Advanced Calculus 2 - HW3

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§ 8.2 #5

Must the boundary of a set have measure zero?
No. Counter example : Set of irrationals in $[0,1]$
This is boundary set but do not have measure zero.

§ 8.3 #4

Since f is continuous, A is open and $f(x_0)>0$,
$\exists$ a neighborhood $D(x_0,\varepsilon)$ of x_0
First of all $\int_A f \geq 0$ is trivial
If $\int_A f = 0$, then $\{x\in A \mid f(x)\neq 0\}$ has measure zero.
Since $D(x_0,\varepsilon) \subset \{x\in A \mid f(x)\neq 0\}$, then $D(x_0,\varepsilon)$ has measure zero.
However, $D(x_0,\varepsilon)$ has measure. Since A does,
so $\{x\in A \mid f(x)\neq 0\}$ has measure.
$\therefore$ $\int_A f \neq 0$, and $\int_A f > 0$.

§ 8.5 #3.

If $-1<p$, then $e^{-x}x^p \leq x^p$
And since $\int_0^1 x^p\,dx$ converges to $\frac{1}{p+1}$,
$\int_0^1 e^{-x}x^p\,dx$ converges also by comparison.

Ex. #7.

For $[a,b]$, $\exists$ partition P s.t $x_i = (a + (b-a)\frac{i}{a})$

for $i = 0, 1, 2, \cdots$

Then $U(f, P) = (f(x_1) + f(x_2) + \cdots + f(x_n))(b-a)\frac{1}{n}$

and $L(f, P) = (f(x_0) + f(x_1) + \cdots + f(x_{n-1}))(b-a)\frac{1}{n}$

Then $U - L = (f(x_n) - f(x_0))(b-a)\frac{1}{n} = \frac{1}{n}(f(b) - f(a))(b-a)$

$\therefore$ $U - L \to 0$ as $n \to \infty$

$\therefore$ By Riemann condition, f is integrable. ■

Ex #20.

Since $f(x) \geq 0$, continuous and monotonically increases.

integrable on $(a, b]$

then $\int_a^x f(x)\,dx$ is convergent where $\forall x \in (a, b]$.

$*$ $\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^x f(x)\,dx + \int_x^b f(x)\,dx$

and $\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x)\,dx$ is convergent.

implies $\lim_{\varepsilon \to 0} \int_{a+\varepsilon}^x f(x)\,dx$ is convergent.

Now for $\forall y \in (a, x]$, $f(y) \geq f(x)$.

Since f is monotonically increases as $x \to 0$.

Hence $\int_a^{a+x} f(t)\,dt \geq \int_a^{a+x} f(x)\,dt = f(x)\,x$

Then for all $\varepsilon > 0$, $\exists \delta$ s.t $0 < x < \delta$.

implies $0 < f(x) \cdot x < \int_x^{a+x} f(t)\,dt$

$\Rightarrow$ $\int_a^{a+x} f(t)\,dt = \int_a^b f(t)\,dt - \int_x^b f(t)\,dt < \varepsilon$

$\therefore$ $x \to 0$. $\Rightarrow$ $x f(x) \to 0$ $\qquad \varepsilon f(\varepsilon) \to 0$ as $\varepsilon \to 0$

Ex #24

Example : $A = [0,1]$ $f = \begin{cases} 0 & : x \in [0,1] \setminus \mathbb{Q} \\ 1 & : x \in [0,1] \cap \mathbb{Q} \end{cases}$

Then $\forall \varepsilon > 0$, $\exists \delta$ s.t $|P| < \delta$ (P: partition).

$\exists x_i \in S_i \cap \mathbb{Q}$ and $\left| \sum_{i=1}^{n} f(x_i) v(S_i) - 1 \right| < \varepsilon$

but f is not integrable.

(since $A \setminus \mathbb{Q}$ boundary does not have measure zero).

Ex # 29.

Let C_k be a left length of Cantor set

then $C_k = (\frac{2}{3})^k$ and $\exists \varepsilon$ s.t $(\frac{2}{3})^k < \varepsilon$.

Moreover, the union of the $2k$ intervals of C_k

cover the Cantor Set. by partition S_i

then $\sum_{i=1}^{N} v(S_i) \leq \sum_{i=1}^{\infty} (\frac{2}{3})^i < \varepsilon$.