- Basic model set:

$D = \{(x_i, y_i)\}_{i=1}^{n}$ with $x_i \in \mathbb{R}^M$

$$\beta_m \sim N(0, \sigma_\beta^2)\pi_\beta + \delta_0(\beta_m)(1-\pi_\beta) \quad m=0,1,\dots,M$$

$$y_i \sim N(\beta_0 + \sum_{m=1}^{M} \beta_m x_{im}, \sigma_e^2) \quad , i=1,\dots,n$$

$$\pi_\beta \sim \mathrm{Beta}(a,b) \quad , a>0, b>0$$

$$\sigma_e^2 \sim \text{Inverse Gamma}(q_1, q_2)$$

$\theta = (\sigma_\beta^2, a, b, \beta_0)$ are hyperparameters optimized using gradient descend

Now we reparametrize β_m as

$$\beta_m = \delta_m \tilde{\beta}_m$$

where $\tilde{\beta}_m \sim N(0, \sigma_\beta^2)$

$$\delta_m \sim \pi_\beta^{\delta_m}(1-\pi_\beta)^{1-\delta_m}$$

$$\Rightarrow f(\tilde{\beta}_m, \delta) \sim N(\tilde{\beta}_m | 0, \sigma_\beta^2)\pi_\beta^{\delta_m}(1-\pi_\beta)^{1-\delta_m}$$

achieving the same likelihood with $\beta_m = \tilde{\beta}_m \delta$

We want to approximate $p(\beta, \pi_\beta, \sigma_e | X, y) = p(\tilde{\beta}, \delta, \pi_\beta, \sigma_e | X, y)$

We select the following approximation:

$$q(\tilde{\beta}, \delta, \pi_\beta, \sigma_e) = \prod_{i=1}^{M} N(\tilde{\beta}_m | \mu_m, \sigma_{\beta_m}^2)\pi_m^{\delta_m}(1-\pi_m)^{(1-\delta_m)} \times$$

$$\mathrm{Beta}(\pi_\beta | c, d) \times IG(q_3, q_4)$$

The variational parameters are $\phi = (\mu_m, \pi_m, \sigma^2_{\beta_m}, c, d, q_3, q_4)$

By introducing δ, we get a smooth variational approximation function once integrating out δ.

Evidence lower bound:

$$ELBO = \mathbb{E}_{\substack{\tilde{\beta}, \delta \sim q_\phi \\ \pi_\beta, \sigma_e}}\left[\ln \frac{P_\theta(\tilde{\beta}, \delta, \pi_\beta, \sigma_e, y \mid X)}{q_\phi(\tilde{\beta}, \delta, \pi_\beta, \sigma_e)}\right]$$

$$= \mathbb{E}_{\substack{\tilde{\beta}, \delta \sim q_\phi \\ \pi_\beta, \sigma_e}}\left[\ln P_\theta(\tilde{\beta}, \delta, \pi_\beta, \sigma_e, y \mid X) - \ln q_\phi(\tilde{\beta}, \delta, \pi_\beta, \sigma_e)\right]$$

$$= \mathbb{E}_{\substack{\tilde{\beta}, \delta \sim q_\phi \\ \pi_\beta, \sigma_e}}\left\{\ln\left[\mathbb{P}_\theta(y \mid \tilde{\beta}, \delta, \pi_\beta, \sigma_e, X)\, \mathbb{P}_\theta(\tilde{\beta}, \delta, \pi_\beta, \sigma_e)\right]\right\}$$

$$- \mathbb{E}_{\substack{\tilde{\beta}, \delta \sim q_\phi \\ \pi_\beta, \sigma_e}}\left[\ln q_\phi(\tilde{\beta}, \delta, \pi_\beta, \sigma_e)\right]$$

$$= \mathbb{E}_{\substack{\tilde{\beta}, \delta \sim q_\phi \\ \pi_\beta, \sigma_e}}\left\{\log \underbrace{\mathbb{P}_\theta(y \mid \tilde{\beta}, \delta, \pi_\beta, \sigma_e, X)}_{\text{data likelihood}} + \log \underbrace{\mathbb{P}_\theta(\tilde{\beta}, \delta, \pi_\beta, \sigma_e)}_{\text{prior}}\right]$$

$$\underbrace{- \mathbb{E}_{\substack{\tilde{\beta}, \delta \sim q_\phi \\ \pi_\beta, \sigma_e}}\left[\log q_\phi(\tilde{\beta}, \delta, \pi_\beta, \sigma_e)\right]}_{\text{entrophy}}$$

① Data likelihood

$$\mathbb{P}_\theta(y \mid \tilde{\beta}_{\pi_\beta, \sigma_e}, \delta, X) = \prod_{i=1}^{n} \frac{1}{\sigma_e \sqrt{2\pi}} \exp\left\{ -\frac{\left(y_i - \sum_{m=1}^{M} x_{im}(\tilde{\beta}_m \delta_m) - \beta_0\right)^2}{2\sigma_e^2} \right\}$$

$$\log \mathbb{P}_\theta(y \mid \tilde{\beta}_{\pi_\beta, \sigma_e}, \delta, X) = \sum_{i=1}^{n} \left[-\log(\sigma_e \sqrt{2\pi}) - \frac{\left(y_i - \sum_{m=1}^{M} x_{im}(\tilde{\beta}_m \delta_m) - \beta_0\right)^2}{2\sigma_e^2} \right]$$

$$= -\frac{n}{2} \log(\sigma_e^2 2\pi) - \frac{1}{2\sigma_e^2} \sum_{i=1}^{n} \left(y_i - \sum_{m=1}^{M} x_{im}(\tilde{\beta}_m \delta_m) - \beta_0\right)^2$$

We can take the expectation for σ_e^2 by independence of the guide

$$\mathbb{E}_{\sigma_e \sim q_\phi}\left[\log \mathbb{P}_\theta(y \mid \tilde{\beta}, \delta, \pi_\beta, \sigma_e)\right]$$

$$= -\frac{n}{2}\left[\ln(q_4) - \psi(q_3) + \log(2\pi)\right] + \frac{1}{2}\frac{q_3}{q_4} \sum_{i=1}^{n} \left(y_i - \sum_{m=1}^{M} x_{im}(\tilde{\beta}_m \delta_m) - \beta_0\right)^2$$

Need to use numerical approximation to expectation of $\tilde{\beta}, \pi_\beta, \delta$

there is no π_β since it is dependent on δ.

② Prior

$$\mathbb{P}_{\theta}(\tilde{\beta},\delta,\pi_\beta,\pi_\beta,\sigma_e^2)=\left[\prod_{m=1}^{M}\pi_\beta^{\delta_m}(1-\pi_\beta)^{1-\delta_m}\frac{1}{\sigma_\beta\sqrt{2\pi}}\exp\left(-\frac{\tilde{\beta}_m^2}{2\sigma_\beta^2}\right)\right]\frac{\pi_\beta^{a-1}(1-\pi_\beta)^{b-1}}{B(a,b)}$$

$$\times\frac{q_2^{q_1}}{\Gamma(q_1)}\left(\frac{1}{\sigma_e^2}\right)^{q_1+1}\exp\left(-\frac{q_2}{\sigma_e^2}\right)$$

$$\log\mathbb{P}_{\theta}(\tilde{\beta},\delta,\pi_\beta,\pi_\beta,\sigma_e^2)=\sum_{m=1}^{M}\left[\delta_m\log(\pi_\beta)+(1-\delta_m)\log(1-\pi_\beta)-\log(\sigma_\beta\sqrt{2\pi})-\frac{\tilde{\beta}_m^2}{2\sigma_\beta^2}\right]+(a-1)\log\pi_\beta+(b-1)\log(1-\pi_\beta)$$

$$-\left(\log(\Gamma(a))+\log(\Gamma(b))-\log(\Gamma(a+b))\right)$$

$$+q_1\log(q_2)-\log(\Gamma(q_1))-(q_1+1)\log(\sigma_e^2)-\frac{q_2}{\sigma_e^2}$$

$$=\log(\pi_\beta)\sum_{m=1}^{M}\delta_m+\log(1-\pi_\beta)\sum_{m=1}^{M}(1-\delta_m)-M\log(\sigma_\beta\sqrt{2\pi})$$

$$-\frac{\sum_{m=1}^{M}\tilde{\beta}_m^2}{2\sigma_\beta^2}+(a-1)\log\pi_\beta+(b-1)\log(1-\pi_\beta)$$

$$-\log(\Gamma(a))-\log(\Gamma(b))+\log(\Gamma(a+b))$$

$$+q_1\log(q_2)-\log(\Gamma(q_1))-(q_1+1)\log(\sigma_e^2)-\frac{q_2}{\sigma_e^2}$$

$$\mathbb{E}_{\pi_\beta\sim\phi}(\log\pi_\beta)=\psi(c)-\psi(c+d)$$

$$\mathbb{E}_{\pi_\beta\sim\phi}(\log(1-\pi_\beta))=\psi(d)-\psi(c+d)$$

$$\mathbb{E}_{\sigma_e^2\sim\phi}(\log(\sigma_e^2))=\ln(q_4)-\psi(q_3)$$

$$\mathbb{E}_{\sigma_e^2\sim\phi}((\sigma^2)^{-1})=\frac{q_3}{q_4}$$

$$\Rightarrow\mathbb{E}\quad\left[\log\mathbb{P}_{\theta}(\tilde{\beta},\delta,\pi_\beta)\right]$$

$$= \mathbb{E}_{\tilde{\beta},\delta,\pi_\beta \sim q_\phi}\left[\log p_\theta(\cdot)\right]$$

$$= \mathbb{E}(\log \pi_\beta) \sum_{m=1}^{M} \pi_m + \mathbb{E}\left[\log(1-\pi_\beta)\right] \sum_{m=1}^{M} (1-\pi_m) - M \log(\sigma_\beta \sqrt{2\pi})$$

$$- \frac{\sum_{m=1}^{M} (\sigma_{\beta_m}^2 + \mu_m^2)}{2\sigma_\beta^2} + (a-1)\,\mathbb{E}(\log \pi_\beta) +$$

$$(b-1)\,\mathbb{E}(\log(1-\pi_\beta)) - \log(\Gamma(a)) - \log(\Gamma(b)) + \log(\Gamma(a+b))$$

$$+ q_1 \log(q_2) - \log(\Gamma(q_1)) - (q_1+1)\,\mathbb{E}(\log(\sigma_e^2)) - q_2\,\mathbb{E}((\sigma_e^2)^{-1})$$

ψ is the digamma function, Γ is the gamma function.
No need to do numerical integration
include all the variational parameters

③ The negative entrophy term

We separate into three parts: $\tilde{\beta}, \delta$; π_β ; σ_e^2

ⓐ For $\tilde{\beta}, \delta$

$$q_\phi(\tilde{\beta}, \delta) = \prod_{m=1}^{M} \frac{1}{\sigma_{\beta_q}\sqrt{2\pi}} \exp\left(\frac{-(\tilde{\beta}_m - \mu_m)^2}{2\sigma_{\beta_m}^2}\right) \pi_m^{\delta_m} (1-\pi_m)^{1-\delta_m}$$

$$\log q_\phi(\tilde{\beta}, \delta) = \sum_{m=1}^{M}\left[-\log(\sigma_{\beta_m}\sqrt{2\pi}) - \frac{(\tilde{\beta}_m - \mu_m)^2}{2\sigma_{\beta_m}^2} + \delta_m \log(\pi_m) + (1-\delta_m)\log(1-\pi_m)\right]$$

$$\mathbb{E}_{\tilde{\beta},\delta \sim q_\phi}\left[\log q_\phi(\tilde{\beta}, \delta)\right]$$

$$= \sum_{m=1}^{M} \mathbb{E}_{\tilde{\beta},\delta}\left[\log q_\phi(\tilde{\beta}_m, \delta_m)\right]$$ by indepent variables entropy

$$= \sum_{m=1}^{M} \mathbb{E}_{\tilde{\beta}, \gamma}\left[\log \tau_m^{b_m}(1-\tau_m)^{1-b_m} N(\tilde{\beta}_m \mid \mu_m, \sigma_{\beta_m})\right]$$

$$= \sum_{m=1}^{M}\Big[\ \mathbb{E}_{\tilde{\beta}}\left[\tau_m \log \tau_m N(\tilde{\beta}_m \mid \mu_m, \sigma_{\beta_m})\right]$$

$$+ \mathbb{E}_{\tilde{\beta}}\left[(1-\tau_m)\log\left[(1-\tau_m) N(\tilde{\beta}_m \mid \mu_m, \sigma_{\beta_m})\right]\right]\Big\}\Big]$$

$$= \sum_{m=1}^{M}\Big[\tau_m\left(\log \tau_m + \mathbb{E}_{\tilde{\beta}}\left[N(\tilde{\beta}_m \mid \mu_m, \sigma_{\beta_m})\right]\right)$$

$$+ (1-\tau_m)\left(\log(1-\tau_m) + \mathbb{E}_{\tilde{\beta}}\left[N(\tilde{\beta}_m \mid \mu_m, \sigma_{\beta_m})\right]\right)\Big]$$

$$= \sum_{m=1}^{M}\Big[\tau_m\left(\log \tau_m - H(N(\tilde{\beta}_m \mid \mu_m, \sigma_{\beta_m}))\right)$$

$$+ (1-\tau_m)\left(\log(1-\tau_m) - H(N(\tilde{\beta}_m \mid \mu_m, \sigma_{\beta_m}))\right)\Big]$$

$$= \sum_{m=1}^{M}\Big[\tau_m\left(\log \tau_m - 0.5\log(2\pi e \sigma_{\beta_m}^2)\right)$$

$$+ (1-\tau_m)\left(\log(1-\tau_m) - 0.5\log(2\pi e \sigma_{\beta_m}^2)\right)\Big]$$

$$= \sum_{m=1}^{M}\Big[\tau_m \log \tau_m - 0.5\tau_m \log(2\pi e \sigma_{\beta_m}^2) +$$

$$(1-\tau_m)\log(1-\tau_m) - 0.5(1-\tau_m)\log(2\pi e \sigma_{\beta_m}^2)\Big]$$

ⓑ For π_β

$$\mathbb{E}_{\pi_\beta \sim q_\phi}\left[-\log q_\phi(\pi_\beta)\right] = \ln\left(\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}\right)$$

$$-(c-1)\psi(c)-(d-1)\psi(d)+(c+d-2)\psi(c+d)$$

ⓒ For σ_e^2

$$\mathbb{E}_{\sigma_e^2 \sim q_\phi}\left[-\log q_\phi(\sigma_e^2)\right]$$

$$= q_3 + \ln(q_4) + \ln(\Gamma(q_3)) - (q_3+1)\psi(q_3)$$