

# Advanced Calculus 2 – Midterm Exam

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1. Let  $A \subset \mathbb{R}^n$  be an open set,  $f: A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a function, and  $x_0 \in A$ . Provide the definition of the differentiability of  $f$  at  $x_0$ .
2. Let  $A \subset \mathbb{R}^n$  be an open set,  $f: A \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a function,  $x_0 \in A$ , and  $v \in \mathbb{R}^n$ . The directional derivative of  $f$  at  $x_0$  along the vector  $v$  is defined by

$$\partial_v f(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + hv) - f(x_0)}{h}.$$

Suppose the function  $f$  is differentiable. Show that

$$\partial_v f(x_0) = \nabla f(x_0) \cdot v.$$

3. Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable function. For  $x_0 \in \mathbb{R}^n$ , consider a level set  $S := \{x \in \mathbb{R}^n : f(x) = f(x_0)\}$ . Show that  $\nabla f(x_0)$  is orthogonal to  $S$ .
4. Let  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Show that there exists  $M \in \mathbb{R}$  such that  $\|L(x)\| \leq M\|x\|$  for all  $x \in \mathbb{R}^n$ .
5. Find the second-order Taylor approximation for  $f(x, y) = e^x \cos y$  around  $(1, \pi)$ .
6. Let  $A \subset \mathbb{R}^n$  be an open set,  $f: A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable on  $A$ . Show that  $f$  is continuous.
7. Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that  $f(x, y) = (x + y)(xy + xy^2)$ . Find all critical points of  $f$  and determine whether the function  $f$  has an extreme point or not.
8. Let  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map and  $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a function such that  $\|g(x)\| \leq M\|x\|^2$  for some  $M$ . Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfy  $f(x) = L(x) + g(x)$ . Prove that  $Df(0) = L$ .
9. Consider the map  $\mathcal{L}^{-1}: GL(\mathbb{R}^n, \mathbb{R}^n) \rightarrow GL(\mathbb{R}^n, \mathbb{R}^n)$  such that  $\mathcal{L}^{-1}(A) = A^{-1}$ . Show that the derivative of this map is given by

$$D\mathcal{L}^{-1}(A) \cdot B = -A^{-1} \circ B \circ A^{-1}.$$