Advanced Calculus 1 – Midterm Exam

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1. Let $A \subset \mathbb{R}$ be a bounded set. Show the following equality:
$$\sup A - \inf A = \sup\{x - y : x, y \in A\}.$$

2. Determine whether the following sequences converge or diverge. It if converges, attain the limit.
   (1) $a_n = \sqrt{n+1} - \sqrt{n}$
   (2) $a_n = \sum_{k=1}^{n} \frac{k}{k^2+1}$
   (3) $a_n = \sum_{k=1}^{n} \frac{k!}{3^k}$

3. Show that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

4. Let $\{a_n\}$ be a sequence such that $a_n > 0$ for all $n \in \mathbb{N}$. Show the following inequality.
$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} (a_n)^{\frac{1}{n}}.$$

5. Prove or disprove the following statements:
   (1) Every inner product space is a normed space.
   (2) Every normed space is an inner product space.

6. Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$ satisfying $2a_n \leq a_{n-1} + a_{n+1}$. Show that
$$\lim_{n\to\infty} (a_{n+1} - a_n) = 0.$$

7. Let M be a metric space. Suppose A is a closed set in M and B is a compact set in M. Show that $A \cap B$ is compact.

8. Let $A \subset \mathbb{R}^n$ be a compact set. Show A is closed and bounded without the Heine–Borel theorem.

9. Show that a closed interval $[a, b] \in \mathbb{R}$ is connected by using the definition of connectedness.