Financial Mathematics 1 – Exam 3

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1. Let $T > 0$. Consider the market model with bank account $(G_t)_{0\le t\le T} = (e^{rt})_{0\le t\le T}$ and risky asset $(S_t)_{0\le t\le T}$ given by SDE
$$dS_t = \mu(t, S_t)S_t\, dt + \sigma(t, S_t)S_t\, dB_t.$$
Assume that this SDE has a unique solution.
   (a) Consider an European option with payoff $g(S_T)$ for a Borel function g having polynomial growth. Use the heuristic argument to derive the Black-Scholes PDE: Let $f(t, x)$ be the function such that the time-t price of option is $f(t, S_t)$. Then
   $$f_t + rxf_x(t, x) + \frac{1}{2}\sigma^2(t, x)x^2 f_{xx}(t, x) - rf(t, x) = 0$$
   with the terminal condition $f(T, x) = g(x)$.
   (b) Apply the Feynman-Kac formula and express $f(t, x)$ as an expectation form.

2. Consider the following market model with bank account $G = (G_t)_{t\ge 0}$ and risky asset $S = (S_t)_{t\ge 0}$ given by
$$\begin{aligned} dG_t &= rtG_t\, dt \\ dS_t &= \mu S_t\, dt + \sigma S_t\, dB_t \end{aligned}$$
for positive constants r, μ, σ. Let $T > 0$.
   (a) Find a risk-neutral measure.
   (b) Calculate the time-0 price of an option whose payoff is $\log S_T$ at maturity T.

3. Fix $T > 0$. Consider the market $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ with bank account $G \equiv 1$ and two stocks $S^{(1)}$, $S^{(2)}$ given as
$$\begin{aligned} \frac{dS_t^{(1)}}{S_t^{(1)}} &= \mu^{(1)}\, dt + v^{(1)}\, dB_t^{(1)} \\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= \mu^{(2)}\, dt + \sigma^{(1)}\, dB_t^{(1)} + \sigma^{(2)}\, dB_t^{(2)} \end{aligned}$$
for $\mu^{(1)}, \mu^{(2)}, \sigma^{(1)} \in \mathbb{R}$ and $v^{(1)}, \sigma^{(2)} \neq 0$.
   (a) Find the time-t price of the option with payoff
   $$S_T^{(1)} S_T^{(2)} \mathbb{I}_{S_T^{(1)} > K}$$
   for $K > 0$ and maturity T. Use the cumulative distribution function $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\, dz$.
   (b) Find the hedging portfolio.

4. Let X be the solution of the SDE
$$dX_t = -\theta X_t\, dt + \sigma\, dB_t$$
for $\theta \neq 0$, $\sigma > 0$ and $X_0 \in \mathbb{R}$. Calculate
$$\mathbb{R}(e^{X_T}(e^{X_T} - K)_+ | \mathcal{F}_t^X)$$
for $0 \le t \le T$.