미분방정식 HW 2-1

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3.1 - 14.

$2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}$

Assume $y = e^{rt}$ is solution.

$(2r^2 - 3r + 1)e^{rt} = 0.$

$2r^2 - 3r + 1 = 0. \Rightarrow r = 1, \frac{1}{2} \Rightarrow y_1 = e^t \quad y_2 = e^{\frac{1}{2}t}$

$W[e^t, e^{\frac{1}{2}t}] = \begin{vmatrix} e^t & e^{\frac{1}{2}t} \\ e^t & \frac{1}{2}e^{\frac{1}{2}t} \end{vmatrix} = e^t(\frac{1}{2}e^{\frac{1}{2}t} - e^{\frac{1}{2}t}) = -\frac{1}{2}e^t e^{\frac{1}{2}t} = -\frac{1}{2}e^{\frac{3}{2}t} \neq 0$

$\therefore y = c_1 e^t + c_2 e^{\frac{1}{2}t}, \quad y' = c_1 e^t + \frac{1}{2}c_2 e^{\frac{1}{2}t}$

$y(0) = c_1 + c_2 = 2 \quad y'(0) = c_1 + \frac{1}{2}c_2 = \frac{1}{2}$

$c_1 + c_2 = 2$

$c_1 + \frac{1}{2}c_2 = \frac{1}{2} \quad \frac{1}{2}c_2 = \frac{3}{2} \quad \therefore c_2 = 3, \quad c_1 = -1.$

$\therefore y = -e^t + 3e^{\frac{1}{2}t}$

$y(t_0) = -e^{t_0} + 3e^{\frac{1}{2}t_0} = 0. \Rightarrow e^{t_0} = 3e^{\frac{1}{2}t_0} \quad t_0 = \frac{1}{2}t_0 + \ln 3$

$y'(t_0') = -e^{t_0'} + \frac{3}{2}e^{\frac{1}{2}t_0'} = 0. \quad t_0 = 2\ln 3 = \ln 9$

$= -e^{t_0'}(1 - \frac{3}{2}e^{-\frac{1}{2}t_0'}) = 0 \Rightarrow e^{-\frac{1}{2}t_0'} = \frac{2}{3} \Rightarrow t_0' = \ln\frac{9}{4}$

$\therefore (\ln 9, 0).$ maximum value. $\Rightarrow y(t_0') = \frac{9}{4}$

$(\ln\frac{9}{4}, \frac{9}{4})$ $(\because \lim_{t\to\infty} y = -\infty)$ $y''(t_0') = -\frac{9}{8} < 0$

3.3 - 13.

$y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0, \quad y'(\frac{\pi}{2}) = 4.$

$r^2 - 2r + 5 = 0. \Rightarrow r = 1 \pm 2i$

$\Rightarrow y = c_1 e^t \cos 2t + c_2 e^t \sin 2t.$

$y(\frac{\pi}{2}) = -c_1 e^{\frac{\pi}{2}} = 0 \Rightarrow c_1 = 0.$

$y' = \{(c_1 + 2c_2)\cos 2t + (-2c_1 + c_2)\sin 2t\}e^t$

$y'(\frac{\pi}{2}) = (-c_1 - 2c_2)e^{\frac{\pi}{2}} = -2c_2 e^{\frac{\pi}{2}} = 4. \Rightarrow c_2 = -2e^{-\frac{\pi}{2}}$

$\therefore y = -2e^{-\frac{\pi}{2}} \cdot e^t \sin 2t$

$= -2e^{t-\frac{\pi}{2}} \sin 2t$

3.4 - 14.

a. $9y'' + 12y' + 4y = 0$, $y(0) = a > 0$, $y'(0) = -1$

$9r^2 + 12r + 4 = 0 \Rightarrow r = -\frac{2}{3} \Rightarrow y_1 = e^{-\frac{2}{3}t}$

By using reduction of order,

$y_2 = V \cdot y_1 = V \cdot e^{-\frac{2}{3}t}$

$y_2' = V' e^{-\frac{2}{3}t} - \frac{2}{3} V e^{-\frac{2}{3}t}$

$y_2'' = V'' e^{-\frac{2}{3}t} - \frac{4}{3} V' e^{-\frac{2}{3}t} + \frac{4}{9} V e^{-\frac{2}{3}t}$

$$9y_2'' + 12y_2' + 4y_2 = \begin{pmatrix} 9V'' - \cancel{12V'} + \cancel{4V} \\ \cancel{12V'} - \cancel{8V} \\ \cancel{4V} \end{pmatrix} e^{-\frac{2}{3}t} = 9V'' e^{-\frac{2}{3}t} = 0.$$

$\Rightarrow V'' = 0.$

$V' = A_1$, $V = A_1 t + A_2$

$y_2 = V \cdot e^{-\frac{2}{3}t} = \underbrace{A_1 t e^{-\frac{2}{3}t}}_{\hookrightarrow 1} + \underbrace{A_2 e^{-\frac{2}{3}t}}_{\hookrightarrow 0.} \Rightarrow y_2 = t e^{-\frac{2}{3}t}$

$\therefore y = c_1 e^{-\frac{2}{3}t} + c_2 t e^{-\frac{2}{3}t}$ $\qquad y' = (-\frac{2}{3}c_1 + c_2 - \frac{2}{3}c_2 t) e^{-\frac{2}{3}t}$

$y(0) = c_1 = a > 0.$ $\qquad y'(0) = -\frac{2}{3}a + c_2 = -1$

$\Rightarrow c_2 = -1 + \frac{2}{3}a$

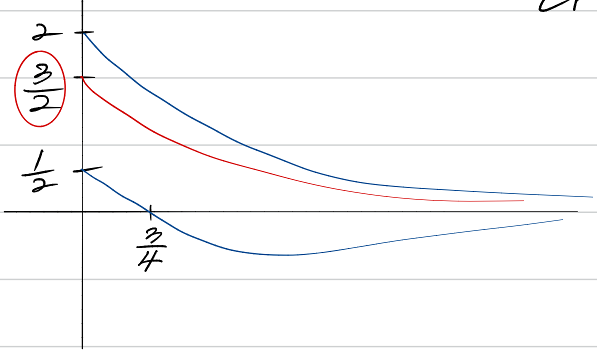
$\therefore y = a e^{-\frac{2}{3}t} + (\frac{2}{3}a - 1) t e^{-\frac{2}{3}t}$

b. If $a = \frac{3}{2}$. $\Rightarrow y = \frac{3}{2} e^{-\frac{2}{3}t}$ $\qquad \lim_{t\to\infty} y = 0$

$a = \frac{1}{2} \Rightarrow y = \frac{1}{2} e^{-\frac{2}{3}t} - \frac{2}{3} t e^{-\frac{2}{3}t}$ $\qquad \lim_{t\to\infty} y = 0$ $\qquad y(\frac{3}{4}) = 0.$

$a = 2 \Rightarrow y = 2e^{-\frac{2}{3}t} + \frac{1}{3} t e^{-\frac{2}{3}t}$ $\qquad \lim_{t\to\infty} y = 0$

$y'(0) = -1 \ \forall a.$



critical value of a : $\frac{3}{2}$

3.4 - 20

$t^2y'' - t(t+2)y' + (t+2)y = 0.\quad t>0,\quad y_1(t) = t.$

Check $y_1(t) = t$ is solution.

$t^2\cdot 0 - t(t+2)\cdot 1 + (t+2)t = 0.$

Let $y_2 = v\cdot t \quad y_2' = v't + v \quad y_2'' = v''t + 2v'$

$t^2y_2'' - t(t+2)y_2' + (t+2)y_2$

$= t^2(v''t + 2v') - t(t+2)(v't+v) + (t+2)vt$

$= t^3v'' + (2t^2 - t^3 - 2t^2)v' + (-t^2 - 2t + t^2 + 2t)v$

$= t^3v'' - t^3v' = 0$

$u := v' \Rightarrow t^3u' - t^3u = 0 \Rightarrow u' = u. \Rightarrow u = ce^t$

$v = ce^t \qquad \therefore y_2 = cte^t \xrightarrow{\text{set } c=1,} \therefore y_2 = te^t$

3.5 - 15.

Step 1). particular solution. y_p.

$y'' + 2y' + 5y = 4e^{-t}\cos 2t,\quad y(0) = 0,\quad y'(0) = 0$

$= Re[4e^{(-1+2i)t}]$

Guess $y_p = Ae^{(-1+2i)t}$

$y_p' = (-1+2i)Ae^{(-1+2i)t}$

$y_p'' = (-1+2i)^2Ae^{(-1+2i)t}$

$y_p'' + 2y_p' + 5y_p = (-3-4i-2+4i+5)Ae^{(-1+2i)t} = 0.$ (X)

Guess $y_p = Ae^{-t}\cos 2t + Be^{-t}\sin 2t.$

$y_p' = (-A+2B)e^{-t}\cos 2t + (-2A-B)e^{-t}\sin 2t$

$y_p'' = (A-2B-4A-2B)e^{t}\cos 2t + (2A-4B+2A+B)e^{t}\sin 2t$

$= (-3A-4B)e^{t}\cos 2t + (4A-3B)e^{t}\sin 2t.$

$y_p'' + 2y_p' + 5y_p = 0.$ (X).

Guess $y_p = Ate^{-t}\cos 2t + Bte^{-t}\sin 2t$

$$y_p' = (A - At + 2Bt)e^{-t}\cos 2t + (-2At + B - Bt)e^{-t}\sin 2t$$

$$\begin{aligned} y_p'' &= (-A + 2B - A + At - 2Bt - 4At + 2B - 2Bt)e^{-t}\cos 2t \\ &\quad + (-2A + 2At - 4Bt - 2A - B + 2At - B + Bt)e^{-t}\sin 2t \\ &= (-2A + 4B - 3At - 4Bt)e^{-t}\cos 2t \\ &\quad + (-4A - 2B + 4At - 3Bt)e^{-t}\sin 2t \end{aligned}$$

$$y_p'' + 2y_p' + 5y_p = 4Be^{-t}\cos 2t - 4Ae^{-t}\sin 2t = 4e^{-t}\cos 2t.$$

$$\Rightarrow A = 0, \quad B = 1.$$

$\therefore y_p = te^{-t}\sin 2t$

Step 2). Complementary solution. y_c

$y'' + 2y' + 5y = 0.$

$W[e^{-t}\cos 2t, e^{-t}\sin 2t] \neq 0.$

$r^2 + 2r + 5 = 0. \Rightarrow r = -1 \pm 2i \Rightarrow y_c = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t.$

Step 3). IVP

solution of $y'' + 2y' + 5y = 4e^{-t}\cos 2t$

$$: y = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + te^{-t}\sin 2t$$

$$y' = (-C_1 + 2C_2 + 2t)e^{-t}\cos 2t + (-2C_1 - 2C_2 + 1 - t)e^{-t}\sin 2t$$

$$y(0) = 0, \quad y'(0) = 0. \Rightarrow C_1 = 0. \quad C_2 = 0.$$

$\therefore y = te^{-t}\sin 2t.$

3.5 - 24.

$$y'' + 2y' + 5y = \begin{cases} 1 & 0 \le t \le \frac{\pi}{2} \\ 0 & t > \frac{\pi}{2} \end{cases} \qquad y(0) = 0,\ y'(0) = 0.$$

y, y' : conti. at $t = \frac{\pi}{2}$

Case 1) $0 \le t \le \frac{\pi}{2}$

i) particular solution y_p.

Guess $y_p = A$. $y_p'' = y_p' = 0$.

$y_p'' + 2y_p' + 5y_p = 1 \Rightarrow y_p = \frac{1}{5}$

ii) complementary solution y_c

$y'' + 2y' + 5y = 0$.

$r^2 + 2r + 5 = 0 \Rightarrow r = -1 \pm 2i \Rightarrow y_c = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

iii) IVP

$y = y_c + y_p = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + \frac{1}{5}$

$y' = (-c_1 + 2c_2)e^{-t}\cos 2t + (-2c_1 - c_2)e^{-t}\sin 2t$

$y(0) = y'(0) = 0 \Rightarrow c_1 = -\frac{1}{5} \quad c_2 = -\frac{1}{10}$

$\therefore y = -\frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t + \frac{1}{5} \quad (0 \le t \le \frac{\pi}{2})$.

$y(\frac{\pi}{2}) = \frac{1}{5}e^{-\frac{\pi}{2}} + \frac{1}{5}, \quad y'(\frac{\pi}{2})$

$y' = \frac{1}{2}e^{-t}\sin 2t \qquad y'(\frac{\pi}{2}) = 0$

Case 2). $t > \frac{\pi}{2}$

$y'' + 2y' + 5y = 0$: homogeneous.

by case 1. ii) $y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$.

Since $y = \phi(t)$, $y' = \phi'(t)$ are continuous on $t = \frac{\pi}{2}$,

$y(\frac{\pi}{2}) = \frac{1}{5}e^{-\frac{\pi}{2}} + \frac{1}{5} = e^{-\frac{\pi}{2}}(\frac{1}{5} + \frac{1}{5}e^{\frac{\pi}{2}}) = -c_1 e^{-\frac{\pi}{2}}$

$\Rightarrow c_1 = -\frac{1}{5} - \frac{1}{5}e^{\frac{\pi}{2}}$

$y' = (-c_1 + 2c_2)e^{-t}\cos 2t + (-2c_1 - c_2)e^{-t}\sin 2t$

$y'(\frac{\pi}{2}) = c_1 - 2c_2 = 0. \Rightarrow c_2 = \frac{1}{2}c_1 = -\frac{1}{10} - \frac{1}{10}e^{\frac{\pi}{2}}$

$$\therefore y = \begin{cases} -\frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t + \frac{1}{5} & (0 \le t \le \frac{\pi}{2}) \\ -(\frac{1}{5} + \frac{1}{5}e^{\frac{\pi}{2}})e^{-t}(\cos 2t + \frac{1}{2}\sin 2t) & (t > \frac{\pi}{2}) \end{cases}$$