Stochastic Calculus Instructor: Hyungbin Park

Lecture note 3: Stochastic calculus

1 Exercises

Problem 1.1. (10 points) Evaluate the variance of $\int_0^T tB_t\,dB_t$.

Problem 1.2. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $f:[0,T]\to\mathbb{R}$ be a Borel function with $\int_0^T f^2(u)\,du<\infty$.

(i) (10 points) Show that as a stochastic process the map

$$f:\Omega\times[0,T]\to\mathbb{R}$$

is progressively measurable. Hint: See the definition of progressively measurable and use the fact that the product σ-algebra $\mathcal{F}_t\otimes\mathcal{B}[0,t]$ is generated by measurable rectangles. For each $t\in[0,T]$, since $f:[0,t]\to\mathbb{R}$ is a Borel function, we know $f^{-1}(A)$ is in $\mathcal{B}[0,t]$ for Borel set $A\subseteq\mathbb{R}$.

(ii) (10 points) Show that the process

$$\Big(\int_0^t f(u)\,dB_u\Big)_{0\leq t\leq T}$$

is a Gaussian process. You may use, without proof, the fact that the limit of normal distributions is normal.

(iii) (5 points) Evaluate $\mathbb{E}(e^{\int_0^T t\,dB_t})$.

Problem 1.3. Solve the following problems.

(i) (10 points) Let $f:[0,T]\to\mathbb{R}$ be a Borel function with $\int_0^T f^2(t)\,dt<\infty$. Show that a process

$$M_t:=e^{\int_0^t f(s)\,dB_s-\frac{1}{2}\int_0^t f^2(s)\,ds},\ 0\leq t\leq T$$

is a martingale.

(ii) (10 points) Let $\theta\in\mathcal{H}^2_{\text{loc}}$. Show that a process

$$M_t:=e^{\int_0^t \theta_s\,dB_s-\frac{1}{2}\int_0^t \theta_s^2\,ds},\ 0\leq t\leq T$$

is a local martingale.

Problem 1.4. Let $B=(B_t^{(1)},B_t^{(2)},B_t^{(3)})_{t\geq 0}$ be a three dimensional Brownian motion, and consider the filtration $(\mathcal{F}_t^B)_{t\geq 0}$ generated by B.

(i) (10 points) For each $T > 0$, show that

$$(B^{(1)}B^{(2)}, tB^{(3)}, 0) \in \mathcal{H}^2(\Omega \times [0,T], \mathbb{R}^3, \mathcal{F}, (\mathcal{F}_t^B), \mathbb{P}),$$

and deduce that

$$\Big(\int_0^t B_s^{(1)} B_s^{(2)}\, dB_s^{(1)} + \int_0^t sB_s^{(3)}\, dB_s^{(2)}\Big)_{t\geq 0}$$

is a martingale.

(ii) (10 points) Find the mean and the variance of

$$\int_0^T B_s^{(1)} B_s^{(2)}\, dB_s^{(1)} + \int_0^T sB_s^{(3)}\, dB_s^{(2)}$$

(iii) (10 points) Show that

$$\Big(B_t^{(1)} - 2B_t^{(2)} + \int_0^t B_s^{(1)}\, dB_s^{(3)}\Big)^2 - \int_0^t (B_s^{(1)})^2\, ds - 5t\,,\ t \geq 0$$

is a martingale.

Problem 1.5. Let $B = (B_t^{(1)}, \cdots, B_t^{(d)})_{t\geq 0}^\top$ be a d-dimensional Brownian motion.

(i) (10 points) Let $g, h \in \mathcal{H}^2([0,T] \times \Omega, \mathbb{R}^d, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Show that

$$\mathbb{E}\Big(\int_0^T g_t\, dB_t \int_0^T h_t\, dB_t\Big) = \mathbb{E}\Big(\int_0^T g_t \cdot h_t\, dt\Big).$$

Hint: Apply the Ito isometry to $\int_0^T (g_t + h_t)\, dB_t$.

(ii) (10 points) Evaluate

$$\mathbb{E}\Big(B_{T/2}^{(1)} \int_0^T (B_t^{(2)})^2\, dB_t^{(1)}\Big)\,,\ \mathbb{E}\Big(B_T^{(1)} B_T^{(2)} \int_0^T e^{B_t^{(2)}}\, dB_t^{(1)}\Big).$$

Problem 1.6. (15 points) Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and define

$$X_t = -t^2 + \sin(B_t^2)$$
$$Y_t = \int_0^t B_s\, ds + \int_0^t sB_s^2\, dB_s$$

Find the quadratic variations and covariations $\langle X\rangle$, $\langle Y\rangle$, and $\langle X, Y\rangle$.

Problem 1.7. (20 points) Let $B = (B_t^{(1)}, \cdots, B_t^{(d)})_{t\geq 0}^\top$ be a d-dimensional Brownian motion. Consider two Ito processes X and Y given as

$$X_t = x + \int_0^t b_u^X\, du + \int_0^t \sigma_u^X\, dB_u$$
$$Y_t = y + \int_0^t b_u^Y\, du + \int_0^t \sigma_u^Y\, dB_u$$

where b^X, b^Y, σ^X, σ^Y are progressively measurable and

$$\mathbb{P}\Big(\int_0^T |b_u^X| + |b_u^Y| + \|\sigma_u^X\|^2 + \|\sigma_u^Y\|^2\, du < \infty\Big) = 1$$

for each $T > 0$. Show that the quadratic covariation is

$$\langle X, Y\rangle_t = \int_0^t \sigma_u^X \cdot \sigma_u^Y\, du\,.$$

Here, $\cdot$ is the usual dot product. Write down the main idea of the proof as we did in class. Do not provide any rigorous proof.

Problem 1.8. Let B be a d-dimensional Brownian motion.

(i) (10 points) Suppose that

$$C + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dB_s = 0\,,\ 0 \le t \le T$$

where C is a constant, and b and σ are 1-dimensional and d-dimensional progressively measurable processes, respectively, satisfying $\int_0^T |b_s| + \|\sigma_s\|^2\, ds < \infty$ almost surely. Show that $C = 0$, and $b = 0$, $\sigma = 0$ almost surely on $\Omega \times [0, T]$. Hint: quadratic variation

(ii) (5 points) Let X be an Ito process. Deduce that the decomposition

$$X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dB_s\,,\ 0 \le t \le T$$

is unique.

References

금융수학 I - HW6 양준우 (2020-91659)

Prob. 1.1.

Claim 1). $tB_t \in H^2 = \{g : \text{prog. m'ble} \mid E\int_0^T g_u^2\,du < \infty\}$.

(i) $(tB_t)_{0\le t\le T}$: adapted + conti $\Rightarrow$ prog m'ble.

(ii). $E\int_0^T (tB_t)^2 dt = E\int_0^T t^2 B_t^2\,dt$

$$= \int_0^T t^2 E(B_t^2)\,dt = \int_0^T t^3\,dt = \frac{T^4}{4} < \infty$$

$\therefore \int_0^T tB_t\,dB_t$ is well-defined.

Claim 2). $E(\int_0^T tB_t\,dB_t) = 0$.

Since $tB_t \in H^2$, $(\int_0^t uB_u\,dB_u)_{0\le t\le T}$: martingale.

$\therefore E(\int_0^T tB_t\,dB_t) = E(\int_0^0 tB_t\,dB_t) = 0$.

We calculate

$$Var(\int_0^T tB_t\,dB_t) = E((\int_0^T tB_t\,dB_t)^2) - (E(\int_0^T tB_t\,dB_t))^2$$

$= E(\int_0^T t^2 B_t^2\,dt)$ (by Ito isometry and claim 2.).

$= \int_0^T t^2 E(B_t^2)\,dt = \int_0^T t^3\,dt = \frac{T^4}{4}$ □.

Prob. 1.2 (i).

Since $f : [0,T] \to \mathbb{R}$ is Borel function,

for $A \in \mathcal{B}(\mathbb{R})$, $f^{-1}(A) \in \mathcal{B}([0,T])$.

$f|_{[0,t]} : [0,t] \to \mathbb{R}$, $u \mapsto f(u)$ for $t \in [0,T]$, $u \in [0,t]$

$(f|_{[0,t]})^{-1}(A) = (f^{-1}(A) \cap [0,t]) \in \mathcal{B}([0,t])$ for $A \in \mathcal{B}(\mathbb{R})$.

$\tilde{f} : \Omega \times [0,T] \to \mathbb{R}$, $(\omega, t) \mapsto f(t)$ for $\omega \in \Omega$, $t \in [0,T]$

$\tilde{f}^{-1}(A) \in \Omega \times \mathcal{B}([0,T])$.

$\tilde{f}|_{[0,t]} : \Omega \times [0,t] \to \mathbb{R}$, $(\omega, u) \mapsto f(u)$ for $\omega \in \Omega$, $t \in [0,T]$, $u \in [0,t]$

$\therefore (\tilde{f}|_{[0,t]})^{-1}(A) = \Omega \times (\tilde{f}^{-1}(A) \cap [0,t]) \in \Omega \times \mathcal{B}([0,t])$

$\in \mathcal{F}_t \times \mathcal{B}([0,t])$

$\therefore \tilde{f}$ is $\mathcal{F}_t \times \mathcal{B}([0,t])$ - measurable,

namely, progressively measurable.

Prob. 1.2 (ii)

$\left(\int_0^t f(u)dB_u\right)_{0\le t\le T}$ is Gaussian process.

$\Leftrightarrow$ $\left(\int_0^{t_1} f(u)dB_u, \cdots, \int_0^{t_n} f(u)dB_u\right)$ is jointly normal $\forall t_1, \cdots, t_n \in [0,T]$

$\Leftrightarrow$ $\sum_{i=1}^{n} \lambda_i \int_0^{t_i} f(u)dB_u$ is normal $\forall \lambda_i \in \mathbb{R}$.

$\Leftrightarrow$ $\int_0^T \left(\sum_{i=1}^n \lambda_i f(u) \mathbb{1}_{[0,t_i]}(u)\right)dB_u$ is normal $\forall \lambda_i \in \mathbb{R}$.

$\int_0^T \left(\sum_{i=1}^n \lambda_i f(u) \mathbb{1}_{[0,t_i]}(u)\right)dB_u$

$= \int_0^T \left((\lambda_1+\cdots+\lambda_n)f(u)\mathbb{1}_{[0,t_1]} + (\lambda_2+\cdots+\lambda_n)f(u)\mathbb{1}_{[t_1,t_2]} + \cdots + \lambda_n f(u)\mathbb{1}_{[t_{n-1},t_n]}\right)dB_u$

$\Rightarrow \int_0^T \sum_{i=0}^{n-1} c_i f(u) \mathbb{1}_{[t_i,t_{i+1}]} dB_u$ where $t_i = \frac{iT}{n}$, $c_i \in \mathbb{R}$

$\therefore$ It suffices to show that $\int_0^T f(t)dB_t$ is normal

where $\int_0^T f^2(t)\,dt < \infty$.

As a stochastic process, $(f(t))_{0\le t\le T}$ is progressively measurable by (i).

Since f is prog. m'ble and $E\int_0^T f^2 dt = \int_0^T f^2 dt < \infty$, $f \in H^2$.

Thus, f can be approximated by a seq. of simple processes.

$(f^{(n)})_{n\ge1}$ s.t $E\int_0^T (f^{(n)}-f)^2 dt \to 0$.

$f_t^{(n)} = \sum_{i=0}^{n-1} f_{t_i} \mathbb{1}_{[t_i,t_{i+1})}$ where $\{0 = t_0 < t_1 < \cdots < t_n = T\}$.

Note that $\int_0^T f(t)dB_t = \lim_{n\to\infty} f^{(n)}(t)\,dB_t$,

$\int_0^T f^{(n)}(t)dB_t = \sum_{i=0}^{n-1} f_{t_i}(B_{t_{i+1}} - B_{t_i})$ is normal.

By the fact that the limit of normal distributions is normal,

$\int_0^T f(t)\,dB_t$ is normal. $\square$

Prob. 1.2 (iii)

Since t is Borel function, as a stochastic process, t is prog. m'ble.

Also, $E\int_0^T t^2 dt = \int_0^T t^2 dt = \frac{T^3}{3} < \infty$. Thus $t \in H^2$ and

$\left(\int_0^t u\,dB_u\right)_{0\le t\le T}$ is martingale.

$\therefore E\left(\int_0^T t\,dB_t\right) = E\left(\int_0^0 t\,dB_t\right) = 0$.

$Var\left(\int_0^T t\,dB_t\right) = E\left(\left(\int_0^T t\,dB_t\right)^2\right) = E\left(\int_0^T t^2 dt\right) = \int_0^T t^2 dt = \frac{T^3}{3}$

Ito isometry. $(t \in H^2)$.

By (ii), $\int_0^T t\,dB_t$ is normal.

$\therefore E\left(e^{\int_0^T t\,dB_t}\right) = e^{E\left(\int_0^T t\,dB_t\right) + \frac{1}{2}Var\left(\int_0^T t\,dB_t\right)} = e^{\frac{T^3}{6}}$ $\square$

Prob. 1.3.(i).

Since f is a Borel function with $\int_0^T f^2(t)dt < \infty$, By Prob. 1.2, $f \in \mathcal{H}^2$ and $\int_0^t f(u)dB_u$ is normal for $0 \le t \le T$.

$(\int_0^t f(u)dB_u)_{0\le t\le T}$ is martingale.

$$
\begin{aligned}
E(M_t|\mathcal{F}_s) &= E(e^{\int_0^t f(u)dB_u - \frac{1}{2}\int_0^t f^2(u)du}|\mathcal{F}_s) \quad \text{for } 0\le s<t \\
&= e^{-\frac{1}{2}\int_0^t f^2(u)du} E(e^{\int_0^t f(u)dB_u}|\mathcal{F}_s) \quad \text{for } 0\le s<t \\
&= e^{-\frac{1}{2}\int_0^t f^2(u)du} E(e^{\int_0^s f(u)dB_u} \cdot e^{\int_s^t f(u)dB_u}|\mathcal{F}_s). \\
&= e^{\int_0^s f(u)dB_u - \frac{1}{2}\int_0^t f^2(u)du} E(e^{\int_s^t f(u)dB_u}) \quad * \\
&= e^{\int_0^s f(u)dB_u - \frac{1}{2}\int_0^t f^2(u)du} \cdot e^{\frac{1}{2}\int_s^t f^2(u)du} \quad (\because \int_s^t f(u)dB_u \text{ is normal}). \\
&= e^{\int_0^s f(u)dB_u - \frac{1}{2}\int_0^s f^2(u)du} = M_s.
\end{aligned}
$$

(*). Since $f \in \mathcal{H}^2$, f can be approximated by a seq. of simple processes.

$$\int_0^t f(u)dB_u = \lim_{n\to\infty}\int_0^t f^{(n)}(u)dB_u = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(u_i)(B_{u_{i+1}}-B_{u_i})$$

(where $u_0=0$, $u_n=t$).

$$e^{\int_0^s f(u)dB_u} = \lim_{n\to\infty} e^{\sum_{i=0}^{n-1} f(u_i)(B_{u_{i+1}}-B_{u_i})} \quad \text{where } u_0=0,\ u_n=s$$

(Then $B_{u_{i+1}}-B_{u_i} \in \mathcal{F}_s$ $\forall i$, each step n).

$$e^{\int_s^t f(u)dB_u} = \lim_{n\to\infty} e^{\sum_{i=0}^{n-1} f(u_i)(B_{u_{i+1}}-B_{u_i})} \quad \text{where } u_0=s,\ u_n=t$$

(Then $B_{u_{i+1}}-B_{u_i} \perp \mathcal{F}_s$ $\forall i$, each step n).

$$\therefore E(e^{\int_0^s f(u)dB_u} \cdot e^{\int_s^t f(u)dB_u}|\mathcal{F}_s) = e^{\int_0^s f(u)dB_u} E(e^{\int_s^t f(u)dB_u})$$

$$M_t = e^{\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds}, \quad 0\le t\le T$$

Prob. 1.3.(ii).

$\theta \in \mathcal{H}^2_{loc} \Rightarrow (\int_0^t \theta_u dB_u)_{0\le t\le T}$ is local martingale.

There exists seq. of stopping times $(\tau_n)_{n\ge1}$ s.t $\tau_n \le \tau_{n+1}$, $\lim_{n\to\infty}\tau_n = T$, $(\int_0^{t\wedge\tau_n}\theta_u dB_u)_{0\le t\le T}$ is a martingale.

WTS $(M_{t\wedge\tau_n})_{0\le t\le T}$ is a martingale.

$E(M_{t\wedge\tau_n}|\mathcal{F}_s)$

$$= E(e^{\int_0^{t\wedge\tau_n}\theta_u dB_u - \frac{1}{2}\int_0^{t\wedge\tau_n}\theta_u^2 du}|\mathcal{F}_s) \overset{\text{by Prob. 1.3.(i).}}{=} e^{\int_0^{s\wedge\tau_n}\theta_u dB_u - \frac{1}{2}\int_0^{s\wedge\tau_n}\theta_u^2 du}$$

$$= M_{s\wedge\tau_n} \quad \square.$$

Prob. 1.4. (i)

$B = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t\geq 0}$, $(\mathcal{F}_t^B)$.

$(B_t^{(1)}B_t^{(2)}, tB_t^{(3)}, 0)$ is continuous and adapted. Thus progressively m'ble.

$$E\int_0^T |(B_t^{(1)}B_t^{(2)}, tB_t^{(3)}, 0)|^2 dt = E\int_0^T (B_t^{(1)^2}B_t^{(2)^2} + t^2 B_t^{(3)^2})dt$$
$$= \int_0^T (E(B_t^{(1)^2})E(B_t^{(2)^2}) + t^2 E(B^{(3)^2}))dt \quad (\because B^{(1)} \perp B^{(2)})$$
$$= \int_0^T t^2 + t^3 \, dt = \frac{T^3}{3} + \frac{T^4}{4} < \infty$$

$\therefore (B_t^{(1)}B_t^{(2)}, tB_t^{(3)}, 0) \in \mathcal{H}^2[\Omega \times [0,T], \mathbb{R}^3, \mathcal{F}, (\mathcal{F}_t^B), P]$.

We know that for $g \in \mathcal{H}^2$, $(\int_0^t g_u dB_u)_{0\leq t\leq T}$ is martingale.

Thus, $(\int_0^t (B_u^{(1)}B_u^{(2)}, uB_u^{(3)}, 0)\cdot(dB_u^{(1)}, dB_u^{(2)}, dB_u^{(3)}))_{0\leq t\leq T}$ is mart. $\forall T>0$.

Hence, $(\int_0^t B_u^{(1)}B_u^{(2)}dB_u^{(1)} + \int_0^t uB_u^{(3)}dB_u^{(2)})_{t\geq 0}$ is martingale w.r.t $(\mathcal{F}_t^B)_{t\geq 0}$.

Prob. 1.4. (ii)

$$E\left(\int_0^T B_s^{(1)}B_s^{(2)}dB_s^{(1)} + \int_0^T sB_s^{(3)}dB_s^{(2)}\right) \overset{\because \text{ martingale.}}{=} E\left(\int_0^0 B_s^{(1)}B_s^{(2)}dB_s^{(1)} + \int_0^0 sB_s^{(3)}dB_s^{(2)}\right) = 0.$$

$$Var\left(\int_0^T B_s^{(1)}B_s^{(2)}dB_s^{(1)} + \int_0^T sB_s^{(3)}dB_s^{(2)}\right) = Var\left(\int_0^T (B_s^{(1)}B_s^{(2)}, sB_s^{(3)}, 0)dB_s\right)$$
$$= E\left(\left(\int_0^T (B_s^{(1)}B_s^{(2)}, sB_s^{(3)}, 0)dB_s\right)^2\right)$$
$$= E\left(\int_0^T (B_s^{(1)^2}B_s^{(2)^2} + s^2B_s^{(3)^2})ds\right) \quad \because \text{Ito isometry.}$$
$$= \int_0^T E(B_s^{(1)^2})E(B_s^{(2)^2}) + s^2E(B_s^{(3)^2})\, ds$$
$$= \int_0^T s^2 + s^3 \, ds = \frac{T^3}{3} + \frac{T^4}{4}$$

Prob. 1.4. (iii)

We know that $g \in \mathcal{H}^2 \Rightarrow ((\int_0^t g_u dB_u)^2 - \int_0^t |g_u|^2 du)_{0\leq t\leq T}$ is martingale.

Let $h = (1, -2, B_s^{(1)})$

Since h is continuous and adapted, h is progressively m'ble.

$$E\int_0^T |h|^2 ds = E\int_0^T 5 + B_s^{(1)^2} ds = \int_0^T 5+s \, ds = 5T + \frac{T^2}{2} < \infty.$$

$\therefore h \in \mathcal{H}^2$ for each $T>0$. $\Rightarrow ((\int_0^t h_s dB_s)^2 - \int_0^t |h_s|^2 ds)_{t\geq 0}$ is martingale.

$\Rightarrow ((\int_0^t 1\cdot dB_s^{(1)} + \int_0^t -2\cdot dB_s^{(2)} + \int_0^t B_s^{(1)}dB_s^{(3)})^2 - \int_0^t (5 + B_s^{(1)^2})ds)_{t\geq 0}$

$\Rightarrow ((B_t^{(1)} - 2B_t^{(2)} + \int_0^t B_s^{(1)}dB_s^{(3)})^2 - \int_0^t B_s^{(1)^2}ds - 5t)_{t\geq 0}$ is martingale. □

Prob. 1.5. (i).

Since $g, h \in \mathcal{H}^2$, $g+h \in \mathcal{H}^2$

Apply the Ito isometry to $\int_0^T (g_t + h_t)\,dB_t$

$$\Rightarrow E\left(\left(\int_0^T (g_t+h_t)\,dB_t\right)^2\right) = E\left(\int_0^T (g_t+h_t)^2\,dt\right)$$

$$\begin{aligned}
(LHS) &= E\left(\left(\int_0^T g_t\,dB_t + \int_0^T h_t\,dB_t\right)^2\right) \\
&= E\left(\left(\int_0^T g_t\,dB_t\right)^2\right) + E\left(\left(\int_0^T h_t\,dB_t\right)^2\right) + 2E\left(\int_0^T g_t\,dB_t \int_0^T h_t\,dB_t\right) \\
&= E\left(\int_0^T g_t^2\,dt\right) + E\left(\int_0^T h_t^2\,dt\right) + 2E\left(\int_0^T g_t\,dB_t \int_0^T h_t\,dB_t\right)
\end{aligned}$$

$$\begin{aligned}
(RHS) &= E\left(\int_0^T (g_t^2 + 2g_t\cdot h_t + h_t^2)\,dt\right) \\
&= E\left(\int_0^T g_t^2\,dt\right) + E\left(\int_0^T h_t^2\,dt\right) + 2E\left(\int_0^T g_t\cdot h_t\,dt\right)
\end{aligned}$$

Hence, $E\left(\int_0^T g_t\,dB_t \int_0^T h_t\,dB_t\right) = E\left(\int_0^T g_t\cdot h_t\,dt\right)$

Prob 1.5. (ii).

Let $B = \left(B^{(1)}_{\frac{t}{2}},\ B^{(1)}_t - B^{(1)}_{\frac{t}{2}}\right)_{t\geq 0}$

Note $B^{(1)}_{\frac{t}{2}} \perp B^{(1)}_t - B^{(1)}_{\frac{t}{2}}$ ($\because$ by def. of B.M). and

$B^{(1)}_t - B^{(1)}_{\frac{t}{2}} \sim N(0, \frac{t}{2}) \sim B^{(2)}_{\frac{t}{2}} \Rightarrow$ one-dim'l B.M.

Thus $B = \left(B^{(1)}_{\frac{t}{2}},\ B^{(1)}_t - B^{(1)}_{\frac{t}{2}}\right)$ is two-dim'l B.M.

$$\begin{aligned}
E\left(B^{(1)}_{\frac{t}{2}} \int_0^T (B_t^{(2)})^2\,dB_t^{(1)}\right) &= E\left(\int_0^T (1,0)\,dB_t \int_0^T \left({B_t^{(2)}}^2, {B_t^{(2)}}^2\right)\cdot dB_t\right) \\
&= E\left(\int_0^T (1,0)\cdot\left({B_t^{(2)}}^2, {B_t^{(2)}}^2\right)dt\right) \\
&= E\left(\int_0^T {B_t^{(2)}}^2\,dt\right) = \int_0^T E\left({B_t^{(2)}}^2\right)dt = \int_0^T t\,dt = \frac{T^2}{2}
\end{aligned}$$

$E\left(B_T^{(1)} B_T^{(2)} \int_0^T e^{B_t^{(2)}}\,dB_t^{(1)}\right).$

$= E\left(\int_0^T (0, B_T^{(1)})\,dB_t \cdot \int_0^T (e^{B_t^{(2)}}, 0)\,dB_t\right)$. (where $B = (B_t^{(1)}, B_t^{(2)}, B_t^{(2)})_{t\geq 0}$)

$= E\left(\int_0^T (0, B_T^{(1)})\cdot(e^{B_t^{(2)}}, 0)\,dt\right) = 0.$

$$\begin{aligned}
(\text{or } &= E\left(\int_0^T (B_T^{(2)}, 0)\,dB_t \int_0^T (e^{B_t^{(2)}}, 0)\,dB_t\right)) \\
&= E\left(\int_0^T B_T^{(2)} e^{B_t^{(2)}}\,dt\right) \\
&= \int_0^T E\left(B_T^{(2)} e^{B_t^{(2)}}\right)dt \\
&= \int_0^T E\left((B_T^{(2)} - B_t^{(2)})e^{B_t^{(2)}} + B_t^{(2)} e^{B_t^{(2)}}\right)dt \\
&= \int_0^T \underbrace{E(B_T^{(2)} - B_t^{(2)})}_{0} E(e^{B_t^{(2)}}) + E(B_t^{(2)} e^{B_t^{(2)}})\,dt \\
&= \int_0^T t e^{\frac{t}{2}}\,dt = \left[t\cdot 2e^{\frac{t}{2}}\right]_0^T - \int_0^T 2e^{\frac{t}{2}}\,dt \qquad (\because B_t^{(2)} \sim \sqrt{t}Z,\ E(Ze^{uZ}) = ue^{\frac{u^2}{2}}) \\
&= 2Te^{\frac{T}{2}} - 4e^{\frac{T}{2}}
\end{aligned}$$

Prob. 1.6.

$X_t = -t^2 + \sin(B_t^2) = \int_0^t -2s\,ds + \int_0^t \cos(B_s^2) 2B_s\,dB_s$

$Y_t = \int_0^t B_s\,ds + \int_0^t sB_s^2\,dB_s$

$\langle X \rangle_t = \int_0^t \cos^2(B_s^2) 4B_s^2\,ds$

$\langle Y \rangle_t = \int_0^t s^2 B_s^4\,ds$

$\langle X, Y \rangle_t = \int_0^t 2sB_s^3 \cos(B_s^2)\,ds$

Prob. 1.7.

$$\begin{aligned}
d\langle X, Y\rangle_t &= dX_t \cdot dY_t \\
&= (b_t^X dt + \sigma_t^X dB_t)(b_t^Y dt + \sigma_t^Y dB_t). \\
&= (b_t^X dt + \sigma_t^{X,(1)} dB_t^{(1)} + \cdots + \sigma_t^{X,(d)} dB_t^{(d)})(b_t^Y dt + \sigma_t^{Y,(1)} dB_t^{(1)} + \cdots + \sigma_t^{Y,(d)} dB_t^{(d)}) \\
&= \sigma_t^{X,(1)} \sigma_t^{Y,(1)} dB_t^{(1)^2} + \cdots + \sigma_t^{X,(d)} \sigma_t^{Y,(d)} dB_t^{(d)^2} \\
&\quad (\because (dt)^2 = dt\,dB_t^{(i)} = dB_t^{(i)} dB_t^{(j)} = 0.) \\
&= (\sigma_t^{X,(1)} \sigma_t^{Y,(1)} + \cdots + \sigma_t^{X,(d)} \sigma_t^{Y,(d)})\,dt \quad (\because dB_t^{(i)^2} \approx dt). \\
&= (\sigma_t^X \cdot \sigma_t^Y)\,dt
\end{aligned}$$

$\therefore \langle X, Y \rangle_t = \int_0^t \sigma_u^X \cdot \sigma_u^Y\,du$

Prob. 1.8.(i)

$Y_t := c + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dB_s = 0 \quad 0 \le t \le T.$

$Y_0 = c + \int_0^0 b_s\,ds + \int_0^0 \sigma_s\,dB_s = c = 0. \Rightarrow c = 0.$

$\langle Y \rangle_t = \int_0^t \|\sigma_s\|^2 ds = \langle 0 \rangle_t = 0. \Rightarrow E\int_0^t \|\sigma_s\|^2 ds = 0 \quad \forall t \in [0, T]$

$\Rightarrow \|\sigma_s\|^2 = 0 \Leftrightarrow \sigma_s = 0.$ as on $\Omega \times [0, T]$

Since σ is prog. m'ble, $E\int_0^T \|\sigma_s\|^2 ds = 0 < \infty \Rightarrow \sigma_s \in \mathcal{H}^2$

By Ito isometry, $E\int_0^T \|\sigma_s\|^2 ds = E(\int_0^T \sigma_s\,dB_s)^2 = 0. \Rightarrow \int_0^T \sigma_s\,dB_s = 0$

Now we get $Y_t = \int_0^t b_s\,ds = 0.$

For any $w \in \Omega$, $Y_t(w) = \int_0^t b_s(w)\,ds = 0 \Rightarrow b_s(w) = 0$ a.e $t \in [0, T]$

$\Rightarrow E\int_0^T |b_t|\,dt = E(0) = 0. \Rightarrow |b| = 0 \Rightarrow b = 0$ a.s on $\Omega \times [0, T]$

Prob. 1.8.(ii)

Suppose $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s$

$= X_0' + \int_0^t b_s' ds + \int_0^t \sigma_s' dB_s.$

Then $X_t - X_t = (X_0 - X_0') + \int_0^t (b_s - b_s') ds + \int_0^t (\sigma_s - \sigma_s') dB_s = 0.$

By Prob. 1.8.(i), $X_0 - X_0' = 0$, $b_s - b_s' = 0$, $\sigma_s - \sigma_s' = 0$ a.s. on $\Omega \times [0,T]$

Thus $X_0 = X_0'$, $\int_0^t b_s ds = \int_0^t b_s' ds$, $\int_0^t \sigma_s dB_s = \int_0^t \sigma_s' dB_s$.

Hence, the decomposition is unique. $\square$.