미분기하학개론 1 - HW 1

수리과학부

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1-3.4. $\alpha(t) = (\sin t,\ \cos t + \log \tan \frac{t}{2})$

a. $\frac{d}{dt}(\sin t) = \cos t,\quad \frac{d}{dt}(\cos t + \log\tan\frac{t}{2}) = -\sin t + \frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}} \quad (t \neq \frac{\pi}{2})$

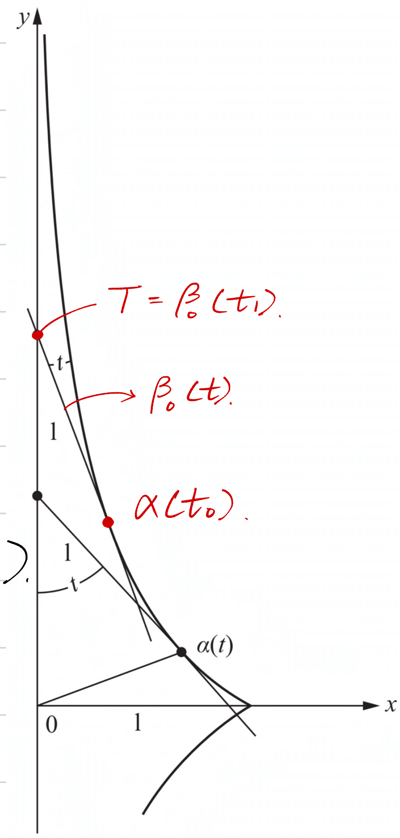
$$\alpha'(t) = (\cos t,\ -\sin t + (2\tan\tfrac{t}{2}\cdot\cos^2\tfrac{t}{2})^{-1})$$
$$= (\cos t,\ -\sin t + (2\sin\tfrac{t}{2}\cdot\cos\tfrac{t}{2})^{-1})$$
$$= (\cos t,\ -\sin t + \frac{1}{\sin t}) \quad (\because \sin(\tfrac{t}{2}+\tfrac{t}{2}) = 2\sin\tfrac{t}{2}\cdot\cos\tfrac{t}{2})$$

This is well-defined on $(0,\pi)$ except at $t = \frac{\pi}{2}$

b. $\alpha(t_0) = (\sin t_0,\ \cos t_0 + \log\tan\frac{t_0}{2})$.

$\alpha'(t_0) = (\cos t_0,\ -\sin t_0 + \frac{1}{\sin t_0})$.

$$\beta_0(t) = (\sin t_0,\ \cos t_0 + \log\tan\tfrac{t_0}{2}) + t(\cos t_0,\ -\sin t_0 + \frac{1}{\sin t_0})$$
$$= (\sin t_0 + t\cos t_0,\ \cos t_0 + \log\tan\tfrac{t_0}{2} - t\sin t_0 + \frac{t}{\sin t_0})$$



$\beta_0(t)$ meets the y-axis at

$\sin t_0 + t\cos t_0 = 0. \Rightarrow t = -\frac{\sin t_0}{\cos t_0} = t_1.$

$$T = (0,\ \cos t_0 + \log\tan\tfrac{t_0}{2} + \frac{\sin^2 t_0}{\cos t_0} - \frac{1}{\cos t_0})$$
$$= (0,\ \cos t_0 + \log\tan\tfrac{t_0}{2} + \frac{-\cos^2 t_0}{\cos t_0})$$
$$= (0,\ \log\tan\tfrac{t_0}{2})$$

$$|\alpha(t_0) - T| = |(\sin t_0,\ \cos t_0)|$$
$$= (\sin^2 t_0 + \cos^2 t_0)^{\frac{1}{2}} = 1$$

1-5. 2. $\alpha(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c})$. $s\in\mathbb{R}$, $c^2 = a^2+b^2$

a. $\alpha'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$

$$|\alpha'(s)|^2 = \frac{a^2}{c^2}(\sin^2\frac{s}{c} + \cos^2\frac{s}{c}) + \frac{b^2}{c^2}$$
$$= \frac{a^2+b^2}{c^2} = 1.$$

$\therefore \int_0^t |\alpha'(s)|\,ds = \int_0^t 1\,ds = t$

b. $\alpha''(s) = (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0)$.

$$k(s) = |\alpha''(s)| = \left\{\frac{a^2}{c^4}(\cos^2\frac{s}{c} + \sin^2\frac{s}{c})\right\}^{\frac{1}{2}} = \frac{|a|}{c^2}$$

$$n(s) = \frac{\alpha''(s)}{k(s)} = \frac{c^2}{|a|}\cdot(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0).$$
$$= \frac{a}{|a|}(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0).$$

$$n'(s) = \frac{a}{|a|c}(\sin\frac{s}{c}, -\cos\frac{s}{c}, 0).$$

$$t(s) = \alpha'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$$

$$b'(s) = t(s) \wedge n'(s).$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ \frac{a}{|a|c}\sin\frac{s}{c} & \frac{-a}{|a|c}\cos\frac{s}{c} & 0 \end{vmatrix}$$

$$= (\frac{ab}{|a|c^2}\cos\frac{s}{c}, \frac{ab}{|a|c^2}\sin\frac{s}{c}, 0).$$

$b' = \tau n.$ $\quad\therefore \tau = -\frac{b}{c^2}$

c. $b(s) = t(s) \wedge n(s)$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\frac{a}{|a|}\cos\frac{s}{c} & -\frac{a}{|a|}\sin\frac{s}{c} & 0 \end{vmatrix}$$

$$= (\frac{ab}{|a|c}\sin\frac{s}{c}, -\frac{ab}{|a|c}\cos\frac{s}{c}, \frac{a^2}{|a|c}).$$

Thus, the osculating plane at s is

$$\frac{ab}{|a|c}\sin\frac{s}{c}x - \frac{ab}{|a|c}\cos\frac{s}{c}y + \frac{a^2}{|a|c}z = 0.$$

$$\sin\frac{s}{c}x - \cos\frac{s}{c}y + \frac{z}{b} = 0 \quad (\text{when } c\neq 0).$$

d. $n(s) = \frac{a}{|a|}(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$.

$\cos\varphi = \frac{n(s)\cdot e_3}{|n(s)|\cdot|e_3|} = \frac{1}{|n(s)|}\frac{a}{|a|}(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)\cdot(0,0,1)$.

$= 0$.

$\therefore \varphi = \frac{\pi}{2}$.

e. $\cos\phi = \frac{t(s)\cdot e_3}{|t(s)|\cdot|e_3|} = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})\cdot(0,0,1) = \frac{c}{b}$

$\phi = \cos^{-1}\frac{b}{c}$

$\therefore$ The angle between the tangent line and z-axis is constant.

1-5.6.

a. $u, v \in \mathbb{R}^3$, ρ : orthogonal transformation.

$|\rho u| = \sqrt{\rho u\cdot\rho u} = \sqrt{u\cdot u} = |u|$.

$\therefore$ norm of a vector $\in\mathbb{R}^3$ is invariant under ρ.

$\varphi(\rho u, \rho v)$: angle between $\rho u, \rho v$.

$\cos\varphi = \frac{\rho u\cdot\rho v}{|\rho u||\rho v|}$

$\varphi(\rho u, \rho v) = \cos^{-1}\frac{\rho u\cdot\rho v}{|\rho u||\rho v|} = \cos^{-1}\frac{u\cdot v}{|u||v|} = \varphi(u,v)$.

$\therefore$ angle is also invariant under ρ.

b. WTS. $\rho u\wedge\rho v = \rho(u\wedge v)$.

$|\rho u\wedge\rho v|^2 = \det\begin{pmatrix}\rho u\cdot\rho u & \rho v\cdot\rho u\\ \rho u\cdot\rho v & \rho v\cdot\rho v\end{pmatrix} = \det\begin{pmatrix}u\cdot u & v\cdot u\\ u\cdot v & v\cdot v\end{pmatrix} = |u\wedge v|^2$

$\therefore |\rho u\wedge\rho v| = |u\wedge v|$.

By a, $|\rho(u\wedge v)| = |u\wedge v|$

$\therefore |\rho u\wedge\rho v| = |\rho(u\wedge v)|$

$(\rho u\wedge\rho v)\cdot\rho(u\wedge v) = \begin{vmatrix}\rho u\\ \rho v\\ \rho(u\wedge v)\end{vmatrix} = \det(\rho)\cdot\begin{vmatrix}u\\ v\\ u\wedge v\end{vmatrix} = \det(\rho)\cdot(u\wedge v\cdot u\wedge v)$

$= \det(\rho)\cdot|u\wedge v|^2$

φ : angle between $\rho u\wedge\rho v$, $\rho(u\wedge v)$.

$$\cos\varphi = \frac{\rho u \wedge \rho v \cdot \rho(u \wedge v)}{|\rho u \wedge \rho v||\rho(u\wedge v)|} = \frac{\det(\rho)\cdot|u\wedge v|^2}{|u\wedge v|^2} = \det \rho.$$

Determinant of matrix of orthogonal transformation is eighter -1 or 1. If $\cos\varphi = 1$, $\rho u \wedge \rho v$ and $\rho(u\wedge v)$ are collinear. Since each norm is same, they are equal.

If determinant is -1, the vectors are opposite.

c. Let $\beta(t) = A \circ \rho(\alpha(t))$. rigid motion

$$\beta'(t) = (\rho\alpha(t) + v)' = \rho\alpha'(t)$$

$$\beta''(t) = \rho\alpha''(t).$$

$$\beta'''(t) = \rho\alpha'''(t).$$

by a.

$$S_\beta = \int_{t_0}^{t} |\beta'(t)|\,dt = \int_{t_0}^{t} |\rho\alpha'(t)|\,dt = \int_{t_0}^{t} |\alpha'(t)|\,dt = S_\alpha$$

$$k_\beta = |\beta''(t)| = |\rho\alpha''(t)| = |\alpha''(t)| = k_\alpha$$

by a

$$\tau_\beta = -\frac{(\rho\alpha' \wedge \rho\alpha'')\cdot\rho\alpha'''}{k_\beta^2} = -\frac{1}{k_\alpha}\cdot\begin{vmatrix}\rho\alpha' \\ \rho\alpha'' \\ \rho\alpha'''\end{vmatrix} = -\frac{1}{k_\alpha}\det(\rho)\cdot\begin{vmatrix}\alpha' \\ \alpha'' \\ \alpha'''\end{vmatrix}$$

$$= -\frac{\alpha'\wedge\alpha''\cdot\alpha'''}{k_\alpha^2} = \tau_\alpha. \quad \blacksquare$$