

# Modern Algebra I – Midterm Exam

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April 24, 2019

1. Show that a group  $G$  with identity  $e$  and such that  $x^2 = e$  for all  $x \in G$  is Abelian.
2. The set  $GL(n, \mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) \mid \det(A) \neq 0\}$  under matrix multiplication is a group. Let  $M = \{A \in GL(n, \mathbb{R}) \mid A^T A = I_n\}$ . Show that  $M$  is a subgroup of  $GL(n, \mathbb{R})$ . (Hint: If  $A \in M$ , then  $A^T = A^{-1}$ ).
3. In  $\mathbb{Z}_{40}$ ,
  - (1) compute the order of 28.
  - (2) how many generators does  $\mathbb{Z}_{40}$  have?
  - (3) find all the subgroups of  $\mathbb{Z}_{40}$  and draw their subgroup diagram (or lattice).
4. Compute  $8^{41} \pmod{13}$ .
5. Let  $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$ .
  - (1) compute the order of  $\alpha$  and  $\beta$ .
  - (2) write  $\alpha\beta$  as product of 2-cycles.
6. Prove that every group of prime order is cyclic.
7. Show that  $\mathbb{R} \approx \mathbb{R}^+$ .
8. Prove that  $\text{Aut}(\mathbb{Z}_8) \approx \text{Aut}(\mathbb{Z}_{12})$ .
9. Let  $G$  be a group. Prove that if  $G$  has only finitely many subgroups, then  $G$  must be a finite group.