

Advanced Calculus I – Midterm Exam

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1. Let $A \subset \mathbb{R}$ be a bounded set. Show the following equality:

$$\sup A - \inf A = \sup\{x - y : x, y \in A\}.$$

2. Determine whether the following sequences converge or diverge. If it converges, attain the limit.

(1) $a_n = \sqrt{n+1} - \sqrt{n}$

(2) $a_n = \sum_{k=1}^n \frac{k}{k^2 + 1}$

(3) $a_n = \sum_{k=1}^n \frac{k!}{3^k}$

3. Show that every bounded sequence in \mathbb{R} has a convergent subsequence.

4. Let $\{a_n\}$ be a sequence such that $a_n > 0$ for all $n \in \mathbb{N}$. Show the following inequality.

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} (a_n)^{\frac{1}{n}}.$$

5. Prove or disprove the following statements:

(1) Every inner product space is a normed space.

(2) Every normed space is an inner product space.

6. Let $\{a_n\}$ be a bounded sequence in \mathbb{R} satisfying $2a_n \leq a_{n-1} + a_{n+1}$. Show that

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0.$$

7. Let M be a metric space. Suppose A is a closed set in M and B is a compact set in M . Show that $A \cap B$ is compact.

8. Let $A \subset \mathbb{R}^n$ be a compact set. Show A is closed and bounded without the Heine-Borel theorem.

9. Show that a closed interval $[a, b] \subset \mathbb{R}$ is connected by using the definition of connectedness.