Linear Algebra I – Final Exam 1

Junwoo Yang

May 10, 2018

1. Mark each of the following true or false.

   (1) If U is a reduced row echelon form of A, then $C(A) = C(U)$ where $C(A)$ and $C(U)$ are column spaces of A and U.

   (2) If $A \sim B$, then $rank(A) = rank(B)$.

   (3) There is a 2×3 matrix A such that $A^{\mathrm{T}}A$ is invertible.

   (4) $\dim(\mathcal{L}(R^2, R^4)) = 6$.

   (5) Let $W = \{(x_1, x_2, x_3, x_4) \,|\, x_1 - 2x_2 + 3x_3 = 0\}$, then $\dim(W) = 2$.

2. Let $B = PA$ where P is an invertible matrix and B is a reduced row echelon form of nonzero matrix $A \in \mathrm{Mat}_{m\times n}$. Prove or disprove that P is unique.

3. Let A be a 4×6 matrix with $rank(A) = 4$. Prove or disprove that $AX = B$ has always infinitely many solutions for any 4×1 matrix B.

4. Prove or disprove that if $A^2 = A$ and $A \neq 0$, then $AX = 0$ has a unique solution.

5. Let $AB = I_n$ for $n \times n$ matrices A and B. Show that $BA = I_n$.

6. Find a rank of A as a function of x : $A = \begin{pmatrix} 2 & 2 & -6 & 8 \\ 3 & 3 & -9 & 8 \\ 1 & 1 & x & 4 \end{pmatrix}$.

7. Let $[T]_\alpha^\beta = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix}$ for $T \in \mathcal{L}(V, W)$ where $\alpha = \{v_1, v_2, v_3, v_4\}$ is a basis of V and $\beta = \{w_1, w_2, w_3\}$ is a basis of W. Find a basis of $\mathrm{im}(T)$ and $\ker(T)$.

8. Let the reduced row echelon form of A be $\begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$. Determine A if the first, second, and fourth columns of A are $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, respectively.

9. Let $\alpha = \{1, x, x^2\}$ be an ordered basis of $P_2(R)$ and $\beta = \{e_1, e_1 + e_2, e_1 + e_2 + e_3\}$ be an ordered basis of R^3. If $T \in \mathcal{L}(P_2(R), R^3)$ is defined as $T(f) = \left[f + f' + f^{(2)}\right]_\alpha$.

   (1) Show that T is an isomorphism.

   (2) Find $[T]_\alpha^\beta$ and $\left[T^{-1}\right]_\beta^\alpha$.

   (3) Find $\left[T^{-1}(a, b, c)\right]_\alpha$.

10. Let $v_1 = (1, 3, -2, 2, 3)$, $v_2 = (1, 4, -3, 4, 2)$, $v_3 = (1, 3, 0, 2, 3)$, $w_1 = (2, 3, -1, -2, 9)$, $w_2 = (1, 5, -6, 6, 1)$, $w_3 = (2, 4, 4, 2, 8)$. For $V = \langle v_1, v_2, v_3\rangle$ and $W = \langle w_1, w_2, w_3\rangle$, find a basis of $V \cap W$.