Mathematical Statistics II

Ch.5 Distributions of Functions of Random Variables

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pop

$X_1 \cdots X_n$ random sample

$X_1, \cdots, X_n$ indep $\sim (\mu_i, \sigma_i^2)$.

$Y = \sum a_i X_i \sim (\sum a_i \mu_i, \sum a_i^2 \sigma_i^2)$.

$\bar{X} = \frac{1}{n} \sum x_i \sim (\mu, \frac{\sigma^2}{n})$.

$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$

$\frac{(n-1)S^2}{\sigma^2} = \sum \frac{(X_i - \bar{X})^2}{\sigma^2}$

$X_1, \cdots, X_n$ iid.

mgf of normal distribution

$\exp\{\mu t + \frac{\sigma^2}{2} t^2\}$

$X_i \sim N(\mu, \sigma^2)$.

$Y = \sum c_i X_i$

$M_Y(t) = E[e^{tY}] = E[e^{t \sum c_i X_i}] = E[e^{tc_1 x_1} \cdots e^{tc_n x_n}]$

$= \prod_{i=1}^{n} E[e^{tc_i x_i}] = \prod_{i=1}^{n} M_{X_i}(c_i t) = \prod_{i=1}^{n} M_X(c_i t)$.

$M_X(t) = \exp\{\mu t + \frac{\sigma^2}{2} t^2\}$

$= \exp\{\sum c_i \mu t + \sum \frac{c_i^2 \sigma^2}{2} t^2\}$

$=$ mgf of $N(\sum c_i \mu, \sum c_i^2 \sigma^2)$.

$X_i \sim iid.$ $\quad X_i \sim N(\mu, \sigma^2).$

$$\frac{X_i - \mu}{\sigma} \sim N(0,1).$$

$$\frac{\Sigma(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1). \qquad \frac{(X_i-\mu)^2}{\sigma^2} \sim \chi^2(1).$$

$$\frac{\Sigma(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n).$$

$$W = \frac{\Sigma(X_i-\mu)^2}{\sigma^2}$$

$$= \frac{\Sigma((X_i-\bar{X})+(\bar{X}-\mu))^2}{\sigma^2}$$

$$= \frac{\Sigma[(X_i-\bar{X})^2 + 2(X_i-\bar{X})(\bar{X}-\mu) + (\bar{X}-\mu)^2]}{\sigma^2}$$

$$= \frac{\Sigma(X_i-\bar{X})^2 + 2(\bar{X}-\mu)\cancel{\Sigma(X_i-\bar{X})}^{\,0} + n(\bar{X}-\mu)^2}{\sigma^2}$$

$$= \frac{\Sigma(X_i-\bar{X})^2}{\sigma^2} + \frac{n(\bar{X}-\mu)^2}{\sigma^2}$$

$$V = \frac{(n-1)s^2}{\sigma^2} \qquad \frac{(\bar{X}-\mu)^2}{\sigma^2/n} \sim \chi^2(1).$$

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

$$\frac{(\bar{X}-\mu)^2}{\sigma^2/n} \sim \chi^2(1)$$

$$(1-2t)^{-\frac{n}{2}} = M_V(t) + (1-2t)^{-\frac{1}{2}}$$

$$M_V(t) = (1-2t)^{-\frac{(n-1)}{2}}$$

$$\therefore V = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1).$$

Ch5.5 Random Functions associated with normal distributions

Thm. 5.5-1

If $X_1, X_2, \ldots, X_n$ are independent random variables and $X_i \sim N(\mu_i, \sigma_i^2)$, for $i = 1, \ldots, n$, then for constant values c_i,

$$Y = \sum_{i=1}^{n} c_i X_i \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$$

pop

$N(\mu, \sigma^2) \longrightarrow X_1, \dots, X_n$

$X_i \sim N(\mu, \sigma^2)$. iid.

$c_i = 1$.

1). $\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$.

$c_i = \frac{1}{n}$

☆ 2). $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \frac{1}{n}\mu = \mu, \sum_{i=1}^{n} \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n}\right)$.

$N(\mu, \frac{\sigma^2}{n})$

(indep.

3) $S^2 = \frac{\Sigma(X_i - \overline{X})^2}{n-1} \sim ?$

$X_i \sim \boxed{?}(\mu_i, \sigma_i^2)$ indep.

$Y = \sum_{i=1}^{n} a_i X_i \sim \boxed{?}\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

Thm $X_i \sim N(\mu_i, \sigma_i^2)$ indep.

$$Y = \sum_{i=1}^{n} c_i X_i \sim N\left(\sum_{i=1}^{n} c_i\mu_i, \sum_{i=1}^{n} c_i^2\sigma_i^2\right).$$

$$M_{X_i}(t) = \exp\left\{\mu_i t + \frac{\sigma_i^2}{2}t^2\right\} \quad t \in \mathbb{R}.$$

$$M_Y(t) = E[e^{tY}] = E[e^{t\Sigma c_i X_i}] = \prod_{i=1}^{n} E[e^{tc_iX_i}]$$

$$= \prod_{i=1}^{n} M_{X_i}(c_i t) = \prod_{i=1}^{n} \exp\left\{\mu_i c_i t + \frac{\sigma_i^2}{2}c_i^2 t^2\right\}$$

$$= \exp\left\{\sum_{i=1}^{n} \mu_i c_i t + \frac{\sum_{i=1}^{n}\sigma_i^2 c_i^2}{2} t^2\right\}$$

: mgf of $N\left(\sum_{i=1}^{n} c_i\mu_i, \sum_{i=1}^{n} c_i^2\sigma_i^2\right)$.

Corollary 5.5-1

If $X_1, X_2, \ldots, X_n$ are observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$, then the sample mean

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \frac{1}{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Thm. 5.5-2

If $X_1, X_2, \ldots, X_n$ are observations of a random sample of size n from $N(\mu, \sigma^2)$, $\overline{X}$ is the sample mean, and

$S^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - \overline{X})^2$ is the sample variance, then

- $\overline{X}$ and S^2 are independent. 일단 증명없이 받아들이기!
- $\dfrac{(n-1)S^2}{\sigma^2} = \dfrac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$

pop. $N(\mu, \sigma^2)$ $\longrightarrow$ $X_i \sim N(\mu, \sigma^2)$. iid.

$$\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(1).$$

$$W \overset{let}{=} \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(n)$$

cf). $S^2 = \frac{\Sigma(X_i-\bar{X})^2}{n-1}$

$$V \overset{let}{=} \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 \sim \chi^2(n-1)$$

$$W = \sum_{i=1}^{n}\left[\frac{(X_i-\bar{X})+(\bar{X}-\mu)}{\sigma}\right]^2$$

$$= \sum_{i=1}^{n}\left[\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 + 2\frac{(X_i-\bar{X})(\bar{X}-\mu)}{\sigma^2} + \frac{(\bar{X}-\mu)^2}{\sigma^2}\right]$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 + 2\frac{(\bar{X}-\mu)}{\sigma^2}\underbrace{\cancel{\sum_{i=1}^{n}(X_i-\bar{X})}}_{0} + \frac{n(\bar{X}-\mu)^2}{\sigma^2}$$

$$= \underbrace{\frac{(n-1)S^2}{\sigma^2}}_{V} + \frac{(\bar{X}-\mu)^2}{\sigma^2/n}$$

Since $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$$\Rightarrow Z^2 = \frac{(\bar{X}-\mu)^2}{\sigma^2/n} \sim \chi^2(1)$$

indep.

$$\therefore W = \underbrace{\frac{(n-1)S^2}{\sigma^2}}_{V} + \underbrace{Z^2}_{Z^2}$$

Since $\bar{X}$ and S^2 are indep., Z^2 and S^2 are indep.

$$\therefore M_W(t) = E[e^{tW}] = E\left[e^{\frac{t(n-1)S^2}{\sigma^2}+tZ^2}\right]$$

$$= \underbrace{E\left[e^{t\frac{(n-1)S^2}{\sigma^2}}\right]}_{e^{tV}} E\left[e^{tZ^2}\right] \quad (\because \text{indep. } S^2, Z^2).$$

$$\therefore (1-2t)^{-\frac{n}{2}} = E\left[e^{t\frac{(n-1)S^2}{\sigma^2}}\right] \times (1-2t)^{-\frac{1}{2}}, \quad t<\frac{1}{2}$$

e^{tV}

$$\therefore E\left[e^{t\frac{(n-1)S^2}{\sigma^2}}\right] = (1-2t)^{-\frac{(n-1)}{2}}, \quad t<\frac{1}{2}$$

$$\therefore \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

cf). $X_i \sim N(\mu, \sigma^2)$.

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(n).$$

$$\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

복습.

$X \sim N(\mu, \sigma^2)$. $\longrightarrow$ $X_1, \cdots, X_n$: random sample.
iid.

$\bar{X} \sim$ (

$E[\bar{X}] = E[\frac{1}{n}\Sigma X_i] = \frac{1}{n}\Sigma E[X_i] = \frac{1}{n}\cdot n\mu. = \mu.$

$Var[\bar{X}] = \Sigma \frac{1}{n^2}\cdot \sigma^2 = \frac{\sigma^2}{n}.$

$\therefore\ \bar{X} \sim ?(\mu, \frac{\sigma^2}{n})$.

WTS $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

mgf of normal distribution : $\exp\{\mu t + \frac{\sigma^2}{2}t^2\}$.

$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E[e^{t\cdot\frac{1}{n}\Sigma X_i}] = E[e^{\frac{t}{n}X_1} \cdots e^{\frac{t}{n}X_n}]$

$= \prod_{i=1}^{n} E[e^{\frac{t}{n}X_i}] = \prod_{i=1}^{n} M_{X_i}(\frac{t}{n}) = \prod_{i=1}^{n} M_X(\frac{t}{n})$.

$= \prod_{i=1}^{n} \exp\{\mu\cdot\frac{t}{n} + \frac{\sigma^2}{2}\cdot\frac{t^2}{n^2}\}$

$= [\exp\{\mu\cdot\frac{t}{n} + \frac{\sigma^2}{2}\cdot\frac{t^2}{n^2}\}]^n$.

$= \exp\{\mu t + \frac{\sigma^2}{2n}t^2\}$

: mgf of $N(\mu, \frac{\sigma^2}{n})$.

$\therefore\ \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

sample variance. $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

WTS $V := \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$.

we assume that $\bar{X}$ and S^2 are independent.

$W := \sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$.

$$
\begin{aligned}
W &= \sum_{i=1}^{n}\left[\frac{X_i-\bar{X}}{\sigma} + \frac{\bar{X}-\mu}{\sigma}\right]^2 \\
&= \sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{\sigma^2} + \frac{2}{\sigma^2}(\bar{X}-\mu)\cdot\cancelto{0.}{\sum_{i=1}^{n}(X_i-\bar{X})} + \frac{n(\bar{X}-\mu)^2}{\sigma^2} \\
&= \sum_{i=1}^{n}\frac{(X_i-\bar{X})^2}{\sigma^2} + \frac{(\bar{X}-\mu)^2}{\sigma^2/n} = V + \frac{(\bar{X}-\mu)^2}{\sigma^2/n}
\end{aligned}
$$

cf) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

$\therefore \frac{(\bar{X}-\mu)^2}{\sigma^2/n} \sim \chi^2(1)$.

$\exp(\beta) = \text{Gamma}(1, \beta)$.

$\chi^2(r) = \text{Gamma}(\frac{r}{2}, 2)$

mgf of $\text{Gamma}(\alpha, \beta)$

: $(1-\beta t)^{-\alpha}$

$$
\begin{aligned}
M_W(t) &= (1-2t)^{-\frac{n}{2}} \\
&= E[e^{(V + \frac{(\bar{X}-\mu)^2}{\sigma^2/n})t}] \\
&= E[e^{Vt}] + (1-2t)^{-\frac{1}{2}}
\end{aligned}
$$

$\therefore E[e^{Vt}] = (1-2t)^{-\frac{(n-1)}{2}}$: mgf of $\chi^2(n-1)$.

$\therefore V = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.

Thm. 5.5-3

If $Z \sim N(0,1)$, $U \sim \chi^2(r)$, and Z and U are independent, then

$$T = \frac{Z}{\sqrt{U/r}} \sim t(r)$$

$Z \sim N(0,1)$, $U \sim \chi^2(r) = \text{Gamma}\left(\frac{r}{2}, 2\right)$

indep

$$\Rightarrow T = \frac{Z}{\underbrace{\sqrt{U/r}}_{>0}} \sim t(r) \qquad t \in \mathbb{R}.$$

pf).

1) joint dist^n of Z & U

$$g(z,u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \times \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} \cdot u^{\frac{r}{2}-1} e^{-\frac{u}{2}}$$

2) cdf of T

$$F_T(t) = P(T \le t) = P\left(\frac{Z}{\sqrt{U/r}} \le t\right) = P\left(Z \le \sqrt{\frac{U}{r}}\, t\right)$$

$$= \int_0^\infty \int_{-\infty}^{\sqrt{\frac{u}{r}}t} g(z,u)\, dz\, du$$

$$= \int_0^\infty \int_{-\infty}^{\sqrt{\frac{u}{r}}t} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{u^{\frac{r}{2}-1} e^{-\frac{u}{2}}}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}}\, dz\, du$$

$$= \int_0^\infty \frac{u^{\frac{r}{2}-1} e^{-\frac{u}{2}}}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} \left[\int_{-\infty}^{\sqrt{\frac{u}{r}}t} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz\right] du \qquad = \Phi\left(\sqrt{\frac{u}{r}}\, t\right).$$

$$= \int_0^\infty \frac{u^{\frac{r}{2}-1} e^{-\frac{u}{2}}}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} \Phi\left(\sqrt{\frac{u}{r}}\, t\right) du$$

$$\int \frac{d}{dt} \square\, du.$$

$$\Uparrow$$

$$f_T(t) = \frac{d}{dt} \int \square\, du$$

By interchange the derivative and integral operations,

$$f_T(t) = \frac{d}{dt} F_T(t) = \int_0^\infty \frac{u^{\frac{r}{2}-1} e^{-\frac{u}{2}}}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} \sqrt{\frac{u}{r}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\frac{u}{r}t^2}{2}}\, du$$

$$= \int_0^\infty \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{(r+1)}{2}} \sqrt{\pi r}} u^{\frac{(r+1)}{2}-1} \cdot e^{-\frac{u\left(1+\frac{t^2}{r}\right)}{2}}\, du$$

Let $y = u\left(1+\frac{t^2}{r}\right) \Rightarrow dy = \left(1+\frac{t^2}{r}\right)du$

$u = \left(1+\frac{t^2}{r}\right)^{-1} y.$

$$f_T(t) = \int_0^\infty \frac{1}{\Gamma(\frac{r}{2})\, 2^{\frac{(r+1)}{2}} \sqrt{\pi r}} \left(1+\frac{t^2}{r}\right)^{-\frac{(r+1)}{2}\cancel{+1}} \times \underline{y^{\frac{(r+1)}{2}-1}} \times \underline{e^{-\frac{y}{2}}}$$

$$\times \left(1+\frac{t^2}{r}\right)^{\cancel{-1}} dy$$

$$= \frac{1}{\Gamma(\frac{r}{2}) \cdot 2^{\frac{(r+1)}{2}} \sqrt{\pi r} \left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \int_0^\infty y^{\frac{(r+1)}{2}-1} \times e^{-\frac{y}{2}}\, dy.$$

$$= \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2}) \sqrt{\pi r} \left(1+\frac{t^2}{r}\right)^{\frac{(r+1)}{2}}} \int_0^\infty \underbrace{\frac{1}{\Gamma(\frac{r+1}{2})\, 2^{\frac{r+1}{2}}} y^{\frac{(r+1)}{2}-1} \cdot e^{-\frac{y}{2}} dy}_{\text{pdf of } \chi^2(r+1).}$$

$\Rightarrow \int — = 1.$

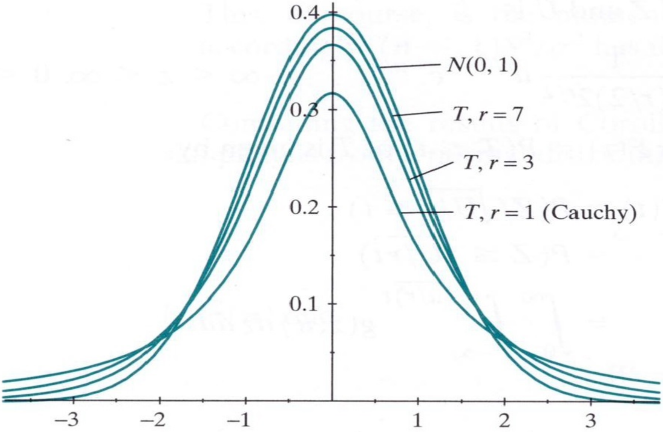
$$\therefore f_T(t) = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2}) \sqrt{\pi r}} \left(\frac{1}{1+\frac{t^2}{r}}\right)^{\frac{(r+1)}{2}}, \quad t \in \mathbb{R}.$$

0에 대해 대칭.

pdf of $t(r)$.

왜도 X, 첨도 O. //

Figure : pdf of t distribution



As *df* $\to \infty$, a pdf of T distribution approximates a pdf of the standard normal distribution.

population mean.

pop

$N(\mu, \sigma^2) \longrightarrow X_1, \cdots, X_n$: iid.

indep.
(accept).

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$.

$U := \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.

compare.

$$T = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{U/(n-1)}} = \frac{\frac{\bar{X}-\mu}{\cancel{\sigma}/\sqrt{n}}}{\sqrt{S^2/\cancel{\sigma^2}}} = \underline{\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)}$$

☆ 퀴즈출제

* To find a C.I of unknown mean "μ"

1) If σ is ~~un~~known, we use a normal dist^n
2) If σ is unknown, we use a t-dist^n with "s" instead of "σ"

현실적으로.
실제로 μ, σ 를 알 수 없음.

$$g(z,u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{1}{\Gamma(\frac{r}{2}) \cdot 2^{\frac{r}{2}}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}}$$

$t_\alpha(r)$: $100(1-\alpha)$th percentile.

$$P(T \le t_\alpha(r)) = 1-\alpha$$

$$P(Z \le z_\alpha) = 1-\alpha.$$

$E\left[\frac{\bar{X}-\mu}{S/\sqrt{n}}\right]$ $\quad X\sim N(\mu,\sigma^2)$, $\quad \bar{X}\sim N(\mu,\frac{\sigma^2}{n})$.

$E[\bar{X}]=\mu$, $\quad Var[\bar{X}]=\frac{\sigma^2}{n}$

$$= E\left[\frac{\bar{X}-\mu}{(S^2/n)^{\frac{1}{2}}}\right] = \frac{E[\bar{X}-\mu]}{E[(S^2/n)^{\frac{1}{2}}]} = 0.$$

$$Var\left[\frac{\bar{X}-\mu}{S/\sqrt{n}}\right] = E\left[\frac{(\bar{X}-\mu)^2}{S^2/n}\right] = \frac{E[(\bar{X}-\mu)^2]}{E[S^2/n]} = \frac{Var[\bar{X}]}{\sigma^2/n} = \frac{\sigma^2/n}{\sigma^2/n} = 1.$$

$$\frac{E[(\bar{X}-\mu)^2]}{\left[\frac{(n-1)S^2}{\sigma^2}\cdot\frac{\sigma^2}{(n-1)n}\right]} = \frac{Var[\bar{X}] = \frac{\sigma^2}{n}}{\frac{\sigma^2}{\cancel{(n-1)}n}\cdot\cancel{n-1}} = 1.$$

$t(r)$.

Example 5.5-4

Let $t_\alpha(r)$ be the $100(1-\alpha)$th percentile for $t(r)$. That means, $P(T \le t_\alpha(r)) = 1-\alpha$, where $T \sim t(r)$.

We have $T \sim t(11)$.

- Find $P(-1.796 \le T \le 1.796)$
- Find $P(T \le 2.201)$
- Find $P(T \le -1.363)$

cf) In R, pt(t,r): P($T \le t$) where $T \sim t(r)$

See Table 6

Ch5.6 Central Limit Theorem (CLT)

$X \sim ?(\mu, \sigma^2)$ $\longrightarrow$ $X_1, \cdots, X_n$.

$\bar{X} \xrightarrow{n \to \infty} N(\mu, \frac{\sigma^2}{n})$.

$$\bar{X} \sim ?\,(\mu, \frac{\sigma^2}{n})$$

$$Var(\bar{X}) = \frac{\sigma^2}{n} \xrightarrow{n \to \infty} 0.$$

$$W = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \xrightarrow{n \to \infty} N(0, 1).$$

$$E[W] = E\left[\frac{\bar{X} - \mu}{\sigma / \sqrt{n}}\right] = \frac{\mu - \mu}{\sigma / \sqrt{n}} = 0.$$

$$Var[W] = E[W^2] = E\left[\left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}}\right)^2\right]$$

$$= \frac{E[(\bar{X} - \mu)^2]}{\sigma^2 / n} = \frac{\sigma^2 / n}{\sigma^2 / n} = 1$$

pop

?

$$X_1, \cdots, X_n \sim (\mu, \sigma^2).$$

$$\bar{X} \sim ?\,(\mu, \frac{\sigma^2}{n}).$$

As $n \to \infty$

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

$$= \frac{\frac{1}{n}\sum X_i - \mu}{\sigma / \sqrt{n}} = \frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma}$$

Central Limit Theorem (CLT)

Using the results:

- $$\lim_{n\to\infty}\left(1+\frac{b}{n}\right)^n = e^b$$
- Let X_i have the cdf $F_{X_i}(x)$ and the mgf $M_{X_i}(t)$, $i = 1, \ldots, n$. If $\lim_{n\to\infty} M_{X_n}(t) = M_Y(t)$ for all t, there is a r.v Y with the cdf $F_Y(y)$ and the mgf $M_Y(t)$ where $\lim_{n\to\infty} F_{X_n}(x) = F_Y(x)$.

Thm. 5.6-1 (CLT)

Suppose that $X_1, \ldots, X_n$ are observed from the population with finite mean μ and variance σ^2. Let $\overline{X}$ be the sample mean.

$$W = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \to N(0,1) \text{ as } n \to \infty.$$

$X_i \sim F_{X_i}(x)$, $M_{X_i}(t)$ $\quad i=1,\cdots,n$

$\lim_{n\to\infty} M_{X_n}(t) = M_Y(t)$: mgf of Y

$\downarrow n\to\infty$

$\exists Y$ s.t $M_Y(t)$

$\lim_{n\to\infty} F_{X_n}(x) = F_Y(t)$

Thm CLT.

$X_i \sim (\mu, \sigma^2)$ $\quad i=1,\cdots,n$

$$W \overset{let}{=} \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\Sigma X_i - n\mu}{\sqrt{n}\,\sigma} \sim N(0,1) \text{ as } n\to\infty$$

pf) ① mgf of W, $M_W(t)$

② Using Taylor's formula,

$$\lim_{n\to\infty} M_W(t) = M_Y(t) = \exp\{\tfrac{t^2}{2}\}$$

③ Since $M_Y(t)$ is the mgf of $N(0,1)$

the cdf of W approximates a cdf of $N(0,1)$

$\bar{X} \xrightarrow{n\to\infty} N(\mu, \frac{\sigma^2}{n})$

$\downarrow$ 0.

이렇게 증명할 수 없음.

$n\to\infty$ 이기 때문.

1) Using the mgf technique

$$M_W(t) = E[e^{tW}] = E[\exp\{\frac{t(\sum X_i - n\mu)}{\sqrt{n}\sigma}\}]$$

$$= E[\exp\{\frac{\sum_{i=1}^{n} t(X_i-\mu)}{\sqrt{n}\ \sigma}\}]$$

Let $Y_i = \frac{X_i - \mu}{\sigma}$

then $E[Y_i] = 0$, $Var(Y_i) = \frac{1}{\sigma^2} Var(X_i) = 1$

Then $M_W(t) = E[\exp\{\frac{\sum_{i=1}^{n} tY_i}{\sqrt{n}}\}]$

$$= E[\exp\{\frac{tY_1}{\sqrt{n}} + \frac{tY_2}{\sqrt{n}} + \cdots + \frac{tY_n}{\sqrt{n}}\}]$$

X_i : iid

$\rightarrow Y_i$: iid.

$$= \prod_{i=1}^{n} E[\exp\{\frac{tY_i}{\sqrt{n}}\}] = [E[\exp\{\frac{tY_1}{\sqrt{n}}\}]]^n \quad (\because X_i \text{ is iid}).$$

$$= [M_{Y_1}(\frac{t}{\sqrt{n}})]^n = [m(\frac{t}{\sqrt{n}})]^n$$

Here, the mgf of Y_1, $M_{Y_1}(t)$, is

$$m(t) \overset{let}{=} M_{Y_1}(t) = E(e^{tY_1})$$

Since $E(Y_1) = 0$, $Var(Y_1) = E(Y_1^2) = 1$,

second moment.

$m(0) = 1$, $m^{(1)}(0) = E(Y_1) = 0$, $m^{(2)}(0) = E(Y_1^2) = 1$

By Taylor's formula, $\exists\, 0 < t_1 < t$ s.t

$$m(t) = M_{Y_1}(t) = \underbrace{m(0)}_{1} + \underbrace{m^{(1)}(0)}_{0} t + \frac{m^{(2)}(t_1)}{2} t^2 = 1 + \frac{m^{(2)}(t_1)}{2} t^2$$

$$m(t) = 1 + \frac{m^{(2)}(t_1)}{2}t^2 + \frac{t^2}{2} - \frac{t^2}{2} = 1 + \frac{t^2}{2} + \frac{(m^{(2)}(t_1)-1)}{2}t^2$$

Again $\exists\, 0 < t_1 < \frac{t}{\sqrt{n}}$ $\left(\lim_{n\to\infty} t_1 = 0\right)$ s.t

바로 여기로 와요ㅎ

$$m\left(\frac{t}{\sqrt{n}}\right) = m(0) + \underbrace{m^{(1)}(0)\left(\frac{t}{\sqrt{n}}\right)}_{=0} + \frac{m^{(2)}(t_1)}{2}\frac{t^2}{n}$$

$$= 1 + \frac{m^{(2)}(t_1)t^2}{2n} + \frac{t^2}{2n} - \frac{t^2}{2n} = 1 + \frac{t^2}{2n} + \frac{\{m^{(2)}(t_1)-1\}}{2n}t^2$$

2) $\lim_{n\to\infty} M_W(t) = \lim_{n\to\infty} \left\{m\left(\frac{t}{\sqrt{n}}\right)\right\}^n = \lim_{n\to\infty}\left[1 + \frac{t^2}{2n} + \frac{\overbrace{\{m^{(2)}(t_1)-1\}}^{n\to\infty,\ \to 0}t^2}{2n}\right]^n$

where $\lim_{n\to\infty} m^{(2)}(t_1) = m^{(2)}(0) = 1$

$$= \lim_{n\to\infty}\left[1 + \frac{t^2}{2n}\right]^n = \exp\left\{\frac{t^2}{2}\right\} : \text{mgf of } N(0,1)$$

3) Since $\lim_{n\to\infty} M_W(t) = \exp\left\{\frac{t^2}{2}\right\}$ is a mgf of $N(0,1)$,

$W \to N(0,1)$ as $n \to \infty$

(i.e. cdf of W approximate a cdf of $N(0,1)$

$M_W(t) \xrightarrow{n\to\infty} M_Z(t)$, $Z \sim N(0,1)$.

$\Rightarrow$ cdf of $W \xrightarrow{n\to\infty}$ cdf of $N(0,1)$.

$$\bar{X} \stackrel{\cdot}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right).$$

$$\left(\text{from } \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \stackrel{\cdot}{\sim} N(0,1)\right).$$

Example 5.6-1

Let $\overline{X}$ be the mean of a random sample of $n = 25$ currents in a strip of wire in which each measurement has a mean of 15 and a variance of 4. Compute the approximate probability of $P(14.4 < \overline{X} < 15.6)$

Example 5.6-2

Let $X_1, \ldots, X_{20} \sim \mathrm{Unif}(0, 1)$ and $Y = \sum_{i=1}^{20} X_i$. Compute the approximate probabilities of $P(Y \leq 9.1)$ and $P(8.5 \leq Y \leq 11.7)$.

5.6.1

pop. $\longrightarrow$ $X_1, \cdots, X_{25}$

$(15, 2^2)$

$$P(14.4 < \overset{\sim ?}{\bar{X}} < 15.6).$$

cf). $X_i \sim (\mu, \sigma^2)$. iid $i=1,\cdots,n$

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\Sigma x_i - n\mu}{\sqrt{n}\,\sigma} \sim N(0,1).$$

$$\bar{X} \sim ?\ (15, \tfrac{4}{25})$$

$$Z := \frac{\bar{X}-15}{2/5} \dot{\sim} N(0,1)$$

$$P(14.4 < \bar{X} < 15.6)$$
$$= P\left(\frac{14.4-15}{2/5} < \frac{\bar{X}-15}{2/5} < \frac{15.6-15}{2/5}\right)$$
$$\approx P(\bigcirc < Z < \bigcirc), \quad Z \sim N(0,1).$$

5.6.2.

$X_i \sim \text{Unif}(0,1)$.

$Y = \sum_{i=1}^{20} X_i \sim ?(\ ,\ )$.

$E(X_i) = \frac{1}{2}$, $Var(X_i) = \frac{1}{12}$

$\rightarrow \Sigma X_i \sim (n\cdot\mu = 10,\ n\sigma^2 = 20\cdot\frac{1}{12} = \frac{5}{3})$

$Z := \frac{\Sigma X_i - 10}{\sqrt{5/3}} \dot{\sim} N(0,1)$

$P(Y \le 9.1) = P\left(\frac{\Sigma X_i - 10}{\sqrt{5/3}} \le \frac{9.1-10}{\sqrt{5/3}}\right)$

Example 5.6-3

Let $\overline{X}$ be the mean of a random sample of $n = 25$ from the distribution whose pdf is $f(x) = x^3/4,\ 0 < x < 2$. Calculate the approximate probability of $P(1.5 \leq \overline{X} \leq 1.65)$.