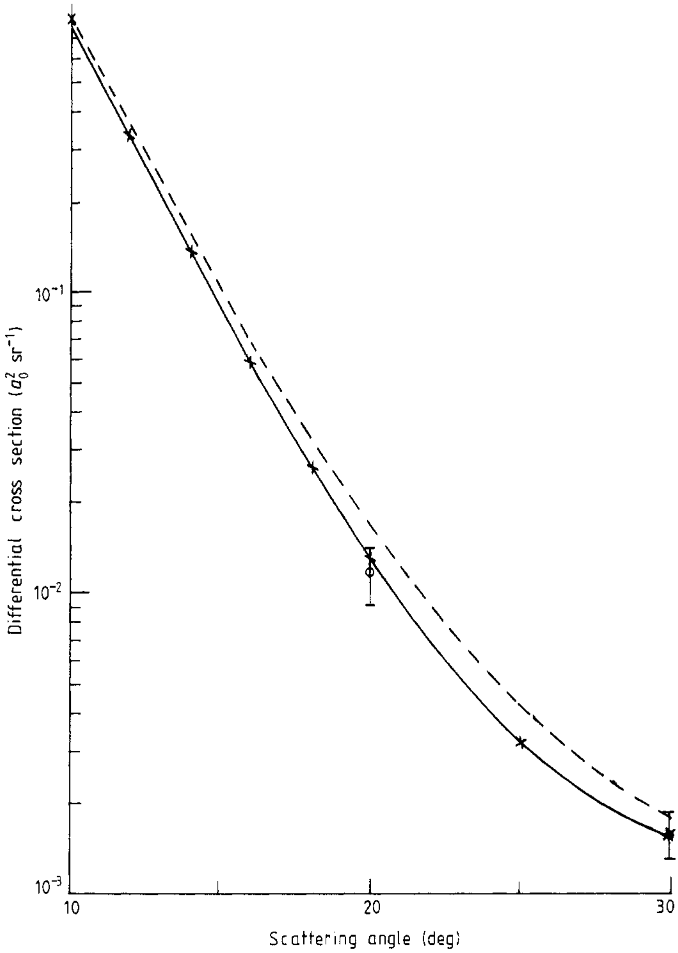

Figure 1. Differential cross sections for the excitation of the $n = 2$ states of atomic hydrogen by electrons of energy 100 eV. ——, second-order eikonal; – – –, Glauber; ×, DWBSA (Kingston and Walters 1980); $\overline{\underset{\perp}{\varphi}}$, experiment (Williams and Willis 1975).

where F is the complete elliptic integral of the first kind, and

$$g = \frac{2}{A+B} \qquad k^2 = \frac{4AB}{(A+B)^2}$$

$$A^2 = \gamma^2 + (1+a)^2 \qquad B^2 = \gamma^2 + (1-a)^2$$

$$\tan\phi = \frac{y+g_1}{1-g_1 y} \qquad g_1 = \frac{4-(A-B)^2}{(A+B)^2-4}.$$

From the expression for $\tan\phi$, it is clear that as y goes from $-\infty$ to ∞, ϕ goes from $-\tan^{-1}(1/g_1)$ to $(\pi - \tan^{-1}(1/g_1))$. Equation (16) of § 2 therefore follows from (A.2) and (A.3) and the definition of $F(\phi, k)$.

Evaluation of the integral I_2

As before, we define

$$J(y, y_1)=\int_{y_1}^{y}\frac{\alpha(\alpha-\gamma)\,\mathrm{d}\alpha}{(1+\alpha^2)^{1/2}[a^2+(\alpha-\gamma)^2]^{1/2}} \tag{A.4}$$

so that (cf equation (15))

$$I_2=\mathrm{Lt}_{y\to\infty}[2y-(J(y, y_1)-J(-y, y_1))]. \tag{A.5}$$

The integral J can be evaluated in terms of Jacobian elliptic functions, using a series of formulae starting from (267.01) of Byrd and Friedman (1971). After some severe cancellations, one is left with

$$J(y, y_1)=\frac{1}{g}\left(\mathrm{d}n\ u_1 tnu_1-\frac{g\ \mathrm{dc}\ u_1 ncu_1}{1+g_1 tnu_1}-E(\phi, k)\right) \tag{A.6}$$

where $u_1=F(\phi, k)$. Evaluating the elliptic functions, we find

$$J(y, y_1)=y-\frac{k^2 g_1}{1-k^2+g_1^2}-\frac{1}{g}E(\phi, k)+\mathrm{O}(y^{-1}). \tag{A.7}$$

Substituting (A.7) in (A.5) and again noting that ϕ has a range of π, equation (17) is obtained.

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