About

After addressing comments from supervisors from the meeting note of Feb 20, 2023 , this document aims to achieve the following:

1. Formulate equations of different NPFG algorithms that can be used for **Multirotor**
2. Define the **criteria** for a good formulation (which one is the best?)

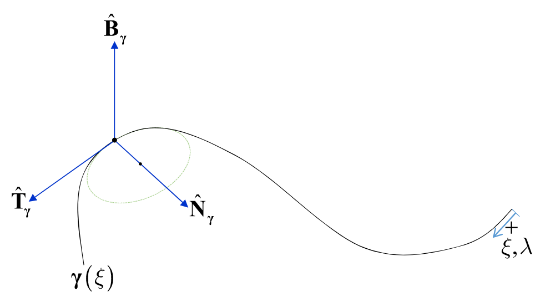
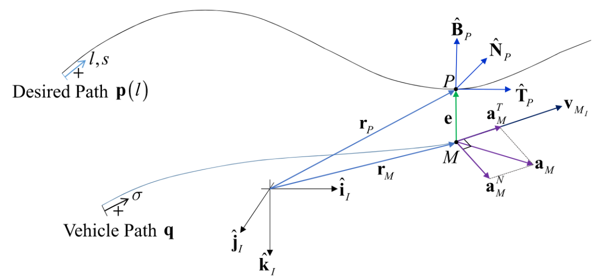
Additionally, it could:

1. Include Jerk limits

🍊 Problem Definition

Definition follows conventions from the paper “Three-Dimensional Nonlinear Differential Geometric Path-Following Guidance Law”.

Basic overview:

 Fig. 1 Space curve and the Frenet–Serret frame.	 Fig. 2 Geometry of 3-D path-following.
Frame definition	Geometry

Problem definition:

Nomenclature

$\boldsymbol{a}$	=	acceleration vector, m/s^2
$\hat{\boldsymbol{B}}$	=	unit binormal vector
d_{shift}	=	radially shifted distance, m
$\boldsymbol{d}$	=	shifted error vector, m
$\boldsymbol{e}$	=	tracking error vector, m
k	=	guidance gain, m^{-1}
$\hat{\boldsymbol{L}}$	=	look-ahead vector
l	=	parameter of desired path
$\hat{\boldsymbol{N}}$	=	unit normal vector
$\boldsymbol{p}$	=	desired path defined as parameterized curve in $\mathbb{R}^3$, m
$\boldsymbol{r}$	=	position vector, m
s	=	arc length along desired path (curvilinear abscissa), m
$\hat{\boldsymbol{T}}$	=	unit tangent vector
t	=	time, s
$\boldsymbol{v}$	=	velocity vector, m/s
v	=	speed, m/s
δ_{BL}	=	boundary-layer thickness, m
θ_L	=	look-ahead angle, rad
κ	=	curvature, m^{-1}
σ	=	arc length along vehicle path, m

Subscripts

a	=	flow-relative
cmd	=	command
I	=	inertial
M	=	vehicle
P	=	closest projection point, which is a reference point on the desired path

Nomenclature

Definition 2 (path-following problem): Design a guidance command $\boldsymbol{a}^N_{M_{\text{cmd}}}$ such that the following conditions are satisfied for a vehicle as $\sigma \to \infty$:

$$\boldsymbol{r}_M(\sigma) \to \boldsymbol{r}_P(s(\sigma)) \tag{11}$$

$$\hat{\boldsymbol{T}}_M(\sigma) \to \hat{\boldsymbol{T}}_P(s(\sigma)) \tag{12}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{T}}_M(\sigma)}{\mathrm{d}\sigma} \to \frac{\mathrm{d}\hat{\boldsymbol{T}}_P(s(\sigma))}{\mathrm{d}s} \tag{13}$$

Original problem formulation

However, for now we:

- **Ignore wind**
- **Only generate Ground Velocity Vector Field**

For the first step.

Further constraint on speed on path

But on top of that, we also define a few more constraints:

$$V_{path}(\xi) \ = \ Desired\ speed\ on\ the\ path\ at\ the\ position\ \gamma(\xi)$$

Which further extends the problem formulation to include:

$$As\ \sigma \rightarrow \infty:\ v(s(\sigma)) \ \rightarrow \ V_{path}(s(\sigma))$$

Which dictates that the speed on the path should be satisfied as defined in our constraint.

🐦 Different Formulations

Note: Week 9 report includes some initial rough formulations of the algorithm.

Nomenclature

$e\ =\ Current\ cross\ track\ error\ to\ path$
$t_{const}\ =\ Track\ error\ boundary\ calculation\ time\ constant$
$e_b\ =\ Track\ error\ boundary$
$\hat{e} = Normalized\ track\ error\ boundary\ (= \frac{e}{e_b})$
$V_{nom}\ =\ Nominal\ speed\ of\ the\ vehicle\ (cruise)$
$\Theta_{la}\ =\ Look\ ahead\ angle\ (away\ from\ orthogonal\ component\ to\ the\ path)$

$V_g^{\perp}\ =\ Orthogonal\ ground\ reference\ velocity\ (towards\ the\ path)$
$V_g^{||}\ =\ Parallel\ ground\ reference\ velocity\ (along\ the\ path)$

$e_{approach}^{min} = Minimum\ track\ error\ boundary\ to\ satisfy\ approaching\ velocity\ constraint$
$e_{path}^{min} = Minimum\ track\ error\ boundary\ to\ satisfy\ on\ path\ velocity\ constraint$
$a_{max}^{\perp} = Maximum\ orthogonal\ (to\ path)\ acceleration\ limit$
$a_{max}^{||} = Maximum\ parallel\ (to\ path)\ acceleration\ limit$
$j_{max} = Maximum\ jerk\ limit$

1 TJ NPFG Original

Original formulation

- **Bearing feasibility is omitted**, as we're only dealing with ground velocity vector field
- **Track keeping / minimum ground speed part is omitted**, as it is dealing with wind.
- E_b will NOT change while approaching the path, since ground velocity magnitude stays at V_nom.

$V_{approach}\ =\ V_{nom}$
$e_b = V_{approach} * t_{const}$
$\theta_{la} = constrain(\frac{\pi}{2}(1 - \hat{e})^2,\ 0,\ \frac{\pi}{2})$
$V_g^{||} = V_{nom} * sin(\theta_{la})$
$V_g^{\perp} = V_{nom} * cos(\theta_{la})$
$return\ [V_g^{||},\ V_g^{\perp}]$

2 TJ NPFG Squashed

This is a variation of the original TJ's NPFG, that squashes (or expands) ground velocity vector field component parallel to path to match V_path as vehicle approaches to allow various range on V_path values for the Multirotor.

- When V_path is higher than V_nom, the vector field draws a unicyclic motion with V_path speed.

$V_{approach} = max(V_{nom}, V_{path})$

$e_b = V_{approach} * t_{const}$

$\theta_{la} = constrain(\frac{\pi}{2}(1 - \hat{e})^2, 0, \frac{\pi}{2})$

$V_g^{||} = V_{path} * sin(\theta_{la})$

$V_g^{\perp} = V_{approach} * cos(\theta_{la})$

$return\ [V_g^{||}, V_g^{\perp}]$

3 Maximum acceleration

This calculates a track error boundary as minimum distance required to reach both V_path in parallel direction & 0 (on-path) in orthogonal direction. This was introduced in Week 10 Report appendix.

- It assumes instantaneous application of maximum acceleration.
- When V_path is higher than V_nom, the approach speed is set to V_path.

$V_{approach} = max(V_{nom}, V_{path})$

$e_{approach}^{min} = \frac{[V_{approach}]^2}{2a_{max}^{\perp}}$

$e_{path}^{min} = [\frac{V_{path}V_{approach}}{2a_{max}^{||}}]^2 * \frac{1}{e_{approach}^{min}} = [\frac{V_{path}}{a_{max}^{||}}]^2 * \frac{a_{max}^{\perp}}{2}$

$if(e_{approach}^{min} < e_{path}^{min})$

$\quad e_{approach}^{min} = e_{path}^{min} = [\frac{V_{path}V_{approach}}{2a_{max}^{||}}]$

$e_b = max(e_{approach}^{min}, e_{path}^{min})$

$V_g^{||} = V_{path} * (1 - \sqrt{\hat{e}})$

$V_g^{\perp} = V_{approach} * \sqrt{\hat{e}}$

$return\ [V_g^{||}, V_g^{\perp}]$

This formulation has a distinct solution for *smallest track error boundary, utilizing the most acceleration capability as possible.

4 Jerk limited

$V_{approach} = max(V_{nom}, V_{path})$

$e_{approach}^{min} = \frac{[V_{approach}]^2}{2a_{max}^{\perp}}$

Vector Field Evaluation Criteria

We evaluate how ideal each vector field formulation is with the following 4 criteria:

Criteria	Description	Evaluation
Time to convergence on a path [s]	Time required by the vehicle to converge to the path when following the ground velocity vector field with no error $t_{conv} = -\int_{e_b}^{0} \frac{1}{V_g^{\perp}} de = e_b \cdot \int_{0}^{1} \frac{1}{V_g^{\perp}} d\hat{e} \, (when\ e_b\ is\ constant)$	Smaller the better
Track error boundary length [m]	e_b	Smaller the better
Converging path parallel distance [m]	Path-parallel distance travelled to converge to path when following ground velocity vector field with no error $d_{conv} = -\int_{e_b}^{0} V_g^{\parallel} \cdot \frac{e}{V_g^{\perp}} de = e_b \cdot \int_{0}^{1} V_g^{\parallel} \cdot \frac{\hat{e}}{V_g^{\perp}} d\hat{e} \, (when\ e_b\ is\ constant)$	Smaller the better
Velocity on Path fulfilled [bool]	$V_g^{\parallel} = V_{path}\ when\ \hat{e} = 0$	Needs to be True
Speed **monotonicity** [bool]	Whether the magnitude of velocity grows or decreases consistently throughout track error boundary $Monotonicity = \{\frac{d}{de} \|\|V_g(e)\|\| \geq 0 \mid for\ all\ e \in [0, e_b]\}$ Or $\{\frac{d}{de} \|\|V_g(e)\|\| \leq 0 \mid for\ all\ e \in [0, e_b]\}$ This can be compared via: $V_g^{\perp}(\frac{d}{d\hat{e}} V_g^{\perp}) + V_g^{\parallel}(\frac{d}{d\hat{e}} V_g^{\parallel})$	Needs to be True
Total **acceleration RMS** [m/s^2]	Root-Mean-Square value of acceleration $acc_{rms} = \frac{1}{e_b} * \sqrt{-\int_{e_b}^{0} (\|\|\frac{d}{dt} V_g(e)\|\|^2)\, de} = \sqrt{\int_{0}^{1} (\|\|\frac{d}{dt} V_g(\hat{e})\|\|^2)\, d\hat{e}} \, (when\ e_b\ is\ constant)$ $= \frac{1}{e_b} \sqrt{\int_{0}^{1} (\|\|V_g^{\perp} \cdot \frac{d}{d\hat{e}} V_g(\hat{e})\|\|^2)\, d\hat{e}} = \frac{1}{e_b} \sqrt{\int_{0}^{1} ([(V_g^{\perp}]^2 \cdot [(\frac{d}{d\hat{e}} V_g^{\perp})^2 + (\frac{d}{d\hat{e}} V_g^{\parallel})^2])\, d\hat{e}}$	Smaller the better
Maximum acceleration total [m/s^2]	Maximum total acceleration $acc_{max} = max\{\frac{d}{dt} V_g(e) \mid for\ e \in [0, e_b]\} = \frac{1}{e_b} \cdot max\{V_g^{\perp} \cdot \sqrt{(\frac{d}{d\hat{e}} V_g^{\perp})^2 + (\frac{d}{d\hat{e}} V_g^{\parallel})^2}\ for\ \hat{e} \in [0, 1]\} \, (when\ e_b\ is\ constant)$	Smaller the better
FW/VTOL Specific:		

Max acceleration parallel to velocity [m/s^2]	Acceleration required in direction parallel to velocity vector (equals to body longitudinal axis for VTOL with no wind) $acc^{\|\|}_{max} = max\{\|(\frac{d}{dt}V_g(e)) \bullet \widehat{V_g}\| \, for \, e \in [0, e_b] = \frac{1}{e_b} \cdot max\{\frac{V^{\perp}_g}{V_g} \cdot [V^{\perp}_g(\frac{d}{d\hat{e}}V^{\perp}_g) + V^{\|\|}_g(\frac{d}{d\hat{e}}V^{\|\|}_g)] \, for \, \hat{e} \in [0, 1]\} \, (when \, e_b \, is \, constant)$	Needs to be smaller than vehicle's longitudinal axis acceleration limit
Max acceleration orthogonal to velocity [m/s^2]	Acceleration required in direction orthogonal to velocity vector (equals to body longitudinal axis for VTOL with no wind) $acc^{\perp}_{max} = max\{\|(\frac{d}{dt}V_g(e)) \times \widehat{V_g}\| \, for \, e \in [0, e_b]\} = \frac{1}{e_b} \cdot max\{\|\frac{V^{\perp}_g}{V_g} \cdot [V^{\perp}_g(\frac{d}{d\hat{e}}V^{\|\|}_g) - V^{\|\|}_g(\frac{d}{d\hat{e}}V^{\perp}_g)]\| \, for \, \hat{e} \in [0, 1]\} \, (when \, e_b \, is \, constant)$	Needs to be smaller than vehicle's lateral axis acceleration limit

Evaluation

Based on the equations given in Formulation section, it is possible to provide definite evaluation values under certain assumptions.

In general, the following assumptions apply:

- $a^{\perp}_{max} = a^{||}_{max} = a_{max}$: Multicopter has uniform acceleration limit in both directions relative to path (for Maximum acceleration formulation)
  - This ensures that $e^{min}_{approach}$ is greater than e^{min}_{path}, so the track error boundary is fixed by V_nom.

Mathematical calculation (V_nom > V_path)

This is the case where we want to achieve a speed lower than V_nom on the path

Algorithm	Time to Convergence [s]	Track error boundary length [m]	Converging path parallel distance [m]	Path Velocity fulfilled [bool]	Speed monotonicity [bool]	Total acceleration RMS [m/s^2]	Maximum acceleration total [m/s^2]	Max acceleration parallel to velocity [m/s^2]	Max acceleration orthogonal to velocity [m/s^2
TJ NPFG Original	~~Source~~	$V_{nom} * t_{const}$	$V_{nom} * t_{const} * 0.18$ Source	False	True	$\frac{V_{nom}}{t_{const}} * 0.8288$ Source	$\frac{V_{nom}}{t_{const}} (\hat{e} = 1)$	0 No longitudinal acceleration	$\frac{V_{nom}}{t_{const}} (\hat{e} = 1)$ Source
TJ NPFG Squashed	~~Source~~	$V_{nom} * t_{const}$	$V_{path} * t_{const} * 0.18$	True	True (Elliptical curve)	Between $\frac{V_{nom}}{t_{const}} * 0.3 \, (V_{path} = 0)$ & $\frac{V_{nom}}{t_{const}} * 0.8288$ Source	Between $\frac{V_{nom}}{2t_{const}} (V_{path} = 0)$ $\hat{e} = \frac{1}{\sqrt{2}}$ & $\frac{V_{nom}}{t_{const}} (V_{path} = V_{nom})$	$\frac{V_{nom}}{t_{const}} * 0.385 * (1 - (\frac{V_{path}}{V_{nom}})^2)$ $(\hat{e} = 0.374)$ Source Reaches max when V_path = 0 (max deceleration)	$\frac{V_{path}}{t_{const}} (\hat{e} = 1)$
Maximum acceleration	$\frac{V_{nom}}{a_{max}} * 0.333$ Source	$\frac{[V_{nom}]^2}{2a_{max}}$	$\frac{V_{path}}{2a_{max}} * \infty$ Doesn't converge Source	True	?				

Jerk limited									
BEST									

Mathematical calculation (V_nom < V_path)

This is the case when we want to achieve a speed higher than V_nom on the path

- Note that e_b for maximum acceleration formulation chooses optimal minimum track error boundary (if- condition satisfied)

Algorithm	Time to Convergence [s]	Track error boundary length [m]	Converging path parallel distance [m]	Path Velocity fulfilled [bool]	Speed monotonicity [bool]	Total acceleration RMS [m/s^2]	Maximum acceleration total [m/s^2]	Max acceleration parallel to velocity [m/s^2]	Max acceleration orthogonal to velocity [m/s^2
TJ NPFG Original		$V_{nom} * t_{const}$		False	True				
TJ NPFG Squashed		$V_{path} * t_{const}$		True	True (Elliptical curve)				
Maximum acceleration		$\frac{[V_{path}]^2}{2a_{max}}$							
Jerk limited									