Stochastic calculus Instructor: Hyungbin Park

Lecture note 3: Stochastic calculus

1 Exercises

Problem 1.1. (10 points) Let S_0 be the stock price and let C_A and P_A be the prices of American call and put options, respectively, for the same maturity and strike K. Show that

$$S_0 - K \le C_A - P_A \,.$$

Problem 1.2. Let C_E and C_A be the prices of European call and American call options, respectively, for the same maturity and strike.

(i) (10 points) Show that $C_E = C_A$ when the short rate $r \ge 0$.

(ii) (10 points) The proof in class is not valid when $r = 0$. Explain why.

Problem 1.3. (15 points) Let $f(K)$ be the price of option with payoff

$$(S_T - K)_+^3$$

for $K > 0$. Show that the map $K \mapsto f(K)$ is convex.

Problem 1.4. (10 points) For a time-index set $\mathbb{T} = \{0, 1, \cdots, T\}$, let $(\Omega, 2^\Omega, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ be a filtered probability space, and $X : \Omega \to \mathbb{R}$ be a random variable. Show that a process M defined by

$$M_t = \mathbb{E}(X|\mathcal{F}_t)\,,\ t = 0, 1, \cdots, T$$

is a martingale.

Problem 1.5. (10 points) Let Ω be a set and $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. For a map $X : \Omega \to \mathbb{R}$, show that

$$\sigma(X) := \{X^{-1}(A) : A \in \mathcal{B}\}$$

is a σ-algebra of Ω.

Problem 1.6. (10 points) Let $X = (X_1, X_2, \cdots, X_d)^\top$ be jointly normal with the characteristic function

$$\phi_X(t) := \mathbb{E}(e^{it^\top X}) = e^{i\mu^\top t - \frac{1}{2}t^\top \Sigma t}\,,\ t \in \mathbb{R}^d$$

for a vector $\mu \in \mathbb{R}^d$ and a nonnegative symmetric matrix Σ. Show that the mean vector is μ and the covariance matrix is Σ.

Problem 1.7. (10 points) Let $X \sim \mathcal{N}(0, 1)$ and let $a > 0$. Define a random variable Y by

$$Y = \begin{cases} X & \text{if } |X| < a \\ -X & \text{if } |X| \ge a \end{cases}$$

Show that $Y \sim \mathcal{N}(0, 1)$. Find the covariance $\mathrm{cov}(X, Y)$ in terms of the density function of the standard normal distribution. Is the random vector (X, Y) a jointly normal?

Problem 1.8. Solve the followings.

(i) (5 points) Show that if $X_1 \cdots, X_n$ are mutually independent, so are $h_1(X_1) \cdots, h_n(X_n)$ for any (possibly complex-valued) Borel functions $h_1, \cdots, h_n$.

(ii) (10 points) Show that if $X_1 \cdots, X_n$ are mutually independent and each X_i is normal, then $(X_1 \cdots, X_n)$ is jointly normal. Hint: Use (i).

References

Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2004.