Advanced Calculus II – Midterm Exam

October 28, 2019

1. Let $A \subset \mathbb{R}^n$ be an open set, $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function, and $x_0 \in A$. Provide the definition of the differentiability of f at x_0.

2. Let $A \subset \mathbb{R}^n$ be an open set, $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$ be a function, $x_0 \in A$, and $v \in \mathbb{R}^n$. The directional derivative of f at x_0 along the vector v is defined by

$$\partial_v f(x_0) = \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}.$$

   Suppose the function f is differentiable. Show that

$$\partial_v f(x_0) = \nabla f(x_0) \cdot v.$$

3. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. For $x_0 \in \mathbb{R}^n$, consider a level set $S := \{x \in \mathbb{R}^n : f(x) = f(x_0)\}$. Show that $\nabla f(x_0)$ is orthogonal to S.

4. Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that there exists $M \in \mathbb{R}$ such that $\|L(x)\| \leq M\|x\|$ for all $x \in \mathbb{R}^n$.

5. Find the second-order Taylor approximation for $f(x, y) = e^x \cos y$ around $(1, \pi)$.

6. Let $A \subset \mathbb{R}^n$ be an open set, $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable on A. Show that f is continuous.

7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) = (x + y)(xy + xy^2)$. Find all critical points of f and determine whether the function f has an extreme point or not.

8. Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $\|g(x)\| \leq M\|x\|^2$ for some M. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfy $f(x) = L(x) + g(x)$. Prove that $Df(0) = L$.

9. Consider the map $\mathcal{L}^{-1}\colon GL(\mathbb{R}^n, \mathbb{R}^n) \to GL(\mathbb{R}^n, \mathbb{R}^n)$ such that $\mathcal{L}^{-1}(A) = A^{-1}$. Show that the derivative of this map is given by

$$D\mathcal{L}^{-1}(A) \cdot B = -A^{-1} \circ B \circ A^{-1}.$$