

Homework 9

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Exercise 3-3.5 Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right)$$

and show that

(a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

(b) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

(c) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

(d) The lines of curvature are the coordinate curves.

(e) The asymptotic curves are $u + v = \text{const.}$, $u - v = \text{const.}$

Proof. (a) $\mathbf{x}_u = (1 - u^2 + v^2, 2uv, 2u)$ and $\mathbf{x}_v = (2uv, 1 - v^2 + u^2, -2v)$. Therefore,

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u = (1 - u^2 + v^2)^2 + (2uv)^2 + (2u)^2 \\ &= 1 + u^4 + v^4 - 2u^2 + 2v^2 - 2u^2v^2 + 4u^2v^2 + 4u^2 \\ &= 1 + u^4 + v^4 + 2u^2 + 2v^2 + 2u^2v^2 = (1 + u^2 + v^2)^2 \\ F &= \mathbf{x}_u \cdot \mathbf{x}_v = (1 - u^2 + v^2) \cdot 2uv + 2uv(1 - v^2 + u^2) + 2u \cdot (-2v) \\ &= 4uv - 4uv = 0 \\ G &= \mathbf{x}_v \cdot \mathbf{x}_v = (2uv)^2 + (1 - v^2 + u^2)^2 + (-2v)^2 = \dots = E \end{aligned}$$

(b) We calculate

$$\begin{aligned} \mathbf{x}_u \wedge \mathbf{x}_v &= \begin{vmatrix} i & j & k \\ 1 - u^2 + v^2 & 2uv & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \end{vmatrix} \\ &= (-4uv^2 - 2u + 2uv^2 - 2u^3)i + (4u^2v + 2v - 2u^2v + 2v^3)j \\ &\quad + (1 - v^2 + u^2 - u^2 + u^2v^2 - u^4 + v^2 - v^4 + v^2u^2 - 4u^2v^2)k \\ &= (-2u(u^2 + v^2 + 1), 2v(u^2 + v^2 + 1), 1 - (u^2 + v^2)^2) \\ |\mathbf{x}_u \wedge \mathbf{x}_v|^2 &= 4u^2(u^2 + v^2 + 1)^2 + 4v^2(u^2 + v^2 + 1)^2 + (1 - (u^2 + v^2)^2)^2 \\ &= (u^2 + v^2 + 1)^4. \end{aligned}$$

Thus,

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{(-2u, 2v, 1 - u^2 - v^2)}{u^2 + v^2 + 1}.$$

Since $\mathbf{x}_{uu} = (-2u, 2v, 2)$, $\mathbf{x}_{uv} = (2v, 2u, 0)$, and $\mathbf{x}_{vv} = (2u, -2v, -2)$,

$$\begin{aligned} e &= \langle N, \mathbf{x}_{uu} \rangle = \frac{4u^2 + 4v^2 + 2(1 - u^2 - v^2)}{u^2 + v^2 + 1} = \frac{2(1 + u^2 + v^2)}{u^2 + v^2 + 1} = 2 \\ f &= \langle N, \mathbf{x}_{uv} \rangle = \frac{-4uv + 4uv + 0 \cdot (1 - u^2 - v^2)}{u^2 + v^2 + 1} = 0 \\ g &= \langle N, \mathbf{x}_{vv} \rangle = \frac{-4u^2 - 4v^2 - 2(1 - u^2 - v^2)}{u^2 + v^2 + 1} = \frac{-2(1 + u^2 + v^2)}{u^2 + v^2 + 1} = -2 \end{aligned}$$

(c) We first calculate

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} = \frac{2 \cdot (-2) - 0^2}{(1 + u^2 + v^2)^4 - 0^2} = \frac{-4}{(1 + u^2 + v^2)^4} \\ H &= \frac{gE - 2fF + eG}{2(EG - F^2)} = \frac{(1 + u^2 + v^2)^2 \cdot 2 - 2 \cdot 0 \cdot 0 + (1 + u^2 + v^2)^2(-2)}{2(1 + u^2 + v^2)^4} = 0 \end{aligned}$$

Specially, the Enneper surface is minimal. Now from the relations

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2},$$

we get a system of equations for k_1 and k_2 . From $H = 0$ we get $k_2 = -k_1$. Now

$$K = -k_1^2 \Rightarrow \frac{-4}{(1 + u^2 + v^2)^4} = -k_1^2 = k_1 = \frac{\pm 2}{(1 + u^2 + v^2)^2}$$

and therefore

$$k_2 = -k_1 = \frac{\mp 2}{(1 + u^2 + v^2)^2}$$

Since the principal curvatures k_1 and k_2 can be switched (the order doesn't matter), we actually have that

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

(d) The curve $c(t) = \mathbf{x}(u(t), v(t))$, which lies on S , is a line of curvature if and only if u and v satisfy the differential equation

$$\begin{vmatrix} v'^2 & -u'v & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

Substituting the values for the Enneper surface we get

$$\begin{aligned} 0 &= \begin{vmatrix} v'^2 & -u'v & u'^2 \\ (1 + u^2 + v^2)^2 & 0 & (1 + u^2 + v^2)^2 \\ 2 & 0 & -2 \end{vmatrix} = (\text{Lapl. exp. 2nd col.}) \\ &= (-1)^4(-u'v') \begin{vmatrix} v'^2 & u'^2 \\ 2 & -2 \end{vmatrix} = -u'v'(-2v'^2 - 2u'^2) = 2u'v'(u'^2 + v'^2) \end{aligned}$$

which is equivalent to $u' = 0$ or $v' = 0$ or $u'^2 + v'^2 = 0$. The last case implies $u' = v' = 0$, which is covered by first two cases. Now we have that $j' = 0 \Rightarrow u = \text{const.}$ or $v' = 0 \Rightarrow$

$v = \text{const.}$, which are exactly the coordinate curves.

(e) The curve c is an asymporic curve if and only if u and v satisfy the differential equation

$$eu'^2 + 2fu'v' + gv'^2 = 0$$

Substituting the values for the Enneper surface we get

$$2u'^2 - 2v'^2 = 0 \Rightarrow 2(u' - v')(u' + v') = 0 \Rightarrow u' \pm v' = 0 \Rightarrow u \pm v = \text{const.}$$

□

Claim 1. Fix $p \in S$. Choose a unit normal vector n at p . Let $h: S \rightarrow \mathbb{R}$ s.t. $q \mapsto \langle q - p, n \rangle$ be the height function relative to p with n . Show that $q \in S$ is a critical point of h if and only if $n \perp T_q S$ (i.e., $n = \pm N(q)$).

Proof. For $q \in S$ and for any $v \in T_q S$, choose $\alpha: (-\varepsilon, \varepsilon) \rightarrow S$ s.t. $\alpha(0) = q$ and $\alpha'(0) = v$. Then,

$$dh_q(v) = \left. \frac{d}{dt} \right|_{t=0} h(\alpha(t)) = \left. \frac{d}{dt} \right|_{t=0} \langle \alpha(t) - p, n \rangle = \langle \alpha'(0), n \rangle = \langle v, n \rangle,$$

and TFAE:

$$\begin{aligned} q \text{ is a critical point} &\Leftrightarrow dh_q \text{ is not surjective} \\ &\Leftrightarrow dh_q \equiv 0 \\ &\Leftrightarrow dh_q(v) = \langle v, n \rangle = 0 \text{ for all } v \in T_q S \\ &\Leftrightarrow v \perp n \text{ for all } v \in T_q S \\ &\Leftrightarrow n \perp T_q S \end{aligned}$$

□

Claim 2. There is no compact regular surface with negative Gauss curvature everywhere.

Proof. Suppose not. Let S be a compact regular surface with negative Gauss curvature everywhere. Choose any point $p \in S$. Then, $h: S \rightarrow \mathbb{R}$ s.t. $q \mapsto \langle q - p, n \rangle$ is continuous. Since S is compact, $h(S)$ is also compact, i.e. $h(S)$ is closed and bounded. Then, $h(S)$ has a maximum with some point $q \in S$ s.t. $h(q) = \max_{q' \in S} \langle q' - p, n \rangle$. Hence, for any $\alpha: (-\varepsilon, \varepsilon) \rightarrow S$ s.t. $\alpha(0) = q$ and $\alpha'(0) = v \in T_q S$, $dh_q(v) = \left. \frac{d}{dt} \right|_{t=0} h(\alpha(t)) = 0$. Thus, $dh_q \equiv 0$, so q is a critical point of h . By Claim 1, $n \perp T_q S$. Since q is a hyperbolic point, by Proposition 1, in each neighborhood of q there exist points $q_1, q_2 \in S$ in both sides of $T_p(S)$. Then,

$$h(q_i) = \langle q_i - p, n \rangle = \langle q_i - q, n \rangle + \langle q - p, n \rangle = h(q) + \langle q_i - q, n \rangle \quad \text{for } i = 1, 2.$$

Since $n \perp T_p S$ and q_1, q_2 are in both sides of $T_p(S)$, signs of $\langle q_1 - q, n \rangle$ and $\langle q_2 - q, n \rangle$ are different. WLOG, assume $\langle q_1 - q, n \rangle > 0$ and $\langle q_2 - q, n \rangle < 0$. Then, $h(q_1) = h(q) + \langle q_1 - q, n \rangle > h(q)$ and it is a contradiction to the maximality of $h(q)$. Therefore, there is no compact regular surface with negative Gauss curvature everywhere. □