Final Project: Heston Model Calibration

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Task Distribution:

Juntao: Collecting and cleaning data, building Heston model, writing report

Yiren: Model Calibration, testing monotone behavior and stability of parameter

Cheng-Yun Tsai: Monte Carlo implementation and Literature Review, writing report

1 Literature Review of Heston Model

Since the Black-Scholes model is proposed as the criterion of the option pricing model, much research has been devoted to identifying its limitation. First of all, the BS model assumes a constant volatility, which is not sufficient to capture the existing term structure of implied volatilities. Moreover, the BS model also find its difficulty to deal with the volatility smile feature of options. Heston (1993) proposed an option pricing model with stochastic volatility. The identical processes for its underlying return and variance equip the model the ability to generate volatility smiles, and the inclusion of mean-reverting CIR process allows the model to capture the term structure.

Since the Heston model being proposed, the model became popular to apply on financial markets according to its feature of stochastic volatility. For example, Remer Mahnke (2004) develop probability density distribution of the logarithmic returns based on the stationary solution derived from the Heston model. They showed that The Heston model fits the center part and the height of the empirical probability density distribution well, but it underestimates the probability of finding large returns.

Besides applications on the original model, many research papers proposed advanced pricing models based on the Heston model. For instance, Christoffersen et al. (2009) proposed to use a two-factor Heston model to explain largely independent fluctuations in its level and slope over time by modeling the two uncorrelated variance processes. Gauthier Possamaï (2010) improved the call option formula of the two-factor Heston model given in Christoffersen et al. (2009) and compared the performance and efficiency of different schemes for Monte Carlo analysis. Furthermore, Benhamou et al. (2010) derived an analytical formula for the price of vanilla options, for any time dependent Heston model using a small vol of vol expansion and the Malliavin calculus techniques.

In terms of Heston model calibration methodologies, Dimitroff et al. (2011) provided an easy-to-implement calibration algorithm in their research. They used the historical asset time series to determine the correlations, and further calibrate the parameters in the model. Zhang et al. (2017) provided an exact formula for the skewness of stock returns implied in the Heston model and apply their skewness formula in calibrating the model. In Mrázek Pospíšil (2017), they proposed a novel calibration procedure that initialize parameters with genetic algorithm and then refined using nonlinear least-squares method. As the result, their calibration method significantly outperforms other existing calibration methods.

In this research project, our aim is to calibrate the parameters in the Heston model. Following the previous research papers, we consider the calibration as a optimization problem that minimizes the error based on nonlinear least-squares method.

2 Math Behind Heston Model

2.1 Heston Model

The stock price follows a Geometric Brownian Motion, which has the following SDE:

$$dS = \mu S dt + \sqrt{(}V_t)SdW_{1t} \tag{1}$$

The Heston Model is used to evaluate volatility of underlying asset.

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dW_{2t} \tag{2}$$

where V_t is the volatility of stock price and σ is the volatility of volatility. The Heston pricing formula for a call option is:

$$C = S_t P_1 - Ke^{-rt}P_2 \tag{3}$$

where

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-isln(K)}f_j]}{is}]ds \tag{4}$$

Thus, the full formula is:

$$C = S_t(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-isln(K)}f_1}{is}]ds) - Ke^{-rt}(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-isln(K)}f_2}{is}]ds) \tag{5}$$

$$= \frac{1}{2}(S_t - Ke^{-rt}) + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-isln(K)}f_1}{is} - \frac{e^{-isln(K)}f_2}{is}]ds \tag{6}$$

f_j is defined as:

$$f_j = e^{C_j + D_j + isln(S_t)} \tag{7}$$

$$C_j = ris\tau + \frac{\kappa\theta}{\sigma^2}[(b_j - \rho * \sigma * i * s + d_j)\tau - 2ln(\frac{1 - g_j e^{d_j\tau}}{1 - g_j})] \tag{8}$$

$$D_j = \frac{b_j - \rho * \sigma * i * s + d_j}{\sigma^2}(\frac{1 - e^{d_j \tau}}{1 - g_j e^{d_j \tau}}) \tag{9}$$

where:

$$d_j = \sqrt{(\rho * \sigma * i * s - h_j)^2 - \sigma^2(2 * u_j * i * s - s^2)} \tag{10}$$

where

$$u_1 = u_2 = 0.5, h_1 = \kappa - \rho\sigma, h_2 = \kappa \tag{11}$$

$$g_j = \frac{b_j - (\rho * \sigma * i * s) + d_j}{b_j - (\rho * \sigma * i * s) - d_j} \tag{12}$$

2.2 Greeks

In this project, we only use the European option. We calculated delta, gamma, and theta. There are many ways to calculate Greeks. For simplicity, we calculate delta as:

$$S_1 = S + ds \tag{13}$$

We calculate the option price using Heston model with input S_1 and S, getting after price and original price. See Figure 1 and Figure 2.

$$delta = \frac{afterprice - originalprice}{ds} \tag{14}$$

we calculate gamma with changes in delta. See Figure 3 and Figure 4.

$$gamma = \frac{afterdelta - originaldelta}{ds} \tag{15}$$

Theta is calculated as:

$$T1 = T + dt \tag{16}$$

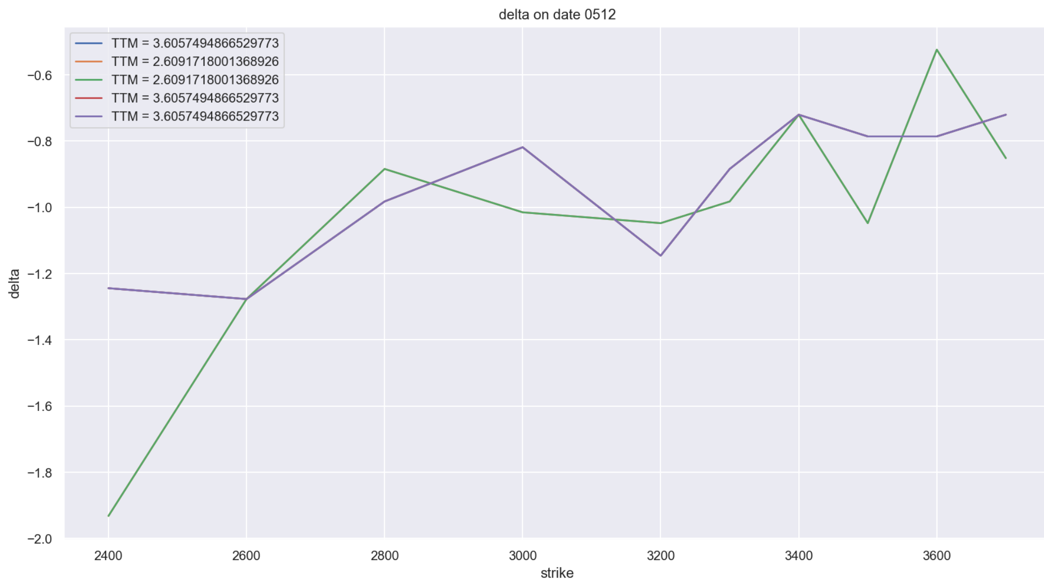


Figure 1: delta on 0512

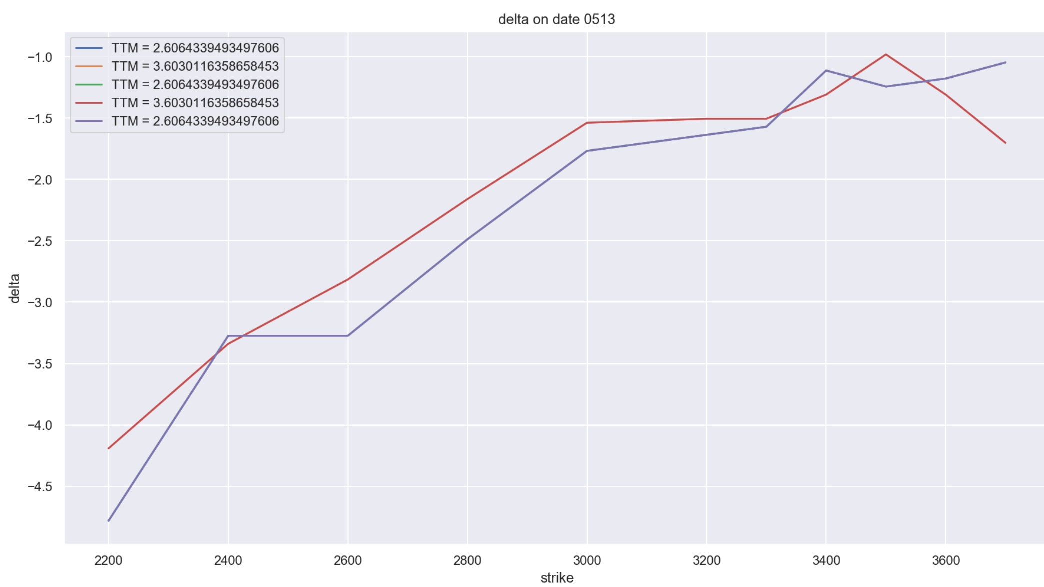


Figure 2: delta on 0513

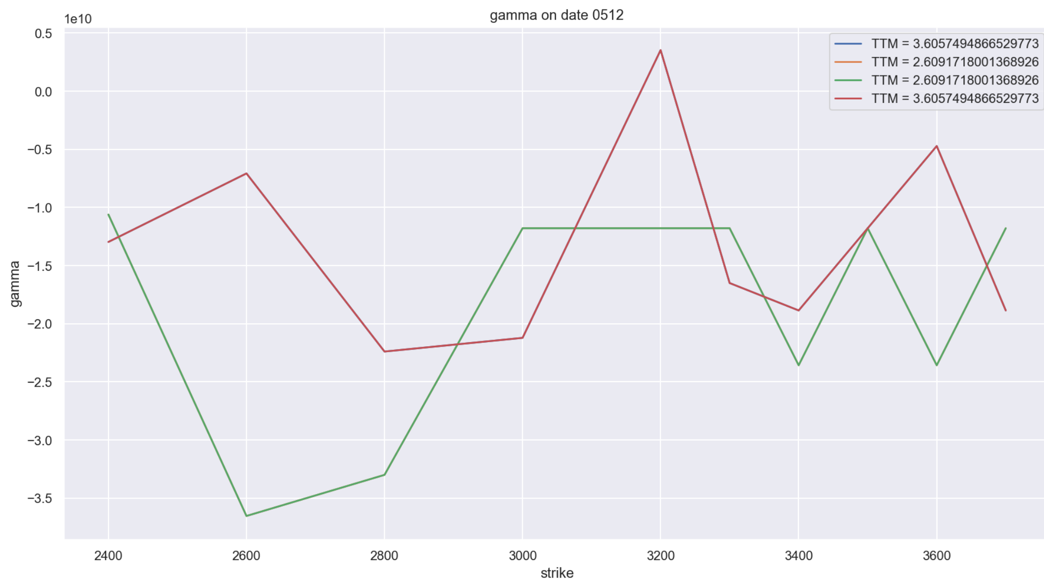


Figure 3: gamma on 0512

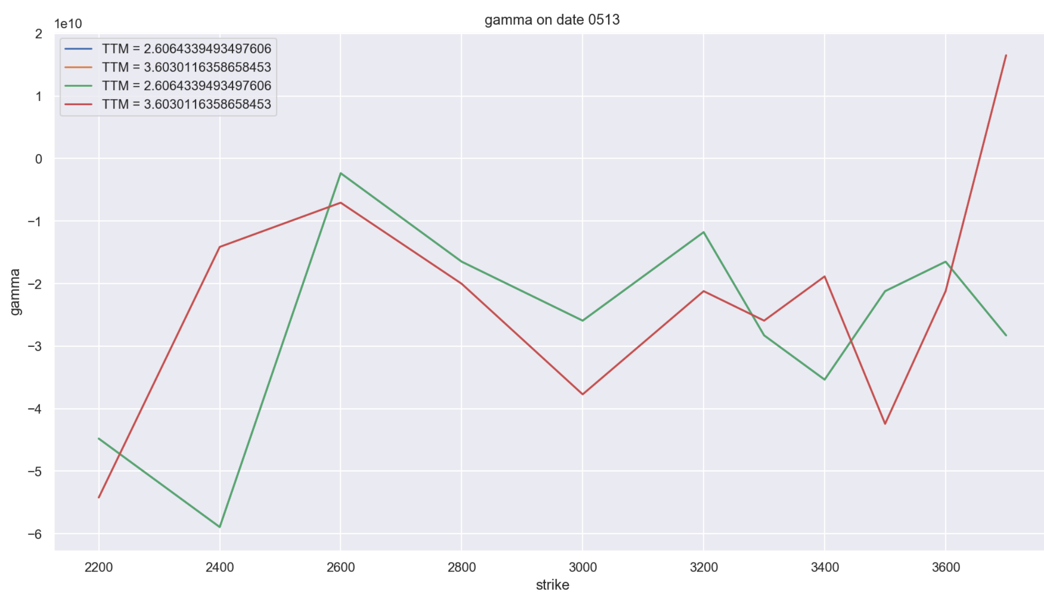


Figure 4: gamma on 0513

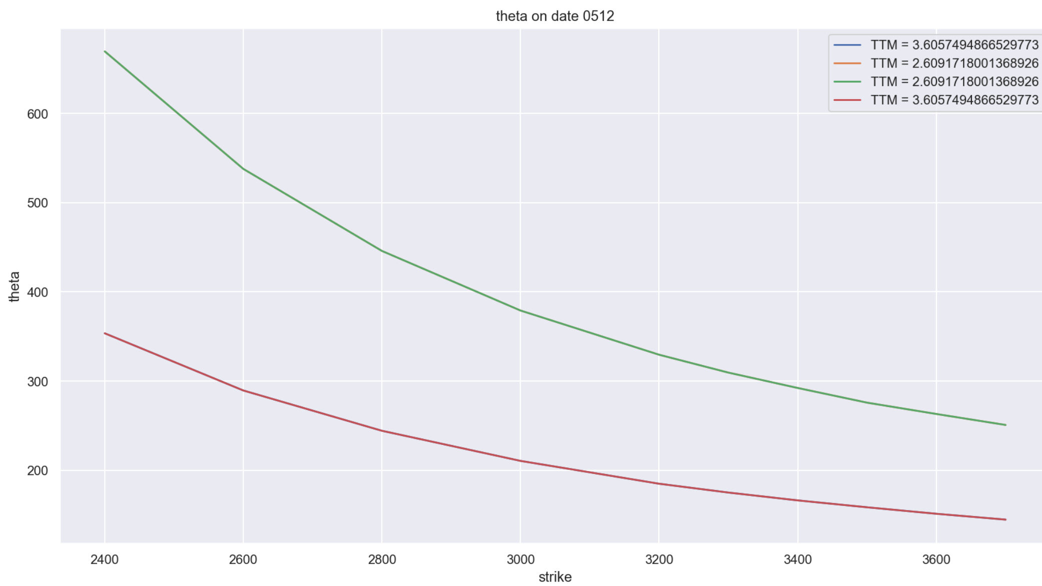


Figure 5: theta on 0512

We calculate the option using Heston model with T1 and T, getting after price and original price. See Figure 5 and Figure 6.

$$theta = \frac{afterprice - originalprice}{dt} \tag{17}$$

3 Model Calibration: finding parameter

3.1 Finding Option data

The first step is to find the data. We use Bloomberg to find the S&P 500 index option with different maturities and different strike price, as of day May 12th and May 13th. We selected options that are in the money, otherwise their price would be close to 0. After exporting all the data into a spreadsheet. We started writing Python codes to

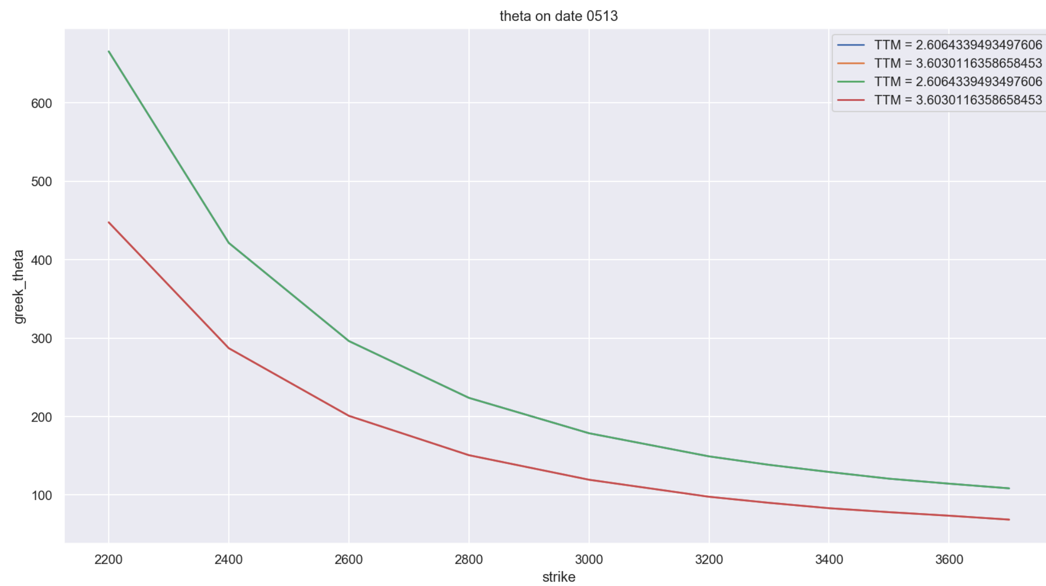


Figure 6: theta on 0512

clean up everything. We only kept strike price, maturity, mid price, implied volatility in the dataframe. Mid price is defined as:

$$midprice = \frac{bid + ask}{2} \tag{18}$$

in all the calculations.

3.2 Finding Treasury rates

Daily Treasury rates are found on the U.S. Department of the Treasury website. To obtain interest rate curve, we use the Nelson Siegel Svensson method. Maturities are

$$\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{6}{12}, 1, 2, 3, 5, 7, 10, 20, 30 \tag{19}$$

3.3 Calibration

We begin by initializing parameter. Our starting values are:

$$V_t = 0.5, \kappa = 0.5, \theta = 0.5, \rho = -0.5, \sigma = 0.5 \tag{20}$$

where σ stands for volatility of volatility.

We also set a lower bound and a upper bound for all the parameters we want to estimate. Hence, we plug in the parameters into the Heston model defined in part 1, and then calculating the difference between the market price and the estimated price. We use lmfit package (Non-linear least squares minimization and curve fitting package in Python) with the least square method to minimize the difference.

We fit the model with market quotes on day May 12th and May 13th. See Figure 7. The results for May 12th:

$$V_t = 0.01001701, \kappa = 0.77535479, \theta = 0.01000000, \rho = -3.5882e{-10}, \sigma = 0.04738423 \tag{21}$$

May 13th:

$$V_t = 0.12800599, \kappa = 1.24624354, \theta = 0.01000000, \rho = -1, \sigma = 0.08973167 \tag{22}$$

From Figure 7, we see that $\kappa, \theta, \sigma, V_t$ are stable. ρ is not very stable.

3.4 Implied Volatility and Volatility Surface

Using the estimated option price and Black-Scholes Model, we calculated the implied volatility on May12 th and May 13th. See Figure 8 and Figure 9

We have also calculated the volatility smile on May 12th. See Figure 10.

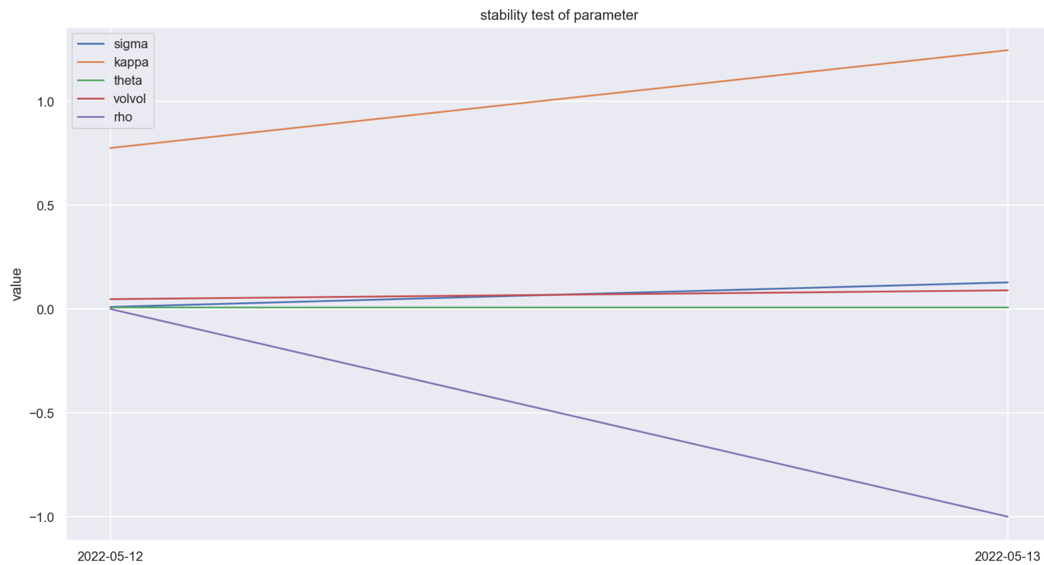


Figure 7: stability of parameter

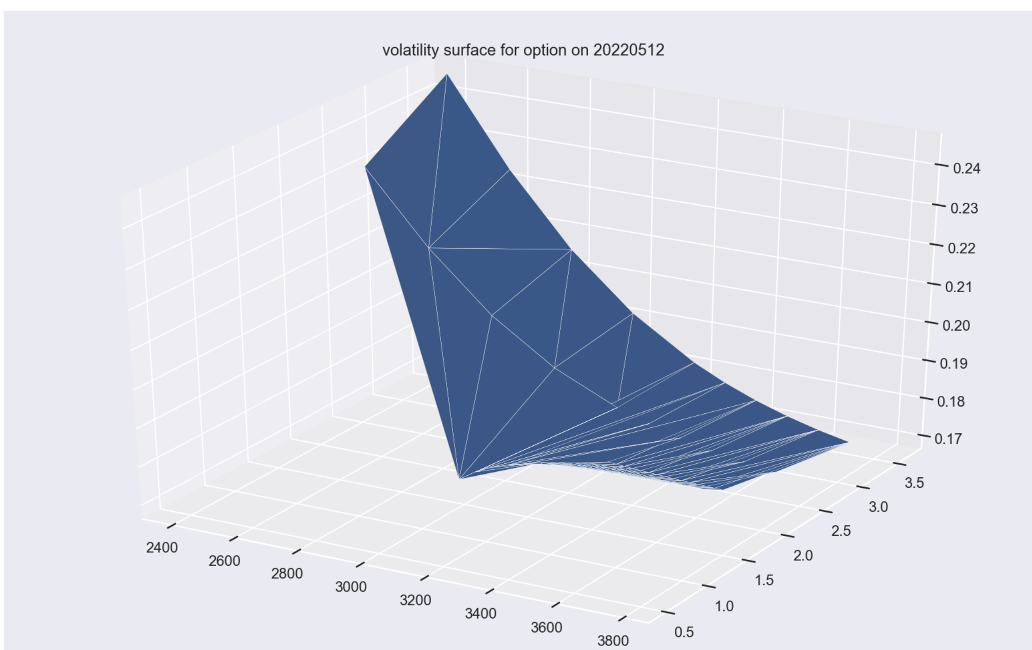


Figure 8: volatility surface on 0512

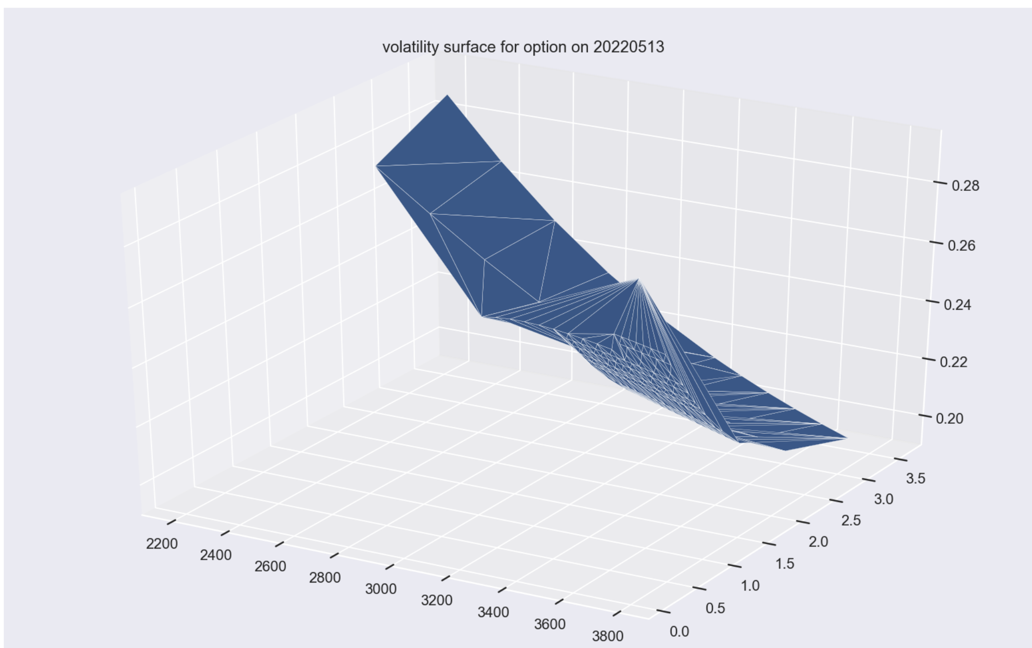
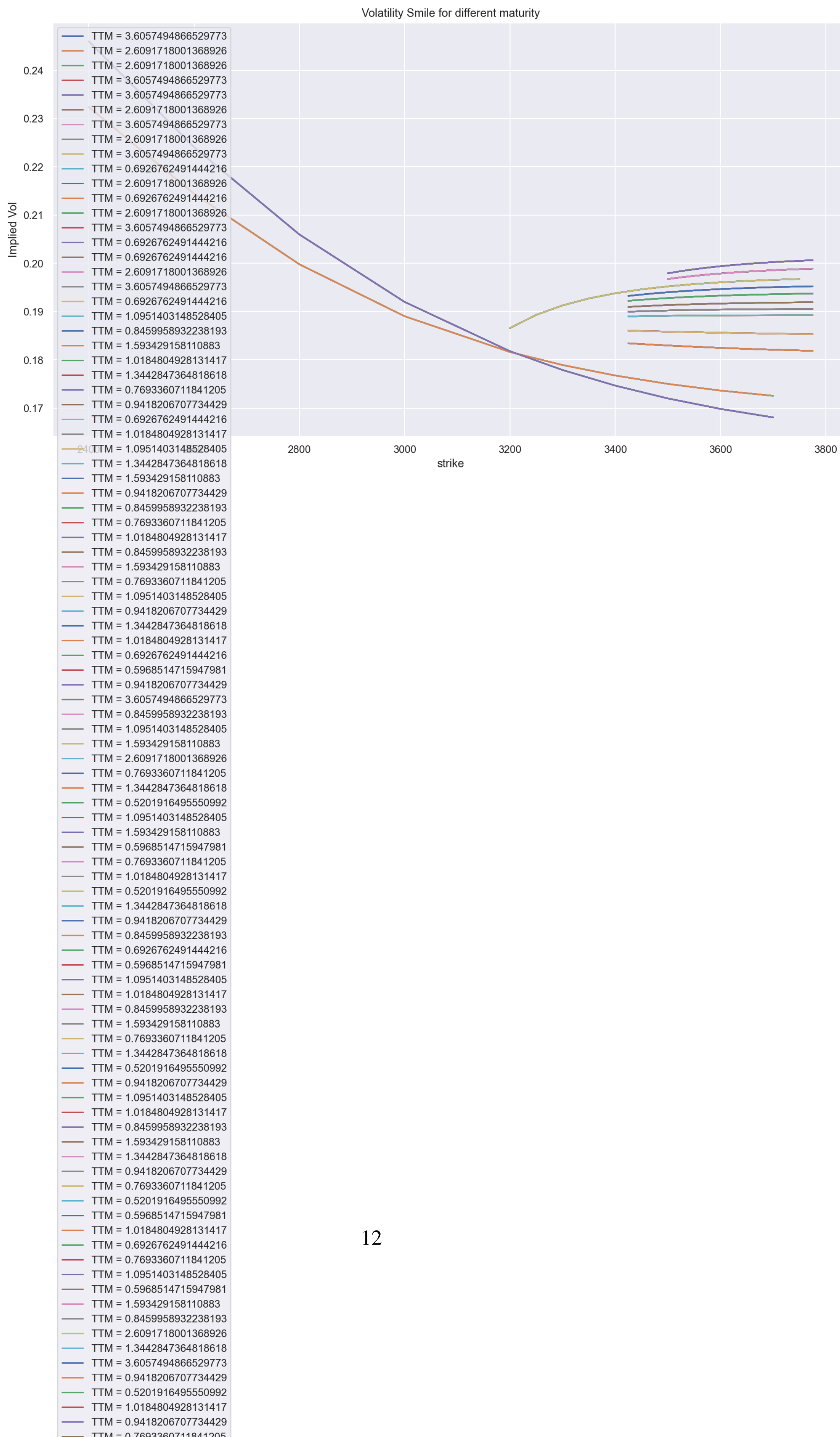


Figure 9: volatility surface on 0513


Volatility Smile for different maturity
Implied Vol
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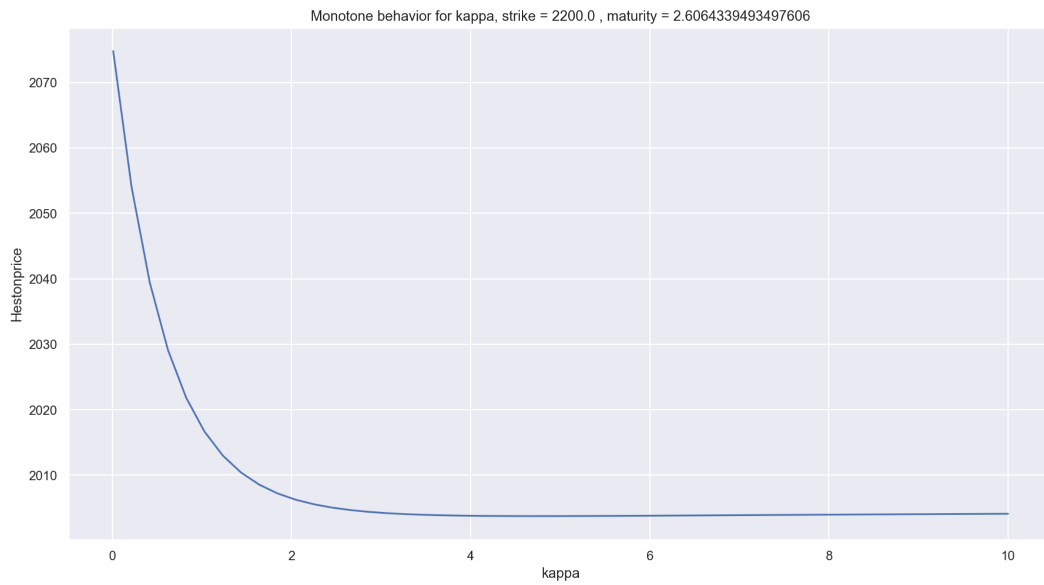

Figure 11: kappa

4 Testing

4.1 Monotone Behavior

Now, we want to check the monotone behavior of parameter. We select a option with strike price of 2200 and maturity of 2.6064339493497606.

First, we observe κ. κ is the mean-reversion factor. The larger the kappa, the sooner the volatility of price returns back to its long-run variance of price. We observe from the graph that option price stabilize after kappa = 2. Smaller volatility also leads to lower option price. See Figure 11.

Then, we want to observe $\sigma.\sigma$ is the volatility of volatility. Holding all else constant, as σ increases, volatility of price increases. Higher volatility leads to higher option price. See Figure 12

Next, we observe the behavior of V_t. V_t is the volatility of index, nad it exhibits a bell-shaped curve. The curve reaches its maximum at 1. This is probably because of

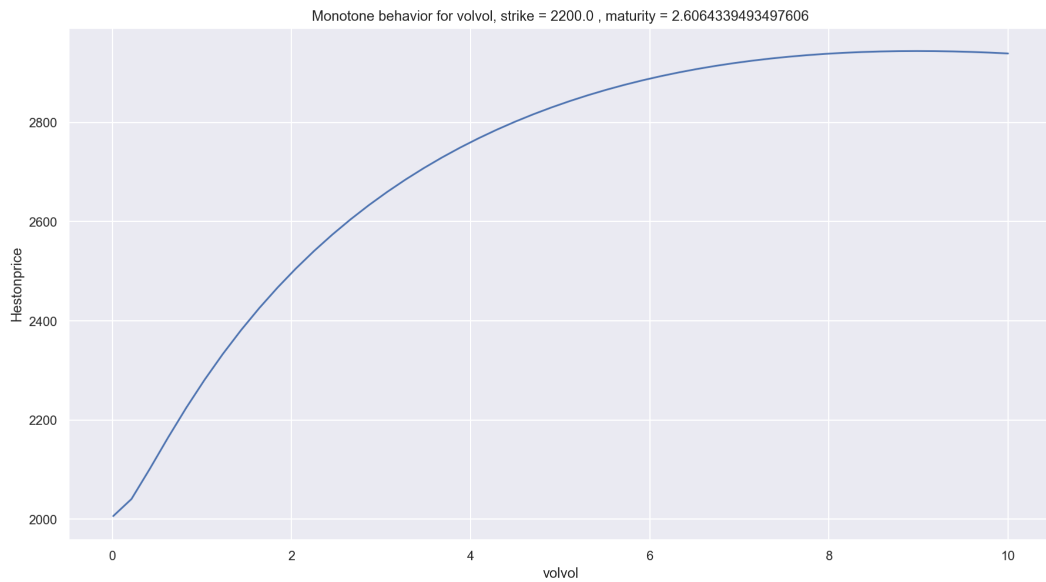

Figure 12: volatility of volatility

our selection of theta, which is approximately equals to 1. As a result, the difference between theta and V_t is very close to 0. dV_t is purely dominated by its diffusion term. See Figure 13

4.2 Monte Carlo

This section uses Monte Carlo Simulation to confirm our Heston model generates the correct pricing results. To implement Monte Carlo for the Heston model, we simulate both Geometric Brownian Motion and CIR process and make the correlation of two Brownian Motions, the drift of the underlying asset process and the coefficients in the CIR mean-reverting process equal to the calibrated parameters. We simulate 10,000 stock price and volatility paths for each option. Figure 14 and Figure 15 show the options price from our Heston model and Monte Carlo simulation on 05/12 and 05/13. The results show that the options prices from Heston model and Monte Carlo are quite similar. Heston model tends to generate higher prices in 05/12 but lower in 05/13.

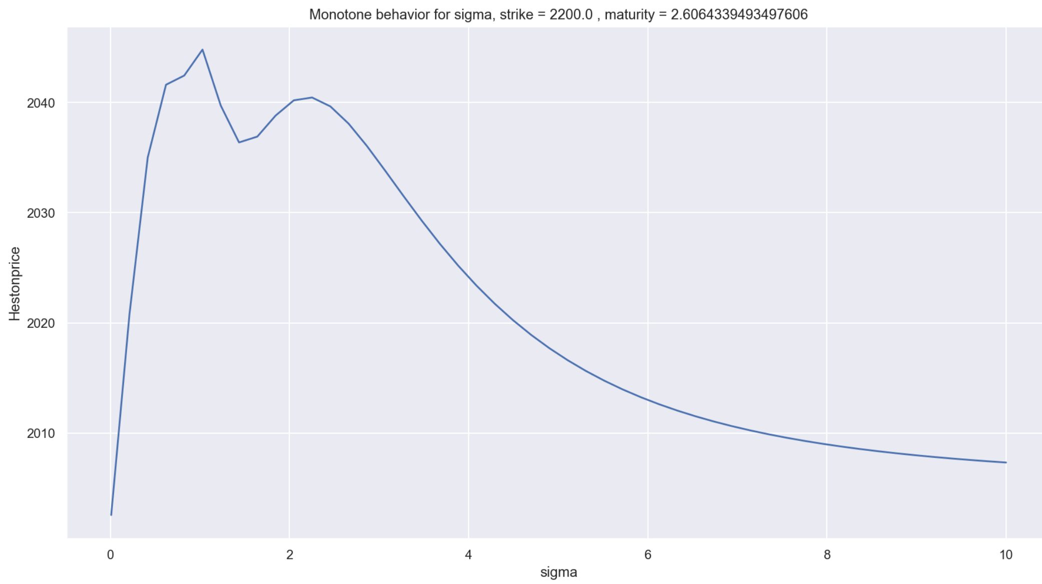

Figure 13: V_t

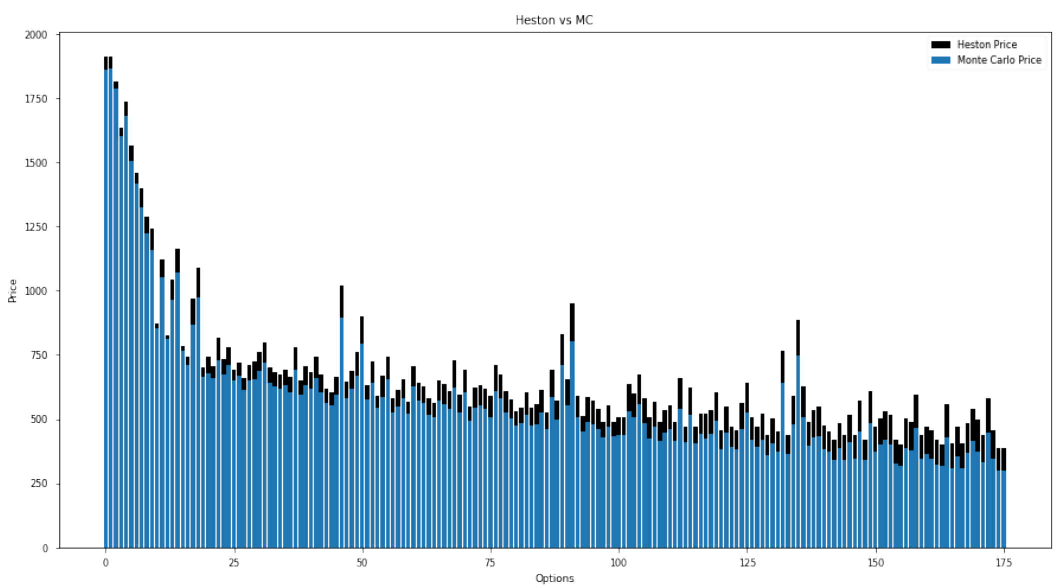

Figure 14: Comparing MC model with Heston model on 0512

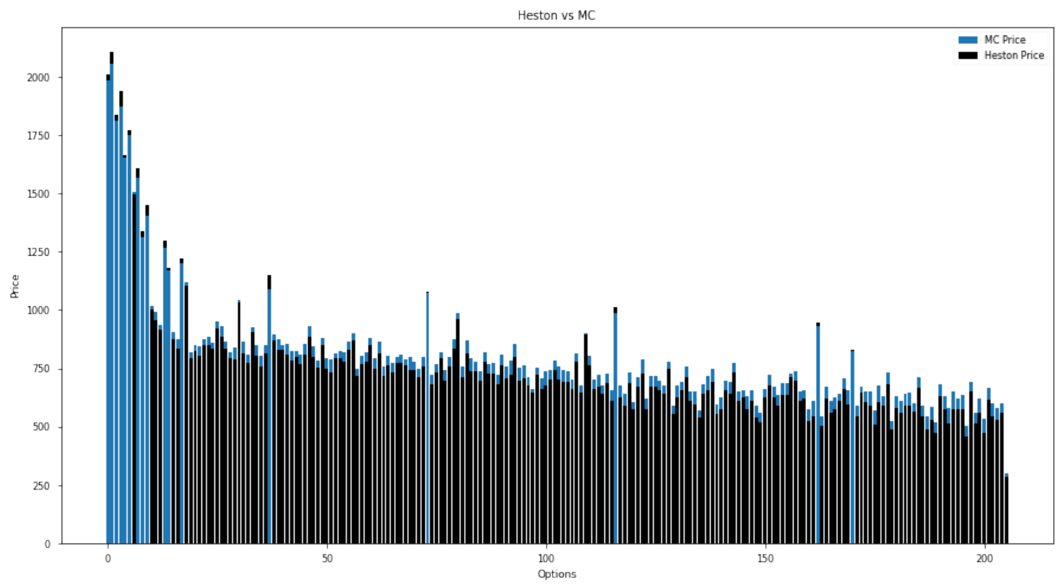

Figure 15: Comparing MC model with Heston model on 0513

5 Limitations

There are several limitations of this project. First, because of limitation of time, we have only try market quotes of day May 12th and May 13th. If possible, we should have run our model on multiple days. Second, we have only try lmfit-least square method in this project. If we have more time, we would try different optimizers. This is because the parameters in the Heston model can be easily affected by the algorithms we choose.

Advantages of Heston model includes:

1. The volatility is mean reverting.

2. It has a closed-form solution.

3. The model considers the negative correlation of stock returns and volatility, and it permits the correlation to be changed.

Disadvantages of Heston model includes:

1. The parameters of this model is hard to estimate. They depends on the algorithm

used in the calibration.

2. The estimated option price is highly sensitive to changes in parameter, as we can observed from the monotone behavior of parameters section.

6 References

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