Project I

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Contents

1 Introduction

1.1 Introduction of the Model and of the Problem

Let $T > 0$. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a Brownian motion $(B_t)_{t\geq 0}$ is defined. We denote by $(\mathcal{F}_t)_{0\leq t\leq T}$ the natural filtration generated by $(B_t)_{t\geq 0}$.

The price process of the risky asset $(S_t)_{0\leq t\leq T}$ is assumed to satisfy

$$dS_t = S_t\big(\mu_t\,dt + \sigma\,dB_t\big),$$

where $(\mu_t)_{0\leq t\leq T}$ is a measurable adapted process verifying $\int_0^T \mu_s^2\,ds < +\infty$, and $\sigma > 0$ is the (constant) volatility.

A trading strategy is described by a pair of predictable processes $(H_t^0, \Delta_t)_{0\leq t\leq T}$, where H_t^0 denotes the number of units held in the money market account at time t, and Δ_t denotes the number of shares of the risky asset.

$$V_t := H_t^0 e^{rt} + \Delta_t S_t.$$

Assume that the strategy is self-financing and that the risk-free interest rate is constant and equal to r. It is equivalent to imposing the following stochastic differential equation (this is an imposed condition, not a mathematical deduction):

$$dV_t = \Delta_t\,dS_t + rH_t^0 e^{rt}\,dt = \Delta_t\,dS_t + r\big(V_t - \Delta_t S_t\big)\,dt.$$

The problem is to determine the value process $(V_t)_{0\leq t\leq T}$ of a European option. In our framework, this value will later be shown to coincide with the value of a self-financing trading strategy that replicates the option payoff at time T.[see Proposition 2.1]

In particular, we are interested in the initial value V_0, which corresponds to the option price.

For a European call option, the payoff is given by

$$V_T = (S_T - K)^+,$$

where $K > 0$ is the strike price. Similarly, for a European put option with the same strike $K > 0$, the payoff is

$$V_T = (K - S_T)^+.$$

1.2 Overview

The aim of this project is to study the pricing of European call and put options within the Black–Scholes model, using both analytical formulas and numerical methods.

We first implement the Black–Scholes closed-form pricing formulas and illustrate how the option price varies with respect to the main model parameters, such as the spot price, volatility, maturity and interest rate.

We then solve the Black–Scholes partial differential equation numerically using finite-difference methods. Three standard time discretisation schemes are considered: an explicit scheme, an implicit scheme and the Crank–Nicolson scheme. We compare the convergence properties of the different numerical schemes.

Finally, we study the binomial tree approach for European option pricing. The CRR binomial tree method is implemented for both call and put options and its convergence towards the Black–Scholes closed-form solution is investigated.

2 Analytic Solution

2.1 Solution

Consider the discounted price process

$$\tilde{S}_t := e^{-rt} S_t.$$

By Itô's formula, we obtain

$$d\tilde{S}_t = e^{-rt} dS_t - re^{-rt} S_t\, dt = \tilde{S}_t\big((\mu_t - r)\, dt + \sigma\, dB_t\big).$$

Define

$$\theta_t := \frac{\mu_t - r}{\sigma}, \qquad W_t := B_t + \int_0^t \theta_s\, ds.$$

Then

$$d\tilde{S}_t = \sigma \tilde{S}_t\, dW_t.$$

By Girsanov's theorem, under the probability measure $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(- \int_0^T \theta_s\, dB_s - \frac{1}{2} \int_0^T \theta_s^2\, ds \right),$$

the process $(W_t)_{0\le t\le T}$ is a Brownian motion. Consequently, the discounted price process $(\tilde{S}_t)_{0\le t\le T}$ is a martingale under $\mathbb{P}^*$.

Proposition 2.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function such that $f(S_T) \in L^2(\mathbb{P}^*)$, then there exists a strategy $(H_t^0, \Delta_t)_{0\le t\le T}$ whose portfolio value process $(V_t)_{0\le t\le T}$ satisfies*

$$V_T = f(S_T).$$

and for all $t \in [0, T]$, the portfolio value is of the form

$$V_t = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} f(S_T) \,\middle|\, \mathcal{F}_t \right].$$

Proof. Assume that there exists a strategy $(H_t^0, \Delta_t)_{0\le t\le T}$ with portfolio value process $(V_t)_{0\le t\le T}$ such that $V_T = f(S_T)$. Define the discounted processes

$$\tilde{S}_t := e^{-rt} S_t, \qquad \tilde{V}_t := e^{-rt} V_t.$$

Then, we have

$$dV_t = \Delta_t\, dS_t + r\big(V_t - \Delta_t S_t\big)\, dt.$$

Applying Itô's formula to $\tilde{V}_t = e^{-rt} V_t$ yields

$$d\tilde{V}_t = -re^{-rt} V_t\, dt + e^{-rt} dV_t = -r\tilde{V}_t\, dt + e^{-rt}\Big(\Delta_t\, dS_t + r(V_t - \Delta_t S_t)\, dt\Big) = \Delta_t\, d\tilde{S}_t = \Delta_t \sigma \tilde{S}_t\, dW_t.$$

So, under the measure $\mathbb{P}^*$ constructed previously, $(\tilde{V}_t)_{0\le t\le T}$ is a $\mathbb{P}^*$-martingale. In particular, for all $t \in [0, T]$,

$$\tilde{V}_t = \mathbb{E}^{\mathbb{P}^*}\left[\tilde{V}_T \,\middle|\, \mathcal{F}_t \right].$$

Since $\tilde{V}_T = e^{-rT} V_T = e^{-rT} f(S_T)$, we obtain

$$V_t = e^{rt} \tilde{V}_t = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} f(S_T) \,\middle|\, \mathcal{F}_t \right].$$

Now we prove the existence of such a strategy. Define the process

$$M_t := \mathbb{E}^{\mathbb{P}^*}\left[e^{-rT} f(S_T) \,\middle|\, \mathcal{F}_t \right], \qquad 0 \le t \le T.$$

Then $(M_t)_{0\le t\le T}$ is a $\mathbb{P}^*$-martingale and $M_T = e^{-rT} f(S_T)$. Since $f(S_T) \in L^2(\mathbb{P}^*)$, the martingale $M_t = \mathbb{E}^{\mathbb{P}^*}[e^{-rT} f(S_T) \mid \mathcal{F}_t]$ is square-integrable. Motivited by the argument before, we would like to find a strategy $(H_t^0, \Delta_t)_{0\le t\le T}$ such that $(M_t)_{0\le t\le T}$ is its associated discounted value process.

Since the filtration $(\mathcal{F}_t)$ is also the natural filtration of (W_t), by the Martingale Representation Theorem, there exists a predictable process $(K_t)_{0\le t\le T}$ such that

$$M_t = M_0 + \int_0^t K_s\, dW_s = M_0 + \int_0^t \frac{K_s}{\sigma \tilde{S}_s}\, d\tilde{S}_s, \qquad 0 \le t \le T.$$

Define

$$\Delta_t := \frac{K_t}{\sigma \tilde{S}_t},$$

$$H_t^0 := M_t - \Delta_t \tilde{S}_t, \qquad 0 \le t \le T.$$

Then the portfolio value at t is

$$V_t = H_t^0 S_t^0 + \Delta_t S_t = e^{rt} M_t.$$

Moreover,

$$dV_t = d(e^{rt} M_t) = r e^{rt} M_t \, dt + e^{rt} dM_t = r\big(V_t - \Delta_t S_t\big)\, dt + \Delta_t \, dS_t,$$

so the strategy (H_t^0, Δ_t) is self-financing.

Finally, we have also

$$V_T = e^{rT} M_T = f(S_T).$$

$\square$

Closed form for the Portfolio Value Proposition 2.1 tells us that the value of the option at time t is given by

$$V_t = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} f(S_T) \,\middle|\, \mathcal{F}_t \right].$$

Since we can deduce an explicit expression for S_t, $\quad 0 \le t \le T$

$$S_t = S_0 \exp\left(\int_0^t \mu_s \, ds - \tfrac{1}{2}\sigma^2 t + \sigma B_t \right) = \exp\big((rt - \tfrac{1}{2}\sigma^2) t + \sigma W_t \big).$$

we can write furthermore

$$V_t = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} f(S_T) \,\middle|\, \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} f(S_t \exp\Big(\Big(r - \tfrac{1}{2}\sigma^2\Big)(T-t) + \sigma(W_T - W_t)\Big)) \,\middle|\, \mathcal{F}_t \right]..$$

Since S_t is $\mathcal{F}_t$-measurable and the increment $W_T - W_t$ is independent of $\mathcal{F}_t$ under $\mathbb{P}^*$, it follows that the option value can be written as

$$V_t = F(t, S_t),$$

where the pricing function F is defined by

$$F(t,x) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)} f\Big(x \exp\Big(\Big(r - \tfrac{1}{2}\sigma^2\Big)(T-t) + \sigma(W_T - W_t)\Big)\Big) \right].$$

Since $W_T - W_t$ is a centered Gaussian random variable with variance $T-t$ under $\mathbb{P}^*$, the computation of F can be carried further with

$$f(x) = (x - K)^+.$$

Let $\theta := T - t$ and introduce the quantities

$$d_1 = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)\theta}{\sigma\sqrt{\theta}}, \qquad d_2 = d_1 - \sigma\sqrt{\theta},$$

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-u^2/2} \, du.$$

With these notations, one obtains

$$F(t,x) = \mathbb{E}\Big[\Big(x e^{\sigma\sqrt{\theta} g - \frac{1}{2}\sigma^2\theta} - K e^{-r\theta} \Big) \mathbf{1}_{\{g + d_2 \ge 0\}} \Big],$$

where $g \sim \mathcal{N}(0,1)$ under $\mathbb{P}$.

Equivalently, this expectation can be written as

$$F(t,x) = \int_{-d_2}^{+\infty} \left(x e^{\sigma\sqrt{\theta} y - \frac{1}{2}\sigma^2\theta} - K e^{-r\theta} \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy = \int_{-\infty}^{d_2} \left(x e^{-\sigma\sqrt{\theta} y - \frac{1}{2}\sigma^2\theta} - K e^{-r\theta} \right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy.$$

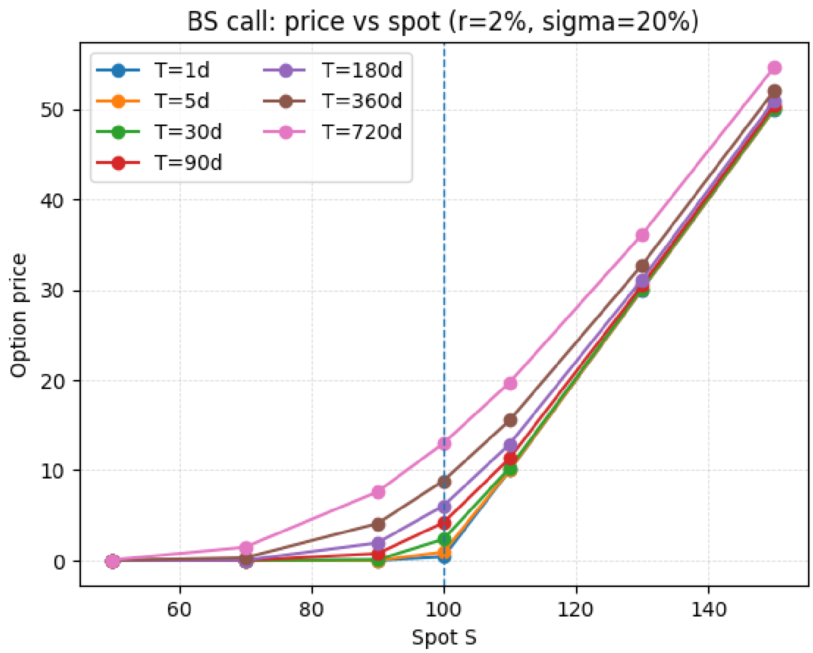
Splitting this expression into the difference of two integrals and performing, in the first one, the change of variable $z = y + \sigma\sqrt{\theta}$, we obtain

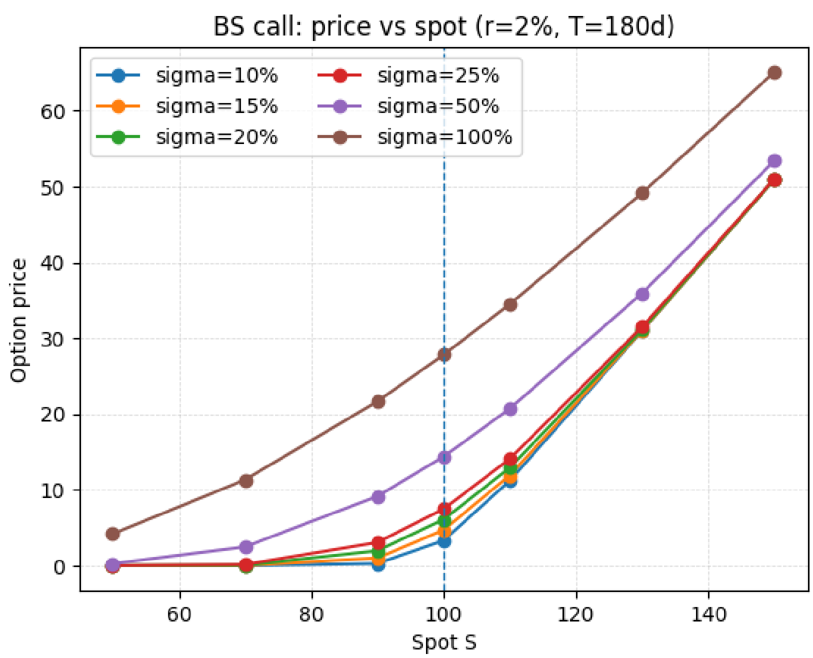
$$F(t,x) = x\,N(d_1) - Ke^{-r\theta}N(d_2),$$

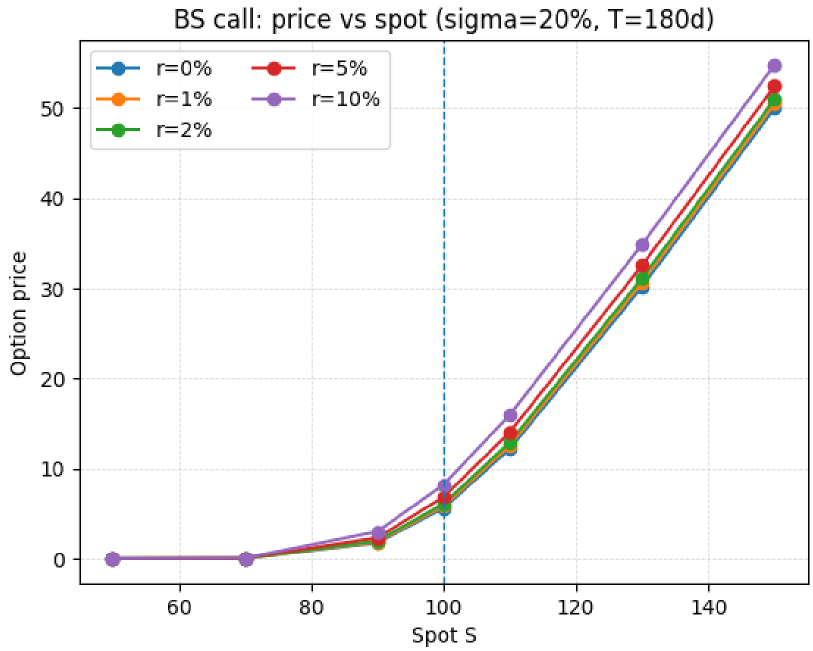
For a European put option, an analogous computation yields, with the same notations,

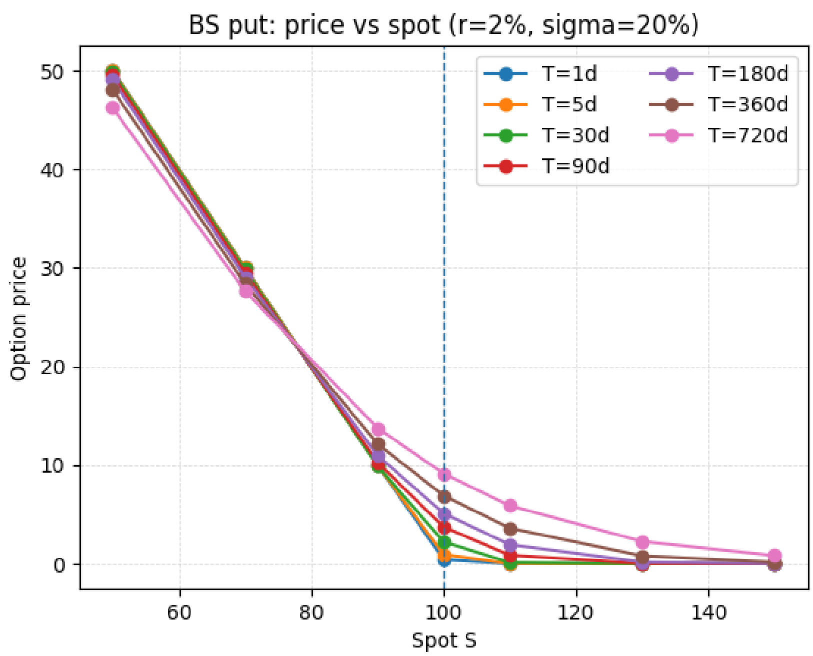
$$F(t,x) = Ke^{-r\theta}N(-d_2) - xN(-d_1).$$

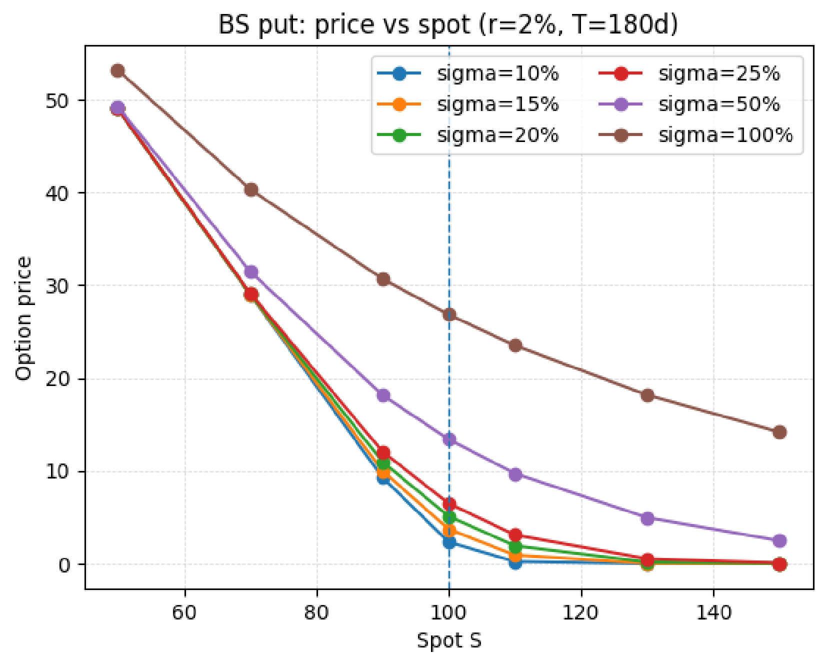
2.2 Graphic Representation of the Option Price

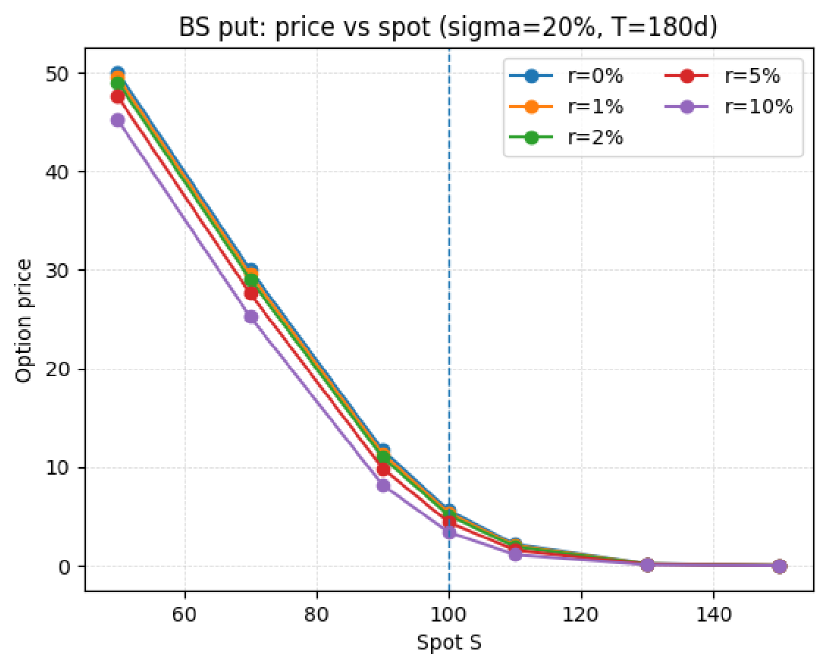





BS call: price vs spot (sigma=20%, T=180d)
r=0%
r=1%
r=2%
r=5%
r=10%
Option price
Spot S
0
10
20
30
40
50
60
80
100
120
140


BS put: price vs spot (r=2%, sigma=20%)
T=1d
T=5d
T=30d
T=90d
T=180d
T=360d
T=720d
Option price
Spot S
0
10
20
30
40
50
60
80
100
120
140





3 Finite-difference Method

3.1 Derivation of the Black–Scholes Equation

In Section 2, we have shown that there exists a function

$$v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}, \qquad (t, S) \mapsto v(t, S),$$

such that

$$V_t = v(t, S_t), \qquad V_T = f(S_T),$$

where

$$f(S) = \begin{cases} (S-K)^+, & \text{for a European call option,} \\ (K-S)^+, & \text{for a European put option.} \end{cases}$$

We assume that $v \in C^{1,2}$ with respect to (t,S). Applying Itô's formula to $v(t,S_t)$ yields

$$\begin{aligned} \mathrm{d}v(t,S_t) &= \Big(\partial_t v(t,S_t) + \mu S_t \partial_S v(t,S_t) + \tfrac{1}{2}\sigma^2 S_t^2 \partial^2_{SS} v(t,S_t)\Big)\,\mathrm{d}t \\ &\quad + \sigma S_t \partial_S v(t,S_t)\,\mathrm{d}B_t. \end{aligned}$$

On the other hand, for a self-financing portfolio with strategy Δ_t, the value process satisfies

$$\mathrm{d}V_t = \big(\Delta_t \mu S_t + rV_t - r\Delta_t S_t\big)\,\mathrm{d}t + \Delta_t \sigma S_t\,\mathrm{d}B_t.$$

By identification of the martingale and finite variation parts, we obtain

$$\begin{cases} \partial_S v(t,S_t) = \Delta_t, \\ \partial_t v(t,S_t) + \mu S_t \partial_S v(t,S_t) + \tfrac{1}{2}\sigma^2 S_t^2 \partial^2_{SS} v(t,S_t) = \Delta_t \mu S_t + r v(t,S_t) - r\Delta_t S_t. \end{cases}$$

We are therefore interested in solving the following Black–Scholes partial differential equation problems:

For a European call option

$$\boxed{\begin{cases} \partial_t v(t,S) + \tfrac{1}{2}\sigma^2 S^2 \partial^2_{SS} v(t,S) + rS\partial_S v(t,S) - rv(t,S) = 0, & (t,S) \in [0,T) \times (0,+\infty), \\ v(T,S) = (S-K)^+,, & S \in (0,+\infty), \\ v(t,0) = 0, & t \in [0,T), \\ v(t,S) \sim S \quad (S \to +\infty), & t \in [0,T). \end{cases}}$$

For a European put option

$$\boxed{\begin{cases} \partial_t v(t,S) + \tfrac{1}{2}\sigma^2 S^2 \partial^2_{SS} v(t,S) + rS\partial_S v(t,S) - rv(t,S) = 0, & (t,S) \in [0,T) \times (0,+\infty), \\ v(T,S) = (K-S)^+, & S \in (0,+\infty), \\ v(t,0) = Ke^{-r(T-t)}, & t \in [0,T), \\ v(t,S) \to 0 \quad (S \to +\infty), & t \in [0,T), \end{cases}}$$

By an appropriate change of variables, the Black–Scholes equation can be transformed into the heat equation, which is a well-posed problem.

We perform the following change of variables:

$$S = Ke^x, \qquad t = T - \frac{2\tau}{\sigma^2}, \qquad q = \frac{2r}{\sigma^2}.$$

We define the function u by

$$u(\tau,x) := \frac{1}{K}\exp\!\left(\frac{1}{2}(q-1)x + \frac{1}{4}(q+1)^2\tau\right) v\!\left(Ke^x,\, T - \frac{2\tau}{\sigma^2}\right) = \frac{1}{K}\exp\!\left(\frac{1}{2}(q-1)x + \frac{1}{4}(q+1)^2\tau\right) v(S,t).$$

With this transformation, the Black–Scholes equation is reduced to the heat equation

$$\boxed{\partial_\tau u(\tau,x) = \partial^2_{xx} u(\tau,x), \qquad (\tau,x) \in (0, \frac{1}{2}\sigma^2 T] \times \mathbb{R}.}$$

The initial condition is given by

$$u(0,x) = h(x), \qquad h(x) = \frac{\exp\big(\frac{1}{2}(q-1)x\big)}{K}\, f(Ke^x)\,,$$

where f denotes the payoff function.

For the asymptotic conditions:

European call option.

$$\begin{cases} u(\tau,x) \to 0, & x \to -\infty, \\ u(\tau,x) \sim \exp\Big(\frac{1}{2}(q+1)x + \frac{1}{4}(q+1)^2\tau\Big), & x \to +\infty, \end{cases} \qquad \tau \in [0, \tfrac{1}{2}\sigma^2 T].$$

European put option.

$$\begin{cases} u(\tau,x) \sim \exp\Big(\frac{1}{2}(q-1)x + \frac{1}{4}(q-1)^2\tau\Big), & x \to -\infty, \\ u(\tau,x) \to 0, & x \to +\infty, \end{cases} \qquad \tau \in [0, \tfrac{1}{2}\sigma^2 T].$$

3.2 Localisation and Finite-Difference Discretisation

3.2.1 Localisation (truncation of the spatial domain)

Since the spatial domain is unbounded, we localise the problem on a finite interval $[x_{\min}, x_{\max}]$ with $x_{\min} < 0$ and $x_{\max} > 0$. We then solve the heat equation on $(0, \Theta] \times [x_{\min}, x_{\max}]$ with Dirichlet boundary conditions obtained from the asymptotic behaviour.

European call option. We impose the truncated boundary conditions

$$u(\tau, x_{\min}) = 0, \qquad u(\tau, x_{\max}) = \exp\Big(\tfrac{1}{2}(q+1)x_{\max} + \tfrac{1}{4}(q+1)^2\tau\Big), \qquad 0 \le \tau \le \Theta.$$

European put option. We impose

$$u(\tau, x_{\min}) = \exp\Big(\tfrac{1}{2}(q-1)x_{\min} + \tfrac{1}{4}(q-1)^2\tau\Big), \qquad u(\tau, x_{\max}) = 0, \qquad 0 \le \tau \le \Theta.$$

3.2.2 Finite-difference discretisation and numerical schemes

Discretisation. Let $\Delta x > 0$ and $\Delta\tau > 0$. Define the spatial grid

$$x_i = x_{\min} + i\Delta x, \qquad i = 0, 1, \dots, N, \qquad (x_0 = x_{\min},\ x_N = x_{\max}),$$

and the time grid

$$\tau_m = m\Delta\tau, \qquad m = 0, 1, \dots, M, \qquad (\tau_M = \Theta).$$

We denote the approximation of $u(\tau_m, x_i)$ by u_i^m.

The initial condition is discretised as

$$u_i^0 = h(x_i), \qquad i = 0, 1, \dots, N.$$

For each $m = 0, 1, \dots, M$ we set the boundary values using the localisation conditions above:

$$u_0^m = u(\tau_m, x_{\min}), \qquad u_N^m = u(\tau_m, x_{\max}).$$

Numerical Schemes. We consider the θ-scheme: for $i = 1, \dots, N-1$ and $m = 0, \dots, M-1$,

$$\frac{u_i^{m+1} - u_i^m}{\Delta\tau} = (1-\theta)\,\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{(\Delta x)^2} + \theta\,\frac{u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1}}{(\Delta x)^2}, \qquad \theta \in [0,1].$$

Equivalently, setting $\alpha = \Delta\tau/(\Delta x)^2$, this can be written as

$$-\theta\alpha\, u_{i-1}^{m+1} + (1+2\theta\alpha)\, u_i^{m+1} - \theta\alpha\, u_{i+1}^{m+1} = (1-\theta)\alpha\, u_{i-1}^m + \big(1 - 2(1-\theta)\alpha\big)\, u_i^m + (1-\theta)\alpha\, u_{i+1}^m.$$

- If $\theta = 0$, the scheme is explicit. In this case, stability requires the CFL condition

$$\alpha = \frac{\Delta\tau}{(\Delta x)^2} \le \frac{1}{2}.$$

- If $\theta = 1$, the scheme is implicit.
- If $\theta = \frac{1}{2}$, the scheme is the Crank–Nicolson scheme.

3.3 Algorithm

Algorithm 1 θ-scheme for $\partial_\tau u = \partial^2_{xx} u$

Require: $[x_{\min}, x_{\max}]$, Θ (final time), N_x, N_τ (grid steps), h, g_L, g_R (boundary functions), $\theta \in [0,1]$
Ensure: $\mathbf{u}^{N_\tau} \in \mathbb{R}^{N_x-1}$ (interior values at $\tau = \Theta$)

$\Delta x \leftarrow (x_{\max} - x_{\min})/N_x, \quad \Delta\tau \leftarrow \Theta/N_\tau$
$\alpha \leftarrow \Delta\tau/(\Delta x)^2$
$x_i \leftarrow x_{\min} + i\Delta x$ for $i = 0, \ldots, N_x$
$\tau_m \leftarrow m\Delta\tau$ for $m = 0, \ldots, N_\tau$
$\mathbf{u}^0 \leftarrow (h(x_1), \ldots, h(x_{N_x-1}))^\top$
$M \leftarrow \mathrm{tridiag}(1, -2, 1)$
$A_\theta \leftarrow I - \theta\alpha M, \quad B_\theta \leftarrow I + (1-\theta)\alpha M$
for $m = 0, \ldots, N_\tau - 1$ **do**
  $u_0^m \leftarrow g_L(\tau_m), \quad u_{N_x}^m \leftarrow g_R(\tau_m)$
  $u_0^{m+1} \leftarrow g_L(\tau_{m+1}), \quad u_{N_x}^{m+1} \leftarrow g_R(\tau_{m+1})$
  $\mathbf{b}^m \leftarrow \mathbf{0},\ \mathbf{b}_1^m \leftarrow \alpha u_0^m,\ \mathbf{b}_{N_x-1}^m \leftarrow \alpha u_{N_x}^m$ ▷ boundary contribution at τ_m
  $\mathbf{b}^{m+1} \leftarrow \mathbf{0},\ \mathbf{b}_1^{m+1} \leftarrow \alpha u_0^{m+1},\ \mathbf{b}_{N_x-1}^{m+1} \leftarrow \alpha u_{N_x}^{m+1}$ ▷ boundary contribution at τ_{m+1}
  $\mathbf{r} \leftarrow B_\theta \mathbf{u}^m + (1-\theta)\mathbf{b}^m + \theta\mathbf{b}^{m+1}$ ▷ right-hand side term
  if $\theta = 0$ **then** ▷ explicit
    $\mathbf{u}^{m+1} \leftarrow \mathbf{r}$
  else ▷ implicit / Crank–Nicolson
    Solve the tridiagonal system $A_\theta\, \mathbf{u}^{m+1} = \mathbf{r}$
  end if
end for
return $\mathbf{u}^{N_\tau}$

3.4 Experiments

3.4.1 Comparison with the Black–Scholes closed-form solution

The numerical experiments are carried out for maturities

$$Tdays \in \{5,\, 360\} \text{ (days)}, \qquad T = Tdays/365$$

while keeping the other model parameters fixed:

$$K = 100, \qquad r = 2\%, \qquad \sigma = 20\%.$$

In order to solve the transformed heat equation on a bounded domain, the log-price variable

$$x = \log(S/K)$$

is truncated to a finite interval. More precisely, the computational domain is chosen as

$$x \in \big[\log(S_{\min}/K) - L,\ \log(S_{\max}/K) + L\big],$$

where $S_{\min}$ and $S_{\max}$ denote the smallest and largest spot values of interest, and $L > 0$ is a localisation parameter. In the numerical experiments, we set

$$L = 3.0,$$

which ensures that the artificial boundaries are placed sufficiently far from the region of interest, so that boundary effects do not significantly affect the solution in the spot range under consideration.

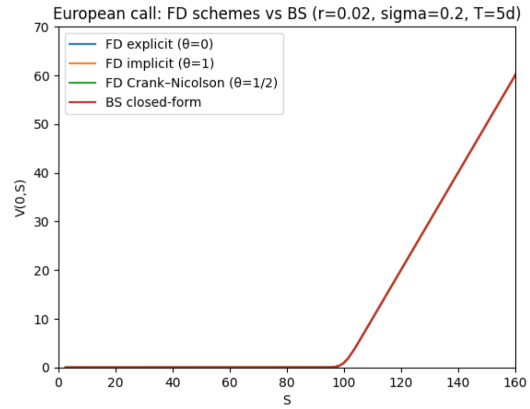
Once the computational domain has been fixed, the spatial and temporal discretisation parameters are chosen as

$$N_x = 400, \qquad N_\tau = 4000,$$

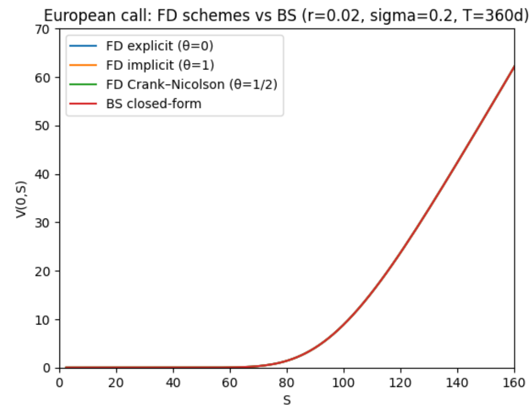
which provides a sufficiently fine grid for all three finite-difference schemes (explicit, implicit and Crank–Nicolson).

For each T, the numerical results are compared with the Black–Scholes closed-form solution over the spot range

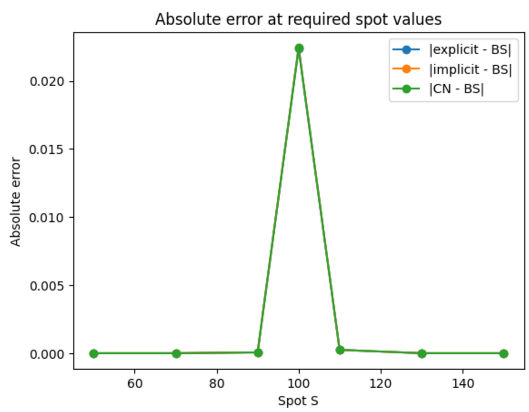
$$S \in \{50, 70, 90, 100, 110, 130, 150\}.$$



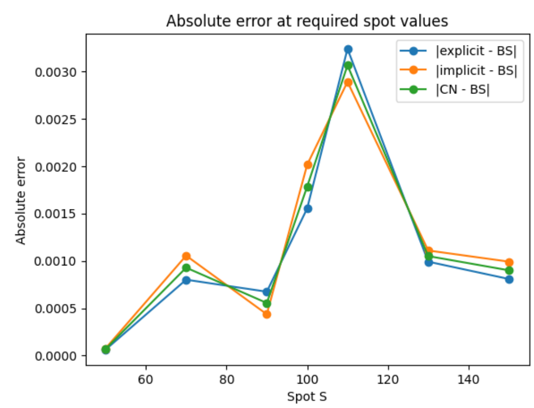
(a) $Tdays = 5$ days



(b) $Tdays = 360$ days

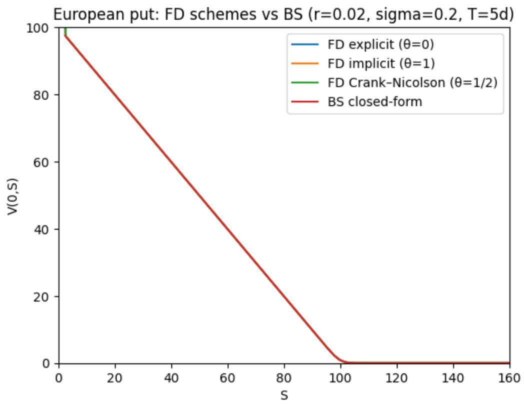


(c) $Tdays = 5$ days

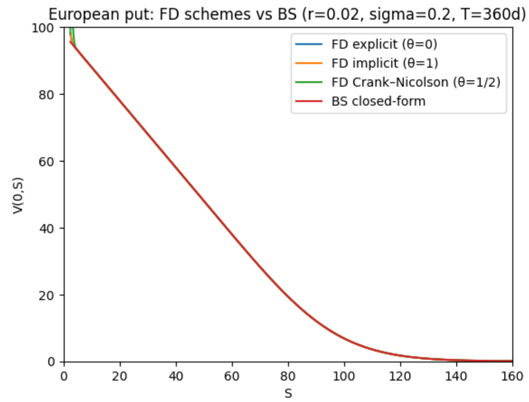


(d) $Tdays = 360$ days

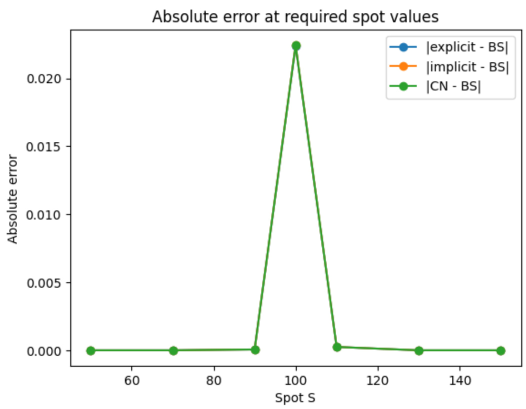
Figure 1: Top row: European call option price $V(0, S)$ computed with the explicit, implicit and Crank–Nicolson finite-difference schemes, compared with the Black–Scholes closed-form solution. Bottom row: Absolute error at selected spot values for the corresponding maturities.



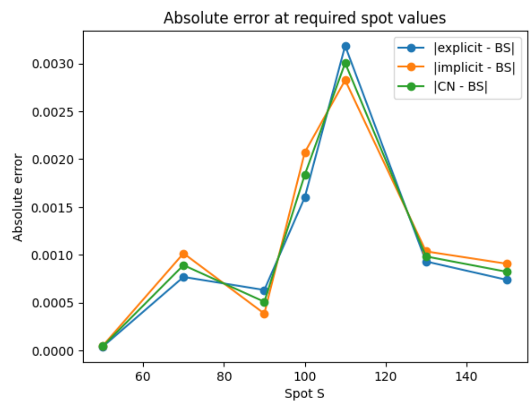
(a) $Tdays = 5$ days



(b) $Tdays = 360$ days



(c) $Tdays = 5$ days



(d) $Tdays = 360$ days

Figure 2: Top row: European put option price $V(0, S)$ computed with the explicit, implicit and Crank–Nicolson finite-difference schemes, compared with the Black–Scholes closed-form solution. Bottom row: Absolute error at selected spot values for the corresponding maturities.

Figure 1 and Figure 2 illustrate the comparison between the finite-difference solutions and the Black–Scholes closed-form prices for European call and put options, for short and long maturities ($T = 5$ days and $T = 360$ days). For the chosen discretisation parameters, the explicit, implicit and Crank–Nicolson schemes yield numerical prices that are visually indistinguishable from the analytical solution over the considered spot range, both for calls and puts.

For a very short maturity ($T = 5$ days), the absolute error is almost entirely concentrated near $S = K$, where the payoff function has a kink. This behaviour is observed for both call and put options and is directly related to the fact that the payoff is continuous but not differentiable at the strike.

When the maturity increases to $T = 360$ days, the error is no longer confined to a very narrow neighbourhood of the strike, but spreads over a wider range of spot values. Nevertheless, the largest errors are still attained close to $S = K$, while away from this region the numerical error remains small.

Finally, for both maturities and payoff types, the explicit, implicit and Crank–Nicolson schemes give very similar error levels on the chosen grid. In practice, this means that changing the time discretisation scheme does not noticeably improve the results, and that the remaining errors are mainly due to the payoff shape near the strike and to the truncation of the computational domain.

3.4.2 Grid refinement and Convergence

We investigate the convergence of the finite-difference schemes by refining the spatial grid and monitoring the corresponding error at time $t = 0$. The experiment is designed to study how the numerical error decreases as the grid resolution increases.

The localisation interval is defined as

$$x \in \left[\log(S_{\min}/K) - L,\ \log(S_{\max}/K) + L\right],$$

with $S_{\min} = 50$, $S_{\max} = 150$ and $L = 4.0$.

The spatial grid is refined by considering a sequence of resolutions

$$N_x \in \{100, 200, 400, 800\}.$$

For each spatial resolution, the corresponding grid size is $\Delta x = (x_{\max} - x_{\min})/N_x$. The number of time steps is then chosen according to a scaling $\Delta\tau \propto (\Delta x)^2$. This choice ensures the stability of the explicit scheme.

For each grid resolution, three time-stepping schemes are applied: the explicit scheme, the implicit Euler scheme, and the Crank–Nicolson scheme. Once the numerical solution is obtained, the option price $V^{\mathrm{FD}}(0, S)$ is reconstructed and compared with the Black–Scholes price $V^{\mathrm{BS}}(0, S)$ at a fixed set of spot values

$$S \in \{50, 70, 90, 100, 110, 130, 150\}.$$

The error is quantified using a discrete L^2 loss defined as the root mean square difference between the numerical and analytical prices,

$$\mathcal{L}_2 = \left(\frac{1}{N_S} \sum_{j=1}^{N_S} \bigl(V^{\mathrm{FD}}(0, S_j) - V^{\mathrm{BS}}(0, S_j) \bigr)^2 \right)^{1/2},$$

where N_S denotes the number of spot values. This loss is computed for each grid resolution and for each numerical scheme, and its evolution under grid refinement is used to assess convergence.

The resulting discrete L^2 losses are then plotted as functions of the spatial step size Δx on a log–log scale, allowing a direct comparison of the convergence behaviour of the explicit, implicit and Crank–Nicolson schemes.

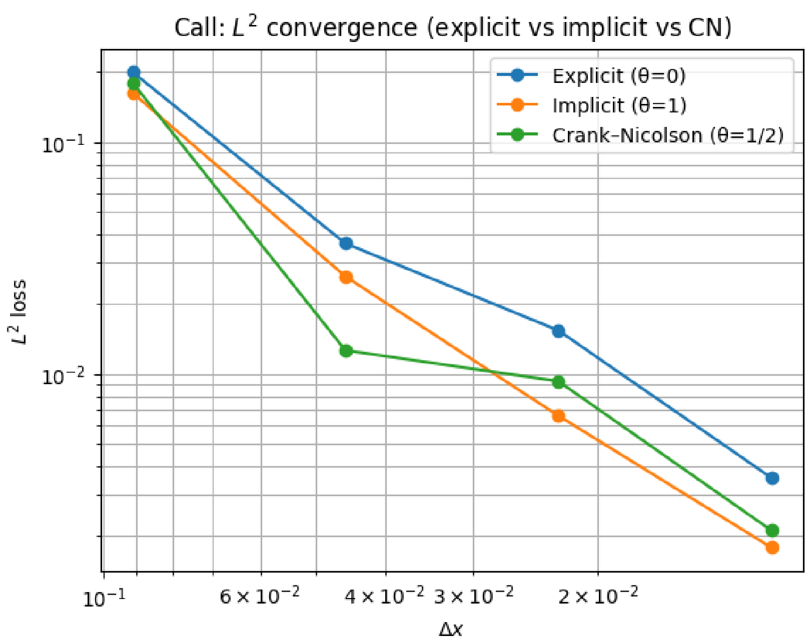


Figure 3: L^2 convergence for the European call option.

Figures 3 and 4 show the discrete L^2 error as a function of the spatial step size Δx for the European call option. As the spatial grid is refined, the error decreases steadily for all three finite-difference schemes, indicating convergence of the numerical solutions towards the Black–Scholes closed-form price.

For all grid resolutions, the explicit scheme yields the largest error, while the implicit Euler and Crank–Nicolson schemes provide significantly improved accuracy. The Crank–Nicolson scheme typically produces the smallest L^2 loss, although its performance remains close to that of the implicit scheme on the grids considered here.

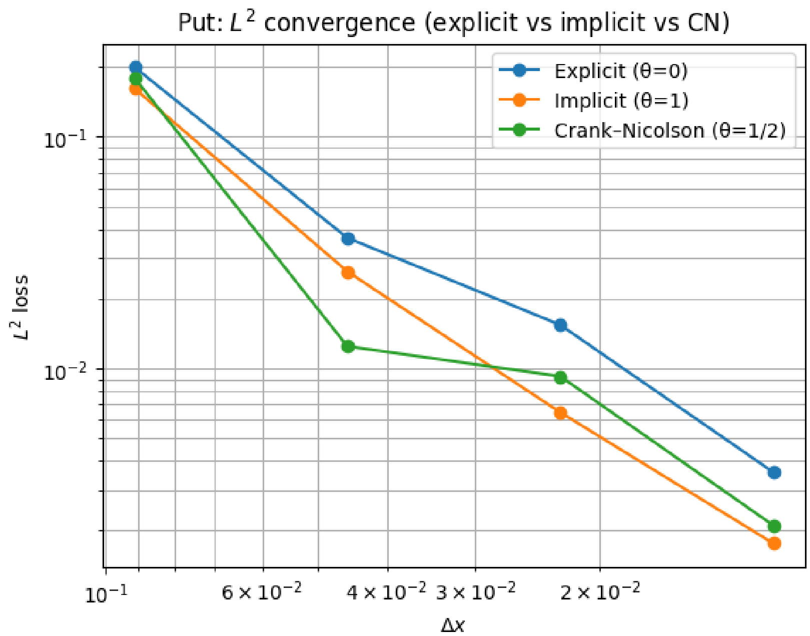


Figure 4: L^2 convergence for the European put option.

Across all grid resolutions, the explicit scheme consistently produces the largest error. The implicit Euler and Crank–Nicolson schemes are both significantly more accurate, and their curves remain close to each other. Depending on the grid size, either of the two can be slightly better: on the finest grids shown here, the implicit scheme achieves the smallest L^2 loss, while Crank–Nicolson is slightly lower on some intermediate resolutions.

4 The Binomial Tree Method

4.1 Intuition and Introduction of the Method

Let $0 < d < u$ and fix an integer $N \geq 1$. We set

$$\Delta t = \frac{T}{N}, \qquad t_i = i\,\Delta t, \quad i = 0, 1, \ldots, N.$$

We consider the measurable space

$$\Omega = \{u, d\}^N, \qquad \mathcal{F} = \mathcal{P}(\Omega).$$

The initial price $S_0 > 0$ is deterministic.

We define the canonical coordinate random variables

$$T_n : \Omega \to \{u, d\}, \qquad T_n(\omega_1, \ldots, \omega_N) = \omega_n, \quad n = 1, \ldots, N,$$

and the natural filtration

$$\mathcal{F}_n = \sigma(T_1, \ldots, T_n), \qquad \mathcal{F}_0 = \{\emptyset, \Omega\} \quad n = 1, \ldots, N.$$

The stock price process $(S_n)_{0 \leq n \leq N}$ is defined recursively by

$$S_{n+1} = T_{n+1}\, S_n, \qquad S_0 \text{ given},$$

that is,

$$S_{n+1} = \begin{cases} u\, S_n, & \text{if } T_{n+1} = u \\ d\, S_n, & \text{if } T_{n+1} = d. \end{cases}$$

Up to now, we have not specified any probability measure on the measurable space $(\Omega, \mathcal{F})$. The idea is to to determine the values of u, d and to construct a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_n))$ such that the discrete-time process (S_n) "behaves" in a similar way as the continuous-time price process (S_t) under the risk-neutral measure.

For the construction of $\mathbb{Q}$, we aim to find a probability measure $\mathbb{Q}$ such that the discounted price process

$$\widetilde{S}_n := e^{-rt_n} S_n$$

is a $\mathbb{Q}$-martingale.

It is easy to show by definition that for a probability measure $\mathbb{Q}$ on the filterd space $(\Omega, \mathcal{F}, (\mathcal{F}_n))$, $(\widetilde{S}_n)_{0 \leq n \leq N}$ is a $\mathbb{Q}$-martingale if and only if

$$\mathbb{E}^{\mathbb{Q}}[T_{n+1} \mid \mathcal{F}_n] = e^{r\Delta t}.$$

With this, we can prove the following proposition.

Proposition 4.1. *Let*

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

For a probability measure $\mathbb{Q}$ *on the filterd space* $(\Omega, \mathcal{F}, (\mathcal{F}_n))$, $(\widetilde{S}_n)_{0 \leq n \leq N}$ *is a* $\mathbb{Q}$*-martingale if and only if the random variables* $T_1, \dots, T_N$ *are i.i.d. under* $\mathbb{Q}$ *and*

$$\mathbb{Q}(T_n = u) = p, \qquad \mathbb{Q}(T_n = d) = 1 - p, \quad n = 1, \dots, N.$$

Proof. Assume first that the random variables $T_1, \dots, T_N$ are i.i.d. under $\mathbb{Q}$ and

$$\mathbb{Q}(T_n = u) = p, \qquad \mathbb{Q}(T_n = d) = 1 - p, \quad n = 1, \dots, N.$$

Then

$$\mathbb{E}^{\mathbb{Q}}[T_{n+1} \mid \mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[T_{n+1}] = pu + (1-p)d = e^{r\Delta t}.$$

Conversely, we now assume that $(\widetilde{S}_n)_{0 \leq n \leq N}$ is a $\mathbb{Q}$-martingale. Since T_{n+1} takes only the values u and d, we may write

$$\begin{cases} e^{r\Delta t} = \mathbb{E}^{\mathbb{Q}}[T_{n+1} \mid \mathcal{F}_n] = u\,\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_{n+1}=u\}} \mid \mathcal{F}_n\right] + d\,\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_{n+1}=d\}} \mid \mathcal{F}_n\right], \\ \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_{n+1}=u\}} \mid \mathcal{F}_n\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_{n+1}=d\}} \mid \mathcal{F}_n\right] = 1. \end{cases}$$

Solving this linear system yields

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_{n+1}=u\}} \mid \mathcal{F}_n\right] = p, \qquad \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_{n+1}=d\}} \mid \mathcal{F}_n\right] = 1 - p,$$

where

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

With this, it is easy to show by induction on n, that for any $(x_1, \dots, x_n) \in \{u, d\}^n$,

$$\mathbb{Q}(T_1 = x_1, \dots, T_n = x_n) = \prod_{i=1}^{n} p_i,$$

where

$$p_i = \begin{cases} p, & \text{if } x_i = u, \\ 1 - p, & \text{if } x_i = d. \end{cases}$$

This means that $T_1, \dots, T_N$ are i.i.d. under $\mathbb{Q}$ and

$$\mathbb{Q}(T_n = u) = p, \qquad \mathbb{Q}(T_n = d) = 1 - p, \quad n = 1, \dots, N.$$

$\square$

From this proposition, we understand that such a probability is unique and it exists. In fact, we can define it formally in this way: for all $\omega = (\omega_1, \dots, \omega_N) \in \Omega$,

$$\mathbb{Q}(\{\omega\}) := \prod_{i=1}^{N} p_i,$$

where

$$p_i := \begin{cases} p, & \text{if } \omega_i = u, \\ 1-p, & \text{if } \omega_i = d. \end{cases}$$

Of course, the probability measure $\mathbb{Q}$ depends on the values of u and d.

We know that for the continue process (S_t) ,

$$\log(S_{t+\Delta t}/S_t) \sim \mathcal{N}\left((r - \sigma^2/2)\Delta t, \sigma^2 \Delta t\right).$$

We 'simplify' this by setting

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}}.$$

In the following paragraph, we denote by

$$S_{i,j} := S_0\, u^{\,j} d^{\,i-j}, \qquad i = 0, 1, \dots, N, \quad j = 0, 1, \dots, i.$$

Here, the index i denotes the time layer, while j denotes the number of upward movements up to time i.

For $0 \le i \le N$ and $0 \le j \le i$, we define

$$V_{i,j} := e^{-r(T-t_i)} \mathbb{E}^{\mathbb{Q}}\left[f\left(S_{i,j} \prod_{n=i+1}^{N} T_n \right) \right],$$

where

$$f(x) = \begin{cases} (x-K)^+, & \text{(call option)}, \\ (K-x)^+, & \text{(put option)}. \end{cases}$$

It can be interpreted as the 'approximated' option price conditional on the information generated by the discrete-time process up to step i.

We hope to calculate $V_{0,0}$. Of course, in the case of the European option, with the result given by Proposition 4.1 (the independence of the increments (T_n)), we can give an explicit expression:

$$\boxed{V_{0,0} = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\left[f\left(S_0 \prod_{n=1}^{N} T_n \right) \right] = e^{-rT} \sum_{j=0}^{N} \binom{N}{j} p^{\,j} (1-p)^{\,N-j} f\big(S_0\, u^{\,j} d^{\,N-j}\big).}$$

$V_{0,0}$ can also be obtained by solving a dynamic system (backward).

Note that by the independence of the increments (T_n), we can write

$$\begin{aligned} V_{i,j} &= e^{-r(T-t_i)} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[f\left(S_{i,j} \prod_{n=i+1}^{N} T_n \right) \middle| \mathcal{F}_{i+1} \right] \right] \\ &= e^{-r(T-t_i)} \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{T_{i+1}=u\}}\, \mathbb{E}^{\mathbb{Q}}\left[f\left(S_{i,j} u \prod_{n=i+2}^{N} T_n \right) \right] + \mathbf{1}_{\{T_{i+1}=d\}}\, \mathbb{E}^{\mathbb{Q}}\left[f\left(S_{i,j} d \prod_{n=i+2}^{N} T_n \right) \right] \right]. \\ &= e^{-r\Delta t}\left(p\, V_{i+1,j+1} + (1-p)\, V_{i+1,j} \right). \end{aligned}$$

So we are interested in solving the following dynamic system (backward):

$$\boxed{\begin{cases} V_{N,j} = f(S_{N,j}), \\ V_{i,j} = e^{-r\Delta t}\big(p\, V_{i+1,j+1} + (1-p)\, V_{i+1,j} \big), \qquad 0 \le j \le i < N. \end{cases}}$$

4.2 Algorithm

Algorithm 2 Binomial Tree Pricing for a European Option

Require: $S_0 > 0$, strike $K > 0$, maturity $T > 0$, rate $r \in \mathbb{R}$, steps $N \in \mathbb{N}$, volatility σ, payoff $f : \mathbb{R}_+ \to \mathbb{R}$
Ensure: Option price $V_{0,0}$

```
Δt ← T/N
disc ← e^{−rΔt}
u ← e^{σ√Δt}
d ← e^{−σ√Δt}
p ← (e^{rΔt} − d) / (u − d)

for j = 0 to N do
    S_T[j] ← S_0 u^j d^{N−j}
end for

for j = 0 to N do
    V[j] ← f(S_T[j])
end for

for i = N − 1 down to 0 do
    for j = 0 to i do
        V[j] ← disc × (p V[j + 1] + (1 − p) V[j])
    end for
end for
return V[0]
```

4.3 Experiments

4.3.1 Comparison between the Binomial Tree and the Black–Scholes Closed Form

In this first numerical experiment, we fix the number of time steps in the binomial tree to $N = 100$ and compare the binomial tree method with the Black–Scholes closed-form solution for pricing European call and put option.

The parameters are chosen as follows:

- maturity: $Tdays = 180, T = Tdays/365$,
- risk-free interest rate: $r = 0.02$,
- volatility: $\sigma = 0.2$,
- strike price: $K = 100$,
- initial stock prices:
$$S_0 \in \{50, 70, 90, 100, 110, 130, 150\}.$$

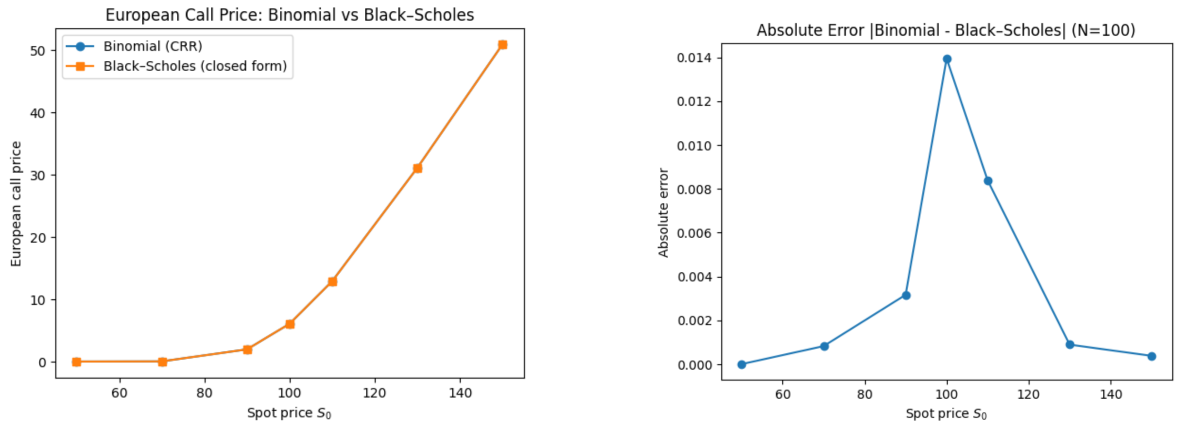


Figure 5: Comparison between the binomial tree method and the Black–Scholes closed-form solution for a European call option with $N = 100$ time steps.

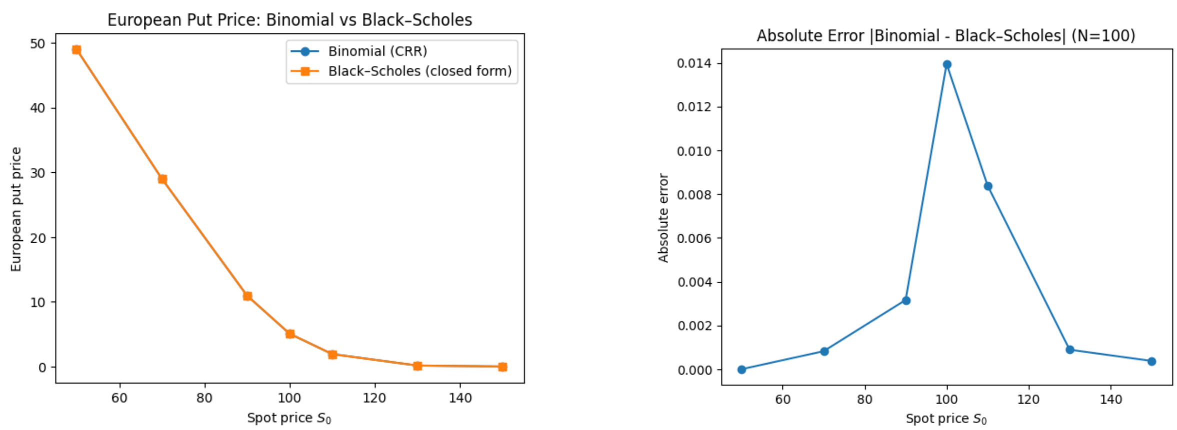


Figure 6: Comparison between the binomial tree method and the Black–Scholes closed-form solution for a European put option with $N = 100$ time steps.

For each value of S_0, we compute the price of European call and put options using both the binomial tree method and the Black–Scholes formula. The resulting option prices are plotted as functions of the initial stock price S_0.

The numerical results show that the two pricing methods are in very good agreement for all considered values of S_0, indicating that the binomial tree method already provides an accurate approximation of the Black–Scholes prices with a moderate number of time steps.

To further analyze the discrepancy between the two methods, we also plot the absolute error between the binomial tree prices and the Black–Scholes prices. The error is small in the deep out-of-the-money and deep in-the-money regions, while it reaches its maximum in the at-the-money region, that is when $S_0 \approx K$. This behavior is consistent with the lack of smoothness of the payoff functions at the strike price and reflects the higher sensitivity of option prices to numerical discretization errors in this region.

4.3.2 L^2 Convergence Analysis with Respect to the Number of Time Steps

In this experiment, we investigate the convergence behavior of the binomial tree method as the number of time steps N increases. The accuracy of the binomial tree prices is assessed by comparing them with the Black–Scholes closed-form solution.

We consider both European call and put options and fix all model parameters except for the number of time steps N. In particular, the parameters are chosen as in Section 4.3.1, and the option prices are evaluated on a fixed set of initial stock prices

$$S_0 \in \{50, 70, 90, 100, 110, 130, 150\}.$$

For each value of N, the option prices are computed using the binomial tree method and the Black–Scholes formula. The discrepancy between the two methods is quantified by a discrete L^2 error, defined as

$$\|V_N^{\text{Binomial}} - V^{\text{BS}}\|_{L^2} = \left(\frac{1}{|\mathcal{S}|} \sum_{S_0 \in \mathcal{S}} \left| V_N^{\text{Binomial}}(S_0) - V^{\text{BS}}(S_0) \right|^2 \right)^{1/2},$$

where $\mathcal{S}$ denotes the set of considered initial stock prices.

The number of time steps N is chosen in a geometrically increasing manner, namely $N = 2^k$ for $k = 4, \ldots, 10$, in order to clearly identify the convergence rate in a log–log scale. For each option type, the L^2 error is plotted as a function of N using logarithmic scales on both axes.

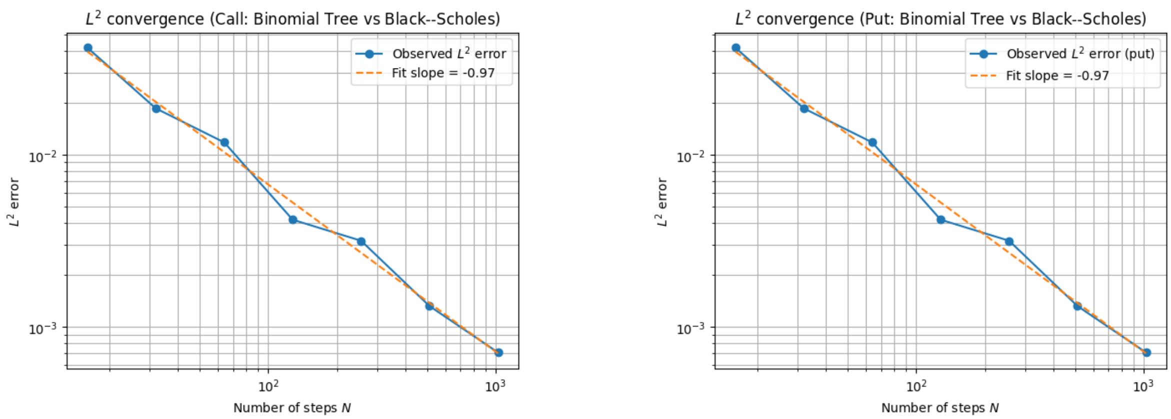


Figure 7: Log–log plots of the L^2 convergence of the binomial tree method towards the Black–Scholes closed-form solution as the number of time steps N increases. **Left:** European call option. **Right:** European put option.

The results are presented in Figure 7. In both the call and put cases, the L^2 error exhibits an approximately linear behavior in the log–log plots, indicating a polynomial convergence of the binomial tree method. A least-squares fit performed in log–log scale yields an estimated convergence rate close to first order for both payoff types.