Homework 2

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8.6.1 Lebesgue's Monotone Convergence Theorem *Let $g_n\colon [0,1] \to \mathbb{R}$ be a sequence of nonnegative functions such that each improper integral $\int_0^1 g_n(x)\,\mathrm{d}x$ exists and is finite. Suppose that $0 \le g_{n+1} \le g_n$ and that $g_n(x) \to 0$ for each $x \in [0,1]$. Then*

$$\lim_{n\to\infty} \int_0^1 g_n(x)\,\mathrm{d}x = 0.$$

Proof. We have $0 \le \int_0^1 g_{n+1} \le \int_0^1 g_n$; that is, the integrals form a bounded decreasing sequence, and so $\lambda = \lim_{n\to\infty} \int_0^1 g_n$ exists, and $\lambda \ge 0$. We wish to show that $\lambda = 0$, and so we assume that $\lambda > 0$. Note that $\int_0^1 g_n \ge \lambda$ for all n. Consider $E_n = \{x \in [0,1] \mid g_n(x) \ge 2\lambda/5\}$. Observe that $E_{n+1} \subset E_n$. We want to apply the lemma, but g_n might not be bounded. However, $g_n \le g_1$; since $\int_0^1 g_1$ exists as an improper integral, $\int_0^1 g_{1M} \to \int_0^1 g_1$ as $M \to \infty$. Here

$$g_{nM}(x) = \begin{cases} g_n(x) & \text{for } g_n(x) < M \\ M & \text{for } g_n(x) \ge M. \end{cases}$$

Choose $M > 2\lambda/5$ and such that $0 \le \int_0^1 (g_1 - g_{1M}) \le \lambda/5$, so that for all n,

$$0 \le \int_0^1 (g_n - g_{nM}) \le \int_0^1 (g_1 - g_{1M}) \le \lambda/5.$$

Therefore, $\int_0^1 g_{nM} \ge 4\lambda/5 = \alpha$. Note that since $M > 2\lambda/5$, $E_n = \{x \in [0,1] \mid g_{nM}(x) \ge 2\lambda/5\}$. Therefore, by the lemma, E_n contains a finite union of intervals of total length $\ge \lambda/5M$. Now define

$$D = \bigcup_{n=1}^{\infty} \{x \in [0,1] \mid g_n \text{ is not continuous at } x\};$$

then D has measure 0. Thus D is contained in the union U of a countable number of disjoint open intervals with total length $< \lambda/5M$. It may be readily seen that E_n is not a subset of U. Observe that if x_0 is an accumulation point of E_n but is not in E_n, then g_n must be discontinuous at x_0, so that $x_0 \in D \subset U$. That is, $\mathrm{cl}(E_n) \subset E_n \cup U$. Define $F_n = \mathrm{cl}(E_n) \setminus U$. By what we have just shown, $F_n \subset E_n$. But F_n is compact and $F_{n+1} \subset F_n$, and so by the nested set property, $\bigcap_{n=1}^{\infty} F_n \ne \emptyset$, and hence $\bigcap_{n=1}^{\infty} E_n \ne \emptyset$. But this means that for some $x \in [0,1]$, $g_n(x) \ge 2\lambda/5 > 0$ for all n, contradicting the hypothesis $g_n(x) \to 0$. $\square$