Mathematical Statistics 1

Ch.4 Bivariate Distributions

Jungsoon Choi

jungsoonchoi@hanyang.ac.kr

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2. Correlation Coefficient

Definition

- The covariance of X and Y is defined by

$$\begin{aligned}\sigma_{XY} = Cov(X, Y) &= E\left[(X - \mu_X)(Y - \mu_Y)\right] \\ &= E[XY] - \mu_X \mu_Y\end{aligned}$$

- The correlation of X and Y is defined by

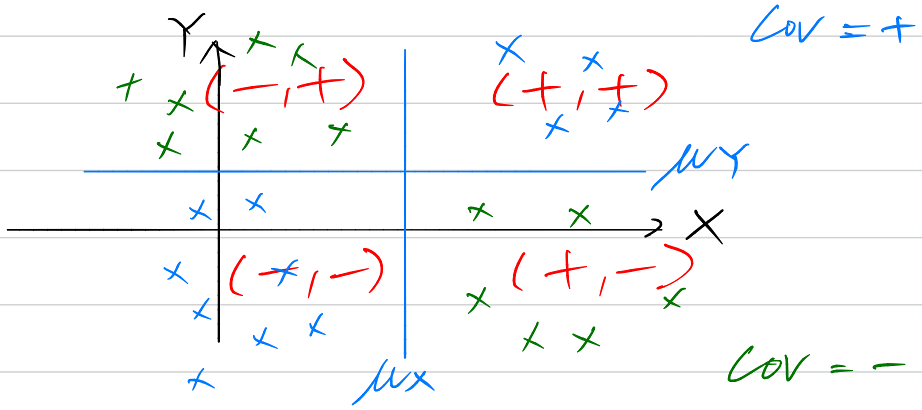
$$Corr(X, Y) = \rho_{XY} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

where ρ_{XY} is called the correlation coefficient. The range is $-1 \leq \rho_{XY} \leq 1$.

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= \sum\sum (x-\mu_X)(y-\mu_Y) f_{XY}(x,y)$$

$$\oplus < \begin{matrix} \oplus & \oplus & \oplus \\ \ominus & \ominus & \end{matrix}$$

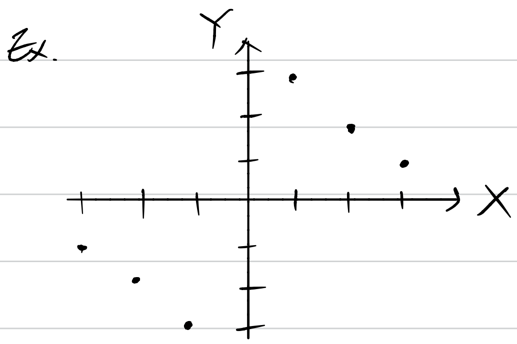
$$\ominus < \begin{matrix} \oplus & \ominus \\ \ominus & \oplus \end{matrix}$$



Cov is not bounded.
⇒ any Real number.

$$E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y]$$
$$= E(XY) - E(X)\mu_Y - \mu_X E(Y) + E(\mu_X\mu_Y)$$
$$= E(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y$$
$$= E(XY) - \mu_X\mu_Y$$

Ex.



$f_{XY}(x,y) = \frac{1}{6}$

$(x,y) = (1,3) \quad (-1,-3)$
$\quad (2,2) \quad (-2,-2)$
$\quad (3,1) \quad (-3,-1)$

$f_X(x) = \frac{1}{6} \quad x = -3, -2, -1, 1, 2, 3$

$f_Y(y) = \frac{1}{6} \quad y = \quad ''$

$E(X) = 0 \quad E(Y) = 0. \quad Cov(X,Y) = \frac{20}{6} > 0$

$$E(g(X,Y)) = \sum_{\forall x}\sum_{\forall y} \overbrace{g(x,y)}^{\ge 0}\, \overbrace{f_{XY}(x,y)}^{\ge 0} \ge 0$$

$g(X,Y) \ge 0$

To show $-1 \le \rho_{XY} \le 1$

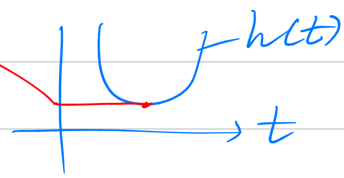
pf) Consider the function

$$h(t) = E[\{(X-\mu_X)t + (Y-\mu_Y)\}^2]$$

$$= E[(X-\mu_X)^2 t^2 + 2(X-\mu_X)(Y-\mu_Y)t + (Y-\mu_Y)^2]$$

$$= \sigma_X^2 t^2 + 2t\,Cov(X,Y) + \sigma_Y^2 \ge 0 \quad \text{for } \forall t$$

$$\Rightarrow \sigma_X^2 \left\{t + \frac{Cov(X,Y)}{\sigma_X^2}\right\}^2 \underbrace{- \frac{\{Cov(X,Y)\}^2}{\sigma_X^2} + \sigma_Y^2}_{\text{min}} \ge 0 \quad \text{for } \forall t$$



$$\therefore\ - \frac{\{Cov(X,Y)\}^2}{\sigma_X^2} + \sigma_Y^2 \ge 0$$

$$\therefore\ \frac{\{Cov(X,Y)\}^2}{\sigma_X^2} \le \sigma_Y^2$$

$$-\sigma_X \sigma_Y \le Cov(X,Y) \le \sigma_X \sigma_Y$$

$$-1 \le \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \le 1$$

$$\Downarrow$$

$$\rho_{XY}$$

Example 4.2-1

Let X_1 and X_2 have the joint pmf

$$f(x_1, x_2) = \frac{x_1 + 2x_2}{18}, \quad x_1 = 1, 2, \quad x_2 = 1, 2.$$

Find the correlation coefficient of X_1 and X_2.

$$f_{X_1}(x_1) = \sum_{\forall x_2} \frac{x_1+2x_2}{18} = \frac{x_1+2}{18} + \frac{x_1+4}{18} = \frac{2x_1+6}{18} = \frac{x_1+3}{9} \quad x_1 = 1, 2$$

$$f_{X_2}(x_2) = \sum_{\forall x_1} \frac{x_1+2x_2}{18} = \frac{3+4x_2}{18} \quad x_2 = 1, 2$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$E(X_1) = \sum_1^2 x_1 \frac{x_1+3}{9} = \frac{4}{9} + \frac{10}{9} = \frac{14}{9}$$
$$E(X_2) = \sum_1^2 x_2 \frac{3+4x_2}{18} = \frac{7}{18} + \frac{22}{18} = \frac{29}{18}$$

$$E(X_1^2) = \sum_1^2 x_1^2 \cdot \frac{x_1+3}{9} = \frac{4}{9} + \frac{20}{9} = \frac{24}{9} = \frac{8}{3}$$
$$E(X_2^2) = \sum_1^2 x_2^2 \frac{3+4x_2}{18} = \frac{7}{18} + \frac{44}{18} = \frac{51}{18} = \frac{17}{6}$$

$$Var(X_1) = \frac{8}{3} - \frac{196}{81} = \frac{216-196}{81} = \frac{20}{81}$$
$$Var(X_2) = \frac{51}{18} - \left(\frac{29}{18}\right)^2 = \frac{918-841}{324} = \frac{77}{324}$$

$$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$$
$$= \sum_{x_1=1}^{2}\sum_{x_2=1}^{2} (xy) \frac{x_1+2x_2}{18} - \frac{14}{9} \cdot \frac{29}{18}$$
$$= \frac{1}{18}(3+10+8+24) - \frac{7}{9} \cdot \frac{29}{9}$$
$$= \frac{1}{18} \cdot 45 - \frac{203}{81} = \frac{5}{2} - \frac{203}{81} = \frac{405-406}{162} = -\frac{1}{162}$$

$$\sigma_{X_1}^2 = \frac{20}{81} \qquad \sigma_{X_1} = \frac{\sqrt{20}}{9}$$
$$\sigma_{X_2}^2 = \frac{77}{324} \qquad \sigma_{X_2} = \frac{\sqrt{77}}{18}$$
$$\sigma_{X_1X_2} = -\frac{1}{162}$$

$$\rho_{X_1X_2} = \frac{\sigma_{X_1X_2}}{\sigma_{X_1}\sigma_{X_2}} = -\frac{1}{162} \cdot \frac{9}{\sqrt{20}} \cdot \frac{18}{\sqrt{77}} = -\frac{1}{\sqrt{20 \cdot 77}} = -\frac{1}{2\sqrt{385}}$$

Thm

If X and Y are independent random variables, then

$Cov(X, Y) = 0$ and $\rho_{XY} = 0$.

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \sum_x \sum_y (xy) f_{XY}(x,y) - E(X)E(Y)$$

by indep

$$= \sum_x \sum_y (xy) f_X(x) f_Y(y) - E(X)E(Y)$$

$$= \sum_x x f_X(x) \sum_y y f_Y(y) - E(X)E(Y)$$

$$= \sum_x x f_X(x) E(Y) - E(X)E(Y)$$

$$= E(X)E(Y) - E(X)E(Y) = 0.$$

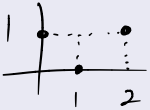
$$\text{Indep} \Rightarrow \mathrm{Cov} = 0.$$

$$\not\Leftarrow$$

Example 4.2-3

Let X and Y have the joint pmf

$$f(x,y) = \frac{1}{3}, \quad (x,y) = (0,1), (1,0), (2,1).$$



- Are X and Y independent?
- Compute $Cov(X,Y)$.

$f_X = \frac{1}{3} \quad x = 0, 1, 2$

$f_Y = \begin{cases} \frac{1}{3} & y = 0 \\ \frac{2}{3} & y = 1 \end{cases}$

$f_{XY}(0,1) = \frac{1}{3}$

$\neq$

$f_X(0) \cdot f_Y(1) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$

Ch.4.3 Conditional Distributions

3.1 Conditional Distributions

Definition (conditional pmf)

Let (X, Y) be a discrete bivariate random vector with joint pmf $f_{XY}(x, y)$ and marginal pmfs $f_X(x)$ and $f_Y(y)$. The conditional probability mass function of X given that $Y = y$, is defined by

$$f_{X|Y}(x|y) = P(X = x|Y = y) = \frac{f_{XY}(x, y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

The conditional probability mass function of Y given that $X = x$, is defined by

$$f_{Y|X}(y|x) = P(Y = y|X = x) = \frac{f_{XY}(x, y)}{f_X(x)},$$

provided that $f_X(x) > 0$.

Definition (conditional pdf)

Let (X, Y) be a continuous bivariate random vector with joint pdf $f_{XY}(x,y)$ and marginal pdfs $f_X(x)$ and $f_Y(y)$. The conditional probability density function of X given that $Y = y$, is defined by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

The conditional probability density function of Y given that $X = x$, is defined by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)},$$

provided that $f_X(x) > 0$.

Conditional probabilty of A given B.

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \text{if } P(B) > 0.$$

Conditional pmf (pdf) of X given Y=y.

$$f_{X|Y}(x|y) = P(X=x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$= \frac{f_{XY}(x,y)}{f_Y(y)} \qquad f_Y(y) > 0.$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \qquad f_X(x) > 0.$$

joint dist

marginal dist

Y\X	1	2	3	
1	1/9	1/9	1/9	3/9
2	1/9	1/9	1/9	3/9
3	1/9	1/9	1/9	3/9
	3/9	3/9	3/9	1

marginal dist of Y

marginal dist of X.

$$f_{X|Y}(x,2) = \frac{1/9}{3/9} = \frac{1}{3}$$

$$f_{X|Y} = \frac{f_{XY}(x,y)}{f_Y(y)}$$

3.2 Conditional Moments

Definition of conditional expectation

The conditional expectation of $g(X)$ given $Y = y$ is

$$E[g(X)|y] = \int_{-\infty}^{\infty} g(x)f(x|y)\,dx \quad \text{or} \quad \sum_{x} g(x)f(x|y).$$

Definition of conditional variance

The conditional variance of X given $Y = y$ is 해보기.

$$Var[X|y] = \boxed{E\left[\{X - E(X|y)\}^2|y\right] = E[X^2|y] - [E(X|y)]^2}$$

Conditional Expectation of $g(X)$ given $Y=y$

$$E[g(X)|y] = \begin{cases} \sum_{\forall x} g(x) \cdot f_{X|Y}(x|y) & \text{discrete} \\ \int g(x) f_{X|Y}(x|y)\, dx & \text{continuous.} \end{cases}$$

Conditional mean of X given $Y=y$

$$E[X|y] = \sum_{\forall x} x f_{X|Y}(x|y)$$

$$\int x f_{X|Y}(x|y)\, dx$$

Conditional Variance of X given $Y=y$

$$Var[X|y] = E[(X - \underbrace{E(X|y)}_{\text{conditional mean}})^2 | y]$$

$\Rightarrow \mu_{X|Y=y}$

$$= \sum_{\forall x} (x - E(X|y))^2 \cdot f_{X|Y}(x|y)$$

$$\sum_{\forall x} \left(x^2 - 2xE(X|y) + \{E(X|y)\}^2\right) f_{X|Y}(x|y)$$

$$\sum_{\forall x} x^2 f_{X|Y}(x|y) = E(X^2|y)$$

$$E(X^2|y) - 2E(X|y) \cdot E(X|y) + E(X|y)^2$$

$$= E[X^2|y] - \{E(X|y)\}^2$$

Example 4.3-1 & 2

Define the joint pmf of (X, Y) by

$$f(x,y) = \frac{x+y}{21}, \quad x = 1,2,3, \quad y = 1,2.$$

- Find the conditional pmf of Y given $X = x$.
- Find the conditional pmf of X given $Y = y$.
- Find $\mu_{Y|3} = E(Y|3)$ and $\sigma^2_{Y|3} = Var[Y|3]$

$$f_{XY}(x,y) = \frac{x+y}{21} \quad x=1,2,3 \quad y=1,2$$

$$f_X(x) = \sum_{\forall y} \frac{x+y}{21} \quad x=1,2,3$$

$$= \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21} \quad x=1,2,3.$$

$$f_Y(y) = \sum_{\forall x} \frac{x+y}{21}, \ y=1,2$$

$$= \frac{1+y}{21} + \frac{2+y}{21} + \frac{3+y}{21} = \frac{6+3y}{21} = \frac{2+y}{7} \quad y=1,2$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{x+y}{21}}{\frac{2+y}{7}} = \frac{x+y}{6+3y}$$ $x=1,2,3$ when $y=1,2.$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{x+y}{21}}{\frac{2x+3}{21}} = \frac{x+y}{2x+3}$$ $y=1,2$ when $x=1,2,3.$

OR
$$\left(\begin{aligned} f_{Y|1} &= \frac{1+y}{5}, \ y=1,2 \\ f_{Y|2} &= \frac{2+y}{7}, \ y=1,2 \\ f_{Y|3} &= \frac{3+y}{9}, \ y=1,2 \end{aligned} \right)$$

$$E[Y|3] = \sum_{\forall y} y \cdot f_{Y|X}(y|3) \quad y=1,2$$

$$= f_{Y|X}(1|3) + 2 \cdot f_{Y|X}(2|3)$$

$$= \frac{4}{9} + 2 \cdot \frac{5}{9} = \frac{14}{9}.$$

$$Var[Y|3] = E[Y - E[Y|3] \mid 3]$$
$$= E[Y^2|3] - \{E[Y|3]\}^2$$

$$E[Y^2|3] = \sum_{\forall y} y^2 f_{Y|X}(y|3) = \sum_{y=1}^{2} y^2 \left(\frac{3+y}{9}\right) = \frac{4}{9} + \frac{20}{9} = \frac{8}{3}$$

$$\Rightarrow \frac{8}{3} - \left(\frac{14}{9}\right)^2 = \frac{216}{81} - \frac{196}{81} = \frac{20}{81}$$

Thm

$$E[E(X|Y)] = E(X)$$

Example

$$X|Y \sim \mathrm{Bin}(Y, p)$$

$$Y \sim \mathrm{Poi}(\lambda)$$

Calculate $E(X)$.

$E_Y(\underbrace{E_{X|Y}(X|Y)}_{f \text{ of } Y}) = E(X).$

Assume that X & Y are continuous. $g(Y)$.

$$
\begin{aligned}
E[E(X|Y)] &= E\left[\int x f_{X|Y}(x|y)\,dx\right] \\
&= \int \left[\int x f_{X|Y}(x|y)\,dx\right] f_Y(y)\,dy \\
&= \iint x\, f_{X|Y}(x|y) f_Y(y)\,dx\,dy \\
&= \iint x\, f_{XY}(x,y)\,dx\,dy \\
&= \iint x\, f_{XY}(x,y)\,dy\,dx \\
&= \int x \int f_{XY}(x,y)\,dy\,dx \\
&= \int x\, f_X(x)\,dx \\
&= E(X).
\end{aligned}
$$

$\frac{f_{XY}(x,y)}{f_Y(y)} \cdot f_Y(y)$

To show : $E(E(XY|X)) = E(XY)$

$g(x)$

pf)
$$
\begin{aligned}
E[E(XY|X)] &= E\left[\int (xy) f_{Y|X}(y|x)\,dy\right] \\
&= \int \left[\int (xy) f_{Y|X}(y|x)\,dy\right] f_X(x)\,dx \\
&= \iint (xy) f_{Y|X}(y|x) f_X(x)\,dy\,dx \\
&= \iint (xy) f_{XY}(x,y)\,dy\,dx \\
&= E[XY]
\end{aligned}
$$

$$
\begin{aligned}
&= E\left[X \int y\, f_{Y|X}(y|x)\,dy\right] \\
&= E[X\,E(Y|X)]
\end{aligned}
$$

$X \sim Bin(n, p) \to E(X) = np \Leftrightarrow E(X|n,p) = np$

$X|Y \sim Bin(Y, p) \to E(X|Y, p) = Yp = E(X|Y).$

Y = # of total trials.

$Y \sim Poi(\lambda)$

$\Downarrow$

$$
\begin{aligned}
E(X) &= E[E(X|Y)] \\
&= E(Yp) \\
&= pE(Y) = p\lambda.
\end{aligned}
$$

$E[E[X|Y]] = E[X]$.

$E[\int x f_{X|Y}(x|y)\,dx]$

$\iint x f_{X|Y}(x|y)\,dx\, f_Y(y)\,dy$

$\iint x f_{X|Y}(x|y) f_Y(y)\,dx\,dy$

$\iint x f_{XY}(x,y)\,dx\,dy$

$\iint x f_{XY}(x,y)\,dy\,dx$

$\int x \int f_{XY}(x,y)\,dy\,dx$

$\int x f_X(x)\,dx = E(X)$.

$E[E[XY|X]] = E[XY]$

$E[\int xy f_{Y|X}(y|x)\,dy]$

$\iint xy f_{Y|X}(y|x)\,dy\, f_X(x)\,dx$

$\iint xy f_{Y|X}(y|x) f_X(x)\,dy\,dx$

$\iint xy f_{XY}(x,y)\,dy\,dx$

$E[XY]$

3.3 Linear Conditional Mean

Linear Conditional Mean

If $u(x) = E[Y|x] = a + bx$ is a linear function of x, then

$$a = \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X$$

$$b = \rho\frac{\sigma_Y}{\sigma_X}.$$

Let $Var[Y|x] = K(x)$. Then

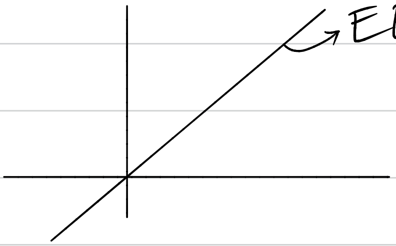
(o).

X

(x)

$$E[K(x)] = \sigma_Y^2(1 - \rho^2)$$

$(x_i, y_i)\quad i=1,\cdots,n$



$E[y_i|x_i] = a + bx_i$

$E[Y|x] = a+bx$

$\Rightarrow f^{tn}$ of x

1) $a = ?\qquad b = ?$

2) $Var[Y|x] = K(x)$

$$E[Var(Y|X)] = E[ker(X)] = \sigma_Y^2(1-\rho_{XY}^2)$$

$$\mu(x) = E[Y|x] = a+bx$$

1) $E[Y] = E[E(Y|X)] = E[a+bX] = a+b\mu_X = \mu_Y \cdots ①$

$$E[XY] = E[E(XY|X)] = E[XE(Y|X)] = E[X(a+bX)]$$
$$= E[aX+bX^2] = \underline{a\mu_X + b(\sigma_X^2+\mu_X^2)}$$

Since $E(XY) = Cov(X,Y) + \mu_X\mu_Y$ // $\because Cov(X,Y) = E(XY) - \mu_X\mu_Y$

$$= \underline{\mu_X\mu_Y + \rho_{XY}\sigma_X\sigma_Y}$$

$$\therefore \begin{cases} a = \mu_Y - b\mu_X \\ \mu_X\mu_Y + \rho_{XY}\sigma_X\sigma_Y = a\mu_X + b(\sigma_X^2+\mu_X^2) \end{cases}$$

$$\therefore b = \rho_{XY}\cdot\frac{\sigma_Y}{\sigma_X},\quad a = \mu_Y - \rho_{XY}\frac{\sigma_Y}{\sigma_X}\mu_X$$

$$\mu(x) = E(Y|x) = \mu_Y - \rho_{XY}\frac{\sigma_Y}{\sigma_X}\mu_X + \rho_{XY}\frac{\sigma_Y}{\sigma_X}x$$
$$= \mu_Y + \rho_{XY}\frac{\sigma_Y}{\sigma_X}(x-\mu_X)$$

$$\frac{\mu_X\mu_Y + \rho_{XY}\sigma_X\sigma_Y - a\mu_X}{\sigma_X^2+\mu_X^2} = b$$

$$\mu_X(\mu_Y - a) + \rho_{XY}\sigma_X\sigma_Y$$

2) $k(x) = Var(Y|x) = E[\{y - E(y|x)\}^2|x] = E[\{y - u(x)\}^2|x]$

$= E[\{Y - \mu_Y + \mu_Y - u(x)\}^2|x]$

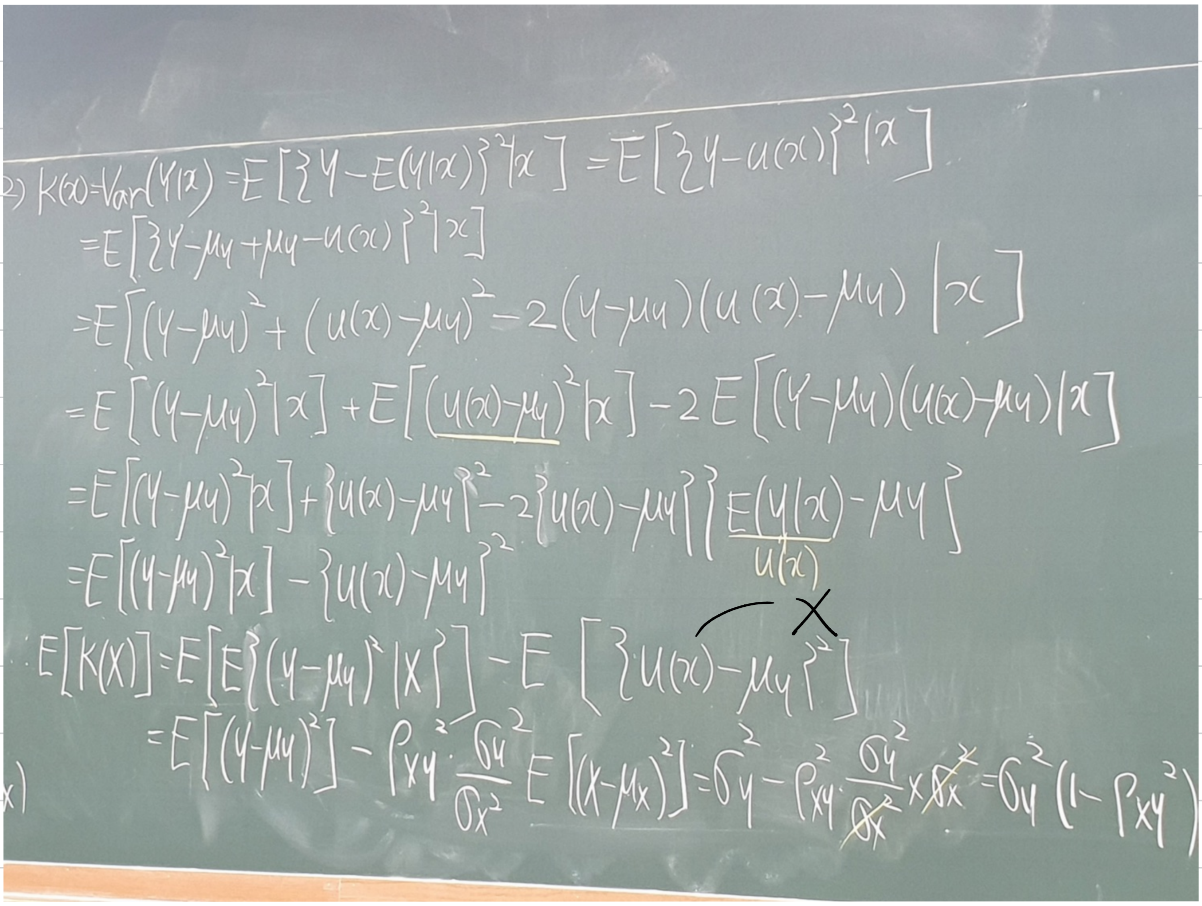
$= E[(Y-\mu_Y)^2 + (u(x)-\mu_Y)^2 - 2(Y-\mu_Y)(u(x)-\mu_Y)|x]$

$= E[(Y-\mu_Y)^2|x] + \underbrace{E[(u(x)-\mu_Y)^2|x]}_{\text{상수}} - 2E[(Y-\mu_Y)((u(x)-\mu_Y)|x]$

$= E[(Y-\mu_Y)^2|x] + \{u(x)-\mu_Y\}^2 - 2\{u(x)-\mu_Y\}\{\underbrace{E(Y|x)}_{=u(x)} - \mu_Y\}$

$= E[(Y-\mu_Y)^2|x] - \{u(x)-\mu_Y\}^2$

$\therefore E[K(X)] = E[E\{$



$$K(x) = Var(Y|x) = E[\{Y - E(Y|x)\}^2 | x] = E[\{Y - u(x)\}^2 | x]$$

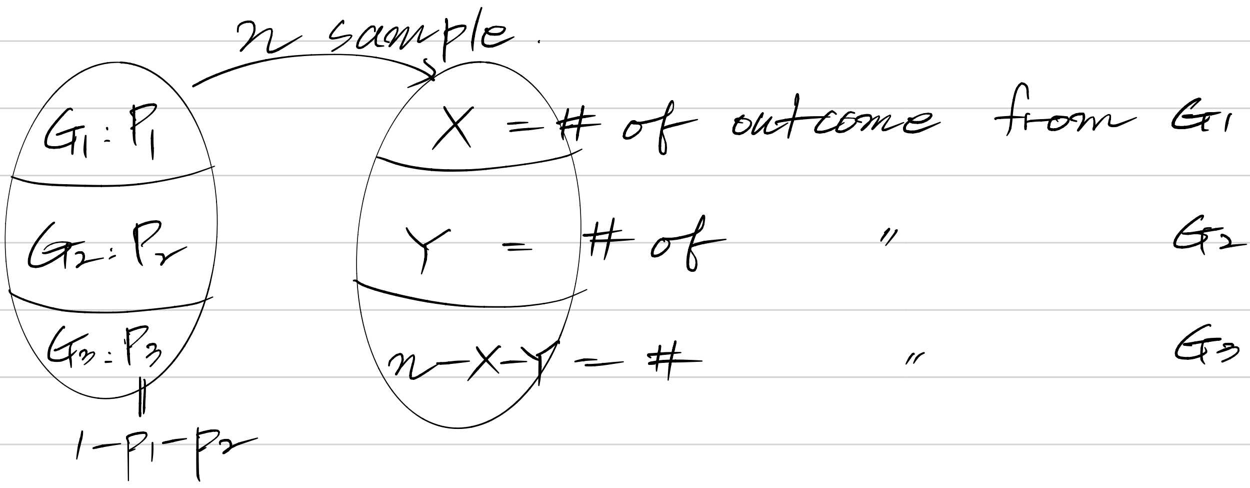
$$= E(\{Y - \mu_Y + \mu_Y - u(x)\}^2 | x]$$

$$= E[(Y-\mu_Y)^2 + (u(x) - \mu_Y)^2 - 2(Y-\mu_Y)(u(x) - \mu_Y) | x]$$

Example 4.3-3

Let X and Y have the trinomial pmf with parameters n, p_1, p_2, and $1 - p_1 - p_2 = p_3$.

- Find the conditional pmf/mean of Y given $X = x$.
- Find the conditional pmf/mean of X given $Y = y$.



$$f_{XY}(x,y) = P(X=x, Y=y)$$

$$= {}_nC_x \times {}_{n-x}C_y \, P_1^x P_2^y (1-P_1-P_2)^{n-x-y}$$

$$= \frac{n!}{x!\,y!\,(n-x-y)!} P_1^x P_2^y (1-P_1-P_2)^{n-x-y} \qquad x+y \le n$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_X(x) = P(X=x) = \sum_{\forall y} f_{XY}(x,y) = \sum_{y=0}^{n-x} f_{XY}(x,y)$$

$$= \sum_{y=0}^{n-x} \frac{n!}{x!\,y!\,(n-x-y)!} P_1^x P_2^y (1-P_1-P_2)^{n-x-y}$$

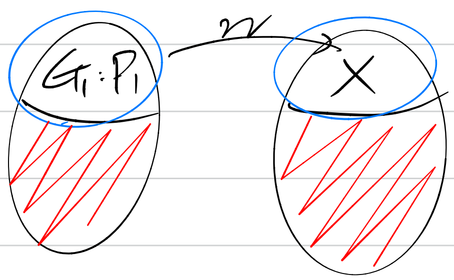
$$= \frac{n!}{x!} P_1^x \frac{1}{(n-x)!} \sum_{y=0}^{n-x} \frac{(n-x)!}{y!\,(n-x-y)!} P_2^y (1-P_1-P_2)^{n-x-y}$$

$$= {}_nC_x \, P_1^x \sum_{y=0}^{n-x} {}_{n-x}C_y \, P_2^y (1-P_1-P_2)^{(n-x)-y}$$

$$= {}_nC_x \, P_1^x (P_2 + 1 - P_1 - P_2)^{n-x}$$

$$= {}_nC_x \, P_1^x (1-P_1)^{n-x}, \quad x = 0, 1, \cdots, n$$

$\therefore X \sim Bin(n, p_1)$



$$f_{Y|X} = \frac{f_{XY}(x,y)}{f_X(x)}$$

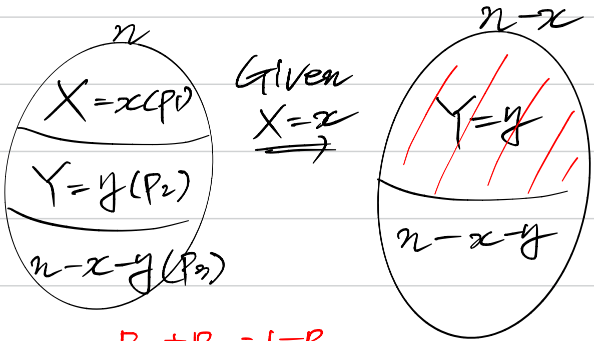
$$= \frac{\frac{n!}{x!\,y!\,(n-x-y)!} p_1^{x} p_2^{y} (1-p_1-p_2)^{n-x-y}}{\frac{n!}{x!\,(n-x)!} p_1^{x} \cdot (1-p_1)^{n-x}}$$

$$= \frac{(n-x)!}{y!\,(n-x-y)!} \frac{p_2^{y}}{(1-p_1)^{y}} \frac{(1-p_1-p_2)^{n-x-y}}{(1-p_1)^{n-x}\,(1-p_1)^{-y}}$$

$$= {}_{n-x}C_y \left(\frac{p_2}{1-p_1}\right)^{y} \left(\frac{1-p_1-p_2}{1-p_1}\right)^{(n-x)-y}$$

$$= {}_{n-x}C_y \left(\frac{p_2}{1-p_1}\right)^{y} \left(1-\frac{p_2}{1-p_1}\right)^{(n-x)-y} \quad \left(\begin{array}{l} y = 0, 1, \cdots, n-x \\ \text{when } x = 0, 1, \cdots, n \end{array} \right.$$

$\therefore Y|X=x \sim Bin\left(n-x, \frac{p_2}{1-p_1}\right)$



$p_2+p_3 = 1-p_1$

$$\frac{p_2}{p_2+p_3} = \frac{p_2}{1-p_1}$$

$$\frac{p_3}{p_2+p_3} = \frac{p_3}{1-p_1} = 1-p_1-p_2$$

$$f_{Y|X}(y|x) = P(Y=y \mid X=x)$$

$$E[Y|X] = n-x \cdot \frac{p_2}{1-p_1}$$

Example 4.4-6

In Example 4.4-3, the joint pdf of X and Y is given by

$$f(x, y) = 2, \quad 0 \le x \le y \le 1.$$

- Find the conditional pdf, mean, and variance of Y given $X = x$.
- Find the conditional pdf, mean, and variance of X given $Y = y$.
- Compute a conditional probability, $P(\frac{3}{4} \le Y \le \frac{7}{8} | X = \frac{1}{4})$.

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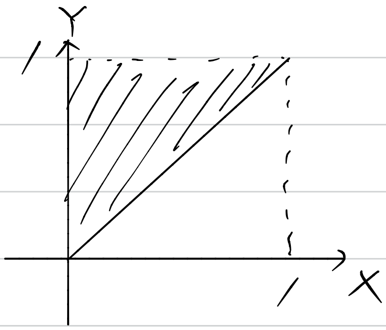

$$f_{XY}(x,y) = 2$$

$$f_X(x) = 2(1-x) \qquad 0 \le x \le 1$$

$$f_Y(y) = 2y \qquad 0 \le y \le 1$$

$$f_{Y|X}(y|x) = \frac{2}{2(1-x)} = \frac{1}{1-x}$$

$$x \le y \le 1 \quad \text{when } 0 \le x < 1.$$