Advanced Calculus I – Final Exam

June 20, 2019

1. Let $f\colon A \subset \mathbb{R} \to \mathbb{R}$ be a function. Write the definitions of the following statements.
   (1) f is continuous.
   (2) f is uniformly continuous.
   (3) f is Lipschitz continuous.
2. Prove or disprove the following statements.
   (1) If f is Lipschitz continuous, then f is continuous.
   (2) If f is uniformly continuous, then f is Lipschitz continuous.
3. Consider a sequence of functions $\{g_n\}$ where $g_n(x) \coloneqq \left(\frac{x^n}{n!}\right)^2$ for $x \in [-a, a]$, $a \in \mathbb{R}$. Prove that $\sum_{n=0}^{\infty} g_n(x)$ is continuous on $[-a, a]$.
4. Prove the following statements.
   (1) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Then, the sum $f + g\colon \mathbb{R} \to \mathbb{R}$ is continuous at a.
   (2) Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Then, the product $f \cdot g\colon \mathbb{R} \to \mathbb{R}$ is continuous at a.
   (3) Let $f\colon A \subset \mathbb{R} \to \mathbb{R}$ and $g\colon B \subset \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(A) \subset B$. Then the composition $g \circ f\colon A \subset \mathbb{R} \to \mathbb{R}$ is continuous.
5. Let M be a complete normed space and $\Phi\colon M \to M$ be a mapping. Suppose that there is a constant $\alpha \in [0, 1)$ such that $\|\Phi(x) - \Phi(y)\| \le \alpha \|x - y\|$ for all $x, y \in M$. Prove that there exists a fixed point in M.
6. Let M be a normed space and $K \subset M$ be a compact set. Suppose that $f\colon M \to M$ is a continuous function. Prove that $f(K)$ is compact.
7. Prove or disprove the following statements.
   (1) Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b] \in \mathbb{R}$. Suppose f_n converges to f uniformly on $[a, b]$. Then, f is Riemann integrable on $[a, b]$.
   (2) Let $\{f_n\}$ be a sequence of Riemann integrable functions on $\mathbb{R}$ such that $f_n(x)$ converges to 0 for all $x \in \mathbb{R}$ as $n \to \infty$. Then, $\int_{-\infty}^{\infty} f_n(x)\,\mathrm{d}x$ converges as $n \to \infty$.