STAT 447C: Bayesian Statistics **Fall 2024**

Exercise 1

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1 sampling from a joint distribution

1. Since flips are indenpendent, then the joint pmf: $p(x, y_1, y_2, y_3, y_4) = p(y_1, y_2, y_3, y_4|x) \cdot p(x) = \prod_{i=1}^{4} p(y_i|x) \cdot p(x)$.

   Let $g(X, Y_1, Y_2, Y_3, Y_4) = (1 + Y_1)^X$, then by LOTUS,

$$
\begin{aligned}
&\mathbb{E}(1+Y_1)^X = \mathbb{E}g(X, Y_1, Y_2, Y_3, Y_4) \\
&= \sum_{x\in\{0,1,2\}} \sum_{y_1\in\{0,1\}} \sum_{y_2} \sum_{y_3} \sum_{y_4} g(x, y_1, y_2, y_3, y_4) \cdot p(x, y_1, y_2, y_3, y_4) \\
&= \sum_{x} \sum_{y_1} (1+y_1)^x \cdot p(y_1|x) \cdot p(x) \sum_{y_2} p(y_2|x) \sum_{y_3} p(y_3|x) \sum_{y_4} p(y_4|x) \\
&= \sum_{x} \sum_{y_1} (1+y_1)^x \cdot p(y_1|x) \cdot p(x) \\
&= \frac{13}{6}
\end{aligned}
$$

2.

```r
library(extraDistr)
set.seed(2024)
# theoretical expectation
X = c(0,1,2)
Y = c(0,1)
p_mtx = matrix(c(1,1/2,0,0,1/2,1), byrow = FALSE, nrow=3)
sum = 0
for (i in (1:length(X))) {
  x = X[i]
  for (j in (1:length(Y))) {
    y = Y[j]
    p = p_mtx[i,j]
    sum = sum + (1+y)^x * p *(1/3)
  }
}
sum #2.16666666666667

# sample function
forward_sample = function() {
  x = rdunif(1, 0, 2)
  Y = rbinom(4, 1, x/2)
  return (c(x, Y))
}
forward_sample()
# 1 1 0 1 0

# function of (x, Y)
f_eval = function(v) ((1 + v[2])^v[1])

# compute expectation
mean(replicate(f_eval(forward_sample()), n = 100000))
# 2.16713

```

3. we use the sample mean of the function values of $g = (1 + y_1)^x$ to approximate the expectation by the LLN
4. From the simulation results, if the number of simulation gets larger, then the sample mean is closer to the true expectation.

2 computing a conditional

1. If $p = \frac{1}{2}$, then $X = 1$, so we want: $\mathbb{P}(X = 1|(Y_1, Y_2, Y_3, Y_4) = (0, 0, 0, 0))$
2. Note that

$$\mathbb{P}(Y_i = 0, \forall i | X = 0) = 1$$

and

$$\mathbb{P}(Y_i = 0, \forall i | X = 1) = (\frac{1}{2})^4$$

and

$$\mathbb{P}(Y_i = 0, \forall i | X = 2) = 0$$

, so by Bayes rule and total probability:

$$\begin{aligned}
&\mathbb{P}(X = 1|(Y_1, Y_2, Y_3, Y_4) = (0, 0, 0, 0)) \\
&= \frac{\mathbb{P}(Y_i = 0, \forall i | X = 1) \cdot \mathbb{P}(X = 1)}{\sum_{j=0}^{2} \mathbb{P}(Y_i = 0, \forall i | X = j) \cdot \mathbb{P}(X = j)} \\
&= \frac{(\frac{1}{2})^4 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + (\frac{1}{2})^4 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} \\
&= \frac{1}{17}
\end{aligned}$$

3 non uniform prior on coin types

1.

$$\begin{aligned}
&X \sim \mathit{Categorical}((0, 1, 2)|(\frac{1}{100}, \frac{98}{100}, \frac{1}{100})) \\
&Y_i|X \sim \text{Bernoulli}(\frac{X}{2})
\end{aligned}$$

2.

$$\begin{aligned}
&\mathbb{P}(X = 1|(Y_1, Y_2, Y_3, Y_4) = (0, 0, 0, 0)) \\
&= \frac{\mathbb{P}(Y_i = 0, \forall i | X = 1) \cdot \mathbb{P}(X = 1)}{\sum_{j=0}^{2} \mathbb{P}(Y_i = 0, \forall i | X = j) \cdot \mathbb{P}(X = j)} \\
&= \frac{(\frac{1}{2})^4 \cdot \frac{98}{100}}{1 \cdot \frac{1}{100} + (\frac{1}{2})^4 \cdot \frac{98}{100} + 0 \cdot \frac{1}{100}} \\
&= \frac{98}{114}
\end{aligned}$$

4 a first posterior inference algorithm

1.

```
posterior_given_four_heads = function(rho) {
  K = length(rho)
  posterior = NULL
  sum = 0
```

```
    for (k in (1 : K)) {
      p_k = (1-(k-1)/(K-1))^4 * rho[k] #joint p(Y, X = k)
      sum = sum + p_k
      posterior = c(posterior, p_k)
    }
    # normalizing
    posterior = posterior / sum
    return (posterior)
  }
```

2.

```
    posterior_given_four_heads(c(1/100,98/100,1/100))[2] - 98/114
  # 0
```

3.

```
    rho = seq(1,10,1)
    rho = rho / sum(rho)
    posterior_given_four_heads(rho)
    # 0.201845869866174, 0.252022765728349, 0.221596677434241, 0.159483156437471,
    0.0961390555299185, 0.0472542685740655, 0.0174434702353484, 0.00393785571450546,
    0.000276880479926166, 0
```

From the posterior distribution, we can infer that when we have four heads: $(0,0,0,0)$, then it is most likely that $X = 2$. i.e. type 2 coin.

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