미분방정식 HW1-1

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양 준 우

1.2-7

a. $\frac{dp}{dt} = 0.5p - 450$

$p' = \frac{p-900}{2}$

$\frac{p'}{p-900} = \frac{1}{2}$

$(\ln|p-900|)' = \frac{1}{2}$

$\ln|p-900| = \frac{t}{2} + c$

$|p-900| = e^{\frac{t}{2}+c}$

$p - 900 = \pm e^c \cdot e^{\frac{t}{2}}$

$p(t) = c \cdot e^{\frac{t}{2}} + 900$

$p(0) = 800.$

$800 = c + 900. \quad \therefore c = -100.$

$\therefore p(t) = 900 - 100e^{\frac{t}{2}}$

We want to find t_* s.t $p(t_*) = 0$

$900 - 100e^{\frac{t_*}{2}} = 0.$

$e^{\frac{t_*}{2}} = 9.$

$\frac{t_*}{2} = \ln 9$

$\therefore t_* = 2\ln 9$

b. $p(0) = p_0 = c + 900 \quad c = p_0 - 900$

$p(t) = (p_0 - 900)e^{\frac{t}{2}} + 900 \quad (0 < p_0 < 900).$

$p(t_*) = (p_0 - 900)e^{\frac{t_*}{2}} + 900 = 0$

$e^{\frac{t_*}{2}} = \frac{900}{900 - p_0}$

$t_* = 2\ln\left(\frac{900}{900-p_0}\right)$

c. $t_* = 12. \quad p(12) = ce^6 + 900 = 0. \quad c = \frac{-900}{e^6}$

$p(t) = -\frac{900}{e^6} \cdot e^{\frac{t}{2}} + 900$

$p(0) = p_0 = -\frac{900}{e^6} + 900 = 900(1 - e^{-6})$

2.1 - 11

$y' + \frac{3}{t}y = \frac{\cos t}{t^3}$, $y(\pi) = 0$, $t > 0$.

$\because$ linear D.E. By using integrating factor

$\mu(t) = e^{\int \frac{3}{t} dt} = e^{3\ln|t|} = |t|^3 = t^3$ $(\because t > 0)$

$t^3 y' + 3t^2 y = \cos t$

$(t^3 y)' = \cos t$, $t^3 y = \sin t$, $\therefore y = \frac{\sin t}{t^3}$

2.1 - 16

b. $ty' + (t+1)y = 4te^{-t}$, $y(1) = a$, $t > 0$

$y' + \frac{t+1}{t}y = 4e^{-t}$

$\mu(t) = e^{\int \frac{t+1}{t} dt} = e^{\int (1+\frac{1}{t}) dt} = e^{t + \ln|t|} = e^t \cdot e^{\ln|t|} = te^t$ $(\because t > 0)$.

$te^t y' + (te^t + e^t)y = 4t$

$(te^t y)' = 4t$

$te^t y = 2t^2 + c$, $y = \frac{2t}{e^t} + \frac{c}{te^t}$

$y(1) = \frac{2+c}{e} = a$ $\therefore c = ae - 2$

$y = \frac{2t}{e^t} + \frac{ae-2}{te^t}$

$a_0 e - 2 = 0$, $a_0 = \frac{2}{e}$

c. $a = a_0 = \frac{2}{e}$

$y = \frac{2t}{e^t}$

$\lim_{t\to\infty} \frac{2t}{e^t} = \lim_{t\to\infty} \frac{2}{e^t} = 0$ (by L'Hôpital's rule).

$\lim_{t\to 0} \frac{2t}{e^t} = 0$.

2.1 - 17

b. $(\sin t)y' + (\cos t)y = 2e^t$, $y(1) = a$, $0 < t < \pi$

$y' + \cot t\, y = \frac{2e^t}{\sin t}$

$\mu(t) = e^{\int \cot t\, dt} = e^{\ln|\sin t|} = |\sin t| = \sin t$ $(\because 0 < t < \pi)$.

$\sin t\, y' + \cos t\, y = 2e^t$

$(\sin t\, y)' = 2e^t$

$\sin t \, y = \int 2e^t dt = 2e^t + C$

$\therefore y = \frac{2e^t}{\sin t} + \frac{C}{\sin t} \quad (0 < t < \pi).$

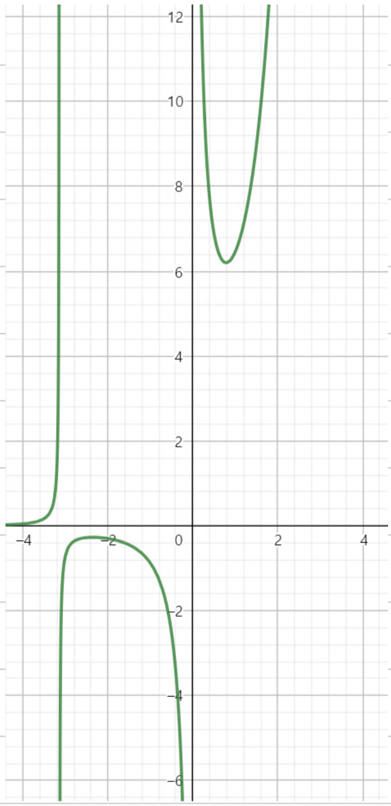
$y(1) = a = \frac{2e + C}{\sin 1} \qquad \therefore C = a\sin 1 - 2e$

$\therefore y = \frac{1}{\sin t}(2e^t + a\sin 1 - 2e)$

$\therefore y = \frac{2e^t}{\sin t} + (a\sin 1 - 2e)\frac{1}{\sin t}$

$a_0 \sin 1 - 2e = 0$

$\therefore a_0 = \frac{2e}{\sin 1}$

$y = \frac{2e^t}{\sin t} \Rightarrow$ 

C. $a = a_0 = \frac{2e}{\sin 1}$

$y = \frac{2e^t}{\sin t}$

2.2 - 21

a. $y' = \frac{ty(4-y)}{3}, \qquad y(0) = y_0$

$= \frac{t}{3} \cdot y(4-y)$: separable

$\frac{y'}{y(4-y)} = \frac{t}{3} \Rightarrow \int \frac{1}{y(4-y)} dy = \int \frac{t}{3} dt$

$\frac{1}{y(4-y)} = \frac{a}{y} + \frac{b}{(4-y)} \qquad a(4-y) + by = 1$

$= \frac{1}{4y} + \frac{1}{4(4-y)} \qquad 4a - ay + by = 1 \qquad a = b.$

$4a = 1. \quad a = \frac{1}{4}$

$\int \frac{1}{y(4-y)} dy = \int \frac{1}{4y} dy + \int \frac{1}{4(4-y)} dy = \frac{1}{4}\ln y - \frac{1}{4}\ln(4-y)$

$= \int \frac{t}{3} dt = \frac{1}{6}t^2 + C$

$\frac{1}{4}\ln \frac{y}{4-y} = \frac{1}{6}t^2 + C$

$\ln \frac{y}{4-y} = \frac{2}{3}t^2 + C$

$\frac{y}{4-y} = C \cdot e^{\frac{2}{3}t^2}$ $\quad y(0) = y_0$

$\frac{y_0}{4-y_0} = C$

$\therefore \frac{y}{4-y} = \frac{y_0}{4-y_0} \cdot e^{\frac{2}{3}t^2}$

$y = (4-y)\frac{y_0}{4-y_0} e^{\frac{2}{3}t^2}$

$= 4\left(\frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}\right) - y\left(\frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}\right)$

$y\left(1 + \frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}\right) = 4\left(\frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}\right).$

$\therefore y = \frac{4\left(\frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}\right)}{1 + \frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}}$

$$y = \frac{4\left(\frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}\right)}{1+\frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}}$$

$$= \frac{4}{\frac{1}{\frac{y_0}{4-y_0}e^{\frac{2}{3}t^2}}+1}$$

If $y_0 > 0$.

$$\frac{4}{\frac{1}{\frac{+}{4-y_0}\infty}+1} \to 4.$$

If $y_0 \to 0$.

$$\frac{4}{\frac{1}{\frac{0}{4-0}\infty}+1} \to \frac{4}{\infty+1} \to 0.$$

If $y_0 < 0$.

$$\frac{4}{\frac{1}{\frac{-}{4+}\infty}+1} \to 4.$$

If $y_0 > 0$, $y \to 4$.
$y_0 \to 0$, $y \to 0$
$y_0 < 0$, $y \to 4$.

2.2-30

b. $\frac{dy}{dx} = \frac{3y^2-2x^2}{2xy}$

$$y' = f(x,y) = \frac{3y^2-2x^2}{2xy}$$

$$f(\lambda x, \lambda y) = \frac{3(\lambda y)^2 - 2(\lambda x)^2}{2(\lambda x)(\lambda y)} = \frac{\lambda^2(3y^2-2x^2)}{2\lambda^2 xy} = \frac{3y^2-2x^2}{2xy} = f(x,y).$$

$\therefore$ homogeneous D.E. of order 0.

$f(\lambda x, \lambda y) = \lambda^0 f(x,y)$. $\quad f(x,y) = f\left(\frac{y}{x}\right)$.

$$y = xv. \qquad \frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{3(xv)^2 - 2x^2}{2x(xv)}$$

$$v + x\frac{dv}{dx} = \frac{3x^2v^2 - 2x^2}{2x^2v} = \frac{3v^2-2}{2v}$$

$$x\frac{dv}{dx} = \frac{3v^2-2-2v^2}{2v} = \frac{v^2-2}{2v}$$

$$\frac{2v}{v^2-2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{2v}{v^2-2}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|v^2-2| = \ln|x| + c$$

$$\ln\left|\frac{v^2-2}{x}\right| = c$$

$$\frac{v^2-2}{x} = e^c = c$$

$$v^2-2 = cx$$

$$v^2 = cx+2$$

$$v(x) = \sqrt{cx+2} \qquad y = vx \qquad v = \frac{y}{x}$$

$$\frac{y}{x} = \sqrt{cx+2}$$

$$\therefore y = x\sqrt{cx+2}$$

2.4-24

$$t^2y' + 2ty - y^3 = 0, \quad t>0$$

$$y' + \frac{2}{t}y = \frac{1}{t^2}y^3$$

$$y^{-3}y' + \frac{2}{t}y^{-2} = \frac{1}{t^2}$$

Let $u = y^{-2}$. $du = -2y^{-3}dy \qquad \frac{du}{dt} = -2y^{-3}\frac{dy}{dt}, \quad \frac{dy}{dt} = -\frac{y^3}{2}\frac{du}{dt}.$

$$y^{-3}\cdot -\frac{y^3}{2}\frac{du}{dt} + \frac{2}{t}u = \frac{1}{t^2}$$

$$\frac{du}{dt} - \frac{4}{t}u = \frac{-2}{t^2}$$

$$u' - \frac{4}{t}u = -\frac{2}{t^2}$$

$$\mu(t) = e^{\int -\frac{4}{t}dt} = e^{-4\ln|t|} = t^{-4}$$

$$(t^{-4}u)' = -\frac{2}{t^2}\cdot t^{-4} = -2t^{-6}$$

$$t^{-4}u = \int -2t^{-6} = \frac{2}{5}t^{-5} + C$$

$$u = t^4\cdot\frac{2}{5}t^{-5} + ct^4 = \frac{2}{5t} + ct^4$$

$$y^{-2} = \frac{2}{5t} + ct^4 = \frac{2+ct^5}{5t}$$

$$y^2 = \frac{5t}{2+ct^5}$$

$$\therefore y = \pm\sqrt{\frac{5t}{2+ct^5}}$$

2.4 - 26.

$y' + 3y = g(t)$, $y(0) = 0$. $\quad g(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$

$\mu(t) \cdot y' + 3\mu(t) y = \mu(t) g(t)$.

$\mu(t) = e^{\int 3 dt} = e^{3t}$

$e^{3t} y' + 3e^{3t} y = e^{3t} g(t)$.

$(e^{3t} y)' = e^{3t} g(t)$.

$e^{3t} y = \int e^{3t} g(t)\, dt$.

$(0 \le t \le 1)$.

$e^{3t} y = \frac{1}{3} e^{3t} + C$

$y = \frac{1}{3} + Ce^{-3t}$

$y(0) = 0. \Rightarrow C = -\frac{1}{3}$

$y = \frac{1}{3} - \frac{1}{3} e^{-3t} = \frac{1}{3}(1 - e^{-3t})$

$(t > 1)$

$e^{3t} y = C$

$y = ce^{-3t}$

$y(1^-) = \frac{1}{3}(1 - e^{3(1)}) = \frac{1}{3}(1 - e^{3})$ $(0 \le t \le 1)$.

$y(1^+) = ce^{-3(1)} = ce^{-3}$ $(t > 1)$.

$\frac{1}{3}(1 - e^{3}) = ce^{-3}$

$c = \frac{1}{3}(e^{3} - 1)$

$y = \frac{1}{3}(e^{3} - 1) e^{-3}$