미분방정식 HW 4-1 양준우 (2020-91659)

5.6-11. (c).

$xy''+y'-y=0$. 0 is singular point.

$\lim_{x\to 0} \frac{1}{x}x = 1$, $\lim_{x\to 0} -\frac{1}{x}x^2 = 0$.

$\Rightarrow$ 0 is a regular singular point.

$xy''+y'=0$.

$r(r-1)+r=0$. $r^2=0$. $r=0$.

By Thm 5.6.1.

$$y_1(x) = |x|^{r_1}\left(1+\sum_{n=1}^{\infty} a_n(r_1)x^n\right) = 1+\sum_{n=1}^{\infty} a_n(0)x^n$$

$$n^2 a_n(0) + \sum_{k=0}^{n-1} a_k(0)(kp_{n-k}+q_{n-k}) = 0 \quad \forall n$$

where $\sum_{n=0}^{\infty} p_n x^n = x\cdot\frac{Q(x)}{P(x)} = 1$ $\quad p_0=1,\ p_n=0\ \forall n.$

$\sum_{n=0}^{\infty} q_n x^n = x^2\frac{Q(x)}{P(x)} = -x$ $\quad q_1=-1,\ q_n=0\ \forall n\in\mathbb{N}\setminus\{1\}.$

$$a_n(0) = \frac{-\sum_{k=0}^{n-1} a_k(0)(kp_{n-k}+q_{n-k})}{n^2}$$

$$a_1(0) = -a_0(0)q_1 = 1.$$

$$a_2(0) = -\frac{1}{4}(a_0(0)q_2 + a_1(0)(p_1+q_1)) = \frac{1}{4}$$

$$a_3(0) = -\frac{1}{9}(a_0(0)q_3 + a_1(0)(p_2+q_2) + a_2(0)(2p_1+q_1)) = \frac{1}{36}$$

$$\therefore y_1(x) = 1+x+\frac{1}{4}x^2+\frac{1}{36}x^3+\cdots$$

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1} + \sum_{n=1}^{\infty} b_n(r_1)x^n = y_1(x)\ln|x| + \sum_{n=1}^{\infty} b_n(0)x^n$$

$$y_2'(x) = y_1'(x)\ln|x| + \frac{y_1(x)}{x} + \sum_{n=1}^{\infty} nb_n(0)x^{n-1}$$

$$y_2''(x) = y_1''(x)\ln|x| + 2\frac{y_1'(x)}{x} - \frac{y_1(x)}{x^2} + \sum_{n=2}^{\infty} n(n-1)b_n(0)x^{n-2}$$

$$\Rightarrow xy_2''+y_2'-y_2 = \ln|x|(xy_1''+y_1'-y_1) + 2\sum_{n=1}^{\infty} na_n(0)x^{n-1} + \sum_{n=2}^{\infty} n(n-1)b_n(0)x^{n-1}$$
$$+ \sum_{n=1}^{\infty} nb_n(0)x^{n-1} - \sum_{n=1}^{\infty} b_n(0)x^n$$
$$= 2\sum_{n=0}^{\infty}(n+1)a_{n+1}(0)x^n + \sum_{n=1}^{\infty} n(n+1)b_{n+1}(0)x^n + \sum_{n=0}^{\infty}(n+1)b_{n+1}(0)x^n$$
$$- \sum_{n=1}^{\infty} b_n(0)x^n = 0.$$

$$2a_1(0)+b_1(0) + \sum_{n=1}^{\infty}\left(2(n+1)a_{n+1}(0) + (n+1)^2 b_{n+1}(0) - b_n(0)\right)x^n = 0.$$

$b_1(0) = -2a_1(0) = -2$, $\quad b_{n+1}(0) = \frac{1}{(n+1)^2}(b_n(0) - 2(n+1)a_{n+1}(0))$ $\quad b_2 = \frac{-2-2\cdot 2\cdot\frac{1}{4}}{4} = \frac{-3}{4}$

$b_3(0) = \frac{-\frac{3}{4}-2\cdot 3\cdot\frac{1}{36}}{9} = \frac{-11}{108}$ $\quad \therefore y_2(x) = y_1\ln|x| - 2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 + \cdots$

5.7 - 3.

$x^2y'' + 3xy' + (1+x)y = 0.$

$\lim_{x\to 0} x\frac{3x}{x^2} = 3 \quad \lim_{x\to 0} x^2\frac{1+x}{x^2} = 1.$

0 is regular singular point.

$r(r-1) + 3r + 1 = (r+1)^2 = 0, \quad r = -1.$

By Thm 5.6.1,

$y_1(x) = |x|^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right) = x^{-1}\left(1 + \sum_{n=1}^{\infty} a_n(-1)x^n\right).$

$n^2 a_n(0) + \sum_{k=0}^{n-1} a_k(0)(kp_{n-k} + q_{n-k}) = 0.$