Probability Theory – Midterm Exam

Junwoo Yang

October 20, 2020

Problem 1. Let $(\Omega, \mathcal{F}, \mu)$ be measure space and $f : \Omega \to \mathbb{R}$ be measurable functions. Show that $\{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$ is σ-algebra on Y. Show also that $\nu(B) = \mu(f^{-1}(B))$ defines a measure on this σ-algebra.

Proof. To show that $\mathcal{G} = \{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$ is σ-algebra, we need to check that $\emptyset \in \mathcal{G}$, $\mathcal{G}$ is closed under complements and countable unions. Since $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$, $\mathcal{G}$ contains $\emptyset$. If $E \in \mathcal{G}$, $f^{-1}(E) \in \mathcal{F}$. Since $\mathcal{F}$ is σ-algebra, $(f^{-1}(E))^c = f^{-1}(E^c) \in \mathcal{F}$. Thus, $E^c \in \mathcal{G}$. If $E_i \in \mathcal{G}$ for $i = 1, 2, \cdots$, then $f^{-1}(E_i) \in \mathcal{F}$. Since $\mathcal{F}$ is closed under countable unions, $\bigcup_{i=1}^\infty f^{-1}(E_i) = f^{-1}(\bigcup_{i=1}^\infty E_i) \in \mathcal{F}$. Therefore, $\bigcup_{i=1}^\infty E_i \in \mathcal{G}$, $\mathcal{G}$ is σ-algebra. To show that ν is a measure, we need to show that $\nu(\emptyset) = 0$ and $\nu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \nu(E_i)$ for pairwise disjoint subsets $E_i \in \Omega$ $(i = 1, 2, \cdots)$. Since μ is a measure, we get that

$$\nu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$$
$$\nu(\bigcup_{i=1}^\infty E_i) = \mu(f^{-1}(\bigcup_{i=1}^\infty E_i)) = \mu(\bigcup_{i=1}^\infty f^{-1}(E_i)) = \sum_{i=1}^\infty \mu(f^{-1}(E_i)) = \sum_{i=1}^\infty \nu(E_i).$$

Hence, ν is a measure on $\mathcal{G}$. □

Problem 2. Let $f : \Omega \to \mathbb{R}$ be measurable and $g : \mathbb{R} \to \mathbb{R}$ be continuous function. Show that $g \circ f$ is measurable.

Proof. Enough to show that

$$(g \circ f)^{-1}((a, \infty)) = f^{-1}(g^{-1}((a, \infty))) = \{\omega \in \Omega : (g \circ f)(\omega) > a\}$$

is measurable for all $a \in \mathbb{R}$. Since g is continuous, inverse image of open set is open. Since any open subset of $\mathbb{R}$ is a countable union of disjoint open intervals, $g^{-1}((a, \infty))$ can be written as $\bigcup_{n=1}^\infty I_n$ where I_n are disjoint open intervals. Then we get

$$f^{-1}(g^{-1}((a, \infty))) = f^{-1}(\bigcup_{n=1}^\infty I_n) = \bigcup_{n=1}^\infty f^{-1}(I_n).$$

Since f is measurable, inverse image of interval is measurable, i.e. $f^{-1}(I_n)$ is measurable. Since countable union of measurable sets is also measurable, $\bigcup_{n=1}^\infty f^{-1}(I_n)$ is measurable. Hence, $g \circ f$ is measurable. □

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be continuous function. Show that if $f = 0$ almost everywhere, then $f = 0$ everywhere.

Proof. Suppose that there exists $c \in [a, b]$ such that $f(c) > 0$. Since f is continuous, for given $\varepsilon = \frac{f(c)}{2}$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{f(c)}{2}$. This implies that whenever $0 < |x - c| < \delta$, we have $0 < \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$. Then, $m(\{x \in [a, b] : f(x) \neq 0\}) > \delta > 0$. This is a contradiction. Similarly, for the case of $f(c) < 0$, proceed as before. Therefore, there is no point c such that $f(c) \neq 0$, so $f = 0$ everywhere. □

Problem 4. Let f be non-negative integrable function and α be positive real number. Show that

$$m(\{x \in E : f(x) > \alpha\}) < \frac{1}{\alpha} \int_E f \, \mathrm{d}m.$$

Proof. Let $A = \{x \in E : f(x) > \alpha\}$ and $\varphi = \alpha \mathbf{1}_A$ be simple function. Note that $f > \alpha$ on A. Then we get

$$\int_E \varphi \, \mathrm{d}m = \int_E \alpha \mathbf{1}_A \, \mathrm{d}m = \int_A \alpha \, \mathrm{d}m = \alpha m(A) < \int_A f \, \mathrm{d}m \leq \int_E f \, \mathrm{d}m.$$

$$\therefore m(A) = m(\{x \in E : f(x) > \alpha\}) < \frac{1}{\alpha} \int_E f \, \mathrm{d}m.$$

□

Problem 5. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove that if H_i are pairwise disjoint events such that $\bigcup_{i=1}^{\infty} H_i = \Omega$, $P(H_i) \neq 0$, then

$$P(A) = \sum_{i=1}^{\infty} P(A|H_i) P(H_i).$$

Proof. Since $A \subset \Omega$, $P(A) = P(A \cap \Omega) = P(A \cap (\bigcup_{i=1}^{\infty} H_i))$. Since H_i are pairwise disjoint, $A \cap H_i$ are also pairwise disjoint. By countable additivity of probability measure and definition of conditional probability,

$$P(A) = P(A \cap (\bigcup_{i=1}^{\infty} H_i)) = P(\bigcup_{i=1}^{\infty} (A \cap H_i)) = \sum_{i=1}^{\infty} P(A \cap H_i) = \sum_{i=1}^{\infty} P(A|H_i) P(H_i).$$

□

Problem 6. Let $X_1, \ldots, X_n$ be random variables and $a_i \in \mathbb{R}$. Show that

$$\mathrm{Var}(a_1 X_1 + \cdots + a_n X_n) = \sum_{j,k} a_j a_k \mathrm{Cov}(X_j, X_k).$$

Proof. Let $Z \coloneqq a_1 X_1 + \cdots + a_n X_n = \sum_{i=1}^{n} a_i X_i$. Then, we get that

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{i=1}^{n} a_i \mathbb{E}(X_i) \\
Z^2 &= \sum_{j,k} a_j a_k X_j X_k \\
\mathbb{E}(Z^2) &= \sum_{j,k} a_j a_k \mathbb{E}(X_j X_k) \\
\mathbb{E}(Z)^2 &= \sum_{j,k} a_j a_k \mathbb{E}(X_j) \mathbb{E}(X_k)
\end{aligned}$$

$$\begin{aligned}\operatorname{Var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 &= \sum_{j,k} a_j a_k (\mathbb{E}(X_j X_k) - \mathbb{E}(X_j)\mathbb{E}(X_k)) \\ &= \sum_{j,k} a_j a_k \operatorname{Cov}(X_j, X_k).\end{aligned}$$

□

Problem 7. Take $\Omega = [0,1]$ with Lebesgue measure and let $X(\omega) = \sin 2\pi\omega$, $Y(\omega) = \cos 2\pi\omega$. Show that X, Y are uncorrelated but not independent.

Proof. Let Lebesgue measure $P \coloneqq m|_{[0,1]}$. Then we get

$$\begin{aligned}\mathbb{E}(X) &= \int_\Omega X \,\mathrm{d}P = \int_\Omega \sin 2\pi\omega \,\mathrm{d}P = \int_0^1 \sin 2\pi\omega \,\mathrm{d}\omega = -\frac{1}{2\pi}\cos 2\pi\omega\bigg|_0^1 = 0 \\ \mathbb{E}(Y) &= \int_\Omega Y \,\mathrm{d}P = \int_\Omega \cos 2\pi\omega \,\mathrm{d}P = \int_0^1 \cos 2\pi\omega \,\mathrm{d}\omega = \frac{1}{2\pi}\sin 2\pi\omega\bigg|_0^1 = 0 \\ \mathbb{E}(XY) &= \int_\Omega XY \,\mathrm{d}P = \int_\Omega \sin 2\pi\omega \cos 2\pi\omega \,\mathrm{d}P \\ &= \int_0^1 \frac{1}{2}(\sin(2\pi\omega + 2\pi\omega) + \sin(2\pi\omega - 2\pi\omega)) \,\mathrm{d}\omega \\ &= \frac{1}{2}\int_0^1 \sin 4\pi\omega \,\mathrm{d}\omega = -\frac{1}{8\pi}\cos 4\pi\omega\bigg|_0^1 = 0 \\ \operatorname{Cov}(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0.\end{aligned}$$

Thus, $\rho_{X,Y} = 0$, i.e. X and Y are uncorrelated. Take $a > 0$ so small that the sets $A = \{\omega : \sin 2\pi\omega < a - 1\}$, $B = \{\omega : \cos 2\pi\omega < a - 1\}$ are disjoint. Then $P(A \cap B) = 0$ but $P(A)P(B) \neq 0$. Thus, X and Y are not independent. □