Homework 9

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Problem 3-3.5 Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, \ v - \frac{v^3}{3} + vu^2, \ u^2 - v^2\right)$$

and show that

(a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

(b) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

(c) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

(d) The lines of curvature are the coordinate curves.

(e) The asymptotic curves are $u + v = \text{const.}, \ u - v = \text{const.}$

Proof. (a) $\mathbf{x}_u = (1 - u^2 + v^2, \ 2uv, \ 2u)$ and $\mathbf{x}_v = (2uv, \ 1 - v^2 + u^2, \ -2v)$. Therefore,

$$\begin{aligned}
E = \mathbf{x}_u \cdot \mathbf{x}_u &= (1 - u^2 + v^2)^2 + (2uv)^2 + (2u)^2 \\
&= 1 + u^4 + v^4 - 2u^2 + 2v^2 - 2u^2v^2 + 4u^2v^2 + 4u^2 \\
&= 1 + u^4 + v^4 + 2u^2 + 2v^2 + 2u^2v^2 = (1 + u^2 + v^2)^2 \\
F = \mathbf{x}_u \cdot \mathbf{x}_v &= (1 - u^2 + v^2) \cdot 2uv + 2uv(1 - v^2 + u^2) + 2u \cdot (-2v) \\
&= 4uv - 4uv = 0 \\
G = \mathbf{x}_v \cdot \mathbf{x}_v &= (2uv)^2 + (1 - v^2 + u^2)^2 + (-2v)^2 = \cdots = E
\end{aligned}$$

(b) We calculate

$$\begin{aligned}
\mathbf{x}_u \wedge \mathbf{x}_v &= \begin{vmatrix} i & j & k \\ 1 - u^2 + v^2 & 2uv & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \end{vmatrix} \\
&= (-4uv^2 - 2u + 2uv^2 - 2u^3)i + (4u^2v + 2v - 2u^2v + 2v^3)j \\
&\quad + (1 - v^2 + u^2 - u^2 + u^2v^2 - u^4 + v^2 - v^4 + v^2u^2 - 4u^2v^2)k \\
&= (-2u(u^2 + v^2 + 1), \ 2v(u^2 + v^2 + 1), \ 1 - (u^2 + v^2)^2) \\
|\mathbf{x}_u \wedge \mathbf{x}_v|^2 &= 4u^2(u^2 + v^2 + 1)^2 + 4v^2(u^2 + v^2 + 1)^2 + (1 - (u^2 + v^2)^2)^2 \\
&= (u^2 + v^2 + 1)^4.
\end{aligned}$$

Thus,

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{(-2u,\ 2v,\ 1 - u^2 - v^2)}{u^2 + v^2 + 1}.$$

Since $\mathbf{x}_{uu} = (-2u,\ 2v,\ 2)$, $\mathbf{x}_{uv} = (2v,\ 2u,\ 0)$, and $\mathbf{x}_{vv} = (2u,\ -2v,\ -2)$,

$$e = \langle N, \mathbf{x}_{uu}\rangle = \frac{4u^2 + 4v^2 + 2(1 - u^2 - v^2)}{u^2 + v^2 + 1} = \frac{2(1 + u^2 + v^2)}{u^2 + v^2 + 1} = 2$$

$$f = \langle N, \mathbf{x}_{uv}\rangle = \frac{-4uv + 4uv + 0 \cdot (1 - u^2 - v^2)}{u^2 + v^2 + 1} = 0$$

$$g = \langle N, \mathbf{x}_{vv}\rangle = \frac{-4u^2 - 4v^2 - 2(1 - u^2 - v^2)}{u^2 + v^2 + 1} = \frac{-2(1 + u^2 + v^2)}{u^2 + v^2 + 1} = -2$$

(c) We first calculate

$$K = \frac{eg - f^2}{EG - F^2} = \frac{2 \cdot (-2) - 0^2}{(1 + u^2 + v^2)^4 - 0^2} = \frac{-4}{(1 + u^2 + v^2)^4}$$

$$H = \frac{gE - 2fF + eG}{2(EG - F^2)} = \frac{(1 + u^2 + v^2)^2 \cdot 2 - 2 \cdot 0 \cdot 0 + (1 + u^2 + v^2)^2(-2)}{2(1 + u^2 + v^2)^4} = 0$$

Specially, the Enneper surface is minimal. Now from the relations

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2},$$

we get a system of equations for k_1 and k_2. From $H = 0$ we get $k_2 = -k_1$. Now

$$K = -k_1^2 \Rightarrow \frac{-4}{(1 + u^2 + v^2)^4} = -k_1^2 = k_1 = \frac{\pm 2}{(1 + u^2 + v^2)^2}$$

and therefore

$$k_2 = -k_1 = \frac{\mp 2}{(1 + u^2 + v^2)^2}$$

Since the principal curvatures k_1 and k_2 can be switched (the order doesn't matter), we actually have that

$$k_1 = \frac{2}{(1 + u^2 + v^2)}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

(d) The curve $c(t) = \mathbf{x}(u(t),\ v(t))$, which lies on S, is a line of curvature if and only if u and v satisfy the differential equation

$$\begin{vmatrix} v'^2 & -u'v & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

Substituting the values for the Enneper surface we get

$$0 = \begin{vmatrix} v'^2 & -u'v' & u'^2 \\ (1 + u^2 + v^2)^2 & 0 & (1 + u^2 + v^2)^2 \\ 2 & 0 & -2 \end{vmatrix} = (\text{Lapl. exp. 2nd col.})$$

$$= (-1)^4(-u'v')\begin{vmatrix} v'^2 & u'^2 \\ 2 & -2 \end{vmatrix} = -u'v'(-2v'^2 - 2u'^2) = 2u'v'(u'^2 + v'^2)$$

which is equivalent to $u' = 0$ or $v' = 0$ or $u'^2 + v'^2 = 0$. The last case implies $u' = v' = 0$, which is covered by first two cases. Now we have that $j' = 0 \Rightarrow u = \text{const.}$ or $v' = 0 \Rightarrow$

$v = $ const., which are exactly the coordinate curves.

(e) The curve c is an asymproric curve if and only if u and v satisfy the differential equation

$$eu'^2 + 2fu'v' + gv'^2 = 0$$

Substituting the values for the Enneper surface we get

$$2u'^2 - 2v'^2 = 0 \Rightarrow 2(u' - v')(u' + v') = 0 \Rightarrow u' \pm v' = 0 \Rightarrow u \pm v = \text{const.}$$

$\square$

Claim 1 Fix $p \in S$. Choose a unit normal vector n at p. Let $h\colon S \to \mathbb{R}$ s.t. $q \mapsto \langle q - p, n \rangle$ be the height function relative to p with n. Show that $q \in S$ is a critical point of h if and only if $n \perp T_q S$ (i.e., $n = \pm N(q)$).

Proof. For $q \in S$ and for any $v \in T_q S$, choose $\alpha\colon (-\varepsilon, \varepsilon) \to S$ s.t. $\alpha(0) = q$ and $\alpha'(0) = v$. Then,

$$dh_q(v) = \frac{d}{dt}\Big|_{t=0} h(\alpha(t)) = \frac{d}{dt}\Big|_{t=0} \langle \alpha(t) - p, n \rangle = \langle \alpha'(0), n \rangle = \langle v, n \rangle,$$

and TFAE:

$$q \text{ is a critical point} \Leftrightarrow dh_q \text{ is not surjective}$$
$$\Leftrightarrow dh_q \equiv 0$$
$$\Leftrightarrow dh_q(v) = \langle v, n \rangle = 0 \text{ for all } v \in T_q S$$
$$\Leftrightarrow v \perp n \text{ for all } v \in T_q S$$
$$\Leftrightarrow n \perp T_q S$$

$\square$

Claim 2 There is no compact regular surface with negative Gauss curvature everywhere.

Proof. Suppose not. Let S be a compact regular surface with negative Gauss curvature everywhere. Choose any point $p \in S$. Then, $h\colon S \to \mathbb{R}$ s.t. $q \mapsto \langle q - p, n \rangle$ is continuous. Since S is compact, $h(S)$ is also compact, i.e. $h(S)$ is closed and bounded. Then, $h(S)$ has a maximum with some point $q \in S$ s.t. $h(q) = \max_{q' \in S} \langle q' - p, n \rangle$. Hence, for any $\alpha\colon (-\varepsilon, \varepsilon)$ s.t. $\alpha(0) = q$ and $\alpha'(0) = v \in T_q S$, $dh_q(v) = \frac{d}{dt}\big|_{t=0} h(\alpha(t)) = 0$. Thus, $dh_q \equiv 0$, so q is a critical point of h. By Claim 1, $n \perp T_q S$. Since q is a hyperbolic point, by Proposition 1, in each neighborhood of q there exist points $q_1, q_2 \in S$ in both sides of $T_p(S)$. Then,

$$h(q_i) = \langle q_i - p, n \rangle = \langle q_i - q, n \rangle + \langle q - p, n \rangle = h(q) + \langle q_i - q, n \rangle \quad \text{for } i = 1, 2.$$

Since $n \perp T_p S$ and q_1, q_2 are in both sides of $T_p(S)$, signs of $\langle q_1 - q, n \rangle$ and $\langle q_2 - q, n \rangle$ are different. WLOG, assume $\langle q_1 - q, n \rangle > 0$ and $\langle q_2 - q, n \rangle < 0$. Then, $h(q_1) = h(q) + \langle q_1 - q, n \rangle > h(q)$ and it is a contradiction to the maximality of $h(q)$. Therefore, there is no compact regular surface with negative Gauss curvature everywhere. $\square$