Topology I – Homework 3

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Problem 3.1 Find the definition of the Cantor set and write it down.

Solution. Let $C_0 \coloneqq [0,1]$ closed unit interval, and $I_1 \coloneqq (\frac{1}{3}, \frac{2}{3})$ the middle third of C_0. Define $C_1 \coloneqq C_0 \setminus I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, we repeat this procedure for each sub-interval of C_1; that is, we delete the middle third open interval. At the second stage we get $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. We repeat this process to define $C_k = C_{k-1} \setminus I_k$. Then, the Cantor set $\mathcal{C}$ is by definition the intersection of all C_k's:

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k.$$

$\square$

Problem 3.2 Show that the Cantor set $\mathcal{C}$ (i) is a closed subset of $\mathbb{R}$ with empty interior. (ii) is uncountable. (iii) has no isolated point.

Solution. (i) $\mathcal{C}$ is an arbitrary intersection of closed set C_k's. So, $\mathcal{C}$ is compact. Since any $c \in \mathcal{C}$ is end-point of some interval C_k, $B(c,r)$ contains points in some I_k. Thus there is no interior point.
(ii) The number in $[0,1]$ can be written with ternary number system (3-adic number system). For $a \in I_k$, $a = 0.\cdots$, the first 1 appears at k-th digit. Thus, $c \in \mathcal{C}$ can be expressed by 0 and 2. Consider the binary number system for [0,1]. $b \in [0,1]$ can be expressed by 0 and 1. Then we can construct a 1 to 1 correspondence between binary number expression of [0,1] and ternary number expression for $c \in \mathcal{C}$ by replacing 1 and 2. Hence $\mathcal{C}$ is uncountable.
(iii) It is enough to show that every point $c \in \mathcal{C}$ is an accumulation point of $\mathcal{C}$. We know that the boundary of C_k belong to $\mathcal{C}$ and the length of C_k goes to 0 as $k \to \infty$. Consider an open interval $B(c,r)$ for any $c \in \mathcal{C}$, $r > 0$. There exsits $N \in \mathbb{N}$ such that the interval of C_k containg c as boundary on one side is contained in $B(c,r)$ for $k \geq N$. The boundary of this interval belongs to $\mathcal{C}$. Therefore, $B(c,r) \cap \mathcal{C}$ has another boundary of C_k besides c. Hence every $c \in \mathcal{C}$ is an accumulation point of $\mathcal{C}$. Since $\mathcal{C}$ is closed and has no isolated point, $\mathcal{C}$ is perfect. $\square$