

Financial Mathematics 1 – Exam 3

Junwoo Yang

June 18, 2020

- Let $T > 0$. Consider the market model with bank account $(G_t)_{0 \leq t \leq T} = (e^{rt})_{0 \leq t \leq T}$ and risky asset $(S_t)_{0 \leq t \leq T}$ given by SDE

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dB_t.$$

Assume that this SDE has a unique solution.

- Consider an European option with payoff $g(S_T)$ for a Borel function g having polynomial growth. Use the heuristic argument to derive the Black-Scholes PDE: Let $f(t, x)$ be the function such that the time- t price of option is $f(t, S_t)$. Then

$$f_t + rx f_x(t, x) + \frac{1}{2} \sigma^2(t, x) x^2 f_{xx}(t, x) - r f(t, x) = 0$$

with the terminal condition $f(T, x) = g(x)$.

- Apply the Feynman-Kac formula and express $f(t, x)$ as an expectation form.
- Consider the following market model with bank account $G = (G_t)_{t \geq 0}$ and risky asset $S = (S_t)_{t \geq 0}$ given by

$$dG_t = rtG_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

for positive constants r, μ, σ . Let $T > 0$.

- Find a risk-neutral measure.
 - Calculate the time-0 price of an option whose payoff is $\log S_T$ at maturity T .
- Fix $T > 0$. Consider the market $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ with bank account $G \equiv 1$ and two stocks $S^{(1)}, S^{(2)}$ given as

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = \mu^{(1)} dt + v^{(1)} dB_t^{(1)}$$

$$\frac{dS_t^{(2)}}{S_t^{(2)}} = \mu^{(2)} dt + \sigma^{(1)} dB_t^{(1)} + \sigma^{(2)} dB_t^{(2)}$$

for $\mu^{(1)}, \mu^{(2)}, \sigma^{(1)} \in \mathbb{R}$ and $v^{(1)}, \sigma^{(2)} \neq 0$.

- Find the time- t price of the option with payoff

$$S_T^{(1)} S_T^{(2)} \mathbb{I}_{S_T^{(1)} > K}$$

for $K > 0$ and maturity T . Use the cumulative distribution function $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz$.

- Find the hedging portfolio.
- Let X be the solution of the SDE

$$dX_t = -\theta X_t dt + \sigma dB_t$$

for $\theta \neq 0, \sigma > 0$ and $X_0 \in \mathbb{R}$. Calculate

$$\mathbb{R}(e^{X_T}(e^{X_T} - K)_+ | \mathcal{F}_t^X)$$

for $0 \leq t \leq T$.