Introduction to Real Analysis – Final Exam

Junwoo Yang

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Problem 1. (1) Let

$$\|f\|_{L^{1,w}(\mathbb{R}^d)} = \sup_{\alpha>0} \alpha \cdot m\left(\left\{x \in \mathbb{R}^d : |f(x)| > \alpha\right\}\right)$$

where m stands for the Lebesgue measure on $\mathbb{R}^d$. Check that

$$\|f\|_{L^{1,w}(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

(2) Give an example of a function g in $(0, \infty)$ such that

$$\|g\|_{L^{1,w}((0,\infty))} = 1 \quad \text{and} \quad \|g\|_{L^1((0,\infty))} = +\infty.$$

Proof. (1) For any $\alpha > 0$, let $A_\alpha = \{x \in \mathbb{R}^d : |f(x)| > \alpha\}$. It's measure is

$$m(A_\alpha) = \int_{A_\alpha} \mathrm{d}m \leq \int_{A_\alpha} \frac{|f(x)|}{\alpha} \,\mathrm{d}x \leq \frac{1}{\alpha}\|f\|_{L^1(\mathbb{R}^d)}.$$

After multiplying by $\sup_{\alpha>0} \alpha$ the result is proved.

$$\|f\|_{L^{1,w}(\mathbb{R}^d)} = \sup_{\alpha>0} \alpha \cdot m(A_\alpha) \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

(2) Let $g(x) = \dfrac{1}{x}$.

$$\begin{cases} \text{For any } \alpha > 0,\ \alpha m(A_\alpha) = \alpha \cdot \dfrac{1}{\alpha} = 1 \Rightarrow \|g\|_{L^{1,w}((0,\infty))} = 1. \\ \|g\|_{L^1((0,\infty))} = \displaystyle\int_0^\infty \frac{1}{x} \,\mathrm{d}x = +\infty. \end{cases}$$

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Problem 2. (1) Suppose that F is a $\mathbb{R}$-valued absolutely continuous function on $[a,b]$. Prove that

$$T_F(a,b) = \int_a^b |F'(t)| \, \mathrm{d}t.$$

(2) Suppose that F is a $\mathbb{R}$-valued continuous function on $[a,b]$. Show that

$$T_F(a,b) = \lim_{\varepsilon \to 0+} T_F(a+\varepsilon, b).$$

(3) Determine whether

$$F(x) = (x-1)^{2022} \sin((x-1)^{-2020}) \quad \text{for } x \in [0,2]$$

is of bounded variation on $[0,2]$ or not.

Proof. (1) Stein, Shakarachi, Chap.3, Prop.4.2

(2)

(3)

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Problem 3. (1) For a fixed number $\xi \in (0,1)$, we construct a subset $\mathcal{C}_\xi$ of $\mathbb{R}$ in the following manner:

- In the first stage of the construction, we remove the middle ξ from $[0,1]$ so that the remaining set is $[0, \frac{1-\xi}{2}] \cup [\frac{1+\xi}{2}, 1]$.
- In the second stage, we remove the middle ξ^2 from each of $[0, \frac{1-\xi}{2}]$ and $[\frac{1+\xi}{2}, 1]$.
- By repeating this process countably many times, we obtain the set $\mathcal{C}_\xi$. Note that $\mathcal{C}_{\frac{1}{3}}$ is the Cantor set.

Compute the (strict) Hausdorff dimension of the set $\mathcal{C}_\xi$.

(2) Prove that there exists a subset of $\mathbb{R}$ having Hausdorff dimension γ for any $\gamma \in (0,1)$.

(3) Compute the Hausdorff dimension and the Minkowski dimension of the compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $\mathbb{R}$.

Proof. (1) By adopting the argument to obtain the Hausdorff dimension of the Cantor set $\mathcal{C}_{\frac{1}{3}}$, we see that the Hausdorff dimension of $\mathcal{C}_\xi$ is $\frac{\log 2}{\log 2 - \log(1-\xi)}$. Stein, Shakarchi, Chap.7, Exercise 8.

(2) Let $f(\xi) = \frac{\log 2}{\log 2 - \log(1-\xi)}$, $\xi \in (0,1)$.

$$\begin{cases} f\text{: continuous in (0,1).} \\ f\text{: monotone decreasing in (0,1).} \\ \lim\limits_{\varepsilon \to 0+} f(\xi) = 1, \ \lim\limits_{\varepsilon \to 1-} f(\xi) = 0. \end{cases}$$

$\{\mathcal{C}_\xi : \xi \in (0,1)\}$ provides subsets of $\mathbb{R}$ having Hausdorff dimension γ for any $\gamma \in (0,1)$.

(3)

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