금융수학 I - HW3 양준우 (2020-91659)

1.1 If not exercised before T

$(t=0)$ $S_0 - K \le C_A - P_A$

$\uparrow$ ($\because$ otherwise, $\exists$ arbitrage).

$(t=T)$ $S_T - Ke^{rT} \le (S_T-K)_+ - (K-S_T)_+ = S_T - K$

If exercised at $t<T$

$(t=0)$ $S_0 - K \le C_A - P_A$

$\uparrow$ ($\because$ otherwise, $\exists$ arbitrage).

$(t=t)$ $S_t - Ke^{rt} \le (S_t-K)_+ - (K-S_t)_+ = S_t - K$

$\therefore$ $S_0 - K \le C_A - C_P$ $\square$.

1.2. (i) We know that $C_A \ge C_E$ trivial.

WTS : $C_A = C_E$ when $r=0$. ($\because$ $C_A = C_E$ when $r>0$ proved in class).

Suppose $C_A > C_E$ i.e. $\alpha := C_A - C_E > 0$.

We can make a transaction as following :

sell American call, Buy European call, Bank α

If C_A is not exercised before T,

$$\begin{array}{lcccl} (t=0) & -C_A & C_E & \alpha & = 0 \\ & \downarrow & \downarrow & \downarrow & \\ (t=T) & -(S_T-K)_+ & (S_T-K)_+ & \alpha & = \alpha > 0 \end{array}$$ (abitrage).

If C_A is exercised at $t<T$

$(t=0)$ $-C_A$ C_E α $= 0$

$(\text{time } t)$ $-(S_t-K)_+$ $(S_t-K)_+$ α $= \alpha$ ($\because r=0$).

$(t=T)$ $= \alpha > 0$ (arbitrage).

This is a contradiction to no-arbitrage.

$\therefore$ $C_A = C_E$

1.2. (ii) In $r=0$ case,

At time $t<T$

$C_A(t) \geq C_E(t) \geq S_t - Ke^{-r(T-t)} = S_t - K$

$\Rightarrow C_A(t) \geq C_E(t) \geq S_t - K$

$\Rightarrow C_A(t) \geq S_t - K.$

It means that C_A could be exercised at $t<T$.

Thus C_A is not exactly same with C_E. $\square$

1.3. WTS : $f(\alpha K + (1-\alpha)K') \leq \alpha f(K) + (1-\alpha) f(K')$ $\alpha \in (0,1)$, $K \neq K'$ $(t=0)$

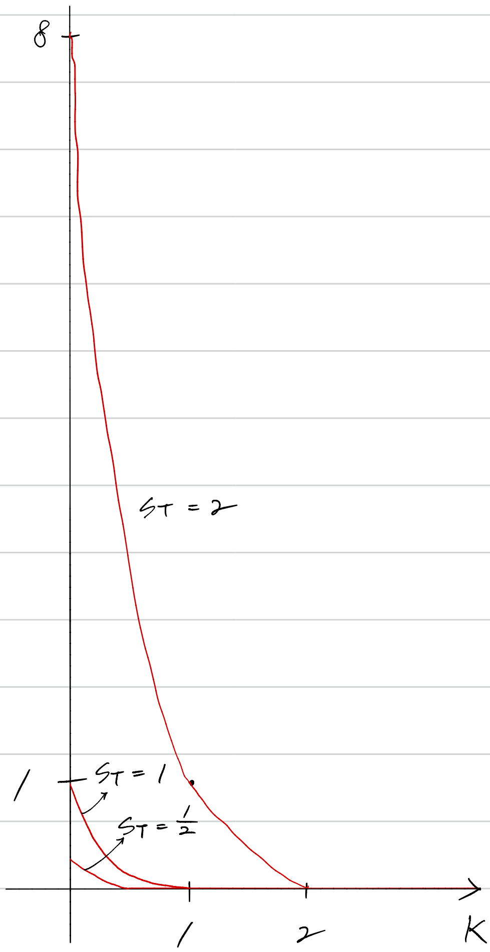
i.e., $(S_T - (\alpha K + (1-\alpha)K'))_+^3 \leq \alpha (S_T - K)_+^3 + (1-\alpha)(S_T - K')_+^3$ $(t=T)$

i.e., $g(K) = (S_T - K)_+^3$ is convex.

The payoff ftn $g(K) = (S_T - K)_+^3$ is convex. (graph).

$\therefore f(\alpha K + (1-\alpha)K') \leq \alpha f(K) + (1-\alpha) f(K')$

$\therefore K \mapsto f(K)$ is convex.



1.4 $(\Omega, 2^{\Omega}, \{\mathcal{F}_t\}_{t\in\mathbb{T}}, P)$: filtered prob. space.

$M_t = E(X|\mathcal{F}_t)$, $t = 0, 1, \cdots, T$.

We need show that ① M is adapted (i.e., M_t is $\mathcal{F}_t$-m'ble for $t\in\mathbb{T}$).

② $E^P(M_{t+1}|\mathcal{F}_t) = M_t$ for $t = 0, 1, \cdots, T-1$

① $E(M_t|\mathcal{F}_t) = E(E(X|\mathcal{F}_t)|\mathcal{F}_t) = E(X|\mathcal{F}_t) = M_t$

$\Rightarrow M_t$ is $\mathcal{F}_t$-measurable for $t\in\mathbb{T}$ $(\because \mathcal{F}_t \subseteq \mathcal{F}_t)$.

$\Rightarrow M_t$ is adapted.

② $E(M_{t+1}|\mathcal{F}_t) = E(E(X|\mathcal{F}_{t+1})|\mathcal{F}_t) = E(X|\mathcal{F}_t) = M_t$ for $t = 0, 1, \cdots, T-1$ □.

$(\because \mathcal{F}_t \subseteq \mathcal{F}_{t+1})$.

1.5. $\mathcal{B}$: Borel σ-alg. of $\mathbb{R}$.

$X : \Omega \to \mathbb{R}$.

$\sigma(X) = \{X^{-1}(A) : A \in \mathcal{B}\}$

We need to show that ① $\Omega \in \sigma(X)$.

② $A \in \sigma(X) \Rightarrow A^c \in \sigma(X)$.

③ $\bigcup_{\alpha=1}^{\infty} A_\alpha \in \sigma(X)$ for $A_\alpha \in \sigma(X)$.

① $\mathbb{R} \in \mathcal{B} \Rightarrow X^{-1}(\mathbb{R}) = \Omega \in \sigma(X)$.

② For $A \in \sigma(X)$, $\exists A' \in \mathcal{B}$ s.t $A = X^{-1}(A')$. $(A')^c \in \mathcal{B}$. $(\because \mathcal{B}$ is σ-alg.$)$.

$X^{-1}((A')^c) = X^{-1}(\mathbb{R}\setminus A') = X^{-1}(\mathbb{R})\setminus X^{-1}(A') = \Omega\setminus A = A^c$

$\therefore A \in \sigma(X) \Rightarrow A^c \in \sigma(X)$.

③ $A_1, A_2, \cdots \in \sigma(X) \Rightarrow \exists A_1', A_2', \cdots \in \mathcal{B}$ s.t $X^{-1}(A_1') = A_1$, $X^{-1}(A_2') = A_2$, $\cdots$

Since $\mathcal{B}$ is σ-alg., $\bigcup_{\alpha=1}^{\infty} A_\alpha' \in \mathcal{B}$. and $X^{-1}(\bigcup_{\alpha=1}^{\infty} A_\alpha') \in \sigma(X)$.

$X^{-1}(\bigcup_{\alpha=1}^{\infty} A_\alpha') = \bigcup_{\alpha=1}^{\infty} X^{-1}(A_\alpha') = \bigcup_{\alpha=1}^{\infty} A_\alpha$

$\therefore A_1, A_2, \cdots \in \sigma(X) \Rightarrow$ countable union $\bigcup_{\alpha=1}^{\infty} A_\alpha \in \sigma(X)$. □

1.6. $\mathbb{X}$ is jointly normal.

Let $E(\mathbb{X}) = \mu_0$ (mean vector) $Var(\mathbb{X}) = \Sigma_0$. (covariance matrix).

$\phi_{\mathbb{X}}(\mathbb{t}) = E(e^{i\mathbb{t}^T\mathbb{X}}) = e^{i\mu^T\mathbb{t} - \frac{1}{2}\mathbb{t}^T\Sigma\mathbb{t}}$

Since $Y := \mathbb{t}^T\mathbb{X}$ is normal,

$\phi_Y(t) = E(e^{itY}) = e^{iE(Y)t - \frac{1}{2}Var(Y)t^2}$

$\phi_Y(1) = E(e^{iY}) = e^{iE(Y) - \frac{1}{2}Var(Y)}$

$\therefore E(e^{iY}) = E(e^{i\mathbb{t}^T\mathbb{X}}) = e^{iE(\mathbb{t}^T\mathbb{X}) - \frac{1}{2}Var(\mathbb{t}^T\mathbb{X})}$

$E(\mathbb{t}^T X) = E(t_1X_1 + \cdots + t_dX_d) = \mu_0^T\mathbb{t}$

$Var(\mathbb{t}^T\mathbb{X}) = Var(t_1X_1 + \cdots + t_dX_d)$

$= t_1^2 Var(X_1) + \cdots + t_d^2 Var(X_d) + \sum_{1\le i<j\le d} 2t_it_j Cov(X_i, X_j)$

$= \sum_{i=1}^{d} t_i^2 Cov(X_i, X_i) + \sum_{1\le i<j\le d} 2t_it_j Cov(X_i, X_j)$

$= \mathbb{t}^T\Sigma_0\mathbb{t}$

$\therefore E(e^{i\mathbb{t}^T\mathbb{X}}) = e^{i\mu^T\mathbb{t} - \frac{1}{2}\mathbb{t}^T\Sigma\mathbb{t}} = e^{i\mu_0^T\mathbb{t} - \frac{1}{2}\mathbb{t}^T\Sigma_0\mathbb{t}}$ for $\forall\mathbb{t} \in \mathbb{R}^d$

$\therefore E(\mathbb{X}) = \mu$, $Var(\mathbb{X}) = \Sigma$ □.

1.7. $X \sim N(0,1) \Rightarrow \phi_X(t) = e^{-\frac{1}{2}t^2}$

$Y = \begin{cases} X & \text{if } |X| < a \\ -X & \text{if } |X| \ge a \end{cases}$

$\phi_Y(t) = E(e^{itY}) = \begin{cases} E(e^{itX}) & \text{if } |X| < a \Rightarrow \phi_Y(t) = e^{-\frac{1}{2}t^2} \\ E(e^{-itX}) & \text{if } |X| \ge a \Rightarrow \phi_Y(t) = e^{-\frac{1}{2}(-t)^2} = e^{-\frac{1}{2}t^2} \end{cases}$

Since characteristic ftn of Y is $\phi_Y(t) = e^{-\frac{1}{2}t^2}$,

$Y \sim N(0,1)$.

$Cov(X,Y) = E(XY) = \begin{cases} E(X^2) & \text{if } |X| < a \\ E(-X^2) & \text{if } |X| \ge a \end{cases}$

$\Rightarrow \int_{|x|<a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_{|x|\ge a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$\Rightarrow \int_{|x|<a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - (1 - \int_{|x|<a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$$\Rightarrow 2\int_{-a}^{a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - 1.$$

Assume (X, Y) is jointly noronal, then $X+Y$ is normal with density function $f_{X+Y}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Since $X+Y$ is continuous r.v., $P(X+Y=0) = 0$.

But $|X| \geq a \Rightarrow X+Y = X+(-X) = 0$.

$P(|X| \geq a) = \int_{|x| \geq a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx > 0$.

$\therefore$ $X+Y$ is not normal. $\divideontimes$ $\square$.

1.8. (i) Since $\{X_i\}$ mutually independent, $\{\sigma(X_i)\}$ mutually independent.

1.8.(ii).