Mathematical Finance I — 7 May 2020

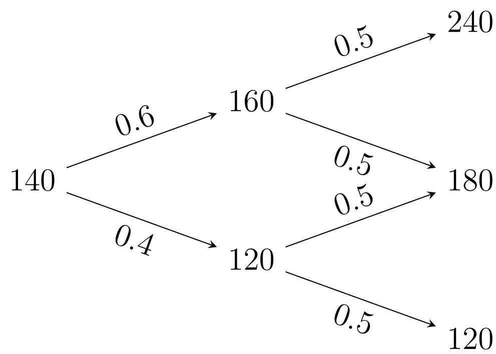
Exam 1 — Seoul National University, Seoul, Korea

Problems (500 points) Show your work for all problems.

1. (150 points) Consider a two-period binomial model with the time index $t = 0, 1, 2$. This model has two underlying assets; a bank account process $(G_t)_{t=0,1,2}$ given by

$$G_0 = G_1 = 1\,, \quad G_2 = 1.2 = \frac{6}{5}$$

and stock with price process given by the following tree.



(0.1)

   (i) (50 points) Construct a sample space Ω, a probability $\mathbb{P}$, a filtration $(\mathcal{F}_t)_{t=0,1,2}$, and stock price process $(S_t)_{t=0,1,2}$ representing this market.

   (ii) Consider an option with payoff

$$X_2 = \begin{cases} 30 & \text{if } S_2 = 240 \\ 15 & \text{if } S_2 = 180 \\ 0 & \text{if } S_2 = 120 \end{cases} \tag{0.2}$$

   and maturity $t = 2$.

      (i) (50 points) Find the price process $(X_t)_{t=0,1,2}$ and the hedging portfolio of this option.

      (ii) (50 points) Find the risk-neutral measure $\mathbb{Q}$.

2. (100 points) Let C_E and C_A be the prices of European call and American call options, respectively, for the same maturity and strike. Show that $C_E = C_A$ when the short rate $r \geq 0$.

3. (100 points) Let B be a Brownian motion. Calculate $\mathbb{E}(B_1 | B_3)$.

4. Solve the following problems.

   (i) (50 points) Let $(X_t)_{t\geq 0}$ be a RCLL Gaussian process. Show that

$$\left(\int_0^t X_s \, ds \right)_{t \geq 0}$$

   is a continuous Gaussian process. You may use, without proof, the fact that the limit (in the sense of convergence in distribution) of a sequence of normal random variables is normal.

   (ii) (100 points) Calculate $\mathbb{E}(e^{\int_0^T B_u \, du})$.