Solution 2: Bayes estimator

1 Q.1: define a Bayesian model

The data corresponds to the result of the first three launches: $Y_i = 1$. The unknowns are

- the success probability of a typical launch p, and
- the result of the fourth launch, Y_4.

The joint distribution of these random variables is

$$\begin{aligned} p &\sim \text{Discrete}(\{0, 1/K, 2/K, \dots, 1\}, \rho) \\ Y_i \mid p &\sim \text{Bern}(p), \quad i \in \{1, \dots, 4\}. \end{aligned}$$

2 Point estimates

To assist in creating the vector `prior_probabilities`, let us define a function that takes $p \in [0, 1]$ and evaluates the unnormalized prior at p

```r
prior_pmf <- function(x) {
  x * (1 - x)
}
```

Now we can simply do

```r
K <- 20
realizations = 0:K/K
prior_probabilities <- prior_pmf(realizations)
prior_probabilities <- prior_probabilities / sum(prior_probabilities)
```

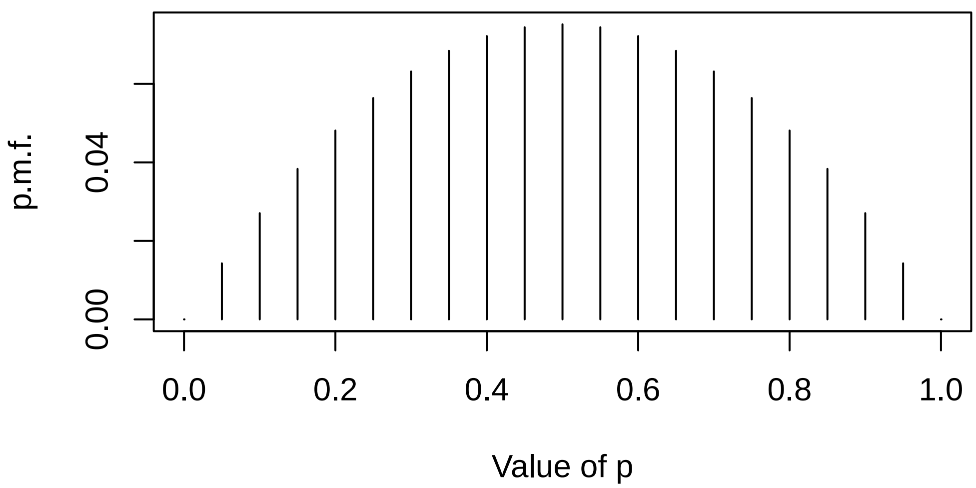
2.1 Plot the prior

Let us declare a function for plotting any distribution over our collection of `realizations`

```
plot_pmf <- function(probabilities) {
  plot(realizations, probabilities, type='h',
       xlab = "Value of p", ylab = "p.m.f.")
}
```

The plot of the prior can be obtained using

```
plot_pmf(prior_probabilities)
```



2.2 Plot the posterior

Define a function to produce posterior probabilities from the data and prior

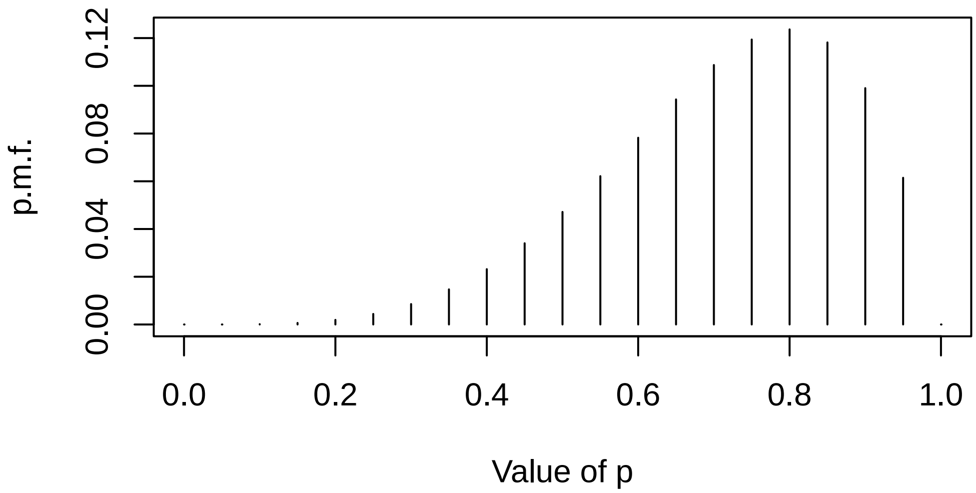
```
posterior = function(prior_probabilities, n_successes, n_observations) {
  gamma = prior_probabilities *
    dbinom(n_successes, n_observations, p = realizations)
  gamma / sum(gamma)
}
```

Compute the posterior probabilities

```
posterior_probabilities = posterior(prior_probabilities, 3, 3)
```

Inspect using the `plot_pmf` function from before

```
plot_pmf(posterior_probabilities)
```



2.3 Posterior statistics

We write a function that computes the mean and mode for any distribution over the vector `realizations`

```
summary_pmf = function(pmf){
  i_mode = which.max(pmf)
  posterior_mode = realizations[i_mode + 1]
  posterior_mean = sum(pmf * realizations)
  c(posterior_mean, posterior_mode)
}
```

The result of applying the above to `posterior_probabilities` is

```
posterior_summary = summary_pmf(posterior_probabilities)
cat(paste0("mean = ", round(posterior_summary[1], 2), ", ",
           "mode = ", posterior_summary[2]))
```

```
mean = 0.71, mode = 0.85
```

3 Bayes action

3.1 Loss function

$$\begin{aligned} L(a, y) &= 100 \cdot \mathbb{1}\{a = y = 0\} + 0 \cdot \mathbb{1}\{a = 0, y = 1\} + 2 \cdot \mathbb{1}\{a = 1\} \\ &= 100 \cdot \mathbb{1}\{a = y = 0\} + 2 \cdot \mathbb{1}\{a = 1\}. \end{aligned}$$

3.2 Expected loss function

$$\begin{aligned} \mathcal{L}(a) &= \mathbb{E}[L(a, y) | Y_{1:3} = (1, 1, 1)] \\ &= 2 \cdot \mathbb{1}\{a = 1\} + 100 \cdot \mathbb{1}\{a = 0\} \mathbb{E}[\mathbb{1}\{Y_4 = 0\} | Y_{1:3} = (1, 1, 1)] \\ &= 2 \cdot \mathbb{1}\{a = 1\} + 100 \cdot \mathbb{1}\{a = 0\} \mathbb{E}[1 - p | Y_{1:3} = (1, 1, 1)] \\ &\approx 2 \cdot \mathbb{1}\{a = 1\} + 30 \cdot \mathbb{1}\{a = 0\}. \end{aligned}$$

3.3 Recommendation

The owner should take the insurance because doing so minimizes the expected loss under the posterior distribution.