

# Topology II

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Based on lectures by Prof. Youngsik Huh

# Preface

These notes are based on the course MAT4004: Topology II taught by Professor Youngsik Huh at Hanyang University in fall 2021. The lectures mainly covered the second part of James Munkres' *Topology*.

December 13, 2021

Junwoo Yang

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# Chapter -1

## Introduction

Lecture 1  
Wed, Sep 1

A fundamental problem in math: to classify objects in the given category.

- Sets:  $|A| = |B|$  (cardinality)
- Groups, Rings, Fields:  $G \cong G'$  (isomorphic)
- Topological spaces:  $X \cong Y$  (homeomorphic)

When two topological spaces are homeomorphic, we may prove it by finding out a homeomorphism. But, in the case that they are not homeomorphic, how can we prove it?

**Example.** Let  $S$  be a 2-dimensional sphere and  $T$  be a torus. Then  $S \not\cong T$ .

**Proof.** Suppose there exists a homeomorphism  $h: T \rightarrow S$ . Let  $c$  be a simple closed curve on  $T$ , as Figure 1. Then  $h(c)$  should be a simple closed curve on  $S$ , and  $h: T - c \rightarrow S - h(c)$  is a homeomorphism. But  $T - c$  is connected and  $S - h(c)$  is not connected, which is a contradiction.  $\nexists$

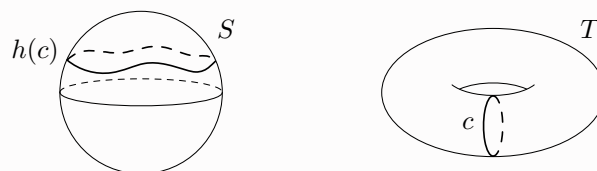


Figure 1:  $S \not\cong T$

In fact, on  $S$ , every loop can be continuously deformed to a point. But  $c$  cannot be on  $T$ . Such loops as  $c$  would be one of our interests in the lecture. From the family of loops on a topological space  $X$ , we will construct a group  $\pi_1(X)$ , called the **fundamental group** of  $X$ .

In fact, if  $X \cong Y$ , then  $\pi_1(X) \cong \pi_1(Y)$ . So we may use the fundamental group to distinguish topological spaces.

# Chapter 0

## Construction of more topological spaces

Consider two topological spaces  $(X_1, \mathcal{T}_1)$  and  $(X_2, \mathcal{T}_2)$  where  $X_1$  and  $X_2$  are disjoint.

**Union of spaces.** Let  $\mathcal{T} = \{U \subset X_1 \sqcup X_2 \mid U \cap X_1 \in \mathcal{T}_1, U \cap X_2 \in \mathcal{T}_2\}$ . Then  $(X_1 \sqcup X_2, \mathcal{T})$  is a topological space such that  $(X_i, \mathcal{T}_i)$  is a subspace.

**Product space.** Let  $\mathcal{B} = \{U_1 \times U_2 \subset X_1 \times X_2 \mid U_i \in \mathcal{T}_i\}$  and  $\mathcal{T} = \{U \subset X_1 \times X_2 \mid U \text{ is a union of some elements of } \mathcal{B}\}$ , i.e.  $\mathcal{B}$  is a base for  $\mathcal{T}$ . Then  $(X_1 \times X_2, \mathcal{T})$  is a topological space, called product space of  $X_1$  and  $X_2$ . Note the projection function  $\pi_i: X_1 \times X_2 \rightarrow X_i$  given by  $(x_1, x_2) \mapsto x_i$  is continuous.

**Example.** Consider  $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$  (subspace of the Euclidean space  $\mathbb{R}^2$ ) and a torus  $T \subset \mathbb{R}^3$ . Then  $S^1 \times S^1 \cong T$ .

**Quotient space.** E.g.,  $\mathbb{Z}/2\mathbb{Z}$  ( $a - b = 2n \Rightarrow a \sim b$ ).

Lecture 2  
Mon, Sep 6

### 0.1 Quotient spaces

**Definition 1.** Let  $X, Y$  be topological spaces and  $p: X \rightarrow Y$  be a surjective map<sup>a</sup>. Then  $p$  is said to be a **quotient map** if

$$U \subset Y \text{ is open in } Y \iff p^{-1}(U) \text{ is open in } X \quad (1)$$

or equivalently,

$$V \subset Y \text{ is closed in } Y \iff p^{-1}(V) \text{ is closed in } X. \quad (2)$$

<sup>a</sup>The map usually means the function between topological spaces.

**Proposition 1.** (1)  $\Leftrightarrow$  (2).

**Proof.** (1)  $\Rightarrow$  (2) Suppose  $p$  is a quotient map by the first definition. For a closed subset  $V$  of  $Y$ ,  $p^{-1}(Y - V) = X - p^{-1}(V)$  is open in  $X$ . Thus,  $p^{-1}(V)$  is closed in  $X$ . If  $p^{-1}(V)$  is closed,  $X - p^{-1}(V) = p^{-1}(Y - V)$  is open in  $X$ . Thus  $Y - V$  is open, hence  $V$  is closed.

(2)  $\Rightarrow$  (1) Similar. □

**Remark.** A quotient map is continuous.

**Remark.** A surjective continuous function  $f: X \rightarrow Y$  is a quotient map if  $f$  is an open map.

**Definition 2.** Suppose  $X$  be a topological space and  $A$  be a set. Let  $f: X \rightarrow A$  be a surjective function and

$$\mathcal{T}_f = \{U \subset A \mid f^{-1}(U) \text{ is open in } X\}.$$

Then  $\mathcal{T}_f$  is a topology for  $A$ , called **quotient topology** induced by  $f$ .

**Remark.**  $f: X \rightarrow (A, \mathcal{T}_f)$  is a quotient map by definition.

Let  $X$  be a topological space and  $\sim$  be an equivalence relation on  $X$ . For  $x \in X$ ,  $[x] = \{x' \in X \mid x \sim x'\}$  is a equivalence class of  $x$ , and  $X/\sim = \{[x] \mid x \in X\}$  is the set of all equivalence classes. Now consider  $q: X \rightarrow X/\sim$  given by  $x \mapsto [x]$ . ( $q$  is clearly surjective by definition.) Then,  $(X/\sim, \mathcal{T}_q)$  is called a **quotient space** of  $X$ .

**Example.** Let  $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ . Define an equivalence relation  $\sim$  on  $X$  by  $(x, y) \sim (x', y')$  iff

- $x = x', y = 0, y' = 1$
- $y = y', x = 0, x' = 1$
- $x = x', y = y'$

A quotient space is obtained by identifying a part with another part!

**Theorem 1 (22.2).** Let  $X, Y, Z$  be topological spaces,  $p: X \rightarrow Y$  a quotient map, and  $g: X \rightarrow Z$  a map s.t.  $p(x_1) = p(x_2)$  implies  $g(x_1) = g(x_2)$ . Then

- (i)  $\exists f: Y \rightarrow Z$  s.t.  $f \circ p = g$ .
- (ii)  $f$  is continuous iff  $g$  is continuous.
- (iii)  $f$  is a quotient map iff  $g$  is a quotient map.

**Proof.** (i) Define  $f$  by  $f(y) = g(x)$  for  $x \in p^{-1}(y)$ . It is well defined.

(ii) If  $f$  is continuous, a composition of continuous functions,  $g = f \circ p$ , is also continuous. Conversely, for an open subset  $u$  of  $Z$ ,  $g^{-1}(u) = p^{-1}(f^{-1}(u))$  is open in  $X$ . Since  $p$  is a quotient map,  $f^{-1}(u)$  is open. Thus  $f$  is continuous.

(iii) DIY. (not HW)

□

**Notation.** For a function  $g: X \rightarrow Z$ , define an equivalence relation  $\sim$  on  $X$  by  $x_1 \sim x_2$  iff  $g(x_1) = g(x_2)$ . Then,  $X/g := X/\sim$ .

**Corollary 1 (22.3).** Let  $g: X \rightarrow Z$  be a surjective continuous map. Then

- (i) There exists a homeomorphism  $f: X/g \rightarrow Z$  iff  $g$  is a quotient map.
- (ii) If  $Z$  is Hausdorff, then so is  $X/g$ .
- (iii) If  $X$  is compact and  $Z$  is Hausdorff, then  $f$  is a homeomorphism.

**Proof.** By Theorem 1.(i),  $g$  induces a continuous function  $f: X/g \rightarrow Z$  s.t.  $f \circ p = g$ . We can immediately see that  $f$  is injective and surjective.

$$\begin{array}{ccc} X & & \\ \downarrow p & \searrow g & \\ X/g & \xrightarrow{f} & Z \end{array}$$

- (i) If  $f$  is a homeomorphism, then  $f$  is a quotient map. Thus,  $g = f \circ p$  is quotient map. Conversely, if  $g$  is a quotient map, then so is  $f$  by Theorem 1.(iii). Since  $f$  is a injective quotient map,  $f$  is a homeomorphism.
- (ii) Let  $w_1, w_2$  be two distinct points of  $X/g$ . Then  $f(w_1) \neq f(w_2)$  and there are two disjoint open sets  $u_1, u_2$  in  $Z$  s.t.  $f(w_1) \in u_1, f(w_2) \in u_2$ .  $f^{-1}(u_1)$  and  $f^{-1}(u_2)$  are disjoint open neighborhoods of  $w_1$  and  $w_2$ , respectively.
- (iii) Recall that  $f$  is injective, surjective and continuous. So, it's enough to show that  $f$  is an open map, which is equivalent to  $f^{-1}$  is continuous. Since  $X$  is compact, so is  $X/g$  by continuity. Note that the closed subset of a compact set is compact. Let  $U$  be an open subset of  $X/g$ . Then  $X/g - U$  is compact, and so is  $f(X/g - U) = f(X/g) - f(U) = Z - f(U)$  in the Hausdorff space  $Z$ . Since every compact subset of a Hausdorff space is closed,  $Z - f(U)$  is closed in  $Z$ . Therefore,  $f(U)$  is open.

□

**Example.** Let  $g: [0, 1] \rightarrow S^1 \subset \mathbb{R}^2$  (or  $\mathbb{C}$ ) be given by  $r \mapsto (\cos 2\pi r, \sin 2\pi r)$

( $= e^{2\pi ir} = \cos 2\pi r + i \sin 2\pi r$ ). Note  $[0, 1]$  is compact and  $S^1$  is Hausdorff. Then,

$$\underbrace{[0, 1]/g = [0, 1]/\{0, 1\}}_{\text{quotient spaces}} \underset{\text{Cor 1.(iii)}}{\cong} \underbrace{S^1 \subset \mathbb{R}^2}_{\text{Euclidean subspace}}.$$

**Example.** Let  $X = [0, 1] \times [0, 1]$  and  $g: X \rightarrow S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$  (or  $\mathbb{C} \times \mathbb{C}$ ) be given by  $(x, y) \mapsto (e^{2\pi ir}, e^{2\pi is})$ . Note that  $g$  is surjective and continuous. Then,

$$X/g = X / \left( \begin{smallmatrix} (0,y) \sim (1,y) \\ (x,0) \sim (x,1) \end{smallmatrix} \right) = \text{Torus} \cong S^1 \times S^1.$$

**Notation.** Let  $X$  be a topological space and  $A$  be a subset of  $X$ . Define an equivalence relation  $\sim$  on  $X$  by  $x_1 \sim x_2$  iff  $x_1, x_2 \in A$  or  $x_1 = x_2$ . Then  $X/A := X/\sim$ .

**Example.** Let  $D = \{re^{i\theta} \mid 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$  and  $g: D \rightarrow S^2 \subset \mathbb{R}^3$  be given by  $re^{i\theta} \mapsto (\sqrt{4r-4r^2} \cos \theta, \sqrt{4r-4r^2} \sin \theta, 2r-1)$ . Then,

$$D/g = D / \partial D (= S^1) \cong S^2.$$

For  $n \geq 0$ ,

- $S^0 = \{-1, 1\} \subset \mathbb{R}$
- $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid \sum x_i^2 = 1\}$
- $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 = 1\}$
- ...
- $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$  ( $n$ -sphere)

Define an equivalence relation  $\sim$  on  $S^n$  by  $x \sim y$  iff  $y = -x$  or  $y = x$ . Then,  $\mathbb{RP}^n := S^n/\sim$  is called the **real  $n$ -dimensional projective space**.

- $\mathbb{RP}^0 = \{\text{a point}\}$
- $\mathbb{RP}^1 \cong [0, 1]/\{0, 1\} \cong S^1$
- $\mathbb{RP}^2 \cong D^2 \cup \mathbb{RP}^1$

In general,  $S^n$  can be decomposed depend upon last coordinate as

$$S^n = \underbrace{\text{upper half of } S^n \cup \text{lower half of } S^n}_{n\text{-dimensional disk } D^n} \cup \underbrace{S^{n-1}}_{\mathbb{RP}^{n-1}},$$

and then,

$$\mathbb{RP}^n \cong \text{attaching } D^n \text{ along } \mathbb{RP}^{n-1}$$

where  $D^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$ ,  $\partial D^n \cong S^{n-1}$ .



**Notation.** Let  $X, Y$  be topological spaces and  $A$  be a subspace of  $X$ . Let  $f: A \rightarrow Y$  be continuous. Define  $\sim$  on  $X \sqcup Y$  by  $a \sim f(a)$  for  $a \in A$ . Then  $X \cup_f Y := X \sqcup Y / \sim$  is the **adjunction space**. In that case,  $f$  is called the **attaching map**.

Now if we define attaching maps as

$$\begin{aligned} f_0: S^0 &\rightarrow \mathbb{RP}^0 \\ f_1: S^1 &\rightarrow \mathbb{RP}^1 \cong D^1 \cup_{f_0} \mathbb{RP}^0 \end{aligned}$$

then,

$$\mathbb{RP}^n \cong \underbrace{\{\text{a point}\} \cup_{f_0} D^1}_{\mathbb{RP}^1} \cup_{f_1} D^2 \cup_{f_2} \dots \cup_{f_{n-2}} D^{n-1} \cup_{f_{n-1}} D^n.$$

$\underbrace{\hspace{10em}}_{\mathbb{RP}^2}$   
 $\underbrace{\hspace{10em}}_{\dots}$   
 $\underbrace{\hspace{10em}}_{\mathbb{RP}^{n-1}}$

$S^n$  represents all the directions in  $\mathbb{R}^{n+1}$ .  $x$  and  $-x$  are on the same line passing through the origin point. Thus we can say that  $\mathbb{RP}^n$  is the space of lines passing through  $O$  in  $\mathbb{R}^{n+1}$ .

**Example.**  $\mathbb{C}^n = \{(z_1, \dots, z_n) \mid z_i \in \mathbb{C}\}$  is  $n$ -dimensional complex vector space. The **complex  $n$ -projective space**  $\mathbb{CP}^n$  is the space of complex lines passing through  $O$  in  $\mathbb{C}^{n+1}$ . Formally,

$$\begin{aligned} \mathbb{CP}^n &= \mathbb{C}^{n+1} - \{O\} / z \sim \lambda z \\ &= \{\text{unit vectors in } \mathbb{C}^{n+1}\} / z \sim \lambda z \\ &= \{z \in \mathbb{C}^{n+1} \mid \|z\| = 1\} / z \sim \lambda z \\ &= S^{2n+1} / z \sim \lambda z \end{aligned}$$

where  $\lambda \in \mathbb{C}$ ,  $\|\lambda\| = 1$ .

# Surfaces

## Surfaces

**Definition 3.** An  $n$ -manifold is a topological space  $X$  s.t.

- (i)  $X$  is Hausdorff.
- (ii)  $X$  has a countable basis for its topology.
- (iii) Every point of  $X$  has an open neighborhood which is homeomorphic to  $\mathbb{R}^n$  (or  $\mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ ).

Especially, a 2-manifold is called a **surface**.

Shortly, an  $n$ -manifold is a second countable, Hausdorff topological space which is locally homeomorphic to  $\mathbb{R}^n$ .

**Definition 4.** An  $n$ -manifold with boundary is a top'al sp  $X$  s.t.

- (i)  $X$  is Hausdorff.
- (ii)  $X$  has a countable basis for its topology.
- (iii) Every point of  $X$  has an open neighborhood homeomorphic to  $\mathbb{R}^n$  or  $H_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$  (or  $\mathring{D}_+^n = \{(x_1, \dots, x_n) \in \mathring{D}^n \mid x_n \geq 0\}$ ).
- (iv)  $\partial X = \{\text{pts whose nbd is homeomorphic to } H_+^n \text{ or } \mathring{D}_+^n\} \neq \emptyset$

**Note.** From now on, the numbering on theorem, corollary, and lemma follows Munkres' book.

**Theorem 2** (36.2, Embedding theorem). A compact  $n$ -manifold  $X$  can be embedded into  $\mathbb{R}^N$  for some  $N \in \mathbb{N}$ , that is, there exists a continuous map  $f: X \rightarrow \mathbb{R}^N$  s.t.  $f: X \rightarrow f(X)$  is a homeomorphism.

**Proof.** Not covered in this course. □

Lecture 5  
Wed, Sep 15

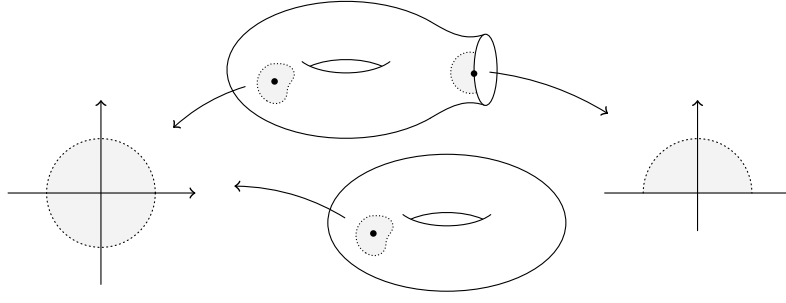


Figure 1: Surface with boundary

**Definition 5.** Let  $S_1, S_2$  be surfaces and  $D_i$  be a 2-dimensional disk in  $S_i$  for  $i = 1, 2$ . Then,  $\partial D_1, \partial D_2 \cong S^1$ , and there exists a homeomorphism  $f: \partial D_1 \rightarrow \partial D_2$ . The **connect sum** of  $S_1$  and  $S_2$  is defined as

$$S_1 \# S_2 = (S_1 - \overset{\circ}{D}_1) \cup_f (S_2 - \overset{\circ}{D}_2).$$

**Notation.** •  $T_0 := S^2$

- $T_1 := \text{Torus}$
- $T_n := T \# \cdots \# T = T_{n-1} \# T_1$

Let  $S := S^2 - \{\text{two open disks}\}$  and  $f: c_1 \rightarrow c_2$ . Then  $S/f \cong T_1$ . Similarly,  $T_{n-1} - \{\text{two open disks}\}/f \cong T_n$ .

$\mathbb{RP}^2 - \text{open disk} \cong \text{Möbius band}$

## Labelling scheme

Assign labels and directions to each edge of polygonal region  $P$ :

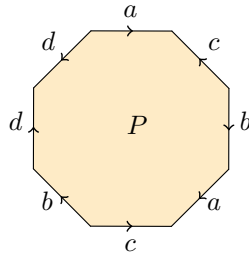


Figure 2: Labelling scheme:  $a^{-1}dd^{-1}b^{-1}ca^{-1}b^{-1}c$  (read counterclockwise)

A labelling scheme gives a surface which is a quotient of  $P$ .

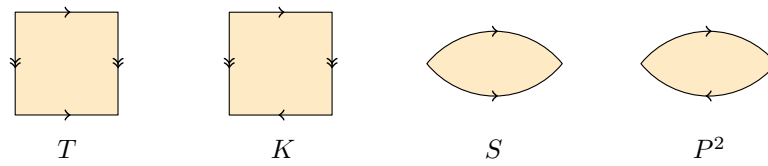


Figure 3: Examples of surfaces

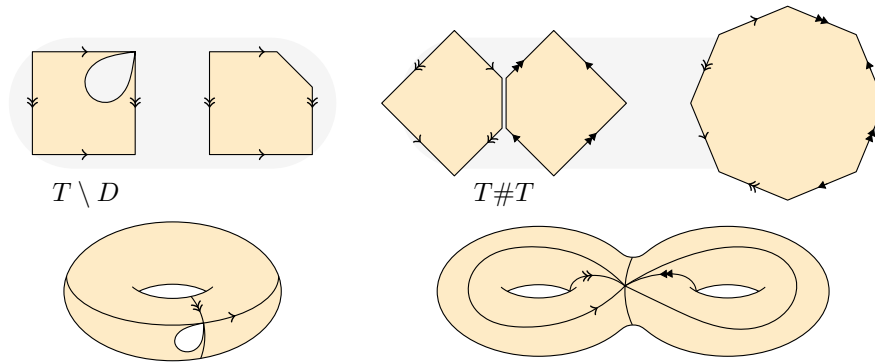


Figure 4: On the left: a torus with a disk removed. On the right: the connected sum of two tori.

## Elementary operations on schemes

Suppose  $\{w_1, \dots, w_n\}$  be a labelling scheme.

Lecture 6  
Mon, Sep 20

**Cut**  $w_i = Y_0 Y_1 \rightarrow \{Y_0 c, c^{-1} Y_1\}$  ( $c$  does not appear elsewhere)

**Paste** Reverse of cut.

**Relabel** Change an alphabet by a new alphabet. Reverse the sign of an alphabet.

**Permute** Cyclically permute alphabets on a word  $w_i$ . E.g.,  $w_i = a_1 a_2 \dots a_n \rightarrow w'_i = a_2 \dots a_n a_1$

**Flip**  $w_i = (a_{i1})^{\varepsilon_1} \dots (a_{in})^{\varepsilon_n} \rightarrow w_i^{-1} = (a_{i1})^{-\varepsilon_1} \dots (a_{in})^{-\varepsilon_n}$

**Cancel**  $Y_0 a a^{-1} Y_1 \rightarrow Y_0 Y_1$

**Uncancel** Reverse of cancel.

**Note.** These operations do not change the topological type of the resulting surfaces.

**Definition 6.** Two labelling schemes are said to be **equivalent** if one can be obtained from the other by applying the elementary operations in finitely many times.

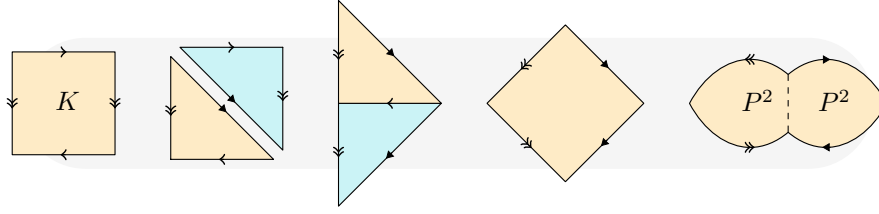


Figure 5:  $K = P \# P$

**Note.** Two equivalent schemes give surfaces of the same homeomorphic type.

## Classification theorem I

**Definition 7.** A scheme is **proper** if each label appears twice in the scheme.

**Note.** proper scheme  $\xrightarrow{\text{elem. oper.}}$  still proper!

**Definition 8.** Let  $w$  be a proper scheme for a single polygonal region  $P$ .  $w$  is of **torus type** if each label appears exactly once with exponent  $+1$ , and once with  $-1$ . Otherwise we say  $w$  is of **projective type**.

**Lemma 1** (77.1). If  $w$  is a proper scheme of the form  $w = Y_0 a Y_1 a Y_2^a$  where  $Y_i$  is a sequence of labels, then  $w \sim aaY_0Y_1^{-1}Y_2$ .

<sup>a</sup>that is to say  $w$  is of projective type

**Proof. Case 1.**  $Y_0$  is empty.

- If  $Y_1$  is also empty, then  $w$  is the desired form itself.
- If  $Y_2$  is empty,

$$aY_1a \xrightarrow{\text{flip}} a^{-1}Y_1^{-1}a^{-1} \xrightarrow{\text{permute}} a^{-1}a^{-1}Y_1^{-1} \xrightarrow{\text{relabel}} aaY_1^{-1}.$$

- If neither is empty,

$$aY_1aY_2 \xrightarrow[\text{paste}]{\text{cut}} ccY_1^{-1}Y_2 \xrightarrow{\text{relabel}} aaY_1^{-1}Y_2.$$

**Case 2.**  $Y_0$  is not empty.

- If both  $Y_1$  and  $Y_2$  are empty, a permutation is enough.
- In general,

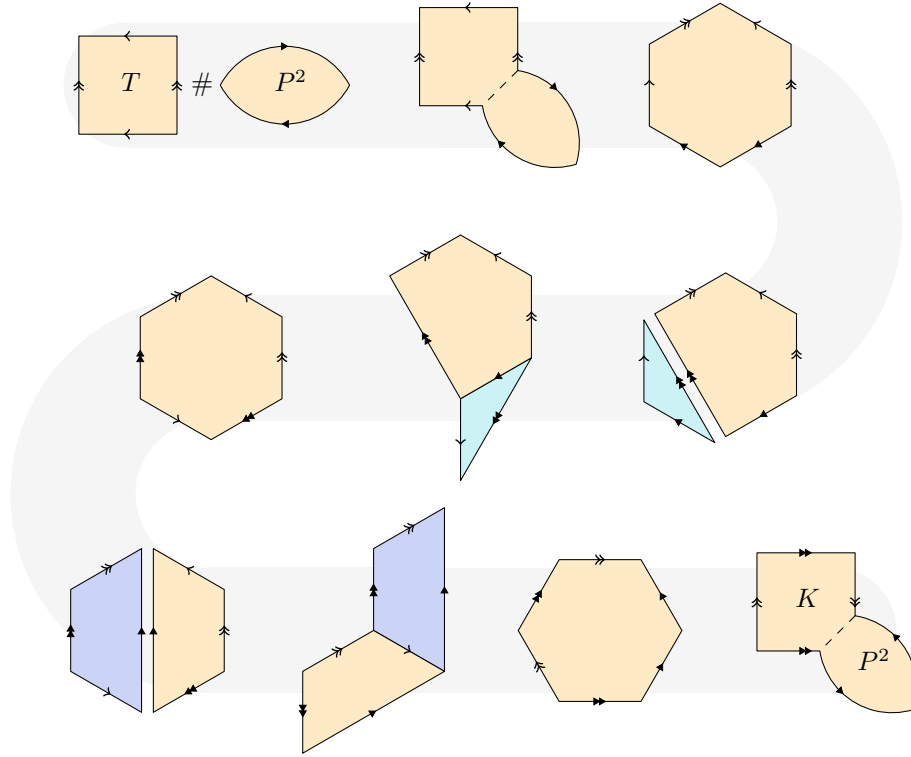


Figure 6:  $T \# P = K \# P$

$$\begin{array}{ccccc}
 Y_0 a Y_1 a Y_2 & \xrightarrow{\text{cut}} & b Y_2 b Y_1 Y_0^{-1} & \xrightarrow{\text{Case 1}} & b b Y_2^{-1} Y_1 Y_0^{-1} \\
 & \text{paste} & & & \downarrow \text{flip} \\
 a a Y_0 Y_1^{-1} Y_2 & \xleftarrow{\text{relabel}} & b^{-1} b^{-1} Y_0 Y_1^{-1} Y_2 & \xleftarrow{\text{permute}} & Y_0 Y_1^{-1} Y_2 b^{-1} b^{-1}
 \end{array}$$

□

**Corollary 2 (77.2).** If  $w$  is projective type, then  $w$  is equivalent to a scheme of the form  $(a_1 a_1)(a_2 a_2) \dots (a_k a_k) w'$ , where the length<sup>a</sup> is unchanged,  $k \geq 1$ , and  $w'$  is empty or of torus type.

<sup>a</sup>the number of alphabets

**Proof.** Since  $w$  is of projective type, it can be written to be  $w = Y_0 a Y_1 a Y_2$ . By Lemma 1,  $w \sim a a w_1$  so that the length is unchanged. If  $w_1$  is empty or of torus type, it's done. Otherwise, we can write  $w_1$  so that  $a a w_1 \sim a a Z_0 b Z_1 b Z_2$ . Again by Lemma 1,  $a a w_1 \sim b b a a Z_0 Z_1^{-1} Z_2$ , length of  $w_2$ . By repeating this process, we obtain the desired form. □

**Lemma 2 (77.3).** Let  $w = w_0w_1$  be a proper scheme, where  $w_1$  is a scheme itself of torus type that does not contain any two adjacent terms having the same label. Then  $w \sim w_0w_2$  s.t.  $w_2 = aba^{-1}b^{-1}w_3$  with same length as  $w_1$ , where  $w_3$  is of torus type or is empty.

**Proof.**  $w$  can be written as  $w = w_0Y_1aY_2bY_3a^{-1}Y_4b^{-1}Y_5$ . □

**Lemma 3 (77.4).** If  $w$  is a proper scheme of the form  $w = w_0ccaba^{-1}b^{-1}w_1$ , then  $w \sim w_0aabbccw_1$

**Proof.** Proceed as follows:

$$\begin{aligned}
 w_0ccaba^{-1}b^{-1}w_1 &\sim ccaba^{-1}b^{-1}w_1w_0 && \text{(permute)} \\
 &= cc(ab)(ba)^{-1}w_1w_0 \\
 &\sim (ab)c(ba)cw_1w_0 && \text{(Lemma 1)} \\
 &= abcb(acw_1w_0) \\
 &\sim bbac^{-1}acw_1w_0 && \text{(Lemma 1)} \\
 &\sim aabbccw_1w_0 && \text{(Lemma 1)} \\
 &\sim w_0aabbccw_1 && \text{(permute)}
 \end{aligned}$$

□

**Theorem 3 (77.5, Classification theorem).** Let  $X$  be a quotient space obtained from a polygonal region  $P$  by glueing its edges in pairs. Then  $X$  is homeomorphic to one of  $S^2$ ,  $T_n$ , and  $(P^2)_n^a$  where  $n \geq 1$ .

<sup>a</sup>connect sum of  $\mathbb{RP}^2$

**Proof.** Let  $w$  be a proper scheme on  $P$  which results in  $X$ . If  $|w| = 2$ ,  $w = aa^{-1}$  ( $S^2$ ) or  $w = aa$  ( $P^2$ ). We may assume that  $|w| \geq 4$  ( $|w|$  is even). In fact we will show that □

**Note.** HW: Exercise 77.1 and 77.4

## Constructing compact surfaces

**Definition 9.** Let  $X$  be a compact Hausdorff space. A subspace  $A$  of  $X$  is a **curved triangle** if there exists a homeomorphism  $h: \Delta \rightarrow A$ , where  $\Delta$  is a closed triangular region in  $\mathbb{R}^2$ .

**Definition 10.** A **triangulation** of  $X$  is a collection of curved triangles  $\{A_\alpha\}$  s.t.

- $\bigcup A_\alpha = X$ .
- For  $\alpha \neq \beta$ ,  $A_\alpha \cap A_\beta = \emptyset$ , single vertex or single edge.
- When  $A_\alpha \cap A_\beta = \text{single edge}$ ,  $h_\beta^{-1} \circ h_\alpha$  is a linear map.

$X$  is said to be **triangulable** if it has a triangulation.

**Theorem 4 (78.1).** If  $X$  is a compact triangulable surface (with or without boundary), then  $X$  is homeomorphic to a quotient space obtained from a collection of disjoint triangular regions by pasting their edges together in pairs.

**Proof.** Let  $\{A_1, \dots, A_n\}$  be a triangulation of  $X$  with homeomorphisms  $\{h_i: \Delta_i \rightarrow A_i \mid i = 1, \dots, n\}$ . Then we have a quotient map  $h: \Delta_1 \sqcup \dots \sqcup \Delta_n \rightarrow X$  s.t.  $h|_{\Delta_i} = h_i$ . There are two things to be proved.

- If two triangles meet at a vertex, then there exists a sequence of triangles. Thus, the quotient is obtained only by edge-pastings.
- For each edge  $e$  of  $A_i$  s.t.  $e \not\subset \partial X$ ,  $\exists! j$  s.t.  $A_i \cap A_j = e$ . Thus, the quotient is obtained by pasting edges in pairs.

□

**Theorem 5 (78.2).** Let  $X$  be a compact connected triangulable surface without boundary. Then  $X$  is homeomorphic to a quotient space obtained from a polygonal region by pasting all the edges together in pairs. That is,  $X$  is homeomorphic to a surface obtained from a proper scheme on a polygonal region.

**Proof.** From Theorem 4,  $\Delta_1 \sqcup \dots \sqcup \Delta_n \xrightarrow{h} X$ . Assemble the triangles  $\{\Delta_i\}$  on the plane as much as possible in the following way: □

**Theorem 6 (A).** Every compact connected surface is triangulable.

**Proof (Sketch of proof).** • surface and compact  $\Rightarrow \exists$  a finite collection  $\{B_1, \dots, B_n\}$  s.t.  $B_i \cong D^2$ ,  $\bigcup B_i = X$ .

- We may assume that no proper subset satisfies  $\bigcup B_i = X$ .
- Let  $C = \bigcup \partial B_i$  and  $D$  be thickening of  $C$  in  $X$ . Then  $X - D \cong \bigcup \mathring{D}^2$ . □

**Theorem 7 (Surface classification theorem).** Every compact connected surface without boundary is homeomorphic to one of  $S^2$ ,  $T_n$ , and  $(P^2)_n$ .

**Proof.** Theorem 3 + Theorem 5 + Theorem 6. □



# Chapter 9

## Fundamental group

Lecture 8  
Mon, Sep 27

### 9.51 Homotopy of paths

**Definition 11.** Let  $X, Y$  be topological spaces and  $f, f': X \rightarrow Y$  be continuous maps. We say,  $f$  is **homotopic** to  $f'$  ( $f \simeq f'$ ) if there is a continuous function  $F: X \times I \rightarrow Y$  s.t.  $F(x, 0) = f(x)$ ,  $F(x, 1) = f'(x)$  for all  $x \in X$ . The function  $F$  is called a **homotopy** from  $f$  to  $f'$  ( $f \simeq^F f'$ ). Especially, if  $f'$  is a constant map, then we say,  $f$  is **null-homotopic**.

**Definition 12.** Let  $f, f': I \rightarrow X$  be two paths in  $X$  s.t.  $f(0) = f'(0) = x_0$  and  $f(1) = f'(1) = x_1$ . We say,  $f$  is **path-homotopic** to  $f'$  ( $f \simeq_p f'$ ) if there is a homotopy  $F: I \times I \rightarrow X$  s.t.

- $F(x, 0) = f(x)$ ,  $F(x, 1) = f'(x)$
- For each  $t$ ,  $F(0, t) = x_0$ ,  $F(1, t) = x_1$

The homotopy  $F$  is called a **path-homotopy** from  $f$  to  $f'$  ( $f \simeq_p^F f'$ ).

**Notation.**

- $\Omega(X, Y) := \{f: X \rightarrow Y \mid f \text{ is continuous}\}$
- $\mathcal{P}(X) := \text{the set of all paths in } X$

**Lemma 4 (51.1).**  $\simeq$  and  $\simeq_p$  are equivalence relations on  $\Omega(X, Y)$  and  $\mathcal{P}(X)$ , respectively.

**Proof. Reflective**  $F(x, t) = f(x)$

**Symmetric** Suppose  $f \simeq f'$ . Then there is a homotopy  $F: X \times I \rightarrow Y$  s.t.  $F(x, 0) = f(x)$ ,  $F(x, 1) = f'(x)$ . Define  $F'(x, t) = F(x, 1 - t)$ . Then,  $F'$  is conti. and  $F'(x, 0) = F(x, 1) = f'(x)$ ,  $F'(x, 1) = F(x, 0) = f(x)$ .

**Transitive** Suppose  $f \simeq^F f'$  and  $f' \simeq^G f''$ . Define  $H: X \times I \rightarrow Y$  by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Note that  $H$  is continuous by pasting lemma: For any closed subset  $U$  of  $Y$ , the preimages  $H^{-1}(U) \cap (X \times [0, \frac{1}{2}])$  and  $H^{-1}(U) \cap (X \times [\frac{1}{2}, 1])$  are closed since each is the preimage of  $H$  when restricted to  $X \times [0, \frac{1}{2}]$  and  $X \times [\frac{1}{2}, 1]$  respectively, which by continuity of  $F$  and  $G$ . Thus, their union  $H^{-1}(U)$  is closed, hence  $H$  is continuous.

$\simeq_p$ : skip.  $\square$

Denote the equivalence class of  $f$  by  $[f] = \{f' \in \Omega(X, Y) \mid f' \simeq f\}$ .

**Example.** Let  $C \subset \mathbb{R}^n$  be a convex subset.

- Any two maps  $f, g: X \rightarrow C$  are homotopic.
- Any two paths  $f, g: I \rightarrow C$  with  $f(0) = g(0)$  and  $f(1) = g(1)$  are path-homotopic.

Choose  $F: X \times I \rightarrow C$  defined by  $(x, t) \mapsto F(x, t) = (1 - t)f(x) + tg(x)$ .

**Example.** Let  $X = \mathbb{R}^2 - \{0\}$  (punctured plane).  $f(x) = (\cos \pi x, \sin \pi x)$ ,  $g(x) = (\cos \pi x, 2 \sin \pi x)$  and  $h(x) = (\cos \pi x, -\sin \pi x)$  are paths in  $X$ . In fact,  $f \simeq_p g \not\simeq_p h$ .

## Product of paths

Let  $f, g: I \rightarrow X$  be paths,  $f(1) = g(0)$ . Define the product  $f * g: I \rightarrow X$  by

$$f * g = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Define the product  $*$  on path-homotopy classes of  $X$  by  $[f] * [g] := [f * g]$ .

**Well-definedness** Suppose  $f' \in [f]$  ( $f \simeq_p^F f'$ ) and  $g' \in [g]$  ( $g \simeq_p^G g'$ ). Define

$$H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq \frac{1}{2} \\ G(2s - 1, t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then  $H(s, 0) = (f * g)(s)$ ,  $H(s, 1) = (f' * g')(s)$  and  $H$  is continuous by pasting lemma again. Thus,  $f * g \simeq_p f' * g'$ ,  $[f * g] = [f' * g']$ .

Lecture 9  
Wed, Sep 29

**Theorem 8 (51.2).** The product  $*$  has the following properties:

- (i) Associative:  $([f] * [g]) * [h] = [f] * ([g] * [h])$
- (ii) Let  $e_x$  denote the constant path  $e_x: I \rightarrow X$  given by  $s \mapsto x$ ,  $x \in X$ . If  $f(0) = x_0$  and  $f(1) = x_1$  then  $[e_{x_0}] * [f] = [f]$  and  $[f] * [e_{x_1}] = [f]$ .
- (iii) Let  $\bar{f}: I \rightarrow X$  given by  $s \mapsto f(1-s)$ . Then  $[f] * [\bar{f}] = [e_{x_0}]$  and  $[\bar{f}] * [f] = [e_{x_1}]$ .

**Proof.** First two observations

- Suppose  $f \simeq_p g$  via homotopy  $H$ ,  $f, g: I \rightarrow X$ . Let  $k: X \rightarrow Y$ . Then  $k \circ f \simeq_p k \circ g$  using  $k \circ H$ .
- If  $f * g$  (not necessarily path homotopic). Then  $k \circ (f * g) = (k \circ f) * (k \circ g)$ .

Now, the proof

- (ii) Take  $e_0: I \rightarrow I$  given by  $s \mapsto 0$ . Take  $i: I \rightarrow I$  given by  $s \mapsto s$ . Then  $e_0 * i$  is a path from 0 to 1 in  $I$ . The path  $i$  is also such a path. Because  $I$  is a convex subset,  $e_0 * i$  and  $i$  are path homotopic,  $e_0 * i \simeq i$ . Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

- (iii) Note that  $i * \bar{i} \simeq_p e_0$ . Now, applying the same rules, we get

$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \bar{f} &\simeq_p e_{x_0} \\ [f] * [\bar{f}] &= [e_{x_0}]. \end{aligned}$$

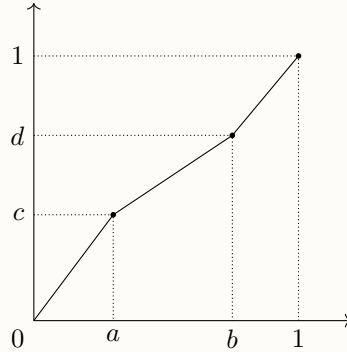
- (i) Remark: Only defined if  $f(1) = g(0)$ ,  $g(1) = h(0)$ . Note that  $f * (g * h) \neq (f * g) * h$ . The trajectory is the same, but the speed is not.

Assume the product is defined. Suppose  $[a, b]$ ,  $[c, d]$  are intervals in  $\mathbb{R}$ . Then there is a unique positive (positive slope), linear map from  $[a, b] \rightarrow [c, d]$ . For any  $a, b \in [0, 1]$  with  $0 < a < b < 1$ , we define a path

$$\begin{aligned} k_{a,b}: [0, 1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 1] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

Then  $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$  and  $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$ .

Let  $\gamma$  be that path  $\gamma: I \rightarrow I$  with the following graphs:



Note that  $\gamma \simeq_p i$ . Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

which is what we wanted to show. □

## 9.52 Fundamental group

**Definition 13.** Let  $X$  be a topological space and  $x_0 \in X$ . A **loop** based at  $x_0$  in  $X$  is a path  $\alpha: I \rightarrow X$  s.t.  $\alpha(0) = \alpha(1) = x_0$ . Then

$$\pi_1(X, x_0) = \{[\alpha] \mid \alpha: \text{loop in } X \text{ based at } x_0\}$$

is the **fundamental group** of  $X$  with base point  $x_0$ .<sup>a</sup>

<sup>a</sup> $\pi_1(X, x_0)$  is a group with the operation  $*$  by Theorem 8. For  $[\alpha], [\beta] \in \pi_1(X, x_0)$ ,  $[\alpha] * [\beta]$  is always defined,  $[e_{x_0}]$  is an identity element,  $*$  is associative and  $[\alpha]^{-1} = [\bar{\alpha}]$ . This makes  $(\pi_1(X, x_0), *)$  a group.

**Example.**  $\pi_1(\mathbb{R}^n, x_0)$  is a trivial group. Any two loops in  $\mathbb{R}^n$  based at  $x_0$  are path-homotopic. Thus,  $\pi_1(\mathbb{R}^n, x_0)$  has only one element.

**Remark.** All groups are a fundamental group of some space.

**Definition 14.** Let  $\alpha$  be a path in  $X$  from  $x_0$  to  $x_1$ . Define a function  $\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$  by  $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$ .

**Theorem 9** (52.1).  $\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$  is a group isomorphism.

**Proof. Homomorphism** To show that  $\hat{\alpha}$  is a group homomorphism, we compute

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * ([f] * [g]) * [\alpha] \\ &= [\bar{\alpha}] * [f] * [e_{x_0}] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g]).\end{aligned}$$

**Bijjective** To show that  $\hat{\alpha}$  is one-to-one and onto function, we show existence of inverse of  $\alpha$ .

$$\begin{aligned}(\hat{\alpha} \circ \hat{\alpha})([h]) &= [\bar{\alpha}] * ([\bar{\alpha}] * [h] * [\bar{\alpha}]) * [\alpha] \\ &= [e_{x_1}] * [h] * [e_{x_1}] = [h].\end{aligned}$$

Thus,  $\hat{\alpha} \circ \hat{\alpha}$  is the identity function. Similarly, we can show that  $\hat{\alpha} \circ \hat{\alpha}$  is the identity function.  $\square$

**Definition 15.** A topological space  $X$  is said to **simply connected** if it is path-connected and  $\pi_1(X, x_0)$  is a trivial group.

**Example.** Any convex subset of  $\mathbb{R}^n$  is simply connected.

**Lemma 5 (52.3).** Suppose  $X$  is simply connected and  $\alpha, \beta: I \rightarrow X$  are paths from  $x_0$  to  $x_1$ . Then  $\alpha \simeq_p \beta$ .

**Proof.**  $\alpha * \bar{\beta}$  is a loop base at  $x_0$ . Since  $X$  is simply connected,  $\alpha * \bar{\beta} \simeq_p e_{x_0}$ . Thus,  $[\alpha] = [\alpha] * [e_{x_1}] = [\alpha] * [\bar{\beta}] * [\beta] = [e_{x_0}] * [\beta] = [\beta]$ .  $\square$

**Definition 16.** Let  $h: (X, x_0) \rightarrow (Y, y_0)$  be a continuous map ( $h(x_0) = y_0$ ). Define  $h_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$  by  $h_*([f]) = [h \circ f]$ . Then  $h_*$  is a **group homomorphism induced from  $h$** .

**Well-definedness** Let  $f' \in [f]$  and  $F$  be a path-homotopy from  $f$  to  $f'$ . Then  $h \circ F: I \times I \rightarrow Y$  is a path-homotopy from  $h \circ f$  to  $h \circ f'$ .

**Homomorphism**  $h_*$  is a homomorphism, because  $(h \circ f) * (h \circ g) = h \circ (f * g)$ . That is,  $h_*([f]) * h_*([g]) = h_*([f * g])$ .

**Theorem 10 (52.4).** (i) For two continuous maps  $h: (X, x_0) \rightarrow (Y, y_0)$  and  $k: (Y, y_0) \rightarrow (Z, z_0)$ ,  $(k \circ h)_* = k_* \circ h_*$ .

(ii) For the identity map  $i: (X, x_0) \rightarrow (X, x_0)$ ,  $i_*$  is the identity homomorphism.

**Proof.** (i)  $(k \circ h)_*([f]) = [(k \circ h) \circ f] = [k \circ (h \circ f)] = k_*([h \circ f]) = k_*(h_*([f])) = (k_* \circ h_*)([f]).$

(ii)  $i_*([f]) = [i \circ f] = [f].$

□

**Corollary 3 (52.5).** If  $h: (X, x_0) \rightarrow (Y, y_0)$  is a homeomorphism, then  $h_*$  is an isomorphism.

**Proof.** Let  $k: (Y, y_0) \rightarrow (X, x_0)$  be the inverse of  $h$ . Then,

$$k_* \circ h_* = (k \circ h)_* = (\text{id}_X)_* = \text{the identity on } \pi_1(X, x_0)$$

$$h_* \circ k_* = (h \circ k)_* = (\text{id}_Y)_* = \text{the identity on } \pi_1(Y, y_0)$$

Thus,  $h_*$  is an isomorphism. □

This corollary says  $\pi_1$  is an topological invariant. We can use the fundamental group to detect that two spaces are not homeomorphic, i.e.  $\pi_1(X, x_0) \not\cong \pi_1(Y, y_0) \Rightarrow X \not\cong Y$ . Note that  $X \not\cong Y \not\Rightarrow \pi_1(X, x_0) \not\cong \pi_1(Y, y_0)$  and  $\pi_1(X, x_0) \cong \pi_1(Y, y_0) \not\Rightarrow X \cong Y$ .

**Exercise (52.6).** Let  $X$  be path-connected and  $h: X \rightarrow Y$  be continuous with  $h(x_0) = y_0$  and  $h(x_1) = y_1$ . Let  $\alpha$  be a path in  $X$  from  $x_0$  to  $x_1$  and  $\beta = h \circ \alpha$ . Then,  $\hat{\beta} \circ h_* = h_* \circ \hat{\alpha}$ , that is, the diagram of maps

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_*} & \pi_1(Y, y_0) \\ \downarrow \hat{\alpha} & & \downarrow \hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{h_*} & \pi_1(Y, y_1) \end{array}$$

commutes.

**Proof.** Let  $[f] \in \pi_1(X, x_0)$ .

$$\begin{aligned} (\hat{\beta} \circ h_*)([f]) &= \hat{\beta}(h_*([f])) = [\bar{\beta}] * h_*([f]) * [\beta] \\ &= h_*([\bar{\alpha}]) * h_*([f]) * h_*([\alpha]) \\ &= h_*([\bar{\alpha}] * [f] * [\alpha]) \\ &= h_*([\hat{\alpha}([f])]) \\ &= (h_* \circ \hat{\alpha})([f]). \end{aligned}$$

Thus, if  $X$  is path-connected, the group homomorphism induced by a continuous map is independent of base point. ◇

**Note.** HW3: Exercise §52 – #1, #2, #3, #4.

Lecture 11  
Wed, Oct 6

## 9.53 Covering spaces

**Definition 17.** Let  $p: E \rightarrow B$  be a continuous surjective map. An open subset  $U$  of  $B$  is said to be **evenly covered** by  $p$  if  $p^{-1}(U)$  is a union of disjoint open subsets  $V_\alpha$  of  $E$  s.t. each  $V_\alpha$  is homeomorphic to  $U$  by  $p$ . That is,  $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$ ,  $V_\alpha \cong U$  by  $p \forall \alpha$ .

Each  $V_\alpha$  is called a *slice*. (The set  $\{V_\alpha\}$  is a partition of  $p^{-1}(U)$  into slices.)

If every point of  $B$  has an open nbh which is evenly covered by  $p$ , then  $p$  is called a **covering map**,  $E$  **covering space**,  $B$  **base space**.

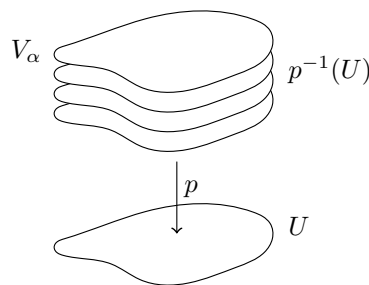


Figure 9.1: evenly covered

**Remark.** If  $U' \subset U$ , also open and  $U$  is evenly covered, then also  $U'$ .

**Example.** Let  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . Take  $p: \mathbb{R} \rightarrow S^1$  given by  $t \mapsto e^{2\pi it}$ . Note that  $\mathbb{R}$  is an easier space than  $S^1$ , and so will be  $\pi_1$  (1 vs  $\mathbb{Z}$ ).

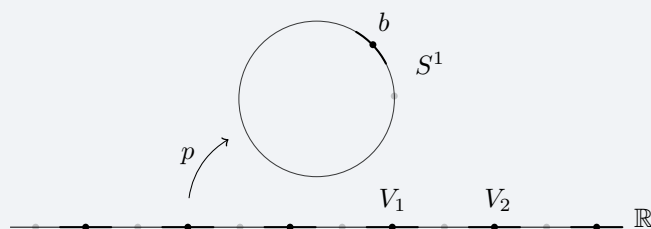


Figure 9.2: example of a covering space

There are also other covering spaces of  $p$ . For example,  $p': S^1 \rightarrow S^1$  given by  $z \mapsto z^3$ .

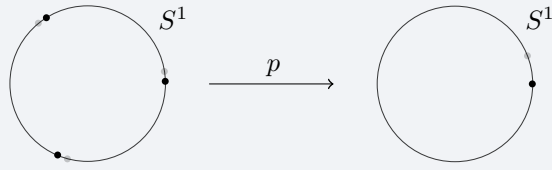


Figure 9.3: second example of a covering space

Here we have three copies for each point. We say that the covering has 3 sheets. Note that this is independent of which point we take. This is always the case! We can show that these are the only coverings of  $S^1$ :  $\mathbb{R}$  and  $z \mapsto z^n$ .

**Proposition 2.** A covering map  $p: E \rightarrow B$  is always an open map.

**Proof.** We want to show that for every  $x \in E$  and any open subset  $A \subset E$  containing  $x$ , there is an open subset of  $B$  contained in  $p(A)$ . Choose an evenly covered open subset  $U$  of  $p(x)$ . Let  $\{V_\alpha\}$  be a partition of  $p^{-1}(U)$  into slices, and  $V_\beta$  be a slice containing  $x$ . Since  $A$  and  $V_\beta$  are open,  $A \cap V_\beta$  is open in  $E$ , hence open in  $V_\beta$ . Since  $V_\beta \cong U$  by  $p$ ,  $p(A \cap V_\beta)$  is open in  $U$ , hence in  $B$ . Thus,  $p(A \cap V_\beta)$  is open in  $B$  and contained in  $p(A)$ .  $\square$

**Theorem 11** (53.2). Let  $p: E \rightarrow B$  be a covering map,  $B_0$  a subspace of  $B$ ,  $E_0 = p^{-1}(B_0)$ . Then,  $p|_{E_0}: E_0 \rightarrow B_0$  is also a covering map.

**Proof.** For each  $b \in B_0$ , there is open nbh  $U$  of  $b$  in  $B$  which is evenly covered by  $p$ . Let  $\{V_\alpha\}$  be a partition of  $p^{-1}(U)$  into slices. Then,

- $U \cap B_0$  is an open nbh of  $b$  in  $B_0$ .
- $\{V_\alpha \cap E_0\}$  is a partition of  $p^{-1}(U \cap B_0)$ .
- $V_\alpha \cap E_0 \cong U \cap B_0$  by  $p$ .

$\square$

**Theorem 12** (53.3). Let  $p_1: E_1 \rightarrow B_1$  and  $p_2: E_2 \rightarrow B_2$  be covering maps. Then,  $p_1 \times p_2: E_1 \times E_2 \rightarrow B_1 \times B_2$  is a covering map.

**Proof.** Let  $(b_1, b_2) \in B_1 \times B_2$  and  $U_1$  be an evenly covered open nbh of  $b_1$  in  $B_1$  for  $p_1$  (same for  $U_2$ ). We claim that  $U_1 \times U_2$  is an evenly covered open nbh of  $(b_1, b_2)$  in  $B_1 \times B_2$  for  $p_1 \times p_2$ .

$$\begin{aligned} (p_1 \times p_2)^{-1}(U_1 \times U_2) &= p_1^{-1}(U_1) \times p_2^{-1}(U_2) \\ &= (\bigsqcup_\alpha V_\alpha) \times (\bigsqcup_\beta W_\beta) = \bigsqcup_{\alpha, \beta} (V_\alpha \times W_\beta). \end{aligned}$$

$V_\alpha \times W_\beta \cong U_1 \times U_2$  by  $p_1 \times p_2$ , since  $V_\alpha \cong U_1$  by  $p_1$  and  $W_\beta \cong U_2$  by  $p_2$ .  $\square$



**Example.** Let  $p: \mathbb{R} \rightarrow S^1$  be the covering map in the previous example.

- $p \times p: \mathbb{R} \times \mathbb{R} \rightarrow S^1 \times S^1$  is a covering map by Theorem 12.
- $p \times p: (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \rightarrow \text{Bouguet with two leaves}$  is a covering map by Theorem 11.

**Exercise (53.3).** Let  $p: E \rightarrow B$  be a covering map; let  $B$  be connected. Show that if  $p^{-1}(b_0)$  has  $k$  elements for some  $b_0 \in B$ , then  $p^{-1}(b)$  has  $k$  elements for every  $b \in B$ . In such a case,  $E$  is called a  **$k$ -fold covering** of  $B$ .

**Proof.** Let  $B_1 = \{b \in B \mid |p^{-1}(b)| = k\}$  and  $B_2 = \{b \in B \mid |p^{-1}(b)| \neq k\}$ . Then  $b_0 \in B_1$ , hence  $B_1 \neq \emptyset$ . Suppose  $B_2 \neq \emptyset$ . For  $b \in B$ , let  $U_b$  be an evenly covered open nbh of  $b$ . And let  $U_1 = \bigcup_{b \in B_1} U_b$ ,  $U_2 = \bigcup_{b \in B_2} U_b$ . Then, both are open non-empty and  $U_1 \cup U_2 = B$ . Since  $B$  is connected,  $U_1 \cap U_2 \neq \emptyset$ . If  $b_1 \in U_1 \cap U_2$ , then we have a contradiction.  $\diamond$

**Note.** HW3: Exercise §53 – #4, #5, #6.

**Remark.** A covering is always local homeomorphism. But there are surjective local homeomorphism which are not covering maps. A covering map is more than a surjective local homeomorphism.

For example,  $p: \mathbb{R}_0^+ \rightarrow S^1$  given by  $t \mapsto e^{2\pi it}$ . Consider the inverse image of a neighborhood around 1. When we restrict  $p$  to the part around 0, it is no longer a homeomorphism (we don't get the whole neighborhood around one).

## Creating new covering spaces out of old ones

- Suppose  $p: E \rightarrow B$  is a covering and  $B_0 \subset B$  is a subspace with the subspace topology. Let  $E_0 = p^{-1}(B_0)$  and  $p_0 = p|_{E_0}$ . Then  $(E_0, p_0)$  is a covering of  $B_0$ .
- Suppose that  $(E, p)$  is a covering of  $B$  and  $(E', p')$  is a covering of  $B'$ , then  $(E \times E', p \times p')$  is a covering of  $B \times B'$ .

**Example.** Let  $T^2 = S^1 \times S^1$ .

- $p: \mathbb{R}^2 \rightarrow S^1 \times S^1$  given by  $(t, s) \mapsto (e^{ait}, e^{bis})$ .
- $p': \mathbb{R} \times S^1 \rightarrow T^2$  given by  $(t, z) \mapsto (e^{ait}, z^n)$ .
- $p: S^1 \times S^1 \rightarrow T^2$  given by  $(z_1, z_2) \mapsto (z_1^n, z_2^m)$ .

These are the only types of coverings of the torus. We'll prove this later on.

## 9.54 $\pi_1(S^1)$

**Definition 18.** Let  $p: E \rightarrow B$  and  $f: X \rightarrow B$  be continuous maps. Then, a **lifting** of  $f$  is a map  $\tilde{f}: X \rightarrow E$  s.t.  $f = p \circ \tilde{f}$ .

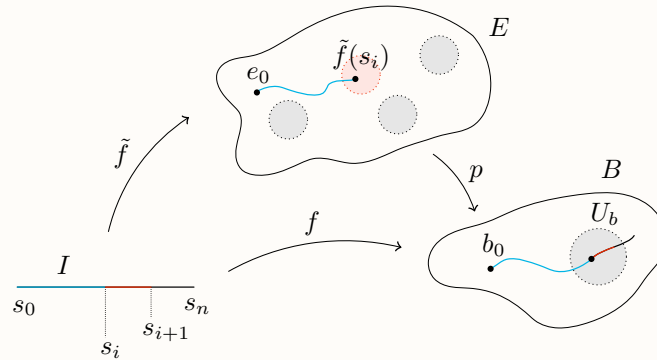
$$\begin{array}{ccc} & & E \\ & \nearrow \tilde{f} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

**Lemma 6** (54.1, Unique path-lifting lemma).

$$\begin{array}{ccc} & & (E, e_0) \\ & \nearrow \exists! \tilde{\gamma} & \downarrow p: \text{c.m.} \\ (I, 0) & \xrightarrow[\text{path}]{\gamma} & (B, b_0) \end{array}$$

**Proof. Existence** Let  $\{U_\alpha\}$  be an open covering of  $B$  consisting of evenly-covered open subsets. Then,  $\{\gamma^{-1}(U_\alpha)\}$  is an open covering of the compact space  $I$ , and there exists a Lebesgue number  $\varepsilon$  (Any open interval of length less than  $\varepsilon$  is contained in some  $\gamma^{-1}(U_\alpha)$ ). Then we have a subdivision  $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$  so that  $\gamma[s_i, s_{i+1}] \subset U_\alpha$  for some  $\alpha$  (by setting  $s_i - s_{i-1} < \varepsilon$ ).

Define  $\tilde{\gamma}(0) = e_0$ . Suppose  $\tilde{\gamma}(s)$  is defined for  $0 \leq s \leq s_i$ . Select  $\alpha_0$  so that  $\gamma[s_i, s_{i+1}] \subset U_{\alpha_0}$ . Let  $\{V_\beta\}$  be the partition of  $p^{-1}(U_{\alpha_0})$  into slices. And let  $V_{\beta_0}$  be the slice s.t.  $\tilde{\gamma}(s_i) \in V_{\beta_0}$ . Since  $V_{\beta_0} \cong U_{\alpha_0}$  by  $p|_{V_{\beta_0}}$ , we have an closed arc  $(p|_{V_{\beta_0}})^{-1}(\gamma[s_i, s_{i+1}])$ . For  $s_i \leq s \leq s_{i+1}$ , defined  $\tilde{\gamma}(s) = (p|_{V_{\beta_0}})^{-1}(\gamma(s))$ . Then  $(p \circ \tilde{\gamma})(s) = \gamma(s)$ .



**Uniqueness** Let  $\tilde{\tilde{\gamma}}$  be another lift of  $\gamma$  s.t.  $\tilde{\tilde{\gamma}}(0) = e_0$ . Since  $\tilde{\tilde{\gamma}}[s_i, s_{i+1}]$  is connected and  $\{V_\beta\}$  are mutually disjoint,  $\tilde{\tilde{\gamma}}[s_i, s_{i+1}] \subset V_{\beta_0}$ . Note that, in  $V_{\beta_0}$ ,  $\tilde{\gamma}(s)$  is a unique point which projects  $\gamma(s)$ . Thus,  $\tilde{\tilde{\gamma}}(s) = \tilde{\gamma}(s) \forall s$ .  $\square$

Lecture 12  
Mon, Oct 11

**Lemma 7** (54.2, Homotopy lifting lemma).

$$\begin{array}{ccc} & & (E, e_0) \\ & \nearrow \exists! \tilde{F} & \downarrow p: \text{c.m.} \\ (I \times I, (0, 0)) & \xrightarrow[\text{conti.}]{F} & (B, b_0) \end{array}$$

Furthermore, if  $F$  is a path-homotopy, then so is  $\tilde{F}$ .

**Proof.** (i) • Define  $\tilde{F}(0, 0) = e_0$ .

- Divide  $I \times I$  into subrectangles so that  $F(I_i \times J_j)$  is contained in an evenly-covered open subset of  $B$ .
- Define  $\tilde{F}$  step by step: Assume that  $\tilde{F}$  is defined on the red-part. Define  $\tilde{F}(x) = (p|_V)^{-1}(F(x))$ ,  $\forall x \in A$ . (Then  $p \circ \tilde{F}(x) = F(x)$ ).

- (ii) Assume that  $F$  is a path-homotopy ( $F(0, t) = b_0$ ,  $F(1, t) = b_1$ ,  $\forall t$ ). Then  $\tilde{F}(\{0\} \times I) \subset p^{-1}(b_0)$  and  $\tilde{F}(\{1\} \times I) \subset p^{-1}(b_1)$ . Since  $\{0\} \times I$  and  $\{1\} \times I$  are connected,  $\tilde{F}(\{0\} \times I) = e_0$ ,  $\tilde{F}(\{1\} \times I) =$  a pt in  $p^{-1}(b_1)$ . □

**Theorem 13** (54.3). Let  $p: (E, e_0) \rightarrow (B, b_0)$  be a covering map. Let  $f, g$  be paths in  $B$  from  $b_0$  to  $b_1$  and  $\tilde{f}, \tilde{g}$  be lifts of  $f$  and  $g$  starting  $e_0$ . Then, if  $f \simeq_p g$ , then  $\tilde{f} \simeq_p \tilde{g}$ .

**Definition 19.** Let  $p: (E, e_0) \rightarrow (B, b_0)$  be a covering map. Then the **lifting correspondence** is the map

$$\begin{aligned} \phi: \pi_1(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1) \end{aligned}$$

where  $\tilde{f}$  is the unique lift of  $f$  starting at  $e_0$ .<sup>a</sup> This is well-defined because  $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$ . This  $\phi$  depends on the choice of  $e_0$ .

<sup>a</sup>There is no guarantee that  $\tilde{f}$  is a loop.

**Theorem 14** (54.4). If  $E$  is path-connected, then  $\phi$  is surjective. If  $E$  is simply-connected, then  $\phi$  is bijective.

**Proof.** For  $e \in p^{-1}(b_0)$ , there is a path  $g$  in  $E$  from  $e_0$  to  $e$ . Then,  $p \circ g$  is a loop based at  $b_0$  ( $p(e_0) = p(e) = b_0$ ). By the uniqueness of path-lifting,  $\widetilde{p \circ g} = g$ . Then,  $\phi([p \circ g]) = (\widetilde{p \circ g})(1) = g(1) = e$ . For any point of  $p^{-1}(b_0)$ , there is a loop homotopy class which is sent to  $e$  by  $\phi$ . Thus,  $\phi$  is surjective.

For  $[f], [g] \in \pi_1(B, b_0)$ , suppose  $\tilde{f}(1) = \tilde{g}(1)$ , that is,  $\phi([f]) = \phi([g])$ . Since  $E$  is simply connected,  $\tilde{f} \simeq_p \tilde{g}$  by Lemma 5. For a homotopy  $\tilde{F}$  between  $\tilde{f}$  and  $\tilde{g}$ ,  $f \simeq_p g$  by  $p \circ \tilde{F}$ . Thus  $[f] = [g]$ , hence  $\phi$  is injective. □

**Theorem 15** (54.5).  $\pi_1(S^1) \cong \langle \mathbb{Z}, + \rangle$ .

**Proof.** We use the covering map  $p: (\mathbb{R}, 0) \rightarrow (S^1, 1)$  defined by  $p(t) = e^{2\pi it}$ . The function  $\phi: \pi_1(S^1, 0) \rightarrow p^{-1}(1) = \mathbb{Z}$  is bijective, because  $\mathbb{R}$  is simply connected. It's enough to show that  $\phi$  is a group homomorphism.

For  $[f], [g] \in \pi_1(S^1, 1)$ , let  $\tilde{f}$  and  $\tilde{g}$  be their lifts starting at 0. Define  $\tilde{g}(s) = \tilde{f}(1) + \tilde{g}(s)$ . Then  $(p \circ \tilde{g})(s) = p(\tilde{f}(1) + \tilde{g}(s)) = p(\tilde{g}(s)) = g(s)$ . Thus,  $\tilde{g}$  is the lift of  $g$  starting at  $\tilde{f}(1)$ .  $\tilde{f} * \tilde{g}$  is a path starting at  $\tilde{f}(0) = 0$ , and  $p(\tilde{f} * \tilde{g}) = (p \circ \tilde{f}) * (p \circ \tilde{g}) = f * g$ , hence  $\tilde{f} * \tilde{g}$  is the lift of  $f * g$  starting at 0. In conclusion,

$$\phi([f] * [g]) = (\tilde{f} * \tilde{g})(1) = \tilde{g}(1) = \tilde{f}(1) + \tilde{g}(1) = \phi([f]) + \phi([g]).$$

□

**Theorem 16** (54.6). Let  $p: (E, e_0) \rightarrow (B, b_0)$  be a covering map.

(i)  $p_*: \pi_1(E, e_0) \rightarrow \pi_1(B, b_0)$  is a monomorphism<sup>a</sup>.

(ii) Let  $H = p_*(\pi_1(E, e_0))$ . Then,

$$\begin{aligned} \Phi: \pi_1(B, b_0) / H &\longrightarrow p^{-1}(b_0) \\ H * [f] &\longmapsto \phi([f]) \end{aligned}$$

is injective. If  $E$  is path-connected, then  $\Phi$  is bijective.

(iii) For a loop based at  $b_0$ ,  $[f] \in H$  iff  $f$  lifts to a loop in  $E$  based at  $e_0$ .

<sup>a</sup>injective homomorphism. \*epimorphism: surjective homomorphism.

**Proof.** (i) Let  $\tilde{h}$  be a loop at  $e_0$  s.t.  $p_*([\tilde{h}]) = [c_{b_0}]$ . Then, there is a path-homotopy  $F$  between  $p \circ \tilde{h}$  and  $c_{b_0}$ , and its lift  $\tilde{F}$  is a homotopy between  $\tilde{h}$  and  $\tilde{c}_{b_0} = c_{e_0}$ . Thus,  $[\tilde{h}] = [c_{e_0}]$ .<sup>a</sup>

(ii) Let  $[f] \in \pi_1(B, b_0)$  and  $[h] \in H$ . Then,  $\phi([h * f]) = \widetilde{h * f}(1) = \tilde{f}(1)$ . Thus, we can define  $\Phi(H * [f]) = \tilde{f}(1) = \phi([f])$ . Suppose  $\phi([f]) = \phi([g])$ , i.e.  $\tilde{f}(1) = \tilde{g}(1)$ .  $\tilde{h} := f * \tilde{g}$  is a loop at  $e_0$ , and  $[\tilde{h} * \tilde{g}] = [\tilde{f}]$ . Thus,

□

<sup>a</sup> $p_*$  is injective iff  $\ker p_*$  is trivial.

## 9.55 Retractions and fixed points

**Definition 20.** A subspace  $A$  of a topological space  $X$  is called a **retract** of  $X$  if there exists a continuous map  $r: X \rightarrow A$  s.t.  $r|_A = \text{id}_A$ . The map  $r$  is called a **retraction** of  $X$  onto  $A$ .

**Example.**  $S^1$  is a retract of  $\mathbb{R}^2 - \{0\}$ .

Lecture 13  
Wed, Oct 13

**Example.** Let  $X$  be the figure 8 space, and denote the right circle by  $A$ . Then it's easy to see that there exists a retract from the whole space to  $A$  by mapping the left circle onto the right

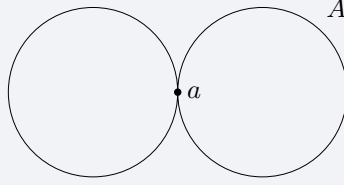


Figure 9.4: Figure 8 space

**Lemma 8** (55.1). Let  $A$  be a retract of  $X$  and  $i: A \hookrightarrow X$  be the inclusion map. Then  $i_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$  is injective.

**Proof.**  $r \circ i: A \hookrightarrow X \rightarrow A$  is the identity map, hence  $(r \circ i)_* = r_* \circ i_*$  is an isomorphism. Thus  $i_*$  is injective.  $\square$

**Theorem 17** (55.2). There is no retraction of  $B^2$  onto  $S^1$ .

**Proof.**  $i_*: \pi_1(S^1, 1) \rightarrow \pi_1(B^2, 1)$  can not be injective because  $\pi_1(S^1, 1) \cong \mathbb{Z}$  has infinitely many elements and  $\pi_1(B^2, 1)$  is a trivial group.  $\square$

**Lemma 9** (55.3). Let  $h: S^1 \rightarrow X$  be a continuous map. Then TFAE:

- (i)  $h$  is null-homotopic.
- (ii)  $h$  extends to a continuous map  $k: B^2 \rightarrow X$ .
- (iii)  $h_*: \pi_1(S^1, b_0) \rightarrow \pi_1(X, x_0)$  is the trivial homomorphism.<sup>a</sup>

<sup>a</sup> $h_*$  maps every element of  $\pi_1(S^1, b_0)$  to the identity in  $\pi_1(X, x_0)$ .

**Proof.** (i)  $\Rightarrow$  (ii) Let  $H$  be a homotopy between  $h$  and a constant map, and  $\pi: S^1 \times I \rightarrow B^2$  be the map  $\pi(x, t) = (1 - t)x$ . Then  $\pi$  is a quotient map. By Theorem 1.(i), there is a continuous map  $k: B^2 \rightarrow X$  s.t.  $H = k \circ \pi$ . For  $x \in S^1 \subset B^2$ ,  $\pi(x, 0) = x$ .  $k(x) = k(\pi(x, 0)) = H(x, 0) = h(x)$ .

(ii)  $\Rightarrow$  (iii) Let  $j: S^1 \hookrightarrow B^2$  be the inclusion map. Then  $h = k \circ j$ ,  $h_* = k_* \circ j_*$ . Since  $\pi_1(B^2)$  is trivial,  $j_*$  is trivial. Thus  $h_*$  is trivial.

(iii)  $\Rightarrow$  (i) Let  $p: [0, 1] \rightarrow S^1$  be the quotient map s.t.  $p(0) = p(1) = b_0$ . Then  $p$  is a loop based at  $b_0$ . Since  $h_*$  is trivial,  $h_*([p]) = [h \circ p]$  is the identity element of  $\pi_1(X, x_0)$ . Let  $F$  be a path-homotopy from  $h \circ p$  to the constant map  $c_{x_0}$ . Applying Theorem 1.(i) to  $F$  and  $p \times \text{id}$ , there is a continuous map  $H: S^1 \times I \rightarrow X$  s.t.  $F = H \circ (p \times \text{id})$ .

$$\bullet H(x, 0) = H(p(y), 0) = F(y, 0) = (h \circ p)(y) = h(x)$$

- $H(x, 1) = H(p(y), 1) = F(y, 1) = c_{x_0}$

Thus  $H$  is a homotopy between  $h$  and  $c_{x_0}$ .  $\square$

**Corollary 4** (55.4). The inclusion map  $j: S^1 \hookrightarrow \mathbb{R}^2 - \{0\}$  is not null-homotopic.

**Proof.**  $r \circ j: S^1 \rightarrow \mathbb{R}^2 - \{0\} \rightarrow S^1$  is the identity map. Thus  $j_*$  is injective,  $\pi_1(S^1) \cong \mathbb{Z}$ , hence  $j_*$  is not trivial. Then by Lemma 9,  $j$  is not null-homotopic.  $\square$

**Definition 21.** A **vector field**  $v$  on  $B^2$  is a continuous map  $v: B^2 \rightarrow \mathbb{R}^2$ .

**Theorem 18** (55.5). Given a non-vanishing vector field  $v$  on  $B^2$ , there are two points  $x_0$  and  $x_1$  on  $S^1$  s.t.  $v(x_0)$  is inward and  $v(x_1)$  is outward.

**Proof.** To say that  $v$  is non-vanishing, we can consider  $v$  as a map  $v: B^2 \rightarrow \mathbb{R}^2 - \{0\}$ .  $v$  is an extension of  $v|_{S^1}: S^1 \rightarrow \mathbb{R}^2 - \{0\}$ , so  $v|_{S^1}$  is null-homotopic by Lemma 9. Now suppose that there is no such  $x_0$ . Then,  $v|_{S^1}$  is path-homotopic to the inclusion  $j: S^1 \rightarrow \mathbb{R}^2 - \{0\}$  by the homotopy  $F: S^1 \times I \rightarrow \mathbb{R}^2 - \{0\}$  defined by  $F(x, t) = tx + (1 - t)v|_{S^1}(x)$ . Indeed,

- $F(x, 0) = v|_{S^1}(x)$ ,  $F(x, 1) = x = j(x)$
- If  $F(x, t) = 0$ , then  $v|_{S^1}(x) = \frac{t}{t-1}x$ ; inward.  $\nmid \nexists x_0$ .

Thus,  $j \simeq v|_{S^1} \simeq \text{constant map}$ .  $\nmid$  Corollary 4.  $\square$

**Theorem 19** (55.6, Brouwer fixed-point theorem for  $B^2$ ). If  $f: B^2 \rightarrow B^2$  is a continuous map, then there exists  $x \in B^2$  s.t.  $f(x) = x$ .

**Proof.** Suppose there is no fixed point. Then  $v(x) = f(x) - x$  is a non-vanishing vector field on  $B^2$ . Hence there is an outward point  $x_0$  on  $S^1$ , i.e.  $v(x_0) = ax_0$  for  $a > 0$ .  $f(x_0) - x_0 = ax_0$ .  $f(x_0) = (a + 1)x_0 \notin B^2$ .  $\nmid \square$

**Corollary 5** (55.7, Application of FPT). Let  $A$  be a  $3 \times 3$  matrix of positive real numbers. Then  $A$  has a positive real eigenvalue.

**Proof.** Consider  $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  to be a linear map. Let  $B = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1, x_1, x_2, x_3 \geq 0\}$  (first octant of  $S^2$ ). Note that  $B \cong B^2$ . For any  $x \in B$ ,  $A(x) \in \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \geq 0\}$ , as  $A$  has positive entries. Then  $F(x) = \frac{A(x)}{\|A(x)\|}$  is a map from  $B$  to  $B$ . By Theorem 19, there exists  $x_0 \in B$  s.t.  $F(x_0) = \frac{A(x_0)}{\|A(x_0)\|} = x_0$ , hence  $A(x_0) = \|A(x_0)\|x_0$ .  $\square$

**Note.** HW4:

- Read and understand the topological proof of fundamental theorem

of algebra.

- Exercise §55 – #1, #2.

## 9.57 Borsuk–Ulam theorem

Lecture 14  
Mon, Oct 18

**Definition 22.** For  $x \in S^n$ , the **antipode**  $x$  is  $-x$ . A map  $h: S^n \rightarrow S^m$  is **antipode-preserving** if  $h(-x) = -h(x)$  for all  $x \in S^n$ .

**Theorem 20** (57.1). If  $h: S^1 \rightarrow S^1$  is continuous and antipode-preserving, then  $h$  is not null-homotopic.

**Proof.** Let  $b_0 = (0, 1)$  and  $\rho: S^1 \rightarrow S^1$  be the rotation of  $S^1$  s.t.  $\rho(h(b_0)) = b_0$ .  $(\rho \circ h)(-x) = \rho(-h(x)) = -\rho(h(x)) = -(\rho \circ h)(x)$  (antipode-preserving). Suppose there is a homotopy between  $h$  and a constant map. Then  $\rho \circ h$  is a homotopy between  $\rho \circ h$  and a constant map. Therefore we may prove the theorem under assumption  $h(b_0) = b_0$ .

**Step 1.** Let  $q: S^1 \subset \mathbb{C} \rightarrow S^1$  be the map  $q(z) = z^2$ . Then  $q$  is a quotient map and  $q(-z) = q(z)$ . For  $x \in S^1$ ,  $q^{-1}(x)$  is two antipodal points.  $h(-x) = -h(x)$ .  $q(h(-z)) = q(-h(z)) = q(h(z))$ . Apply Theorem 1 to  $q$  and  $q \circ h$ . Then there exists  $k: S^1 \rightarrow S^1$  s.t.  $k \circ q = q \circ h$ .  $k(b_0) = k(b_0^2) = k(q(b_0)) = q(h(b_0)) = q(b_0) = b_0^2 = b_0$ .

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ \downarrow q & & \downarrow q \\ S^1 & \xrightarrow{k} & S^1 \end{array}$$

**Step 2.** We claim  $k_*: \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, b_0)$  is non-trivial. We can check that  $q$  is a covering map. If  $\tilde{f}$  is a path from  $b_0$  to  $-b_0$  in  $S^1$ , then  $[f = q \circ \tilde{f}] \neq 1$  in  $\pi_1(S^1, b_0)$ .  $k_*([f]) = [k \circ q \circ \tilde{f}] = [q \circ h \circ \tilde{f}] \neq 1$ .

**Step 3.**  $h_*$  is nontrivial. We will prove  $h$  is not null-homotopic.

$$\begin{aligned} q_*: \pi_1(S^1, b_0) \cong \mathbb{Z} &\longrightarrow \pi_1(S^1, b_0) \cong \mathbb{Z} \\ n &\longmapsto 2n \end{aligned}$$

Thus  $q_*$  is injective,  $k_* \circ q_*$  is injective, so is  $q_* \circ h_*$ . Hence  $h_*$  is injective.

□

**Theorem 21** (57.2). There is no continuous antipode-preserving map  $g: S^2 \rightarrow S^1$ .

**Proof.** Suppose  $g$  is such a map. Then  $h = g|_{S^1}$  is continuous and antipode-

preserving map  $S^1 \rightarrow S^1$ . Then, by Theorem 20,  $h$  is not null-homotopic. But  $g|_{\text{upper-hemi-sphere}}$  is an extension of  $h$ .  $\nmid$  to Lemma 9.  $\square$

**Theorem 22** (57.3, Borsuk–Ulam theorem for  $S^2$ ). Given a continuous map  $f: S^2 \rightarrow \mathbb{R}^2$ , there is a point  $x \in S^2$  s.t.  $f(x) = f(-x)$ .

**Proof.** Suppose there is no such point. Then we can define  $g: S^2 \rightarrow S^1$  by  $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$ .  $g(-x) = \frac{f(-x)-f(x)}{\|f(-x)-f(x)\|} = -g(x)$ .  $\nmid$  to Theorem 21.  $\square$

**Theorem 23** (57.4, Bisection theorem). For two bounded polygonal regions in  $\mathbb{R}^2$ , there exists a line that bisects each of them.

**Proof.** Let  $A_1, A_2$  be bounded polygonal regions in  $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$ . Given a point  $u \in S^2$ , let  $P$  be the plane s.t.  $O \in P$ ,  $u \perp P$ . Let  $f_i(u)$  equal the area of the portion of  $A_i$  that lies on the same side of  $P$  as does the vector  $u$ . If  $u = (0, 0, 1)$ , then  $f_i(u) = \text{area } A_i$ , and if  $u = (0, 0, -1)$ , then  $f_i(u) = 0$ .  $f_i(u) + f_i(-u) = \text{area } A_i$  for all  $u \in S^2$ . Define a map  $F: S^2 \rightarrow \mathbb{R}^2$  by  $F(u) = (f_1(u), f_2(u))$ . By Theorem 22, there exists  $u_0 \in S^2$  s.t.  $F(u_0) = F(-u_0)$ . Then  $f_i(u_0) = f_i(-u_0) = \frac{1}{2} \text{area } A_i$ . Hence,  $P_{u_0} \cap \mathbb{R}^2 \times \{1\}$  bisects  $A_1$  and  $A_2$ .  $\square$

**Note.** HW5: Exercise §57 – #1, #2, #3.

## 9.58 Deformation retracts and homotopy type

**Lemma 10** (58.1). Let  $h, k: (X, x_0) \rightarrow (Y, y_0)$  be continuous maps. If there is a homotopy  $H$  between  $h$  and  $k$  s.t.  $H(x_0, t) = y_0$  for all  $t$ , then  $h_* = k_*$ .

**Proof.** Let  $f$  be a loop in  $X$  based at  $x_0$ . Consider the map

$$\begin{aligned} F: I \times I &\xrightarrow{f \times \text{id}} X \times I \xrightarrow{H} Y \\ (s, t) &\longmapsto (f(s), t) \mapsto H(f(s), t) \end{aligned} .$$

- Then  $F$  is continuous.
- $F(s, 0) = H(f(s), 0) = (h \circ f)(s)$
- $F(s, 1) = H(f(s), 1) = (k \circ f)(s)$
- $F(0, t) = H(f(0), t) = H(x_0, t) = y_0$
- $F(1, t) = H(f(1), t) = H(x_0, t) = y_0$

Thus  $F$  is a path-homotopy between  $h \circ f$  and  $k \circ f$  so that  $h_*([f]) = [h \circ f] = [k \circ f] = k_*([f])$ .  $\square$



**Theorem 24** (58.2). The inclusion map  $j: S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$  induces an isomorphism between fundamental groups.

**Proof.** Let  $X = \mathbb{R}^{n+1} - \{0\}$ ,  $b_0 = (1, 0, \dots, 0)$ . There exists a retraction  $r: X \rightarrow S^n$  defined by  $r(x) = \frac{x}{\|x\|}$ . Then  $r \circ j: S^n \rightarrow X \rightarrow S^n$  is the identity map, hence  $r_* \circ j_* = \text{id}_{S^n}^*$ . Now consider  $j \circ r: X \rightarrow S^n \rightarrow X$  which maps  $X$  to itself. This map is not the identity map  $\text{id}_X$ , but it is homotopic to. Indeed,  $H: X \times I \rightarrow X$  given by  $H(x, t) = (1-t)x + t \frac{x}{\|x\|}$  is a homotopy between  $H(x, 0) = x = \text{id}_X(x)$  and  $H(x, 1) = \frac{x}{\|x\|} = (j \circ r)(x)$ . Note that  $H(b_0, t) = b_0$ . Then by Lemma 10,  $(j \circ r)_* = j_* \circ r_* = \text{id}_X^*$ . Thus  $j_*$  has the right and left inverse homomorphism.  $\square$

**Definition 23.** A subspace  $A$  of  $X$  is a **deformation retract** of  $X$  if there is a continuous map  $H: X \times I \rightarrow X$  s.t.  $H(x, 0) = x$ ,  $H(x, 1) \in A$   $\forall x \in X$ , and  $H(a, t) = a$   $\forall a \in A, \forall t \in I$ .<sup>a</sup> The homotopy  $H$  is called a **deformation retraction** of  $X$  onto  $A$ .

<sup>a</sup> $r(x) = H(x, 1)$  is a retraction of  $X$  onto  $A$ .  $H$  is a homotopy between  $\text{id}_X$  and  $j \circ r$ .

$H$  shows a continuous shrinking of  $X$  onto  $A$ . During the shrinking, points of  $A$  stay where they are.

**Theorem 25** (58.3). Let  $A$  be a deformation retract of  $X$ , and  $j: A \hookrightarrow X$  the inclusion map. Then  $j_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$  is an isomorphism.

**Proof.** Similar with Theorem 24.  $\square$

**Example.**  $(\mathbb{R}^2 - \{0\}) \times \{0\} \subset \mathbb{R}^3 - \{z\text{-axis}\}$ .  $H((x, y, z), t) = (x, y, (1-t)z)$ . Thus,  $\pi_1(\mathbb{R}^3 - \{z\text{-axis}\}) \cong \pi_1(\mathbb{R}^2 - \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$ .

**Example.**  $\pi_1(\mathbb{R}^2 - \{\text{two points}\}) \cong \pi_1(\text{Bouquet with two leaves})$ . Can you write down the deformation retractions concretely?

**Example.**  $S^1 \cup (\{0\} \times [-1, 1])$  (theta space) is deformation retract of  $\mathbb{R}^2 - \{\text{two points}\}$ .

**Definition 24.** Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  be continuous maps.  $f$  and  $g$  are called **homotopy equivalences (maps)** if  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ . ( $f$  is a homotopy inverse of  $g$ ).  $X$  is homotopically equivalent to  $Y$ .

**Definition 25.** Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  be continuous maps.  $X$  and  $Y$  are said to be of the same **homotopy type** if  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ . We say that  $f, g$  are **homotopy equivalences** and are **homotopy inverses** of each other.

**Note.** The relation of homotopy equivalence is an equivalence relation.

**Remark.** Suppose  $A$  is a deformation retract of  $X$ . Then for the retraction  $r(x) = H(x, 1)$  and inclusion  $j: A \hookrightarrow X$ ,  $r \circ j = \text{id}_A$ ,  $j \circ r \simeq \text{id}_X$  by  $H$ . Thus  $r$  and  $j$  are homotopy equivalence maps.

**Example.** Bouquet with two leaves and theta space are deformation retract of  $\mathbb{R}^2 - \{\text{two points}\}$ . Thus they are homotopically equivalent to each other. Can you find a homotopy equivalence map between them?

**Lemma 11 (58.4).** Let  $h, k: X \rightarrow Y$  be continuous maps.  $h(x_0) = y_0$ ,  $k(x_0) = y_1$ . If  $h \simeq k$ , then there is a path  $\alpha$  in  $Y$  from  $y_0$  to  $y_1$  s.t.  $k_* = \hat{\alpha} \circ h_*$ .

**Proof.** Let  $f$  be a loop based at  $x_0$ . Then we have to show:

$$k_*([f]) = \hat{\alpha}(h_*([f])), [k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha], [\alpha] * [k \circ f] = [h \circ f] * [\alpha].$$

Let  $f_0(s) = (f(s), 0) \subset X \times \{0\}$ ,  $f_1(s) = (f(s), 1) \subset X \times \{1\}$ ,  $c(t) = (x_0, t) \in X \times I$ . If  $H$  is a homotopy between  $h$  and  $k$ , then  $(H \circ f_0)(s) = H(f(s), 0) = (h \circ f)(s)$ ,  $(H \circ f_1) = k \circ f$ . Define  $F: I \times I \rightarrow X \times I$  by  $F(s, t) = (f(s), t)$ . Label  $\square$

**Example.** Let  $S^1 \subset \mathbb{R}^2 \setminus \{(0,0)\}$ . Then  $S^1$  is a deformation retract of  $\mathbb{R}^2 \setminus \{(0,0)\}$ . Using homotopy  $H: \mathbb{R}_0^2 \times I \rightarrow \mathbb{R}_0^2$  given by  $x \mapsto (1-t)x + t\frac{x}{\|x\|}$ . (The same for  $S^n$  and  $\mathbb{R}_0^n$ )

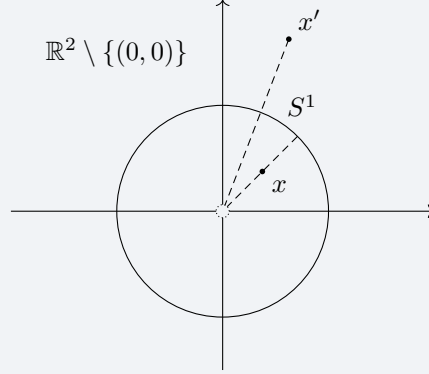


Figure 9.5: Example of a deformation retract

**Example.** Consider the figure 8 space. Claim:  $A$  is not a deformation retract of  $X$ . We'll prove this later on.

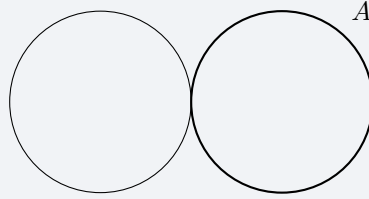


Figure 9.6: Example of a deformation retract

**Example.** Consider the torus and a circle on the torus. Then it is a retract, but not a deformation retract.

**Theorem 26.** If  $A$  is a deformation retract of  $X$ , then  $i: A \rightarrow X$  induces an *isomorphism*  $i_*$ . I.e. if you have a deformation retract, it's not only injective but also surjective.

**Proof.** Let  $i: A \rightarrow X$  be the inclusion and  $r: X \rightarrow A$  be the deformation retraction using  $H$ . Then  $r \circ i = 1_A$ , which gives  $r_* \circ i_* = 1_{\pi(A, a_0)}$ .

Now,  $i \circ r \simeq_p 1_X$  using the homotopy of the previous lemma, i.e.  $H$  with  $H(a_0, t) = a_0$ . Call  $h = i \circ r$ ,  $k = 1_X$ , and using the previous lemma,  $(i \circ r)_* = (1_X)_*: \pi(X, x_0) \rightarrow \pi(X, x_0)$ , which shows that  $i_* \circ r_* = 1_{\pi(X, x_0)}$ .

We conclude that both  $i_*$  and  $r_*$  are isomorphisms.  $\square$

**Remark.** This means that the fundamental group of  $\mathbb{R}_0^2$  is the same as the one of  $S^1$ , which is  $\mathbb{Z}$ .

**Example.** The fundamental group of the figure 8 space and the  $\theta$ -space are isomorphic. These spaces are not deformations of each other, but we can show that they are deformation retracts of  $\mathbb{R}^2 \setminus \{p, q\}$ . We say that these spaces are of the same homotopy type.

**Definition 26.** Let  $X, Y$  be two spaces, then  $X$  and  $Y$  are said to be of the same **homotopy type** if there exists  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  such that  $g \circ f \simeq 1_X$  and  $f \circ g \simeq 1_Y$ . We say that  $f, g$  are **homotopy equivalences** and are **homotopy inverses** of each other.

**Remark.** This is an equivalence relation.

We'll prove that spaces of the same homotopy type have the same fundamental group. For that, we'll prove the previous lemma in a more general form, not preserving the base point.

**Lemma 12 (58.4).** Suppose  $h, k: X \rightarrow Y$  with  $h(x_0) = y_0$  and  $k(x_0) = y_1$ . Assume that  $h \simeq k$  via a homotopy  $H: X \times I \rightarrow X$ , ( $H(x, 0) = h(x)$ ,  $H(x, 1) = k(x)$ ). Then  $\alpha: I \rightarrow X$  given by  $s \mapsto H(x_0, s)$  is a path starting in  $y_0$  and ending in  $y_1$  such that the following diagram commutes

$$\begin{array}{ccc} & \pi(X, x_0) & \\ h_* \swarrow & & \searrow k_* \\ \pi(Y, y_0) & \xrightarrow{\hat{\alpha}} & \pi(Y, y_1) \\ [g] \longmapsto & & [\bar{\alpha}] * [g] * [\alpha] \end{array} .$$

**Proof.** We need to show that  $\hat{\alpha}(h_*[f]) = k_*[f]$ , or  $[\bar{\alpha}] * [h \circ f] * [\alpha] = [k \circ f]$ , or  $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$ . We'll prove that these paths are homotopic. Using the picture, we see that  $\beta_0 * \gamma_2 \simeq_p \gamma_1 * \beta_1$ , because they are loops in a path connected space,  $I \times I$ . Therefore,  $F \circ (\beta_0 * \gamma_2) \simeq_p F \circ (\gamma_1 * \beta_1)$ . This is  $f_0 * c \simeq_p c * f_1$ . Now, if we apply  $H$ , we get  $H \circ (f_0 * c) \simeq_p H \circ (c * f_1)$ , so  $(h \circ f) * \alpha \simeq_p \alpha * (k \circ f)$ , which implies that  $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$ .  $\square$

**Theorem 27.** Let  $f: X \rightarrow Y$  be a homotopy equivalence, with  $f(x_0) = y_0$ . Then  $f_*: \pi(X, x_0) \rightarrow \pi(Y, y_0)$  is an isomorphism.

**Proof.** Let  $g$  be a homotopy inverse of  $f$ .

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1) \cdots$$

$$\begin{array}{ccc} \pi(X, x_0) & \xrightarrow{f_*, x_0} \pi(Y, y_0) & \xrightarrow{g_*, x_0} \pi(X, x_1) \\ & \searrow 1_{\pi(X, x_0) = (1_X)_*} & \downarrow \hat{\alpha} \\ & & \pi(X, x_0) \end{array}$$

$$\begin{array}{ccc} \pi(Y, y_0) & \xrightarrow{g_*, x_0} \pi(X, x_1) & \xrightarrow{f_*, x_1} \pi(Y, y_1) \\ & \searrow 1_{\pi(Y, y_0) = (1_Y)_*} & \downarrow \hat{\beta} \\ & & \pi(Y, y_0) \end{array}$$

From the first diagram,  $g_{y_0,*} \circ f_{x_0,*}$  is an isomorphism,  $g_{y_0,*}$  is surjective. The second diagram gives that  $f_{x_1,*} \circ g_{y_0,*}$  is an isomorphism, so  $g_{y_0,*}$  is injective, so  $g_{y_0,*}$  is an isomorphism. Now composing, we find that  $g_{y_0,*}^{-1} \circ (g_{y_0,*} \circ f_{x_0,*}) = f_{x_0,*}$  is an isomorphism.  $\square$

## 9.59 $\pi_1(S^n)$

Lecture 16  
Mon, Oct 25

**Theorem 28** (59.1, Special version of van Kampen theorem). Let  $X = U \cup V$ , where  $U, V$  are open subsets of  $X$ , and  $U \cap V$  is path connected. Let  $i: U \hookrightarrow X$  and  $j: V \hookrightarrow X$  denote the inclusions and consider  $x_0 \in U \cap V$ . Then the images of  $i_*$  and  $j_*$  generate the whole group  $\pi(X, x_0)$ .<sup>a</sup>

<sup>a</sup>In other words, every element of  $\pi_1(X, x_0)$  is a product of the elements of the subgroups.

**Proof.** Let  $f$  be a loop in  $X$  based at  $x_0$ . Need to show that  $f$  is a product of loops in  $U$  or  $V$ .

- (i) We can divide  $[0, 1]$  into subintervals  $0 = a_0 < a_1 < \cdots < a_n = 1$  so that  $f(a_i) \in U \cap V$  and  $f([a_{i-1}, a_i]) \subset U$  or  $V$ .
- (ii)  $U \cap V$  is path-connected, so we can choose a path  $\alpha_i$  from  $x_0$  to  $f(a_i)$ . Let  $f_i$  be a path s.t.  $f(I) = f([a_{i-1}, a_i])$ . Then,

$$\begin{aligned} [f] &= [f_1] * \cdots * [f_n] \\ &= [f_1] * [\bar{\alpha}_1 * \alpha_1] * [f_2] * \cdots * [\bar{\alpha}_{n-1} * \alpha_{n-1}] * [f_n] \\ &= [f_1 * \bar{\alpha}_1] * [\alpha_1 * f_2 * \bar{\alpha}_2] * \cdots * [\alpha_{n-2} * f_{n-1} * \bar{\alpha}_{n-1}] * [\alpha_{n-1} * f_n] \end{aligned}$$

Each factor in the product is a loop in  $U$  or  $V$ .  $\square$

**Proof.** Let  $[f] \in \pi(X, x_0)$  denote  $f: I \rightarrow X$  is a loop based at  $x_0$ .

Claim: there exists a subdivision of  $[0, 1]$  such that  $f[a_i, a_{i+1}]$  lies entirely

inside  $U$  or  $V$  and  $f(a_i) \in U \cap V$ . Proof of the claim: Lebesgue number lemma says that such a subdivision  $b_i$  exists. Now assume  $b_j$  is such that  $f(b_j) \notin U \cap V$ , for  $0 < j < m$ . Then either  $f(b_j) \in U \setminus V$ , or  $f(b_j) \in V \setminus U$ . The first one would imply that  $f([b_{j-1}, b_j]) \subset U$  and  $f([b_j, b_{j+1}]) \subset U$ . So  $f[b_{j-1}, b_{j+1}] \subset U$ , so we can discard  $b_j$ . Same for the second possibility.

Let  $\alpha_i$  be a path from  $x_0$  to  $f(a_i)$  and  $\alpha_0$  the constant path  $t \mapsto x_0$ , inside  $U \cap V$  (which is possible, as it is path connected). Now define

$$f_i: I \rightarrow X \text{ given by } I \xrightarrow{\text{p.l.m.}} [a_{i-1}, a_i] \xrightarrow{f} X.$$

Then  $[f] = [f_1] * [f_2] * \cdots * [f_n]$ . Note that all  $f_i$  have images inside  $U$  or  $V$ . Now,

$$\begin{aligned} [f] &= [a_0] * [f_1] * [\overline{\alpha_1}] * [\alpha_1] * [f_2] * [\overline{\alpha_2}] * [\alpha_2] * [f_3] * \cdots * [\alpha_{n-1}] * [f_n] * [\overline{\alpha_n}] \\ &= [\alpha_0 * (f_1 * \overline{\alpha_1})] * [\alpha_1 * (f_2 * \overline{\alpha_2})] * \cdots . \end{aligned}$$

Every path of the form  $\alpha_{i-1} * (f_i * \overline{\alpha_i})$  is a loop based at  $x_0$  lying entirely inside  $U$  or  $V$ . This means that

$$[f] \in \text{grp}\{i_*(\pi(U, x_0)), j_*(\pi(V, x_0))\}.$$

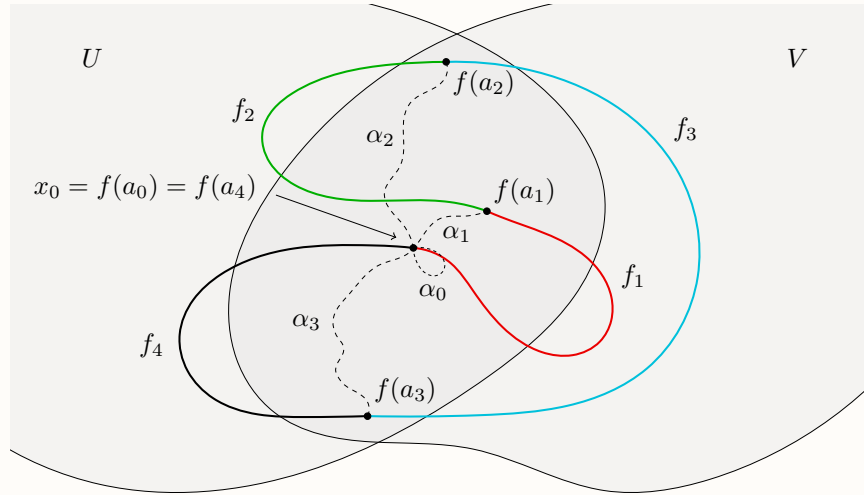


Figure 9.7: Proof of Theorem 59.1

□

**Corollary 6** (59.2). If  $U$  and  $V$  are simply connected, then so is  $X$ .

**Theorem 29** (59.3). For  $n \geq 2$ ,  $S^n$  is simply connected.

**Proof.** Consider  $S^n$  and  $N, S$  the north and south pole. Let  $U = S^n \setminus \{N\}$

and  $V = S^n \setminus \{S\}$ . Then  $U, V \approx \mathbb{R}^n$  and  $U \cap V$  is path connected, which is easy to prove as it is simply homeo to  $\mathbb{R}^n$  with points removed. Then  $\pi(S^n, x_0)$  is generated by  $i_*(\pi(U, x_0))$  and  $j_*(\pi(V, x_0))$ , which both are trivial. This proof doesn't work for  $S^1$  because then the intersection is not path connected anymore!  $\square$

**Note.** HW6:

- Prove (i) of Proof of Theorem 28 in detail.
- Exercise §59 – #1, #3.

## 9.60 Fundamental groups of some surfaces

**Definition 27.** Given groups  $(G, \cdot)$  and  $(H, *)$ , the **direct product**  $G \times H$  is the set  $\{(g, h) \mid g \in G, h \in H\}$  where  $(g_1, h_1)(g_2, h_2) = (g_1 \cdot g_2, h_1 * h_2)$ .

**Theorem 30 (60.1).**  $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ .

**Proof.** Let  $p, q$  be projection mappings from  $X \times Y$  to  $X$  and  $Y$ , respectively. With given base points, we have induced homomorphisms  $p_*$  and  $q_*$ .

**Homomorphism** Define a map  $\Phi: \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$  by  $\Phi([f]) = (p_*([f]), q_*([f])) = ([p \circ f], [q \circ f])$ . For two loops  $f, g$  in  $X \times Y$  based at  $(x_0, y_0)$ ,

$$\begin{aligned} \Phi([f] * [g]) &= \Phi([f * g]) = (p_*([f * g]), q_*([f * g])) \\ &= (p_*([f]) * p_*([g]), q_*([f]) * q_*([g])) \\ &= (p_*([f]), q_*([f])) \cdot (p_*([g]), q_*([g])) \\ &= \Phi([f]) \cdot \Phi([g]). \end{aligned}$$

Thus  $\Phi$  is a group homomorphism.

**Surjective** Let  $\mathcal{L}(Z, z_0)$  denote the set of all loops in  $Z$  based at  $z_0$ . For  $g \in \mathcal{L}(X, x_0)$  and  $h \in \mathcal{L}(Y, y_0)$ , let  $f \in \mathcal{L}(X \times Y, (x_0, y_0))$  s.t.  $f(s) = (g(s), h(s))$ . Then  $\Phi([f]) = ([p \circ f], [q \circ f]) = ([g], [h])$ .

**Injective** Let  $f \in \mathcal{L}(X \times Y, (x_0, y_0))$  such that  $\Phi([f]) = ([c_{x_0}], [c_{y_0}])$ . Then  $p \circ f \simeq_p^G c_{x_0}$  and  $q \circ f \simeq_p^H c_{y_0}$ . Define a map  $F: I \times I \rightarrow X \times Y$  by  $F(s, t) = (G(s, t), H(s, t))$ . Then,

- $F(s, 0) = (G(s, 0), H(s, 0)) = (p \circ f, q \circ f) = f$
- $F(s, 1) = (G(s, 1), H(s, 1)) = (c_{x_0}, c_{y_0}) = c_{(x_0, y_0)}$
- $F(0, t) = (G(0, t), H(0, t)) = (x_0, y_0)$
- $F(1, t) = (G(1, t), H(1, t)) = (x_0, y_0)$

Thus,  $F$  is a path-homotopy between  $f$  and  $c_{(x_0, y_0)}$ , hence  $[f]$  is the identity element of  $\pi_1(X \times Y, (x_0, y_0))$ .

□

**Example.**  $\pi_1(T^2, x_0) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$ . We know that  $\pi(S^2, x_0) = 1$ , so the torus and the two sphere are not homeomorphic to each other, they aren't even homotopically equivalent.

**Example.**  $\mathbb{RP}^2 = S^2/\sim$ . Then  $p: S^2 \rightarrow \mathbb{RP}^2$ , which is continuous by definition of the topology on the projective plane. This means that  $(S, p)$  is a covering of the projective plane. The lifting correspondence says that

$$\Phi: \pi(\mathbb{RP}^2, x_0) \rightarrow p^{-1}(x_0) = \{\tilde{x}_0, -\tilde{x}_0\}$$

is a isomorphism. Therefore,  $\pi_1(\mathbb{RP}^2, x_0)$  is a group with 2 elements, so  $\mathbb{Z}_2$ .

This means, there exists loops which we cannot deform to the trivial loop, but when going around twice, they do deform to the trivial loop. E.g. consider the loop  $a$ . This is not homotopic equivalent with the trivial loop, as  $e_1 \neq e_0$ . (Or also you can see it because  $\alpha = \bar{\alpha}$ .) But pasting the loop it twice, we see that is possible. This means that the fundamental group of the projective space is different from all the one we've seen before.

**Example.**  $T^2$  is the torus.  $T^2 \# T^2$  is the connected sum of two tori (Remove small disc of both tori and glue together), in Dutch: 'tweeling zwemband'. This space has yet another fundamental group.

**Example.** Figure eight space: fundamental group is not abelian. Indeed,  $[b * a] \neq [a * b]$ .

**Example.** Tweeling zwemband. The space retracts to the figure 8 situation, which shows that the group of the tweeling zwemband has a nonabelian component.



# Chapter 10

## Separation theorems in the plane

Lecture 17  
Wed, Oct 27

### Review on connectedness

**Definition 28.** A topological space  $X$  is **disconnected** if there are two non-empty open subsets  $U$  and  $V$  (called **separation** of  $X$ ) s.t.  $U \cap V = \emptyset$ ,  $U \cup V = X$ .<sup>a</sup>  $X$  is said to be **connected** if it is not disconnected.<sup>b</sup>

<sup>a</sup>Hence,  $U$  and  $V$  are open and closed.

<sup>b</sup>Iff  $\emptyset, X$  are only sets which are both open and closed.

**Theorem 31 (23.3).** Let  $\{E_\alpha\}_{\alpha \in A}$  be a family of connected subsets of a topological space  $X$  s.t.  $E_\alpha \cap E_\beta \neq \emptyset$  for every  $\alpha, \beta \in A$ . Then,  $\bigcup_{\alpha \in A} E_\alpha$  is connected.

**Proof.** Let  $\bigcup_{\alpha \in A} E_\alpha = A \cup B$  be a separation. For  $x \in A$ ,  $x \in E_{\alpha_0}$  for some  $\alpha_0$ .  $A \cap E_{\alpha_0}$  is a non-empty, open and closed subset of  $E_{\alpha_0}$ . By connectedness of  $E_{\alpha_0}$ ,  $A \cap E_{\alpha_0} = E_{\alpha_0}$ ,  $A \supset E_{\alpha_0}$ . For any  $\beta$ ,  $A \cap E_\beta$  is an open and closed subset of  $E_\beta$ . Note that  $A \cap E_\beta \supset E_{\alpha_0} \cap E_\beta \neq \emptyset$ . Thus,  $A \cap E_\beta = E_\beta$ ,  $A = \bigcup_{\alpha \in A} E_\alpha$ ,  $B = \emptyset$ .  $\square$

**Definition 29.** A **connected component** of  $X$  is a maximal connected subset of  $X$ .<sup>a</sup>

<sup>a</sup> $A$  is a connected component of  $X$  if there is no connected subset of  $X$  which contain  $A$ .

Assume that  $C$  is a connected component of  $X$  and  $U$  is a connected subset. If  $C \cap U \neq \emptyset$ ,  $C \cup U$  is a connected (by Theorem 31) subset which contains  $C$ .  
 $\nmid$ . Thus we have only two possible cases:  $C \cap U = \emptyset$  or  $C \subset U$ . This implies, two connected components are disjoint and  $X$  can be partitioned into a disjoint union of connected components.

**Definition 30.** A space  $X$  is **path-connected** if for every  $x, y \in X$ , there is a path from  $x$  to  $y$ .

**Note.** The existence of a path between two points is an equivalence relation on the points of  $X$ . The equivalence classes of such a relation are called **path-components**.

A path-connected space is connected. Thus a connected component is split into path-components.

**Theorem 32 (25.5).** If a space  $X$  is locally path-connected, then connected components and path-components of  $X$  are the same.

**Proof.** Let  $C$  be a connected component and  $P$  be a path-component s.t.  $C \cap P \neq \emptyset$ . Since  $P$  is connected,  $C \supset P$ . Suppose that  $C \neq P$ . Let  $Q$  be the union of path-components other than  $P$ .  $C = P \sqcup Q$ . Since  $X$  is locally path-connected, each path-component is open. Thus  $P, Q$  are open, hence  $P \sqcup Q$  is a separation of  $C$ .  $\nmid$   $\square$

## 10.61 Jordan separation theorem

**Definition 31.** Let  $A$  be a subspace of a connected space  $X$ . We say  $A$  **separates**  $X$  if  $X - A$  is not connected.  $A$  is an **arc** if  $A \cong [0, 1]$ , that is, there is a continuous map  $\alpha: [0, 1] \rightarrow X$  s.t.  $\alpha$  is injective,  $\alpha(I) = A$ .  $A$  is a **simple closed curve** if  $A \cong S^1$ , i.e. there is a continuous map  $\alpha: [0, 1] \rightarrow X$  s.t.  $\alpha(I) = A$ ,  $\alpha(0) = \alpha(1)$ ,  $\alpha$  is injective on  $(0, 1)$ .

The main content of this section is the proof of the following theorem.

**Theorem 33 (61.3, Jordan separation theorem).** Any simple closed curve in  $S^2$  separates  $S^2$ .

**Lemma 13 (61.1).** Let  $C$  be a compact subspace of  $S^2$ ,  $b \in S^2 - C$ ,  $h: S^2 - \{b\} \rightarrow \mathbb{R}^2$  be a homeomorphism, and  $U$  be a component of  $S^2 - C$ . If  $b \notin U$ , then  $h(U)$  is a bounded component of  $\mathbb{R}^2 - h(C)$ . If  $b \in U$ , then  $h(U - \{b\})$  is an unbounded component of  $\mathbb{R}^2 - h(C)$ .

**Proof.** (i)  $U - \{b\}$  is connected. If  $b \notin U$ ,  $U - \{b\} = U$  is connected. Assume  $b \in U$ . Let  $U - \{b\} = A \sqcup B$  be a separation. Choose an open nbh  $W$  of  $b$  in  $S^2$  so that  $W \cong$  open disk in  $\mathbb{R}^2$ ,  $W \cap C = \emptyset$ . Then  $W - \{b\}$  is connected, hence we may say  $W - \{b\} \subset A$ . Thus  $(A \cup \{b\}) \sqcup B$  is a separation of  $U$ .  $\nmid$

(ii) Let  $\{U_\alpha\}$  be the collection of all components of  $S^2 - C$ .  $S^2 - C$  ( $\cong$  open subset of  $\mathbb{R}^2$ ) is locally connected, hence each  $U_\alpha$  is open in  $S^2 - C$ . By (i),  $\{U_\alpha - \{b\}\}$  is the collection of open, disjoint, and connected subsets. The homeomorphism  $h$  preserves such properties.

Thus  $\{h(U_\alpha - \{b\})\}$  is the collection of all components of  $\mathbb{R}^2 - h(C)$ .

- (iii) If  $b \in U_{\alpha_0}$ , then  $h(U_{\alpha_0} - \{b\})$  is an unbounded component of  $\mathbb{R}^2 - h(C)$ .  $(S^2 - C) - U_{\alpha_0}$  is bounded and closed, hence compact in  $S^2 - C$ .  $h((S^2 - C) - U_{\alpha_0}) = \bigsqcup_{\alpha \neq \alpha_0} h(U_\alpha - \{b\})$ . Each  $h(U_\alpha - \{b\})$  ( $\alpha \neq \alpha_0$ ) is bounded. □

**Lemma 14** (61.2). Let  $a, b \in S^2$ ,  $A$  be a compact space,  $f: A \rightarrow S^2 - \{a, b\}$  be a continuous map. If  $a, b$  are in the same component of  $S^2 - f(A)$ , then  $f$  is null-homotopic.

**Proof.** Note that there is a homeomorphism  $h: S^2 - \{a, b\} \rightarrow \mathbb{R}^2 - \{O\}$ . If  $h \circ f: A \xrightarrow{f} S^2 - \{a, b\} \xrightarrow{h} \mathbb{R}^2 - \{O\}$  is null-homotopic, then  $h \circ f \simeq$  constant map, hence  $f \simeq h^{-1} \circ$  constant map.

Therefore, it's enough to show: For a continuous map  $g: A \rightarrow \mathbb{R}^2 - \{O\}$ , if  $O$  lies in the unbounded component of  $\mathbb{R}^2 - g(A)$ , then  $g$  is null-homotopic. (If  $a, b$  are in the same component of  $S^2 - f(A)$ , then  $O$  is in the unbounded component of  $\mathbb{R}^2 - (h \circ f)(A)$  by Lemma 13.)

Choose a disk  $B$  centered at  $O$  in  $\mathbb{R}^2$  so that  $g(A) \subset B$ . And choose a point  $p \in B$ . Then  $O$  and  $p$  are in the unbounded component of  $\mathbb{R}^2 - g(A)$ .  $\mathbb{R}^2$  is locally path-connected, hence so is the open subset  $\mathbb{R}^2 - g(A)$ . (Thus the components and the path-components of  $\mathbb{R}^2 - g(A)$  are the same.)  $O$  and  $p$  are in the same path-component of  $\mathbb{R}^2 - g(A)$ . So, there is a path  $\alpha$  in  $\mathbb{R}^2 - g(A)$  from  $O$  to  $p$ .

Define a homotopy  $G: A \times I \rightarrow \mathbb{R}^2 - \{O\}$  by  $G(x, t) = g(x) - \alpha(t)$ .  $G(x, t) \neq O$  because of  $g(A) \cap \alpha(I) = \emptyset$ . ( $G(x, 0) = g(x)$ ,  $G(x, 1) = g(x) - p$ ) Also, define a homotopy  $H: A \times I \rightarrow \mathbb{R}^2 - \{O\}$  by  $H(x, t) = tg(x) - p$ . ( $H(x, 0) = -p$ ,  $H(x, 1) = g(x) - p = G(x, 1)$ ) Therefore, by  $G$  and  $H$ ,  $g(x)$  is null-homotopic. □

**Proof (of Theorem 33).**  $S^2 - C$  is open, hence it is locally path-connected,  $\{\text{path-components}\} = \{\text{connected components}\}$ . Suppose that  $S^2 - C$  is path-connected. Let  $C = A_1 \cup A_2$  and  $X = S^2 - \{a, b\}$ . And let  $U = S^2 - A_1$ ,  $V = S^2 - A_2$ ,  $x_0 \in U \cap V$ ,  $i: U \hookrightarrow X$ , and  $j: V \hookrightarrow X$ . (Then,  $X = U \cup V$ ,  $U \cap V = S^2 - (A_1 \cup A_2) = S^2 - C$  (path-connected).) By special van-Kampen theorem,  $i_*(\pi_1(U, x_0))$  and  $j_*(\pi_1(V, x_0))$  generate  $\pi_1(X, x_0)$ .

Claim:  $i_*$  and  $j_*$  is trivial homomorphisms. The claim implies,  $\pi_1(X, x_0)$  should be trivial, but,  $\pi_1(X) \cong \pi_1(\mathbb{R}^2 - \{\text{a point}\}) \cong \mathbb{Z}$  is not trivial.  $\nmid$

Proof of claim: Let  $p: I \rightarrow S^1$  be the loop  $p(t) = e^{2\pi it}$ . Then  $[p]$  generates  $\pi_1(S^1, b_0)$ . For a loop  $f \in \mathcal{L}(U, x_0)$ , Let  $h: S^1 \rightarrow U$  be the loop s.t.  $h \circ p = f$ . Consider the map  $i \circ h: S^1 \xrightarrow{h} U \xrightarrow{i} X = S^2 - \{a, b\}$ .  $i(h(S^1)) = h(S^1) \cap A_1 = \emptyset$ , hence  $a$  and  $b$  are in the same path-component (= conn. comp.) of  $S^2 - i(h(S^1))$ . Applying Lemma 14 to  $i \circ h: S^1 \rightarrow X$ , we know that  $i \circ h$  is null-homotopic. By Lemma 9,  $(i \circ h)_*$  is the trivial

homomorphism. Thus,

$$(i \circ h)_*([p]) = [i \circ h \circ p] = [i \circ f] = i_*([f]) = i_*([e_{x_0}]),$$

hence,  $i_*$  is trivial. Similarly, so is  $j_*$ .  $\square$

**Theorem 34** (61.4, A general separation theorem). If  $A_1$  and  $A_2$  are closed connected subsets of  $S^2$  s.t.  $A_1 \cap A_2 = \{\text{two points}\}$ , then  $A_1 \cap A_2$  separates  $S^2$ .

**Proof.**  $A_1 \cup A_2 \neq S^2$ . Because  $S^2 - \{a, b\}$  is connected.  $(A_1 \cup A_2) - \{a, b\} = (A_1 - \{a, b\}) \sqcup (A_2 - \{a, b\})$  (both are open). Thus  $(A_1 \cup A_2) - \{a, b\}$  is disconnected. The remainder of proof is same with that of Theorem 33.  $\square$

**Note.** HW7: Exercise §61 – #1, #2.

## 10.62 Invariance of domain

**Theorem 35** (62.3). If  $U$  is an open subset of  $\mathbb{R}^n$  and  $f: U \rightarrow S^n$  is continuous and injective, then  $f(U)$  is open in  $S^n$  and the inverse function  $f^{-1}: f(U) \rightarrow U$  is continuous. ( $\therefore U \cong f(U)$  by  $f$ )

In this section, we prove this theorem for  $n = 2$ .

**Lemma 15** (62.1, Homotopy extension lemma). Let  $X$  be a space s.t.  $X \times I$  is normal,  $A$  be a closed subset of  $X$ ,  $Y$  be an open subset of  $\mathbb{R}^n$ , and  $f: A \rightarrow Y$  be a continuous map. If  $f$  is null-homotopic, then  $f$  can be extended to a continuous map  $g: X \rightarrow Y$  that is null-homotopic.

**Proof.** Let  $F: A \times I \rightarrow Y$  be a homotopy between  $f$  and  $c_{y_0}$ . Define  $\bar{F}: (A \times I) \cup (X \times \{1\}) \rightarrow \mathbb{R}^n$  by  $\bar{F}|_{A \times I} = F$  and  $\bar{F}(x, 1) = y_0$ . Applying the Tietze extension theorem  $n$ -times,  $\bar{F}$  can be extended to a continuous map  $G: X \times I \rightarrow \mathbb{R}^n$ . The map  $x \mapsto G(x, 0)$  is an extension of  $f$ , but it may map  $X$  into  $\mathbb{R}^n$ , rather than  $Y$ . So, we need to do something more.

Let  $U = G^{-1}(Y)$ . Then  $U \supset (A \times I) \cup (X \times \{1\})$ . By the Tube lemma (26.8), there is an open subset  $W$  of  $X$  s.t.  $W \times I \subset U$ ,  $A \subset W$ . Apply the Urysohn lemma to  $(X, A, W^c)$ , we have a continuous map  $\phi: X \rightarrow [0, 1]$  s.t.  $\phi(x) = 0 \ \forall x \in A$ ,  $1 \ \forall x \in X - W$ . Then the map  $x \mapsto (x, \phi(x))$  carries  $X$  into  $(W \times I) \cup (X \times \{1\}) \subset U$ .  $g(x) = G(x, \phi(x))$  is a continuous map from  $X$  to  $Y$ . For  $x \in A$ ,  $g(x) = G(x, \phi(x)) = G(x, 0) = f(x)$  (extension of  $f$ ). Define  $H: X \times I \rightarrow Y$  by  $H(x, t) = G(x, (1 - t)\phi(x) + t)$ . Then,

- $H(x, 0) = G(x, \phi(x)) = g(x)$  (extension of  $f$ )
- $H(x, 1) = G(x, 1) = c_{y_0}$

Thus,  $H$  is a homotopy between  $g$  and  $c_{y_0}$ .  $\square$

Lecture 19  
Wed, Nov 3

**Lemma 16** (62.2, Borsuk lemma). Let  $a, b \in S^2$ ,  $A$  be a compact space, and  $f: A \rightarrow S^2 - \{a, b\}$  be a continuous and injective map. If  $f$  is null-homotopic, then  $a$  and  $b$  are in the same component of  $S^2 - f(A)$ .

**Proof.** Because  $A$  is compact,  $S^2 - \{a, b\}$  is Hausdorff, and  $f$  is injective, by Theorem 26.6,  $A \cong f(A)$  by  $f$ . Let  $h: S^2 - \{b\} \rightarrow \mathbb{R}^2$  be a homeomorphism s.t.  $h(a) = O$  ( $\because f^{-1}: f(A) \rightarrow A$  is continuous). By Lemma 14, if  $O$  is in unbounded component of  $\mathbb{R}^2 - h(f(A))$ , then  $a, b$  are in the same component of  $S^2 - f(A)$ .

Consider the map  $A \xrightarrow{h \circ f} h(f(A)) \xrightarrow{j} \mathbb{R}^2 - \{O\}$ . Let  $H: A \times I \rightarrow S^2 - \{a, b\}$  be a homotopy between  $f$  and  $c_{y_0}$  ( $y_0 \neq a, b$ ). Define  $J: h(f(A)) \times I \rightarrow S^2 - \{O\}$  by  $J(z, t) = (h \circ H)((h \circ f)^{-1}(z), t)$ . Then,

- $J(z, 0) = h \circ f \circ (h \circ f)^{-1}(z) = z = j(z)$
- $J(z, 1) = (h \circ c_{y_0})((h \circ f)^{-1}(z)) = h(y_0) = c_{h(y_0)}$

Thus,  $j$  is null-homotopic. Now, it's enough to show: If  $X$  is compact subspace of  $\mathbb{R}^2 - \{O\}$  and  $j: X \hookrightarrow \mathbb{R}^2 - \{O\}$  is null-homotopic, then  $O$  is in the unbounded component of  $\mathbb{R}^2 - X$ .

Let  $C$  be the component of  $\mathbb{R}^2 - X$  containing  $O$ . Suppose  $C$  is bounded. Let  $D$  be the union of the other components ( $\mathbb{R}^2 - X = C \sqcup D$ ).  $\mathbb{R}^2 - X$  is open in  $X$ , and  $C, D$  are open in  $\mathbb{R}^2 - X$ , hence  $C, D$  are open in  $\mathbb{R}^2$ . Thus  $X \cup C$  is closed and normal in  $\mathbb{R}^2$ .

Apply Lemma 15 to  $(X \cup C, X, j: X \hookrightarrow \mathbb{R}^2 - \{O\})$ . Then  $j$  can be extended to a map  $k: X \cup C \rightarrow \mathbb{R}^2 - \{O\}$ . And extend  $k$  to a map  $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2 - \{O\}$  by  $g(x) = x$  for all  $x \in D$ . Let  $B$  be a closed ball in  $\mathbb{R}^2$  centered at  $O$  s.t.  $\text{Int } B \supset X \cup C$ . Note that  $g(x) = x$  for all  $x \in \partial B$ . Define  $g_1: B \rightarrow \partial B$  by  $g_1(x) = (\text{Radius of } B) \times \frac{g(x)}{\|g(x)\|}$ . Then  $g_1$  is a retraction of  $B$  onto  $\partial B$ , which is a contradiction to Theorem 17.  $\square$

**Proof** (of Theorem 35 for  $n = 2$ ). (i) For a closed ball  $B$  in  $\mathbb{R}^2$  containing  $U$ ,  $f(B)$  does not separate  $S^2$ . Let  $a, b \in S^2 - f(B)$ . The identity map  $i: B \rightarrow B$  is null-homotopic ( $i \simeq^H c_{x_0}$ ). Consider the map  $f \circ H: B \times I \xrightarrow{H} B \xrightarrow{f} S^2 - \{a, b\}$ . Then,

- $(f \circ H)(x, 0) = (f \circ i)(x) = f(x)$
- $(f \circ H)(x, 1) = f(x_0)$

for all  $x \in B$ . Thus,  $f|_B: B \rightarrow S^2 - \{a, b\}$  is null-homotopic, hence by Lemma 16,  $a, b$  are in the same component of  $S^2 - f(B)$ .

(ii) If  $B$  is a closed disk of  $\mathbb{R}^2$  s.t.  $B \subset U$ , then  $f(\text{Int } B)$  is open in  $S^2$ . Let  $C = f(\partial B)$ . Since  $f$  is continuous and injective,  $C$  is a simple closed curve on  $S^2$ , hence it separates  $S^2$ . Let  $V$  be the component of  $S^2 - C$  s.t.  $V \supset f(\text{Int } B)$  and  $W$  be the union of the other components. Because  $S^2$  is locally connected,  $V$  and  $W$  are open in  $S^2$ . In fact,  $V = f(\text{Int } B)$  (open). Otherwise, select a point

$a \in V$  s.t.  $a \notin f(\text{Int } B)$  and another point  $b \in W$ . By (i),  $S^2 - f(B)$  is a connected and contains both  $a$  and  $b$ . But  $S^2 - f(B) \subset S^2 - C$ ,  $S^2 - f(B)$  is contained in a component of  $S^2 - C$ . Both  $a$  and  $b$  are also in the component.  $\nmid$

(iii) Since  $U$  is open, for any  $x \in U$ , we can select a closed ball  $B_x$  so that  $B_x \subset U$ . By (ii),  $f(\text{Int } B_x)$  is open in  $S^2$ .  $U = \bigcup_{x \in U} \text{Int } B_x$ .  $f(U) = \bigcup_{x \in U} f(\text{Int } B_x)$  is open.  $\square$

**Note.** HW7: Exercise §62 – #6.

## 10.63 Jordan curve theorem

**Theorem 36** (63.4, Jordan curve theorem). Let  $C$  be a simple closed curve in  $S^2$ . Then,

- (i)  $C$  separates  $S^2$  into precisely two components  $W_1$  and  $W_2$ .
- (ii)  $\partial W_1 = C = \partial W_2$ .

**Theorem 37** (Schöflies theorem).  $\overline{W}_1 \cong B^2 \cong \overline{W}_2$ .

**Theorem 38** (General version). For a subspace  $C$  of  $S^n$ , if  $C \cong S^{n-1}$ , then  $C$  separates  $S^n$  into precisely two components  $W_1, W_2$ , and  $\partial W_1 = C = \partial W_2$ .

**Remark.** For  $S^3$ , the Schöflies theorem is true if  $C$  is a smooth manifold. Otherwise, there exist counterexamples.

**Example.** Alexander's horned sphere which is homeomorphic to  $S^2$  separates  $S^3$  into  $W_1$  and  $W_2$  s.t.  $\overline{W}_1 \cong B^3$  but  $W_2$  is not simply connected.

In this section, we prove Theorem 36.

Lecture 20  
Mon, Nov 8

**Theorem 39** (63.1). Assume that

- $X = U \cup V$  s.t.  $U, V$  are open,  $U \cap V = A \sqcup B$ ,  $A, B$  are open.
- There is a path  $\alpha$  in  $U$  from a point  $a \in A$  to a point  $b \in B$ , and there is a path  $\beta$  in  $V$  from  $b$  to  $a$ .
- $f = \alpha * \beta$

Then,

- (i)  $[f]$  generates an infinite cyclic subgroup of  $\pi_1(X, a)$ .
- (ii) If  $\pi_1(X, a) \cong \mathbb{Z}$ , then  $[f]$  generates  $\pi_1(X, a)$ .
- (iii) If there is a path  $\gamma$  in  $U$  from  $a$  to a point  $a' \in A$ , and there is a path  $\delta$  in  $V$  from  $a'$  to  $a$ , then the subgroups of  $\pi_1(X, a)$  generated by  $[f]$  and  $[\gamma * \delta]$  intersect in the identity element alone.

**Theorem 40** (63.2, Non-separation theorem). A compact contractible subspace  $D$  of  $S^2$  does not separate  $S^2$ .

**Proof.**  $D$  is contractible, that is, there is a homotopy  $H: D \times I \rightarrow D$  between the identity map  $i: D \rightarrow D$  and a constant map  $c_{x_0}: D \rightarrow D$ . For any  $a, b \in S^2 - D$ , the inclusion map  $j: D \hookrightarrow S^2 - \{a, b\}$  is null-homotopic. (Consider the map  $j \circ H: D \times I \rightarrow D \hookrightarrow S^2 - \{a, b\}$ .  $(j \circ H)(x, 0) = (j \circ i)(x) = x$ ,  $(j \circ H)(x, 1) = j(x_0) = x_0$ .) By Lemma 16,  $a$  and  $b$  are in the same component of  $S^2 - D$ .  $\square$

**Corollary 7.** An arc in  $S^2$  does not separate  $S^2$ .

**Theorem 41** (63.3, General non-separation theorem). Let  $D_1, D_2$  be closed subsets of  $S^2$  s.t.  $S^2 - (D_1 \cap D_2)$  is simply connected. If neither  $D_1$  nor  $D_2$  separates  $S^2$ , then  $D = D_1 \cup D_2$  does not.

**Proof.** Since  $S^2$  is locally path-connected, every open subset is also locally path-connected. Thus, for  $S^2 - D_i$ ,  $S^2 - (D_1 \cap D_2)$  and  $S^2 - D$ ,  $\{\text{conn. comps.}\} = \{\text{path-comps.}\}$ . Suppose that  $S^2 - D$  is not connected, equivalently, there are  $a, b \in S^2 - D$  s.t. they are not joined by any path in  $S^2 - D$ . Let  $U = S^2 - D_1$ ,  $V = S^2 - D_2$  and  $X = U \cup V$ . Then  $X = S^2 - (D_1 \cap D_2)$ ,  $U \cap V = S^2 - D$ . Let  $A$  be the path-component of  $U \cap V$  s.t.  $a \in A$ , and  $B$  be the union of the other path-components. Since  $U \cap V$  is locally path-connected, every path-component is open, hence  $A$  and  $B$  are open in  $X$ . Note that  $a$  and  $b$  can be joined by a path in  $U$ , also a path in  $V$ . By Theorem 39.(i),  $\pi_1(X, a)$  is not trivial, which contradicts  $X = S^2 - (D_1 \cap D_2)$  is simply connected.  $\square$

**Proof (of Theorem 36).** (i) WTS:  $S^2 - C$  has precisely two components. Let  $C = C_1 \cup C_2$  s.t.  $C_1 \cap C_2$  is the set of two points  $p, q$ ,  $X =$

$S^2 - \{p, q\}$ ,  $U = S^2 - C_1$ ,  $V = S^2 - C_2$ . Then,  $X = U \cup V$  and  $U \cap V = S^2 - C$ . By the Jordan separation theorem,  $U \cap V$  has at least two components. Let  $A_1, A_2$  be components of  $U \cap V$ , and  $B$  be the union of the others. (They are open, because  $S^2 - C$  is locally connected.) Choose three points  $a \in A_1$ ,  $a' \in A_2$  and  $b \in B$ . By Theorem 40, we know, there are paths  $\alpha$  in  $U$  from  $a$  to  $b$ ,  $\gamma$  in  $U$  from  $a$  to  $a'$ ,  $\beta$  in  $V$  from  $b$  to  $a$ , and  $\delta$  in  $V$  from  $a'$  to  $a$ . Let  $f = \alpha * \beta$ ,  $g = \gamma * \delta$ . Considering  $U \cap V = (A_1 \cup A_2) \sqcup B$ , by Theorem 39.(i), we know,  $[f]$  is a nontrivial element of  $\pi_1(X, a)$ . Similarly,  $U \cap V = A_1 \cup (A_2 \sqcup B)$ ,  $[g]$  is a nontrivial element of  $\pi_1(X, a)$ . Since  $\pi_1(X, a)$  is infinite cyclic,  $[f]^m = [g]^k$  for some  $m, k$ , which contradicts Theorem 39.(iii).

- (ii) WTS:  $\partial W_1 = C = \partial W_2$ . Because  $S^2$  is locally connected, each  $W_i$  is open. Recall the definition of  $\partial W_i = \overline{W_i} \cap (\overline{S^2 - W_i}) = \overline{W_i} \cap (S^2 - W_i) = \overline{W_i} - W_i$ .  $S^2 = W_1 \sqcup C \sqcup W_2$ , hence  $\partial W_i \subset C$ . Now, we will show, if  $x \in C$ , then every nbh  $U$  of  $x$  intersects the closed set  $\overline{W_1} - W_1$ . (Then  $x \in \overline{W_1} - W_1 = \partial W_1$ . Also similarly  $x \in \partial W_2$ .) Take two arcs  $C_1, C_2$  so that  $C = C_1 \cup C_2$ ,  $C_1 \cap C_2 = \{\text{two pts}\}$ ,  $C_1 \subset U$  (use Lebesgue lemma). Let  $\alpha(I) \cap \overline{W_1} - W_1 \neq \emptyset$ . (Otherwise, the connected set  $\alpha(I) \subset W_1 \sqcup S^2 - \overline{W_1}$ , that is,  $\alpha(I)$  is a union of non-empty disjoint open subsets.  $\nmid$ ) Let  $y \in \alpha(I) \cap \overline{W_1} - W_1$ . Then  $y \in \overline{W_1} - W_1 \subset C$ ,  $\alpha(I) \cap C_2 = \emptyset$ . Thus  $y \in C_1 \subset U$ ,  $y \in U \cap (\overline{W_1} - W_1)$ . □

Now, let's prove Theorem 39.

Lecture 21  
Wed, Nov 10

**Proof** (of Theorem 39). (i) □

**Note.** HW8:

- Prove Theorem 63.5.
- Exercise §63 – #3.



# Review of groups

Lecture 24  
Mon, Nov 29

## 10.67 Direct sums

**Definition 32.** Let  $G$  be an abelian group<sup>a</sup> and  $\{G_\alpha\}_{\alpha \in J}$  be a family of subgroups of  $G$ . We say,  $G$  is a **direct sum** of  $\{G_\alpha\}_{\alpha \in J}$  and we write  $G = \bigoplus_{\alpha \in J} G_\alpha$ <sup>b</sup> if

- $\{G_\alpha\}_{\alpha \in J}$  generates  $G$ , that is, if  $x \in G$ ,  $x = \sum_{\alpha \in J} x_\alpha$  s.t.  $x_\alpha \in G_\alpha$  for all  $\alpha$ , and  $x_\alpha = 0$  for all but finitely many  $\alpha$ .
- $\sum_{\alpha \in J} x_\alpha = \sum_{\alpha \in J} x'_\alpha \Rightarrow x_\alpha = x'_\alpha$  for all  $\alpha \in J$ .

<sup>a</sup>operation:  $+$ , identity:  $0$ , inverse:  $a \leftrightarrow -a$

<sup>b</sup>If  $|J| < \infty$ , then  $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ .

**Lemma 17** (67.1, Extension condition). Let  $G$  be an abelian group and  $\{G_\alpha\}$  be a family of subgroups of  $G$ .

- (i)  $G = \bigoplus_{\alpha} G_\alpha$
- (ii) For any abelian  $H$  and a family of homomorphisms  $\{h_\alpha: G_\alpha \rightarrow H\}$ , there exists a homomorphism  $h: G \rightarrow H$  s.t. the diagram

$$\begin{array}{ccc} G_\alpha & \hookrightarrow & G \\ & \searrow h_\alpha & \downarrow h \\ & & H \end{array}$$

commutes. In fact,  $h$  is unique.

- (iii)  $\{G_\alpha\}$  generates  $G$ .

Then, (i)  $\Rightarrow$  (ii) and (ii) + (iii)  $\Rightarrow$  (i).

**Proof.** Given  $h_\alpha: G_\alpha \rightarrow H$ , define  $h: G \rightarrow H$  by  $h(x) = \sum_{\alpha} h_\alpha(x_\alpha)$  for  $x = \sum_{\alpha} x_\alpha$ .

Conversely,  $\{G_\alpha\}$  generates  $G$ . For  $x \in G$ ,  $x$  can be written as a finite sum  $x = \sum_{\alpha} x_\alpha$ . Suppose  $\sum_{\alpha} x_\alpha = \sum_{\alpha} x'_\alpha$ . Fix  $\beta \in J$ . And for each  $\alpha$ , define  $h_\alpha: G_\alpha \rightarrow G_\beta$  by  $h_\alpha(g) = g$  if  $\alpha = \beta$ ,  $0$  if  $\alpha \neq \beta$ . By extension

condition, there is a homomorphism  $h: G \rightarrow G_\beta$  s.t. the diagram

$$\begin{array}{ccc} G_\alpha & \hookrightarrow & G \\ & \searrow h_\alpha & \downarrow h \\ & & G_\beta \end{array}$$

commutes. For each  $\beta$ ,  $h(\sum_\alpha x_\alpha) = \sum_\alpha h_\alpha(x_\alpha) = x_\beta$  and  $h(\sum_\alpha x'_\alpha) = x'_\beta$ . Thus,  $G = \bigoplus_\alpha G_\alpha$ .  $\square$

**Corollary 8** (67.2). If  $G = G_1 \oplus G_2$ ,  $G_1 = \bigoplus_{\alpha \in J} H_\alpha$ ,  $G_2 = \bigoplus_{\beta \in K} H_\beta$ ,  $J \cap K = \emptyset$ , then  $G = \bigoplus_{r \in J \cup K} H_r$ .

**Corollary 9** (67.3). If  $G = G_1 \oplus G_2$ , then  $G/G_2 \cong G_1$ .

**Definition 33.** Let  $\{G_\alpha\}_{\alpha \in J}$  be a family of abelian groups. An abelian group  $G$  is an **external direct sum** of  $\{G_\alpha\}$  if there is a family of monomorphisms  $\{i_\alpha: G_\alpha \rightarrow G\}$  s.t.  $G = \bigoplus_\alpha i_\alpha(G_\alpha)$ .

## 10.68 Free products

**Note.** From now on, groups may not be abelian.

Lecture 25  
Wed, Dec 1

# Seifert–van Kampen theorem

Lecture 26  
Mon, Dec 6

## Group presentation

A group presentation is a method to represent a group. For a group  $G$ ,

- generators: a set of alphabets s.t. every element of  $G$  except for the identity is written as a finite sequence of these alphabets
- relators: words corresponding to relations in  $G$ .

We then say  $G$  has presentation  $G = \langle \text{generators} \mid \text{relators} \rangle$ .

### Rules

- (i) the operation of group corresponds to the join of two words, i.e.  $W_1, W_2 \rightarrow W_1 W_2$
- (ii) 1: identity element  $a1 = 1a = a$ .
- (iii)  $a^{-1}$ : inverse of  $a$
- (iv)  $aa^{-1} = a^{-1}a = 1$
- (v)  $a^n = \underbrace{a \dots a}_n$
- (vi)  $(ab)^{-1} = b^{-1}a^{-1}$
- (vii)  $a = b \Rightarrow ac = bc$

**Example.** •  $\mathbb{Z} \cong \langle a \rangle$  ( $0 \rightarrow 1, 1 \rightarrow a, n \rightarrow a^n$ )

- $\mathbb{Z}_2 \cong \langle a \mid a^2 \rangle$  or  $\langle a \mid a^2 = 1 \rangle$
- $\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$  or  $\langle a, b \mid ab = ba \rangle$
- $K_4 \cong \langle a, b \mid a^2 = 1, b^2 = 1, aba^{-1}b^{-1} = 1 \rangle$
- Free group: a group with no relations

$$F_1 \cong \langle a \rangle \cong \mathbb{Z}, F_2 \cong \langle a_1, a_2 \rangle, F_3 \cong \langle a_1, a_2, a_3 \rangle$$

(viii)  $G \cong \langle a_1, \dots, a_n \mid R_1, \dots, R_m \rangle \Leftrightarrow G \cong \langle a_1, \dots, a_n, c \mid R_1, \dots, R_m, c = W \rangle$   
where  $W$  is a word written in  $a_1, \dots, a_n$ . E.g.

$$\langle a, b \mid a^2 = 1, b^2 = 1, ab = ba \rangle \cong \langle a, b, c \mid a^2 = 1, b^2 = 1, ab = ba, c = ab \rangle.$$

(ix) Let  $G_1 \cong \langle a_1, \dots, a_n \mid R_1, \dots, R_p \rangle$ ,  $G_2 \cong \langle b_1, \dots, b_m \mid S_1, \dots, S_q \rangle$ . Then, the free product  $G_1 * G_2 \cong \langle a_1, \dots, a_n, b_1, \dots, b_m \mid R_1, \dots, R_p, S_1, \dots, S_q \rangle$ . E.g. for  $F_3 = \langle a_1, a_2, a_3 \rangle$  and  $F_2 = \langle b_1, b_2 \rangle$ ,

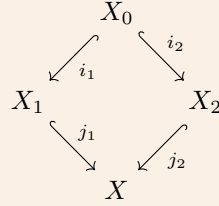
$$F_3 * F_2 \cong \langle a_1, a_2, a_3, b_1, b_2 \rangle \cong \langle a_1, a_2, a_3, a_4, a_5 \rangle \cong F_5.$$

(x) Let  $G \cong \langle a_1, \dots, a_n \mid R_1, \dots, R_p \rangle$  and  $N$  be the smallest normal subgroup which contains  $W_1, \dots, W_q$ , each of which is a word in  $a_1, \dots, a_n$ . Then,  $G/N \cong \langle a_1, \dots, a_n \mid R_1, \dots, R_p, W_1, \dots, W_q \rangle$ . E.g.

$$\mathbb{Z}/2\mathbb{Z} \cong \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}_2.$$

## Van Kampen theorem

**Theorem 42** (Van Kampen theorem (special version)). Suppose  $X = X_1 \cup X_2$  is a topological space where  $X_1$  and  $X_2$  are open, path-connected subsets of  $X$ ;  $X_0 = X_1 \cap X_2$  is nonempty and path-connected.



Given group presentations:

- $\pi_1(X_1) \cong \langle x_1, \dots, x_k \mid R_1, \dots, R_l \rangle$
- $\pi_1(X_2) \cong \langle y_1, \dots, y_m \mid S_1, \dots, S_n \rangle$
- $\pi_1(X_0) \cong \langle z_1, \dots, z_p \mid T_1, \dots, T_q \rangle$

$\pi_1(X)$  can be represented as

$$\langle x_1, \dots, x_k, y_1, \dots, y_m \mid R_1, \dots, R_l, S_1, \dots, S_n, \\ i_{1*}(z_1) = i_{2*}(z_1), \dots, i_{1*}(z_p) = i_{2*}(z_p) \rangle.$$

Furthermore, if  $X_0$  is simply connected, then  $\pi_1(X) \cong \pi_1(X_1) * \pi_1(X_2)$ .

All fundamental groups are based at  $x_0 \in X_0$ .  
 $i_{1*}$  and  $i_{2*}$  are group homomorphisms induced by inclusions  $i_1: X_0 \hookrightarrow X_1$  and  $i_2: X_0 \hookrightarrow X_2$ , respectively.

**Proof.** We do not prove this theorem in class. cf. Theorem 28.  $\square$

**Example.** Let  $X$  be bouquet with two leaves.  $\pi_1(X_1) = \langle a \rangle$ ,  $\pi_1(X_2) = \langle b \rangle$ ,  $\pi_1(X_0) = 1$ . Thus,  $\pi_1(X) \cong \langle a, b \rangle = F_2$ . In general,  $\pi_1(n\text{-bouquet}) \cong F_n$ .

**Example.** Let  $X$  be a torus. Select a simple closed curve. Then  $X_1$  has a deformation retraction onto bouquet with two leaves, hence  $\pi_1(X_1) \cong \langle a, b \rangle$ .  $X_2$  is contractible, so  $\pi_1(X_2)$  is trivial.  $\pi_1(X_0) \cong \langle c \rangle$ .  $c \sim aba^{-1}b^{-1}$  in  $X_1$  and  $c \sim 1$  in  $X_2$ . Thus,  $\pi_1(X) \cong \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$  (or  $\mathbb{Z} \oplus \mathbb{Z}$ )

**Example.** Let  $X$  be a connected sum of two tori.  $\pi_1(X_1) \cong \langle a, b \rangle$ ,  $\pi_1(X_2) \cong \langle c, d \rangle$ , and  $\pi_1(X_0) \cong \langle e \rangle$ . Thus,

$$\begin{aligned}\pi_1(X) &\cong \langle a, b, c, d \mid aba^{-1}b^{-1} = cdc^{-1}d^{-1} \rangle \\ &\cong \langle a, b, c, d \mid aba^{-1}b^{-1}dcd^{-1}c^{-1} = 1 \rangle \\ &\cong \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle.\end{aligned}$$

In general, let  $S_n$  be the surface with genus  $n$ .<sup>a</sup> Then,

$$\pi_1(S_n) \cong \langle a_1, b_1, a_2, b_2, \dots, a_n, b_n \mid [a_1, b_1][a_2, b_2] \dots [a_n, b_n] = 1 \rangle.$$

<sup>a</sup>Intuitively, the genus is the number of holes of a surface.

**Example.** Let  $X = \mathbb{RP}^2 \cong S^2/\sim$ . Then,  $X_1$  is homeomorphic to Möbius band, so  $\pi_1(X_1) \cong \langle a \rangle$ . And  $\pi_1(X_2)$  is trivial,  $\pi_1(X_0) \cong \langle b \rangle$ . Since  $b = a^2$  in  $X_1$  and  $b = 1$  in  $X_2$ ,  $\pi_1(X) \cong \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}_2$ .

**Example.** Remove an open disk from  $S_2$ .  $\pi_1(S_2 - \text{open disk}) \cong F_4$ . In general,  $\pi_1(S_n - \text{open disk}) \cong F_{2n}$ . Thus,

$$n \neq m \Rightarrow F_{2n} \not\cong F_{2m} \Rightarrow S_n \not\cong S_m.$$

This implies that  $S_n$  in classification theorem are different each other. Note that isomorphism between free group is easy, but for general group, it is not easy. For that reason, we used  $S_n - \text{open disk}$ , not  $S_n$  itself which produces non-free group presentation.

**Note.** HW9: Let  $(\mathbb{RP}^2)_n = \underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_n$

- (i) Find  $\pi_1((\mathbb{RP}^2)_n)$ .
- (ii) Show that  $(\mathbb{RP}^2)_n \not\cong (\mathbb{RP}^2)_m$  when  $n \neq m$ .
- (iii) Show that  $(\mathbb{RP}^2)_n \not\cong S_m$  for all  $n, m \in \mathbb{N}$ .
- (iv) Find  $\pi_1(S_1 \# \mathbb{RP}^2)$ .

## Knot group (optional)

Lecture 28  
Mon, Dec 13

**Definition 34.** A **knot**  $K$  is a simple closed curve in  $\mathbb{R}^3$  (or  $S^3$ ).

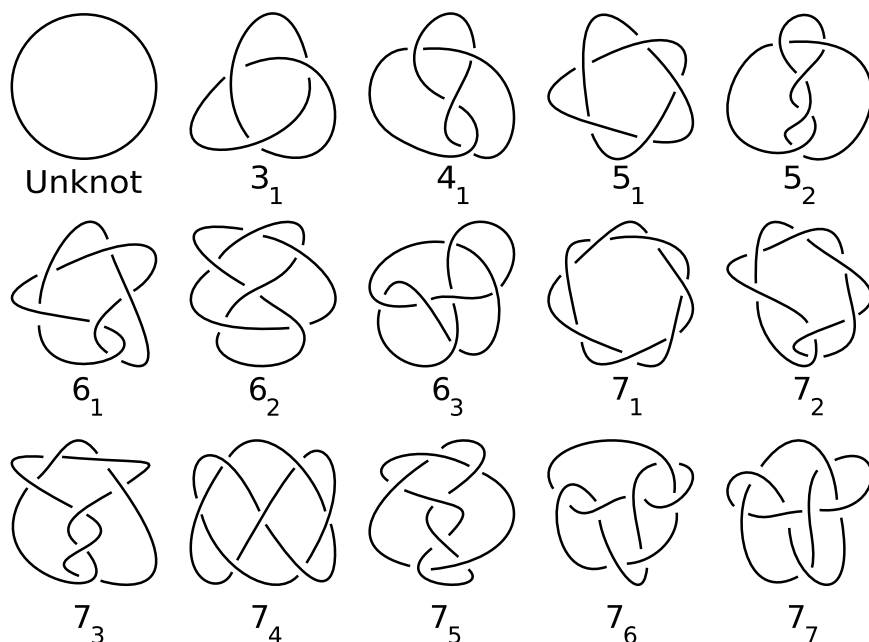


Figure 10.1: A table of prime knots up to seven crossings. The knots are labeled with Alexander-Briggs notation.

**Definition 35.** Two knots  $K_1$  and  $K_2$  are **equivalent** ( $K_1 \sim K_2$ ) if there exists a homeomorphism  $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  s.t.  $h(K_1) = K_2$ .

**Definition 36.** The **exterior** of a knot  $K$  is  $E(K) = \text{cl}(\mathbb{R}^3 - K)$ .

**Definition 37.** The **knot group** of  $K$  is  $G(K) = \pi_1(E(K))$ .

**Theorem 43.** If  $K_1 \sim K_2$ , then  $G(K_1) \cong G(K_2)$ .

**Proof.** If  $K_1 \sim K_2$ , there is a homeomorphism  $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  s.t.  $h(K_1) = K_2$ . Then,  $\mathbb{R}^3 - K_1 \cong \mathbb{R}^3 - K_2$  by  $h$ , hence  $E(K_1) \cong E(K_2)$ . Thus,  $G(K_1) \cong G(K_2)$ .  $\square$

## Wirtinger presentation

The Wirtinger presentation is a method to calculate  $G(K)$  from a diagram of  $K$ . An over-passing of  $D$  is a subarc from an undercrossing to the next undercrossing. Given an orientation to  $K$ , fix a base point  $b_0 \in \mathbb{R}^3$  in a high position w.r.t  $z$ . Let  $x_i$  be a loop based at  $b_0$  which wraps around the  $i$ -th over-passing  $P_i$  positively. Then  $\{x_1, \dots, x_n\}$  generates  $\pi_1(E(K))$ . Relations come from near crossings:

$$x_j x_i = x_{i+1} x_j \quad (x_{i+1} = x_j x_i x_j^{-1}) \quad \text{or} \quad x_i x_j = x_j x_{i+1} \quad (x_{i+1} = x_j^{-1} x_i x_j).$$

For example, let  $K$  be a trefoil knot. Then,

$$\begin{aligned} G(K) &\cong \langle x_1, x_2, x_3 \mid x_1 = x_2^{-1} x_3 x_2, x_2 = x_3^{-1} x_1 x_3, x_3 = x_1^{-1} x_2 x_1 \rangle \\ &\cong \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle \\ &\cong \langle x_2 x_1^2, x_2 x_1 \mid (x_2 x_1^2)^2 = (x_2 x_1)^3 \rangle \\ &\cong \langle a, b \mid a^2 = b^3 \rangle. \end{aligned}$$

The trefoil knot has knot group isomorphic to the braid group  $B_3$ . The unknot (standard circle) has knot group isomorphic to  $\mathbb{Z}$ , i.e.  $G(O) \cong \mathbb{Z} \cong \langle c \rangle$ . How can we prove that unknot is not homeomorphic to trefoil knot?

Suppose  $f : G(K) \rightarrow \mathbb{Z}$  is an isomorphism s.t.  $f(a) = n$  and  $f(b) = m$ . Since  $f$  is surjective,  $\{n, m\}$  generates  $\mathbb{Z}$ , hence  $n$  and  $m$  are relatively prime. But  $f(a^2) = f(b^3) \Rightarrow 2n = 3m \Rightarrow n = 3k$  where  $m = 2k$ .  $n$  and  $m$  have common divisor  $k$ .  $\nmid$ . Thus,  $G(O) \not\cong G(K)$ .

In knot theory, more easy-to-calculate and powerful invariants are developed.



Figure 10.2: A trefoil knot