

# Topology 2

Notes taken by Junwoo Yang

Based on a lecture by Youngsik Huh in fall 2021

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# Chapter 0

## Introduction

### 0.1 Quotient topology

**Definition 1.** An *equivalence relation* is a relation  $x \sim y$  so that  $x \sim x$ ; if  $x \sim y$  then  $y \sim x$ ; and if  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ . Given an equivalence relation defined on  $X$ ,  $X/\sim$  is the set of *equivalence classes*.

**Definition 2.** Let  $f: X \rightarrow Y$  be a surjective map from the topological space  $X$  to the set  $Y$ . Then, we define a topology on  $Y$ , called the *quotient topology*, by requiring that  $O \subset Y$  be open if and only if  $f^{-1}(O)$  is actually an open set of  $X$ . One checks trivially that this defines a topology on  $Y$ .

**Example.** Let  $X$  be the closed unit ball,  $\{(x, y) : x^2 + y^2 \leq 1\}$ , in  $\mathbb{R}^2$  and  $X^*$  be the partition of  $X$  consisting of all the one-point sets  $\{(x, y)\}$  for which  $x^2 + y^2 < 1$ , along with the set  $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ . Then  $X^*$  is homeomorphic with the subspace of  $\mathbb{R}^3$  called the unit 2-sphere.

### 0.2 What is algebraic topology?

Functor from category of topological spaces to the category of groups.

- Category: set of spaces and morphisms.
- Functor:  $X \rightsquigarrow G_X$  and  $f: X \rightarrow Y \rightsquigarrow f_*: G_X \rightarrow G_Y$  such that
  - $(f \circ g)_* = f_* \circ g_*$
  - $(1_X)_* = 1_{G_X}$

Two systems we'll discuss:

- fundamental groups
- homology groups

**Example.** Suppose we have a functor. If  $G_X \not\cong G_Y$ , then  $X$  and  $Y$  are not homeomorphic. If ‘shadows’ are different, then objects themselves are different too.

**Proof.** Suppose  $X$  and  $Y$  are homeomorphic. Then  $\exists f: X \rightarrow Y$  and  $g: Y \rightarrow X$ , maps (maps are always continuous in this course), such that  $g \circ f = 1_X$  and  $f \circ g = 1_Y$ . Then  $f_*: G_X \rightarrow G_Y$  and  $g_*: G_Y \rightarrow G_X$  such that  $(g \circ f)_* = (1_X)_*$  and  $(f \circ g)_* = (1_Y)_*$ . Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that  $f_*: G_X \rightarrow G_Y$  is an isomorphism.  $\diamond$

### 0.3 Fundamental group

Pick a base point  $x_0$  and consider it fixed. (The fundamental group will not depend on it. We assume all spaces are path connected)  $X \rightsquigarrow \pi(X)$ .

- A loop based at  $x_0 \in X$  is a map  $f: I = [0, 1] \rightarrow X$ ,  $f(0) = f(1) = x_0$ .
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

**Example.** Let  $X = B^2$  (2 dimensional disk). Then  $\pi(B^2) = 1$ , because every loop is equivalent to the ‘constant’ loop.

**Example.** Let  $X = S^1$  and pick  $x_0$  on the circle. Two options:

- The loop is trivial equivalent to the constant loop
- The loop goes around the circle.
- The loop goes around the circle, twice.
- The loop goes around the circle, clockwise, once
- ...

$\pi(S^1) \cong \mathbb{Z}$  (proof will follow)

The composition of loops is simply pasting them. In the case of the circle, the loop  $-1 \circ$  the loop  $2$  is the loop  $1$ .

Suppose  $\alpha: I \rightarrow X$  and  $f: X \rightarrow Y$ . Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

**Theorem 1** (Fixed point theorem of Brouwer). Any continuous map from a rectangle to itself has at least one fixed point.

**Proof.** Suppose there is no fixed point, so  $f(x) \neq x$  for all  $x \in B^2$ . Then we can construct map  $r: B^2 \rightarrow S^1$  as follows: take the intersection of the boundary and half ray between  $f(x)$  and  $x$ . If  $x$  lies on the boundary, we have the identity map. This map is continuous. Then we have  $S^1 \rightarrow B^2 \rightarrow S^1$ , via the inclusion and  $r$ . Looking at the fundamental groups:

$$\pi(S^1) = \mathbb{Z} \rightarrow \pi(B^2) = 1 \rightarrow \pi(S^1) = \mathbb{Z}.$$

The map from  $\pi(S^1) \rightarrow \pi(S^1)$  is the identity map, but the first map maps everything on 1.  $\square$

## Chapter 9

# Fundamental group

### 9.51 Homotopy of paths

**Definition 3.** Let  $f, g: X \rightarrow Y$  be continuous maps. Then a **homotopy** between  $f$  and  $g$  is a continuous map  $H: X \times I \rightarrow Y$  such that

- $H(x, 0) = f(x)$ ,  $H(x, 1) = g(x)$
- For all  $t \in I$ , define  $f_t: X \rightarrow Y$  given by  $x \mapsto H(x, t)$

We say that  $f$  is **homotopic** with  $g$ , we write  $f \simeq g$ . If  $g$  is a constant map, we say that  $f$  is **null homotopic**.

**Definition 4.** Let  $f, g: I \rightarrow X$  be two paths such that  $f(0) = g(0) = x_0$  and  $f(1) = g(1) = x_1$ . Then  $H: I \times I \rightarrow X$  is a **path homotopy** between  $f$  and  $g$ , if and only if

- $H(s, 0) = f(s)$  and  $H(s, 1) = g(s)$  (homotopy between maps)
- $H(0, t) = x_0$  and  $H(1, t) = x_1$  (start and end points fixed)

Notation:  $f \simeq_p g$ .

**Lemma 1.**  $\simeq$  and  $\simeq_p$  are equivalence relations.

**Proof.** • Reflective:  $F(x, t) = f(x)$

- Symmetric:  $G(x, t) = H(x, 1 - t)$
- Transitive: Suppose  $f \simeq g$  and  $g \simeq h$ , with  $H_1, H_2$  resp.

$$H(x, t) = \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

□

**Example (Trivial, but important).** Let  $C \subset \mathbb{R}^n$  be a convex subset.

- Any two maps  $f, g: X \rightarrow C$  are homotopic.
- Any two paths  $f, g: I \rightarrow C$  with  $f(0) = g(0)$  and  $f(1) = g(1)$  are path homotopic.

Choose  $H: X \times I \rightarrow C$  defined by  $(x, t) \mapsto H(x, t) = (1 - t)f(x) + tg(x)$ .

## Product of paths

Let  $f: I \rightarrow X$ ,  $g: I \rightarrow X$  be paths,  $f(1) = g(0)$ . Define

$$f * g: I \rightarrow X \text{ given by } s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

**Remark.** If  $f$  is path homotopic to  $f'$  and  $g$  path homotopic to  $g'$  (which means that  $f(1) = f'(1) = g(0) = g'(0)$ ), then  $f * g \simeq_p f' * g'$ .

So we can define  $[f] * [g] := [f * g]$  with  $[f] := \{g: I \rightarrow X \mid g \simeq_p f\}$ .

- Theorem 2.**
1.  $[f] * ([g] * [h])$  is defined iff  $([f] * [g]) * [h]$  is defined and in that case, they are equal.
  2. Let  $e_x$  denote the constant path  $e_x: I \rightarrow X$  given by  $s \mapsto x$ ,  $x \in X$ . If  $f(0) = x_0$  and  $f(1) = x_1$  then  $[e_{x_0}] * [f] = [f]$  and  $[f] * [e_{x_1}] = [f]$ .
  3. Let  $\bar{f}: I \rightarrow X$  given by  $s \mapsto f(1 - s)$ . Then  $[f] * [\bar{f}] = [e_{x_0}]$  and  $[\bar{f}] * [f] = [e_{x_1}]$ .

**Proof.** First two observations

- Suppose  $f \simeq_p g$  via homotopy  $H$ ,  $f, g: I \rightarrow X$ . Let  $k: X \rightarrow Y$ . Then  $k \circ f \simeq_p k \circ g$  using  $k \circ H$ .
- If  $f * g$  (not necessarily path homotopic). Then  $k \circ (f * g) = (k \circ f) * (k \circ g)$ .

Now, the proof

2. Take  $e_0: I \rightarrow I$  given by  $s \mapsto 0$ . Take  $i: I \rightarrow I$  given by  $s \mapsto s$ . Then  $e_0 * i$  is a path from 0 to  $1 \in I$ . The path  $i$  is also such a path. Because  $I$  is a convex subset,  $e_0 * i$  and  $i$  are path homotopic,  $e_0 * i \simeq_p i$ . Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

3. Note that  $i * \bar{i} \simeq_p e_0$ . Now, applying the same rules, we get

$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \bar{f} &\simeq_p e_{x_0} \\ [f] * [\bar{f}] &= [e_{x_0}]. \end{aligned}$$

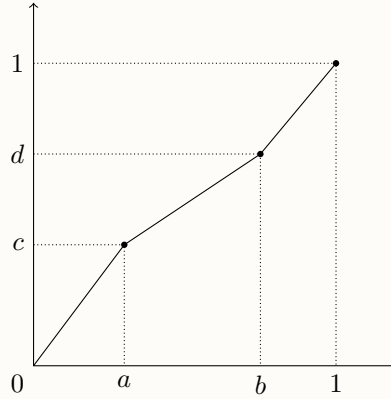
1. Remark: Only defined if  $f(1) = g(0)$ ,  $g(1) = h(0)$ . Note that  $f * (g * h) \neq (f * g) * h$ . The trajectory is the same, but the speed is not.

Assume the product is defined. Suppose  $[a, b]$ ,  $[c, d]$  are intervals in  $\mathbb{R}$ . Then there is a unique positive (positive slope), linear map from  $[a, b] \rightarrow [c, d]$ . For any  $a, b \in [0, 1]$  with  $0 < a < b < 1$ , we define a path

$$\begin{aligned} k_{a,b}: [0, 1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 1] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

Then  $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$  and  $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$ .

Let  $\gamma$  be that path  $\gamma: I \rightarrow I$  with the following graphs:



Note that  $\gamma \simeq_p i$ . Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

which is what we wanted to show. □

## 9.52 Fundamental group



**Definition 5.** Let  $X$  be a space and  $x_0 \in X$ , then the **fundamental group** of  $X$  based at  $x_0$  is

$$\pi(X, x_0) = \{[f] \mid f: I \rightarrow X, f(0) = f(1) = x_0\}.$$

(Also  $\pi_1(X, x_0)$  is used, first homotopy group of  $X$  based at  $x_0$ )

For  $[f], [g] \in \pi(X, x_0)$ ,  $[f] * [g]$  is always defined,  $[e_{x_0}]$  is an identity element,  $*$  is associative and  $[f]^{-1} = [\bar{f}]$ . This makes  $(\pi(X, x_0), *)$  a group.

**Example.** If  $C \subset \mathbb{R}^n$ , convex then  $\pi(X, x_0) = 1$ . E.g.  $\pi(B^2, x_0) = 1$ .

**Remark.** All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

**Theorem 3 (52.1).** Let  $X$  be a space,  $x_0, x_1 \in X$  and  $\alpha: I \rightarrow X$  a path from  $x_0$  to  $x_1$ . Then

$$\begin{aligned} \hat{\alpha}: \pi(X, x_0) &\longrightarrow \pi(X, x_1) \\ [f] &\longmapsto [\bar{\alpha}] * [f] * [\alpha]. \end{aligned}$$

is an isomorphism of groups. Note however that this isomorphism depends on  $\alpha$ .

**Proof.** Let  $[f], [g] \in \pi(X, x_0)$ . Then

$$\begin{aligned} \hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * [f] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}[f] * \hat{\alpha}[g]. \end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic.

□

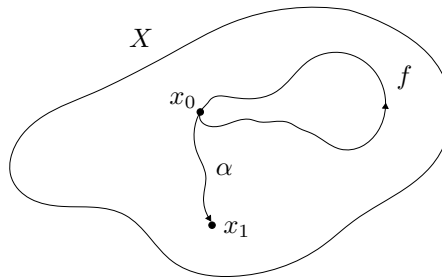


Figure 9.1: Construction of the group homeomorphism

**Remark.** If  $f : (x, x_0) \rightarrow (Y, y_0)$  is a map of pointed topology spaces ( $f : X \rightarrow Y$  continuous and  $f(x_0) = y_0$ ). Then

$$f_* : \pi(X, x_0) \rightarrow \pi(Y, y_0) \text{ given by } [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two ‘rules’ discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \quad (1_X)_* = 1_{\pi(X, x_0)}.$$

**Definition 6.** Let  $X$  be a topological space, then  $X$  is *simply connected* if  $X$  is path connected and  $\pi_1(X, x_0) = 0$  for some  $x_0 \in X$ .

**Remark.** If trivial for one base point, it’s trivial for all base points.

**Example.** Any convex subset  $C \subset \mathbb{R}^n$  is simply connected.

**Example (Wrong proof of  $\pi(S^2)$  being trivial).** Let  $f$  be a path from  $[0, 1] \rightarrow S^2$ . Then pick  $y_0 \in \text{Im}(f)$ . Then  $S^2 \setminus \{y_0\} \approx \mathbb{R}^2$ . Then use  $\mathbb{R}^2$ .

This is wrong because we cannot always find  $y_0 \in \text{Im}(f)$ . Space filling loops! We’ll see the correct proof later on.

**Lemma 2 (52.3).** Suppose  $X$  is simply connected and  $\alpha, \beta : I \rightarrow X$  two paths with same start and end points. Then  $\alpha \simeq_p \beta$ .

**Proof.** Simply connected implies loops are homotopic? Consider  $\alpha * \bar{\beta} \simeq_p e_{x_0}$ , since the space is simply connected.

$$([\alpha] * [\bar{\beta}]) * [\beta] = [e_{x_0}] * [\beta] = [\beta]$$

$$[\alpha] * ([\bar{\beta}] * [\beta]) = [\alpha] * [e_{x_0}] = [\alpha].$$

And therefore  $\alpha \simeq_p \beta$ . (Note: make sure end and start point matches when using  $*$ )  $\square$

## 9.53 Covering spaces

**Definition 7.** Let  $p : E \rightarrow B$  be continuous surjective map. The open set  $U \subset B$  is *evenly covered* if  $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$  with

- $V_\alpha$  open in  $E$
- $V_\alpha \cap V_\beta = \emptyset$  if  $\alpha \neq \beta$
- $p|_{V_\alpha} : V_\alpha \rightarrow U$  is a homeomorphism.

**Remark.** If  $U' \subset U$ , also open and  $U$  is evenly covered, then also  $U'$ .

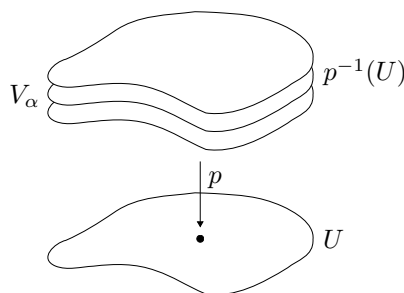


Figure 9.2: Evenly covered

**Definition 8.** Let  $p: E \rightarrow B$  be continuous and surjective. Then  $p$  is a **covering map** if  $\forall b \in B, \exists U \subset B$  open, containing  $b$  such that  $U$  is evenly covered by  $p$ . Then  $(E, p)$  is called a **covering space**.

**Example.** Let  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . Take  $p: \mathbb{R} \rightarrow S^1$  given by  $t \mapsto e^{2\pi it}$ . Note that  $\mathbb{R}$  is an easier space than  $S^1$ , and so will be  $\pi_1$  (1 vs  $\mathbb{Z}$ ).

**Proposition 1.** A covering map is always a open map.

**Proof.** Exercise. □

**Proposition 2.** For any  $b \in B$ ,  $p^{-1}(b)$  is a discrete subset of  $E$ . (No accumulation point)

**Proof.** Indeed for any  $\alpha \in I$ ,  $V_\alpha \cap p^{-1}(b)$  is exactly one point. □

**Remark.** A covering is always local homeomorphism. But there are surjective local homeomorphism which are not covering maps. A covering map is more than a surjective local homeomorphism.

For example,  $p: \mathbb{R}_0^+ \rightarrow S^1$  given by  $t \mapsto e^{2\pi it}$ . Consider the inverse image of a neighborhood around 1. When we restrict  $p$  to the part around 0, it is no longer a homeomorphism (we don't get the whole neighborhood around one).

### Creating new covering spaces out of old ones

- Suppose  $p: E \rightarrow B$  is a covering and  $B_0 \subset B$  is a subspace with the subspace topology. Let  $E_0 = p^{-1}(B_0)$  and  $p_0 = p|_{E_0}$ . Then  $(E_0, p_0)$  is a covering of  $B_0$ .
- Suppose that  $(E, p)$  is a covering of  $B$  and  $(E', p')$  is a covering of  $B'$ , then  $(E \times E', p \times p')$  is a covering of  $B \times B'$ .

**Example.** Let  $T^2 = S^1 \times S^1$ .

- $p: \mathbb{R}^2 \rightarrow S^1 \times S^1$  given by  $(t, s) \mapsto (e^{ait}, e^{bis})$ .
- $p': \mathbb{R} \times S^1 \rightarrow T^2$  given by  $(t, z) \mapsto (e^{ait}, z^n)$ .
- $p: S^1 \times S^1 \rightarrow T^2$  given by  $(z_1, z_2) \mapsto (z_1^n, z_2^m)$ .

These are the only types of coverings of the torus. We'll prove this later on.

## 9.54 Fundamental group of the circle

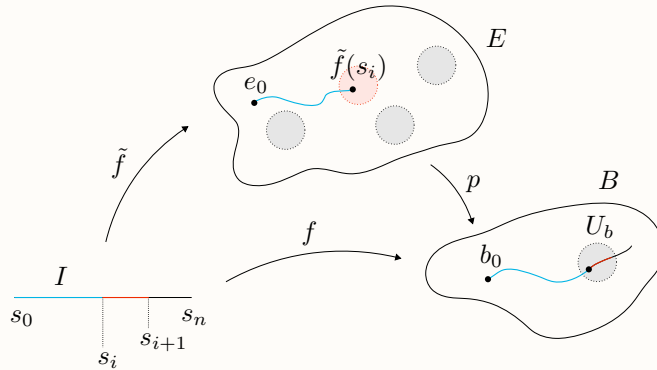
**Definition 9.** Let  $p: E \rightarrow B$  be a map. If  $f$  is a continuous mapping of some space  $X$  into  $B$ , a **lifting** of  $f$  is a map  $\tilde{f}: X \rightarrow E$  such that  $p \circ \tilde{f} = f$ .

$$\begin{array}{ccc} & & E \\ & \nearrow \tilde{f} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Given  $f$ , when can  $f$  be lifted to  $E$ ? In this section, we'll only consider  $X = [0, 1]$ ,  $X = [0, 1]^2$ .

**Lemma 3** (54.1, Important result). Suppose  $(E, p)$  is a covering of  $B$ ,  $b_0 \in B$ ,  $e_0 \in p^{-1}(b_0)$ . Suppose that  $f: I \rightarrow B$  is a path starting at  $b_0$ . Then there exists a unique lift  $\tilde{f}: I \rightarrow E$  of  $f$  with  $\tilde{f}(0) = e_0$ .

**Proof.** For any  $b$  of  $B$ , we choose an open  $U_b$  such that  $U_b$  is evenly covered by  $p$ . Then  $\{f^{-1}(U_b) \mid b \in B\}$  is an open cover of  $I$ , which is compact. There is a number  $\delta > 0$  such that any subset of  $I$  of diameter  $\leq \delta$  is contained entirely in one of these opens  $f^{-1}(U_b)$ . (Lebesgue number lemma). Now, we divide the interval into pieces  $0 = s_0 < s_1 < \dots < s_n = 1$  such that  $|s_{i+1} - s_i| \leq \delta$ . For any  $i$ , we have that  $f([s_i, s_{i+1}]) \subset U_b$  for some  $b$ .



We now construct  $\tilde{f}$  by induction on  $[0, s_i]$ .

- $\tilde{f}(0) = e_0$
- Assume  $\tilde{f}$  has been defined on  $[0, s_i]$ . Let  $U$  be an open such that  $f[s_i, s_{i+1}] \subset U_b$ .  
There is exactly one slice  $V_\alpha$  in  $p^{-1}(U_b)$  containing  $\tilde{f}(s_i)$ . We define  $\forall s \in [s_i, s_{i+1}]: \tilde{f}(s) = (p|_{V_\alpha})^{-1} \circ f(s)$ . By the pasting lemma,  $\tilde{f}$  is continuous.
- In this way, we can construct  $\tilde{f}$  on the whole of  $I$ .

Uniqueness works in exactly the same way, by induction.  $\square$

**Lemma 4** (54.2).  $(E, p)$  is a covering of  $B$ ,  $b_0 \in B$ ,  $e_0 \in E$ , with  $p(e_0) = b_0$ . Suppose  $F: I \times I \rightarrow B$  is a continuous map with  $f(0, 0) = b_0$ , then there is a unique  $\tilde{F}: I \times I \rightarrow E$ . Moreover, if  $F$  is a path homotopy, then also  $\tilde{F}$  is a path homotopy.

**Proof.** Same as in the one dimensional case.  $\square$

**Theorem 4** (54.3). Let  $(E, p)$  be a covering of  $B$ ,  $b_0 \in B$ ,  $e_0 \in E$  with  $p(e_0) = b_0$ . Let  $f, g$  be two paths in  $B$  starting in  $b_0$  s.t.  $f \simeq_p g$  (so  $f$  and  $g$  end at the same point). Let  $\tilde{f}, \tilde{g}$  be the unique lifts of  $f, g$  starting at  $e_0$ . Then  $\tilde{f} \simeq_p \tilde{g}$ , and so  $\tilde{f}(1) = \tilde{g}(1)$ .

**Proof.**  $F: I \times I \rightarrow B$  is a path homotopy between  $f$  and  $g$ . Then  $\tilde{F}: I \times I \rightarrow E$  with  $\tilde{F}(0, 0) = e_0$ . Then  $\tilde{F}$  is a path homotopy, by the previous result, between  $\tilde{F}(\cdot, 0)$  and  $\tilde{F}(\cdot, 1)$ . Note that  $p \circ \tilde{F}(t, 0) = F(t, 0) = f(t)$  and  $p \circ \tilde{F}(t, 1) = F(t, 1) = g(t)$ . By uniqueness  $\tilde{F}(\cdot, 0) = \tilde{f}$ ,  $\tilde{F}(\cdot, 1) = \tilde{g}$ .  $\square$

We've shown that homotopy from below lifts to above. The converse is easy. Now we're ready to discuss the relation between groups and covering spaces.

**Definition 10.** Let  $(E, p)$  be a covering of  $B$ .  $b_0 \in B$ ,  $e_0 \in E$  and  $p(e_0) = b_0$ . Then the **lifting correspondence** is the map

$$\begin{aligned} \phi: \pi(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1) \end{aligned}$$

where  $\tilde{f}$  is the unique lift of  $f$ , starting at  $e_0$ . This is well-defined because  $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$ . This  $\phi$  depends on the choice of  $e_0$ .

**Theorem 5** (54.4). If  $E$  is path connected, then  $\phi$  is a surjective map. If  $E$  is simply connected, then  $\phi$  is a bijective map.

**Proof.** Suppose  $E$  is path connected, and let  $e_0, e_1 \in p^{-1}(b_0)$ . Consider a path  $\tilde{f}: I \rightarrow E$  with  $\tilde{f}(0) = e_0$  and  $\tilde{f}(1) = e_1$ . This is possible because  $E$  is path connected. Let  $f = p \circ \tilde{f}: I \rightarrow B$  with  $f(0) = p(e_0) = b_0$  and

$f(1) = p(e_1) = b_0$ , so  $f$  is a loop based at  $b_0$ . So  $f$  is a loop at  $b_0$  and its unique lift to  $E$  starting at  $e_0$  is  $\tilde{f}$ . Hence  $\phi[f] = \tilde{f}(1) = e_1$ , which shows that  $\phi$  is surjective.

Now assume that  $E$  is simply connected (group is trivial). Consider  $[f], [g] \in \pi(B_0)$  with  $\phi[f] = \phi[g]$ . This implies  $\tilde{f}(1) = \tilde{g}(1)$ . These start at  $e_0$ . It follows from Lemma 2 that  $\tilde{f} \simeq_p \tilde{g}$ .  $\square$

**Example.** Take the circle and the real line as covering space. Then  $p^{-1}(1) = \mathbb{Z}$ . So we know that as a set  $\pi(S^1)$  is countable. Therefore,  $p \circ \tilde{f} \simeq_p p \circ \tilde{g}$ . This implies that  $f \simeq_p g$ , and therefore  $[f] = [g]$ .

**Theorem 6 (54.5).**  $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$ .

**Proof.** Take  $b_0 = 1$  and  $e_0 = 0$  and  $p: \mathbb{R} \rightarrow S^1$  given by  $t \mapsto e^{2\pi it}$ . Then  $p^{-1}(b_0) = \mathbb{Z}$ . And since,  $\mathbb{R}$  is simply connected, we have that  $\pi: \pi(S, 1) \rightarrow \mathbb{Z}$  given by  $[f] \mapsto \tilde{f}(1)$  is a bijection.

Now we'll show that it's a morphism. Let  $[f]$  and  $[g]$  elements of the fundamental group of  $S^1$  and assume that  $\phi[f] = f(1) = m$  and  $\phi[g] = \tilde{g}(1) = n$ .

We're going to prove that  $\phi([f] * [g]) = \phi([f]) + \phi([g]) = n + m$ . Define  $\tilde{g}: I \rightarrow \mathbb{R}$  given by  $t \mapsto \tilde{g}(t) + m$ . Then  $p \circ \tilde{g} = g$ , as  $p(s + m) = p(s)$  for all  $m$ . Now, look at  $\tilde{f} * \tilde{g}$ . This is a lift of  $p \circ (\tilde{f} * \tilde{g}) = (p \circ \tilde{f}) * (p \circ \tilde{g}) = f * g$ , which starts at 0. Hence,  $\phi([f] * [g]) = \phi([f * g])$  = the end point of  $\tilde{f} * \tilde{g}$ , so  $\tilde{g}(1) = \tilde{g}(1) + m = n + m$ .  $\square$

The following lemma makes the fact that the covering space is simpler than the space itself exact.

**Lemma 5 (54.6).** Let  $(E, p)$  be a covering of  $B$ ,  $b_0 \in B$ ,  $e_0 \in E$  and  $p(e_0) = b_0$ . Then

1.  $p_*: \pi(E, e_0) \rightarrow \pi(B, b_0)$  is a monomorphism (injective).
2. Let  $H = p_*(\pi_1(E, e_0))$ . The lifting correspondence induces a well defined map

$$\begin{aligned} \Phi: \pi_1(B, b_0)/H &\longrightarrow p^{-1}(b_0) \\ H * [f] &\longmapsto \phi[f], \end{aligned}$$

so  $\phi$  is constant on right cosets. Dividing by  $H$  makes  $\phi$  always bijective, even when  $E$  is not simply connected.

3. Let  $f$  be a loop based at  $b_0$ , then  $\tilde{f}$  is a loop at  $e_0$  iff  $[f] \in H$ .

**Proof.** 1. Let  $\tilde{f}: I \rightarrow E$  be a loop at  $e_0$  and assume that  $p_*[\tilde{f}] = 1$ . (Then we'd like to show that  $f$  itself is trivial.) This implies  $p \circ \tilde{f} \simeq_p e_{b_0}$ . This implies that  $\tilde{f} \simeq_p \tilde{e}_{b_0} = e_{e_0}$ , or  $[\tilde{f}] = 1$ .

2. We have to prove two things:

**Well defined**  $H * [f] = H[g] \Rightarrow \phi(f) = \phi(g)$ .

Assume  $[f] \in H * [g]$ , or  $H * [f] = H * [g]$ . This means that  $[f] = [h] * [g]$ , where  $h = p \circ \tilde{h}$  for some loop  $\tilde{h}$  at  $e_0$ . In other words  $[f] = [h * g]$ , or  $f \simeq_p h * g$ . Let  $\tilde{f}$  be the unique lift of  $f$  starting at  $e_0$ . Let  $\tilde{g}$  be the unique lift of  $g$  starting at  $e_0$ . Then  $\tilde{h} * \tilde{g}$  (which is allowed,  $\tilde{h}$  is a loop) the unique lift of  $h * g$  starting at  $e_0$ .

$\tilde{f}(1) = \phi(f) = \phi(h * g) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1) = \phi(g)$ . If the cosets are the same, then the end points of the lifts are also the same.

**Injective**  $H * [f] = H * [g] \Leftarrow \phi(f) = \phi(g)$ .

The end points of  $f$  and  $g$  are the same. Now consider  $\tilde{h} = \tilde{f} * \tilde{g}$ . Then  $[\tilde{h}] * [\tilde{g}] = [\tilde{f}] * [\tilde{g}] * [\tilde{g}] = [\tilde{f}]$ . By applying  $p_*$ ,  $[h] * [g] = [f]$ .

3. Trivial. Exercise. Apply 2 with the constant path. □

**Remark.**  $k: X \rightarrow Y$  induces a morphism  $k_*$ , we've proved that earlier. Here we only showed injectiveness.

## 9.55 Retractions and fixed points

**Definition 11.** Let  $A \subset X$ , then  $A$  is a **retract** of  $X$  iff there exists a map  $r: X \rightarrow A$  such that  $r|_A = 1|_A$ , i.e.  $r(a) = a$  for all  $a \in A$ . The map  $r$  is called a **retraction**.

## Chapter 10

# Separation theorems in the plane

### 10.63 Jordan curve theorem

[https://en.wikipedia.org/wiki/Jordan\\_curve\\_theorem](https://en.wikipedia.org/wiki/Jordan_curve_theorem)



## Chapter 11

# Seifert–Van Kampen theorem

[https://en.wikipedia.org/wiki/Seifert%E2%80%93Van\\_Kampen\\_theorem](https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem)

**Note.** This doesn't follow the book very well.

**Definition 12.** A **free group** on a set  $X$  consists of a group  $F_X$  and a map  $i: X \rightarrow F_X$  such that the following holds: For any group  $G$  and any map  $f: X \rightarrow G$ , there exists a unique morphism of groups  $\phi: F_X \rightarrow G$  such that

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow f & \downarrow \exists! \phi \\ & & G \end{array} .$$

**Note.** The free group of a set is unique. Suppose  $i: X \rightarrow F_X$  and  $j: X \rightarrow F'_X$  are free groups.

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow j & \downarrow \exists \phi \\ & & F'_X \end{array} \quad \begin{array}{ccc} X & \xrightarrow{j} & F'_X \\ & \searrow i & \downarrow \exists \psi \\ & & F_X \end{array} .$$

Then

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow i & \downarrow \psi \circ \phi \\ & & F_X \end{array} .$$

Then by uniqueness,  $\psi \circ \phi$  is  $1_{F_X}$ , and likewise for  $\phi \circ \psi$ .

**Note.** The free group on a set always exists. You can construct it using “irreducible words”.

**Example.** Consider  $X = \{a, b\}$ . An example of a word is  $aaba^{-1}baa^{-1}bbb^{-1}a$ . This is not a irreducible word. The reduced form is  $aaba^{-1}bba = a^2ba^{-1}b^2a$ . Then  $F_X$  is the set of irreducible words.

**Example.** If  $X = \{a\}$ , then  $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$ . Exercise: check that  $\mathbb{Z}$  satisfies the universal property.

**Example.** If  $X = \emptyset$ , then  $F_X = 1$ .

**Definition 13.** Let  $G_i$  with  $i \in I$ , be a set of groups. Then the **free product** of these groups denoted by  $*_{i \in I} G_i$  is a group  $G$  together with morphisms  $j_i: G_i \rightarrow G$  such that the following universal property holds: Given any group  $H$  and a collection of morphisms  $f_i: G_i \rightarrow H$ , then there exists a unique morphism  $f: G \rightarrow H$ , such that for all  $i \in I$ , the following diagram commutes:

$$\begin{array}{ccc} G_i & \xrightarrow{j_i} & G \\ & \searrow f_i & \downarrow \exists! f \\ & & H \end{array} .$$

**Note.** Again,  $*_{i \in I} G_i$  is unique.

**Example.** Construction is similar to the construction of a free group. Let  $I = \{1, 2\}$  and  $G_1 = G$ ,  $G_2 = H$ . Then  $G * H$ . Elements of  $G * H$  are “words” of the form  $g_1 h_1 g_2 h_2 g_3$ ,  $g_1 h_1 g_2 h_2$ , or  $h_1 g_1 h_2 g_2 h_3 g_3$  or  $h_1 g_1 h_2$ ,  $\dots$  with  $g_j \in G$ ,  $h_j \in H$ .

**Note.**  $G * H$  is always infinite and nonabelian if  $G \neq 1 \neq H$ . Even if  $G, H$  are very small, for example  $\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, t\} * \{1, s\}$ . Then  $ts \neq st$  and the order of  $ts$  is infinite.

**Note.**  $\mathbb{Z} * \mathbb{Z} = F_{a,b}$ . In general:  $F_X = *_{x \in X} \mathbb{Z}$ .

## 11.70 The Seifert–Van Kampen theorem

**Theorem 7** (70.1, Seifert–Van Kampen theorem). Let  $X = U \cup V$  where  $U, V, U \cap V$  are open and path connected.<sup>a</sup> Let  $x_0 \in U \cap V$ . For any group  $H$  and 2 morphisms  $\Phi_1: \pi(U, x_0) \rightarrow H$  and  $\Phi_2: \pi(V, x_0) \rightarrow H$  such that  $\Phi_1 \circ i_1$  and  $\Phi_2 \circ i_2$ , there exists exactly one  $\Phi: \pi(X, x_0) \rightarrow H$  making the diagram commute

$$\begin{array}{ccccc}
 & & \pi(U, x_0) & & \\
 & \nearrow i_1 & \downarrow j_1 & \searrow \Phi_1 & \\
 \pi(U \cap V, x_0) & \xrightarrow{i} & \pi(x, x_0) & \xrightarrow{\Phi} & H \\
 & \searrow i_2 & \uparrow j_2 & \nearrow \Phi_2 & \\
 & & \pi(V, x_0) & &
 \end{array}$$

$i_1, i_2, i, j_1, j_2$  are induced by inclusions.

<sup>a</sup>Note that  $U, V$  should also be path connected!

**Theorem 8** (70.2, Seifert–Van Kampen theorem (classical version)). Assume the hypotheses of the Theorem 7. Let  $j: \pi(U, x_0) * \pi(V, x_0) \rightarrow \pi(X, x_0)$  (induced by  $j_1$  and  $j_2$ ). On elements of  $\pi(U, x_0)$  it acts like  $j_1$ , on elements of  $\pi(V, x_0)$  it acts like  $j_2$ .

$$\begin{array}{ccc}
 G_1 & & \\
 \downarrow & \searrow f_1 & \\
 G_1 * G_2 & \xrightarrow{f} & H \\
 \uparrow & \nearrow f_2 & \\
 G_2 & &
 \end{array}$$

Then  $j$  is surjective<sup>a</sup> and the kernel of  $j$  is the normal subgroup of  $\pi(U, x_0) * \pi(V, x_0)$  generated by all elements of the form  $i_1(g)^{-1}i_2(g)$ , where  $g \in \pi(U \cap V, x_0)$ .

<sup>a</sup>This is the only place where algebraic topology is used. We've proved this last week. The groups  $U$  and  $V$  generate the whole group. The rest of this theorem follows from the previous theorem.

**Proof.** •  $j$  is surjective. (later)

- Let  $N$  be the normal subgroup generated by  $i_1(g)^{-1}i_2(g)$ . Then we claim that  $N \subset \ker(j)$ . This means we have to show that  $i_1(g)^{-1}i_2(g) \in \ker j$ .  $j(i_1(g)) = j_1(i_1(g))$  by definition of  $j$ . Looking at the diagram, we find that  $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$ . Therefore  $j(i_1(g)^{-1}i_2(g)) = 1$ , which proves that elements of the form  $i_1(g)^{-1}i_2(g)$  are in the kernel.

- Since  $N \subset \ker j$ , there is an induced morphism

$$\begin{aligned}
 k: (\pi_1(U, x_0) * \pi_1(V, x_0))/N &\longrightarrow \pi_1(X, x_0) \\
 gN &\longmapsto j(g).
 \end{aligned}$$

To prove that  $N = \ker j$ , we have to show that  $k$  is injective. Because this would mean that we've divided out the whole kernel of  $j$ .

Now we're ready to use the previous theorem. Let  $H = (\pi(U) * \pi(V))/N$ . Repeating the diagram:

$$\begin{array}{ccccc}
 & & \pi(U, x_0) & & \\
 & \nearrow i_1 & \downarrow j_1 & \searrow \Phi_1 & \\
 \pi(U \cap V, x_0) & \xrightarrow{i} & \pi(x, x_0) & \xrightarrow[\quad k \quad]{\quad \Phi \quad} & H \\
 & \searrow i_2 & \uparrow j_2 & \nearrow \Phi_2 & \\
 & & \pi(V, x_0) & &
 \end{array}$$

Now, we define  $\Phi_1: \pi(U, x_0) \rightarrow H$  given by  $g \mapsto gN$ , and  $\Phi_2: \pi(V, x_0) \rightarrow H$  given by  $g \mapsto gN$ . For the theorem to work, we needed that  $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$ . This is indeed the case: let  $g \in \pi(U \cap V)$ . Then  $\Phi_1(i_1(g)) = i_1(g)N$  and  $\Phi_2(i_2(g)) = i_2(g)N$  and  $i_1(g)N = i_2(g)N$  because  $i_1(g)^{-1}i_2(g) \in N$ .

The conditions of the previous theorem are satisfied, so there exists a  $\Phi$  such that the diagram commutes.

Note that we also have  $k: H \rightarrow \pi(X)$ . We claim that  $\Phi \circ k = 1_H$ , which would mean that  $k$  is injective, concluding the proof. It's enough to prove that

□

**Corollary 8.1.** Suppose  $U \cap V$  is simply connected, so  $\pi_1(U \cap V, x_0)$  is the trivial group. In this case  $N = \ker j = 1$ , hence  $\pi(U, x_0) * \pi(V, x_0) \rightarrow \pi(X, x_0)$  is an isomorphism.

**Corollary 8.2.** Suppose  $U$  is simply connected. Then  $\pi(X, x_0) \cong \pi(V, x_0)/N$  where  $N$  is the normal subgroup generated by the image of  $i_2: \pi(U \cap V) \rightarrow \pi(V, x_0)$ .

**Example.** Let  $X$  be the figure 8 space.

## Chapter 12

# Classification of surfaces

## Chapter 13

# Classification of covering spaces

**Lemma 6** (79.1, General lifting lemma). Let  $p: E \rightarrow B$  be a covering,  $Y$  a space. Assume  $B, E, Y$  are path connected, and locally path connected.<sup>a</sup> Let  $f: Y \rightarrow B$ ,  $y_0 \in Y$ ,  $b_0 = f(y_0)$ . Let  $e_0 \in E$  such that  $p(e_0) = b_0$ . Then  $\exists \tilde{f}: Y \rightarrow E$  with  $\tilde{f}(y_0) = e_0$  and  $p \circ \tilde{f} = f$

$$\begin{array}{ccc} & (E, e_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (Y, y_0) & \xrightarrow{f} & (B, b_0) \end{array}$$

iff  $f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$ . If  $\tilde{f}$  exists then it is unique.

<sup>a</sup>From now on, all spaces are locally path connected: Every neighborhood contains an open that is path connected.

**Proof.** Suppose  $\tilde{f}$  exists. Then  $p \circ \tilde{f} = f$ , so  $(p \circ \tilde{f})_*\pi(Y, y_0) = \pi(Y, y_0)$ . The left hand side is of course  $p_*(\tilde{f}_*(\pi(Y, y_0))) \subset p_*(\pi(E, e_0))$ , so  $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$ .

Conversely, we'll show the uniqueness first. Suppose  $\tilde{f}$  exists.

$p \circ (\tilde{f} \circ \alpha) = f \circ \alpha$ , so  $\tilde{f} \circ \alpha$  is the unique lift of  $f \circ \alpha$  starting at  $e_0$ . Hence  $\tilde{f}(y)$  the endpoint of the unique lift of  $f \circ \alpha$  to  $E$  starting at  $e_0$ .

This also shows how you can define  $\tilde{f}$ : choose a path  $\alpha$  from  $y_0$  to  $y$ . Lift  $f \circ \alpha$  to a path starting at  $e_0$ . Define  $\tilde{f}(y)$  = the end point of this lift. Is this well defined? Is  $\tilde{f}$  continuous?

**Well defined** As  $[\alpha] * [\bar{\beta}] \in \pi(Y, y_0)$ ,

$$f_*([\alpha] * [\bar{\beta}]) = ([f \circ \alpha] * [f \circ \bar{\beta}]) \in f_*(\pi_1(Y, y_0))$$

which is by assumption a subgroup of  $p_*(\pi(E, e_0)) = H$ .

And now, by Lemma 3, a loop in the base space lifts to a loop in  $E$  if the loop is in  $H$ . This lift is of course just  $\gamma * \delta$ , so the end points in

the drawing should be connected! this means that  $\bar{\delta}$  is the lift of  $f \circ \beta$  starting at  $e_0$ , so the endpoint of the lift of  $f \circ \beta$  is the endpoint of the lift of  $f \circ \alpha$ . Therefore  $\tilde{f}(y)$  is well defined.

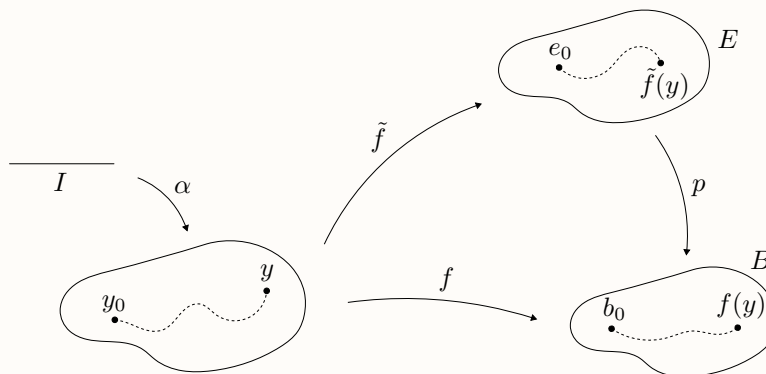


Figure 13.1: General lifting lemma

**Continuity** We prove that  $\tilde{f}$  is continuous.

- Choose a neighborhood of  $\tilde{f}(y_1)$ , say  $N$ .
- Take  $U$ , a path connected open neighborhood of  $f(y_1)$  which is evenly covered, such that the slice  $p^{-1}(U)$  containing  $\tilde{f}(y_1)$  is completely contained in  $N$ .

Can we do this? The inverse image of  $U$  is a pile of pancakes. One of these pancakes contains  $\tilde{f}(y_1)$ . Then, because  $N$  is a neighborhood of  $\tilde{f}(y_1)$ , we can shrink the pancake such that it is contained in  $N$ .

- Choose a path connected open which contains  $y_1$  such that  $f(W) \subset U$ . We can do this because of continuity of  $f$ .
- Take  $y \in W$ . Take a path  $\beta$  in  $W$  from  $y_1$  to  $y$ . (Here we use that  $W$  is path connected.) Now consider  $p|_V$  and defined

□

**Example.** Take  $Y = [0, 1]$ . Then  $f$  is a path, then we showed that every map can be lifted. And indeed, the condition holds:  $f_*(\pi(Y, y_0)) = 1$ , the trivial group, which is a subgroup of all groups.

**Lemma 7** (General lifting lemma, short statement). Short statement:

$$\begin{array}{ccc} & (E, e_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (Y, y_0) & \xrightarrow{f} & (B, b_0) \end{array}$$

$$\exists! \tilde{f} \iff f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0).$$

**Definition 14.** Let  $(E, p)$  and  $(E', p')$  be two coverings of a space  $B$ . An **equivalence** between  $(E, p)$  and  $(E', p')$  is a homeomorphism  $h: E \rightarrow E'$  such that

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ & \searrow p & \downarrow p' \\ & & B \end{array}$$

is commutative.  $p' \circ h = p$ .

**Theorem 9 (79.2).** Let  $p: (E, e_0) \rightarrow (B, b_0)$  and  $p': (E', e'_0) \rightarrow (B, b_0)$  be coverings, and  $H_0 = p_*\pi(E, e_0)$  and  $H'_0 = p'_*\pi(E', e'_0) \leq \pi(B, b_0)$ . Then there exists an equivalence  $h: (E, p) \rightarrow (E', p')$  with  $h(e_0) = e'_0$  iff  $H_0 = H'_0$ . Not isomorphic, but really the same as a subgroup of  $\pi(B, b_0)$ . In that case,  $h$  is unique.

**Proof.**  $\Rightarrow$  Suppose  $h$  exists. Then

$$\begin{array}{ccc} (E, e_0) & \xrightarrow{h} & (E', e'_0) \\ & \searrow p & \downarrow p' \\ & & (B, b_0) \end{array}$$

Therefore  $p_*\pi(E, e_0) = p'_*(h_*\pi(E, e_0))$ . Since  $h$  is a homeomorphism, it induces an isomorphism, so  $p'_*(h_*\pi(E, e_0)) = p'_*(\pi(E', e'_0))$ .

$\Leftarrow$

$$\begin{array}{ccc} & & (E', e'_0) \\ & \nearrow k & \downarrow p' \\ (E, e_0) & \xrightarrow{p} & (B, b_0) \end{array}$$

By the previous lemma, there exists a unique  $k$  iff  $p_*\pi(E, e_0) \subset p'_*\pi(E', e'_0)$  or equivalently  $H_0 \subset H'_0$ , which is the case. Reversing the roles, we get

$$\begin{array}{ccc} & & (E, e_0) \\ & \nearrow l & \downarrow p \\ (E', e'_0) & \xrightarrow{p'} & (B, b_0) \end{array}$$

for the same reasoning,  $l$  exists. Now, composing the diagrams

$$\begin{array}{ccc} & & (E, e_0) \\ & \nearrow l \circ k & \downarrow p \\ (E, e_0) & \xrightarrow{p} & (B, b_0) \end{array} \quad \begin{array}{ccc} & & (E', e'_0) \\ & \nearrow k \circ l & \downarrow p' \\ (E', e'_0) & \xrightarrow{p'} & (B, b_0) \end{array}$$

But placing the identity in place of  $l \circ k$  or  $k \circ l$ , this diagram also commutes! By unicity, we have that  $l \circ k = 1_E$  and  $k \circ l = 1_{E'}$ . Therefore,  $k$  and  $l$  are homeomorphism  $k(e_0) = e'_0$ .



Uniqueness is trivial, because of the general lifting theorem.  $\square$

Note that this doesn't answer the question 'is there an equivalence between two coverings', it only answers the question 'is there an equivalence between two coverings mapping  $e_0 \rightarrow e'_0$ '. So now, we seek to understand the dependence of  $H_0$  on the base point.

**Lemma 8 (79.3).** Let  $(E, p)$  be a covering of  $B$ . Let  $e_0, e_1 \in p^{-1}(b_0)$ . Let  $H_0 = p_*\pi(E, e_0)$ ,  $H_1 = p_*\pi(E, e_1)$ .

- Let  $\gamma$  be a path from  $e_0$  to  $e_1$  and let  $p \circ \gamma = \alpha$  be the induced *loop* at  $b_0$ . Then  $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$ , so  $H_0$  and  $H_1$  are conjugate inside  $\pi(B, b_0)$ .
- Let  $H$  be a subgroup of  $\pi(B, b_0)$  which is conjugate to  $H_0$ , then there is a point  $e \in p^{-1}(b_0)$  such that  $H = p_*\pi(E, e)$ .

So a covering space induces a conjugacy class of a subgroup of  $\pi(B, b_0)$ .

This completely answers the question: when are two covering spaces equivalent?

**Corollary 9.1.** Let  $(E, p)$  and  $(E', p')$  be two coverings,  $e_0 \in E$ ,  $e'_0 \in E'$  with  $p(e_0) = p'(e'_0) = b_0$ . Let  $H_0 = p_*\pi(E, e_0)$ ,  $H'_0 = p'_*\pi(E', e'_0)$ . Then  $(E, p)$  and  $(E', p')$  are equivalent iff  $H_0$  and  $H'_0$  are conjugate inside  $\pi(B, b_0)$ .

Question: can we reach every possible subgroup? Answer: yes, in some conditions.

## 13.80 Universal covering space

**Definition 15.** Let  $B$  be a path connected and locally path connected space. A covering space  $(E, p)$  of  $B$  is called a **universal covering space** if  $E$  is simply connected, so  $\pi(E, e_0) = 1$ .