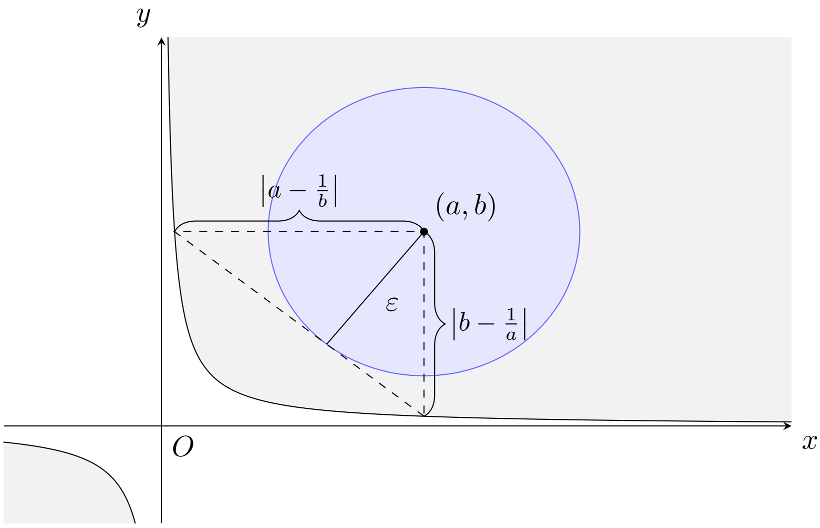
Advanced Calculus I – Assignment 2

Junwoo Yang

April 18, 2019

§2.1 #2. Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy > 1\}$. Show that S is open.

Proof. The set in question is the set of points "outside" the hyperbola $xy = 1$. We need to show that each point in S is surrounded by some small disk completely contained in S. This seems reasonably clear from Figure 2-2. Any radius shorter than the distance from our point to the closest point on the curve will do.



One could actually try to find the closest point on the hyperbola to the origin and its distance using the methods of beginning calculus. This would give the largest radius which will do. Any smaller radius will also work. Here is a geometric argument which shows that such a radius exists. It assumes that the branch of the hyperbola in the first quadrant is the graph of a decreasing function and is concave up. This is easily checked for $f(x) = 1/x$. The argument is based on the figures sketched in Figure 2-3.

$$\begin{aligned}\varepsilon &= \frac{|a - \frac{1}{b}| \cdot |b - \frac{1}{a}|}{\sqrt{(a - \frac{1}{b})^2 + (b - \frac{1}{a})^2}} = \frac{|ab - 2 + \frac{1}{ab}|}{\sqrt{a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} - \frac{2a}{b} - \frac{2b}{a}}} \\ &= \frac{|ab - 2 + \frac{1}{ab}|}{\sqrt{a^2 + b^2 + \frac{a^2+b^2}{a^2+b^2} - \frac{2}{ab}(a^2 + b^2)}} = \frac{ab - 1}{\sqrt{a^2 + b^2}}.\end{aligned}$$

$\square$

§2.3 #5. Let $S = \{x \in \mathbb{R} \,|\, x \text{ is irrational}\}$. Is S closed?

Proof. $\square$

§2.4 #3. Find the accumulation points of the following sets in $\mathbb{R}^2$:

a. $\{(m, n) \,|\, m, n \text{ integers}\}$

b. $\{(p, q) \,|\, p, q \text{ rational}\}$

§2.6 #5. Let $A \subset \mathbb{R}$ be bounded and nonempty and let $x = \sup(A)$. Is $x \in bd(A)$?

Proof. $\square$

§2.8 #2. Let (M, d) be a metric space with the property that every bounded sequence has a convergent subsequence. Prove that M is complete.

Proof. $\square$

Exercises for Chapter 2

#18. If $x, y \in M$ and $x \neq y$, then prove that there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V \neq \emptyset$.

Proof. Use disks of radius no larger that half the distance between x and y. Since x and y are not equal, we know that $d(x, y) > 0$. Let $r = d(x, y)/3$, and set $U = D(x, r)$ and $V = D(y, r)$. Then U and V are open by Proposition 2.1.2, and we certainly have $x \in U$ and $y \in V$. If z were in $D(x, r) \cap D(y, r)$, we would have

$$0 < r = d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{3} + \frac{r}{3} = \frac{2}{3}r.$$

But this is impossible, A positive number cannot be smaller that $\frac{2}{3}$ of itself. Thus there can be no such z, and we must have $U \cap V = D(x, r) \cap D(y, r) = \emptyset$ as desired. The property proved here is called the Hausdorff separation property. $\square$

#29. Let $A, B \subset \mathbb{R}^n$ and x be an accumulation point of $A \cup B$. Must x be an accumulation point of either A or B?

Proof. Yes. If x is not an accumulation point of A or of B, then there are small disks centered at x which intersect A and B respectively in at most the point x. The smaller of these two disks intersects $A \cup B$ in at most the point x, so x is not an accumulation point of $A \cup B$.
Let x be an accumulation point of $A \cup B$. Suppose x were not an accumulation point of either A or B. Since it is not an accumulation point of A, there is a radius $r_1 > 0$ such that $D(x, r_1) \cap (A \setminus \{x\}) = \emptyset$. $\square$