# ADVANCED CALCULUS 1
# ASSIGNMENT # 3 : 2019 SPRING

§4.1. # 1.
(a) Let $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$. Prove that $f$ is continuous.
(b) Let $f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x$. Prove that $f$ is continuous.

§4.5. # 3. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that $f$ has a fixed point.

§4.6. # 3. Must a bounded continuous function on $\mathbb{R}$ be uniformly continuous?

§4.6. # 6.
(a) Show that $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous iff there exist an $\varepsilon > 0$ and sequences $x_n$ and $y_n$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Generalize this statement to metric spaces.
(b) Use (a) on $\mathbb{R}$ to prove that $f(x) = x^2$ is not uniformly continuous.

§4.7. # 5. Let $f$ be continuous on $[3, 5]$ and differentiable on $(3, 5)$, and suppose that $f(3) = 6$ and $f(5) = 10$. Prove that, for some point $x_0$ in the open interval $(3, 5)$, the tangent line to the graph of $f$ at $x_0$ passes through the origin. Illustrate your result with a sketch.

§4.8. # 7. Let $f : [0, 1] \to \mathbb{R},\ f(x) = 1$ if $x = \frac{1}{n}$, $n$ an integer, and $f(x) = 0$ otherwise.
(a) Prove that $f$ is integrable.
(b) Show that $\int_0^1 f(x)\, dx = 0$.

(Exercises for Chapter 4)

# 12.
(a) A map $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz on $A$ if there is a constant $L \geq 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$, for all $x, y \in A$. Show that a Lipschitz map is uniformly continuous.
(b) Find a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous and hence is not Lipschitz.
(c) Is the sum (product) of two Lipschitz functions again a Lipschitz function?
(d) Is the sum (product) of two uniformly continuous functions again uniformly continuous?
(e) Let $f$ be defined and have a continuous derivative on $(a - \varepsilon, b + \varepsilon)$ for some $\varepsilon > 0$. Show that $f$ is a Lipschitz function $[a, b]$.

# Advanced Calculus 1 - HW3

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§4.1 - #1.

a. $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2$

we want to find $\delta$ s.t

$\forall \varepsilon > 0, \ |x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \varepsilon$

$$|x^2 - x_0^2| = |x + x_0||x - x_0| < |x + x_0|\delta = |x - x_0 + 2x_0|\delta$$
$$< (|x - x_0| + 2|x_0|)\delta < (\delta + 2|x_0|)\delta < \varepsilon$$

$$\delta^2 + 2|x_0|\delta + x_0^2 - x_0^2 < \varepsilon$$
$$(\delta + |x_0|)^2 < \varepsilon + x_0^2$$
$$\delta + |x_0| < (\varepsilon + x_0^2)^{\frac{1}{2}}$$
$$\delta < (\varepsilon + x_0^2)^{\frac{1}{2}} - |x_0|$$

$\therefore$ If we choose $\delta < (\varepsilon + x_0^2)^{\frac{1}{2}} - |x_0|$,

then $|x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \varepsilon$

$\therefore$ $f(x) = x^2$ is continuous in $\mathbb{R}$

b. $f : \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x$

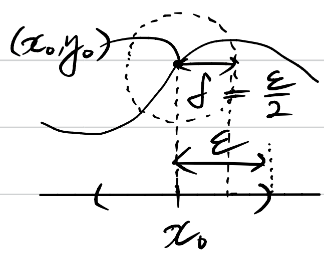
we want to find $\delta$ s.t

$\forall \varepsilon > 0, \ \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow \|x - x_0\| < \varepsilon$

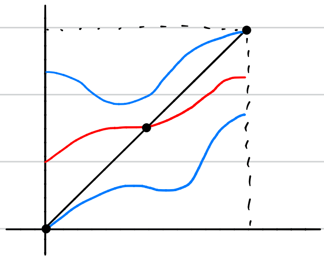
If we choose $\delta = \frac{\varepsilon}{2}$,

then $\forall \varepsilon > 0, \ \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow \|x - x_0\| < \varepsilon$

$\therefore$ $f$ is continuous.



§4.5 — #3.



$f : [0,1] \to [0,1]$ is continuous.

If $f(0) = 0$ or $f(1) = 1$, it is done.

Suppose $f(0) \neq 0$ and $f(1) \neq 1$,

that is, $f(0) \in (0,1]$, $f(1) \in [0,1)$

Let $g(x) = x - f(x)$.

then $g(0) = 0 - f(0) = -f(0) < 0$ $(\because 0 < f(0) \leq 1)$

$g(1) = 1 - f(1) > 0$ $(\because 0 \leq f(1) < 1)$

Since $g$ is continuous and $[0,1]$ is connected,

by Intermidiate value theorem,

$\exists x_0 \in [0,1]$ s.t $g(x_0) = 0$.

$\therefore f$ has fixed point $x_0$.

§4.6 — #3.

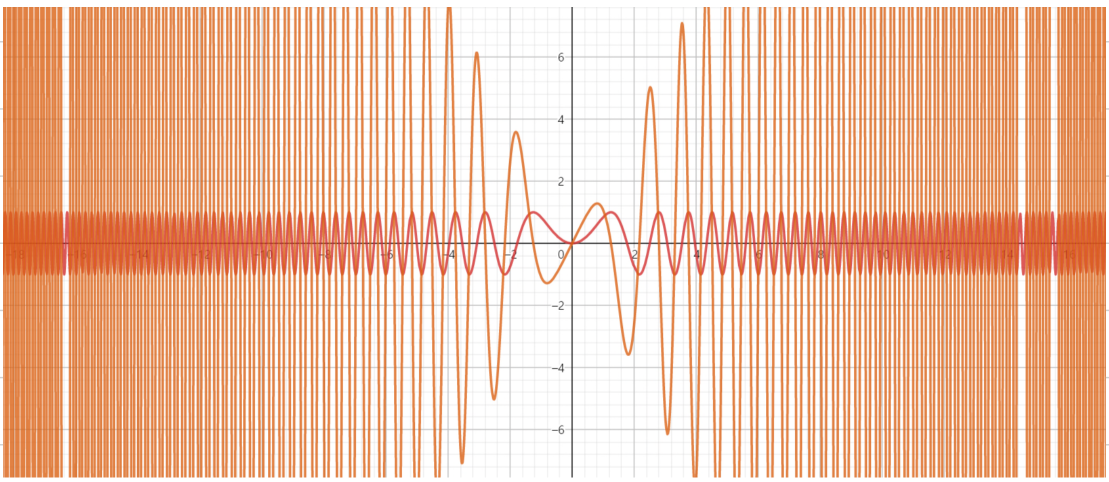
$f$ : bounded continuous on $\mathbb{R}$ $\overset{?}{\Rightarrow}$ uniformly continuous.

consider $f(x) = \sin x^2$

$\sin x^2$ is bounded i.e. $-1 \leq \sin x^2 \leq 1$

But it's periodicity is small as $|x| \to \infty$

while oscillation range is constant




Thus $f'(x) = 2x \cdot \cos x^2$ is oscillate largely as $|x| \to \infty$

Once we choose $\delta > 0$ as possible as small for

given $\varepsilon > 0$, there must exsist infinitely $x_0$ s.t
$\exists x \in B(x_0, \delta)$ s.t $|f(x) - f(x_0)| > \varepsilon$

For instance, WTS given $\varepsilon = \frac{1}{2}$, for any $\delta > 0$,
there are points $x$ and $y$ s.t
$|x-y| < \delta$ and $|f(x) - f(y)| > \frac{1}{2}$

Let $x = \sqrt{2\pi n}$, $y = \sqrt{2\pi n + \frac{1}{2}\pi}$

$$y - x = \sqrt{2\pi n + \frac{1}{2}\pi} - \sqrt{2\pi n} = \frac{\frac{\pi}{2}}{\sqrt{2\pi n + \frac{1}{2}\pi} + \sqrt{2\pi n}} < \frac{\frac{\pi}{2}}{2\sqrt{2n\pi}}$$

Since $y - x \to 0$ as $n \to \infty$
We can choose $n$ s.t $|y-x| < \frac{\pi/2}{2\sqrt{2\pi n}} < \delta$
But $|f(x) - f(y)| = |\sin(2\pi n) - \sin(2\pi n + \frac{1}{2}\pi)|$
$= |0 - 1| = 1 > \frac{1}{2}$

So we can pick $n$ for any $\delta > 0$.
Thus there is no $\delta$ s.t $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
$\therefore f$ is not uniformly continuous.

§4.6 – #6.

a. not uniforomly continuous
$\Leftrightarrow \exists x_n, y_n$ s.t $|x_n - y_n| < \frac{1}{n}$, $|f(x_n) - f(y_n)| \geq \varepsilon$

($\Rightarrow$) Since $f$ is not uniformly continuous,
there is no $\delta > 0$ s.t $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
In particular, $\delta = \frac{1}{n}$ will not satisfy
definition of uniform continuity.
Thus there must be $x_n, y_n$ s.t
$|x_n - y_n| < \frac{1}{n}$, $|f(x_n) - f(y_n)| > \varepsilon$

($\Leftarrow$) Since always there exist $x_n, y_n$ s.t
$|x_n - y_n| < \frac{1}{n} < \delta$, $|f(x_n) - f(y_n)| \geq \varepsilon$,
for given $\varepsilon > 0$, there no $\delta$ s.t
$|x_n - y_n| < \delta \Rightarrow |f(x_n) - f(y_n)| < \varepsilon$.

That is, for given $\varepsilon > 0$, no matter what $\delta$ is, we can pick $n$ s.t $\frac{1}{n} < \varepsilon$, and corresponding $x_n, y_n$.

$\therefore f$ is not uniformly continuous.

Let $(M,d)$, $(N,\rho)$ are metric spaces. $A \subset M$.

$f : A \subset M \to N$ is not uniformly continuous.

$\Leftrightarrow \exists x_n, y_n$ s.t $d(x_n, y_n) < \frac{1}{n}$, $\rho(f(x_n), f(y_n)) \geq \varepsilon$.

$(\Rightarrow)$ Since $f$ is not uniforomly continuous, there be no $\delta > 0$ s.t

$d(x_n, y_n) < \delta \Rightarrow \rho(f(x_n), f(y_n)) < \varepsilon$.

Let $\delta = \frac{1}{n}$. then there must be $x_n, y_n$ s.t $d(x_n, y_n) < \frac{1}{n}$ but $\rho(f(x_n), f(y_n)) \geq \varepsilon$.

$(\Leftarrow)$ Since always there exist $x_n, y_n$ s.t $d(x_n, y_n) < \frac{1}{n}$, $\rho(f(x_n), f(y_n)) \geq \varepsilon$.

For any $\delta > 0$, we can pick $n$ s.t $\frac{1}{n} < \varepsilon$.

then, by assumption, there must be $x_n, y_n$ s.t $d(x_n, y_n) < \frac{1}{n} < \delta$, $\rho(f(x_n), f(y_n)) \geq \varepsilon$

Thus $f$ is not uniformly continuous.

b. $f(x) = x^2$

By a, there exist $x_n, y_n$ s.t

$|x_n - y_n| < \frac{1}{n}$, $|x_n^2 - y_n^2| \geq \varepsilon$.

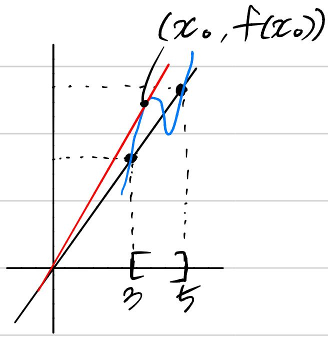
Let $\varepsilon = \frac{1}{2}$, $x_n = n + \frac{1}{2n}$, $y_n = n$

Then $|x_n - y_n| = |(n + \frac{1}{2n}) - n| = \frac{1}{2n} < \frac{1}{n}$

$|x_n^2 - y_n^2| = |(n + \frac{1}{2n})^2 - n^2| = 1 + \frac{1}{4n^2} > \frac{1}{2}$

Thus $f$ is not uniformly continuous.

## §4.7 — #5




$y = f'(x_0)(x - x_0) + f(x_0)$

$0 = -x_0 \cdot f'(x_0) + f(x_0)$

$x_0 f'(x_0) = f(x_0)$.

$g(x) = \frac{f(x)}{x}$ is continuous because $f$, $x$ is continuous.

$g$ is continuous and differentiable on $[3, 5]$

$g(3) = \frac{6}{3} = 2$, $g(5) = \frac{10}{5} = 2$

By Rolle's theorem, there is a point $g'(x_0) = 0$

$g'(x) = (x f'(x) - f(x)) / x^2$

$g'(x_0) = (x_0 f'(x_0) - f(x_0)) / x_0^2 = 0$. ■

## §4.8 — #7

(a) Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ s.t $\frac{1}{n} < \frac{\varepsilon}{2}$ for $n > N$

Let $P$ be a partition of $[0, 1]$ where,

$0 = x_0 < x_1 = \frac{1}{N+1} < x_2 < \cdots < x_K = 1$ and s.t

$x_i - x_{i-1} < \frac{\varepsilon}{4N}$ for $i = 2, 3, \cdots, K$

So, $U(P, f) = \sum_{i=1}^{K} M_i \Delta x_i \le \frac{1}{N+1} + N \times \frac{\varepsilon}{4N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Since at most $2N$ of the numbers $M_2, \cdots, M_K$ can be 1.

and the rest have to be 0.

$\because$ $(x_1, 1] = [\frac{1}{N+1}, 1]$ contains only $\frac{1}{N}, \frac{1}{N-1}, \cdots, \frac{1}{2}, 1$

where $f$ takes the value 1.

$L(p, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = 0$ since $m_\lambda = 0$, $\lambda = 1, \cdots, K$

$\therefore U(f) = L(f) = 0$ $\cdots$ ①

$\therefore$ $f$ is Riemann integrable.

(b) $\int_0^1 f(x)\,dx = 0$ is clear by ①.

Exercise for Ch4 − #12.

a. Show that Lipschitz map is uniformly continuous.

Since $f$ is Lipschitz continuous, $\exists L$ s.t

$\|f(x)-f(y)\| \leq L\|x-y\|$.

For given $\varepsilon>0$, Let $\delta = \frac{\varepsilon}{L}$ then,

$\|x-y\| < \delta \Rightarrow \|f(x)-f(y)\| \leq L\|x-y\| < L\cdot\delta = L\cdot\frac{\varepsilon}{L} = \varepsilon$.

$\therefore f$ is uniformly continuous.

b. Consider $f: \mathbb{R}\to\mathbb{R}$, $x \mapsto x^2$ continuous.

By §4.6 − #6.b. $f$ is not uniformly continuous.

Also, $f$ is not Lipschitz continuous.

c. suppose $(M,d)$, $(N,\rho)$ are metric spaces.

$f,g : A\subset M \to N$, $x,y\in A$

$\rho(f(x), f(y)) \leqslant L\cdot d(x,y)$,

$\rho(g(x), g(y)) \leqslant M\cdot d(x,y)$.

$h(x) = f(x)+g(x)$.

$\rho(h(x), h(y)) = \rho(f(x)+g(x), f(y)+g(y)) \leqslant \rho(f(x),f(y)) + \rho(g(x),g(y))$

$\leqslant L\cdot d(x,y) + M\cdot d(x,y) = (L+M)\cdot d(x,y)$

$\therefore f(x)+g(x)$ is Lipschitz continuous.

$f(x) = g(x) = x$, $x^2$ is product of two Lipschitz continuous function.

Let $x_n = n+\frac{1}{n}$, $y_n = n$, $\varepsilon = 1$. then

$d(x,y) = |n+\frac{1}{n}-n| = \frac{1}{n}$

$\rho(f\cdot g(x), f\cdot g(y)) = |(n+\frac{1}{n})^2 - n^2| = 2+\frac{1}{n^2} > 2 > 1 = \varepsilon$.

$\therefore f\cdot g$ is not uniformly continuous,

also not Lipschitz continuous.

d. $f$ and $g$ is uniformly continuous.

There is $\delta_1, \delta_2$ s.t

$\|x-y\| < \delta_1 \Rightarrow \|f(x)-f(y)\| < \frac{\varepsilon}{2}$,

$\|x-y\| < \delta_2 \Rightarrow \|g(x)-g(y)\| < \frac{\varepsilon}{2}$

Let $\delta := \min(\delta_1, \delta_2)$, $\|x-y\| < \delta$

$$\begin{aligned}\|(f+g)(x)-(f+g)(y)\| &= \|(f(x)-f(y))+(g(x)-g(y))\| \\ &\leq \|f(x)-f(y)\| + \|g(x)-g(y)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

$\therefore$ The sum of two uniformly continuous function is uniformly continuous.

If $f(x) = g(x) = x$ is uniformly continuous,

$x^2$ is product of two uniformly continuous.

But $x^2$ is not uniformly continuous.

e. $f'$ is continuous on compact set $[a,b]$

it is bounded on $[a,b]$.

There is a constant $L$ s.t $|f'(c)| < L$ for $c \in [a,b]$

For $a \leq x < y \leq b$, by mean value theorem,

there is $c$ s.t $f(y)-f(x) = c(y-x)$

so, $|f(y)-f(x)| \leq L|y-x|$

$\therefore$ $f$ is Lipschitz function on $[a,b]$