Topology I – Homework 5

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Problem 5.1 Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a compact metric space X. Show that there exists $\varepsilon > 0$ such that for each x, $B(x, \varepsilon)$ is contained in one of U_α's.

Proof. It's fine that $\{U_\alpha\}$ is finite because X is compact. For $x \in X$, there exists ε_x such that $B(x, \varepsilon_x) \subset U_\alpha$ for some α. $\{B(x, \frac{1}{2}\varepsilon_x) | x \in X\}$ is open cover of X. Since X is compact, there is $Y = \{x_1, x_2, \cdots, x_n\} \subset X$ such that $X \subset B(x_1, \frac{1}{2}\varepsilon_{x_1}) \cup \cdots \cup B(x_n, \frac{1}{2}\varepsilon_{x_n})$. Let $\varepsilon \coloneqq \frac{1}{2}\min\{\varepsilon_{x_1}, \cdots, \varepsilon_{x_n}\}$. We want to show that for all $x \in X$, $B(x, \varepsilon) \subset U_\alpha$ for some α. Consider for all $t \in B(x, \varepsilon)$, there exists $B(x_k, \frac{1}{2}\varepsilon_{x_k})$ containg x for some $x_k \in Y$. Then, by triangle inequality,

$$d(t, x_k) \leq d(t, x) + d(x, x_k) < \varepsilon + \frac{\varepsilon_{x_k}}{2} \leq \frac{\varepsilon_{x_k}}{2} + \frac{\varepsilon_{x_k}}{2} = \varepsilon_{x_k}.$$

Thus t belongs to $B(x_k, \varepsilon_{x_k})$. i.e., $B(x, \varepsilon) \subset B(x_k, \varepsilon_{x_k})$. For $x_k \in Y$, $B(x_k, \varepsilon_{x_k}) \subset U_\alpha$ for some α. Hence, $B(x, \varepsilon)$ is containted in some U_α. This is Lebesgue number Lemma. □

Problem 5.2 Show that $X_1 \times X_2$ is compact, if X_1 and X_2 are compact.

Proof. □