Mathematical Statistics 1

Ch.3 Continuous Distributions

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Table of Contents

Gamma $_{//}$.

general.

(Continuous) Uniform Distribution

2.1 Uniform Distribution, Unif(a, b)

interval.

- Let r.v. X denote the outcome when a point is selected at random from an interval $[a,\ b]$.

$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{o.w.} \end{cases}$

- *pdf*

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b$$

- *cdf*

$$F(x) = \begin{cases} 0 & x < a, \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & b \le x. \end{cases}$$

$X \sim$ discrete Uniform distribution.

$P(X=x) = \frac{1}{6} \quad x=1, \dots, 6.$

$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$

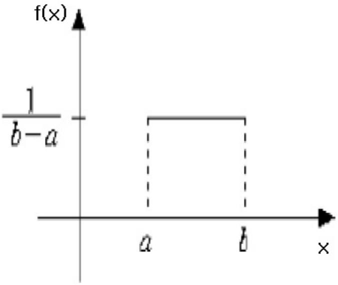
$x < a \rightarrow 0$

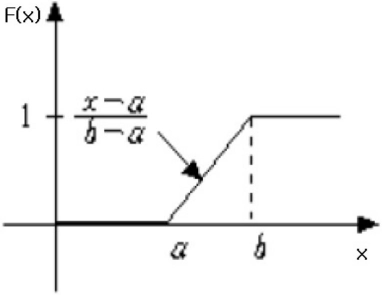
$a \le x \le b \rightarrow \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{a} f_X(t)\,dt + \int_{a}^{x} f_X(t)\,dt$

$x > b \rightarrow 1$

$= \int_{a}^{x} \frac{1}{b-a}\,dt = \frac{x-a}{b-a}$

Uniform pdf and distribution function





- Mean and Variance

$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{a+b}{2}$$

$$E(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{a^2+ab+b^2}{3}$$

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{(b-a)^2}{12}$$

- *mgf*

$$M(t) = E\left(e^{tX}\right) = \int_a^b \frac{e^{tx}}{b-a}\,dx = \left.\frac{e^{tx}}{t(b-a)}\right|_a^b = \frac{e^{tb}-e^{ta}}{t(b-a)}$$

Therefore,

$$M(t) = \begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & t \neq 0, \\ 1 & t = 0. \end{cases}$$

$X \sim Unif(a,b)$

$f_X(x) = \frac{1}{b-a}$

$E(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$

$E(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{1}{b-a}\left[\frac{1}{3}x^3\right]_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b^2+ab+a^2)}{3}$

$$Var(X) = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4} = \frac{1}{12}(4a^2+4ab+4b^2-3a^2-6ab-3b^2)$$

$$= \frac{1}{12}(a^2+b^2-2ab)$$

$$= \frac{1}{12}(a-b)^2$$

$$M_X(t) = E(e^{tX}) = \int_a^b \frac{e^{tx}}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b e^{tx}\,dx = \frac{1}{b-a}\cdot\frac{1}{t}\left[e^{tx}\right]_a^b$$

$$= \frac{e^{tb}-e^{ta}}{t(b-a)}$$

$$M(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & t \neq 0 \\ 1 & t=0 \end{cases}$$

definition 사용

$$M_X^{(1)}(0) = \left.\frac{(be^{tb}-ae^{ta})\cdot t(b-a) - (e^{tb}-e^{ta})(b-a)}{t^2(b-a)^2}\right|_{t=0}$$

$=$

$$f_X(x) = \frac{1}{b-a} \quad (a \le x \le b).$$
$$F_X(x) = \frac{x-a}{b-a}$$

$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b = \frac{b^2-a^2}{2(b-a)}$$
$$= \frac{a+b}{2}$$

$$E(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{1}{3(b-a)}\left[x^3\right]_a^b = \frac{b^3-a^3}{3(b-a)}$$
$$= \frac{a^2+ab+b^2}{3}$$

$$Var(X) = \frac{a^2+ab+b^2}{3} - \frac{(a^2+2ab+b^2)}{4}$$
$$= \frac{4a^2+4ab+4b^2-3(a^2+2ab+b^2)}{12}$$
$$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12} = \frac{(b-a)^2}{12}$$

$$M(t) = E(e^{tX}) = \frac{1}{b-a}\int_a^b e^{tx}\,dx = \frac{1}{t(b-a)}\left[e^{tx}\right]_a^b$$
$$= \frac{e^{tb}-e^{ta}}{t(b-a)}$$

$$\begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & t \ne 0 \\ 1 & t = 0 \end{cases}$$

$$M'(t) = \frac{d}{dt}\left(\frac{e^{tb}-e^{ta}}{t(b-a)}\right)$$

$$= \frac{(be^{tb}-ae^{ta})(t(b-a)) - (e^{tb}-e^{ta})(b-a)}{t^2(b-a)^2}$$

$X \sim Unif(a,b)$

$$M_X(t) = \begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t=0 \end{cases}$$

$$g(t) \overset{\text{let}}{=} e^{tb} = 1 + bt + \frac{b^2t^2}{2!} + \frac{b^3t^3}{3!} + \cdots$$

For $t \neq 0$, $M_X(t) = \dfrac{1}{t(b-a)} \left\{ \left(\cancel{1} + bt + \dfrac{b^2t^2}{2!} + \dfrac{b^3t^3}{3!} + \cdots\right) - \left(\cancel{1} + at + \dfrac{a^2t^2}{2!} + \dfrac{a^3t^3}{3!} + \cdots\right) \right\}$

$$= \frac{1}{t(b-a)} \left\{ (b-a)t + \frac{(b^2-a^2)t^2}{2!} + \frac{(b^3-a^3)t^3}{3!} + \cdots \right\}$$

$$= \frac{1}{b-a} \left\{ (b-a) + \frac{(b^2-a^2)t}{2!} + \frac{(b^3-a^3)t^2}{3!} + \cdots \right\}$$

$$\Rightarrow M_X^{(1)}(t) = \frac{1}{b-a} \left\{ \frac{(b^2-a^2)}{2!} + \frac{(b^3-a^3)2t}{3!} + \cdots \right\}$$

$$M_X^{(1)}(0) = \lim_{t \to 0} \frac{M_X(t) - M_X(0)}{t-0}$$

$$= \lim_{t \to 0} \frac{\cancel{1} + \frac{1}{b-a}\left\{ \frac{(b^2-a^2)t}{2!} + \frac{(b^3-a^3)t^2}{3!} + \cdots \right\} - \cancel{1}}{t}$$

$$= \lim_{t \to 0} \frac{\frac{(b^2-a^2)}{2!} + \frac{(b^3-a^3)t}{3!} + \cdots}{b-a}$$

$$= \frac{a+b}{2} = E(X).$$

$$E(x^2) = M_x^{(2)}(0) = \lim_{t \to 0} \frac{M_x^{(1)}(t) - M_x^{(1)}(0)}{t - 0}$$

$E(X)$

$$= \lim_{t \to 0} \frac{\frac{1}{b-a}\left\{\cancel{\frac{(b^2-a^2)}{2}} + \frac{(b^3-a^3)(2t)}{3!} + \cdots\right\} - \cancel{\frac{(a+b)}{2}}}{t}$$

$$= \lim_{t \to 0} \frac{\frac{(b^3-a^3)(2t)}{3!} + \frac{(b^4-a^4)(3t^2)}{4!} + \cdots}{t(b-a)}$$

$$= \lim_{t \to 0} \frac{\frac{(b^3-a^3)}{3} + \frac{(b^4-a^4)(3t)}{4!} + \cdots}{b-a}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$Var(x) = E(x^2) - \{E(x)\}^2 = \frac{(a-b)^2}{12}$$

Exponential Distribution

2.2 Exponential Distribution, $\mathrm{Exp}(\beta)$

- Let r.v X be the waiting time until the first event occurs.
- *pdf*

$$\frac{1}{\beta} e^{-\frac{x}{\beta}}$$

$$f(x) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 \le x < \infty, \quad 0 < \beta.$$

- *cdf*

$$\frac{1}{\beta}\int_0^x e^{-\frac{x}{\beta}} dx = \frac{1}{\beta}\left[-\beta e^{-\frac{x}{\beta}}\right]_0^x$$

$$F(x) = \int_0^x \frac{1}{\beta} e^{-x/\beta}\, dx = \begin{cases} 0, & x < 0, \\ 1 - e^{-x/\beta}, & 0 \le x < \infty. \end{cases} \quad 1 - e^{-\frac{x}{\beta}}$$

- dexp(x,rate=$1/\beta$): $f_X(x)$, $X \sim$Exp(β) in R
- pexp(x,rate=$1/\beta$): P(X$\le$x), $X \sim$Exp(β) in R

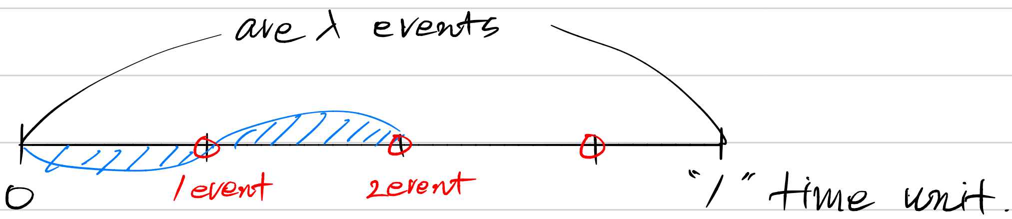
In Poisson, T: # of events in "1" time unit with a mean of λ

$\Rightarrow T \sim Poi(\lambda)$, $f_T(t) = \dfrac{e^{-\lambda}\lambda^t}{t!}$, $t = 0, 1, 2, 3, \cdots$

Next, the waiting time between successive events is of interest.

$\Updownarrow$

the waiting time unitl the first event happens.



X: the waiting time until the first event happens

$F_X(x) = P(X \le x) = 1 - P(X > x)$

$= 1 - P($waiting time until the 1^{st} event happens is grater than x $)$

$= 1 - P($No event in the time interval $[0, x])$.

ave λ events / 1 time unit $\quad$ $X \sim Exp(\beta)$

$\Leftrightarrow$ ave λx events / x time unit.

Assume T_2 : # of events in "x" time unit

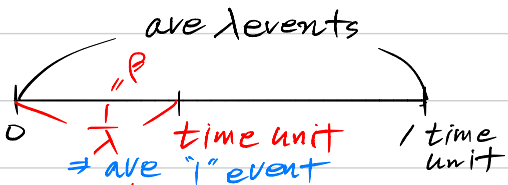
$T_2 \sim Poi(\lambda x) \rightarrow f_{T_2}(t) = \dfrac{e^{-\lambda x}(\lambda x)^t}{t!}$, $t = 0, 1, 2, \cdots$

$P($No event in $[0, x]) = P(T_2 = 0) = e^{-\lambda x}$

$\therefore F_X(x) = 1 - e^{-\lambda x}$

$f_X(x) = F_X'(x) = \lambda e^{-\lambda x}$, $x \ge 0$

let $\frac{1}{\lambda} = \beta$ $\quad f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$, $x \ge 0$



β: average waiting time until the 1^{st} event happens.

- The first two moments

$$
\begin{aligned}
E(X) &= \int_0^\infty \frac{x}{\beta} \exp\left(-\frac{x}{\beta}\right)\, dx = \int_0^\infty \exp\left(-\frac{x}{\beta}\right)\, dx \\
&= -\beta \exp\left(-\frac{x}{\beta}\right)\Big|_0^\infty = \beta. \\
E(X^2) &= \int_0^\infty \frac{x^2}{\beta} \exp\left(-\frac{x}{\beta}\right)\, dx = 2\beta \int_0^\infty \exp\left(-\frac{x}{\beta}\right)\, dx \\
&= -2\beta^2 \exp\left(-\frac{x}{\beta}\right)\Big|_0^\infty = 2\beta^2.
\end{aligned}
$$

- Mean: $E(X) = \beta$
- Variance:

$$
\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \beta^2
$$

- β: the average waiting time until the first event occurs

- *mgf*

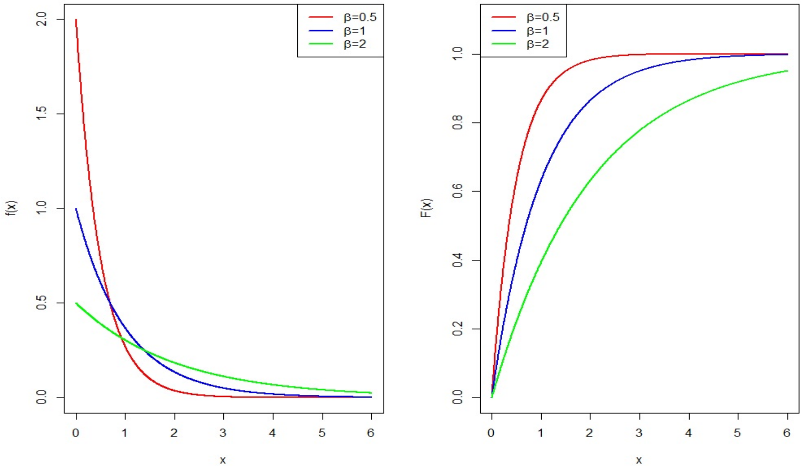
$$
\begin{aligned}
M(t) &= \int_0^\infty \frac{e^{tx}}{\beta} \exp\left(-\frac{x}{\beta}\right)\, dx = \int_0^\infty \frac{1}{\beta} \exp\left[-\left(\frac{1-\beta t}{\beta}\right) x\right]\, dx \\
&= \frac{1}{1-\beta t} \int_0^\infty \frac{1-\beta t}{\beta} \exp\left[-\left(\frac{1-\beta t}{\beta}\right) x\right]\, dx \\
&= \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta}.
\end{aligned}
$$

$$
M^{(1)}(0) = \frac{d}{dt} M(t)\Big|_{t=0} = \frac{d}{dt}(1-\beta t)^{-1}\Big|_{t=0} = \beta(1-\beta t)^{-2}\Big|_{t=0} = \beta
$$

$$
\begin{aligned}
M^{(2)}(0) &= \frac{d^2}{dt^2} M(t)\Big|_{t=0} = \frac{d^2}{dt^2}(1-\beta t)^{-1}\Big|_{t=0} = \frac{d}{dt}\beta(1-\beta t)^{-2}\Big|_{t=0} \\
&= 2\beta^2(1-\beta t)^{-3}\Big|_{t=0} = 2\beta^2
\end{aligned}
$$

$$
\mathrm{Var}(X) = M^{(2)}(0) - \left\{M^{(1)}(0)\right\}^2 = \beta^2.
$$

Exponential pdf and distribution function



Example 3.2-1

Let X have an exponential distribution with a mean of 20. Find the probability that X is less than 18. Also, find the value of m satisfying $F(m) = 0.5$.

$\lambda e^{-\lambda x}$.

$\frac{e^{-\lambda} \lambda^{x}}{x!}$

$\frac{1}{\beta} e^{-\frac{x}{\beta}}$

Relationship between Poisson and Exponential distributions

- $T \sim$Poi(λ), T: the number of events occurring in a given time interval

$$f(t) = \frac{e^{-\lambda}\lambda^t}{t!}, \quad t = 0, 1, \cdots ; \quad 0 \le \lambda < \infty$$

- The waiting times between successive events are also a random variable.
- Let X denote the waiting time until the first change occurs when we know that the mean number of events in the unit interval, λ.
- Then the random variable X has an Exponential distribution with parameter $\beta = 1/\lambda$, which is the mean waiting time for the first event.

Example 3.2-2

Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer? Also, find the median time until the first arrival.

$\lambda = \frac{1}{3}$ $\frac{1}{\lambda} = 3.$

$X \sim Exp(20)$

$f_X(x) = \frac{1}{20} e^{-\frac{x}{20}}, \quad x \geq 0$

1) $P(X < 18) = \int_0^{18} \frac{1}{20} e^{-\frac{x}{20}} dx =$

2) $F_X(m) = 1 - e^{-\frac{m}{20}} = 0.5$

$\Rightarrow m = -20 \ln(0.5).$

$\frac{1}{5}$

X : # of customers per "1" hour

$X \sim Poi(20)$

Y : waiting time (min) until the first customer arrives.

ave 20 customers / 60 min

$\Leftrightarrow$ ave 1 customer / 3 min

$Y \sim Exp(3)$

$P(Y > 5)$, $F_Y(m) = 0.5$

$\frac{20}{12}$ $\frac{1}{3}$ 5 min

$f(x) = \frac{1}{3} e^{-\frac{1}{3}x}, \quad 0 \leq x < \infty.$

$\frac{1}{3}$

$P(X > 5) = \int_5^{\infty} \frac{1}{3} e^{-\frac{1}{3}x} dx$

$-\left[e^{-\frac{1}{3}x}\right]_5^{\infty} = e^{-\frac{5}{3}}$

$e^{-\frac{5}{3}}$

$m =$

$\int_0^m \frac{1}{3} e^{-\frac{1}{3}x} dx = 0.5.$

$-\ln 2 = -\frac{m}{3}$

$-\left(e^{-\frac{m}{3}} - 1\right).$

$m = 3\ln 2.$

$1 - e^{-\frac{m}{3}} = \frac{1}{2}$

$\frac{1}{2} = e^{-\frac{m}{3}}$

Gamma and Chi-Square Distributions

2.3.1 Gamma Distribution, $\mathrm{Gamma}(\alpha, \beta)$

- Gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt > 0, \quad \alpha > 0$$

- If $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

$$\Gamma(\alpha) = \left[-t^{\alpha-1}e^{-t}\right]_0^\infty + \int_0^\infty (\alpha-1)t^{\alpha-2}e^{-t}dt$$
$$= (\alpha - 1)\int_0^\infty t^{\alpha-2}e^{-t}dt = (\alpha-1)\Gamma(\alpha-1).$$

- For integer n, $\Gamma(n) = (n-1)!$, $\Gamma(1) = 1$.

- Let r.v. X be the waiting time until the αth event occurs.
- *pdf*

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, 0 < \alpha, \beta,$$

where $\Gamma(\alpha) = (\alpha - 1)!$ if α is an interger.

- $Exp(\beta) \equiv Gamma(1, \beta)$
- dgamma(x,shape=α, scale=β): $f_X(x)$ in R
- pgamma(x,shape=α, scale=β): P(X$\leq$x) in R

- The first two moments

$$
\begin{aligned}
E(X) &= \int_0^\infty x\, f(x)\, dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{(\alpha+1)-1} \exp\left(-\frac{x}{\beta}\right)\, dx \\
&= \alpha\beta \int_0^\infty \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{(\alpha+1)-1} \exp\left(-\frac{x}{\beta}\right)\, dx = \alpha\beta. \\
E(X^2) &= \int_0^\infty x^2\, f(x)\, dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{(\alpha+2)-1} \exp\left(-\frac{x}{\beta}\right)\, dx \\
&= \alpha(\alpha+1)\beta^2 \int_0^\infty \frac{1}{\Gamma(\alpha+2)\beta^{\alpha+2}} x^{(\alpha+2)-1} \exp\left(-\frac{x}{\beta}\right)\, dx \\
&= \alpha(\alpha+1)\beta^2.
\end{aligned}
$$

- Variance

$$
\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2.
$$

$\sum f_X(x) = 1.$ $\int f(x)dx = 1$

X : waiting time until the $\underset{\parallel}{\overset{2}{\alpha}}{}^{th}$ event happens

$$E(X) = \int_0^\infty \frac{1}{\Gamma(\alpha)\cdot\beta^\alpha} x^\alpha \cdot e^{-\frac{x}{\beta}} dx$$

$\alpha \cdot \Gamma(\alpha)$

$$= \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\cdot\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} \underbrace{x^{(\alpha+1)-1} e^{-\frac{x}{\beta}} dx}$$

pdf of Gamma $(\alpha+1, \beta)$

$= \alpha\beta$ $E(X) = 2\beta$

$X \sim Exp(\beta)$

X : waiting time until 1^{st} event happens.

$\Rightarrow E(X) = \beta$

average waiting time until the 1^{st} event.

$\int$ ____ dx

pdf of Gamma

$1 - \beta t > 0$

$1 > \beta t \rightarrow t < \frac{1}{\beta}$

$$\left(\underset{>0}{\alpha}, \frac{\beta^{>0}}{\underset{>0}{1-\beta t}}\right).$$

$X \sim Gamma(\alpha, \beta)$

α : shape parameter

β : scale "

$X > 0$ $\sim \chi^2(r)$

$\parallel$

$Gamma(\frac{r}{2}, 2)$

- *mgf*

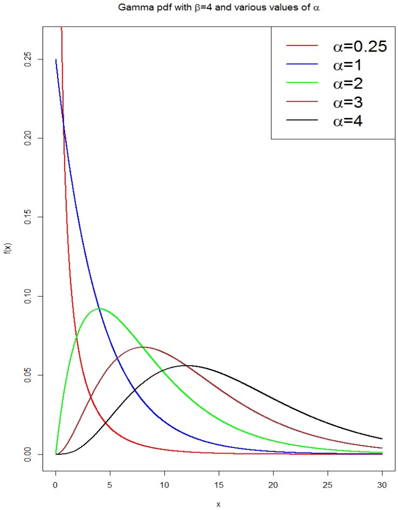
$$
\begin{aligned}
M(t) &= \int_0^\infty \frac{e^{tx}}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)\, dx \\
&= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left[-\left(\frac{1-\beta t}{\beta}\right)x\right]\, dx \\
&= \left(\frac{1}{1-\beta t}\right)^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} \left(\frac{1-\beta t}{\beta}\right)^\alpha x^{\alpha-1} \exp\left[-\left(\frac{1-\beta t}{\beta}\right)x\right]\, dx \\
&= \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}.
\end{aligned}
$$

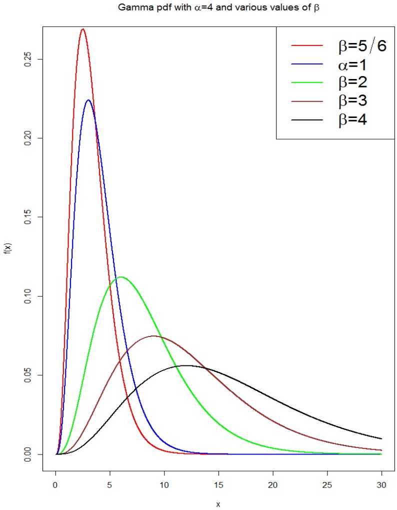
$$
M^{(1)}(0) = \frac{d}{dt}M(t)\Big|_{t=0} = \frac{d}{dt}(1-\beta t)^{-\alpha}\Big|_{t=0} = \alpha\beta(1-\beta t)^{-\alpha-1}\Big|_{t=0} = \alpha\beta
$$

$$
\begin{aligned}
M^{(2)}(0) &= \frac{d^2}{dt^2}M(t)\Big|_{t=0} = \frac{d^2}{dt^2}(1-\beta t)^{-\alpha}\Big|_{t=0} = \frac{d}{dt}\alpha\beta(1-\beta t)^{-\alpha-1}\Big|_{t=0} \\
&= \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}\Big|_{t=0} = \alpha(\alpha+1)\beta^2
\end{aligned}
$$

$$
\mathrm{Var}(X) = M^{(2)}(0) - \left\{M^{(1)}(0)\right\}^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2.
$$

Gamma pdf


Gamma pdf with β=4 and various values of α
α=0.25
α=1
α=2
α=3
α=4
f(x)
x


Gamma pdf with α=4 and various values of β
β=5/6
α=1
β=2
β=3
β=4
f(x)
x

Example 3.2-4

Suppose the number of customers per hour arriving at a shop follows a Poisson process with mean 30. What is the probability that the shopkeeper will wait more than 5 minutes before both of the first two customers arrive?

X: # of customers per hour (= 60 min) ~ Poi (30)

Y: waiting time until the first two customers arrive

$\alpha = 2$

are 30 customers / 60 min

⇒ are 1 customer / 2 min

⇒ $\beta = 2$

$\therefore Y \sim \text{Gamma}(2,2)$

$$P(Y>5) = \int_5^\infty \frac{1}{\Gamma(2)2^2} y e^{-\frac{y}{2}} dy = \cdots = \frac{7}{2} e^{-\frac{5}{2}}$$

2.3.2 Chi-Square Distribution, $\chi^2(r)$

- Special case of Gamma distribution: $\chi^2(r) \equiv Gamma(r/2, 2)$
- r: the number of degrees of freedom

positive
1.2.3.4.5 ...

- *pdf*

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad 0 < x < \infty,\ 0 < r.$$

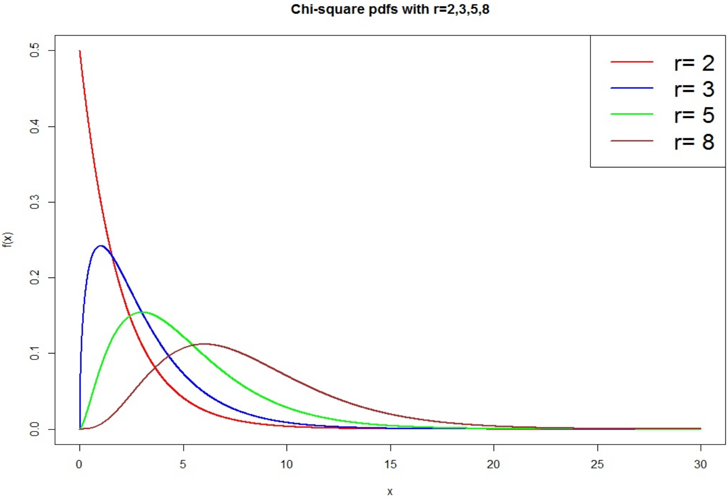
- The mean and variance

$$E(X) = \alpha\beta = r, \quad \mathrm{Var}(X) = \alpha\beta^2 = 2r.$$

- mgf

$$M(t) = (1-2t)^{-r/2}, \quad t < 1/2$$

Chi-square pdf



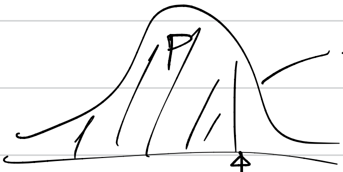
- $\chi^2_\alpha(r)$: the $100(1-\alpha)$th percentile of the chi-square distribution with r degrees of freedom such that

$$P[X \le \chi^2_\alpha(r)] = 1 - \alpha$$

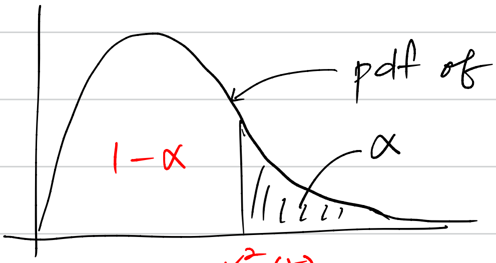
π_p : $100\,p^{th}$ percentile of $X \sim f_X(x)$

$F_X(\pi_p) = p$

$P(X \le \pi_p) = p$



$\chi^2_\alpha(r)$: $100(1-\alpha)^{th}$ percentile of $X \sim \chi^2(r)$



$\chi^2(r)$.

$F_X(\chi^2_\alpha(r)) = 1-\alpha$

$\Updownarrow$

$P(X \le \chi^2_\alpha(r)) = 1-\alpha$

χ^2

$\Leftrightarrow P(X > \chi^2_\alpha(r)) = \alpha$.

$\chi^2(r)$. one parameter. $(r > 0)$.

Example 3.2-8

Let X have a chi-square distribution with five degrees of freedom. Find the values of $\chi^2_{0.10}(5)$ and $\chi^2_{0.90}(5)$.

Example 3.2-9

If customers arrive at a shop on the average of 30 per hour in accordance with a Poisson process, what is the probability that the shopkeeper will have to wait longer than 9.390 minutes for the first nine customers to arrive?

X : # of customers per 1 hour (= 60 min).

$X \sim Poi(30)$.

ave 30 customers / 60 min

→ 〃 1 〃 / 2 〃

Y : waiting time (min) for the first $\underset{\alpha.}{\underline{\text{nine}}}$ customers to arrive

$Y \sim Gamma(9, 2) \equiv \chi^2(18)$

$P(Y > 9.39) = 0.95$