실변수함수론 HW4

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Brunn-Minkowski inequality

$A+B = \{x \in \mathbb{R}^d : x = x' + x'' \text{ with } x' \in A \text{ and } x'' \in B\}$

$(m(A+B))^\alpha \geq C_\alpha (m(A)^\alpha + m(B)^\alpha)$

where α is positive number, C^α is constant

If A is convex set, $\forall x, y \in A$, line segment containing x, y contained in A.

A : convex $\Rightarrow A + \lambda A = (1+\lambda) A \quad \forall \lambda > 0.$

$m(A+\lambda A)^\alpha \geq C_\alpha (m(A)^\alpha + m(\lambda A)^\alpha)$

$m((1+\lambda)A)^\alpha \geq C_\alpha (m(A)^\alpha + \lambda^{\alpha d} m(A))$.

$(1+\lambda)^{\alpha d} m(A)^\alpha \geq C_\alpha m(A)^\alpha (1 + \lambda^{\alpha d})$

$(1+\lambda)^{\alpha d} \geq 1 + \lambda^{\alpha d} \quad (\text{when } C_\alpha = 1)$.

Since $(a+b)^r \geq a^r + b^r$ holds when $r \geq 1$, $a, b > 0$.

$\alpha d \geq 1$, $\alpha \geq \frac{1}{d} \Rightarrow (1+\lambda)^{\frac{1}{d}} \geq 1 + \lambda^{\frac{1}{d}}$

$\Rightarrow m(A+B)^{\frac{1}{d}} \geq m(A)^{\frac{1}{d}} + m(B)^{\frac{1}{d}}$

However, A, B measurable $\not\Rightarrow A+B$ measurable.

Thm. If $A, B, A+B$ are measurable sets in $\mathbb{R}^d$,
then $m(A+B)^{\frac{1}{d}} \geq m(A)^{\frac{1}{d}} + m(B)^{\frac{1}{d}}$

Let A, B are rectangles with side lengths $\{a_j\}_1^d$, $\{b_j\}_1^d$

$\left(\prod_{j=1}^d (a_j + b_j)\right)^{\frac{1}{d}} \geq \left(\prod_{j=1}^d a_j\right)^{\frac{1}{d}} + \left(\prod_{j=1}^d b_j\right)^{\frac{1}{d}}$

$a_j \to \lambda_j a_j \quad b_j \to \lambda_j b_j \quad \lambda_j > 0.$

$\left(\prod_{j=1}^d \lambda_j (a_j + b_j)\right)^{\frac{1}{d}} \geq \left(\prod_{j=1}^d \lambda_j a_j\right)^{\frac{1}{d}} + \left(\prod_{j=1}^d \lambda_j b_j\right)^{\frac{1}{d}}$

$(\lambda_1 \lambda_2 \cdots \lambda_d)^{\frac{1}{d}} \left(\prod_{j=1}^d (a_j + b_j)\right)^{\frac{1}{d}} \geq (\lambda_1 \lambda_2 \cdots \lambda_d)^{\frac{1}{d}} \left\{ \left(\prod_{j=1}^d a_j\right)^{\frac{1}{d}} + \left(\prod_{j=1}^d b_j\right)^{\frac{1}{d}} \right\}$

$\Rightarrow$ homogeneity.

When $\lambda_j = (a_j + b_j)^{-1}$, $\frac{1}{d} \sum_{j=1}^d x_j \geq \left(\prod_{j=1}^d x_j\right)^{\frac{1}{d}}$, $\forall x_j \geq 0$.

Consider A, B are union of finitely many rectangles s.t

$\mathrm{int}(A_\pm)$, $\mathrm{int}(B_\pm)$: disjoint.

Replace $A, B, A+B$ by $A+h, B+h', A+B+h+h'$, where

masures don't changes.

Choose rectangle R_1 in $A_- = A \cap \{x_d \leq 0\}$

R_2 in $A_+ = A \cup \{x_d \geq 0\}$

$\Rightarrow \quad A = A_- \cup A_+$

$B_- = B \cup \{x_d \leq 0\}$, $B_+ = B \cap \{x_d \geq 0\}$

$$\frac{m(B_\pm)}{m(B)} = \frac{m(A_\pm)}{m(A)}.$$

However, $A+B \supset (A_+ + B_+) \cup (A_- + B_-)$.

disjoint monotonicity.

$$\begin{aligned} m(A+B) &\geq m(A_+ + B_+) + m(A_- + B_-) \\ &\geq \left(m(A_+)^{\frac{1}{d}} + m(B_+)^{\frac{1}{d}}\right)^d + \left(m(A_-)^{\frac{1}{d}} + m(B_-)^{\frac{1}{d}}\right)^d \\ &= m(A_+)\left[1 + \left(\frac{m(B)}{m(A)}\right)^{\frac{1}{d}}\right]^d + m(A_-)\left[1 + \left(\frac{m(B)}{m(A)}\right)^{\frac{1}{d}}\right]^d \\ &= \left(m(A)^{\frac{1}{d}} + m(B)^{\frac{1}{d}}\right)^d \end{aligned}$$

It's valid for open set $A_\varepsilon \subset A$ s.t $m(A) \leq m(A_\varepsilon) + \varepsilon$

$B_\varepsilon \subset B$ s.t $m(B) \leq m(B_\varepsilon) + \varepsilon$.

Thus $\forall$ arbitrary compact set A, B,

$A+B$ compact , $A^\varepsilon := \{x : d(x, A) < \varepsilon\}$ $\quad A^\varepsilon \searrow A$ as $\varepsilon \to 0$.

$A+B \subset A^\varepsilon + B^\varepsilon \subset (A+B)^{2\varepsilon}$

$m(A+B)^{\frac{1}{d}} \geq m(A)^{\frac{1}{d}} + m(B)^{\frac{1}{d}}$

1.19.(a). If either A or B is open, then $A+B$ is open, so measurable.

pf). Let $A, B \subseteq G$ be subsets of topological group.

$A+B = \bigcup_{b \in B} A+b$

If A is open, then so is $A+x$ for all $x \in G$

$\Rightarrow b \in B \subseteq G$

$\Rightarrow$ $A+B$ is a union of open sets $\Rightarrow$ open.

Since every open set is measurable,

$A+B$ is measurable.

1.19.(b). If A and B are closed, then $A+B$ is an F_σ set, so it is measurable.

(It may not be closed though).

pf). Assume A, B : closed.

Then $A = \bigcup_{k=1}^{\infty} A_k$ and $B = \bigcup_{j=1}^{\infty} B_j$, where the A_k and B_j are compact sequences of sets.

$\Rightarrow$ $A+B = \bigcup_{k,j} A_k + B_j$ where the union is taken over all combinations of k, j.

Each set $A_k + B_j$ is compact.

: Let $x_n + y_n$ be a sequence in $A_k + B_j$

Since A_k is compact, there is subsequence $x_{n(p)}$ of x_n s.t $x_{n(p)} \to x \in A_k$.

Similary, $\exists$ subsequence $y_{n(p)}$ of y_n s.t $y_{n(p)} \to y \in B_j$

$\therefore$ $x_n + y_n$ has convergent subsequence

$x_{n(p)} + y_{n(p)} \to x+y \in A_k + B_j \Rightarrow$ compact.

$\Rightarrow$ $A+B$ is countable union of compact sets.

$\therefore$ $A+B$ is F_σ. so it's measurable.

1.20. (a). $\exists$ closed set A, B s.t $m(A+B) \geq m(A) + m(B)$.

Consider $A = \{(x,0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$, $B = \{(0,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$

Then we can easily check that $m(A) = m(B) = 0$,

$m(A+B) = m(\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}) = 1$.

Let check two cantor sets case, $A, B \subset \mathbb{R}$

$A = (\text{Cantor set})$. $B = (\text{Cantor set})/2$.

We know that $m(A) = m(B) = 0$.

WTS $A+B \supset [0,1]$, then $m(A+B) \geq 1$, we are done.

$\forall x \in [0,1]$ (expressed in ternary number) can be decomposed to two parts of $0,2$ and $0,1$

For instance, $x = 0.01210212 = \underbrace{0.00200202}_{=:p} + \underbrace{0.01010010}_{=:q}$

Then $p \in A$, $q \in B$.

$\because$ $\forall x \in A$ can be expressed ternary number with only $0, 2$ and $\forall y \in B$ can be expressed ternary number with only $0, 1$ by dividing some $x_y \in A$ by 2.
(ex. $q = 0.01010010$, then $\exists\, 2q = 0.02020020 \in A$).

$\therefore$ $A + B \supset [0,1]$ $\square$