미분방정식 HW5-1. 양준우 (2020-91659)

6.6-9.

$$F(s) = \frac{s}{(s+1)(s^2+9)}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} \quad \text{(by convolution thm).}$$

$$= (e^{-t} * \cos 3t)(t)$$

$$= \int_0^t e^{-(t-s)} \cos 3s \, ds$$

$$= e^{-t} \int_0^t \overset{v'}{e^s} \overset{u}{\cos 3s} \, ds. \cdots ①$$

$$= e^{-t}\left([e^s \cos 3s]_0^t + 3\int_0^t \overset{v'}{e^s} \overset{u}{\sin 3s} \, ds \right)$$

$$= e^{-t}\left\{ e^t \cos 3t - 1 + 3\left([e^s \sin 3s]_0^t - 3\int_0^t e^s \cos 3s \, ds \right) \right\}$$

$$= e^{-t}\left\{ e^t \cos 3t - 1 + 3e^t \sin 3t - 9\int_0^t e^s \cos 3s \, ds \right\} \cdots ②$$

$$\Rightarrow \int_0^t e^s \cos 3s \, ds = \frac{1}{10}\left\{ e^t \cos 3t - 1 + 3e^t \sin 3t \right\} \quad \text{by ① and ②}$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = \frac{1}{10}(\cos 3t + 3\sin 3t - e^{-t}) \quad \square.$$

6.6-12.

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin 2t \quad \text{(Volterra integral equation).}$$

a. $\phi(t) + (t * \phi(t))(t) = \sin 2t.$

$$\overset{\mathcal{L}}{\Rightarrow} \Phi(s) + \frac{1}{s^2}\cdot\Phi(s) = \frac{2}{s^2+4}$$

$$\Phi(s)\left(1 + \frac{1}{s^2}\right) = \frac{2}{s^2+4}$$

$$\Phi(s) = \frac{2s^2}{(s^2+4)(s^2+1)}$$

$$= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} = \frac{8}{3}\cdot\frac{1}{s^2+4} - \frac{2}{3}\cdot\frac{1}{s^2+1}$$

$$\left\{ \begin{array}{l} (As+B)(s^2+1) + (Cs+D)(s^2+4) = 2s^2 \\ A + C = 0 \Rightarrow C = -A \\ B + D = 2 \Rightarrow D = 2 - B \\ A + 4C = 0 \Rightarrow A - 4A = 0 \Rightarrow A = C = 0. \\ B + 4D = 0 \Rightarrow B + 4(2-B) = 0 \Rightarrow B = \frac{8}{3},\ D = -\frac{2}{3} \end{array} \right\}$$

$$= \frac{4}{3}\cdot\frac{2}{s^2+4} - \frac{2}{3}\cdot\frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{\Phi(s)\} = \phi(t) = \frac{4}{3}\sin 2t - \frac{2}{3}\sin t \quad \square.$$

b. $\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin(2t)$.

$\frac{d}{dt} \Rightarrow \phi'(t) + \int_0^t \phi(\xi)\,d\xi + t\cdot\frac{d}{dt}\int_0^t \phi(\xi)\,d\xi - \frac{d}{dt}\int_0^t \xi\phi(\xi)\,d\xi = 2\cos 2t$

by FTC $\Rightarrow \phi'(t) + \int_0^t \phi(\xi)\,d\xi + t\phi(t) - t\phi(t) = 2\cos 2t$

$\phi'(t) + \int_0^t \phi(\xi)\,d\xi = 2\cos 2t$

$\frac{d}{dt} \Rightarrow \phi''(t) + \frac{d}{dt}\int_0^t \phi(\xi)\,d\xi = -4\sin 2t$

$\Rightarrow \phi''(t) + \phi(t) = -4\sin 2t$ //

$\phi(t) = \frac{4}{3}\sin 2t - \frac{2}{3}\sin t$ (by a).

$\phi(0) = \frac{4}{3}\cdot 0 - \frac{2}{3}\cdot 0 = 0$.

$\phi'(t) = \frac{8}{3}\cos 2t - \frac{2}{3}\cos t$

$\phi'(0) = \frac{8}{3} - \frac{2}{3} = 2$. $\square$.

c. $\phi''(t) + \phi(t) = -4\sin 2t$. $\phi(0) = 0$, $\phi'(0) = 2$.

$\overset{\mathcal{L}}{\Rightarrow} (s^2+1)\Phi(s) = 2 - 4\cdot\frac{2}{s^2+4}$

$\Phi(s) = 2\cdot\frac{1}{s^2+1} + \frac{-8}{(s^2+1)(s^2+4)}$

$= 2\cdot\frac{1}{s^2+1} - \frac{8}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$

$= 2\cdot\frac{1}{s^2+1} - \frac{8}{3}\left(\frac{1}{s^2+1} - \frac{1}{2}\frac{2}{s^2+4}\right)$

$= -\frac{2}{3}\cdot\frac{1}{s^2+1} + \frac{4}{3}\frac{2}{s^2+4}$

$\Rightarrow \mathcal{L}^{-1}\{\Phi(s)\} = \phi(t) = -\frac{2}{3}\sin t + \frac{4}{3}\sin 2t$

This is same as the solution in a.

7.1-9.(b)

$$x_1' = -0.5x_1 + 2x_2 \qquad x_1(0) = -1, \quad x_2(0) = 2.$$
$$x_2' = -2x_1 - 0.5x_2$$

$$x_2 = \frac{1}{2}(x_1' + 0.5x_1).$$
$$x_2' = \frac{1}{2}(x_1'' + 0.5x_1') = -2x_1 - \frac{1}{4}(x_1' + \frac{1}{2}x_1).$$
$$\Rightarrow \quad 2x_1'' + x_1' = -8x_1 - x_1' - \frac{1}{2}x_1.$$
$$2x_1'' + 2x_1' + \frac{17}{2}x_1 = 0.$$
$$4x_1'' + 4x_1' + 17x_1 = 0.$$
$$4r^2 + 4r + 17 = 0. \quad \Rightarrow r = -\frac{1}{2} \pm 2i$$

$$x_1 = e^{-\frac{1}{2}t}(c_1\cos 2t + c_2\sin 2t).$$
$$x_1(0) = c_1 = -1.$$
$$x_1 = e^{-\frac{1}{2}t}(-\cos 2t + c_2\sin 2t).$$

$$x_1' = -\frac{1}{2}e^{-\frac{1}{2}t}(-\cos 2t + c_2\sin 2t) + e^{-\frac{1}{2}t}(2\sin 2t + 2c_2\cos 2t).$$
$$= e^{-\frac{1}{2}t}\left((2c_2 + \tfrac{1}{2})\cos 2t + (-\tfrac{c_2}{2} + 2)\sin 2t\right)$$

$$x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1$$
$$= e^{-\frac{1}{2}t}\left\{(c_2 + \tfrac{1}{4} - \tfrac{1}{4})\cos 2t + (-\tfrac{c_2}{4} + 1 + \tfrac{c_2}{4})\sin 2t\right\}$$
$$= e^{-\frac{1}{2}t}(c_2\cos 2t + \sin 2t).$$
$$x_2(0) = c_2 = 2.$$

$$\therefore \; x_1 = e^{-\frac{1}{2}t}(-\cos 2t + 2\sin 2t).$$
$$x_2 = e^{-\frac{1}{2}t}(2\cos 2t + \sin 2t). \quad \square.$$

7.2 - 11.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 7 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 3 & 7 & 6 & 0 & 0 & 1 \end{array}\right) \quad (-2R_1 + R_2)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -3 & 0 & 1 \end{array}\right) \quad (-3R_1 + R_3)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right) \quad (R_2 \leftrightarrow R_3).$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 7 & 0 & -2 \\ 0 & 1 & -3 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \quad (-2R_2 + R_1,\ -R_3).$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 9 & -2 \\ 0 & 1 & 0 & 3 & -3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right) \quad (-9R_3 + R_1,\ 3R_3 + R_2).$$

$\therefore$ A is nonsingular, $A^{-1} = \begin{pmatrix} -11 & 9 & -2 \\ 3 & -3 & 1 \\ 2 & -1 & 0 \end{pmatrix}$

7.3 - 19.

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)\{\lambda(\lambda-3) - 4\} - 2\{(6-2\lambda) - 8\} + 4\{4 + 4\lambda\}$$

$$= (3-\lambda)(\lambda^2 - 3\lambda - 4) + 4\lambda + 4 + 16\lambda + 16$$

$$= (3-\lambda)(\lambda-4)(\lambda+1) + 20(\lambda+1)$$

$$= (\lambda+1)\{(3-\lambda)(\lambda-4) + 20\}$$

$$= (\lambda+1)^2(8-\lambda) = 0. \quad \Rightarrow \lambda = -1 \text{ or } 8$$

$\lambda = -1.$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 8$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ eigenvalue : $-1,\ 8$

eigenvector for $\lambda = -1$: $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for $\lambda = 8$: $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$