Advanced Calculus 2 - HW1

파이낸스경영학과
2017004093
양 준 우

§6.1 - 4

$f: \mathbb{R}^n \to \mathbb{R}^m$, $\exists M$ s.t $\|f(x)\| \le M\|x\|^2$. for $x \in \mathbb{R}^n$

prove f is differentiable at $x_0 = 0$ and $Df(x_0) = 0$.

For $x_0 = 0$, $\|f(x_0)\| \le M\|x_0\|^2 = 0$ $\therefore f(x_0) = 0$.

WTS $\displaystyle\lim_{x\to 0} \frac{\|f(x) - f(0) - Df(0)\cdot x\|}{\|x - 0\|} = \lim_{x\to 0} \frac{\|f(x) - f(0)\|}{\|x - 0\|} = 0$

$$\lim_{x\to 0} \frac{\|f(x) - f(0)\|}{\|x - 0\|} = \lim_{x\to 0} \frac{\|f(x)\|}{\|x\|} \leq \lim_{x\to 0} \frac{M\|x\|^2}{\|x\|} = 0.$$

Since $0 < \frac{\|f(x)\|}{\|x\|} < \frac{M\|x\|^2}{\|x\|}$, by sandwich lemma

$$\lim_{x\to 0} 0 \leq \lim_{x\to 0} \frac{\|f(x)\|}{\|x\|} \leq \lim_{x\to 0} M\|x\|$$

$$\therefore \lim_{x\to 0} \frac{\|f(x) - f(0)\|}{\|x - 0\|} = 0 = \lim_{x\to 0} \frac{\|f(x) - f(0) - Df(0)\cdot x\|}{\|x - 0\|} = 0.$$

$\therefore f$ is differentiable at 0 and $Df(0) = 0$ ▨

§ 6.3 − 1.

$f(x) = \begin{cases} x^2 & (x \notin \mathbb{Q}) \\ 0 & (x \in \mathbb{Q}) \end{cases}$

① continuity at 0.

WTS $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t $|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$

Choose $\delta = \frac{1}{2}\sqrt{\varepsilon}$

then, $|x| < \frac{1}{2}\sqrt{\varepsilon} \Rightarrow |f(x)| = |x^2| < \frac{1}{4}\varepsilon < \varepsilon$

whether x is rational or not.

$\therefore$ f is continuous at 0.

② differentiability at 0.

$$\lim_{x\to 0} \frac{|f(x) - f(0)|}{x - 0} = \lim_{x\to 0} \frac{|x^2|}{x} = \lim_{x\to 0} x = 0.$$

$\exists\ Df(x_0) = 0.$

$\therefore$ f is differentiable at 0.

§ 6.4 − 4.

$z = x^3 + y^4 \quad (1, 3).\quad f(1,3) = 82$

$Df = (3x^2, 4y^3),\quad Df(1,3) = (3, 108)$

tangent plane at $(1,3)$

$$\begin{aligned} z &= 82 + (3, 108)\begin{pmatrix} x-1 \\ y-3 \end{pmatrix} \\ &= 82 + 3(x-1) + 108(y-3) \\ &= 3x + 108y + 82 - 3 - 324 \end{aligned}$$

$\therefore\ 3x + 108y - z = 245$

§6.5 - 2.

$u(x,y,z) = xe^y$, $v(x,y,z) = (\sin x)yz$

$f(u,v) = u^2 + v\sin u$

$h(x,y,z) = f(u(x,y,z), v(x,y,z))$

$= x^2 e^{2y} + (\sin x)yz \cdot \sin(xe^y)$.

$h'(x) = 2x \cdot e^{2y} + (\cos x)yz\sin(xe^y) + (\sin x)yz\cos(xe^y)e^y$

On the other hand, the chain rule gives

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= (2u + v\cos u)e^y + \sin u \cdot (\cos x)yz$$

$$= (2xe^y + (\sin x)yz \cdot \cos(xe^y))e^y + \sin(xe^y)(\cos x)yz$$

$$= 2xe^{2y} + (\sin x)yz \cdot \cos(xe^y) \cdot e^y + \sin(xe^y)(\cos x)yz$$

which is the same result.

The formula for $\frac{\partial h}{\partial y}$, $\frac{\partial h}{\partial z}$ can be checked similarly.

§ 6.6 - 5

$f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$

WTS $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t $\|x - x_0\| < \delta$ implies

$\| fg(x) - fg(x_0) - f(x_0) Dg(x_0)(x - x_0) - Df(x_0)(x - x_0) g(x_0) \|$

$< \varepsilon \|x - x_0\|$

Choose $\delta > 0$ s.t $\|x - x_0\| < \delta$ implies

① $|f(x)| < |f(x_0)| + 1 = M$

② $|g(x) - g(x_0) - Dg(x_0)(x - x_0)| \leq \frac{\varepsilon}{3M} \|x - x_0\|$

③ $|f(x) - f(x_0)| < \frac{\varepsilon}{3M}$ where $|Dg(x_0) \cdot y| \leq M \|y\|$

④ $|f(x) - f(x_0) - Df(x_0)(x - x_0)| \leq \frac{\varepsilon}{3|g(x_0)|} \|x - x_0\|$

$$\| f(x)g(x) - f(x_0)g(x_0) - f(x_0) Dg(x_0)(x - x_0) - Df(x_0)(x - x_0) g(x_0) \|$$

$$\leq \| f(x)g(x) - f(x)g(x_0) - f(x) Dg(x_0)(x - x_0)$$

$$+ f(x) Dg(x_0)(x - x_0) - f(x_0) Dg(x_0)(x - x_0)$$

$$+ f(x)g(x_0) - f(x_0)g(x_0) - Df(x_0)(x - x_0) g(x_0) \|$$

$$\leq \|f(x)\| \cdot \| g(x) - g(x_0) - Dg(x_0)(x - x_0) \|$$

$$+ \| f(x) - f(x_0) \| \cdot \| Dg(x_0)(x - x_0) \|$$

$$+ \| f(x) - f(x_0) - Df(x_0)(x - x_0) \| \cdot \| g(x_0) \|$$

$$\leq M \cdot \frac{\varepsilon}{3M} \|x - x_0\| + \frac{\varepsilon}{3M} \cdot M \|x - x_0\| + \frac{\varepsilon}{3\|g(x_0)\|} \|x - x_0\| \cdot \|g(x_0)\|$$

$$= \varepsilon \|x - x_0\|$$

§ 6.7 - 2

l'Hôpital's rule.

: If f', g' exist at x_0, $g'(x_0) \neq 0$, $f(x_0) = 0 = g(x_0)$

then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$

1) By definition

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{(f(x) - f(x_0)) / x - x_0}{(g(x) - g(x_0)) / x - x_0}$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \Big/ \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= \frac{f'(x_0)}{g'(x_0)}$$

2) By MVT

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(a)(x - x_0)}{g'(b)(x - x_0)}$$

Note that $a, b \to x_0$ as $x \to x_0$

$$\lim_{x \to x_0} \frac{f'(a)(x - x_0)}{g'(b)(x - x_0)} = \lim_{x \to x_0} \frac{f'(a)}{g'(b)} = \frac{f'(x_0)}{g'(x_0)}$$

§ 6.8 - 5.

$f(x,y) = e^x \cdot \cos y$

second order Taylor formula around $(0,0)$.

$$f(x,y) = f(0,0) + Df(0,0)(x,y) + \frac{1}{2} D^2 f(0,0)(x,y)(x,y) + R_2((x,y),(0,0)).$$

where $R_2((x,y),(0,0)) = \frac{1}{3!} D^3 f(c)(x,y)(x,y)(x,y)$.

c lies on line segment joining (x,y), $(0,0)$.

$$Df(x,y) = \nabla f(x,y) = (e^x\cos y,\ -e^x\sin y).$$
$$Df(0,0) = (1,0).$$

$$D^2f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} e^x\cos y & -e^x\sin y \\ -e^x\sin y & -e^x\cos y \end{pmatrix}$$

$$D^2f(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore f(x,y) = 1 + (1\ 0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x\ y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + R_2((x,y),(0,0))$$
$$= 1 + x + \frac{1}{2}(x^2 - y^2) + R_2((x,y),(0,0)).$$

§ 6.9 − 6

$$f(x,y) = x^3 + 2xy^2 - y^4 + x^2 + 3xy + y^2 + 10.$$
$$f(0,0) = 10.$$
$$Df(x,y) = \nabla f(x,y) = (3x^2 + 2y^2 + 2x + 3y,\ -4y^3 + 4xy + 3x + 2y).$$
$$Df(0,0) = (0,0)$$
$$D^2f(x,y) = \begin{pmatrix} 6x+2 & 4y+3 \\ 4y+3 & -12y^2 + 4x + 2 \end{pmatrix}$$
$$D^2f(0,0) = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

Here $\Delta_1 = 2 > 0$ and $\Delta_2 = 4 - 9 = -5 < 0$ and so that Hessian is neither positive nor negative definite.
Thus f can have neither a (local) maximum nor (local) minimum at $(0,0)$, and $(0,0)$ must be saddle point of f.