금융수학 1 - HW5 양준우 (2020-91659)

1.1. Since M is martingale w.r.t $(\mathcal{F}_t)_{t\geq 0}$,

$E(M_t | \mathcal{F}_s) = M_s$ for all $0 \leq s < t$

Take $E(\cdot | \mathcal{G}_s)$

$E(E(M_t | \mathcal{F}_s) | \mathcal{G}_s) = E(M_s | \mathcal{G}_s)$.

$\Rightarrow E(M_t | \mathcal{G}_s) = M_s$ for all $0 \leq s < t$.

($\because \mathcal{G}_s \subseteq \mathcal{F}_s$, M is adapted to $(\mathcal{G}_t)_{t\geq 0}$.)

This implies that M is martingale w.r.t $(\mathcal{G}_t)_{t\geq 0}$.

1.2. (i). $M_t = B_t^2 - t$

<u>Claim 1.</u> M_t is martingale w.r.t $(\mathcal{F}_t^B)_{t\geq 0}$

It is clear that $M_t = B_t^2 - t$ is adapted to $\mathcal{F}_t^B$.

For $0 \leq s < t$,

$$E(M_t | \mathcal{F}_s^B) = E(B_t^2 - t | \mathcal{F}_s^B) = E((B_t - B_s)^2 + 2B_tB_s - B_s^2 - t | \mathcal{F}_s^B)$$
$$= E((B_t - B_s)^2 | \mathcal{F}_s^B) + 2E(B_tB_s | \mathcal{F}_s^B) - E(B_s^2 | \mathcal{F}_s^B) - t$$
$$= E((B_t - B_s)^2) + 2E((B_t - B_s)B_s + B_s^2 | \mathcal{F}_s^B) - B_s^2 - t$$

($\because B_t - B_s \perp \mathcal{F}_s^B = \sigma(B_u | 0 \leq u \leq s)$, $B_s^2 : \mathcal{F}_s^B$-m'ble).

$$= Var(B_t - B_s) + 2E((B_t - B_s)B_s | \mathcal{F}_s^B) + 2E(B_s^2 | \mathcal{F}_s^B) - B_s^2 - t$$

($\because B_t - B_s \sim N(0, t-s)$).

$$= t - s + 2B_s E(B_t - B_s | \mathcal{F}_s^B) + 2B_s^2 - B_s^2 - t$$
$$= 2B_s E(B_t - B_s) + B_s^2 - s$$
$$= B_s^2 - t. \quad (\because E(B_t - B_s) = 0).$$

<u>Claim 2.</u> M is a martingale w.r.t $(\mathcal{F}_t^M)_{t\geq 0}$

Since M is a martingale w.r.t $(\mathcal{F}_t^B)_{t\geq 0}$,

$E(M_t | \mathcal{F}_s^B) = M_s$. for all $0 \leq s < t$

Since the natural filteration $(\mathcal{F}_t^M)_{t\geq 0}$ is a family of smallest σ-alg. which make M adapted, $\mathcal{F}_t^M \subseteq \mathcal{F}_t^B$ for all $t \in [0, \infty)$.

Then take $E(\cdot | \mathcal{F}_s^M)$

$$E(E(M_t|\mathcal{F}_s^B)|\mathcal{F}_s^M) = E(M_s|\mathcal{F}_s^M).$$

$\Rightarrow E(M_t|\mathcal{F}_s^M) = M_s$ ($\because \mathcal{F}_s^M \subseteq \mathcal{F}_s^B$, M is adapted to $(\mathcal{F}_t^M)_{t\geq 0}$).

$\therefore$ $M_t = B_t^2 - t$ is a martingale w.r.t $(\mathcal{F}_t^M)_{t\geq 0}$. $\square$.

1.2.(ii). By 1.2.(i) claim 2, it suffices to show that w.r.t $(\mathcal{F}_t^B)_{t\geq 0}$.

$$M_t = B_t^3 - 3\int_0^t B_u du$$

It is clear that $B_t^3 - 3\int_0^t B_u du$ is $\mathcal{F}_t^B$ - m'ble.

For $0 \leq s < t$,

$E(B_t^3 - 3\int_0^t B_u du | \mathcal{F}_s^B)$

$= E((B_t - B_s)^3 + 3B_t^2 B_s - 3B_t B_s^2 + B_s^3 | \mathcal{F}_s^B) - 3E(\int_0^s B_u du + \int_s^t B_u du | \mathcal{F}_s^B)$

$= E((B_t - B_s)^3) + E(3(B_t - B_s)^2 B_s + 6B_t B_s^2 - 3B_s^3 - 3B_t B_s^2 + B_s^3 | \mathcal{F}_s^B)$

$\quad - 3\int_0^s B_u du - 3E(\int_s^t B_u du)$

($\because B_t - B_s \perp \mathcal{F}_s^B$, $\int_0^s B_u du$: $\mathcal{F}_s^B$ - m'ble, $\int_s^t B_u du \perp \mathcal{F}_s^B$).

$= 3B_s E((B_t - B_s)^2) + E(3(B_t - B_s) B_s^2 | \mathcal{F}_s^B) + B_s^3 - 3\int_0^s B_u du - 3\int_s^t E(B_u) du$

($\because E((B_t - B_s)^3) = 0$, B_s^3 : $\mathcal{F}_s^B$ - m'ble, $E(\int_s^t B_u du) = \int_s^t E(B_u) du$ by Fubini's thm).

$= \underline{3B_s(t-s)} + B_s^3 - 3\int_0^s B_u du$

???

pf for $E((B_t - B_s)^3)$

$B := B_t - B_s \sim N(0, t-s)$

$\Rightarrow \phi_B(t) = E(e^{iBt}) = e^{-\frac{1}{2}\sigma^2 t^2}$ where $\sigma^2 = t - s$.

$\frac{d}{dt}\phi_B(t) = \phi_B'(t) = E(iBe^{iBt}) = (-\sigma^2 t)e^{-\frac{1}{2}\sigma^2 t^2}$

$\phi_B''(t) = E(-B^2 e^{iBt}) = \{(-\sigma^2 t)^2 - \sigma^2\} e^{-\frac{1}{2}\sigma^2 t^2}$

$\phi_B'''(t) = E(-iB^3 e^{iBt}) = \{(-\sigma^2 t)^3 - \sigma^2(-\sigma^2 t) - 2\sigma^2(-\sigma^2 t)\} e^{-\frac{1}{2}\sigma^2 t^2}$

$\phi_B'''(0) = E(-iB^3) = 0. \Rightarrow E(B^3) = 0.$

1.3.(i). $B = (B_t^{(1)}, B_t^{(2)})_{t\geq 0}$: 2-dim'l B.M.

$P(B_t^{(1)} > \sqrt{3} B_s^{(2)}) = P(B_t^{(1)} - \sqrt{3} B_s^{(2)} > 0)$.

Since $B_t^{(1)} \sim N(0,t) \approx \sqrt{t} Z^{(1)}$, $B_s^{(2)} \sim N(0,s) \approx \sqrt{s} Z^{(2)}$,

$B_t^{(1)} \perp B_t^{(2)}$,

$\Rightarrow P(\sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)} > 0)$.

$\sqrt{t} Z^{(1)} \perp \sqrt{3s} Z^{(2)} \Rightarrow \sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)}$ is normal.

$E(\sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)}) = 0$.

$Var(\sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)}) = Var(\sqrt{t} Z^{(1)}) + Var(\sqrt{3s} Z^{(2)}) = t + 3s$.

$\Rightarrow P(\sqrt{t+3s}\, Z > 0) = \frac{1}{2}$.

1.3.(ii). $E(e^{B_t^{(1)} + \int_0^s B_u^{(2)} du}) = E(e^{B_t^{(1)}}) \cdot E(e^{\int_0^s B_u^{(2)} du})$ $(\because B_t^{(1)} \perp B_t^{(2)})$.

Since $B_t^{(1)} \sim N(0,t)$, $\int_0^s B_u^{(2)} du$: normal.

$E(\int_0^s B_u^{(2)} du) = \int_0^s E(B_u^{(2)}) du = 0$.

$Var(\int_0^s B_u^{(2)} du) = \int_0^s \int_0^s E(B_u^{(2)} B_v^{(2)}) du dv$.

$= \int_0^s \int_0^s \min(u,v)\, du dv. = \frac{s^3}{2}$

$E(e^{B_t^{(1)}}) \cdot E(e^{\int_0^s B_u^{(2)} du}) = e^{\frac{t}{2}} \cdot e^{\frac{s^3}{4}} = e^{\frac{t}{2} + \frac{s^3}{4}}$

1.3.(iii). Let $M_t = (B_t^{(1)} \cdot B_t^{(2)})_{t \geq 0}$ $\quad (\mathcal{F}_t^B)_{t\geq 0}$.

It is clear that $B_t^{(1)} B_t^{(2)}$ is adapted to $(\mathcal{F}_t^B)_{t\geq 0}$.

$E(M_t \mid \mathcal{F}_s^B) = E(B_t^{(1)} B_t^{(2)} \mid \mathcal{F}_s^B)$

$= E((B_t^{(1)} - B_s^{(1)}) B_t^{(2)} \mid \mathcal{F}_s^B) + E(B_s^{(1)} B_t^{(2)} \mid \mathcal{F}_s^B)$

$= E((B_t^{(1)} - B_s^{(1)})(B_t^{(2)} - B_s^{(2)}) \mid \mathcal{F}_s^B) + E((B_t^{(1)} - B_s^{(1)}) B_s^{(2)} \mid \mathcal{F}_s^B) + B_s^{(1)} E(B_t^{(2)} \mid \mathcal{F}_s^B)$

$= \underbrace{E((B_t^{(1)} - B_s^{(1)})(B_t^{(2)} - B_s^{(2)}))}_{0} + \underbrace{B_s^{(2)} E(B_t^{(1)} - B_s^{(1)} \mid \mathcal{F}_s^B)}_{0} + B_s^{(1)} B_s^{(2)}$

$= B_s^{(1)} B_s^{(2)} = M_s \quad \square$.

1.3.(iv). $\left(t^2 B_t^{(1)} B_t^{(2)} - 2\int_0^t u B_u^{(1)} B_u^{(2)} du\right)_{t\geq 0}$.

$E(t^2 B_t^{(1)} B_t^{(2)} \mid \mathcal{F}_s^B) = t^2 B_s^{(1)} B_s^{(2)}$ by 1.3.(iii).

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\begin{aligned}
E\left(-2\int_0^t u B_u^{(1)} B_u^{(2)} du \mid \mathcal{F}_s^B\right) &= E\left(-2\int_0^s u B_u^{(1)} B_u^{(2)} du - 2\int_s^t u B_u^{(1)} B_u^{(2)} du \mid \mathcal{F}_s^B\right) \\
&= -2\int_0^s u B_u^{(1)} B_u^{(2)} du - 2\int_s^t E(u B_u^{(1)} B_u^{(2)} du \mid \mathcal{F}_s^B) \\
&= -2\int_0^s u B_u^{(1)} B_u^{(2)} du - 2(t-s) B_u^{(1)} B_u^{(2)}
\end{aligned}
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