STAT 447C: Bayesian Statistics **Fall 2024**

Exercise 5

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1 sequential updating

1. Denote $\theta|x^{(n)} \sim \pi$. Since $x_i|\theta$ is i.i.d, then $x^{(n)}|\theta \sim v_\theta^n$. Suppose μ is the common dominating measure of ν and ρ, i.e. $p(x_i|\theta) = \frac{d\nu}{d\mu}(x_i)$ and $p(\theta) = \frac{d\rho}{d\mu}(\theta)$, then by Bayes' Theorem, we have:

$$\frac{d\pi}{d\mu}(\theta) = p(\theta|x^{(n)}) = \frac{p(x^{(n)}|\theta) \cdot p(\theta)}{\int_\Theta p(x^{(n)}|\theta) \cdot p(\theta) d\mu}$$

Hence

$$\pi(d\theta) = \frac{p(x^{(n)}|\theta) \cdot \rho(d\theta)}{\int_\Theta p(x^{(n)}|\theta) \cdot \rho(d\theta)} = \frac{\nu_\theta^n(dx) \cdot \rho(d\theta)}{\int_\Theta \nu_\theta^n(dx) \cdot \rho(d\theta)}$$

2. Using π as the new prior, then by Bayes's Theorem, the posterior distribution of $\theta|x_{n+1}$ is given by:

$$\begin{aligned} &\frac{p(x_{n+1}|\theta) \cdot \pi(d\theta)}{\int_\Theta p(x_{n+1}|\theta) \cdot \pi(d\theta)} \\ &= \frac{\nu_\theta(dx_{n+1}) \cdot \nu_\theta^n(dx) \cdot \rho(d\theta)}{\int_\Theta \nu_\theta(dx_{n+1}) \cdot \nu_\theta^n(dx) \cdot \rho(d\theta)} \\ &= \frac{\nu_\theta^{n+1}(dx) \cdot \rho(d\theta)}{\int_\Theta \nu_\theta^{n+1}(dx) \cdot \rho(d\theta)} \end{aligned}$$

We can see that is equivalent of using ρ as prior with $(x_1, \ldots, x_{n+1})$ observations.

2 Bayesian inference in the limit of increasing data

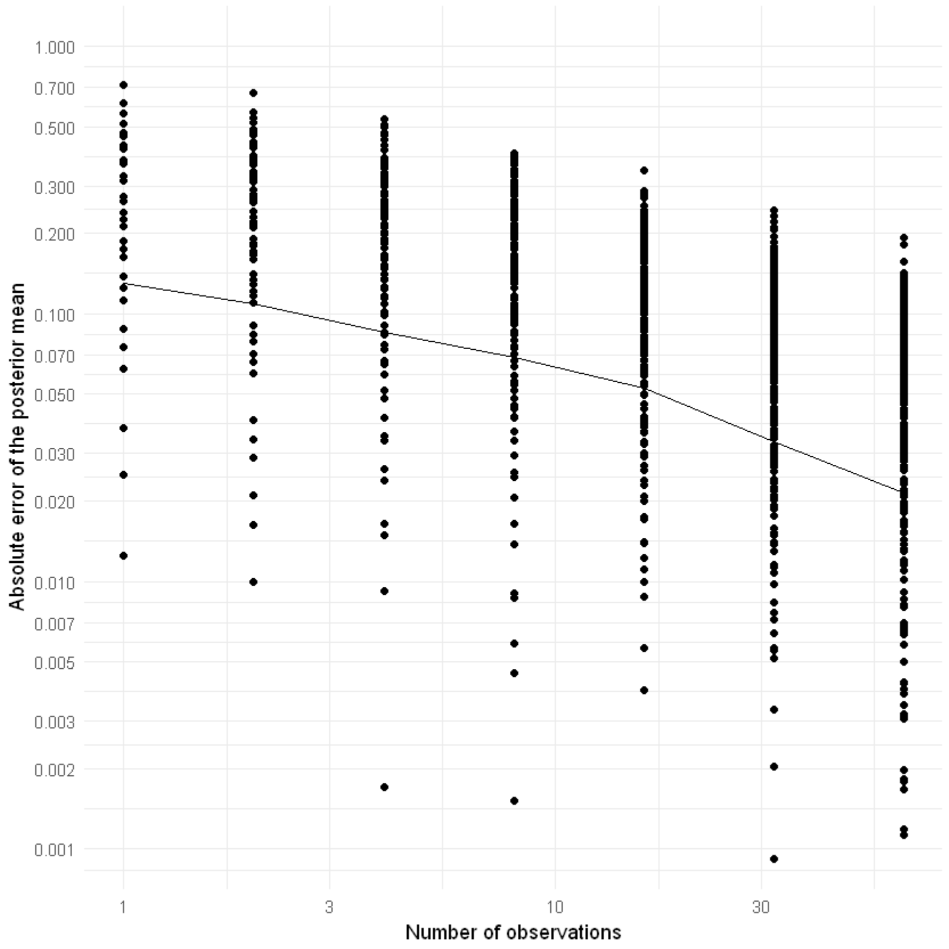
1.
```r
# global
library(ggplot2)
suppressPackageStartupMessages(library(extraDistr))
suppressPackageStartupMessages(library(distr))
source("./simple.R")
source("./simple_utils.R")
set.seed(2025)
K = 20

# 1
posterior_distribution = function(rho, n_successes, n_observations) {
  K = length(rho) - 1
  gamma = rho * dbinom(n_successes, n_observations, (0:K)/K)
  normalizing_constant = sum(gamma)
  gamma/normalizing_constant
}
```

2.
```r
# 2
posterior_mean = function(post_dist) {
    return (sum((seq(0, K, 1)/K) * post_dist))
}
```

3.
```r
# 3
simulate_posterior_mean_error = function(rho_true, rho_prior, n_observations){
    dist_p = DiscreteDistribution(supp = (1/K)*(0:K), prob = rho_true/sum(rho_true))
    p_true = simulate(dist_p)
    Y = replicate(n_observations, simulate(Bern(p_true)))
    post_dist = posterior_distribution(rho_prior, sum(Y), n_observations)
    post_mean = posterior_mean(post_dist)
    return (abs(p_true - post_mean))
}
```

4.
```r
# 4
rho_true = rho_prior = 1:(K+1)
n_obs_vector <- 2^(0:6)
experiment_results = data.frame()
for (n_obs in n_obs_vector) {
    errors = replicate(1000, simulate_posterior_mean_error(rho_true, rho_prior, n_obs))
    df = data.frame(n_observations=rep(n_obs, 1000), replication=(1:1000), errors=errors
    )
    experiment_results = rbind(experiment_results, df)
}
head(experiment_results)
tail(experiment_results)
```

5.
```r
# 5
ggplot(experiment_results, aes(x=n_observations, y=errors+1e-9)) + # avoid log(0)
    stat_summary(fun = mean, geom="line") + # Line averages over 1000 replicates
    scale_x_log10() +  # Show result in log-log scale
    scale_y_log10(n.breaks=16) +
    coord_cartesian(ylim = c(1e-3, 1)) +
    theme_minimal() +
    geom_point() +
    labs(x = "Number of observations",
        y = "Absolute error of the posterior mean")
```



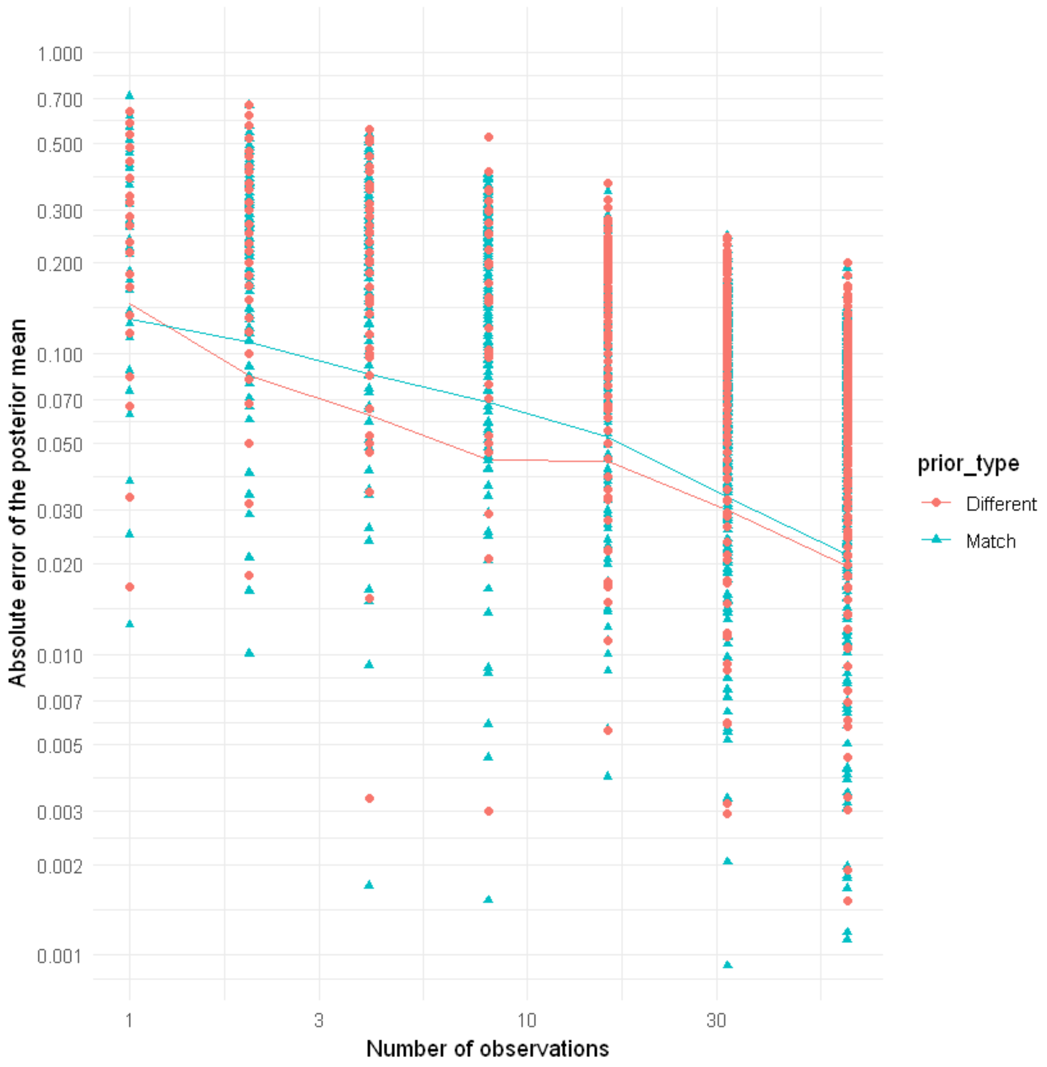
6.
```r
# 6
y7 = mean(experiment_results[experiment_results$n_observations == 2^6, ]$errors)
y5 = mean(experiment_results[experiment_results$n_observations==2^4, ]$errors)
(log10(y7) - log10(y5)) / (log10(2^6) - log10(2^4))
# -0.496968079903831
```

We fetch the corresponding value from the data frame we had in the previous part, and get:

$$\frac{y_7 - y_5}{x_7 - x_5} = \frac{\log_{10}(0.037) - \log_{10}(0.074)}{\log_{10}(2^6) - \log_{10}(2^4)} = -0.497$$

Let $k = \frac{y_j - y_i}{x_j - x_i}, j > i$, then $\frac{\log(\varepsilon_j/\varepsilon_i)}{\log(2^{j-i})} = k$, where ε_i is the ith error. So that means $2^{k(j-i)} = \frac{\varepsilon_j}{\varepsilon_i}$. So the error will be scaled by a factor of $2^{k(j-i)}$ between two errors ε_j and ε_i.

7.
```r
# 7
rho_true = 1:(K+1)
rho_prior = rep(1, K + 1)
new_results = data.frame()
for (n_obs in n_obs_vector) {
    errors = replicate(1000, simulate_posterior_mean_error(rho_true, rho_prior, n_obs))
    df = data.frame(n_observations = rep(n_obs, 1000), replication = (1:1000), errors =
    errors)
    new_results = rbind(new_results, df)
}
new_results$prior_type = rep("Different", 1000*length(n_obs_vector))
experiment_results$prior_type = rep("Match", 1000 * length(n_obs_vector))
all_results = rbind(experiment_results, new_results)

ggplot(all_results, aes(x=n_observations, y=errors+1e-9, # avoid log(0)
                        color=prior_type, shape=prior_type)) +
  stat_summary(fun = mean, geom="line") + # Line averages over 1000 replicates
  scale_x_log10() +  # Show result in log-log scale
  scale_y_log10(n.breaks=16) +
  coord_cartesian(ylim = c(1e-3, 1)) +
  theme_minimal() +
  geom_point() +
  labs(x = "Number of observations",
       y = "Absolute error of the posterior mean")
```



From the plot above, we can see that when number of observation is small, then the error from the wrong prior is larger than from the true prior. But as the number of observation increases, the error tends to decrease regardless of the initial choice of prior. In addition, the decreasing trend from the wrong prior seems to coincide with the trend from the true prior in the long term.