Probability Theory – Exercise 1

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Problem 1

Let $\{\mathcal{M}_\alpha\}$ be an arbitrary collection of σ-fields of E. Show that $\bigcap_\alpha \mathcal{M}_\alpha$ is a σ-field.

Proof. Let $\mathcal{M} = \bigcap_\alpha \mathcal{M}_\alpha$. To show $\mathcal{M}$ is σ-field, we need to check that $\mathcal{M}$ contains $\emptyset$ and closed under countable unions and complements. Since $\mathcal{M}_\alpha$ is σ-field, $\emptyset \in \mathcal{M}_\alpha$ for all α. So, $\emptyset \in \mathcal{M} = \bigcap_\alpha \mathcal{M}_\alpha$. Now let $A_i \in \mathcal{M}$ for $i = 1, 2, 3, \cdots$. If we show that $\bigcup_i A_i \in \mathcal{M}$ and $A_i^c \in \mathcal{M}$ for any i, we are done. Since $A_i \in \mathcal{M}$, $A_i \in \mathcal{M}_\alpha$ for all α. Then $\bigcup_i A_i$ and A_i^c are contained in $\mathcal{M}_\alpha$ for all α, i. Hence $\{\bigcup_i A_i, A_i^c\} \subset \mathcal{M}$ for any i. $\square$

Problem 2

Show that if E_1 and E_2 are measurable sets in $\mathbb{R}$, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

Proof. If either E_1 or E_2 have infinite measure, then given equality holds since $m(E_1 \cup E_2) + m(E_1 \cap E_2) = \infty = m(E_1) + m(E_2)$. Without loss of generality, assume E_1 and E_2 have finite measure. Note that $\mathcal{M}$ is closed under countable unions, countable intersections, complements and we can write $E_1 \cup E_2$ as a union of pairwise disjoint sets $E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_1)$. Thus by countable additivity,

$$\begin{aligned} m(E_1) + m(E_2) &= m((E_1 \setminus E_2) \cup (E_1 \cap E_2)) + m((E_2 \setminus E_1) \cup (E_1 \cap E_2)) \\ &= m(E_1 \setminus E_2) + m(E_1 \cap E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \\ &= m((E_1 \setminus E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1 \cup E_2) + m(E_1 \cap E_2) \end{aligned}$$

$\square$

Problem 3

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove that if H_i are pairwise disjoint events such that $\bigcup_{i=1}^{\infty} H_i = \Omega$, $P(H_i) \neq 0$, then

$$P(A) = \sum_{i=1}^{\infty} P(A|H_i)P(H_i).$$

Proof. Since $A \subset \Omega$, $P(A) = P(A \cap \Omega) = P(A \cap (\bigcup_{i=1}^{\infty} H_i))$. Since H_i are pairwise disjoint, $A \cap H_i$ are also pairwise disjoint. By countable additivity of probability measure and definition of conditional probability,

$$P(A) = P(A \cap (\bigcup_{i=1}^{\infty} H_i)) = P(\bigcup_{i=1}^{\infty}(A \cap H_i)) = \sum_{i=1}^{\infty} P(A \cap H_i) = \sum_{i=1}^{\infty} P(A|H_i)P(H_i). \qquad \square$$

Problem 4

Let $\{f_n\}$ be a sequence of measurable functions defined on an interval [a,b]. Suppose that there exists an itegrable function g on $[a, b]$ such that $f_n \le g$ for all n. Show that

$$\int_a^b \limsup_{n\to\infty} f_n \, \mathrm{d}m \ge \limsup_{n\to\infty} \int_a^b f_n \, \mathrm{d}m.$$

Proof. Let E be an interval $[a, b]$. Since $g - f_n$ is non-negative measurable functions, by Fatou's lemma, we get

$$\liminf_{n\to\infty} \int_E (g - f_n) \, \mathrm{d}m \ge \int_E \Big(\liminf_{n\to\infty}(g - f_n)\Big) \, \mathrm{d}m.$$

By the fact that $\liminf(-f_n) = -\limsup f_n$, we can write LHS as

$$\liminf_{n\to\infty} \int_E (g - f_n) \, \mathrm{d}m = \liminf_{n\to\infty} \Big(\int_E g \, \mathrm{d}m - \int_E f_n \, \mathrm{d}m\Big) = \int_E g \, \mathrm{d}m - \limsup_{n\to\infty} \int_E f_n \, \mathrm{d}m$$

and RHS as

$$\int_E \Big(\liminf_{n\to\infty}(g - f_n)\Big) \, \mathrm{d}m = \int_E \Big(g - \limsup_{n\to\infty} f_n\Big) \, \mathrm{d}m = \int_E g \, \mathrm{d}m - \int_E \limsup_{n\to\infty} f_n \, \mathrm{d}m.$$

Hence, we get

$$\int_E \limsup_{n\to\infty} f_n \, \mathrm{d}m \ge \limsup_{n\to\infty} \int_E f_n \, \mathrm{d}m. \qquad \square$$

Problem 5

Let $\{f_n\}$ be a sequence of integrable functions on a set E such that $f_n \to f$ a.e. with f integrable. Show that

$$\int_E |f_n - f| \, \mathrm{d}m \to 0 \text{ if and only if } \int_E |f_n| \, \mathrm{d}m \to \int_E |f| \, \mathrm{d}m.$$

Proof. By triangle inequality, $|f_n| = |f + f_n - f| \le |f| + |f_n - f|$. Let $g_n = |f_n| - |f_n - f| - |f| \le 0$. Note that g_n converges to 0 a.e. since $f_n \to f$ a.e. Then $|g_n| = |f| + |f_n - f| - |f_n| \le |f| + |f_n| + |f| - |f_n| = 2|f|$. Since f is integrable, $|g_n|$ is also integrable. Then, by the LDCT,

$$\lim_{n\to\infty} \int_E |g_n| \, \mathrm{d}m = \int_E \lim_{n\to\infty} |g_n| \, \mathrm{d}m = \int_E 0 \, \mathrm{d}m = 0.$$

Thus we get,

$$\begin{aligned}
\lim_{n\to\infty}\int_E |g_n|\,\mathrm{d}m &= \lim_{n\to\infty}\int_E (|f|+|f_n-f|-|f_n|)\,\mathrm{d}m \\
&= \lim_{n\to\infty}\Big(\int_E |f|\,\mathrm{d}m+\int_E |f_n-f|\,\mathrm{d}m-\int_E |f_n|\,\mathrm{d}m\Big) \\
&= \int_E |f|\,\mathrm{d}m+\lim_{n\to\infty}\int_E |f_n-f|\,\mathrm{d}m-\lim_{n\to\infty}\int_E |f_n|\,\mathrm{d}m = 0.
\end{aligned}$$

This exactly proves our claim. □

Problem 6

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable. Define a distribution function of X by

$$F_X(y) = P(\{\omega : X(\omega) \le y\}) = P_X((-\infty, y]).$$

Prove that F_X is continuous if and only if $P_X(\{y\}) = 0$ for all $y \in \mathbb{R}$.

Proof. Since F_X is right continuous, to show F_X is continuous, ETS F_X is left continuous. Let $\{a_n\}$ be an arbitrary sequences such that $a_n \ge 0$ and $\lim_{n\to\infty} a_n = 0$. If F_X is continuous, F_X is left continuous. Then $F_X(y) - \lim_{n\to\infty} F_X(y - a_n) = 0$, which is $P_X(\{y\})$. Therefore, $P_X(\{y\}) = 0$ for all $y \in \mathbb{R}$.
Conversely, if $P_X(\{y\}) = 0$, $P_X(\{y\}) = \lim_{n\to\infty} P_X((y - a_n, y]) = F_X(y) - \lim_{n\to\infty} F_X(y - a_n) = 0$. Thus, F_X is left continuous. □

Problem 7

Find the distribution function F_X and the expectation for a random variable X on a probability space $([0,1], \mathcal{M}_{[0,1]}, m_{[0,1]})$ where $X(\omega) = \min\{\omega, 1-\omega\}$.

Proof. Let $P = m_{[0,1]}$ and $P_X(B) = P(X^{-1}(B))$ for Borel set B.

$$\begin{aligned}
F_X(y) &= P(\{\omega \in [0,1] : \min\{\omega, 1-\omega\} \le y\}) = P_X([-\infty, y]) = 2\int_{-\infty}^{y} \mathbf{1}_{[0,\frac{1}{2}]}(\omega)\,\mathrm{d}\omega \\
&= \begin{cases} 0 & \text{if } y < 0 \\ 2y & \text{if } 0 \le y \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < y \end{cases} \\
f_X(y) &= \frac{\mathrm{d}F_X(y)}{\mathrm{d}y} = 2\mathbf{1}_{[0,\frac{1}{2}]}(y) \\
\mathbb{E}(X) &= \int_0^1 X\,\mathrm{d}P = \int_{\mathbb{R}} x\,\mathrm{d}P_X(x) = \int x f_X(x)\,\mathrm{d}x = \int x 2\mathbf{1}_{[0,\frac{1}{2}]}(x)\,\mathrm{d}x = \int_0^{\frac{1}{2}} 2x\,\mathrm{d}x = x^2\Big|_0^{\frac{1}{2}} = \frac{1}{4}
\end{aligned}$$

Since the density $f_X(x) = 2\mathbf{1}_{[0,\frac{1}{2}]}(x)$, X follows uniform distribution within an interval $[0, \frac{1}{2}]$. □