

# Invariance principle of random projection for the norm

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## Abstract

Johnson-Lindenstrauss guarantees certain topological structure is preserved under random projections when embedding high dimensional deterministic vectors to low dimensional vectors. In this work, we try to understand how random projections affect norms of random vectors. In particular we prove the distribution of norm of random vectors  $X \in \mathbb{R}^n$ , whose entries are i.i.d. random variables, is preserved by random projection  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . More precisely,

$$\frac{X^T S^T S X - mn}{\sqrt{\sigma^2 m^2 n + 2mn^2}} \xrightarrow[m/n \rightarrow 0]{m, n \rightarrow \infty} \mathcal{N}(0, 1)$$

**Keywords:** random projection; Johnson-Lindenstrauss lemma; norm; invariance;

## 1 Introduction

Due to the internet boom and computer technology advancement in the last few decades, data collection and storage have been growing exponentially. With 'gold' mining demand on the enormous amount of data reaches to a new level, we are facing many technical challenges in understanding the information we have collected. In many different cases, including text and images, data can be represented as points or vectors in high dimensional space. On one hand, it is very easy to collect more and more information about the object so that the dimensionality grows quickly. On the other hand it is very difficult to analyze and create useful models for high dimensional data due to several reasons including computational difficulty as a result of curse of dimensionality and high noise to signal ratio. It is therefore necessary to reduce the dimensionality of the data while preserving the relevant structures.

The celebrated Johnson-Lindenstrauss lemma [6] states that random projections can be used as a general dimension reduction technique to embed topological structures in high dimensional Euclidean space into a low dimensional space without distorting its topology. Let us first recall the Johnson-Lindenstrauss lemma [4].

**Lemma 1** (Johnson and Lindenstrauss). *Given a set of vectors  $\{u_1, \dots, u_k\}$  in  $\mathbb{R}^n$ , for any  $m \geq 8\varepsilon^{-2} \log k$ , there exists a linear map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that*

$$(1 - \varepsilon)\|u_i - u_j\| \leq \|Au_i - Au_j\| \leq (1 + \varepsilon)\|u_i - u_j\|$$

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Given two fixed vectors  $X_1, X_2 \in \mathbb{R}^n$ , by Johnson-Lindenstrauss lemma, we can find a random projections  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that the projected distance  $\|AX_1 - AX_2\|$  has only a small distortion of the original distance  $\|X_1 - X_2\|$ . More precisely,

$$\left[1 - O\left(\frac{1}{\sqrt{m}}\right)\right] \|X_1 - X_2\|^2 \leq \|A(X_1 - X_2)\|^2 \leq \left[1 + O\left(\frac{1}{\sqrt{m}}\right)\right] \|X_1 - X_2\|^2 \quad (1.1)$$

Equivalently, this property can be reformulated as random projections preserves the inner product of two vectors (equivalence can be obtained by elementary computation and polarization identity). Namely given  $X_1, X_2$  two vectors in the unit ball of  $\mathbb{R}^n$  ( $\|X_1\| \leq 1, \|X_2\| \leq 1$ ), then there is a random projection  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$|\langle AX_1, AX_2 \rangle - \langle X_1, X_2 \rangle| \leq O\left(\frac{1}{\sqrt{m}}\right) \quad (1.2)$$

For general vectors not in the unit ball, the bound on the right hand side has the norms as a factor

$$|\langle AX_1, AX_2 \rangle - \langle X_1, X_2 \rangle| \leq O\left(\frac{1}{\sqrt{m}}\right) \|X_1\| \|X_2\|$$

The natural extension is to consider random vectors  $X_1, X_2$ . The question becomes how random projections affect random vectors. [5] showed there is an invariance phenomenon, the distribution of inner product is preserved. In particular, the following central limit theorems and Berry-Essen type rate of convergence were proved.

**Theorem 1** (Random projection CLT). *Given two independent random vectors in  $\mathbb{R}^n$ :  $X = [x_1, \dots, x_n]^T$ ,  $Z = [z_1, \dots, z_n]^T$  with  $\mathbb{E}x_i = \mathbb{E}z_i = 0$ ,  $\mathbb{E}x_i^2 = \mathbb{E}z_i^2 = 1$ ,  $\mathbb{E}|x_i|^3 \vee \mathbb{E}|z_i|^3 < A < \infty$ . Consider a random matrix  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with independent entries and  $\mathbb{E}s_{i,j} = 0$  and  $\mathbb{E}s_{i,j}^2 = 1$ . Further assume  $S, X, Z$  are all independent and  $\mathbb{E}s_{1,1}^8 \vee \mathbb{E}z_1^4 < c < \infty$ , then we have*

$$\frac{1}{\sqrt{m^2n + mn^2}} X^T S^T S Z \rightarrow \mathcal{N}(0, 1) \quad \text{as } m, n \rightarrow \infty$$

**Theorem 2** (Random projection invariance). *Given the same moments assumption as in Theorem 1, the following bounds hold.*

$$\begin{aligned} \sup_t \left| \mathbb{P}\left(\frac{1}{\sqrt{m^2n + mn^2}} X^T S^T S Z < t\right) - \mathbb{P}(G < t) \right| &\leq O\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right) \\ \sup_t \left| \mathbb{P}\left(\frac{1}{\sqrt{m^2n + mn^2}} X^T S^T S Z < t\right) - \mathbb{P}\left(\frac{1}{\sqrt{n}} X^T Z < t\right) \right| &\leq O\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right) \end{aligned}$$

In this work we try to extend the previous result to the case  $Z = X$ . Therefore obtain an invariance of distribution of the randomly projected norm. To put this in a high level perspective, we shall introduce the full inner product structure.

## 1.1 The full inner product structure

Suppose there are  $p$  independent random vectors  $X_1, \dots, X_p$  with i.i.d. entries (say  $x$ ), and a random projection matrix  $S$  of dimension  $m \times n$  with i.i.d. entries (say  $s$ ). The full inner product structure of those vectors and  $S$ -projected vectors concerns two random matrices each collecting  $p^2$  inner products:

$$\Sigma_{i,j} = \langle X_i, X_j \rangle, \quad \Sigma_{i,j}^S = \langle SX_i, SX_j \rangle \quad \forall 1 \leq i, j \leq p$$

In the previous work ([5]), the cases that  $i \neq j$  are addressed. Summarizing Theorem 1 and Theorem 2, so far we know with proper moments control, namely  $\mathbb{E}x = \mathbb{E}s = 0, \mathbb{E}x^2 = \mathbb{E}s^2 = 1, \mathbb{E}s^8 \wedge x^4 < \infty$ , then the properly centered and scaled entries of  $\Sigma^S$  converges to standard normal distribution.

$$\frac{\Sigma_{i,j}^S}{\sqrt{m^2n + mn^2}} \xrightarrow{m,n \rightarrow \infty} \mathcal{N}(0,1), \quad \forall i \neq j$$

Also the rate of convergence

$$\sup_t \left| \mathbb{P}\left(\frac{\Sigma_{i,j}^S}{\sqrt{m^2n + mn^2}} < t\right) - \mathbb{P}\left(\frac{1}{\sqrt{n}}\Sigma_{i,j} < t\right) \right| \leq O\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right), \quad \forall i \neq j$$

So we have a good understanding the off diagonal terms. We will focus on the remaining  $p$  diagonal terms in this work. Namely, we will try to understand how random projection affects the distribution of the norm of a random vector  $X$ ,

$$\langle X, X \rangle, \langle SX, SX \rangle$$

Before getting into technicality, we will first use techniques similar to the proof of Johnson-Lindenstrauss lemma to obtain a loose concentration result (Theorem 3) in Section 2.

$$\mathbb{P}\left(\frac{1}{\sqrt{m^2n}} |\Sigma_{i,i}^S - mn| > 2t\right) < 5e^{\{-\frac{1}{2048} \min(t^2(1 \wedge C), t\sqrt{n}(1 \wedge C), n)\}}, \quad C = \frac{m}{n}$$

Such concentration properties allow one to analyze the problem from error control perspective. In particular, this shows the randomly projected norm admits a sub-Gaussian behavior.

In Section 3, we would go one step further to deal with the distribution directly and we show the distribution of norm is invariant under random projections (Theorem 4), namely the random projected norm converges to normal distribution after properly centered and scaled.

$$\frac{\Sigma_{i,i}^S - mn}{\sqrt{\sigma^2 m^2 n + 2mn^2}} \xrightarrow[m/n \rightarrow 0]{m,n \rightarrow \infty} \mathcal{N}(0,1), \quad \forall i; \text{ where } \sigma^2 = \mathbb{E}x^4 - 1$$

Lastly, we outline open questions and possible future work in Section 4.

## 2 Concentration of projected norm for sub-Gaussian variables

The purpose of this section is to show the randomly projected norm is concentrated around both the original random norm and expectation of the norm. Concentration inequalities concern the tails of a random quantity deviates from its mean, which are very powerful tools in many applications ([8, 2, 15]). Johnson-Lindenstrauss Lemma 1 itself is a result of concentration inequality for sub-Gaussian random matrix over fixed vectors (see [2, 15]).

Concentration properties of random quadratic forms involving either deterministic vectors or deterministic matrix have been well-studied in the literature (see [12, 10, 14, 15]). Most of the existing results control the tail probability of the distortion by the matrix or expected distortion quantitatively. In Section 2, concentration result in our setting, random matrix acting on random vectors, will be easily obtained by carrying out a similar argument of proving Johnson-Lindenstrauss lemma. First let us recall some properties of sub-Gaussian and sub-exponential random variables.

**Definition 2.** We say  $X$  ( $\mathbb{E}X = \mu_X$ ) is a sub-Gaussian random variable with variance  $v := \mathbb{E}(X - \mu_X)^2$  if

$$\mathbb{P}(|X - \mu_X| > t) \leq 2e^{-\frac{t^2}{2v}}, \quad \forall t \geq 0$$

$Y$  ( $\mathbb{E}Y = \mu_Y$ ) is a sub-exponential random variable with parameter  $b$

$$\mathbb{P}(|Y - \mu_Y| > t) \leq 2e^{-\frac{t}{b}}$$

which admits a sub-Gaussian behavior when  $t$  is small since

$$2e^{-\frac{t}{b}} \leq \begin{cases} 2e^{-\frac{t^2}{c}}, & 0 \leq t \leq c/b \\ 2e^{-\frac{t}{b}}, & t \geq c/b \end{cases}$$

It is not hard to verify sub-Gaussian tail property is equivalent to

$$\mathbb{E} \exp \left\{ \frac{(X - \mu_X)^2}{2v} \right\} \leq 2, \quad \text{or} \quad \mathbb{E} e^{(X - \mu_X)t} \leq e^{\frac{vt^2}{2}}$$

Similarly sub-exponential tail property is equivalent to

$$\mathbb{E} \exp \left\{ \frac{|Y - \mu_Y|}{b} \right\} \leq 2, \quad \text{or} \quad \mathbb{E} e^{(Y - \mu_Y)t} \leq e^{\frac{4b^2 t^2}{1 - bt}} \quad \forall 0 \leq t \leq \frac{1}{b}$$

By further restricting  $t$ , we see

$$\mathbb{E} e^{(Y - \mu_Y)t} \leq e^{8b^2 t^2} \quad \forall 0 \leq t \leq \frac{1}{2b} \quad (2.1)$$

For a centered sub-Gaussian  $X$  with variance  $v$ , it is easy to see  $X^2$  has sub-exponential tail decay. Because  $\mathbb{P}(X^2 > t) = \mathbb{P}(|X| > \sqrt{t}) \leq 2e^{-\frac{t}{2v}}$ . The centered version  $X^2 - v$  has tails shifted by a constant  $v$  thus again admits sub-exponential decay. One can also calculate

$$\mathbb{E} e^{|X^2 - v|/4v} \leq \mathbb{E} e^{(X^2 + v)/4v} \leq e^{\frac{1}{4}} \mathbb{E} \sqrt{e^{X^2/2v}} \leq e^{\frac{1}{4}} \sqrt{\mathbb{E} e^{X^2/2v}} < 2 \quad (2.2)$$

Thus we conclude  $X^2 - v$  is sub-exponential with  $b \leq 4v$ . For extensive detailed discussions and proofs of the properties, we refer to [15, 2].

**Theorem 3.** Given  $X = [x_1, \dots, x_n]^T$ . Let  $A = \frac{1}{\sqrt{m}}S$  be a  $m \times n$  random matrix. Let all random variables  $x_i, S_{i,j}$  are independent identically distributed with mean 0 and variance 1. Suppose all random variables are sub-Gaussian, then

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \|AX\|^2 - n \right| > 2t \right) < 5e^{\{-\frac{1}{2048} \min(t^2(1 \wedge C), t\sqrt{n}(1 \wedge C), n)\}} \quad (2.3)$$

where  $C = \frac{m}{n}$ .

**Remarks.** It is clear

$$\frac{1}{\sqrt{n}} (\|AX\|^2 - n) = \frac{1}{m\sqrt{n}} (X^T S S X - mn)$$

which we will see later has the same scaling as in the CLT type result. This concentration result actually suggest the centered and rescaled projected norm has a tail that decays at a sub-Gaussian rate when  $t \leq \sqrt{n}$ . Thus a Gaussian limit which we will prove in Section 3 is not surprising.

**Proof:** For any random variables  $a, b, c$ , if  $|a - b| > 2\varepsilon$  then either  $|a - c| > \varepsilon$  or  $|b - c| > \varepsilon$ . That implies  $\mathbb{P}(|a - b| > 2\varepsilon) \leq \mathbb{P}(|a - c| > \varepsilon) + \mathbb{P}(|b - c| > \varepsilon)$ . Therefore plug in  $a, b, c$  with corresponding quantities, we find

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} \left| \|AX\|^2 - n \right| > 2t\right) \leq \mathbb{P}\left(\frac{1}{\sqrt{n}} \left| \|AX\|^2 - \|X\|^2 \right| > t\right) + \mathbb{P}\left(\frac{1}{\sqrt{n}} \left| n - \|X\|^2 \right| > t\right) \quad (2.4)$$

Notice  $x_i$  are i.i.d. sub-Gaussian with variance  $v = 1$ , then  $(x_i^2 - 1)$  are i.i.d. sub-exponential random variables with parameter  $b \leq 4$  (see Eq. (2.2)). Then we use Chernoff type argument (or apply Bernstein's concentration inequality directly) to calculate

$$\begin{aligned} \mathbb{P}\left(\frac{1}{\sqrt{n}} (\|X\|^2 - n) > t\right) &= \mathbb{P}\left(e^{\frac{\lambda}{\sqrt{n}}(\|X\|^2 - n)} > e^{\lambda t}\right) \\ &\leq e^{-\lambda t} \mathbb{E}\left[e^{\frac{\lambda}{\sqrt{n}}(\|X\|^2 - n)}\right] \\ &= e^{-\lambda t} \prod_{i=1}^n \left[\mathbb{E} e^{\frac{\lambda}{\sqrt{n}}(x_i^2 - 1)}\right] \end{aligned}$$

which holds for any  $\lambda \geq 0$ . We know for any  $\lambda/\sqrt{n} \leq \frac{1}{8}$ , we have  $\mathbb{E} e^{\frac{\lambda}{\sqrt{n}}(1 - x_i^2)} \leq e^{128\lambda^2/n}$ , then we obtain

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} (n - \|X\|^2) > t\right) \leq e^{-\lambda t + 128\lambda^2}$$

Minimize the right hand under the constraint  $\lambda \leq \frac{\sqrt{n}}{8}$ , we find optimal  $\lambda^* = \min(\frac{\sqrt{n}}{8}, \frac{t}{256})$ . Therefore we obtain

$$\begin{aligned} \mathbb{P}\left(\frac{1}{\sqrt{n}} (\|X\|^2 - n) > t\right) &\leq \begin{cases} \exp\{-\frac{t^2}{512}\}, & \text{if } t < 32\sqrt{n} \\ \exp\{2n - \frac{t\sqrt{n}}{8}\} \leq \exp\{-\frac{t\sqrt{n}}{16}\}, & \text{if } t \geq 32\sqrt{n} \end{cases} \\ &= \exp\{-\min(\frac{t^2}{512}, \frac{t\sqrt{n}}{16})\} \end{aligned} \quad (2.5)$$

Repeat the same argument for  $\frac{1}{\sqrt{n}} (n - \|X\|^2)$ , we find the other half admit same tail bound, thus we conclude

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} |n - \|X\|^2| > t\right) < 2 \exp\{-\min(\frac{t^2}{512}, \frac{t\sqrt{n}}{16})\} \quad (2.6)$$

We see as  $n \rightarrow \infty$ , the tail has a Gaussian behavior namely  $e^{-ct^2}$  which coincide with the CLT of  $\frac{1}{\sqrt{n}} (n - \|X\|^2) \rightarrow \mathcal{N}(0, \mathbb{E}(x_1^2 - 1)^2)$ .

Now we only need to look at how much  $\|AX\|^2$  deviates from  $\|X\|^2$ .

$$\|AX\|^2 = \sum_{i=1}^m (A_{i,\cdot} X)^2 = \sum_{i=1}^m \left( \sum_{j=1}^n A_{i,j} x_j \right)^2 = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n A_{i,j} A_{i,k} x_j x_k$$

Now we can use conditioning on  $X$  and only deal with the conditional probability.

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} \left| \|AX\|^2 - \|X\|^2 \right| > t\right) = \mathbb{E}\left[\mathbb{P}\left(\frac{1}{\sqrt{n}} \left| \|AX\|^2 - \|X\|^2 \right| > t \middle| X\right)\right]$$

In which case, we can think of  $X$  being fixed when computing conditional probability. To simplify notation, define

$$y_i := \left[ \left( \sum_{j=1}^n A_{i,j} x_j \right) \middle| X \right], \quad 1 \leq i \leq m$$

Therefore  $\|AX\|^2 | X = \sum_{i=1}^m y_i^2$ . Notice  $A_{i,j}, A_{i,k}$  are independent if  $j \neq k$ , we find

$$\begin{aligned} \mathbb{E} y_i^2 &= \mathbb{E} \left[ \left( \sum_{j=1}^n A_{i,j} x_j \right)^2 \middle| X \right] \\ &= \mathbb{E} \left[ \sum_{j=1}^n \sum_{k=1}^n A_{i,j} A_{i,k} x_j x_k \middle| X \right] \\ &= \sum_{j=1}^n x_j^2 \mathbb{E} A_{i,j}^2 \\ &= \sum_{j=1}^n x_j^2 \frac{1}{m} \mathbb{E} S_{i,j}^2 = \frac{1}{m} \|X\|^2 \end{aligned}$$

And this also shows conditional expectation of projected norm is the original norm

$$\mathbb{E} [\|AX\|^2 | X] = \sum_{i=1}^m \mathbb{E} [y_i^2 | X] = \|X\|^2$$

We may rewrite the tail of norms in terms of random variable  $y_i$ ,

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} |\|AX\|^2 - \|X\|^2| > t \right) = \mathbb{E} \left[ \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) \right]$$

Now we prove  $y_i^2 - \mathbb{E} y_i^2$  is sub-exponential random variables so that we can obtain Bernstein's type inequality again for the sum. Then it suffices to prove  $y_i$  is sub-Gaussian.

$$\begin{aligned} \mathbb{E}[e^{y_i t}] &= \mathbb{E}[e^{(A_{i,\cdot} X) t} | X] = \mathbb{E} e^{t \sum_{j=1}^n A_{i,j} x_j} = \prod_{j=1}^n \mathbb{E} e^{x_j t A_{i,j}} = \prod_{j=1}^n \mathbb{E} e^{x_j t \frac{S_{i,j}}{\sqrt{m}}} \\ &\leq \prod_{j=1}^n e^{x_j^2 t^2 / 2m} \\ &= e^{(\sum_{j=1}^n x_j^2 t^2 / 2m)} \\ &= e^{(\|X\|^2 / m) \frac{t^2}{2}} \end{aligned}$$

This shows  $y_i$  is sub-Gaussian with variance  $v \leq \|X\|^2 / m$ . Therefore  $y_i^2 - \mathbb{E} y_i^2$  is sub-exponential with  $b \leq 4\|X\|^2 / m$ . Then with Chernoff's method, we calculate

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) \leq 2e^{-\lambda t} \prod_{i=1}^m \mathbb{E} e^{\frac{\lambda}{\sqrt{n}} (y_i^2 - \mathbb{E} y_i^2)}$$

which holds for any  $\lambda \geq 0$ . We know for any  $\lambda/\sqrt{n} \leq \frac{m}{8\|X\|^2}$ , we have (see Eq. (2.1))

$$\mathbb{E} e^{\frac{\lambda}{\sqrt{n}}(y_i^2 - \mathbb{E} y_i^2)} \leq \exp \left\{ 128 \frac{(\|X\|^2)^2}{m^2} \lambda^2 / n \right\}$$

then we obtain

$$\begin{aligned} \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) &\leq 2 \exp \left\{ -\lambda t + m 128 \frac{(\|X\|^2)^2}{m^2} \lambda^2 / n \right\} \\ &= 2 \exp \left\{ -\lambda t + 128 \frac{(\|X\|^2)^2}{mn} \lambda^2 \right\} \end{aligned}$$

Optimize the right hand side under the constraint  $\lambda \leq \frac{\sqrt{nm}}{8\|X\|^2}$ , we find optimal  $\lambda^* = \min(\frac{\sqrt{nm}}{8\|X\|^2}, \frac{t mn}{256(\|X\|^2)^2})$ . Therefore we obtain

$$\begin{aligned} \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) &\leq \begin{cases} 2 \exp\{-\frac{t^2 mn}{512(\|X\|^2)^2}\}, & \text{if } t < 32\|X\|^2/\sqrt{n} \\ 2 \exp\{2m - \frac{t\sqrt{n} m}{8\|X\|^2}\}, & \text{if } t \geq 32\|X\|^2/\sqrt{n} \end{cases} \\ &\leq \begin{cases} 2 \exp\{-\frac{t^2 C n^2}{512(\|X\|^2)^2}\}, & \text{if } t < 32\|X\|^2/\sqrt{n} \\ 2 \exp\{2Cn - \frac{t\sqrt{n} C n}{8\|X\|^2}\}, & \text{if } t \geq 32\|X\|^2/\sqrt{n} \end{cases} \end{aligned}$$

where  $C := \frac{m}{n} \geq 0$  is constant. Notice when  $t \geq 32\|X\|^2/\sqrt{n}$ , we have  $1 \leq \frac{t\sqrt{n}}{32\|X\|^2}$ . We calculate  $2Cn - \frac{t\sqrt{n} C n}{8\|X\|^2} \leq 2Cn \frac{t\sqrt{n}}{32\|X\|^2} - \frac{t\sqrt{n} C n}{8\|X\|^2} = -\frac{t\sqrt{n} C n}{16\|X\|^2}$ . Therefore

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) \leq 2 \exp \left\{ -\min \left( \frac{t^2 C n^2}{512(\|X\|^2)^2}, \frac{t\sqrt{n} C n}{16\|X\|^2} \right) \right\}$$

Then we need to take expectation with respect to  $X$  since  $y_i$  are conditional random variables. In that procedure, we shall split into two sets  $\mathcal{A} := \{|\|X\|^2 - n| \leq n\}$  and  $\mathcal{A}^c := \{|\|X\|^2 - n| > n\}$ . For event  $\mathcal{A}$ , we have  $0 \leq \|X\|^2 \leq 2n$ , and  $\frac{n}{\|X\|^2} \geq \frac{1}{2}$ . Then plug in this into the tail bound, we see

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \middle| \mathcal{A} \right) \leq 2 \exp \left\{ -\min \left( \frac{t^2 C}{2048}, \frac{t\sqrt{n} C}{32} \right) \right\}$$

Finally, split the tail probability in two events  $\mathcal{A}$  and  $\mathcal{A}^c$ ,

$$\begin{aligned} &\mathbb{P} \left( \frac{1}{\sqrt{n}} |\|AX\|^2 - \|X\|^2| > t \right) \\ &= \mathbb{E} \left[ \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) \right] \\ &= \mathbb{E} \left[ 1_{\mathcal{A}} \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) \right] + \mathbb{E} \left[ 1_{\mathcal{A}^c} \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \right) \right] \\ &= \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \middle| \mathcal{A} \right) \mathbb{P}(\mathcal{A}) + \mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \sum_{i=1}^m (y_i^2 - \mathbb{E} y_i^2) \right| > t \middle| \mathcal{A}^c \right) \mathbb{P}(\mathcal{A}^c) \end{aligned}$$

$$\leq 2 \exp \left\{ -\min \left( \frac{t^2 C}{2048}, \frac{t\sqrt{n} C}{32} \right) \right\} + \mathbb{P}(\mathcal{A}^c)$$

By Eq. (2.5), we bound event  $\mathcal{A}^c$  (take  $t = \sqrt{n}$ )

$$\mathbb{P}(\mathcal{A}^c) = \mathbb{P}(\|X\|^2 - n > n) \leq e^{-\frac{n}{512}}$$

Therefore, we obtain

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \|AX\|^2 - \|X\|^2 \right| > t \right) \leq 2e^{\left\{ -\min \left( \frac{t^2 C}{2048}, \frac{t\sqrt{n} C}{32} \right) \right\}} + e^{-n/512} \quad (2.7)$$

Combining Eq. (2.4), Eq. (2.6) and Eq. (2.7), we conclude

$$\mathbb{P} \left( \frac{1}{\sqrt{n}} \left| \|AX\|^2 - n \right| > 2t \right) \leq 2e^{\left\{ -\min \left( \frac{t^2}{512}, \frac{t\sqrt{n}}{16} \right) \right\}} + 2e^{\left\{ -\min \left( \frac{t^2 C}{2048}, \frac{t\sqrt{n} C}{32} \right) \right\}} + e^{-\frac{n}{512}}$$

Then  $\min \left( \frac{t^2}{512}, \frac{t\sqrt{n}}{16}, \frac{t^2 C}{2048}, \frac{t\sqrt{n} C}{32}, \frac{n}{512} \right) \geq \frac{1}{2048} \min (t^2(1 \wedge C), t\sqrt{n}(1 \wedge C), n)$  gives the desired result.  $\square$

### 3 Random projection: invariance of norm

To understand if the random projected norm  $\|SX\|^2$  is convergent, it is necessary to find proper center and scale which is corresponding to first and second moments. Let us compute the mean first.

$$\begin{aligned} \mathbb{E} \|SX\|^2 - mn &= \mathbb{E} X^T S^T S X - mn \\ &= \mathbb{E} \text{tr}(X X^T S^T S) - mn \\ &= \text{tr}(\mathbb{E} X X^T \mathbb{E} S^T S) - mn \\ &= \text{tr}(I_n m I_n) - mn \\ &= 0 \end{aligned}$$

For the variance,

$$\begin{aligned} &\mathbb{E}(X^T S^T S X - mn)^2 \\ &= \mathbb{E}(X^T S^T S X)^2 - m^2 n^2 \\ &= \mathbb{E}(X^T S^T S X X^T S^T S X) - m^2 n^2 \\ &= \mathbb{E} \left( \sum x_{i_1} S_{i_2, i_1} S_{i_2, i_3} x_{i_3} x_{j_1} S_{j_2, j_1} S_{j_2, j_3} x_{j_3} \right) - m^2 n^2 \end{aligned}$$

The surviving terms must have even powers since first moments of the random variables are all 0. Therefore we only need to count  $\{i_1 = i_3 = j_1 = j_3 := i\}$ ,  $\{i_1 = i_3 := i \neq j_1 = j_3 := j\}$ ,  $\{i_1 = j_1 := i \neq i_3 = j_3 := j\}$ ,  $\{i_1 = j_3 := i \neq i_3 = j_1 := j\}$ .

$$\begin{aligned} &\mathbb{E} \left( \sum x_{i_1} S_{i_2, i_1} S_{i_2, i_3} x_{i_3} x_{j_1} S_{j_2, j_1} S_{j_2, j_3} x_{j_3} \right) \\ &= \mathbb{E} \sum_{i_2, j_2=1}^m \sum_{i=1}^n x_i S_{i_2, i} S_{i_2, i} x_i x_i S_{j_2, i} S_{j_2, i} x_i + \mathbb{E} \sum_{i_2, j_2=1}^m \sum_{\substack{i, j=1 \\ i \neq j}}^n x_i S_{i_2, i} S_{i_2, i} x_i x_j S_{j_2, j} S_{j_2, j} x_j \end{aligned}$$



$$\begin{aligned}
& + \mathbb{E} \sum_{i_2, j_2=1}^m \sum_{\substack{i, j=1 \\ i \neq j}}^n x_i S_{i_2, i} S_{i_2, j} x_j x_i S_{j_2, i} S_{j_2, j} x_j + \mathbb{E} \sum_{i_2, j_2=1}^m \sum_{\substack{i, j=1 \\ i \neq j}}^n x_i S_{i_2, i} S_{i_2, j} x_j x_j S_{j_2, j} S_{j_2, i} x_i \\
& = \mathbb{E} \sum_{i_2, j_2=1}^m \sum_{i=1}^n x_i^4 S_{i_2, i}^2 S_{j_2, i}^2 + \mathbb{E} \sum_{i_2, j_2=1}^m \sum_{\substack{i, j=1 \\ i \neq j}}^n x_i^2 x_j^2 S_{i_2, i}^2 S_{j_2, j}^2 \\
& + \mathbb{E} \sum_{k=1}^m \sum_{\substack{i, j=1 \\ i \neq j}}^n x_i^2 x_j^2 S_{k, i}^2 S_{k, j}^2 + \mathbb{E} \sum_{k=1}^m \sum_{\substack{i, j=1 \\ i \neq j}}^n x_i^2 x_j^2 S_{k, i}^2 S_{k, j}^2
\end{aligned}$$

where the last two terms dropped the zero terms  $i_2 \neq j_2$ ,

$$\begin{aligned}
& = (m^2 n - mn + mn \mathbb{E} S_{11}^4) \mathbb{E} x_1^4 + m^2(n^2 - n) + 2m(n^2 - n) \\
& = m^2 n^2 + (\mathbb{E} x_1^4 - 1) m^2 n + 2mn^2 + mn[(\mathbb{E} S_{11}^4 - 1) \mathbb{E} x_1^4 - 2] \\
& = m^2 n^2 + \sigma^2 m^2 n + 2mn^2 + \xi mn
\end{aligned}$$

where  $\xi := [(\mathbb{E} S_{11}^4 - 1) \mathbb{E} x_1^4 - 2]$ . Therefore

$$\mathbb{E} \left[ \left( \frac{X^T S^T S X - mn}{\sqrt{\sigma^2 m^2 n + 2mn^2 + \xi mn}} \right)^2 \right] = 1 \quad (3.1)$$

Next we will show the centered and scaled projected norm actually also converges to a normal. That means distribution of the norm of a vector with independent entries is also invariant under random projection.

### 3.1 CLT for Random projection of norm

**Theorem 4.** *Given a random vector  $X$  in  $\mathbb{R}^n$  with i.i.d. entries*

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

*Let  $\mathbb{E} x_1 = 0, \mathbb{E} x_1^2 = 1, \mathbb{E} x_1^4 = 1 + \sigma^2 (0 \leq \sigma < \infty)$ . Consider a random matrix  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with independent identically distributed entries  $S_{i,j}$  with  $\mathbb{E} S_{i,j} = 0, \mathbb{E} S_{i,j}^2 = 1$  and  $\mathbb{E} S_{1,1}^4 < c < \infty$ . Further assume  $S, X$  are all independent. Define*

$$A(m, n) := \frac{\|SX\|^2 - mn}{\sqrt{\sigma^2 m^2 n + 2mn^2 + \xi mn}}$$

*where  $\xi = [(\mathbb{E} S_{11}^4 - 1) \mathbb{E} x_1^4 - 2]$ . If  $\frac{m}{n} \rightarrow 0$ , then*

$$A(m, n) \xrightarrow{m, n \rightarrow \infty} \mathcal{N}(0, 1) \quad (3.2)$$

**Remarks.** *Before we proceed with the proof, it is worth mentioning that the random norm is a complicated sum of  $mn^2$  correlated terms.*

$$X^T S^T S X = \sum_{k=1}^m \sum_{i, j=1}^n x_i S_{k, i} S_{k, j} x_j$$

Therefore most analytical methods and tools fails to treat the quantity properly. For example, characteristic function need independent property, Lindeberg swapping needs martingale property, and Stein's method needs exchangeable structure and precise control of first and second moments of conditional perturbed differences. So we are constrained to use a robust and universal approach, the moment method, which can be used to prove convergence to a limit law with known moments.

In the moment method we present below, we need to control the order of  $m$  by  $m \leq o(n)$  because the counting procedure would be impossible to carry out if this is not the case. The scaling in this case is dominated by  $\sqrt{2mn^2}$  which allow us to limit the significant terms in the moment calculation. However in simulations, we see convergence to normal even when  $m > n$ . But we could not find a proof for the general case yet due to the highly complex correlations.

**Proof:** We will first note that a truncation argument will show it is sufficient to prove the same CLT result for bounded random variables. Details can be found in many standard moment method proof for CLT in many standard textbook (see for example [12] 2.2).

From now on, we assume all random variables are bounded, so that they have finite moments of all order which is very important in moment method. We will compute all moments of  $A(m, n)$  in the limit and we expect all odd moments vanish and all even moments match with standard normal random variable.

The key idea is to separate the random norm into  $m$  identically distributed but dependent random variables. Let  $S_{k,\cdot}$  be  $k$ -th row of  $S$ . Then  $\|SX\|^2 = \sum_{k=1}^m (S_{k,\cdot}X)^2$ . Define

$$\begin{aligned} L_k &:= \frac{(S_{k,\cdot}X)^2 - n}{\sqrt{\sigma^2 mn + 2n^2 + \xi n}} = \frac{(\sum_{i,j=1}^n x_i S_{k,i} S_{k,j} x_j - n)}{\sqrt{\sigma^2 mn + 2n^2 + \xi n}} \\ \Rightarrow A(m, n) &= \frac{1}{\sqrt{m}} \sum_{k=1}^m L_k \\ \Rightarrow \lim_{m, n \rightarrow \infty} \mathbb{E} A(m, n)^t &= \lim_{m, n \rightarrow \infty} \frac{1}{m^{t/2}} \mathbb{E} \left( \sum_{k=1}^m L_k \right)^t \end{aligned}$$

Let us first record some moments properties of these identically distributed  $L_k$ .

- (1)  $\mathbb{E} L_k = 0$ .
- (2)  $\mathbb{E} L_k^2 = 1 - O(\frac{m}{n})$ .
- (3)  $|\mathbb{E} L_k^t| \leq C_t < \infty$
- (4) The order of the expectation of a polynomial determined by number of singletons.

$$\mathbb{E}[L_1^{q_1} \cdots L_r^{q_r}] \leq O\left(\frac{1}{(\sigma^2 m + 2n)^{d/2}}\right), \quad \text{where } d = \sum_{i=1}^r 1_{(q_i=1)}$$

- (5) For any fixed  $r \in \mathbb{R}$ ,

$$\mathbb{E}(L_1^2 \cdots L_r^2) = \left(\frac{2n}{\sigma^2 m + 2n}\right)^r + O\left(\frac{1}{n}\right)$$

which converges to 1 if  $\frac{m}{n} \rightarrow 0$ .

We will prove one by one.

- (1) Obviously,  $\mathbb{E} L_k = 0$ .

(2) The variance  $\mathbb{E} L_k^2 = 1 - O(\frac{mn}{\sigma^2 mn + 2n^2 + \xi n})$  since

$$\begin{aligned}
\mathbb{E}(\sum_{i,j=1}^n x_i S_{k,i} S_{k,j} x_j - n)^2 &= \mathbb{E}(\sum_{i,j=1}^n x_i S_{k,i} S_{k,j} x_j)^2 - n^2 \\
&= \mathbb{E}(\sum_{i=1}^n x_i^2 S_{k,i}^2)^2 + 2 \mathbb{E}[(\sum_{i=1}^n x_i^2 S_{k,i}^2)(\sum_{\substack{i,j=1 \\ i \neq j}}^n x_i S_{k,i} S_{k,j} x_j)] \\
&\quad + \mathbb{E}(\sum_{\substack{i,j=1 \\ i \neq j}}^n x_i S_{k,i} S_{k,j} x_j)^2 - n^2 \\
&= n \mathbb{E} x_1^4 S_{1,1}^4 + n(n-1) + 0 + 2n(n-1) - n^2 \\
&= 2n^2 + O(n)
\end{aligned}$$

(3) It is also true that  $L_k$  has finite moments of all order which also hold in the limit. More precisely, for any fixed  $t \in \mathbb{R}$ , there is a constant  $C_t$  independent of  $m$  and  $n$  that

$$\lim_{m,n \rightarrow \infty} |\mathbb{E} L_k^t| < C_t < \infty \quad (3.3)$$

We will use a careful counting procedure. First we notice

$$\begin{aligned}
|\mathbb{E} L_k^t| &= \left| \mathbb{E} \left[ \frac{(\sum_{i,j=1}^n x_i S_{k,i} S_{k,j} x_j - n)}{\sqrt{\sigma^2 mn + 2n^2 + \xi n}} \right]^t \right| \\
&\leq n^{-t} \left| \mathbb{E} \left[ \left( \sum_{i,j=1}^n x_i S_{k,i} S_{k,j} x_j - n \right)^t \right] \right| \quad (3.4)
\end{aligned}$$

Since  $n = \sum_{i,j} \delta_{i,j}$  where  $\delta_{i,j} = 1$  when  $i = j$  and 0 otherwise. Then we expand the  $t$ -th moment on the right as polynomials.

$$\begin{aligned}
&\mathbb{E} \left[ \left( \sum_{i,j=1}^n x_i S_{k,i} S_{k,j} x_j - n \right)^t \right] \\
&= \mathbb{E} \left[ \left( \sum_{i_1, j_1=1}^n (x_{i_1} S_{k,i_1} S_{k,j_1} x_{j_1} - \delta_{i_1, j_1}) \right) \cdots \left( \sum_{i_t, j_t=1}^n (x_{i_t} S_{k,i_t} S_{k,j_t} x_{j_t} - \delta_{i_t, j_t}) \right) \right] \\
&= \sum_{i_1, j_1=1}^n \cdots \sum_{i_t, j_t=1}^n \mathbb{E}[(x_{i_1} S_{k,i_1} S_{k,j_1} x_{j_1} - \delta_{i_1, j_1}) \cdots (x_{i_t} S_{k,i_t} S_{k,j_t} x_{j_t} - \delta_{i_t, j_t})]
\end{aligned}$$

Notice  $\mathbb{E}[(x_{i_1} S_{k,i_1} S_{k,j_1} x_{j_1} - \delta_{i_1, j_1}) \cdots (x_{i_t} S_{k,i_t} S_{k,j_t} x_{j_t} - \delta_{i_t, j_t})]$  vanishes if cardinality of  $\{i_1, j_1, \dots, i_t, j_t\}$  is greater than  $t$  since that will have at least one singleton factor  $\mathbb{E} S_{k,i} = 0$ . So the index set must collapse to a set of at most  $t$  distinct indices. This means total number of nonzero terms is exactly  $n^t$ . As we assume all variables are truncated to have bounded moments, we find

$$\left| \mathbb{E} \left[ \left( \sum_{i,j=1}^n x_i S_{k,i} S_{k,j} x_j - n \right)^t \right] \right| \leq O(n^t)$$

Therefore plug into Eq. (3.4) we find  $|\mathbb{E} L_k^t| \leq O(1)$ . This proves Eq. (3.3).

(4) Another important property concerns the product of  $L_k$ 's. Given integer  $q_1, \dots, q_r \geq 0$ ,

$$\mathbb{E}[L_1^{q_1} \dots L_r^{q_r}] \leq O\left(\frac{1}{(\sigma^2 m + 2n)^{d/2}}\right), \quad \text{where } d = \sum_{i=1}^r 1_{(q_i=1)} \quad (3.5)$$

Here  $d$  is the total number of variables with multiplicity 1. What this means is for any product of  $L_k$  involving  $d$  term of power 1, then the expected value is of order  $(m+n)^{-d/2}$ . In other words, each power 1 term contribute a factor of  $(m+n)^{-1/2}$ . we use an argument by conditioning and careful counting. First, noticing  $L_k$  conditioning on  $X$  are independent,

$$\begin{aligned} \mathbb{E}[L_1^{q_1} L_2^{q_2} \dots L_r^{q_r}] &= \mathbb{E}[(L_2^{q_2} \dots L_r^{q_r}) \mathbb{E}(L_1^{q_1} | X, S_{2,\dots}, \dots S_{r,\dots})] \\ &= \mathbb{E}[(L_2^{q_2} \dots L_r^{q_r}) \mathbb{E}(L_1^{q_1} | X)] \\ &= \mathbb{E}[\mathbb{E}((L_2^{q_2} \dots L_r^{q_r}) | X, S_{3,\dots}, \dots S_{r,\dots}) \mathbb{E}(L_1^{q_1} | X)] \\ &= \mathbb{E}[(L_3^{q_3} \dots L_r^{q_r}) \mathbb{E}(L_2^{q_2} | X) \mathbb{E}(L_1^{q_1} | X)] \\ &\dots \\ &= \mathbb{E}[\mathbb{E}(L_1^{q_1} | X) \mathbb{E}(L_2^{q_2} | X) \dots \mathbb{E}(L_r^{q_r} | X)] \end{aligned}$$

Without loss of generality, assume the first  $d$  variables are of multiplicity 1, namely

$$q_1 = q_2 = \dots = q_d = 1$$

To simplify notation, we denote  $\mathbb{E}(L_i^{q_i} | X) := \mu_i$ . We would also only need the above conditioning argument up to  $d$ 's variable  $L_d$ . That is

$$\mathbb{E}[L_1 \dots L_d L_{d+1}^{q_{d+1}} \dots L_r^{q_r}] = \mathbb{E}[\mu_1 \dots \mu_d L_{d+1}^{q_{d+1}} \dots L_r^{q_r}]$$

Then we apply Cauchy-Schwarz inequality, we find

$$\mathbb{E}[L_1 \dots L_d L_{d+1}^{q_{d+1}} \dots L_r^{q_r}] \leq (\mathbb{E}[\mu_1^2 \dots \mu_d^2])^{\frac{1}{2}} \cdot (\mathbb{E}[L_{d+1}^{q_{d+1}} \dots L_r^{q_r}])^{\frac{1}{2}} \quad (3.6)$$

Then we notice the random variables  $\mu_1, \dots, \mu_d$  (conditional expectation is also a random variable) are identical random variables (not just identically distributed).

$$\begin{aligned} \mu_1 &= \mathbb{E}(L_1 | X) = \mathbb{E}\left(\frac{(\sum_{i,j=1}^n x_i S_{1,i} S_{1,j} x_j - n)}{\sqrt{\sigma^2 m n + 2n^2 + \xi n}} \middle| X\right) \\ &= \frac{1}{\sqrt{\sigma^2 m n + 2n^2 + \xi n}} \left( \sum_{i=1}^n x_i^2 \mathbb{E} S_{1,i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i x_j \mathbb{E}(S_{1,i} S_{1,j}) - n \right) \\ &= \frac{\sum_{i=1}^n x_i^2 - n}{\sqrt{\sigma^2 m n + 2n^2 + \xi n}} \xrightarrow{a.s.} 0 \end{aligned}$$

This shows  $\mu_1$  does not depend on the index 1 and indeed  $\mu_1, \dots, \mu_d$  are identical. As a side note,  $\mu_1$  converges to 0 almost surely due to the strong law of large number. We would not need this fact though. Now we can analyze first term on the right hand side of Eq. (3.6),

$$\mathbb{E}[\mu_1^2 \dots \mu_d^2] = \mathbb{E}[\mu_1^{2d}]$$

$$\begin{aligned}
&= \frac{\mathbb{E} \left( \sum_{i=1}^n x_i^2 - n \right)^{2d}}{(\sigma^2 mn + 2n^2 + \xi n)^d} \\
&= \frac{1}{(\sigma^2 m + 2n + \xi)^d} \mathbb{E} \left( \frac{\sum_{i=1}^n x_i^2 - n}{\sqrt{n}} \right)^{2d} \\
&= O \left( \frac{1}{(\sigma^2 m + 2n)^d} \right)
\end{aligned} \tag{3.7}$$

The last step, we used the fact that independent sums  $(\sum_{i=1}^n x_i^2 - n)$  in CLT has moments of  $O(n^{t/2})$  if  $x_i$  has finite moments of all order (see [3, 16]), which holds true due to its boundedness by truncation argument.

$$\mathbb{E} \left( \frac{\sum_{i=1}^n x_i^2 - n}{\sqrt{n}} \right)^t \leq C_t < \infty$$

In fact, if we do not use boundedness, instead we impose sub-Gaussian property, the variables  $x_i^2 - 1$  are sub-exponential. Then Bernstein concentration inequality implies the scaled sum is sub-exponential therefore we conclude it has bounded moments (see Chapter 2 in [2]).

On the other hand, applying Cauchy-Schwarz inequality repeatedly for the second term on the right hand side of Eq. (3.6), we can bound it by a constant depend only on  $t$ . Namely

$$\begin{aligned}
\mathbb{E}[L_{d+1}^{q_{d+1}} \cdots L_r^{q_r}] &\leq \left[ \mathbb{E} L_{d+1}^{2q_{d+1}} \mathbb{E} [L_{d+2}^{2q_{d+2}} \cdots L_r^{2q_r}] \right]^{1/2} \\
&\leq \left[ \mathbb{E} L_{d+1}^{2q_{d+1}} \right]^{\frac{1}{2}} \left[ \mathbb{E} L_{d+2}^{4q_{d+2}} \right]^{\frac{1}{4}} \cdots \left[ \mathbb{E} L_r^{2^{r-d} q_r} \right]^{\frac{1}{2^{r-d}}} \\
&\leq c_t < \infty
\end{aligned} \tag{3.8}$$

Therefore combined with Eq. (3.6) and Eq. (3.7), we obtain Eq. (3.5).

(5) To compute  $\mathbb{E}(L_1^2 \cdots L_r^2)$  we will use conditioning argument again.

$$\mathbb{E}(L_1^2 \cdots L_r^2) = \mathbb{E} [\mathbb{E}(L_1^2 | X) \cdots \mathbb{E}(L_r^2 | X)]$$

Same as before,  $\mathbb{E}(L_1^2 | X), \dots, \mathbb{E}(L_r^2 | X)$  does not depend on the indices  $1, \dots, r$ , and they are all identical random variables not just with same distribution. By definition,

$$\mathbb{E}(L_1^2 | X) = \frac{\mathbb{E} \left( (\sum_{i,j=1}^n x_i S_{1,i} S_{1,j} x_j - n)^2 \middle| X \right)}{\sigma^2 mn + 2n^2 + \xi n}$$

We shall simplify the numerator (denoted as  $Q$ ),

$$Q := \mathbb{E} \left( \left( \sum_{i,j=1}^n x_i S_{1,i} S_{1,j} x_j - n \right)^2 \middle| X \right) = \mathbb{E} \left( \left( \left( \sum_{i=1}^n (x_i^2 S_{1,i}^2 - 1) \right) + \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i x_j S_{1,i} S_{1,j} \right)^2 \middle| X \right)$$

Expand the quadratic we find the cross terms

$$2 \mathbb{E} \left( \left( \sum_{k=1}^n (x_k^2 S_{1,k}^2 - 1) \right) \left( \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i x_j S_{1,i} S_{1,j} \right) \middle| X \right)$$

$$\begin{aligned}
&= 2 \sum_{k=1}^n \sum_{\substack{i,j=1 \\ i \neq j}}^n \mathbb{E} [(x_k^2 S_{1,k}^2 - 1) x_i x_j S_{1,i} S_{1,j} | X] \\
&= 2 \sum_{k=1}^n \sum_{\substack{i,j=1 \\ i \neq j}}^n (x_k^2 x_i x_j \mathbb{E}[S_{1,k}^2 S_{1,i} S_{1,j}] - x_i x_j \mathbb{E}[S_{1,i} S_{1,j}])
\end{aligned}$$

vanish, since each term  $\mathbb{E}[S_{1,k}^2 S_{1,i} S_{1,j}] = \mathbb{E}[S_{1,i} S_{1,j}] = 0$  as  $i \neq j$ . So we are left with

$$\begin{aligned}
Q &= \mathbb{E} \left( \left( \sum_{i=1}^n x_i^2 S_{1,i}^2 - n \right)^2 \middle| X \right) + \mathbb{E} \left( \left( \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i x_j S_{1,i} S_{1,j} \right)^2 \middle| X \right) \\
&= \left( \sum_{i=1}^n \sum_{j=1}^n x_i^2 x_j^2 \mathbb{E} S_{1,i}^2 S_{1,j}^2 - 2n \sum_{i=1}^n x_i^2 \mathbb{E} S_{1,i}^2 + n^2 \right) \\
&\quad + \left( \sum_{\substack{i,j=1 \\ i \neq j}}^n \sum_{\substack{i',j'=1 \\ i' \neq j'}}^n x_i x_j x_{i'} x_{j'} \mathbb{E} S_{1,i} S_{1,j} S_{1,i'} S_{1,j'} \right)
\end{aligned}$$

Notice the surviving terms in the second half are  $\{i = i' \neq j = j'\}$  and  $\{i = j' \neq j = i'\}$ . Therefore,

$$\begin{aligned}
Q &= \left( \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i^2 x_j^2 + \sum_{i=1}^n x_i^4 \mathbb{E} S_{1,1}^4 - 2n \sum_{i=1}^n x_i^2 + n^2 \right) + \left( 2 \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i^2 x_j^2 \right) \\
&= 3 \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i^2 x_j^2 - 2n \sum_{i=1}^n x_i^2 + n^2 + \sum_{i=1}^n x_i^4 \mathbb{E} S_{1,1}^4
\end{aligned}$$

Since  $\mathbb{E}(L_k^2 | X) = Q/(\sigma^2 mn + 2n^2 + \xi n)$  for all  $k$ , we simplify

$$\mathbb{E}(L_1^2 \cdots L_r^2) = \mathbb{E}[Q^r] / (\sigma^2 mn + 2n^2 + \xi n)^r$$

To prove  $\mathbb{E}(L_1^2 \cdots L_r^2) \rightarrow 1$  it suffices to prove  $\mathbb{E}[Q/(2n^2)]^r \rightarrow 1$  since  $(\sigma^2 mn + 2n^2 + \xi n)^r$  is dominated by  $(2n^2)^r$  as  $m/n \rightarrow 0$ .

$$\begin{aligned}
\frac{Q}{2n^2} &= \frac{1}{2n^2} (3 \sum_{\substack{i,j=1 \\ i \neq j}}^n x_i^2 x_j^2 - 2n \sum_{i=1}^n x_i^2 + n^2 + \sum_{i=1}^n x_i^4 \mathbb{E} S_{1,1}^4) \\
&= \frac{1}{2} P + \frac{1}{2} + P_0
\end{aligned}$$

where (to simplify notation) we denoted

$$P := \left( \frac{1}{n^2} \sum_{i=1}^n x_i^2 \left( \sum_{j=1, j \neq i}^n 3x_j^2 - 2n \right) \right), \quad P_0 = \frac{1}{2n^2} \sum_{i=1}^n x_i^4 \mathbb{E} S_{1,1}^4$$

First of all it is fairly easy to see  $\mathbb{E} \left( \frac{Q}{2n^2} \right)^r < \infty$  for all  $r \in \mathbb{R}$ . This is because the total number of nonzero polynomial terms  $(x_i^2, x_i^4, x_i^2 x_j^2)$  in  $Q$  is  $O(n^2)$  which will produce  $O(n^{2r})$  polynomial terms for  $Q^r$ ; and  $x_i$  has finite moment of all order. Now let us use induction to prove the moments are actually constant 1 in the limit. Suppose we have proved for all  $k \leq r$ ,

$$\mathbb{E} \left( \frac{Q}{2n^2} \right)^k \rightarrow 1, \quad \text{and} \quad \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^{k-1} P \right] \rightarrow 1$$

It is easy to see this holds for the initial steps. Then we try to prove the next induction step namely

$$\mathbb{E} \left( \frac{Q}{2n^2} \right)^{r+1} \rightarrow 1, \quad \text{and} \quad \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r P \right] \rightarrow 1$$

Notice by linearity of expectation and Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r P_0 \right] &= \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r \left( \frac{1}{2n^2} \sum_{i=1}^n x_i^4 \mathbb{E} S_{1,1}^4 \right) \right] \\ &= \frac{1}{2n} \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r x_1^4 \mathbb{E} S_{1,1}^4 \right] \\ &\leq \frac{1}{2n} \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^{2r} \right]^{1/2} \mathbb{E} [x_1^8 \mathbb{E} S_{1,1}^8]^{1/2} \rightarrow 0 \end{aligned}$$

We find

$$\begin{aligned} \mathbb{E} \left( \frac{Q}{2n^2} \right)^{r+1} &= \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r \left[ \frac{1}{2} P + \frac{1}{2} + P_0 \right] \right] \\ &\rightarrow \frac{1}{2} \lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r P \right] + \frac{1}{2} + 0 \end{aligned}$$

Now it suffices to show  $\mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r P \right] \rightarrow 1$ . We expand again

$$\mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^r P \right] = \mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^{r-1} \left( \frac{1}{2} P^2 + \frac{1}{2} P + P P_0 \right) \right]$$

By induction hypothesis  $\mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^{r-1} \frac{1}{2} P \right] = \frac{1}{2}$ . And by Cauchy-Schwarz we know  $\mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^{r-1} P P_0 \right] \leq O(\frac{1}{2n}) \rightarrow 0$ . So it suffices to show  $\mathbb{E} \left[ \left( \frac{Q}{2n^2} \right)^{r-1} P^2 \right] \rightarrow 1$ . Repeat this argument  $r$  times, we find it suffices to prove  $\mathbb{E} P^{r+1} \rightarrow 1$ . And since this has to be true for every induction step, we indeed need to show

$$\mathbb{E} P^k \rightarrow 1, \quad \forall k \in \mathbb{R}$$

$$\begin{aligned} \mathbb{E} P^k &= \mathbb{E} \left( \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i^2 (3x_j^2 - \frac{2n}{n-1}) \right)^k \\ &= \frac{1}{n^{2k}} \sum_{i_1, \dots, i_k=1}^n \sum_{j_1, \dots, j_k=1, j_t \neq i_t}^n \mathbb{E} \left( \prod_{t=1}^k x_{i_t}^2 (3x_{j_t}^2 - \frac{2n}{n-1}) \right) \end{aligned}$$

The two summations produce total  $[n(n-1)]^k$  terms. Among them, there are total

$$\binom{n(n-1)}{2k}$$

terms that all indices are distinct, that is the cardinality  $|\{i_1, \dots, i_k, j_1, \dots, j_k\}| = 2k$ . So that we can evaluate the expectation directly by independence. For  $i_t \neq j_t, i_t \neq i'_t, j_t \neq j'_t$

$$\mathbb{E} \left( \prod_{t=1}^k x_{i_t}^2 (3x_{j_t}^2 - \frac{2n}{n-1}) \right) = \prod_{t=1}^k \mathbb{E} x_{i_t}^2 \mathbb{E} (3x_{j_t}^2 - \frac{2n}{n-1}) = (1 - \frac{2}{n-1})^k \rightarrow 1$$

For the remaining  $n^k(n-1)^k - \binom{n(n-1)}{2k}$  terms, the expectations of correlated variables are still bounded by some constant  $c_k$ . So we find

$$\mathbb{E} P^k = \frac{1}{n^{2k}} \binom{n(n-1)}{2k} + \frac{1}{n^{2k}} c_k \left( n^k(n-1)^k - \binom{n(n-1)}{2k} \right) \rightarrow 1$$

After this we can start the standard moments method. Since

$$\lim_{m, n \rightarrow \infty} \mathbb{E} A(m, n)^t = \lim_{m, n \rightarrow \infty} \frac{1}{m^{t/2}} \mathbb{E} \left( \sum_{k=1}^m L_k \right)^t$$

We first expand  $\mathbb{E} (\sum_{k=1}^m L_k)^t$ . Any term will have form  $L_{p_1}^{q_1} \dots L_{p_r}^{q_r}$  with the number of distinct indices  $r : 1 \leq r \leq t$  and positive integer powers satisfy  $q_1 + \dots + q_r = t$ . If we group the terms by total number of distinct indices

$$\mathbb{E} \left( \sum_{k=1}^m L_k \right)^t = \sum_{r=1}^t \sum_{\substack{q_1 + \dots + q_r = t \\ \{p_1, \dots, p_r\} \subset \{1, \dots, m\}}} c_{t, q_1, \dots, q_r} \mathbb{E} (L_{p_1}^{q_1} \dots L_{p_r}^{q_r})$$

where  $c_{t, q_1, \dots, q_r}$  is the total number of orderings when we order  $\{q_1 \text{ number of index } p_1, \dots, q_r \text{ number of index } p_r, q_1 + \dots + q_r = t\}$  all together, which is

$$c_{t, q_1, \dots, q_r} = \frac{t!}{q_1! \dots q_r!}$$

This constant only depend on  $t$  and  $q_1, \dots, q_r$ , and it may be upper bounded by  $t^t$ .

Now we are going to analyze how much the terms contribute for each fixed  $r$ . In particular, we will show the only significant terms which will survive after scaling are when  $r = \frac{t}{2}$  (if  $t$  is odd, that means no surviving terms).

For any term  $L_1^{q_1} \dots L_r^{q_r}$  with  $r > \frac{t}{2}$ , there are at least

$$2 \left( r - \frac{t}{2} \right) = 2r - t \tag{3.9}$$

variables  $L_i$  have multiplicity 1. It is true because increasing the length  $r$  by 1 will create at least two singleton terms. For example if we want to increase length of  $L_1^2 L_2^2$  from 2 to 3, we would end up breaking a square term into two singletons so that we have  $L_1^2 L_2 L_3$  or  $L_1 L_2^2 L_3$ . Formally,



suppose that's not the case. Namely suppose there are only  $s \leq 2r - t - 1$  variables  $L_i$  of multiplicity 1. Then adding all the multiplicity we get

$$t = q_1 + \cdots + q_r \geq s + 2(r - s) = 2r - s \geq t + 1$$

This is a contradiction.

Combining Eq. (3.9) and Eq. (3.5), each of the term when  $r > \frac{t}{2}$  contribute at most

$$O\left(\frac{1}{(\sigma^2 m + 2n)^{d/2}}\right), \quad \text{where } d \geq 2r - t$$

For each fixed  $r$ , the total number of possible choices of  $\{p_1, \dots, p_r\} \subset \{1, \dots, m\}$  is  $\binom{m}{r} \leq O(m^r)$ . Then we also need to count total number of ways to generate  $q_1 + \cdots + q_r = t$ . That is we are looking at separating the integer  $t$  into  $r$  nonzero integers  $q_1, \dots, q_r$ . Total number will be  $\binom{t-1}{r-1} \leq O(t^r)$  which only depend on  $t$  (we can model it as separating  $t$  stones into  $r$  piles.  $\binom{t-1}{r-1}$  is due to the fact we can select  $r - 1$  separating positions out of  $t - 1$  spaces.).

Therefore the total contribution for each fixed  $r > \frac{t}{2}$  is

$$\begin{aligned} & \frac{1}{m^{t/2}} \sum_{\substack{q_1 + \cdots + q_r = t \\ \{p_1, \dots, p_r\} \subset \{1, \dots, m\}}} c_{t, q_1, \dots, q_r} \mathbb{E}(L_{p_1}^{q_1} \cdots L_{p_r}^{q_r}) \\ & \leq m^{-\frac{t}{2}} O(m^r) O(t^r) c_{t, q_1, \dots, q_r} O\left(\frac{1}{(\sigma^2 m + 2n)^{d/2}}\right) \\ & \leq O\left(m^{r-\frac{t}{2}} (\sigma^2 m + 2n)^{-d/2}\right) \\ & \leq O\left(m^{r-\frac{t}{2}} (\sigma^2 m + 2n)^{-r+\frac{t}{2}}\right) \end{aligned}$$

Last step we used the fact  $d \geq 2r - t$ . Since we assumed  $\frac{m}{n} \rightarrow 0$ , we find the total contribution is bounded by

$$O\left(\frac{m}{\sigma^2 m + 2n}\right)^{r-\frac{t}{2}} \rightarrow 0, \quad \forall r > \frac{t}{2}$$

Since  $\lceil \frac{t}{2} \rceil - \frac{t}{2} - 1 = -1$  when  $t$  is even, and  $-0.5$  when  $t$  is odd. Therefore we can safely drop all cases of  $r > \frac{t}{2}$ .

For all cases  $r < \frac{t}{2}$ , each term  $\mathbb{E}(L_{p_1}^{q_1} \cdots L_{p_r}^{q_r}) = O(1)$  depend only on  $t$  by a repeated Cauchy-Schwarz argument similar to Eq. (3.8). We find total contribution is

$$m^{-\frac{t}{2}} O(m^r) O(t^r) c_{t, q_1, \dots, q_r} O(1) \rightarrow 0, \quad \forall r < \frac{t}{2}$$

This implies for  $t$  odd

$$\lim_{m, n \rightarrow \infty} \frac{1}{m^{t/2}} \mathbb{E}\left(\sum_{k=1}^m L_k\right)^t \rightarrow 0$$

Now we only consider even moments. The above analysis shows we have to force  $r = \frac{t}{2}$ . In this case, We are looking at separate the integer  $t$  into  $r = \frac{t}{2}$  nonzero integers  $q_1, \dots, q_r$ . Total number will be  $\binom{t-1}{r-1}$  which only depend on  $t$  ( $\binom{t-1}{r-1}$  is due to the fact we can select  $r - 1$  separating positions out of  $t - 1$  spaces). There are still many terms not significant. By Eq. (3.5), any term has a  $L_k$

of multiplicity 1 will contribute at most  $O(n^{-\frac{1}{2}})$ , thus we may drop these terms. Among these  $\binom{t-1}{r-1}$  terms, there is only one term that every  $L_k$  has multiplicity at least 2, which will survive the scaling, which is

$$q_1 = \cdots = q_r = 2$$

In other words,

$$\begin{aligned} \lim_{m,n \rightarrow \infty} \frac{1}{m^{t/2}} \mathbb{E} \left( \sum_{k=1}^m L_k \right)^t &= \lim_{m,n \rightarrow \infty} \frac{1}{m^{t/2}} \sum_{\substack{r=t/2 \\ \{p_1, \dots, p_r\} \subset \{1, \dots, m\}}} c_{t,2,\dots,2} \mathbb{E}(L_{p_1}^2 \cdots L_{p_r}^2) \\ &= \lim_{m,n \rightarrow \infty} \frac{1}{m^{t/2}} \binom{m}{t/2} c_{t,2,\dots,2} \mathbb{E}(L_1^2 \cdots L_{t/2}^2) \\ &= \lim_{m \rightarrow \infty} \frac{1}{m^{t/2}} \binom{m}{t/2} \frac{t!}{2^{t/2}} \\ &= \frac{t!}{2^{t/2}} \end{aligned}$$

Which is exactly the even moments of standard normal random variable  $\mathcal{N}(0, 1)$ . One can see this by finding a recurrent relation between moments using moment generating function.

$$\mathbb{E}[\mathcal{N}(0, 1)]^{2k+2} = (2k+1) \mathbb{E}[\mathcal{N}(0, 1)]^{2k}$$

□

## 4 Discussion and open questions

### 4.1 Possibility of extending to large row dimension

First, in the proof of the diagonal elements convergence to normal distribution require  $m/n \rightarrow 0$ , it is highly interesting to see if there is a proof for the diagonal elements  $\Sigma_{i,i}^S$  when  $m \geq O(n)$ . In the latter case, the dependency turns out to be much more complex, and the scaling  $\sigma^2 m^2 n$  would play a key role.

If we view the random projected norm  $\|SX\|^2$  as a trace function on the spectral of product of random matrices, there are potential ways from random matrix theory to overcome the difficult of too many correlated random variables when  $m \geq O(n)$ .

$$X^T S^T S X = \text{tr}(X^T S^T S X) = \text{tr}(S^T S X X^T)$$

It is clear the random matrix  $\frac{1}{\sqrt{n}} X X^T$  has one nonzero eigenvalue which is  $\lambda_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i^2 = O(\sqrt{n})$ .

From random matrix analysis [9], it is known that the empirical spectral distribution of  $A_n := \frac{1}{m} S^T S$  converges to the celebrated Marčenko-Pastur law depends on the parameter  $\frac{m}{n} \rightarrow c$ ,  $c \in (0, \infty)$ .

$$\frac{1}{n} \sum_i \delta_{\lambda_i(A_m)} \xrightarrow{m \rightarrow \infty} \mu_{M,c}$$

where  $\lambda_i(A_n)$  are eigenvalues of  $A_n$  and  $\delta_{\lambda_i(A_n)}$  is the Dirac delta function,  $\mu_{M,c}$  is the Marčenko-Pastur probability measure. Moreover, [17, 11, 1] showed if a non-negative definite  $n \times n$  random

matrix  $B_n$  has a deterministic limiting distribution  $F^B$ , then one can characterize the limiting spectral distribution of the product,  $A_n B_n$ , converges in distribution to probability distribution  $F$ .

$$\frac{1}{m} \sum_i \delta_{\lambda_i(A_n B_n)} \xrightarrow{n \rightarrow \infty} \mu_F$$

One may start thinking if it is possible to apply the result of product of random matrices to our problem. Obviously, one would replace  $B_n$  with  $\frac{1}{\sqrt{n}} X X^T$ . However the spectral distribution of  $B_n$  does not converge properly since  $\lambda_1 = O(\sqrt{n})$ . And our CLT result is actually on another level of details. One has to first center and standardize spectral of  $B_n$ , then see how the fluctuation is interacting with the spectral of  $A_n$ . In fact  $A_n B_n$  has only one nonzero eigenvalue, we are actually looking at distribution of this single eigenvalue, which usually requires very different techniques to compute. The extreme eigenvalues of full rank random matrices usually converges to Tracy-Widom distribution ([13, 7]). In our case, we are looking at a version of this type but the random matrix has certain structure of rank one.

## 4.2 Rate of convergence

Another challenging part would be to find the rate of convergence for the diagonal elements  $\Sigma_{i,i}^S$ , the projected norm. Based on some calculation, we believe the following conjecture should be true.

**Conjecture 5** (Random projection of norm invariance: rate of convergence). *Given a random vector  $X$  in  $\mathbb{R}^n$  with i.i.d. entries*

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

*Let  $\mathbb{E} x_1 = 0, \mathbb{E} x_1^2 = 1, \mathbb{E} x_1^4 = 1 + \sigma^2 (0 < \sigma < \infty)$ . Consider a random matrix  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with independent entries and  $\mathbb{E} S_{i,j} = 0$  and  $\mathbb{E} S_{i,j}^2 = 1$ . Further assume  $S, X$  are all independent and  $\mathbb{E} S_{1,1}^8 \vee \mathbb{E} |x_1|^6 < c < \infty$ . Also let  $G$  be a standard normal random variable. Then we have*

$$\sup_t \left| \mathbb{P} \left( \frac{(X^T S^T S X - mn)}{\sqrt{\sigma^2 m^2 n + 2mn^2}} < t \right) - \mathbb{P}(G < t) \right| \leq O \left( \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}} \right) \quad (4.1)$$

$$\sup_t \left| \mathbb{P} \left( \frac{(X^T S^T S X - mn)}{\sqrt{\sigma^2 m^2 n + 2mn^2}} < t \right) - \mathbb{P} \left( \frac{(X^T X - n)}{\sigma \sqrt{n}} < t \right) \right| \leq O \left( \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}} \right) \quad (4.2)$$

**Remarks.** *If we use Berry-Essen theorem for random sequence  $X^2 - 1 = [x_1^2 - 1, \dots, x_n^2 - 1]$  (the assumptions in Berry-Essen are satisfied since  $x_i^2 - 1$  are i.i.d.,  $\mathbb{E} x_1^2 - 1 = 0$ ,  $\mathbb{E} (x_1^2 - 1)^2 = \sigma^2$  and  $\mathbb{E} |x_1^2 - 1|^3 < \infty$ .) we find*

$$\sup_t \left| \mathbb{P}(G < t) - \mathbb{P} \left( \frac{(X^T X - n)}{\sigma \sqrt{n}} < t \right) \right| \leq O \left( \frac{1}{\sqrt{n}} \right)$$

*Then it is tempting to use techniques similar with the proof of Theorem 2 to prove Eq. (4.1), then by triangle inequality one concludes Eq. (4.2). However, the techniques in the proof of Theorem 2 heavily relies on the fact that we can separate the quantity of interests into two independent parts  $X$  and  $S^T S Z$ . For this conjecture on the rate of norm invariance, there is no such luxury property that we can exploit.*

Now let us analyze in detail to see how much distortion is introduced in the projected norm. From the variance calculation Eq. (3.1), we know there is an error term at least the order  $O(\frac{1}{\sqrt{m+n}})$ . Then let us analyze two special cases  $m = 1, n \rightarrow \infty$  and  $m \rightarrow \infty, n = 1$ .

For  $m = 1, n \rightarrow \infty$ , we find

$$\begin{aligned} \frac{(X^T S^T S X - mn)}{\sqrt{\sigma^2 m^2 n + 2mn^2}} &= \frac{(\sum_{i=1}^n S_{1,i} x_i)^2 - n}{\sqrt{m} \sqrt{\sigma^2 n + 2n^2}} \\ &= \frac{1}{\sqrt{m}} \sqrt{\frac{n^2}{\sigma^2 n + 2n^2}} \left[ \left( \frac{\sum_{i=1}^n S_{1,i} x_i}{\sqrt{n}} \right)^2 - 1 \right] \\ &\approx \frac{1}{\sqrt{m}} \sqrt{\frac{n^2}{\sigma^2 n + 2n^2}} [\mathcal{N}(0, 1)^2 - 1] \\ &= \frac{1}{\sqrt{m}} [\mathcal{N}(0, 1) + O(1)] \end{aligned}$$

Since  $m = 1$ , we see the error term is  $O(\frac{1}{\sqrt{m}})$ .

For  $m \rightarrow \infty, n = 1$ , similarly we compute

$$\begin{aligned} \frac{(X^T S^T S X - mn)}{\sqrt{\sigma^2 m^2 n + 2mn^2}} &= \frac{x_1^2 \left( \sum_{i=1}^m S_{i,1}^2 \right) - m}{\sqrt{n} \sqrt{\sigma^2 m^2 + 2m}} \\ &= \frac{x_1^2 \left( \sum_{i=1}^m S_{i,1}^2 - m \right) + m(x_1^2 - 1)}{\sqrt{n} \sqrt{\sigma^2 m^2 + 2m}} \\ &\approx \frac{x_1^2}{\sqrt{n}} \mathcal{N}(0, 1) + \frac{x_1^2 - 1}{\sqrt{\sigma n}} \\ &= \mathcal{N}(0, 1) + O\left(\frac{1}{\sqrt{n}}\right) \end{aligned}$$

In this case the error term is on the scale of  $O(\frac{1}{\sqrt{n}})$ .

In general cases of  $m$  and  $n$ , we believe both  $O(\frac{1}{\sqrt{m}})$  and  $O(\frac{1}{\sqrt{n}})$  are necessary.

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