Topology 2

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Based on lectures by Youngsik Huh in fall 2021

Contents

Chapter 0

Introduction

0.1 Topological spaces

Definition 1. A ***topological space*** is $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ a family of subsets of X, called open sets, such that

- $\emptyset, X \in \mathcal{T}$
- $\bigcup_{i \in I} U_i \in \mathcal{T}$ whenever $U_i \in \mathcal{T}$ for all i
- $\bigcap_{i<n} U_i \in \mathcal{T}$ whenever $U_i \in \mathcal{T}$ for all i.

Let (X, σ) be a topological space.

Definition 2. An open subset that contains $p \in X$ is called a ***(open) neighborhood*** of p.

Definition 3. If $Y \subset X$ then $(Y, \mathcal{T}_Y)$ is a topological space, where

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}.$$

We call $\mathcal{T}_Y$ the ***subspace topology***.

Example. Endowing $\mathbb{R}^2$ with the Euclidean topology, the subspace topology on $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is also Euclidean topology.

Definition 4. An ***equivalence relation*** is a relation $x \sim y$ so that $x \sim x$; if $x \sim y$ then $y \sim x$; and if $x \sim y$ and $y \sim z$, then $x \sim z$. Given an equivalence relation defined on X, $X/_\sim$ is the set of ***equivalence classes***.

Definition 5. Let $\sim$ be an equivalence relation on X. Consider a surjective map $\pi \colon X \to X/_\sim$ given by $x \mapsto [x]$. Then $X/_\sim$ equipped with ***quotient topology*** is a topological space, where the open sets are the subsets $U \subset X/_\sim$ such that $\pi^{-1}(U)$ is open in X.

Example. Let X be the closed unit ball, $\{(x, y) : x^2 + y^2 \leq 1\}$, in $\mathbb{R}^2$ and X^* be the partition of X consisting of all the one-point sets $\{(x, y)\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Then X^* is homeomorphic to $S^2(r)$.

Definition 6. A function $f \colon X_1 \to X_2$ is ***continuous*** if $f^{-1}(U)$ is open in X_1 for every open set $U \subset X_2$.

Definition 7. A topological space X is ***Hausdorff*** if $\forall x, y \in X$, there exists neighborhoods U of x, V of y such that $U \cap V = \emptyset$.

Definition 8. Let $(X, \mathcal{T})$ be a topological space. A ***basis*** for $\mathcal{T}$ is a subset $\mathcal{B} \subset \mathcal{T}$ such that every open set of X is a union of elements of $\mathcal{B}$.

Definition 9. A topological space $(X, \mathcal{T})$ is ***second countable*** if there exists a countable basis.

Example. $\mathbb{R}^n$ is second countable. Indeed $\{B_{\frac{1}{m}}(x) \mid x \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a countable basis for the topology. Here $B_r(x)$ is the open ball with radius r around x.

Definition 10. A ***topological manifold*** M of dimension of m is a second countable, Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^m$.

Remark. 'Locally homeomorphic to $\mathbb{R}^m$' means that $\forall p \in M$, there exists a neighborhood U of p and a homeomorphism $\phi \colon U \to V \subset \mathbb{R}^m$. Recall that homeomorphism means: bijective map that is continuous in both directions.

0.2 What is algebraic topology?

Functor from category of topological spaces to the category of groups.

- Category: set of spaces and morphisms.
- Functor: $X \rightsquigarrow G_X$ and $f \colon X \to Y \rightsquigarrow f_* \colon G_X \to G_Y$ such that
  - $(f \circ g)_* = f_* \circ g_*$
  - $(1_X)_* = 1_{G_X}$

Two systems we'll discuss:

- fundamental groups
- homology groups

Example. Suppose we have a functor. If $G_X \ncong G_Y$, then X and Y are not homeomorphic. If 'shadows' are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f \colon X \to Y$ and $g \colon Y \to X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* \colon G_X \to G_Y$ and $g_* \colon G_Y \to G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* \colon G_X \to G_Y$ is an isomorphism. ◇

0.3 Fundamental group

Pick a base point x_0 and consider it fixed. (The fundamental group will not depend on it. We assume all spaces are path connected) $X \rightsquigarrow \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f \colon I = [0,1] \to X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

Example. Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the 'constant' loop.

Example. Let $X = S^1$ and pick x_0 on the circle. Two options:

- The loop is trivial equivalent to the constant loop
- The loop goes around the circle.
- The loop goes around the circle, twice.
- The loop goes around the circle, clockwise, once
- ...

$\pi(S^1) \cong \mathbb{Z}$ (proof will follow)

The composition of loops is simply pasting them. In the case of the circle, the loop $-1\circ$ the loop 2 is the loop 1.

Suppose $\alpha \colon I \to X$ and $f \colon X \to Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

Theorem 1 (Fixed point theorem of Brouwer)**.** Any continuous map from a rectangle to itself has at least one fixed point.

Proof. Suppose there is no fixed point, so $f(x) \neq x$ for all $x \in B^2$. Then we can construct map $r\colon B^2 \to S_1$ as follows: take the intersection of the boundary and half ray between $f(x)$ and x. If x lies on the boundary, we have the identity map. This map is continuous. Then we have $S^1 \to B^2 \to S^1$, via the inclusion and r. Looking at the fundamental groups:

$$\pi(S^1) = \mathbb{Z} \to \pi(B^2) = 1 \to \pi(S^1) = \mathbb{Z}.$$

The map from $\pi(S^1) \to \pi(S^1)$ is the identity map, but the first map maps everything on 1. $\square$

Chapter 9

The fundamental group

9.51 Homotopy of paths

Definition 11. Let $f, g\colon X \to Y$ be continuous maps. Then a ***homotopy*** between f and g is a continuous map $H\colon X \times I \to Y$ such that

- $H(x, 0) = f(x)$, $H(x, 1) = g(x)$
- For all $t \in I$, define $f_t\colon X \to Y$ given by $x \mapsto H(x, t)$

We say that f is ***homotopic*** to g and write $f \simeq g$. If g is a constant map, we say that f is ***null homotopic***.

Definition 12. Let $f, g\colon I \to X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H\colon I \times I \to X$ is a ***path homotopy*** between f and g if

- $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ (homotopy between maps)
- $H(0, t) = x_0$ and $H(1, t) = x_1$ (start and end points fixed)

We say that f is ***path homotopic*** to g and write $f \simeq_p g$.

Lemma 1. $\simeq$ and $\simeq_p$ are equivalence relations.

Proof.
- Reflective: $F(x, t) = f(x)$
- Symmetric: $G(x, t) = H(x, 1 - t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x, t) = \begin{cases} H_1(x, 2t) & 0 \le t \le \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

□

Example (Trivial, but important). Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g \colon X \to C$ are homotopic.
- Any two paths $f, g \colon I \to C$ with $f(0) = g(0)$ and $g(1) = f(1)$ are path homopotic.

Choose $H \colon X \times I \to C$ defined by $(x, t) \mapsto H(x, t) = (1 - t)f(x) + tg(x)$.

Product of paths

Let $f \colon I \to X$, $g \colon I \to X$ be paths, $f(1) = g(0)$. Define

$$f * g \colon I \to X \text{ given by } s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g \colon I \to X \,|\, g \simeq_p f\}$.

Theorem 2.

1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
2. Let e_x denote the constant path $e_x \colon I \to X$ given by $s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Let $\overline{f} \colon I \to X$ given by $s \mapsto f(1 - s)$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Proof. First two observations

- Suppose $f \simeq_p g$ via homotopy H, $f, g \colon I \to X$. Let $k \colon X \to Y$. Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.
- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

2. Take $e_0 \colon I \to I$ given by $s \mapsto 0$. Take $i \colon I \to I$ given by $s \mapsto s$. Then $e_0 * i$ is a path from 0 to $1 \in I$. The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 * \simeq i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

3. Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

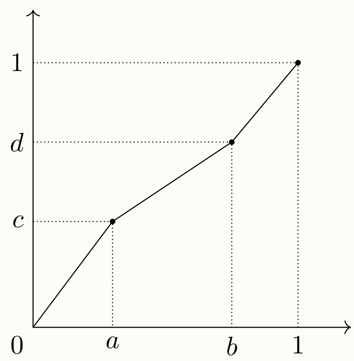
$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \overline{f} &\simeq_p e_{x_0} \\ [f] * [\overline{f}] &= [e_{x_0}]. \end{aligned}$$

1. Remark: Only defined if $f(1) = g(0)$, $g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

   Assume the product is defined. Suppose $[a, b]$, $[c, d]$ are intervals in $\mathbb{R}$. Then there is a unique positive (positive slope), linear map from $[a, b] \to [c, d]$. For any $a, b \in [0, 1)$ with $0 < a < b < 1$, we define a path

$$\begin{aligned} k_{a,b} \colon [0,1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 0] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

   Then $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$.
   Let γ be that path $\gamma \colon I \to I$ with the following graphs:



   Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

   which is what we wanted to show.

□

9.52 The fundamental group

Definition 13. Let X be a space and $x_0 \in X$, then the ***fundamental group*** of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f\colon I \to X,\ f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\overline{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

Theorem 3 (52.1)**.** Let X be a space, $x_0, x_1 \in X$ and $\alpha\colon I \to X$ a path from x_0 to x_1. Then

$$\begin{aligned} \hat{\alpha}\colon \pi(X, x_0) &\longrightarrow \pi(x, x_1) \\ [f] &\longmapsto [\overline{\alpha}] * [f] * [\alpha]. \end{aligned}$$

is an isomorphisms of groups. Note however that this isomorphism depends on α.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned} \hat{\alpha}([f] * [g]) &= [\overline{\alpha}] * [f] * [g] * [\alpha] \\ &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}[f] * \hat{\alpha}[g]. \end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic. □

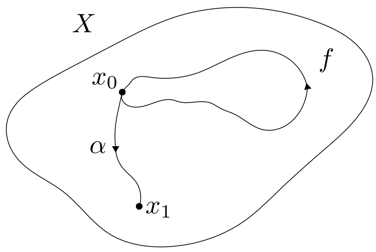


Figure 9.1: construction of the group homeomorphism

Remark. If : $(x, x_0) \to (Y, y_0)$ is a map of pointed topology spaces ($f\colon X \to$

Y continuous and $f(x_0) = y_0$). Then

$$f_*\colon \pi(X, x_0) \to \pi(Y, y_0) \text{ given by } [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two 'rules' discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \quad (1_X)_* = 1_{\pi(X,x_0)}.$$

Definition 14. Let X be a topological space, then X is ***simply connected*** if X is path connected and $\pi_1(X, x_0) = 0$ for some $x_0 \in X$.

Remark. If trivial for one base point, it's trivial for all base points.

Example. Any convex subset $C \subset \mathbb{R}^n$ is simply connected.

Example (Wrong proof of $\pi(S^2)$ being trivial)**.** Let f be a path from $[0, 1] \to S^2$. Then pick $y_0 \in \operatorname{Im}(f)$. Then $S^2 \setminus \{y_0\} \approx \mathbb{R}^2$. Then use $\mathbb{R}^2$.

This is wrong because we cannot always find $y_0 \in \operatorname{Im}(f)$. Space filling loops! We'll see the correct proof later on.

Lemma 2 (52.3)**.** Suppose X is simply connected and $\alpha, \beta\colon I \to X$ two paths with same start and end points. Then $\alpha \simeq_p \beta$.

Proof. Simply connected implies loops are homotopic? Consider $\alpha * \overline{\beta} \simeq_p e_{x_0}$, since the space is imply connected.

$$([\alpha] * [\overline{\beta}]) * [\beta] = [e_{x_0}] * [\beta] = [\beta]$$
$$[\alpha] * ([\overline{\beta}] * [\beta]) = [\alpha] * [e_{x_0}] = [\alpha].$$

And therefore $\alpha \simeq_p \beta$. (Note: make sure end and start point matchs when using $*$) □

9.53 Covering spaces

Definition 15. Let $p\colon E \to B$ be continuous surjective map. The open set $U \subset B$ is ***evenly covered*** if $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$ with

- V_α open in E
- $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$
- $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism.

Remark. If $U' \subset U$, also open and U is evenly covered, then also U'.

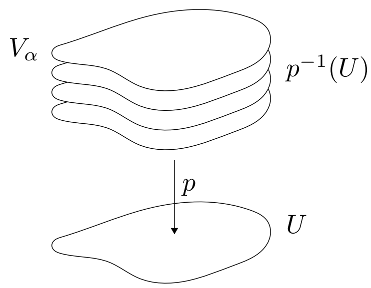


Figure 9.2: evenly covered

Definition 16. Let $p\colon E \to B$ be continuous and surjective. Then p is a ***covering map*** if $\forall b \in B$, $\exists U \subset B$ open, containing b such that U is evenly covered by p. Then (E, p) is called a ***covering space***.

Example. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Take $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. Note that $\mathbb{R}$ is an easier space than S^1, and so will be π_1 (1 vs $\mathbb{Z}$).

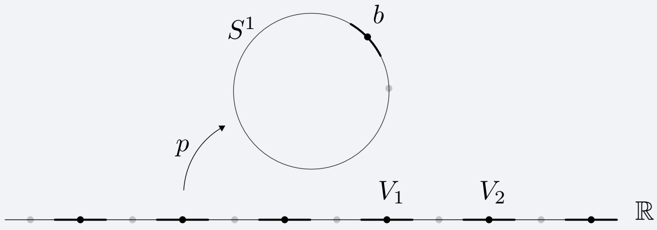


Figure 9.3: example of a covering space

There are also other covering spaces of p. For example, $p'\colon S^1 \to S^1$ given by $z \mapsto z^3$.

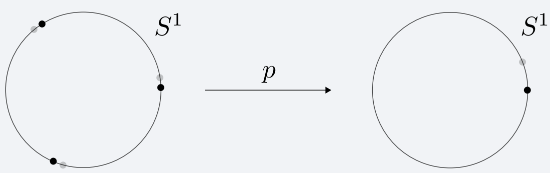


Figure 9.4: second example of a covering space

Here we have three copies for each point. We say that the covering has 3 sheets. Note that this is independent of which point we take. This is always the case! We can show that these are the only coverings of S^1: $\mathbb{R}$ and $z \mapsto z^n$.

Proposition 1. A covering map is always a open map.

Proof. Exercise. □

Proposition 2. For any $b \in B$, $p^{-1}(b)$ is a discrete subset of E. (No accumulation point)

Proof. Indeed for any $\alpha \in I$, $V_\alpha \cap p^{-1}(b)$ is exactly one point. □

Remark. A covering is always local homeomorphism. But there are surjective local homeomorphism which are not covering maps. A covering map is more than a surjective local homeomorphism.

For example, $p\colon \mathbb{R}_0^+ \to S^1$ given by $t \mapsto e^{2\pi i t}$. Consider the inverse image of a neighborhood around 1. When we restrict p to the part around 0, it is no longer a homeomorphism (we don't get the whole neighborhood around one).

Creating new covering spaces out of old ones

- Suppose $p\colon E \to B$ is a covering and $B_0 \subset B$ is a subspace with the subspace topology. Let $E_0 = p^{-1}(B_0)$ and $p_0 = p|_{E_0}$. Then (E_0, p_0) is a covering of B_0.
- Suppose that (E, p) is a covering of B and (E', p') is a covering of B', then $(E \times E', p \times p')$ is a covering of $B \times B'$.

Example. Let $T^2 = S^1 \times S^1$.

- $p\colon \mathbb{R}^2 \to S^1 \times S^1$ given by $(t, s) \mapsto (e^{ait}, e^{bis})$.
- $p'\colon \mathbb{R} \times S^1 \to T^2$ given by $(t, z) \mapsto (e^{ait}, z^n)$.
- $p\colon S^1 \times S^1 \to T^2$ given by $(z_1, z_2) \mapsto (z_1^n, z_2^m)$.

These are the only types of coverings of the torus. We'll prove this later on.

9.54 The fundamental group of the circle

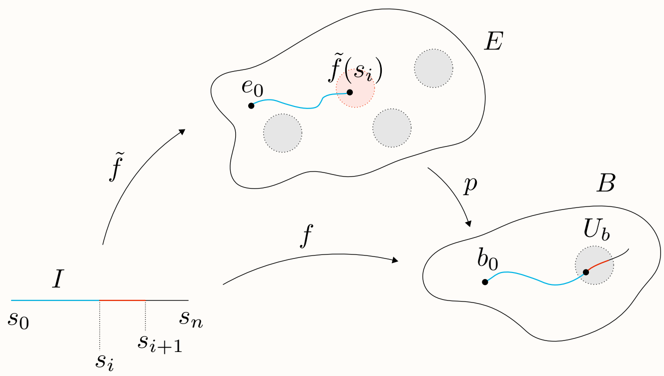
Definition 17. Let $p\colon E \to B$ be a map. If f is a continuous mapping of some space X into B, a ***lifting*** of f is a map $\tilde{f}\colon X \to E$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Given f, when can f be lifted to E? In this section, we'll only consider $X = [0, 1]$, $X = [0, 1]^2$.

Lemma 3 (54.1, Important result)**.** Suppose (E, p) is a covering of B, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$. Suppose that $f \colon I \to B$ is a path starting at b_0. Then there exists a unique lift $\tilde{f} \colon I \to E$ of f with $\tilde{f}(0) = e_0$.

Proof. For any b of B, we choose an open U_b such that U_b is evenly covered by p. Then $\{f^{-1}(U_b) \mid b \in B\}$ is an open cover of I, which is compact. There is a number $\delta > 0$ such that any subset of I of diameter $\leq \delta$ is contained entirely in one of these opens $f^{-1}(U_b)$. (Lebesgue number lemma). Now, we divide the interval into pieces $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $|s_{i+1} - s_i| \leq \delta$. For nay i, we have that $f([s_i, s_{i+1}]) \subset U_b$ for some b.



We now construct $\tilde{f}$ by induction on $[0, s_i]$.

- $\tilde{f}(0) = e_0$
- Assume $\tilde{f}$ has been defined on $[0, s_i]$. Let U be an open such that $f[s_i, s_{i+1}] \subset U_b$.

  There is exactly one slice V_α in $p^{-1}(U_b)$ containing $\tilde{f}(s_i)$. We define $\forall s \in [s_i, s_{i+1}]$: $\tilde{f}(s) = (p|_{V_\alpha})^{-1} \circ f(s)$. By the pasting lemma, $\tilde{f}$ is continuous.
- In this way, we can construct $\tilde{f}$ on the whole of I.

Uniqueness works in exactly the same way, by induction. □

Lemma 4 (54.2)**.** (E, p) is a covering of B, $b_0 \in B$, $e_0 \in E$, with $p(e_0) = b_0$. Suppose $F \colon I \times I \to B$ is a continuous map with $f(0, 0) = b_0$, then there is a unique $\tilde{F} \colon I \times I \to E$. Moreover, if F is a path homotopy, then also $\tilde{F}$ is a path homotopy.

Proof. Same as in the one dimensional case. □

Theorem 4 (54.3)**.** Let (E, p) be a covering of B, $b_0 \in B$, $e_0 \in E$ with $p(e_0) = b_0$. Let f, g be two paths in B starting in b_0 s.t. $f \simeq_p g$ (so f and g end at the same point). Let $\tilde{f}, \tilde{g}$ be the unique lifts of f, g starting at e_0. Then $\tilde{f} \simeq_p \tilde{g}$, and so $\tilde{f}(1) = \tilde{g}(1)$.

Proof. $F\colon I \times I \to B$ is a path homotopy between f and g. Then $\tilde{F}\colon I \times I \to E$ with $\tilde{F}(0,0) = e_0$. Then $\tilde{F}$ is a path homotopy, by the previous result, between $\tilde{F}(\cdot, 0)$ and $\tilde{F}(\cdot, 1)$. Note that $p \circ \tilde{F}(t, 0) = F(t, 0) = f(t)$ and $p \circ \tilde{F}(t, 1) = F(t, 1) = g(t)$. By uniqueness $\tilde{F}(\cdot, 0) = \tilde{f}$, $\tilde{F}(\cdot, 1) = \tilde{g}$. $\square$

We've shown that homotopy from below lifts to above. The converse is easy. Now we're ready to discuss the relation between groups and covering spaces.

Definition 18. Let (E, p) be a covering of B. $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then the ***lifting correspondence*** is the map

$$\begin{aligned} \phi\colon \pi(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1) \end{aligned}$$

where $\tilde{f}$ is the unique lift of f, starting at e_0. This is well-defined because $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$. This ϕ depends on the choice of e_0.

Theorem 5 (54.4)**.** It E is path connected, then ϕ is a surjective map. If E is simply connected, then ϕ is a bijective map.

Proof. Suppose E is path connected, and let $e_0, e_1 \in p^{-1}(b_0)$. Consider a path $\tilde{f}\colon I \to E$ with $\tilde{f}(0) = e_0$ and $\tilde{f}(1) = e_1$. This is possible because E is path connected. Let $f = p \circ \tilde{f}\colon I \to B$ with $f(0) = p(e_0) = b_0$ and $f(1) = p(e_1) = b_0$, so f is a loop based at b_0. So f is a loop at b_0 and its unique lift to E starting at e_0 is $\tilde{f}$. Hence $\phi[f] = \tilde{f}(1) = e_1$, which shows that ϕ is surjective.

Now assume that E is simply connected (group is trivial). Consider $[f], [g] \in \pi(B_0)$ with $\phi[f] = \phi[g]$. This implies $\tilde{f}(1) = \tilde{g}(1)$. These start at e_0. It follows from Lemma 2 that $\tilde{f} \simeq_p \tilde{g}$. $\square$

Example. Take the circle and the real line as covering space. Then $p^{-1}(1) = \mathbb{Z}$. So we know that as a set $\pi(S^1)$ is countable. Therefore, $p \circ \tilde{f} \simeq_p p \circ \tilde{g}$. This implies that $f \simeq_p g$, and therefore $[f] = [g]$.

Theorem 6 (54.5)**.** $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$.

Proof. Take $b_0 = 1$ and $e_0 = 0$ and $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. Then $p^{-1}(b_0) = \mathbb{Z}$. And since, $\mathbb{R}$ is simply connected, we have that $\pi\colon \pi(S, 1) \to \mathbb{Z}$ given by $[f] \mapsto \tilde{f}(1)$ is a bijection.

Now we'll show that it's a morphism. Let $[f]$ and $[g]$ elements of the fundamental group of S^1 and assume that $\phi[f] = \tilde{f}(1) = m$ and $\phi[g] = \tilde{g}(1) = n$.

We're going to prove that $\phi([f] * [g]) = \phi([f]) + \phi([g]) = n + n$. Define $\tilde{\tilde{g}}\colon I \to \mathbb{R}$ given by $t \mapsto \tilde{g}(t) + m$. Then $p \circ \tilde{\tilde{g}} = g$, as $p(s+m) = p(s)$ for all m. Now, look at $\tilde{f} * \tilde{\tilde{g}}$. This is a lift of $p \circ (\tilde{f} * \tilde{\tilde{g}}) = (p \circ \tilde{f}) * (p \circ \tilde{\tilde{g}}) = f * g$, which starts at 0. Hence, $\phi([f] * [g]) = \phi([f * g]) =$ the end point of $\tilde{f} * \tilde{\tilde{g}}$, so $\tilde{\tilde{g}}(1) = \tilde{g}(1) + m = n + m$. □

The following lema makes the fact that the covering space is simpler than the space itself exact.

Lemma 5 (54.6). Let (E, p) be a covering of B, $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then

1. $p_*\colon \pi(E, e_0) \to \pi(B, b_0)$ is a monomorphism (injective).
2. Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence induces a well defined map

$$\begin{aligned} \Phi\colon \pi_1(B, b_0)/H &\longrightarrow p^{-1}(b_0) \\ H * [f] &\longmapsto \phi[f], \end{aligned}$$

   so ϕ is constant on right cosets. Dividing by H makes ϕ always bijective, even when E is not simply connected.
3. Let f be a loop based at b_0, then $\tilde{f}$ is a loop at e_0 iff $[f] \in H$.

Proof.
1. Let $\tilde{f}\colon I \to E$ be a loop at e_0 and assume that $p_*[\tilde{f}] = 1$. (Then we'd like to show that f itself is trivial.) This implies $p \circ \tilde{f} \simeq_p e_{b_0}$. This implies that $\tilde{f} \simeq_p \tilde{e}_{b_0} = e_{e_0}$, or $[\tilde{f}] = 1$.
2. We have to prove two things:

   Well defined $H * [f] = H[g] \Rightarrow \phi(f) = \phi(g)$.

   Assume $[f] \in H * [g]$, or $H * [f] = H * [g]$. This means that $[f] = [h] * [g]$, were $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ at e_0. In other words $[f] = [h * g]$, or $f \simeq_p h * g$. Let $\tilde{f}$ be the unique lift of f starting at e_0. Let $\tilde{g}$ be the unique lift of g starting at e_0. Then $\tilde{h} * \tilde{g}$ (which is allowed, $\tilde{h}$ is a loop) the unique lift of $h * g$ starting at e_0.
   $\tilde{f}(1) = \phi(f) = \phi(h * g) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1) = \phi(g)$. If the cosets are the same, then the end points of the lifts are also the same.

   Injective $H * [f] = H * [g] \Leftarrow \phi(f) = \phi(g)$.

   The end points of f and g are the same. Now consider $\tilde{h} = \tilde{f} * \overline{\tilde{g}}$. Then $[\tilde{h}] * [\tilde{g}] = [\tilde{f}] * [\overline{\tilde{g}}] * [\tilde{g}] = [\tilde{f}]$. By applying p_*, $[h] * [g] = [f]$.
3. Trivial. Exercise. Apply 2 with the constant path.

□

Remark. $k\colon X \to Y$ induces a morphism k_*, we've proved that earlier. Here we only showed injectiveness.

9.55 Retractions and fixed points

Definition 19. Let $A \subset X$, then A is a ***retract*** of X iff there exists a map $r \colon X \to A$ such that $r|_A = 1|_A$, i.e. $r(a) = a$ for all $a \in A$. The map r is called a ***retraction***.

Example. Let X be the figure 8 space, and denote the right circle by A. Then it's easy to see that there exists a retract from the whole space to A by mapping the left circle onto the right

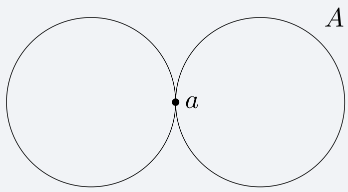


Figure 9.5: Figure 8 space

Lemma 6 (55.1)**.** If A is a retract of X, then $i \colon A \to X$ given by $a \mapsto a$ induces a monomorphism $i_* \colon \pi(A, a_0) \to \pi(X, a_0)$ with $a_0 \in A$.

Proof. Let $r \colon X \to A$ be a retraction. Then $r \circ i = 1_A$.

$$(A, a_0) \xrightarrow{i} (X, x_0) \xrightarrow{r} (A, a_0).$$
$$\pi(A, a_0) \xrightarrow{i_*} \pi(X, x_0) \xrightarrow{r_*} \pi(A, a_0).$$

As $r \circ i = 1_A$, we get that $r_* \circ i_* = (r \circ i)_* = (1_A)_* = 1_{\pi(A,a_0)}$. So i_* is injective, r_* is surjective, which completes the proof. □

Example (Theorem 55.2)**.** Let S^1 be the boundary of B^2. Then S^1 is *not* a retract of B^2. There is a surjective map from B^2 to S^1, but not one that is the identity on S^1.

Proof. Suppose S^1 is a retract. Then $i_* \colon \pi(S^1, x_0) \to \pi(B^2, x_0)$ is injective, but $i_* \colon \mathbb{Z} \to 1$. ◇

Theorem 7 (Brouwer fixed point theorem)**.** For any map $f \colon B^2 \to B^2$, there exists at least one fixed point.

Proof. Look at the proof of the first lecture. Now that we've actually proven that $\pi(S_1) = \mathbb{Z}$ and $\pi(B_2) = 1$, the proof is complete. □

Example. Let A be a 3×3 matrix with strict positive real entries. Then A has a positive real eigenvalue.

Proof. Let $B = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid 0 \leq x_1, x_2, x_3 \leq 1 \wedge x_1^2 + x_2^2 + x_3^2 = 1\}$, an octant of a 2-sphere. Note that $B \approx B^2$, a disk. Now, define $f\colon B \to B$ given by $x \mapsto \frac{Ax}{\|Ax\|}$. Note that this maps vectors from B to vectors of B, as A has positive entries. Note that f is continuous. By Brouwer fixed point theorem, there exists $x_0 \in B$, such that $f(x_0) = x_0$, or $Ax_0 = \|Ax_0\|x_0$. $\diamond$

9.58 Deformation retracts and homotopy type

Lemma 7. Suppose $h, k\colon (X, x_0) \to (Y, y_0)$ and assume $H\colon X \times I \to Y$ is a homotopy with

- $H(x, 0) = h(x)$, $H(x, 1) = k(x)$ (definition of homotopy)
- $H(x_0, t) = y_0$, for all $t \in I$

Then $h_* = k_*\colon \pi(X, x_0) \to \pi_1(Y, y_0)$.

Proof. We have to show that for all $f\colon I \to X$ with $f(0) = f(1) = x_0$ that $h \circ f \simeq_p k \circ f$, i.e. $h_*[f] = k_*[f]$.

$$\begin{aligned} G\colon I \times I &\longrightarrow X \times I \xrightarrow{\ H\ } Y \\ (s, t) &\longmapsto (f(s), t) \longmapsto H(f(s), t) \end{aligned}.$$

- Then G is continuous.
- $G(s, 0) = H(f(s), 0) = (h \circ f)(s)$
- $G(s, 1) = H(f(s), 1) = (k \circ f)(s)$
- $G(0, t) = H(f(0), t) = H(x_0, t) = y_0$
- $G(1, t) = H(f(1), t) = H(x_0, t) = y_0$

So G is a homotopy, and a path homotopy between the two loops. $\square$

Definition 20. Let $A \subset X$, then A is a ***deformation retract*** of X if there exists

- $r\colon X \to A$, such that $r(a) = a$ for all $a \in A$. (normal retract)
- homotopy $H\colon X \times I \to X$ such that
  - $H(x, 0) = x$
  - $H(x, 1) = r(x)$
  - $H(a, t) = a$ for all $a \in A$

This means that 1_X is homotopic to $i \circ r$ via a homotopy leaving A invariant.

Example. Let $S^1 \subset \mathbb{R}^2 \setminus \{(0, 0)\}$. Then S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{(0, 0)\}$. Using homotopy $H\colon \mathbb{R}^2_0 \times I \to \mathbb{R}^2_0$ given by $x \mapsto (1-t)x + t\frac{x}{\|x\|}$.

(The same for S^n and $\mathbb{R}^n_0$)

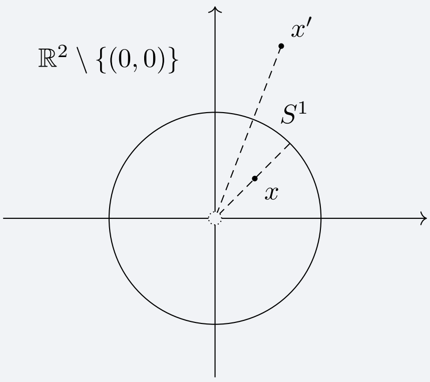


Figure 9.6: Example of a deformation retract

Example. Consider the figure 8 space. Claim: A is not a deformation retract of X. We'll prove this later on.

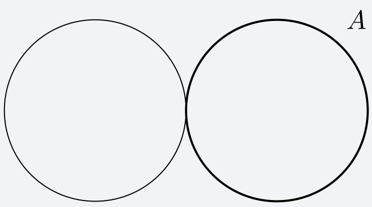


Figure 9.7: Example of a deformation retract

Example. Consider the torus and a circle on the torus. Then it is a retract, but not a deformation retract.

Theorem 8. If A is a deformation retract of X, then $i\colon A \to X$ induces an *isomorphism* i_*. I.e. if you have a deformation retract, it's not only injective but also surjective.

Proof. Let $i\colon A \to X$ be the inclusion and $r\colon X \to A$ be the deformation retraction using H. Then $r \circ i = 1_A$, which gives $r_* \circ i_* = 1_{\pi(A,a_0)}$.

Now, $i \circ r \simeq_p 1_X$ using the homotopy of the previous lemma, i.e. H with $H(a_0, t) = a_0$. Call $h = i \circ r$, $k = 1_X$, and using the previous lemma, $(i \circ r)_* = (1_X)_* \colon \pi(X, x_0) \to \pi(X, x_0)$, which shows that $i_* \circ r_* = 1_{\pi(X,x_0)}$.

We conclude that both i_* and r_* are isomorphisms. $\square$

Remark. This means that the fundamental group of $\mathbb{R}^2_0$ is the same as the one of S^1, which is $\mathbb{Z}$.

Example. The fundamental group of the figure 8 space and the θ-space are isomorphic. These spaces are not deformations of each other, but we can show that they are deformation retracts of $\mathbb{R}^2 \setminus \{p, q\}$. We say that these spaces are of the same homotopy type.

Definition 21. Let X, Y be two spaces, then X and Y are said to be of the same ***homotopy type*** if there exists $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. We say that f, g are ***homotopy equivalences*** and are ***homotopy inverses*** of each other.

Remark. This is an equivalence relation.

We'll prove that spaces of the same homotopy type have the same fundamental group. For that, we'll prove the previous lemma in a more general form, not preserving the base point.

Lemma 8 (58.4)**.** Suppose $h, k\colon X \to Y$ with $h(x_0) = y_0$ and $k(x_0) = y_1$. Assume that $h \simeq k$ via a homotopy $H\colon X \times I \to X$, $(H(x, 0) = h(x)$, $H(x, 1) = k(x))$. Then $\alpha\colon I \to X$ given by $s \mapsto H(x_0, s)$ is a path starting in y_0 and ending in y_1 such that the following diagram commutes

$$\begin{tikzcd} & \pi(X, x_0) \arrow[dl, "h_*"'] \arrow[dr, "k_*"] & \\ \pi(Y, y_0) \arrow[rr, "\hat{\alpha}"] & & \pi(Y, y_1) \\ {[g]} \arrow[rr, mapsto] & & {[\overline{\alpha}] * [g] * [\alpha]} \end{tikzcd}.$$

Proof. We need to show that $\hat{\alpha}(h_*[f]) = k_*[f]$, or $[\overline{\alpha}] * [h \circ f] * [\alpha] = [k \circ f]$, or $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$. We'll prove that these paths are homotopic. Using the picture, we see that $\beta_0 * \gamma_2 \simeq_p \gamma_1 * \beta_1$, because they are loops in a path connected space, $I \times I$. Therefore, $F \circ (\beta_0 * \gamma_2) \simeq_p F \circ (\gamma_1 * \beta_1)$. This is $f_0 * c \simeq_p c * f_1$. Now, if we apply H, we get $H \circ (f_0 * c) \simeq_p H \circ (c * f_1)$, so $(h \circ f) * \alpha \simeq_p \alpha * (k \circ f)$, which implies that $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$. $\square$

Theorem 9. Let $f\colon X \to Y$ be a homotopy equivalence, with $f(x_0) = y_0$. Then $f_*\colon \pi(X, x_0) \to \pi(Y, y_0)$ is an isomorphism.

Proof. Let g be a homotopy inverse of f.

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1) \cdots$$

$$\begin{tikzcd}
\pi(X, x_0) \arrow[r, "f_{*,x_0}"] \arrow[drr, "1_{\pi(X,x_0)=(1_X)_*}"'] & \pi(Y, y_0) \arrow[r, "g_{*,x_0}"] & \pi(X, x_1) \arrow[d, "\hat{\alpha}"] \\
& & \pi(X, x_0)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi(Y, y_0) \arrow[r, "g_{*,x_0}"] \arrow[drr, "1_{\pi(Y,y_0)=(1_Y)_*}"'] & \pi(X, x_1) \arrow[r, "f_{*,x_1}"] & \pi(Y, y_1) \arrow[d, "\hat{\beta}"] \\
& & \pi(Y, y_0)
\end{tikzcd}$$

From the first diagram, $g_{y_0,*} \circ f_{x_0,*}$ is an isomorphism, $g_{y_0,*}$ is surjective. The second diagram gives that $f_{x_1,*} \circ g_{y_0,*}$ is an isomorphism, so $g_{y_0,*}$ is injective, so $g_{y_0,*}$ is an isomorphism. Now composing, we find that $g_{y_0,*}^{-1} \circ (g_{y_0,*} \circ f_{x_0,*}) = f_{x_0,*}$ is an isomorphism. ☐

9.59 The fundamental group of S^n

Theorem 10 (59.1)**.** Let $X = U \cup V$, where U, V are open subsets of X, such that $U \cap V$ is path connected. Let $i\colon U \to X$ and $j\colon V \to X$ denote the natural inclusions and consider $x_0 \in U \cap V$. Then the images of i_* and j_* generate the whole group $\pi(X, x_0)$. In other words: any loop based at x_0 can be written as a product of loops inside U and V.

Proof. Let $[f] \in \pi(X, x_0)$ denote $f\colon I \to X$ is a loop based at x_0.

Claim: there exists a subdevision of $[0, 1]$ such that $f[a_i, a_{i+1}]$ lies entirely inside U or V and $f(a_i) \in U \cap V$. Proof of the claim: Lebesgue number lemma says that such a subdivision b_i exists. Now assume b_j is such that $f(b_j) \notin U \cap V$, for $0 < j < m$. Then either $f(b_j) \in U \setminus V$, or $f(b_j) \in V \setminus U$. The first one would imply that $f([b_{j-1}, b_j]) \subset U$ and $f([b_j, b_{j+1}]) \subset U$. So $f[b_{j-1}, b_{j+1}] \subset U$, so we can discard b_j. Same for the second possibility.

Let α_i be a path from x_0 to $f(a_i)$ and α_0 the constant path $t \mapsto x_0$, inside $U \cap V$ (which is possible, as it is path connected). Now define

$$f_i\colon I \to X \text{ given by } I \xrightarrow{\text{p.l.m.}} [a_{i-1}, a_i] \xrightarrow{f} X.$$

Then $[f] = [f_1] * [f_2] * \cdots * [f_n]$. Note that all f_i have images inside U or V. Now,

$$\begin{aligned}
[f] &= [a_0] * [f_1] * [\overline{\alpha_1}] * [\alpha_1] * [f_2] * [\overline{\alpha_2}] * [\alpha_2] * [f_3] * \cdots * [\alpha_{n-1}] * [f_n] * [\overline{\alpha_n}] \\
&= [\alpha_0 * (f_1 * \overline{\alpha_1})] * [\alpha_1 * (f_2 * \overline{\alpha_2})] * \cdots .
\end{aligned}$$

Every path of the form $\alpha_{i-1} * (f_i * \overline{\alpha_i})$ is a loop based at x_0 lying entirely inside U or V. This means that

$$[f] \in \mathrm{grp}\{i_*(\pi(U, x_0)), j_*(\pi(V, x_0))\}.$$

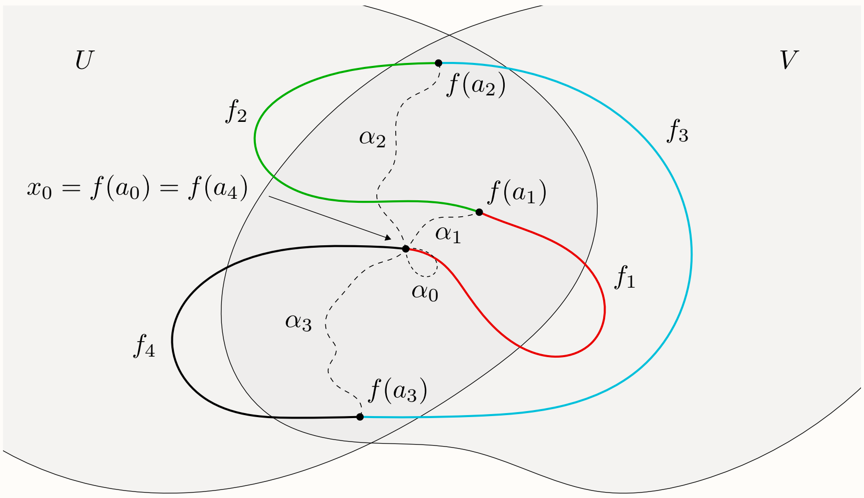


Figure 9.8: Proof of Theorem 59.1

□

Corollary 10.1. Let $n \geq 2$, then $\pi(S^n, x_0) = 1$.

Proof. Consider S^n and N, S the north and south pole. Let $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$. Then $U, V \approx \mathbb{R}^n$ and $U \cap V$ is path connected, which is easy to prove as it is simply homeo to $\mathbb{R}^n$ with points removed. Then $\pi(S^n, x_0)$ is generated by $i_*(\pi(U, x_0))$ and $j_*(\pi(V, x_0))$, which both are trivial. This proof doesn't work for S^1 because then the intersection is not path connected anymore! □

9.60 Fundamental groups of some surfaces

Definition 22. A ***surface*** is compact two-dimensional topological manifold.

Theorem 11. Let X be a space and $x_0 \in X$. Let Y be a space and $y_0 \in Y$. Then $\pi(X \times Y, (x_0, y_0)) \cong \pi(X, x_0) \times \pi(Y, y_0)$.

Proof. Exercise. Idea: Let $f \colon I \to X \times Y$ be a loop based at (x_0, y_0). Then $f(s) = (g(s), h(s))$ where g is a loop in X based at x_0, similar for h, and conversely. □

Example. $\pi_1(T^2, x_0) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$. We know that $\pi(S^2, x_0) = 1$, so the torus and the two sphere are not homeomorphic to each other, they aren't even homotopically equivalent.

Example. $\mathbb{R}\mathrm{P}^2 = S^2/_\sim$. Then $p\colon S^2 \to \mathbb{R}P^2$, which is by definition continuous by definition of the topology on the projective plane.

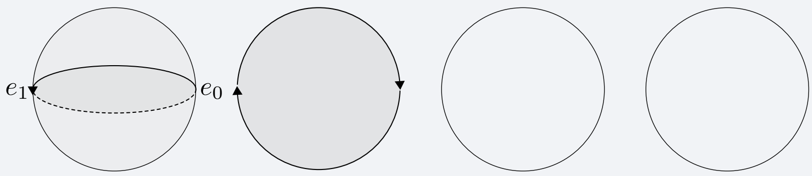


Figure 9.9: projective plane

This means that (S, p) is a covering of the projective plane. The lifting correspondence says that

$$\Phi\colon \pi(\mathbb{R}\mathrm{P}^2, x_0) \to p^{-1}(x_0) = \{\tilde{x_0}, -\tilde{x_0}\}$$

is a isomorphism. Therefore, $\pi_1(\mathbb{R}\mathrm{P}^2, x_0)$ is a group with 2 elements, so $\mathbb{Z}_2$.

This means, there exists loops which we cannot deform to the trivial loop, but when going around twice, they do deform to the trivial loop. E.g. consider the loop a. This is not homotopic equivalent with the trivial loop, as $e_1 \neq e_0$. (Or also you can see it because $\alpha = \overline{\alpha}$.) But pasting the loop it twice, we see that is possible. This means that the fundamental group of the projective space is different from all the one we've seen before.

Example. T^2 is the torus. $T^2 \# T^2$ is the connected sum of two tori (Remove small disc of both tori and glue together), in Dutch: 'tweeling zwemband'. This space has yet another fundamental group.

Example. Figure eight space: fundamental group is not abelian. Indeed, $[b * a] \neq [a * b]$.

Example. Tweeling zwemband. The space retracts to the figure 8 situation, which shows that the gorup of the tweeling zwemband has a nonabelian component.

Chapter 10

Separation theorems in the plane

10.63 Jordan curve theorem

https://en.wikipedia.org/wiki/Jordan_curve_theorem

Chapter 11

Seifert–Van Kampen theorem

https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem

Note. This doesn't follow the book very well.

Definition 23. A ***free group*** on a set X consists of a group F_X and a map $i\colon X \to F_X$ such that the following holds: For any group G and any map $f\colon X \to G$, there exists a unique morphism of groups $\phi\colon F_X \to G$ such that

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F_X \arrow[d, "\exists!\phi"] \\ & G \end{tikzcd}.$$

Note. The free group of a set is unique. Suppose $i\colon X \to F_X$ and $j\colon X \to F'_X$ are free groups.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"'] & F_X \arrow[d, "\exists\phi"] \\ & F'_X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "i"'] & F'_X \arrow[d, "\exists\psi"] \\ & F_X \end{tikzcd}.$$

Then

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i"'] & F_X \arrow[d, "\psi\circ\phi"] \\ & F_X \end{tikzcd}.$$

Then by uniqueness, $\psi \circ \phi$ is 1_{F_X}, and likewise for $\phi \circ \psi$.

Note. The free group on a set always exists. You can construct it using "irreducible words".

Example. Consider $X = \{a, b\}$. An example of a word is $aaba^{-1}baa^{-1}bbb^{-1}a$. This is not a irreducible word. The reduced form is $aaba^{-1}bba = a^2ba^{-1}b^2a$. Then F_X is the set of irreducible words.

Example. If $X = \{a\}$, then $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. Exercise: check that $\mathbb{Z}$ satisfies the universal property.

Example. If $X = \emptyset$, then $F_X = 1$.

Definition 24. Let G_i with $i \in I$, be a set of groups. Then the ***free product*** of these groups denoted by $*_{i\in I}G_i$ is a group G together with morphisms $j_i \colon G_i \to G$ such that the following universal property holds: Given any group H and a collection of morphisms $f_i \colon G_i \to H$, then there exists a unique morphism $f \colon G \to H$, such that for all $i \in I$, the following diagram commutes:

$$\begin{tikzcd} G_i \arrow[r, "j_i"] \arrow[dr, "f_i"'] & G \arrow[d, "\exists! f"] \\ & H \end{tikzcd} .$$

Note. Again, $*_{i\in I}G_i$ is unique.

Example. Construction is similar to the construction of a free group. Let $I = \{1, 2\}$ and $G_1 = G$, $G_2 = H$. Then $G * H$. Elements of $G * H$ are "words" of the form $g_1h_1g_2h_2g_3$, $g_1h_1g_2h_2$, or $h_1g_1h_2g_2h_3g_3$ or $h_1g_1h_2$, ... with $g_j \in G$, $h_j \in H$.

Note. $G * H$ is always infinite and nonabelian if $G \neq 1 \neq H$. Even if G, H are very small, for example $\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, t\} * \{1, s\}$. Then $ts \neq st$ and the order of ts is infinite.

Note. $\mathbb{Z} * \mathbb{Z} = F_{a,b}$. In general: $F_X = *_{x\in X}\mathbb{Z}$.

11.70 The Seifert–Van Kampen theorem

Theorem 12 (70.1, Seifert–Van Kampen theorem)**.** Let $X = U \cup V$ where $U, V, U \cap V$ are open and path connected.[a] Let $x_0 \in U \cap V$. For any group H and 2 morphisms $\Phi_1 \colon \pi(U, x_0) \to H$ and $\Phi_2 \colon \pi(V, x_0) \to H$ such that $\Phi_1 \circ i_1$ and $\Phi_2 \circ i_2$, there exists exactly one $\Phi \colon \pi(X, x_0) \to H$ making the diagram commute

$$\begin{tikzcd}
 & \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, "\Phi"] & H \\
 & \pi(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}.$$

i_1, i_2, i, j_1, j_2 are induced by inclusions.

[a]Note that U, V should also be path connected!

Theorem 13 (70.2, Seifert–Van Kampen theorem (classical version))**.** Assume the hypotheses of the Theorem 12. Let $j \colon \pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ (induced by j_1 and j_2). On elements of $\pi(U, x_0)$ it acts like j_1, on elements of $\pi(V, x_0)$ it acts like j_2.

$$\begin{tikzcd}
G_1 \arrow[d] \arrow[dr, "f_1"] & \\
G_1 * G_2 \arrow[r, dashed, "f"] & H \\
G_2 \arrow[u] \arrow[ur, "f_2"'] &
\end{tikzcd}.$$

Then j is surjective[a] and the kernel of j is the normal subgroup of $\pi(U, x_0) * \pi(U, x_0)$ generated by all elements of the form $i_1(g)^{-1} i_2(g)$, were $g \in \pi(U \cap V, x_0)$.

[a]This is the only place where algebraic topology is used. We've proved this last week. The groups U and V generate the whole group. The rest of this theorem follows from the previous theorem.

Proof.

- j is surjective. (later)
- Let N be the normal subgroup generated by $i_1(g)^{-1} i_2(g)$. Then we claim that $N \subset \ker(j)$. This means we have to show that $i_1(g)^{-1} i_2(g) \in \ker j$. $j(i_1(g)) = j_1(i_1(g))$ by definition of j. Looking at the diagram, we find that $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$. Therefore $j(i_1(g)^{-1} i_2(g)) = 1$, which proves that elements of the form $i_1(g)^{-1} i_2(g)$ are in the kernel.
- Since $N \subset \ker j$, there is an induced morphism

$$\begin{aligned} k \colon (\pi_1(U, x_0) * \pi_1((V, x_0))/N &\longrightarrow \pi_1(X, x_0) \\ gN &\longmapsto j(g). \end{aligned}$$

To prove that $N = \ker j$, we have to show that k is injective. Because this would mean that we've divided out the whole kernel of j.

Now we're ready to use the previous theorem. Let $H = (\pi(U) * \pi(V))/N$. Repeating the diagram:

$$
\begin{tikzcd}
 & \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, shift left, "\Phi"] & H \arrow[l, shift left, "k"] \\
 & \pi(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}
.
$$

Now, we define $\Phi_1 \colon \pi(U, x_0) \to H$ given by $g \mapsto gN$, and $\Phi_2 \colon \pi(V, x_0) \to H$ given by $g \mapsto gN$. For the theorem to work, we needed that $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$. This is indeed the case: let $g \in \pi(U \cap V)$. Then $\Phi_1(i_1(g)) = i_1(g)N$ and $\Phi_2(i_2(g)) = i_2(g)N$ and $i_1(g)N = i_2(g)N$ because $i_1(g)^{-1}i_2(g) \in N$.

The conditions of the previous theorem are satisfied, so there exists a Φ such that the diagram commutes.

Note that we also have $k \colon H \to \pi(X)$. We claim that $\Phi \circ k = 1_H$, which would mean that k is injective, concluding the proof. It's enough to prove that

□

Corollary 13.1. Suppose $U \cap V$ is simply connected, so $\pi_1(U \cap V, x_0)$ is the trivial group. In this case $N = \ker j = 1$, hence $\pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ is an isomorphism.

Corollary 13.2. Suppose U is simply connected. Then $\pi(X, x_0) \cong \pi(V, x_0)/N$ where N is the normal subgroup generated by the image of $i_2 \colon \pi(U \cap V) \to \pi(V, x_0)$.

Example. Let X be the figure 8 space.

Chapter 12

Classification of surfaces

Chapter 13

Classification of covering spaces

Lemma 9 (79.1, General lifting lemma). Let $p\colon E \to B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.[a] Let $f\colon Y \to B$, $y_0 \in Y$, $b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f}\colon Y \to E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd}
 & (E, e_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0)
\end{tikzcd}$$

iff $f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$. If $\tilde{f}$ exists then it is unique.

[a] From now on, all spaces are locally path connected: Every neighborhood contains an open that is path connected.

Proof. Suppose $\tilde{f}$ exists. Then $p \circ \tilde{f} = f$, so $(p \circ \tilde{f})_* \pi(Y, y_0) = \pi(Y, y_0)$. The left hand side is of course $p_*(\tilde{f}_*(\pi(Y, y_0)) \subset p_*(\pi(E, e_0))$, so $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$.

Conversely, we'll show the uniqueness first. Suppose $\tilde{f}$ exists.

$p \circ (\tilde{f} \circ \alpha) = f \circ \alpha$, so $\tilde{f} \circ \alpha$ is the unique lift of $f \circ \alpha$ starting at e_0. Hence $f(y)$ the endpoint of the unique lift of $f \circ \alpha$ to E starting at e_0.

This also shows how you can define $\tilde{f}$: choose a path α from y_0 to y. Lift $f \circ \alpha$ to a path starting at e_0. Define $\tilde{f}(y) =$ the end point of this lift. Is this well defined? Is $\tilde{f}$ continuous?

Well defined As $[\alpha] * [\overline{\beta}] \in \pi(Y, y_0)$,

$$f_*([\alpha] * [\overline{\beta}]) = ([f \circ \alpha] * [f \circ \overline{\beta}) \in f_*(\pi_1(Y, y_0))$$

which is by assumption a subgroup of $p_*(\pi(E, e_0)) = H$.

And now, by Lemma 3, a loop in the base space lifts to a loop in E if the loop is in H. This lift is of course just $\gamma * \delta$, so the end points in the drawing should be connected! this means that $\overline{\delta}$ is the lift of $f \circ \beta$ starting at e_0, so the endpoint of the lift of $f \circ \beta$ is the endpoint of the lift of $f \circ \alpha$. Therefore $\tilde{f}(y)$ is well defined.

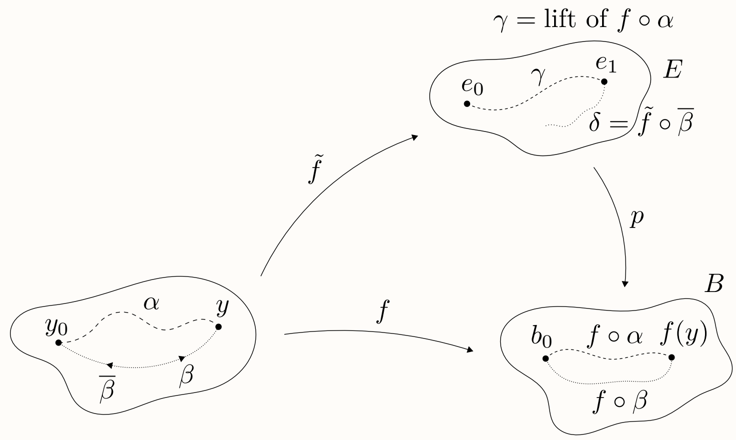


Figure 13.1: well defined general lifting lemma

Continuity We prove that $\tilde{f}$ is continuous.

- Choose a neighborhood of $\tilde{f}(y_1)$, say N.
- Take U, a path connected open neighborhood of $f(y_1)$ which is evenly covered, such that the slice $p^{-1}(U)$ containing $\tilde{f}(y_1)$ is completely contained in N.

  Can we do this? The inverse image of U is a pile of pancakes. One of these pancakes contains $\tilde{f}(y_1)$. Then, because N is a neighborhood of $\tilde{f}(y_1)$, we can shrink the pancake such that it is contained in N.
- Choose a path connected open which contains y_1 such that $f(W) \subset U$. We can do this because of continuity of f.
- Take $y \in W$. Take a path β in W from y_1 to y. (Here we use that W is path connected.) Now consider $p|_V$ and defined Then $\alpha * \beta$ is path fro y_0 to y, $f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$. Then $\widetilde{f \circ \alpha} * (p^{-1}|_V \circ f \circ \beta)$ is the lift of $f \circ (\alpha * \beta)$ starting at y_0. So by definition of $\tilde{f}$, we have that $\tilde{f}(y)$ is the endpoint of that lift, which belongs to $V \subset N$. This means that $\tilde{f}(W) \subset N$, which proves continuity.

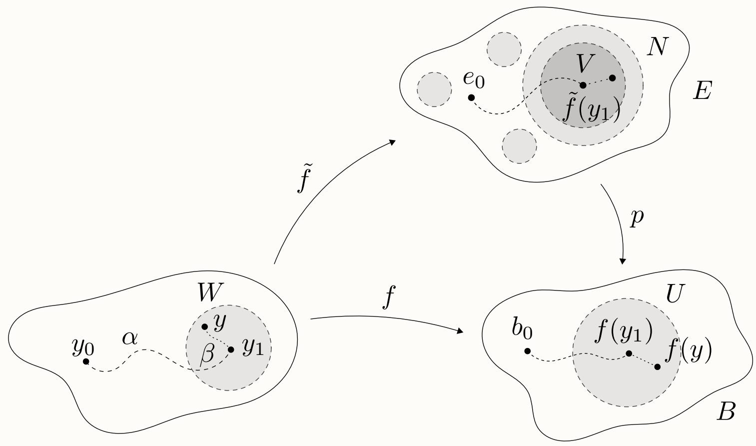


Figure 13.2: Proof of the continuity of the general lifting lemma

□

Example. Take $Y = [0, 1]$. Then f is a path, then we showed that every map can be lifted. And indeed, the condition holds: $f_*(\pi(Y, y_0)) = 1$, the trivial group, which is a subgroup of all groups.

Lemma 10 (General lifting lemma, short statement)**.** Short statement:

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

$\exists! \tilde{f} \iff f_*(\pi(Y, y_0)) \subset p_* \pi(E, e_0)$.

Definition 25. Let (E, p) and (E', p') be two coverings of a space B. An ***equivalence*** between (E, p) and (E', p') is a homeomorphism $h \colon E \to E'$ such that

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[dr, "p"] & E' \arrow[d, "p'"] \\ & B \end{tikzcd}$$

is commutative. $p' \circ h = p$.

Theorem 14 (79.2)**.** Let $p \colon (E, e_0) \to (B, b_0)$ and $p' \colon (E', e_0') \to (B, b_0)$ be coverings, and $H_0 = p_* \pi(E, e_0)$ and $H_0' = p_*' \pi(E', e_0') \leq \pi(B, b_0)$. Then there exists and equivalence $h \colon (E, p) \to (E', p')$ with $h(e_0) = e_0'$ iff $H_0 - H_0'$. Not isomorphic, but really the same as a subgroup of $\pi(B, b_0)$. In that case, h is unique.

Proof. $\boxed{\Rightarrow}$ Suppose h exists. Then

$$\begin{tikzcd} (E,e_0) \arrow[r, "h"] \arrow[dr, "p"'] & (E',e_0') \arrow[d, "p'"] \\ & (B,b_0) \end{tikzcd}.$$

Therefore $p_*\pi(E,e_0) = p'_*(h_*\pi(E,e_0))$. Since h is a homeomorphism, it induces an isomorphism, so $p'_*(h_*\pi(E,e_0)) = p'_*(\pi(E',e_0'))$.

$\boxed{\Leftarrow}$

$$\begin{tikzcd} & (E',e_0') \arrow[d, "p'"] \\ (E,e_0) \arrow[ur, dashed, "k"] \arrow[r, "p"] & (B,b_0) \end{tikzcd}$$

By the previous lemma, there exists a unique k iff $p_*\pi(E,e_0) \subset p'_*\pi(E',e_0')$ or equivalently $H_0 \subset H_0'$, which is the case. Reversing the roles, we get

$$\begin{tikzcd} & (E,e_0) \arrow[d, "p"] \\ (E',e_0') \arrow[ur, "l"] \arrow[r, "p'"] & (B,b_0) \end{tikzcd}$$

for the same reasoning, l exists. Now, composing the diagrams

$$\begin{tikzcd} & (E,e_0) \arrow[d, "p"] \\ (E,e_0) \arrow[ur, dashed, "l \circ k"] \arrow[r, "p"] & (B,b_0) \end{tikzcd} \qquad \begin{tikzcd} & (E',e_0') \arrow[d, "p'"] \\ (E',e_0') \arrow[ur, dashed, "k \circ l"] \arrow[r, "p'"] & (B,b_0) \end{tikzcd}$$

But placing the identity in place of $l \circ k$ or $k \circ l$, this diagram also commutes! By unicity, we have that $l \circ k = 1_E$ and $k \circ l = 1_{E'}$. Therefore, k and l are homeomorphism $k(e_0) = e_0'$.

Uniqueness is trivial, because of the general lifting theorem.

□

Note that this doesn't answer the question 'is there a equivalence between two coverings', it only answers the question 'is there an equivalence between two coverings mapping $e_0 \to e_0'$'. So now, we seek to understand the dependence of H_0 on the base point.

Lemma 11 (79.3)**.** Let (E, p) be a covering of B. Let $e_0, e_1 \in p^{-1}(b_0)$. Let $H_0 = p_*\pi(E, e_0)$, $H_1 = p_*\pi(E, e_1)$.

- Let γ be a path from e_0 to e_1 and let $p \circ \gamma = \alpha$ be the induced *loop* at b_0. Then $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$, so H_0 and H_1 are conjugate inside $\pi(B, b_0)$.
- Let H be a subgroup of $\pi(B, b_0)$ which is conjugate to H_0, then there is a point $e \in p^{-1}(b_0)$ such that $H = p_*\pi(E, e)$.

So a covering space induces a conjugacy class of a subgroup of $\pi(B, b_0)$.

This completely answers the question: when are two covering spaces equivalent?

Corollary 14.1. Let (E, p) and (E', p') be two coverings, $e_0 \in E$, $e_0' \in E'$ with $p(e_0) = p(e_0') = b_0$. Let $H_0 = p_*\pi(E, e_0)$, $H_0' = p'_*\pi(E', e_0')$. Then (E, p) and (E', p') are equivalent iff H_0 and H_0' are conjugate inside $\pi(B, b_0)$.

Question: can we reach every possible subgroup? Answer: yes, in some conditions.

13.80 Universal covering space

Definition 26. Let B be a path connected and locally path connected space. A covering space (E, p) of B is called a ***universal covering space*** if E is simply connected, so $\pi(E, e_0) = 1$.

Chapter 14

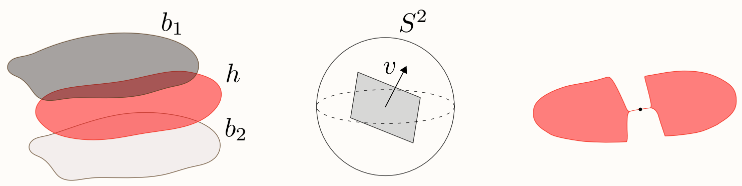
Singular homology

Theorem 15 (Ham sandwich theorem). *Suppose you give me two pieces of bread and 1 slice of ham. Then it is possible to divide both the pieces of bread and the slice of ham in equal pieces by 1 straight cut of knife.*

Proof. Consider for each $v \in S^2$ a plane $P_v \subset \mathbb{R}^3$. $P_v \perp v$ and P_v cuts the slice of ham exactly in two. We defined the *upper side* of the plane to be the half to which v is pointing to.

If you have some weird ham which you can cut in multiple places in half, then you take the middle of the line segment. This makes it unique.

Note that $P_v = P_{-v}$.



Now, consider

$$f \colon S^2 \to \mathbb{R}^2 \text{ given by } v \mapsto (f_1(v), f_2(v)).$$

Then $f_1(v)$ is the volume of bread b_1 above P_v. Then $f_2(v)$ is the volume of bread b_2 above P_v.

Now, you should believe that f_1 and f_2 are continuous. (Proving this precisely needs measure theory etc.) So, now, we can use the Borsak Ulam theorem. So there exists a $v \in S^2$ such that $f(v) = f(-v)$. So $f_1(v) = f_1(-v)$, so volume of bread b_1 above P_v is the volume of bread b_1 below P_v, and similar for f_2. This proves the Ham sandwich theorem. $\square$