

Topology II

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Based on lectures by Prof. Youngsik Huh

Preface

These notes are based on the course MAT4004: Topology II taught by Professor Youngsik Huh at Hanyang University in fall 2021. The lectures mainly covered algebraic topology which is the second part of James Munkres' *Topology*.

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Chapter -1

Introduction

Lecture 1
Wed, Sep 1

A fundamental problem in math: to classify objects in the given category.

- Sets: $|A| = |B|$ (cardinality)
- Groups, Rings, Fields: $G \cong G'$ (isomorphic)
- Topological spaces: $X \cong Y$ (homeomorphic)

When two topological spaces are homeomorphic, we may prove it by finding out a homeomorphism. But, in the case that they are not homeomorphic, how can we prove it?

Example. Let S be a 2-dimensional sphere and T be a torus. Then $S \not\cong T$.

Proof. Suppose there exists a homeomorphism $h: T \rightarrow S$. Let c be a simple closed curve on T , as Figure 1. Then $h(c)$ should be a simple closed curve on S , and $h: T - c \rightarrow S - h(c)$ is a homeomorphism. But $T - c$ is connected and $S - h(c)$ is not connected, which is a contradiction. \nexists

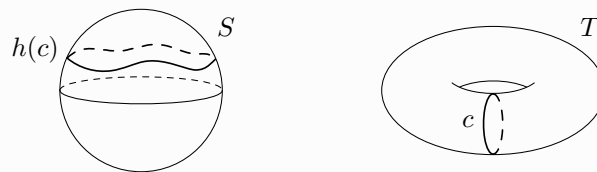


Figure 1: $S \not\cong T$

In fact, on S , every loop can be continuously deformed to a point. But c cannot be on T . Such loops as c would be one of our interests in the lecture. From the family of loops on a topological space X , we will construct a group $\pi_1(X)$, called the **fundamental group** of X .

In fact, if $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$. So we may use the fundamental group to distinguish topological spaces.

Chapter 0

Construction of more topological spaces

Consider two topological spaces (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) where X_1 and X_2 are disjoint.

Union of spaces. Let $\mathcal{T} = \{u \subset X_1 \sqcup X_2 \mid u \cap X_1 \in \mathcal{T}_1, u \cap X_2 \in \mathcal{T}_2\}$. Then $(X_1 \sqcup X_2, \mathcal{T})$ is a topological space such that (X_i, \mathcal{T}_i) is a subspace.

Product space. Let $\mathcal{B} = \{u_1 \times u_2 \subset X_1 \times X_2 \mid u_i \in \mathcal{T}_i\}$ and $\mathcal{T} = \{u \subset X_1 \times X_2 \mid u \text{ is a union of some elements of } \mathcal{B}\}$, i.e. \mathcal{B} is a base for \mathcal{T} . Then $(X_1 \times X_2, \mathcal{T})$ is a topological space, called product space of X_1 and X_2 . Note the projection function $\pi_i: X_1 \times X_2 \rightarrow X_i$ given by $(x_1, x_2) \mapsto x_i$ is continuous.

Example. Consider $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ (subspace of the Euclidean space \mathbb{R}^2) and a torus $T \subset \mathbb{R}^3$. Then $S^1 \times S^1 \cong T$.

Quotient space. E.g., $\mathbb{Z}/2\mathbb{Z}$ ($a - b = 2n \Rightarrow a \sim b$).

Lecture 2
Mon, Sep 6

0.1 Quotient spaces

Definition 1. Let X, Y be topological spaces and $p: X \rightarrow Y$ be a surjective map^a. Then p is said to be a **quotient map** if

$$u \subset Y \text{ is open in } Y \iff p^{-1}(u) \text{ is open in } X \quad (1)$$

or equivalently,

$$v \subset Y \text{ is closed in } Y \iff p^{-1}(v) \text{ is closed in } X. \quad (2)$$

^aThe map usually means the function between topological spaces.

Proposition 1. (1) \Leftrightarrow (2).

Proof. (1) \Rightarrow (2) Suppose p is a quotient map by the first definition. For a closed subset v of Y , $p^{-1}(Y - v) = X - p^{-1}(v)$ is open in X . Thus, $p^{-1}(v)$ is closed in X . If $p^{-1}(v)$ is closed, $X - p^{-1}(v) = p^{-1}(Y - v)$ is open in X . Thus $Y - v$ is open, hence v is closed.

(2) \Rightarrow (1) Similar. □

Remark. A quotient map is continuous.

Remark. A surjective continuous function $f: X \rightarrow Y$ is a quotient map if f is an open map.

Definition 2. Suppose X be a topological space and A be a set. Let $f: X \rightarrow A$ be a surjective function and

$$\mathcal{T}_f = \{u \subset A \mid f^{-1}(u) \text{ is open in } X\}.$$

Then \mathcal{T}_f is a topology for A , called **quotient topology** induced by f .

Remark. $f: X \rightarrow (A, \mathcal{T}_f)$ is a quotient map by definition.

Let X be a topological space and \sim be an equivalence relation on X . For $x \in X$, $[x] = \{x' \in X \mid x \sim x'\}$ is a equivalence class of x , and $X/\sim = \{[x] \mid x \in X\}$ is the set of all equivalence classes. Now consider $q: X \rightarrow X/\sim$ given by $x \mapsto [x]$. (q is clearly surjective by definition.) Then, $(X/\sim, \mathcal{T}_q)$ is called a **quotient space** of X .

Example. Let $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Define an equivalence relation \sim on X by $(x, y) \sim (x', y')$ iff

- $x = x', y = 0, y' = 1$
- $y = y', x = 0, x' = 1$
- $x = x', y = y'$

A quotient space is obtained by identifying a part with another part!

Theorem 1 (22.2). Let X, Y, Z be topological spaces, $p: X \rightarrow Y$ a quotient map, and $g: X \rightarrow Z$ a map s.t. $p(x_1) = p(x_2)$ implies $g(x_1) = g(x_2)$. Then

- (i) $\exists f: Y \rightarrow Z$ s.t. $f \circ p = g$.
- (ii) f is continuous iff g is continuous.
- (iii) f is a quotient map iff g is a quotient map.

Proof. (i) Define f by $f(y) = g(x)$ for $x \in p^{-1}(y)$. It is well defined.

(ii) If f is continuous, a composition of continuous functions, $g = f \circ p$, is also continuous. Conversely, for an open subset u of Z , $g^{-1}(u) = p^{-1}(f^{-1}(u))$ is open in X . Since p is a quotient map, $f^{-1}(u)$ is open. Thus f is continuous.

(iii) DIY. (not HW)

□

Notation. For a function $g: X \rightarrow Z$, define an equivalence relation \sim on X by $x_1 \sim x_2$ iff $g(x_1) = g(x_2)$. Then, $X/g := X/\sim$.

Corollary 1 (22.3). Let $g: X \rightarrow Z$ be a surjective continuous map. Then

- (i) There exists a homeomorphism $f: X/g \rightarrow Z$ iff g is a quotient map.
- (ii) If Z is Hausdorff, then so is X/g .
- (iii) If X is compact and Z is Hausdorff, then f is a homeomorphism.

Proof. By Theorem 1.(i), g induces a continuous function $f: X/g \rightarrow Z$ s.t. $f \circ p = g$. We can immediately see that f is injective and surjective.

$$\begin{array}{ccc} X & & \\ \downarrow p & \searrow g & \\ X/g & \xrightarrow{f} & Z \end{array}$$

- (i) If f is a homeomorphism, then f is a quotient map. Thus, $g = f \circ p$ is quotient map. Conversely, if g is a quotient map, then so is f by Theorem 1.(iii). Since f is a injective quotient map, f is a homeomorphism.
- (ii) Let w_1, w_2 be two distinct points of X/g . Then $f(w_1) \neq f(w_2)$ and there are two disjoint open sets u_1, u_2 in Z s.t. $f(w_1) \in u_1, f(w_2) \in u_2$. $f^{-1}(u_1)$ and $f^{-1}(u_2)$ are disjoint open neighborhoods of w_1 and w_2 , respectively.
- (iii) Recall that f is injective, surjective and continuous. So, it's enough to show that f is an open map, which is equivalent to f^{-1} is continuous. Since X is compact, so is X/g by continuity. Note that the closed subset of a compact set is compact. Let u be an open subset of X/g . Then $X/g - u$ is compact, and so is $f(X/g - u) = f(X/g) - f(u) = Z - f(u)$ in the Hausdorff space Z . Since every compact subset of a Hausdorff space is closed, $Z - f(u)$ is closed in Z . Therefore, $f(u)$ is open.

□

Example. Let $g: [0, 1] \rightarrow S^1 \subset \mathbb{R}^2$ (or \mathbb{C}) be given by $r \mapsto (\cos 2\pi r, \sin 2\pi r)$ ($= e^{2\pi i r} = \cos 2\pi r + i \sin 2\pi r$). Note $[0, 1]$ is compact and S^1 is Hausdorff.

Then,

$$\underbrace{[0, 1] / g = [0, 1] / \{0, 1\}}_{\text{quotient spaces}} \underset{\text{Cor 1.(iii)}}{\cong} \underbrace{S^1 \subset \mathbb{R}^2}_{\text{Euclidean subspace}}.$$

Example. Let $X = [0, 1] \times [0, 1]$ and $g: X \rightarrow S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ (or $\mathbb{C} \times \mathbb{C}$) be given by $(x, y) \mapsto (e^{2\pi i r}, e^{2\pi i r})$. Note that g is surjective and continuous. Then,

$$X/g = X / \left(\begin{smallmatrix} (0,y) \sim (1,y) \\ (x,0) \sim (x,1) \end{smallmatrix} \right) = \text{Torus} \cong S^1 \times S^1.$$

Notation. Let X be a topological space and A be a subset of X . Define an equivalence relation \sim on X by $x_1 \sim x_2$ iff $x_1, x_2 \in A$ or $x_1 = x_2$. Then $X/A := X/\sim$.

Example. Let $D = \{re^{i\theta} \mid 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$ and $g: D \rightarrow S^2 \subset \mathbb{R}^3$ be given by $re^{i\theta} \mapsto (\sqrt{4r - 4r^2} \cos \theta, \sqrt{4r - 4r^2} \sin \theta, 2r - 1)$. Then,

$$D/g = D / \partial D (= S^1) \cong S^2.$$

For $n \geq 0$,

- $S^0 = \{-1, 1\} \subset \mathbb{R}$
- $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid \sum x_i^2 = 1\}$
- $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 = 1\}$
- ...
- $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$ (n -sphere)

Define an equivalence relation \sim on S^n by $x \sim y$ iff $y = -x$ or $y = x$. Then, $\mathbb{RP}^n := S^n/\sim$ is called the **real n -dimensional projective space**.

- $\mathbb{RP}^0 = \{\text{a point}\}$
- $\mathbb{RP}^1 \cong [0, 1] / \{0, 1\} \cong S^1$
- $\mathbb{RP}^2 \cong D^2 \cup \mathbb{RP}^1$

In general, S^n can be decomposed depend upon last coordinate as

$$S^n = \underbrace{\text{upper half of } S^n \cup \text{lower half of } S^n}_{n\text{-dimensional disk } D^n} \cup \underbrace{S^{n-1}}_{\mathbb{RP}^{n-1}},$$

and then,

$$\mathbb{RP}^n \cong \text{attaching } D^n \text{ along } \mathbb{RP}^{n-1}$$

where $D^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$, $\partial D^n \cong S^{n-1}$.

Notation. Let X, Y be topological spaces and A be a subspace of X . Let $f: A \rightarrow Y$ be continuous. Define \sim on $X \sqcup Y$ by $a \sim f(a)$ for $a \in A$. Then $X \cup_f Y := X \sqcup Y / \sim$ is the **adjunction space**. In that case, f is called the **attaching map**.

Now if we define attaching maps as

$$\begin{aligned} f_0: S^0 &\rightarrow \mathbb{RP}^0 \\ f_1: S^1 &\rightarrow \mathbb{RP}^1 \cong D^1 \cup_{f_0} \mathbb{RP}^0 \end{aligned}$$

then,

$$\mathbb{RP}^n \cong \underbrace{\{\text{a point}\} \cup_{f_0} D^1}_{\mathbb{RP}^1} \cup_{f_1} D^2 \cup_{f_2} \dots \cup_{f_{n-2}} D^{n-1} \cup_{f_{n-1}} D^n.$$

$\underbrace{\hspace{10em}}_{\mathbb{RP}^2}$
 $\underbrace{\hspace{10em}}_{\dots}$
 $\underbrace{\hspace{10em}}_{\mathbb{RP}^{n-1}}$

S^n represents all the directions in \mathbb{R}^{n+1} . x and $-x$ are on the same line passing through the origin point. Thus we can say that \mathbb{RP}^n is the space of lines passing through O in \mathbb{R}^{n+1} .

Example. $\mathbb{C}^n = \{(z_1, \dots, z_n) \mid z_i \in \mathbb{C}\}$ is n -dimensional complex vector space. The **complex n -projective space** \mathbb{CP}^n is the space of complex lines passing through O in \mathbb{C}^{n+1} . Formally,

$$\begin{aligned} \mathbb{CP}^n &= \mathbb{C}^{n+1} - \{O\} / z \sim \lambda z \\ &= \{\text{unit vectors in } \mathbb{C}^{n+1}\} / z \sim \lambda z \\ &= \{z \in \mathbb{C}^{n+1} \mid \|z\| = 1\} / z \sim \lambda z \\ &= S^{2n+1} / z \sim \lambda z \end{aligned}$$

where $\lambda \in \mathbb{C}$, $\|\lambda\| = 1$.

Chapter 12

Surfaces

12.1 Surfaces

Definition 3. An **n -manifold** is a topological space X s.t.

- (i) X is Hausdorff.
- (ii) X has a countable basis for its topology.
- (iii) Every point of X has an open neighborhood which is homeomorphic to \mathbb{R}^n (or $\mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$).

Especially, a 2-manifold is called a **surface**.

Shortly, an n -manifold is a second countable, Hausdorff topological space which is locally homeomorphic to \mathbb{R}^n .

Definition 4. An **n -manifold with boundary** is a top'al sp X s.t.

- (i) X is Hausdorff.
- (ii) X has a countable basis for its topology.
- (iii) Every point of X has an open neighborhood homeomorphic to \mathbb{R}^n or $H_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ (or $\mathring{D}_+^n = \{(x_1, \dots, x_n) \in \mathring{D}^n \mid x_n \geq 0\}$).
- (iv) $\partial X = \{\text{pts whose nbd is homeomorphic to } H_+^n \text{ or } \mathring{D}_+^n\} \neq \emptyset$

Note. From now on, the numbering on theorem, corollary, and lemma follows Munkres' book.

Theorem 2 (36.2, Embedding theorem). A compact n -manifold X can be embedded into \mathbb{R}^N for some $N \in \mathbb{N}$, that is, there exists a continuous map $f: X \rightarrow \mathbb{R}^N$ s.t. $f: X \rightarrow f(X)$ is a homeomorphism.

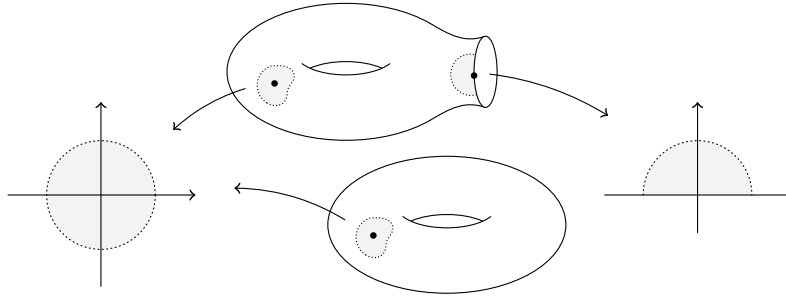


Figure 12.1: Surface with boundary

Lecture 5
Wed, Sep 15

Proof. Not covered in this course. \square

Definition 5. Let S_1, S_2 be surfaces and D_i be a 2-dimensional disk in S_i for $i = 1, 2$. Then, $\partial D_1, \partial D_2 \cong S^1$, and there exists a homeomorphism $f: \partial D_1 \rightarrow \partial D_2$. The **connect sum** of S_1 and S_2 is defined as

$$S_1 \# S_2 = (S_1 - \mathring{D}_1) \cup_f (S_2 - \mathring{D}_2).$$

Notation. • $T_0 := S^2$

- $T_1 := \text{Torus}$
- $T_n := T \# \cdots \# T = T_{n-1} \# T_1$

Let $S := S^2 - \{\text{two open disks}\}$ and $f: c_1 \rightarrow c_2$. Then $S/f \cong T_1$. Similarly,
 $T_{n-1} - \{\text{two open disks}\}/f \cong T_n$.
 $\mathbb{RP}^2 - \text{open disk} \cong \text{Möbius band}$

12.2 Labelling scheme

Assign labels and directions to each edge of polygonal region P :

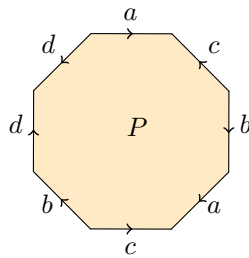


Figure 12.2: Labelling scheme: $a^{-1}dd^{-1}b^{-1}ca^{-1}b^{-1}c$ (read counterclockwise)

A labelling scheme gives a surface which is a quotient of P .

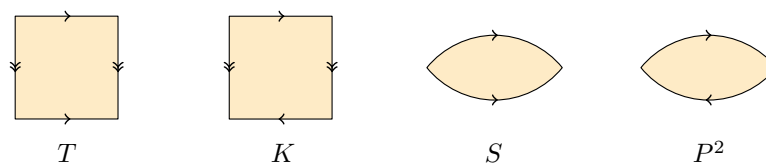


Figure 12.3: Examples of surfaces

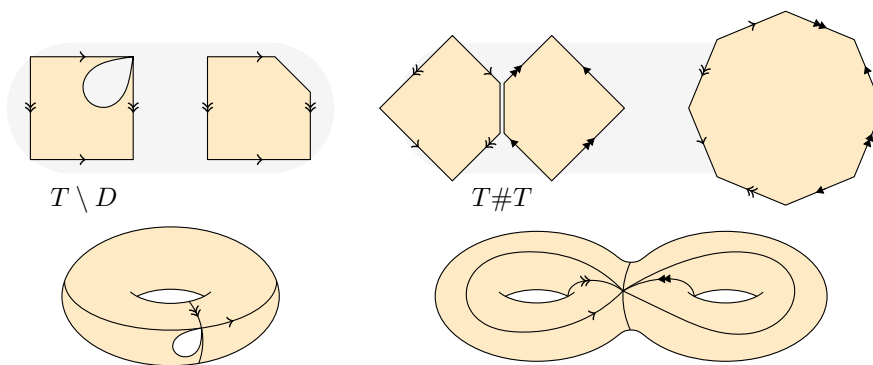


Figure 12.4: On the left: a torus with a disk removed. On the right: the connected sum of two tori.

12.3 Elementary operations on schemes

Suppose $\{w_1, \dots, w_n\}$ be a labelling scheme.

Lecture 6
Mon, Sep 20

Cut $w_i = Y_0 Y_1 \rightarrow \{Y_0 c, c^{-1} Y_1\}$ (c does not appear elsewhere)

Paste Reverse of cut.

Relabel Change an alphabet by a new alphabet. Reverse the sign of an alphabet.

Permute Cyclically permute alphabets on a word w_i . E.g., $w_i = a_1 a_2 \dots a_n \rightarrow w'_i = a_2 \dots a_n a_1$

Flip $w_i = (a_{i1})^{\varepsilon_1} \dots (a_{in})^{\varepsilon_n} \rightarrow w_i^{-1} = (a_{i1})^{-\varepsilon_1} \dots (a_{in})^{-\varepsilon_n}$

Cancel $Y_0 a a^{-1} Y_1 \rightarrow Y_0 Y_1$

Uncancel Reverse of cancel.

Note. These operations do not change the topological type of the resulting surfaces.

Definition 6. Two labelling schemes are said to be **equivalent** if one can be obtained from the other by applying the elementary operations in finitely many times.

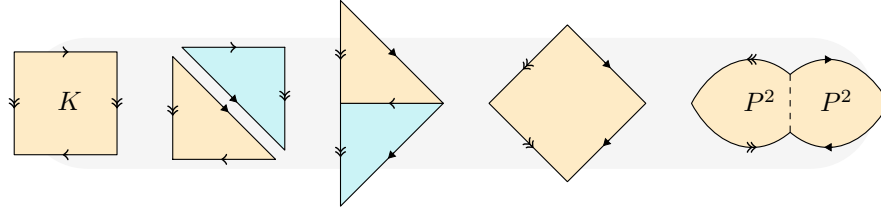


Figure 12.5: $K = P \# P$

Note. Two equivalent schemes give surfaces of the same homeomorphic type.

12.4 Classification theorem I

Definition 7. A scheme is **proper** if each label appears twice in the scheme.

Note. proper scheme $\xrightarrow{\text{elem. oper.}}$ still proper!

Definition 8. Let w be a proper scheme for a single polygonal region P . w is of **torus type** if each label appears exactly once with exponent $+1$, and once with -1 . Otherwise we say w is of **projective type**.

Lemma 1 (77.1). If w is a proper scheme of the form $w = Y_0 a Y_1 a Y_2^a$ where Y_i is a sequence of labels, then $w \sim aaY_0Y_1^{-1}Y_2$.

^athat is to say w is of projective type

Proof. Case 1. Y_0 is empty.

- If Y_1 is also empty, then w is the desired form itself.
- If Y_2 is empty,

$$aY_1a \xrightarrow{\text{flip}} a^{-1}Y_1^{-1}a^{-1} \xrightarrow{\text{permute}} a^{-1}a^{-1}Y_1^{-1} \xrightarrow{\text{relabel}} aaY_1^{-1}.$$

- If neither is empty,

$$aY_1aY_2 \xrightarrow[\text{paste}]{\text{cut}} ccY_1^{-1}Y_2 \xrightarrow{\text{relabel}} aaY_1^{-1}Y_2.$$

Case 2. Y_0 is not empty.

- If both Y_1 and Y_2 are empty, a permutation is enough.
- In general,

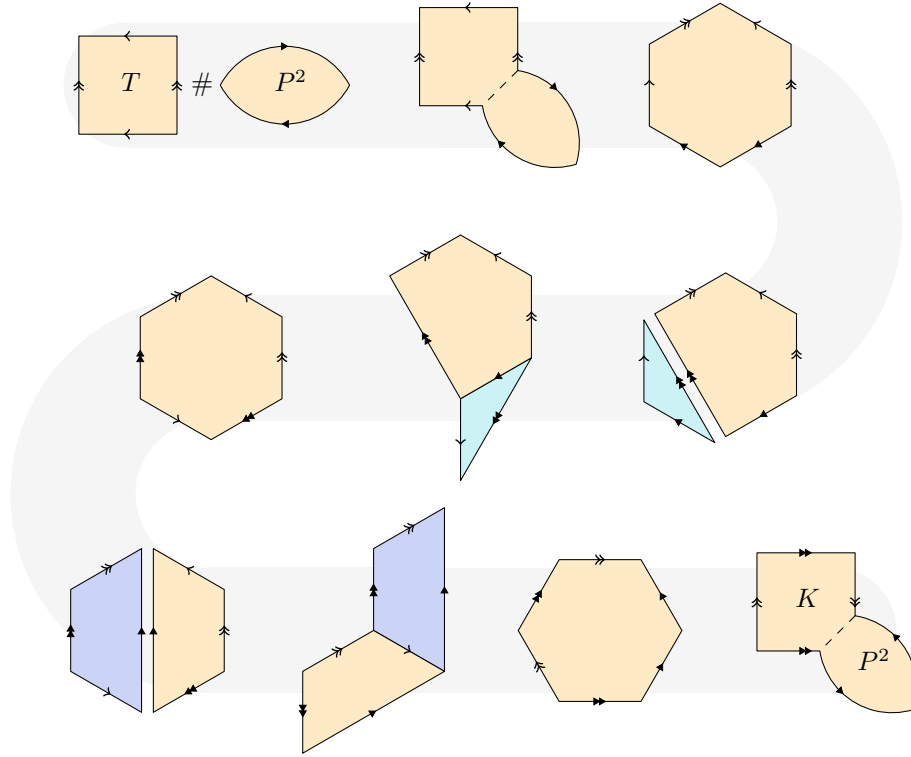


Figure 12.6: $T \# P = K \# P$

$$\begin{array}{c}
 Y_0 a Y_1 a Y_2 \xrightarrow{\text{cut/paste}} b Y_2 b Y_1 Y_0^{-1} \xrightarrow{\text{Case 1}} b b Y_2^{-1} Y_1 Y_0^{-1} \\
 \downarrow \text{flip} \\
 a a Y_0 Y_1^{-1} Y_2 \xleftarrow{\text{relabel}} b^{-1} b^{-1} Y_0 Y_1^{-1} Y_2 \xleftarrow{\text{permute}} Y_0 Y_1^{-1} Y_2 b^{-1} b^{-1}
 \end{array}$$

□

Corollary 2 (77.2). If w is projective type, then w is equivalent to a scheme of the form $(a_1 a_1)(a_2 a_2) \dots (a_k a_k) w'$, where the length^a is unchanged, $k \geq 1$, and w' is empty or of torus type.

^athe number of alphabets

Proof. Since w is of projective type, it can be written to be $w = Y_0 a Y_1 a Y_2$. By Lemma 1, $w \sim a a w_1$ so that the length is unchanged. If w_1 is empty or of torus type, it's done. Otherwise, we can write w_1 so that $a a w_1 \sim a a Z_0 b Z_1 b Z_2$. Again by Lemma 1, $a a w_1 \sim b b a a Z_0 Z_1^{-1} Z_2$, length of w_2 . By repeating this process, we obtain the desired form. □

Lemma 2 (77.3). Let $w = w_0w_1$ be a proper scheme, where w_1 is a scheme itself of torus type that does not contain any two adjacent terms having the same label. Then $w \sim w_0w_2$ s.t. $w_2 = aba^{-1}b^{-1}w_3$ with same length as w_1 , where w_3 is of torus type or is empty.

Proof. w can be written as $w = w_0Y_1aY_2bY_3a^{-1}Y_4b^{-1}Y_5$. □

Lemma 3 (77.4). If w is a proper scheme of the form $w = w_0ccaba^{-1}b^{-1}w_1$, then $w \sim w_0aabbccw_1$

Proof. Proceed as follows:

$$\begin{aligned}
 w_0ccaba^{-1}b^{-1}w_1 &\sim ccaba^{-1}b^{-1}w_1w_0 && \text{(permute)} \\
 &= cc(ab)(ba)^{-1}w_1w_0 \\
 &\sim (ab)c(ba)cw_1w_0 && \text{(Lemma 1)} \\
 &= abcb(acw_1w_0) \\
 &\sim bbac^{-1}acw_1w_0 && \text{(Lemma 1)} \\
 &\sim aabbccw_1w_0 && \text{(Lemma 1)} \\
 &\sim w_0aabbccw_1 && \text{(permute)}
 \end{aligned}$$

□

Theorem 3 (77.5, Classification theorem). Let X be a quotient space obtained from a polygonal region P by glueing its edges in pairs. Then X is homeomorphic to one of S^2 , T_n , and $(P^2)_n^a$ where $n \geq 1$.

^aconnect sum of \mathbb{RP}^2

Proof. Let w be a proper scheme on P which results in X . If $|w| = 2$, $w = aa^{-1}$ (S^2) or $w = aa$ (P^2). We may assume that $|w| \geq 4$ ($|w|$ is even). In fact we will show that □

Note. HW: Exercise 77.1 and 77.4

12.5 Constructing compact surfaces

Definition 9. Let X be a compact Hausdorff space. A subspace A of X is a **curved triangle** if there exists a homeomorphism $h: \Delta \rightarrow A$, where Δ is a closed triangular region in \mathbb{R}^2 .

Definition 10. A **triangulation** of X is a collection of curved triangles $\{A_\alpha\}$ s.t.

- $\bigcup A_\alpha = X$.
- For $\alpha \neq \beta$, $A_\alpha \cap A_\beta = \emptyset$, single vertex or single edge.
- When $A_\alpha \cap A_\beta = \text{single edge}$, $h_\beta^{-1} \circ h_\alpha$ is a linear map.

X is said to be **triangulable** if it has a triangulation.

Theorem 4 (78.1). If X is a compact triangulable surface (with or without boundary), then X is homeomorphic to a quotient space obtained from a collection of disjoint triangular regions by pasting their edges together in pairs.

Proof. Let $\{A_1, \dots, A_n\}$ be a triangulation of X with homeomorphisms $\{h_i: \Delta_i \rightarrow A_i \mid i = 1, \dots, n\}$. Then we have a quotient map $h: \Delta_1 \sqcup \dots \sqcup \Delta_n \rightarrow X$ s.t. $h|_{\Delta_i} = h_i$. There are two things to be proved.

- If two triangles meet at a vertex, then there exists a sequence of triangles. Thus, the quotient is obtained only by edge-pastings.
- For each edge e of A_i s.t. $e \not\subset \partial X$, $\exists! j$ s.t. $A_i \cap A_j = e$. Thus, the quotient is obtained by pasting edges in pairs.

□

Theorem 5 (78.2). Let X be a compact connected triangulable surface without boundary. Then X is homeomorphic to a quotient space obtained from a polygonal region by pasting all the edges together in pairs. That is, X is homeomorphic to a surface obtained from a proper scheme on a polygonal region.

Proof. From Theorem 4, $\Delta_1 \sqcup \dots \sqcup \Delta_n \xrightarrow{h} X$. Assemble the triangles $\{\Delta_i\}$ on the plane as much as possible in the following way: □

Theorem 6 (A). Every compact connected surface is triangulable.

Proof (Sketch of proof). • surface and compact $\Rightarrow \exists$ a finite collection $\{B_1, \dots, B_n\}$ s.t. $B_i \cong D^2$, $\bigcup B_i = X$.

- We may assume that no proper subset satisfies $\bigcup B_i = X$.
- Let $C = \bigcup \partial B_i$ and D be thickening of C in X . Then $X - D \cong \bigcup \mathring{D}^2$. □

Theorem 7 (Surface classification theorem). Every compact connected surface without boundary is homeomorphic to one of S^2 , T_n , and $(P^2)_n$.

Proof. Theorem 3 + Theorem 5 + Theorem 6. □

Chapter 9

Fundamental group

Lecture 8
Mon, Sep 27

9.51 Homotopy of paths

Definition 11. Let X, Y be topological spaces and $f, f': X \rightarrow Y$ be continuous maps. We say, f is **homotopic** to f' ($f \simeq f'$) if there is a continuous function $F: X \times I \rightarrow Y$ s.t. $F(x, 0) = f(x)$, $F(x, 1) = f'(x)$ for all $x \in X$. The function F is called a **homotopy** from f to f' ($f \simeq^F f'$). Especially, if f' is a constant map, then we say, f is **null-homotopic**.

Definition 12. Let $f, f': I \rightarrow X$ be two paths in X s.t. $f(0) = f'(0) = x_0$ and $f(1) = f'(1) = x_1$. We say, f is **path-homotopic** to f' ($f \simeq_p f'$) if there is a homotopy $F: I \times I \rightarrow X$ s.t.

- $F(x, 0) = f(x)$, $F(x, 1) = f'(x)$
- For each t , $F(0, t) = x_0$, $F(1, t) = x_1$

The homotopy F is called a **path-homotopy** from f to f' ($f \simeq_p^F f'$).

Notation.

- $\Omega(X, Y) := \{f: X \rightarrow Y \mid f \text{ is continuous}\}$
- $\mathcal{P}(X) := \text{the set of all paths in } X$

Lemma 4 (51.1). \simeq and \simeq_p are equivalence relations on $\Omega(X, Y)$ and $\mathcal{P}(X)$, respectively.

Proof. Reflective $F(x, t) = f(x)$

Symmetric Suppose $f \simeq f'$. Then there is a homotopy $F: X \times I \rightarrow Y$ s.t. $F(x, 0) = f(x)$, $F(x, 1) = f'(x)$. Define $F'(x, t) = F(x, 1 - t)$. Then, F' is conti. and $F'(x, 0) = F(x, 1) = f'(x)$, $F'(x, 1) = F(x, 0) = f(x)$.

Transitive Suppose $f \simeq^F f'$ and $f' \simeq^G f''$. Define $H: X \times I \rightarrow Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Note that H is continuous by pasting lemma.
 \simeq_p : skip. □

Denote the equivalence class of f by $[f] = \{f' \in \Omega(X, Y) \mid f' \simeq f\}$.

Example. Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g: X \rightarrow C$ are homotopic.
- Any two paths $f, g: I \rightarrow C$ with $f(0) = g(0)$ and $f(1) = g(1)$ are path-homotopic.

Choose $F: X \times I \rightarrow C$ defined by $(x, t) \mapsto F(x, t) = (1 - t)f(x) + tg(x)$.

Example. Let $X = \mathbb{R}^2 - \{0\}$ (punctured plane). $f(x) = (\cos \pi x, \sin \pi x)$, $g(x) = (\cos \pi x, 2 \sin \pi x)$ and $h(x) = (\cos \pi x, -\sin \pi x)$ are paths in X . In fact, $f \simeq_p g \not\simeq_p h$.

Product of paths

Let $f, g: I \rightarrow X$ be paths, $f(1) = g(0)$. Define the product $f * g: I \rightarrow X$ by

$$f * g = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Define the product $*$ on path-homotopy classes of X by $[f] * [g] := [f * g]$.

Well-definedness Suppose $f' \in [f]$ ($f \simeq_p^F f'$) and $g' \in [g]$ ($g \simeq_p^G g'$). Define

$$H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq \frac{1}{2} \\ G(2s - 1, t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then $H(s, 0) = (f * g)(s)$, $H(s, 1) = (f' * g')(s)$ and H is continuous by pasting lemma again. Thus, $f * g \simeq_p f' * g'$, $[f * g] = [f' * g']$.

Lecture 9
Wed, Sep 29

Theorem 8 (51.2). The product $*$ has the following properties:

- Associative: $([f] * [g]) * [h] = [f] * ([g] * [h])$
- Let e_x denote the constant path $e_x: I \rightarrow X$ given by $s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
- Let $\bar{f}: I \rightarrow X$ given by $s \mapsto f(1 - s)$. Then $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

Proof. First two observations

- Suppose $f \simeq_p g$ via homotopy H , $f, g: I \rightarrow X$. Let $k: X \rightarrow Y$. Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.
- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

- (ii) Take $e_0: I \rightarrow I$ given by $s \mapsto 0$. Take $i: I \rightarrow I$ given by $s \mapsto s$. Then $e_0 * i$ is a path from 0 to 1 in I . The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 \simeq i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

- (iii) Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \bar{f} &\simeq_p e_{x_0} \\ [f] * [\bar{f}] &= [e_{x_0}]. \end{aligned}$$

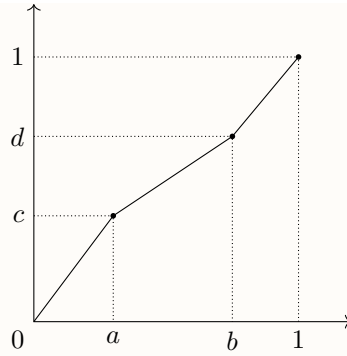
- (i) Remark: Only defined if $f(1) = g(0)$, $g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

Assume the product is defined. Suppose $[a, b]$, $[c, d]$ are intervals in \mathbb{R} . Then there is a unique positive (positive slope), linear map from $[a, b] \rightarrow [c, d]$. For any $a, b \in [0, 1]$ with $0 < a < b < 1$, we define a path

$$\begin{aligned} k_{a,b}: [0, 1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 1] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

Then $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$.

Let γ be that path $\gamma: I \rightarrow I$ with the following graphs:



Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

which is what we wanted to show. \square

9.52 Fundamental group

Definition 13. Let X be a topological space and $x_0 \in X$. A **loop** based at x_0 in X is a path $\alpha: I \rightarrow X$ s.t. $\alpha(0) = \alpha(1) = x_0$. Then

$$\pi_1(X, x_0) = \{[\alpha] \mid \alpha: \text{loop in } X \text{ based at } x_0\}$$

is the **fundamental group** of X with base point x_0 .^a

^a $\pi_1(X, x_0)$ is a group with the operation $*$ by Theorem 8. For $[\alpha], [\beta] \in \pi_1(X, x_0)$, $[\alpha] * [\beta]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[\alpha]^{-1} = [\bar{\alpha}]$. This makes $(\pi_1(X, x_0), *)$ a group.

Example. $\pi_1(\mathbb{R}^n, x_0)$ is a trivial group. Any two loops in \mathbb{R}^n based at x_0 are path-homotopic. Thus, $\pi_1(\mathbb{R}^n, x_0)$ has only one element.

Remark. All groups are a fundamental group of some space.

Definition 14. Let α be a path in X from x_0 to x_1 . Define a function $\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ by $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$.

Theorem 9 (52.1). $\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is a group isomorphism.

Proof. Homomorphism To show that $\hat{\alpha}$ is a group homomorphism, we

compute

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * [f] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [e_{x_0}] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}[f] * \hat{\alpha}[g].\end{aligned}$$

Bijjective To show that $\hat{\alpha}$ is one-to-one and onto function, we show existence of inverse of α .

$$\begin{aligned}(\hat{\alpha} \circ \hat{\alpha})([h]) &= [\bar{\alpha}] * ([\bar{\alpha}] * [h] * [\bar{\alpha}]) * [\alpha] \\ &= [e_{x_1}] * [h] * [e_{x_1}] = [h].\end{aligned}$$

Thus, $\hat{\alpha} \circ \hat{\alpha}$ is the identity function. Similarly, we can show that $\hat{\alpha} \circ \hat{\alpha}$ is the identity function. \square

Definition 15. A topological space X is said to **simply connected** if it is path-connected and $\pi_1(X, x_0)$ is a trivial group.

Example. Any convex subset of \mathbb{R}^n is simply connected.

Lemma 5 (52.3). Suppose X is simply connected and $\alpha, \beta: I \rightarrow X$ are paths from x_0 to x_1 . Then $\alpha \simeq_p \beta$.

Proof. $\alpha * \bar{\beta}$ is a loop base at x_0 . Since X is simply connected, $\alpha * \bar{\beta} \simeq_p e_{x_0}$. Thus, $[\alpha] = [\alpha] * [e_{x_1}] = [\alpha] * [\bar{\beta}] * [\beta] = [e_{x_0}] * [\beta] = [\beta]$. \square

Definition 16. Let $h: (X, x_0) \rightarrow (Y, y_0)$ be a continuous map ($h(x_0) = y_0$). Define $h_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ by $h_*([f]) = [h \circ f]$. Then h_* is a **group homomorphism induced from h** .

Well-definedness Let $f' \in [f]$ and F be a path-homotopy from f to f' . Then $h \circ F: I \times I \rightarrow Y$ is a path-homotopy from $h \circ f$ to $h \circ f'$.

Homomorphism h_* is a homomorphism, because $(h \circ f) * (h \circ g) = h \circ (f * g)$. That is, $h_*([f]) * h_*([g]) = h_*([f * g])$.

Theorem 10 (52.4). (i) For two continuous maps $h: (X, x_0) \rightarrow (Y, y_0)$ and $k: (Y, y_0) \rightarrow (Z, z_0)$, $(k \circ h)_* = k_* \circ h_*$.

(ii) For the identity map $i: (X, x_0) \rightarrow (X, x_0)$, i_* is the identity homomorphism.

Proof. (i) $(k \circ h)_*([f]) = [(k \circ h) \circ f] = [k \circ (h \circ f)] = k_*([h \circ f]) = k_*(h_*([f])) = (k_* \circ h_*)([f])$.

$$(ii) \ i_*([f]) = [i \circ f] = [f].$$

□

Corollary 3 (52.5). If $h: (X, x_0) \rightarrow (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism.

Proof. Let $k: (Y, y_0) \rightarrow (X, x_0)$ be the inverse of h . Then,

$$\begin{aligned} k_* \circ h_* &= (k \circ h)_* = (\text{id}_X)_* = \text{the identity on } \pi_1(X, x_0) \\ h_* \circ k_* &= (h \circ k)_* = (\text{id}_Y)_* = \text{the identity on } \pi_1(Y, y_0) \end{aligned}$$

Thus, h_* is an isomorphism. □

This corollary says π_1 is a topological invariant. We can use the fundamental group to detect that two spaces are not homeomorphic, i.e. $\pi_1(X, x_0) \not\cong \pi_1(Y, y_0) \Rightarrow X \not\cong Y$. Note that $X \not\cong Y \not\Rightarrow \pi_1(X, x_0) \not\cong \pi_1(Y, y_0)$ and $\pi_1(X, x_0) \cong \pi_1(Y, y_0) \not\Rightarrow X \cong Y$.

Exercise (52.6). Let X be path-connected and $h: X \rightarrow Y$ be continuous with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1 and $\beta = h \circ \alpha$. Then, $\hat{\beta} \circ h_* = h_* \circ \hat{\alpha}$, that is, the diagram of maps

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_*} & \pi_1(Y, y_0) \\ \downarrow \hat{\alpha} & & \downarrow \hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{h_*} & \pi_1(Y, y_1) \end{array}$$

commutes.

Proof. Let $[f] \in \pi_1(X, x_0)$.

$$\begin{aligned} (\hat{\beta} \circ h_*)([f]) &= \hat{\beta}(h_*([f])) = [\hat{\beta}] * h_*([f]) * [\beta] \\ &= h_*([\hat{\alpha}]) * h_*([f]) * h_*([\alpha]) \\ &= h_*([\hat{\alpha}] * [f] * [\alpha]) \\ &= h_*(\hat{\alpha}[f]) \\ &= (h_* \circ \hat{\alpha})([f]). \end{aligned}$$

Thus, if X is path-connected, the group homomorphism induced by a continuous map is independent of base point. ◇

Note. HW3: Exercise §52 – #1, #2, #3, #4.

Lecture 11
Wed, Oct 6

9.53 Covering spaces

Definition 17. Let $p: E \rightarrow B$ be a continuous surjective map. An open subset U of B is said to be **evenly covered** by p if $p^{-1}(U)$ is a union of disjoint open subsets V_α of E s.t. each V_α is homeomorphic to U by p . That is, $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$, $V_\alpha \cong U$ by $p \forall \alpha$.

Each V_α is called a *slice*. (The set $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices.)

If every point of B has an open nbh which is evenly covered by p , then p is called a **covering map**, E **covering space**, B **base space**.

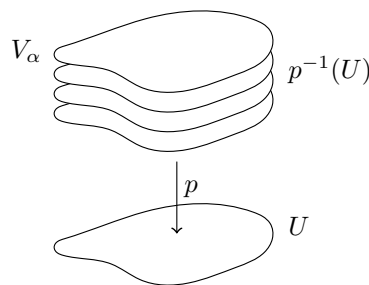


Figure 9.1: evenly covered

Remark. If $U' \subset U$, also open and U is evenly covered, then also U' .

Example. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Take $p: \mathbb{R} \rightarrow S^1$ given by $t \mapsto e^{2\pi it}$. Note that \mathbb{R} is an easier space than S^1 , and so will be π_1 (1 vs \mathbb{Z}).

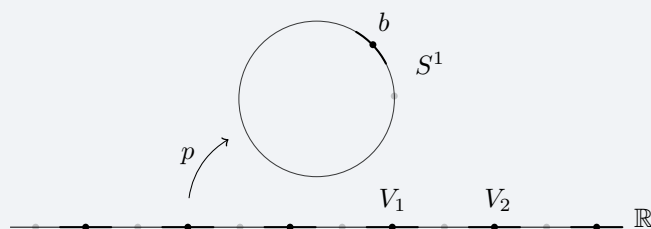


Figure 9.2: example of a covering space

There are also other covering spaces of p . For example, $p': S^1 \rightarrow S^1$ given by $z \mapsto z^3$.

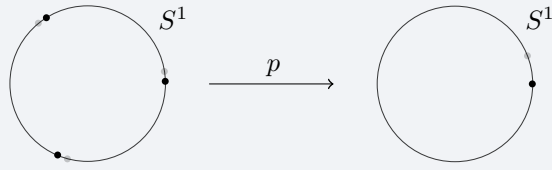


Figure 9.3: second example of a covering space

Here we have three copies for each point. We say that the covering has 3 sheets. Note that this is independent of which point we take. This is always the case! We can show that these are the only coverings of S^1 : \mathbb{R} and $z \mapsto z^n$.

Theorem 11 (53.2). Let $p: E \rightarrow B$ be a covering map, B_0 a subspace of B , $E_0 = p^{-1}(B_0)$. Then, $p|_{E_0}: E_0 \rightarrow B_0$ is also a covering map.

Proof. For each $b \in B_0$, there is open nbh U of b in B which is evenly covered by p . Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Then,

- $U \cap B_0$ is an open nbh of b in B_0 .
- $\{V_\alpha \cap E_0\}$ is a partition of $p^{-1}(U \cap B_0)$.
- $V_\alpha \cap E_0 \cong U \cap B_0$ by p .

□

Theorem 12 (53.3). Let $p_1: E_1 \rightarrow B_1$ and $p_2: E_2 \rightarrow B_2$ be covering maps. Then, $p_1 \times p_2: E_1 \times E_2 \rightarrow B_1 \times B_2$ is a covering map.

Proof. Let $(b_1, b_2) \in B_1 \times B_2$ and U_1 be an evenly covered open nbh of b_1 in B_1 for p_1 (same for U_2). We claim that $U_1 \times U_2$ is an evenly covered open nbh of (b_1, b_2) in $B_1 \times B_2$ for $p_1 \times p_2$.

$$\begin{aligned} (p_1 \times p_2)^{-1}(U_1 \times U_2) &= p_1^{-1}(U_1) \times p_2^{-1}(U_2) \\ &= (\bigsqcup_\alpha V_\alpha) \times (\bigsqcup_\beta W_\beta) = \bigsqcup_{\alpha, \beta} (V_\alpha \times W_\beta). \end{aligned}$$

$V_\alpha \times W_\beta \cong U_1 \times U_2$ by $p_1 \times p_2$, since $V_\alpha \cong U_1$ by p_1 and $W_\beta \cong U_2$ by p_2 . □

Example. Let $p: \mathbb{R} \rightarrow S^1$ be the covering map in the previous example.

- $p \times p: \mathbb{R} \times \mathbb{R} \rightarrow S^1 \times S^1$ is a covering map by Theorem 12.
- $p \times p: (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \rightarrow \text{Bouguet with two leaves}$ is a covering map by Theorem 11.

Exercise (53.3). Let $p: E \rightarrow B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k

elements for every $b \in B$. In such a case, E is called a **k -fold covering** of B .

Proof. Let $B_1 = \{b \in B \mid |p^{-1}(b)| = k\}$ and $B_2 = \{b \in B \mid |p^{-1}(b)| \neq k\}$. Then $b_0 \in B_1$, hence $B_1 \neq \emptyset$. Suppose $B_2 \neq \emptyset$. For $b \in B$, let U_b be an evenly covered open nbh of b . And let $U_1 = \bigcup_{b \in B_1} U_b$, $U_2 = \bigcup_{b \in B_2} U_b$. Then, both are open non-empty and $U_1 \cup U_2 = B$. Since B is connected, $U_1 \cap U_2 \neq \emptyset$. If $b_1 \in U_1 \cap U_2$, then we have a contradiction. \diamond

Note. HW3: Exercise §53 – #4, #5, #6.

Remark. A covering is always local homeomorphism. But there are surjective local homeomorphism which are not covering maps. A covering map is more than a surjective local homeomorphism.

For example, $p: \mathbb{R}_0^+ \rightarrow S^1$ given by $t \mapsto e^{2\pi it}$. Consider the inverse image of a neighborhood around 1. When we restrict p to the part around 0, it is no longer a homeomorphism (we don't get the whole neighborhood around one).

Creating new covering spaces out of old ones

- Suppose $p: E \rightarrow B$ is a covering and $B_0 \subset B$ is a subspace with the subspace topology. Let $E_0 = p^{-1}(B_0)$ and $p_0 = p|_{E_0}$. Then (E_0, p_0) is a covering of B_0 .
- Suppose that (E, p) is a covering of B and (E', p') is a covering of B' , then $(E \times E', p \times p')$ is a covering of $B \times B'$.

Example. Let $T^2 = S^1 \times S^1$.

- $p: \mathbb{R}^2 \rightarrow S^1 \times S^1$ given by $(t, s) \mapsto (e^{ait}, e^{bis})$.
- $p': \mathbb{R} \times S^1 \rightarrow T^2$ given by $(t, z) \mapsto (e^{ait}, z^n)$.
- $p: S^1 \times S^1 \rightarrow T^2$ given by $(z_1, z_2) \mapsto (z_1^n, z_2^m)$.

These are the only types of coverings of the torus. We'll prove this later on.

9.54 $\pi_1(S^1)$

Definition 18. Let $p: E \rightarrow B$ and $f: X \rightarrow B$ be continuous maps. Then, a **lifting** of f is a map $\tilde{f}: X \rightarrow E$ s.t. $f = p \circ \tilde{f}$.

$$\begin{array}{ccc} & & E \\ & \nearrow \tilde{f} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Lemma 6 (54.1, Unique path-lifting lemma).

$$\begin{array}{ccc} & (E, e_0) & \\ \exists! \tilde{\gamma} \nearrow & \downarrow p: \text{c.m.} & \\ (I, 0) & \xrightarrow[\text{path}]{\gamma} & (B, b_0) \end{array}$$

Proof. Existence Let $\{U_\alpha\}$ be an open covering of B consisting of evenly-covered open subsets. Then, $\{\gamma^{-1}(U_\alpha)\}$ is an open covering of the compact space I , and there exists a Lebesgue number ε (Any open interval of length less than ε is contained in some $\gamma^{-1}(U_\alpha)$). Then we have a subdivision $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$ so that $\gamma[s_i, s_{i+1}] \subset U_\alpha$ for some α (by setting $s_i - s_{i-1} < \varepsilon$).

Define $\tilde{\gamma}(0) = e_0$. Suppose $\tilde{\gamma}(s)$ is defined for $0 \leq s \leq s_i$. Select α_0 so that $\gamma[s_i, s_{i+1}] \subset U_{\alpha_0}$. Let $\{V_\beta\}$ be the partition of $p^{-1}(U_\alpha)$ into slices. And let V_{β_0} be the slice s.t. $\tilde{\gamma}(s_i) \in V_{\beta_0}$. Since $V_{\beta_0} \cong U_{\alpha_0}$ by $p|_{V_{\beta_0}}$, we have an closed arc $(p|_{V_{\beta_0}})^{-1}(\gamma[s_i, s_{i+1}])$. For $s_i \leq s \leq s_{i+1}$, defined $\tilde{\gamma}(s) = (p|_{V_{\beta_0}})^{-1}(\gamma(s))$. Then $(p \circ \tilde{\gamma})(s) = \gamma(s)$.

Uniqueness Let $\tilde{\tilde{\gamma}}$ be another lift of γ s.t. $\tilde{\tilde{\gamma}}(0) = e_0$. Since $\tilde{\tilde{\gamma}}[s_i, s_{i+1}]$ is connected and $\{V_\beta\}$ are mutually disjoint, $\tilde{\tilde{\gamma}}[s_i, s_{i+1}] \subset V_{\beta_0}$. Note that, in V_{β_0} , $\tilde{\gamma}(s)$ is a unique point which projects $\gamma(s)$. Thus, $\tilde{\tilde{\gamma}}(s) = \tilde{\gamma}(s) \forall s$. \square

Lecture 12
Mon, Oct 11

Lemma 7 (54.2, Homotopy lifting lemma).

$$\begin{array}{ccc} & (E, e_0) & \\ \exists! \tilde{F} \nearrow & \downarrow p: \text{c.m.} & \\ (I \times I, (0, 0)) & \xrightarrow[\text{conti.}]{F} & (B, b_0) \end{array}$$

Furthermore, if F is a path-homotopy, then so is \tilde{F} .

Proof. (i) • Define $\tilde{F}(0, 0) = e_0$.

- Divide $I \times I$ into subrectangles so that $F(I_i \times J_j)$ is contained in an evenly-covered open subset of B .
- Define \tilde{F} step by step: Assume that \tilde{F} is defined on the red-part. Define $\tilde{F}(x) = (p|_V)^{-1}(F(x))$, $\forall x \in A$. (Then $p \circ \tilde{F}(x) = F(x)$).

- (ii) Assume that F is a path-homotopy ($F(0, t) = b_0$, $F(1, t) = b_1$, $\forall t$). Then $\tilde{F}(\{0\} \times I) \subset P^{-1}(b_0)$ and $\tilde{F}(\{1\} \times I) \subset P^{-1}(b_1)$. Since $\{0\} \times I, \{1\} \times I$ are connected, $\tilde{F}(\{0\} \times I) = e_0$, $\tilde{F}(\{1\} \times I) =$ a pt in $P^{-1}(b_1)$. \square

Theorem 13 (54.3). Let $p: (E, e_0) \rightarrow (B, b_0)$ be a covering map. Let f, g be paths in B from b_0 to b_1 and \tilde{f}, \tilde{g} be lifts of f and g starting e_0 . Then, if $f \simeq_p g$, then $\tilde{f} \simeq_p \tilde{g}$.

Definition 19. Let $p: (E, e_0) \rightarrow (B, b_0)$ be a covering map. Then the **lifting correspondence** is the map

$$\begin{aligned}\phi: \pi_1(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1)\end{aligned}$$

where \tilde{f} is the unique lift of f starting at e_0 .^a This is well-defined because $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$. This ϕ depends on the choice of e_0 .

^aThere is no guarantee that \tilde{f} is a loop.

Theorem 14 (54.4). If E is path-connected, then ϕ is surjective. If E is simply-connected, then ϕ is bijective.

Theorem 15 (54.5). $\pi_1(S^1) \simeq \langle \mathbb{Z}, + \rangle$.

Theorem 16 (54.6). Let $p: (E, e_0) \rightarrow (B, b_0)$ be a covering map.

- (i) $p_*: \pi_1(E, e_0) \rightarrow \pi_1(B, b_0)$ is a monomorphism^a.
- (ii) Let $H = p_*(\pi_1(E, e_0))$. Then,

$$\begin{aligned}\Phi: \pi_1(B, b_0) / H &\longrightarrow p^{-1}(b_0) \\ H * [f] &\longmapsto \phi([f])\end{aligned}$$

is injective. If E is path-connected, then Φ is bijective.

- (iii) For a loop based at b_0 , $[f] \in H$ iff f lifts to a loop in E based at e_0 .

^ainjective homomorphism. *epimorphism: surjective homomorphism.

pagebreak

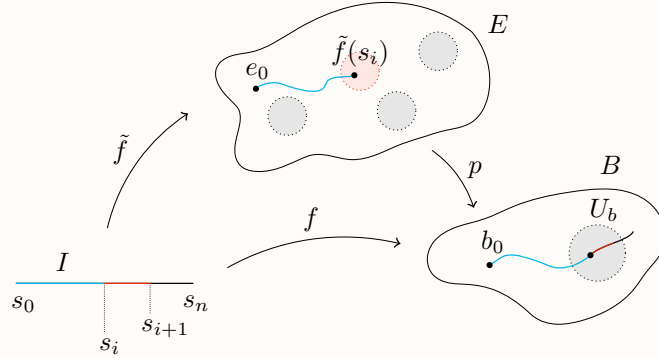
Definition 20. Let $p: E \rightarrow B$ be a map. If f is a continuous mapping of some space X into B , a **lifting** of f is a map $\tilde{f}: X \rightarrow E$ such that $p \circ \tilde{f} = f$.

$$\begin{array}{ccc} & E & \\ \tilde{f} \nearrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Given f , when can f be lifted to E ? In this section, we'll only consider $X = [0, 1]$, $X = [0, 1]^2$.

Lemma 8 (54.1, Important result). Suppose (E, p) is a covering of B , $b_0 \in B$, $e_0 \in p^{-1}(b_0)$. Suppose that $f: I \rightarrow B$ is a path starting at b_0 . Then there exists a unique lift $\tilde{f}: I \rightarrow E$ of f with $\tilde{f}(0) = e_0$.

Proof. For any b of B , we choose an open U_b such that U_b is evenly covered by p . Then $\{f^{-1}(U_b) \mid b \in B\}$ is an open cover of I , which is compact. There is a number $\delta > 0$ such that any subset of I of diameter $\leq \delta$ is contained entirely in one of these opens $f^{-1}(U_b)$. (Lebesgue number lemma). Now, we divide the interval into pieces $0 = s_0 < s_1 < \dots < s_n = 1$ such that $|s_{i+1} - s_i| \leq \delta$. For any i , we have that $f([s_i, s_{i+1}]) \subset U_b$ for some b .



We now construct \tilde{f} by induction on $[0, s_i]$.

- $\tilde{f}(0) = e_0$
- Assume \tilde{f} has been defined on $[0, s_i]$. Let U be an open such that $f[s_i, s_{i+1}] \subset U_b$.

There is exactly one slice V_α in $p^{-1}(U_b)$ containing $\tilde{f}(s_i)$. We define $\forall s \in [s_i, s_{i+1}]: \tilde{f}(s) = (p|_{V_\alpha})^{-1} \circ f(s)$. By the pasting lemma, \tilde{f} is continuous.

- In this way, we can construct \tilde{f} on the whole of I .

Uniqueness works in exactly the same way, by induction. \square

Lemma 9 (54.2). (E, p) is a covering of B , $b_0 \in B$, $e_0 \in E$, with $p(e_0) = b_0$. Suppose $F: I \times I \rightarrow B$ is a continuous map with $f(0, 0) = b_0$, then there is a unique $\tilde{F}: I \times I \rightarrow E$. Moreover, if F is a path homotopy, then also \tilde{F} is a path homotopy.

Proof. Same as in the one dimensional case. \square

Theorem 17 (54.3). Let (E, p) be a covering of B , $b_0 \in B$, $e_0 \in E$ with $p(e_0) = b_0$. Let f, g be two paths in B starting in b_0 s.t. $f \simeq_p g$ (so f and g end at the same point). Let \tilde{f}, \tilde{g} be the unique lifts of f, g starting at e_0 . Then $\tilde{f} \simeq_p \tilde{g}$, and so $\tilde{f}(1) = \tilde{g}(1)$.

Proof. $F: I \times I \rightarrow B$ is a path homotopy between f and g . Then $\tilde{F}: I \times I \rightarrow E$ with $\tilde{F}(0, 0) = e_0$. Then \tilde{F} is a path homotopy, by the previous result, between $\tilde{F}(\cdot, 0)$ and $\tilde{F}(\cdot, 1)$. Note that $p \circ \tilde{F}(t, 0) = F(t, 0) = f(t)$ and $p \circ \tilde{F}(t, 1) = F(t, 1) = g(t)$. By uniqueness $\tilde{F}(\cdot, 0) = \tilde{f}$, $\tilde{F}(\cdot, 1) = \tilde{g}$. \square

We've shown that homotopy from below lifts to above. The converse is easy. Now we're ready to discuss the relation between groups and covering spaces.

Definition 21. Let (E, p) be a covering of B . $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then the **lifting correspondence** is the map

$$\begin{aligned} \phi: \pi(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1) \end{aligned}$$

where \tilde{f} is the unique lift of f , starting at e_0 . This is well-defined because $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$. This ϕ depends on the choice of e_0 .

Theorem 18 (54.4). If E is path connected, then ϕ is a surjective map. If E is simply connected, then ϕ is a bijective map.

Proof. Suppose E is path connected, and let $e_0, e_1 \in p^{-1}(b_0)$. Consider a path $\tilde{f}: I \rightarrow E$ with $\tilde{f}(0) = e_0$ and $\tilde{f}(1) = e_1$. This is possible because E is path connected. Let $f = p \circ \tilde{f}: I \rightarrow B$ with $f(0) = p(e_0) = b_0$ and $f(1) = p(e_1) = b_0$, so f is a loop based at b_0 . So f is a loop at b_0 and its unique lift to E starting at e_0 is \tilde{f} . Hence $\phi[f] = \tilde{f}(1) = e_1$, which shows that ϕ is surjective.

Now assume that E is simply connected (group is trivial). Consider $[f], [g] \in \pi(B_0)$ with $\phi[f] = \phi[g]$. This implies $\tilde{f}(1) = \tilde{g}(1)$. These start at e_0 . It follows from Lemma 5 that $\tilde{f} \simeq_p \tilde{g}$. \square

Example. Take the circle and the real line as covering space. Then $p^{-1}(1) = \mathbb{Z}$. So we know that as a set $\pi(S^1)$ is countable. Therefore, $p \circ \tilde{f} \simeq_p p \circ \tilde{g}$. This implies that $f \simeq_p g$, and therefore $[f] = [g]$.

Theorem 19 (54.5). $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$.

Proof. Take $b_0 = 1$ and $e_0 = 0$ and $p: \mathbb{R} \rightarrow S^1$ given by $t \mapsto e^{2\pi it}$. Then $p^{-1}(b_0) = \mathbb{Z}$. And since, \mathbb{R} is simply connected, we have that $\pi: \pi(S, 1) \rightarrow \mathbb{Z}$ given by $[f] \mapsto \tilde{f}(1)$ is a bijection.

Now we'll show that it's a morphism. Let $[f]$ and $[g]$ elements of the fundamental group of S^1 and assume that $\phi[f] = \tilde{f}(1) = m$ and $\phi[g] = \tilde{g}(1) = n$.

We're going to prove that $\phi([f] * [g]) = \phi([f]) + \phi([g]) = n + m$. Define $\tilde{g}: I \rightarrow \mathbb{R}$ given by $t \mapsto \tilde{g}(t) + m$. Then $p \circ \tilde{g} = g$, as $p(s + m) = p(s)$ for all m . Now, look at $\tilde{f} * \tilde{g}$. This is a lift of $p \circ (\tilde{f} * \tilde{g}) = (p \circ \tilde{f}) * (p \circ \tilde{g}) = f * g$, which starts at 0. Hence, $\phi([f] * [g]) = \phi([f * g]) =$ the end point of $\tilde{f} * \tilde{g}$, so $\tilde{g}(1) = \tilde{g}(1) + m = n + m$. \square

The following lemma makes the fact that the covering space is simpler than the space itself exact.

Lemma 10 (54.6). Let (E, p) be a covering of B , $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then

- (i) $p_*: \pi(E, e_0) \rightarrow \pi(B, b_0)$ is a monomorphism (injective).
- (ii) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence induces a well defined map

$$\begin{aligned} \Phi: \pi_1(B, b_0)/H &\longrightarrow p^{-1}(b_0) \\ H * [f] &\longmapsto \phi[f], \end{aligned}$$

so ϕ is constant on right cosets. Dividing by H makes ϕ always bijective, even when E is not simply connected.

- (iii) Let f be a loop based at b_0 , then \tilde{f} is a loop at e_0 iff $[f] \in H$.

Proof. (i) Let $\tilde{f}: I \rightarrow E$ be a loop at e_0 and assume that $p_*[\tilde{f}] = 1$. (Then we'd like to show that f itself is trivial.) This implies $p \circ \tilde{f} \simeq_p e_{b_0}$. This implies that $\tilde{f} \simeq_p \tilde{e}_{b_0} = e_{e_0}$, or $[\tilde{f}] = 1$.

- (ii) We have to prove two things:

Well defined $H * [f] = H * [g] \Rightarrow \phi(f) = \phi(g)$.

Assume $[f] \in H * [g]$, or $H * [f] = H * [g]$. This means that $[f] = [h] * [g]$, where $h = p \circ \tilde{h}$ for some loop \tilde{h} at e_0 . In other words $[f] = [h * g]$, or $f \simeq_p h * g$. Let \tilde{f} be the unique lift of f starting at e_0 . Let \tilde{g} be the unique lift of g starting at e_0 . Then $\tilde{h} * \tilde{g}$ (which is allowed, \tilde{h} is a loop) the unique lift of $h * g$ starting at e_0 .

$\tilde{f}(1) = \phi(f) = \phi(h * g) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1) = \phi(g)$. If the cosets are the same, then the end points of the lifts are also the same.

Injective $H * [f] = H * [g] \Leftarrow \phi(f) = \phi(g)$.

The end points of f and g are the same. Now consider $\tilde{h} = \tilde{f} * \tilde{g}$. Then $[\tilde{h}] * [\tilde{g}] = [\tilde{f}] * [\tilde{g}] * [\tilde{g}] = [\tilde{f}]$. By applying p_* , $[h] * [g] = [f]$.

(iii) Trivial. Exercise. Apply 2 with the constant path. \square

Remark. $k: X \rightarrow Y$ induces a morphism k_* , we've proved that earlier. Here we only showed injectiveness.

9.55 Retractions and fixed points

Definition 22. A subspace A of a topological space X is called a **retract** of X if there exists a continuous map $r: X \rightarrow A$ s.t. $r|_A = \text{id}_A$. The map r is called a **retraction** of X onto A .

Example. S^1 is a retract of $\mathbb{R}^2 - \{0\}$.

Example. Let X be the figure 8 space, and denote the right circle by A . Then it's easy to see that there exists a retract from the whole space to A by mapping the left circle onto the right

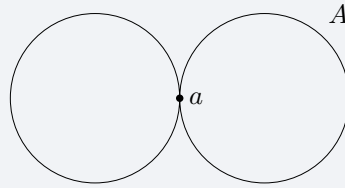


Figure 9.4: Figure 8 space

Lemma 11 (55.1). Let A be a retract of X and $i: A \hookrightarrow X$ be the inclusion map. Then $i_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ is injective.

Proof. $r \circ i: A \hookrightarrow X \rightarrow A$ is the identity map, hence $(r \circ i)_* = r_* \circ i_*$ is an isomorphism. Thus i_* is injective. \square

Theorem 20 (55.2). There is no retraction of B^2 onto S^1 .

Proof. $i_*: \pi_1(S^1, 1) \rightarrow \pi_1(B^2, 1)$ can not be injective because $\pi_1(S^1, 1) \cong \mathbb{Z}$ has infinitely many elements and $\pi_1(B^2, 1)$ is a trivial group. \square

Lemma 12 (55.3). Let $h: S^1 \rightarrow X$ be a continuous map. Then TFAE:

- (i) h is null-homotopic.
- (ii) h extends to a continuous map $k: B^2 \rightarrow X$.
- (iii) $h_*: \pi_1(S^1, b_0) \rightarrow \pi_1(X, x_0)$ is the trivial homomorphism.^a

^a h_* maps every element of $\pi_1(S^1, b_0)$ to the identity in $\pi_1(X, x_0)$.

Proof. (i) \Rightarrow (ii) Let H be a homotopy between h and a constant map, and $\pi: S^1 \times I \rightarrow B^2$ be the map $\pi(x, t) = (1 - t)x$. Then π is a quotient map. By Theorem 1.(i), there is a continuous map $k: B^2 \rightarrow X$ s.t. $H = k \circ \pi$. For $x \in S^1 \subset B^2$, $\pi(x, 0) = x$. $k(x) = k(\pi(x, 0)) = H(x, 0) = h(x)$.

(ii) \Rightarrow (iii) Let $j: S^1 \hookrightarrow B^2$ be the inclusion map. Then $h = k \circ j$, $h_* = k_* \circ j_*$. Since $\pi_1(B^2)$ is trivial, j_* is trivial. Thus h_* is trivial.

(iii) \Rightarrow (i) Let $p: [0, 1] \rightarrow S^1$ be the quotient map s.t. $p(0) = p(1) = b_0$. Then p is a loop based at b_0 . Since h_* is trivial, $h_*([p]) = [h \circ p]$ is the identity element of $\pi_1(X, x_0)$. Let F be a path-homotopy from $h \circ p$ to the constant map c_{x_0} . Applying Theorem 1.(i) to F and $p \times \text{id}$, there is a continuous map $H: S^1 \times I \rightarrow X$ s.t. $F = H \circ (p \times \text{id})$.

- $H(x, 0) = H(p(y), 0) = F(y, 0) = (h \circ p)(y) = h(x)$
- $H(x, 1) = H(p(y), 1) = F(y, 1) = c_{x_0}$

Thus H is a homotopy between h and c_{x_0} . □

Corollary 4 (55.4). The inclusion map $j: S^1 \hookrightarrow \mathbb{R}^2 - \{0\}$ is not null-homotopic.

Proof. $r \circ j: S^1 \rightarrow \mathbb{R}^2 - \{0\} \rightarrow S^1$ is the identity map. Thus j_* is injective, $\pi_1(S^1) \cong \mathbb{Z}$, hence j_* is not trivial. Then by Lemma 12, j is not null-homotopic. □

Definition 23. A **vector field** v on B^2 is a continuous map $v: B^2 \rightarrow \mathbb{R}^2$.

Theorem 21 (55.5). Given a non-vanishing vector field v on B^2 , there are two points x_0 and x_1 on S^1 s.t. $v(x_0)$ is inward and $v(x_1)$ is outward.

Proof. To say that v is non-vanishing, we can consider v as a map $v: B^2 \rightarrow \mathbb{R}^2 - \{0\}$. v is an extension of $v|_{S^1}: S^1 \rightarrow \mathbb{R}^2 - \{0\}$, so $v|_{S^1}$ is null-homotopic by Lemma 12. Now suppose that there is no such x_0 . Then, $v|_{S^1}$ is path-homotopic to the inclusion $j: S^1 \rightarrow \mathbb{R}^2 - \{0\}$ by the homotopy $F: S^1 \times I \rightarrow \mathbb{R}^2 - \{0\}$ defined by $F(x, t) = tx + (1 - t)v|_{S^1}(x)$. Indeed,

- $F(x, 0) = v|_{S^1}(x)$, $F(x, 1) = x = j(x)$

- If $F(x, t) = 0$, then $v|_{S^1}(x) = \frac{t}{t-1}x$; inward. $\nexists x_0$.

Thus, $j \simeq v|_{S^1} \simeq \text{constant map}$. \nexists Corollary 4. \square

Theorem 22 (55.6, Brouwer fixed-point theorem for B^2). If $f: B^2 \rightarrow B^2$ is a continuous map, then there exists $x \in B^2$ s.t. $f(x) = x$.

Proof. Suppose there is no fixed point. Then $v(x) = f(x) - x$ is a non-vanishing vector field on B^2 . Hence there is an outward point x_0 on S^1 , i.e. $v(x_0) = ax_0$ for $a > 0$. $f(x_0) - x_0 = ax_0$. $f(x_0) = (a+1)x_0 \notin B^2$. \nexists \square

Corollary 5 (55.7, Application of FPT). Let A be a 3×3 matrix of positive real numbers. Then A has a positive real eigenvalue.

Proof. Consider $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to be a linear map. Let $B = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1, x_1, x_2, x_3 \geq 0\}$ (first octant of S^2). Note that $B \cong B^2$. For any $x \in B$, $A(x) \in \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \geq 0\}$, as A has positive entries. Then $F(x) = \frac{A(x)}{\|A(x)\|}$ is a map from B to B . By Theorem 22, there exists $x_0 \in B$ s.t. $F(x_0) = \frac{A(x_0)}{\|A(x_0)\|} = x_0$, hence $A(x_0) = \|A(x_0)\|x_0$. \square

Note. HW4:

- Read and understand the topological proof of fundamental theorem of algebra.
- Exercise §55 – #1, #2.

9.57 Borsuk–Ulam theorem

Definition 24. For $x \in S^n$, the **antipode** x is $-x$. A map $h: S^n \rightarrow S^m$ is **antipode-preserving** if $h(-x) = -h(x)$ for all $x \in S^n$.

Theorem 23 (57.1). If $h: S^1 \rightarrow S^1$ is continuous and antipode-preserving, then h is not null-homotopic.

Proof. Let $b_0 = (0, 1)$ and $\rho: S^1 \rightarrow S^1$ be the rotation of S^1 s.t. $\rho(h(b_0)) = b_0$. $(\rho \circ h)(-x) = \rho(-h(x)) = -\rho(h(x)) = -(\rho \circ h)(x)$ (antipode-preserving). Suppose there is a homotopy between h and a constant map. Then $\rho \circ h$ is a homotopy between $\rho \circ h$ and a constant map. Therefore we may prove the theorem under assumption $h(b_0) = b_0$.

Step 1. Let $q: S^1 \subset \mathbb{C} \rightarrow S^1$ be the map $q(z) = z^2$. Then q is a quotient map and $q(-z) = q(z)$. For $x \in S^1$, $q^{-1}(x)$ is two antipodal points. $h(-x) = -h(x)$. $q(h(-x)) = q(-h(x)) = q(h(x))$. Apply Theorem 1 to q and $q \circ h$. Then there exists $k: S^1 \rightarrow S^1$ s.t. $k \circ q = q \circ h$. $k(b_0) = k(b_0^2) = k(q(b_0)) = q(h(b_0)) = q(b_0) = b_0^2 = b_0$.

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ \downarrow q & & \downarrow q \\ S^1 & \xrightarrow{k} & S^1 \end{array}$$

Step 2. We claim $k_*: \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, b_0)$ is non-trivial. We can check that q is a covering map. If \tilde{f} is a path from b_0 to $-b_0$ in S^1 , then $[f = q \circ \tilde{f}] \neq 1$ in $\pi_1(S^1, b_0)$. $k_*([f]) = [k \circ q \circ \tilde{f}] = [q \circ h \circ \tilde{f}] \neq 1$.

Step 3. h_* is nontrivial. We will prove h is not null-homotopic.

$$\begin{aligned} q_*: \pi_1(S^1, b_0) &\cong \mathbb{Z} \longrightarrow \pi_1(S^1, b_0) \cong \mathbb{Z} \\ n &\longmapsto 2n \end{aligned}$$

Thus q_* is injective, $k_* \circ q_*$ is injective, so is $q_* \circ h_*$. Hence h_* is injective. □

Theorem 24 (57.2). There is no continuous antipode-preserving map $g: S^2 \rightarrow S^1$.

Proof. Suppose g is such a map. Then $h = g|_{S^1}$ is continuous and antipode-preserving map $S^1 \rightarrow S^1$. Then, by Theorem 23, h is not null-homotopic. But $g|_{\text{upper-hemi-sphere}}$ is an extension of h . \nexists to Lemma 12. □

Theorem 25 (57.3, Borsuk–Ulam theorem for S^2). Given a continuous map $f: S^2 \rightarrow \mathbb{R}^2$, there is a point $x \in S^2$ s.t. $f(x) = f(-x)$.

Proof. Suppose there is no such point. Then we can define $g: S^2 \rightarrow S^1$ by $g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$. $g(-x) = \frac{f(-x) - f(x)}{\|f(-x) - f(x)\|} = -g(x)$. \nexists to Theorem 24. □

Theorem 26 (57.4, Bisection theorem). For two bounded polygonal regions in \mathbb{R}^2 , there exists a line that bisects each of them.

Proof. Let A_1, A_2 be bounded polygonal regions in $\mathbb{R} \times \{1\} \subset \mathbb{R}^3$. Given a point $u \in S^2$, let P be the plane s.t. $O \in P$, $u \perp P$. Let $f_i(u)$ equal the area of the portion of A_i that lies on the same side of P as does the vector u . If $u = (0, 0, 1)$, then $f_i(u) = \text{area } A_i$, and if $u = (0, 0, -1)$, then $f_i(u) = 0$. $f_i(u) + f_i(-u) = \text{area } A_i$ for all $u \in S^2$. Define a map $F: S^2 \rightarrow \mathbb{R}^2$ by $F(u) = (f_1(u), f_2(u))$. By Theorem 25, there exists $u_0 \in S^2$ s.t. $F(u_0) = F(-u_0)$. Then $f_i(u_0) = f_i(-u_0) = \frac{1}{2} \text{area } A_i$. Hence, $P_{u_0} \cap \mathbb{R}^2 \times \{1\}$ bisects A_1 and A_2 . □

Note. HW5: Exercise §57 – #1, #2, #3.

9.58 Deformation retracts and homotopy type

Lemma 13 (58.1). Let $h, k: (X, x_0) \rightarrow (Y, y_0)$ be continuous maps. If there is a homotopy H between h and k s.t. $H(x_0, t) = y_0$ for all t , then $h_* = k_*$.

Proof. Let f be a loop in X based at x_0 . Consider the map

$$\begin{aligned} F: I \times I &\xrightarrow{f \times \text{id}} X \times I \xrightarrow{H} Y \\ (s, t) &\longmapsto (f(s), t) \mapsto H(f(s), t) \end{aligned} .$$

- Then F is continuous.
- $F(s, 0) = H(f(s), 0) = (h \circ f)(s)$
- $F(s, 1) = H(f(s), 1) = (k \circ f)(s)$
- $F(0, t) = H(f(0), t) = H(x_0, t) = y_0$
- $F(1, t) = H(f(1), t) = H(x_0, t) = y_0$

Thus F is a path-homotopy between $h \circ f$ and $k \circ f$ so that $h_*([f]) = [h \circ f] = [k \circ f] = k_*([f])$. \square

Theorem 27 (58.2). The inclusion map $j: S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$ induces an isomorphism between fundamental groups.

Proof. Let $X = \mathbb{R}^{n+1} - \{0\}$, $b_0 = (1, 0, \dots, 0)$. There exists a retraction $r: X \rightarrow S^n$ defined by $r(x) = \frac{x}{\|x\|}$. Then $r \circ j: S^n \rightarrow X \rightarrow S^n$ is the identity map, hence $r_* \circ j_* = \text{id}_{S^n}^*$. Now consider $j \circ r: X \rightarrow S^n \rightarrow X$ which maps X to itself. This map is not the identity map id_X , but it is homotopic to. Indeed, $H: X \times I \rightarrow X$ given by $H(x, t) = (1-t)x + t\frac{x}{\|x\|}$ is a homotopy between $H(x, 0) = x = \text{id}_X(x)$ and $H(x, 1) = \frac{x}{\|x\|} = (j \circ r)(x)$. Note that $H(b_0, t) = b_0$. Then by Lemma 13, $(j \circ r)_* = j_* \circ r_* = \text{id}_*^X$. Thus j_* has the right and left inverse homomorphism. \square

Definition 25. A subspace A of X is a **deformation retract** of X if there is a continuous map $H: X \times I \rightarrow X$ s.t. $H(x, 0) = x$, $H(x, 1) \in A$ $\forall x \in X$, and $H(a, t) = a$ $\forall a \in A$, $\forall t \in I$.^a The homotopy H is called a **deformation retraction** of X onto A .

^a $r(x) = H(x, 1)$ is a retraction of X onto A . H is a homotopy between id_X and $j \circ r$.

H shows a continuous shrinking of X onto A . During the shrinking, points of A stay where they are.

Theorem 28 (58.3). Let A be a deformation retract of X , and $j: A \hookrightarrow X$ the inclusion map. Then $j_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ is an isomorphism.

Proof. Similar with Theorem 27. \square

Example. $(\mathbb{R}^2 - \{0\}) \times \{0\} \subset \mathbb{R}^3 - \{z\text{-axis}\}$. $H((x, y, z), t) = (x, y, (1-t)z)$. Thus, $\pi_1(\mathbb{R}^3 - \{z\text{-axis}\}) \cong \pi_1(\mathbb{R}^2 - \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Example. $\pi_1(\mathbb{R}^2 - \{\text{two points}\}) \cong \pi_1(\text{Bouquet with two leaves})$. Can you write down the deformation retractions concretely?

Example. $S^1 \cup (\{0\} \times [-1, 1])$ (theta space) is deformation retract of $\mathbb{R}^2 - \{\text{two points}\}$.

Definition 26. Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be continuous maps. f and g are called **homotopy equivalences (maps)** if $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. (f is a homotopy inverse of g). X is homotopically equivalent to Y .

Definition 27. Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be continuous maps. X and Y are said to be of the same **homotopy type** if $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. We say that f, g are **homotopy equivalences** and are **homotopy inverses** of each other.

Note. The relation of homotopy equivalence is an equivalence relation.

Remark. Suppose A is a deformation retract of X . Then for the retraction $r(x) = H(x, 1)$ and inclusion $j: A \hookrightarrow X$, $r \circ j = \text{id}_A$, $j \circ r \simeq \text{id}_X$ by H . Thus r and j are homotopy equivalence maps.

Example. Bouquet with two leaves and theta space are deformation retract of $\mathbb{R}^2 - \{\text{two points}\}$. Thus they are homotopically equivalent to each other. Can you find a homotopy equivalence map between them?

Lemma 14 (58.4). Let $h, k: X \rightarrow Y$ be continuous maps. $h(x_0) = y_0$, $k(x_0) = y_1$. If $h \simeq k$, then there is a path α in Y from y_0 to y_1 s.t. $k_* = \hat{\alpha} \circ h_*$.

Proof. Let f be a loop based at x_0 . Then we have to show:

$$k_*([f]) = \hat{\alpha}(h_*([f])), [k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha], [\alpha] * [k \circ f] = [h \circ f] * [\alpha].$$

Let $f_0(s) = (f(s), 0) \subset X \times \{0\}$, $f_1(s) = (f(s), 1) \subset X \times \{1\}$, $c(t) = (x_0, t) \in X \times I$. If H is a homotopy between h and k , then $(H \circ f_0)(s) = H(f(s), 0) = (h \circ f)(s)$, $(H \circ f_1) = k \circ f$. Define $F: I \times I \rightarrow X \times I$ by $F(s, t) = (f(s), t)$. Label \square

Example. Let $S^1 \subset \mathbb{R}^2 \setminus \{(0,0)\}$. Then S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{(0,0)\}$. Using homotopy $H: \mathbb{R}_0^2 \times I \rightarrow \mathbb{R}_0^2$ given by $x \mapsto (1-t)x + t\frac{x}{\|x\|}$. (The same for S^n and \mathbb{R}_0^n)

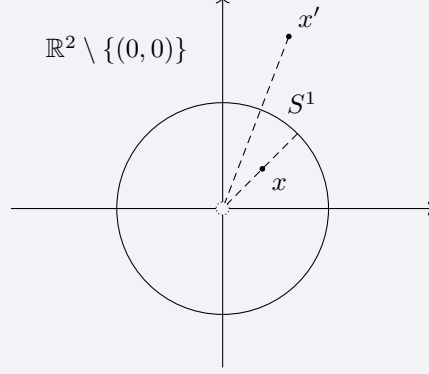


Figure 9.5: Example of a deformation retract

Example. Consider the figure 8 space. Claim: A is not a deformation retract of X . We'll prove this later on.

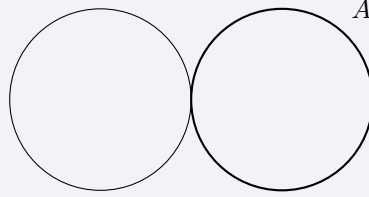


Figure 9.6: Example of a deformation retract

Example. Consider the torus and a circle on the torus. Then it is a retract, but not a deformation retract.

Theorem 29. If A is a deformation retract of X , then $i: A \rightarrow X$ induces an *isomorphism* i_* . I.e. if you have a deformation retract, it's not only injective but also surjective.

Proof. Let $i: A \rightarrow X$ be the inclusion and $r: X \rightarrow A$ be the deformation retraction using H . Then $r \circ i = 1_A$, which gives $r_* \circ i_* = 1_{\pi(A, a_0)}$.

Now, $i \circ r \simeq_p 1_X$ using the homotopy of the previous lemma, i.e. H with $H(a_0, t) = a_0$. Call $h = i \circ r$, $k = 1_X$, and using the previous lemma, $(i \circ r)_* = (1_X)_*: \pi(X, x_0) \rightarrow \pi(X, x_0)$, which shows that $i_* \circ r_* = 1_{\pi(X, x_0)}$.

We conclude that both i_* and r_* are isomorphisms. \square

Remark. This means that the fundamental group of \mathbb{R}_0^2 is the same as the one of S^1 , which is \mathbb{Z} .

Example. The fundamental group of the figure 8 space and the θ -space are isomorphic. These spaces are not deformations of each other, but we can show that they are deformation retracts of $\mathbb{R}^2 \setminus \{p, q\}$. We say that these spaces are of the same homotopy type.

Definition 28. Let X, Y be two spaces, then X and Y are said to be of the same **homotopy type** if there exists $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. We say that f, g are **homotopy equivalences** and are **homotopy inverses** of each other.

Remark. This is an equivalence relation.

We'll prove that spaces of the same homotopy type have the same fundamental group. For that, we'll prove the previous lemma in a more general form, not preserving the base point.

Lemma 15 (58.4). Suppose $h, k: X \rightarrow Y$ with $h(x_0) = y_0$ and $k(x_0) = y_1$. Assume that $h \simeq k$ via a homotopy $H: X \times I \rightarrow Y$, ($H(x, 0) = h(x)$, $H(x, 1) = k(x)$). Then $\alpha: I \rightarrow Y$ given by $s \mapsto H(x_0, s)$ is a path starting in y_0 and ending in y_1 such that the following diagram commutes

$$\begin{array}{ccc} & \pi(X, x_0) & \\ h_* \swarrow & & \searrow k_* \\ \pi(Y, y_0) & \xrightarrow{\hat{\alpha}} & \pi(Y, y_1) \\ [g] \longmapsto & & [\bar{\alpha}] * [g] * [\alpha] \end{array} .$$

Proof. We need to show that $\hat{\alpha}(h_*[f]) = k_*[f]$, or $[\bar{\alpha}] * [h \circ f] * [\alpha] = [k \circ f]$, or $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$. We'll prove that these paths are homotopic. Using the picture, we see that $\beta_0 * \gamma_2 \simeq_p \gamma_1 * \beta_1$, because they are loops in a path connected space, $I \times I$. Therefore, $F \circ (\beta_0 * \gamma_2) \simeq_p F \circ (\gamma_1 * \beta_1)$. This is $f_0 * c \simeq_p c * f_1$. Now, if we apply H , we get $H \circ (f_0 * c) \simeq_p H \circ (c * f_1)$, so $(h \circ f) * \alpha \simeq_p \alpha * (k \circ f)$, which implies that $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$. \square

Theorem 30. Let $f: X \rightarrow Y$ be a homotopy equivalence, with $f(x_0) = y_0$. Then $f_*: \pi(X, x_0) \rightarrow \pi(Y, y_0)$ is an isomorphism.

Proof. Let g be a homotopy inverse of f .

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1) \cdots$$

$$\begin{array}{ccccc} \pi(X, x_0) & \xrightarrow{f_*, x_0} & \pi(Y, y_0) & \xrightarrow{g_*, x_0} & \pi(X, x_1) \\ & \searrow 1_{\pi(X, x_0) = (1_X)_*} & & \downarrow \hat{\alpha} & \\ & & & \pi(X, x_0) & \end{array}$$

$$\begin{array}{ccccc} \pi(Y, y_0) & \xrightarrow{g_*, x_0} & \pi(X, x_1) & \xrightarrow{f_*, x_1} & \pi(Y, y_1) \\ & \searrow 1_{\pi(Y, y_0) = (1_Y)_*} & & \downarrow \hat{\beta} & \\ & & & \pi(Y, y_0) & \end{array}$$

From the first diagram, $g_{y_0,*} \circ f_{x_0,*}$ is an isomorphism, $g_{y_0,*}$ is surjective. The second diagram gives that $f_{x_1,*} \circ g_{y_0,*}$ is an isomorphism, so $g_{y_0,*}$ is injective, so $g_{y_0,*}$ is an isomorphism. Now composing, we find that $g_{y_0,*}^{-1} \circ (g_{y_0,*} \circ f_{x_0,*}) = f_{x_0,*}$ is an isomorphism. \square

9.59 $\pi_1(S^n)$

Lecture 16
Mon, Oct 25

Theorem 31 (59.1, Special version of van Kampen theorem). Let $X = U \cup V$, where U, V are open subsets of X , and $U \cap V$ is path connected. Let $i: U \hookrightarrow X$ and $j: V \hookrightarrow X$ denote the inclusions and consider $x_0 \in U \cap V$. Then the images of i_* and j_* generate the whole group $\pi(X, x_0)$.^a

^aIn other words, every element of $\pi_1(X, x_0)$ is a product of the elements of the subgroups.

Proof. Let f be a loop in X based at x_0 . Need to show that f is a product of loops in U or V .

- (i) We can divide $[0, 1]$ into subintervals $0 = a_0 < a_1 < \cdots < a_n = 1$ so that $f(a_i) \in U \cap V$ and $f([a_{i-1}, a_i]) \subset U$ or V .
- (ii) $U \cap V$ is path-connected, so we can choose a path α_i from x_0 to $f(a_i)$. Let f_i be a path s.t. $f(I) = f([a_{i-1}, a_i])$. Then,

$$\begin{aligned} [f] &= [f_1] * \cdots * [f_n] \\ &= [f_1] * [\bar{\alpha}_1 * \alpha_1] * [f_2] * \cdots * [\bar{\alpha}_{n-1} * \alpha_{n-1}] * [f_n] \\ &= [f_1 * \bar{\alpha}_1] * [\alpha_1 * f_2 * \bar{\alpha}_2] * \cdots * [\alpha_{n-2} * f_{n-1} * \bar{\alpha}_{n-1}] * [\alpha_{n-1} * f_n] \end{aligned}$$

Each factor in the product is a loop in U or V . \square

Proof. Let $[f] \in \pi(X, x_0)$ denote $f: I \rightarrow X$ is a loop based at x_0 .

Claim: there exists a subdivision of $[0, 1]$ such that $f[a_i, a_{i+1}]$ lies entirely inside U or V and $f(a_i) \in U \cap V$. Proof of the claim: Lebesgue number

lemma says that such a subdivision b_i exists. Now assume b_j is such that $f(b_j) \notin U \cap V$, for $0 < j < m$. Then either $f(b_j) \in U \setminus V$, or $f(b_j) \in V \setminus U$. The first one would imply that $f([b_{j-1}, b_j]) \subset U$ and $f([b_j, b_{j+1}]) \subset U$. So $f[b_{j-1}, b_{j+1}] \subset U$, so we can discard b_j . Same for the second possibility.

Let α_i be a path from x_0 to $f(a_i)$ and α_0 the constant path $t \mapsto x_0$, inside $U \cap V$ (which is possible, as it is path connected). Now define

$$f_i: I \rightarrow X \text{ given by } I \xrightarrow{\text{p.l.m.}} [a_{i-1}, a_i] \xrightarrow{f} X.$$

Then $[f] = [f_1] * [f_2] * \cdots * [f_n]$. Note that all f_i have images inside U or V . Now,

$$\begin{aligned} [f] &= [a_0] * [f_1] * [\overline{\alpha_1}] * [\alpha_1] * [f_2] * [\overline{\alpha_2}] * [\alpha_2] * [f_3] * \cdots * [\alpha_{n-1}] * [f_n] * [\overline{\alpha_n}] \\ &= [\alpha_0 * (f_1 * \overline{\alpha_1})] * [\alpha_1 * (f_2 * \overline{\alpha_2})] * \cdots . \end{aligned}$$

Every path of the form $\alpha_{i-1} * (f_i * \overline{\alpha_i})$ is a loop based at x_0 lying entirely inside U or V . This means that

$$[f] \in \text{grp}\{i_*(\pi(U, x_0)), j_*(\pi(V, x_0))\}.$$

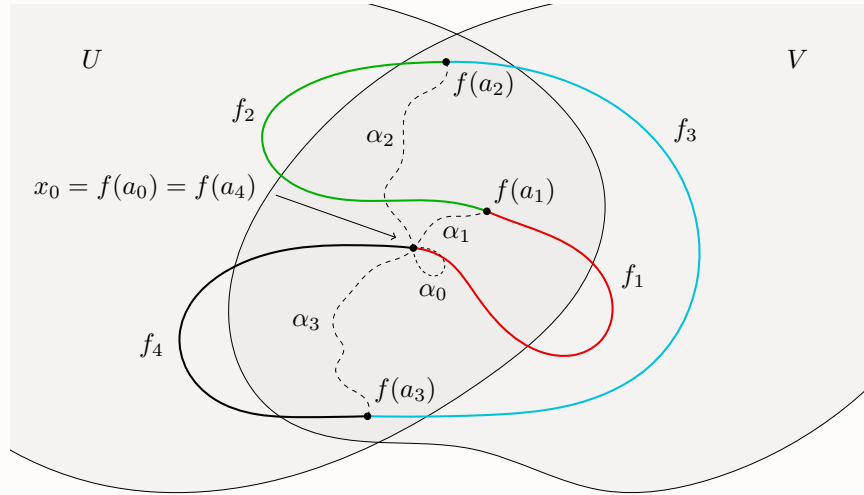


Figure 9.7: Proof of Theorem 59.1

□

Corollary 6 (59.2). If U and V are simply connected, then so is X .

Theorem 32 (59.3). For $n \geq 2$, S^n is simply connected.

Proof. Consider S^n and N, S the north and south pole. Let $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$. Then $U, V \approx \mathbb{R}^n$ and $U \cap V$ is path connected, which

is easy to prove as it is simply homeo to \mathbb{R}^n with points removed. Then $\pi(S^n, x_0)$ is generated by $i_*(\pi(U, x_0))$ and $j_*(\pi(V, x_0))$, which both are trivial. This proof doesn't work for S^1 because then the intersection is not path connected anymore! \square

Note. HW6:

- Prove (i) of Proof of Theorem 31 in detail.
- Exercise §59 – #1, #3.

9.60 Fundamental groups of some surfaces

Definition 29. Given groups (G, \cdot) and $(H, *)$, the **direct product** $G \times H$ is the set $\{(g, h) \mid g \in G, h \in H\}$ where $(g_1, h_1)(g_2, h_2) = (g_1 \cdot g_2, h_1 * h_2)$.

Theorem 33 (60.1). $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. Let p, q be projection mappings from $X \times Y$ to X and Y , respectively. With given base points, we have induced homomorphisms p_* and q_* .

Homomorphism Define a map $\Phi: \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $\Phi([f]) = (p_*([f]), q_*([f])) = ([p \circ f], [q \circ f])$. For two loops f, g in $X \times Y$ based at (x_0, y_0) ,

$$\begin{aligned} \Phi([f] * [g]) &= \Phi([f * g]) = (p_*([f * g]), q_*([f * g])) \\ &= (p_*([f]) * p_*([g]), q_*([f]) * q_*([g])) \\ &= (p_*([f]), q_*([f])) \cdot (p_*([g]), q_*([g])) \\ &= \Phi([f]) \cdot \Phi([g]). \end{aligned}$$

Thus Φ is a group homomorphism.

Surjective Let $\mathcal{L}(Z, z_0)$ denote the set of all loops in Z based at z_0 . For $g \in \mathcal{L}(X, x_0)$ and $h \in \mathcal{L}(Y, y_0)$, let $f \in \mathcal{L}(X \times Y, (x_0, y_0))$ s.t. $f(s) = (g(s), h(s))$. Then $\Phi([f]) = ([p \circ f], [q \circ f]) = ([g], [h])$.

Injective Let $f \in \mathcal{L}(X \times Y, (x_0, y_0))$ such that $\Phi([f]) = ([c_{x_0}], [c_{y_0}])$. Then $p \circ f \simeq_p^G c_{x_0}$ and $q \circ f \simeq_p^H c_{y_0}$. Define a map $F: I \times I \rightarrow X \times Y$ by $F(s, t) = (G(s, t), H(s, t))$. Then,

- $F(s, 0) = (G(s, 0), H(s, 0)) = (p \circ f, q \circ f) = f$
- $F(s, 1) = (G(s, 1), H(s, 1)) = (c_{x_0}, c_{y_0}) = c_{(x_0, y_0)}$
- $F(0, t) = (G(0, t), H(0, t)) = (x_0, y_0)$
- $F(1, t) = (G(1, t), H(1, t)) = (x_0, y_0)$

Thus, F is a path-homotopy between f and $c_{(x_0, y_0)}$, hence $[f]$ is the identity element of $\pi_1(X \times Y, (x_0, y_0))$.

□

Example. $\pi_1(T^2, x_0) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$. We know that $\pi(S^2, x_0) = 1$, so the torus and the two sphere are not homeomorphic to each other, they aren't even homotopically equivalent.

Example. $\mathbb{RP}^2 = S^2/\sim$. Then $p: S^2 \rightarrow \mathbb{RP}^2$, which is continuous by definition of the topology on the projective plane. This means that (S, p) is a covering of the projective plane. The lifting correspondence says that

$$\Phi: \pi(\mathbb{RP}^2, x_0) \rightarrow p^{-1}(x_0) = \{\tilde{x}_0, -\tilde{x}_0\}$$

is a isomorphism. Therefore, $\pi_1(\mathbb{RP}^2, x_0)$ is a group with 2 elements, so \mathbb{Z}_2 .

This means, there exists loops which we cannot deform to the trivial loop, but when going around twice, they do deform to the trivial loop. E.g. consider the loop a . This is not homotopic equivalent with the trivial loop, as $e_1 \neq e_0$. (Or also you can see it because $\alpha = \bar{\alpha}$.) But pasting the loop it twice, we see that is possible. This means that the fundamental group of the projective space is different from all the one we've seen before.

Example. T^2 is the torus. $T^2 \# T^2$ is the connected sum of two tori (Remove small disc of both tori and glue together), in Dutch: 'tweeling zwemband'. This space has yet another fundamental group.

Example. Figure eight space: fundamental group is not abelian. Indeed, $[b * a] \neq [a * b]$.

Example. Tweeling zwemband. The space retracts to the figure 8 situation, which shows that the group of the tweeling zwemband has a nonabelian component.

Chapter 10

Separation theorems in the plane

Lecture 17
Wed, Oct 27

Review on connectedness

Definition 30. A topological space X is **disconnected** if there are two non-empty open subsets U and V (called **separation** of X) s.t. $U \cap V = \emptyset$, $U \cup V = X$.^a X is said to be **connected** if it is not disconnected.^b

^aHence, U and V are open and closed.

^bIff \emptyset, X are only sets which are both open and closed.

Theorem 34 (23.3). Let $\{E_\alpha\}_{\alpha \in A}$ be a family of connected subsets of a topological space X s.t. $E_\alpha \cap E_\beta \neq \emptyset$ for every $\alpha, \beta \in A$. Then, $\bigcup_{\alpha \in A} E_\alpha$ is connected.

Proof. Let $\bigcup_{\alpha \in A} E_\alpha = A \cup B$ be a separation. For $x \in A$, $x \in E_{\alpha_0}$ for some α_0 . $A \cap E_{\alpha_0}$ is a non-empty, open and closed subset of E_{α_0} . By connectedness of E_{α_0} , $A \cap E_{\alpha_0} = E_{\alpha_0}$, $A \supset E_{\alpha_0}$. For any β , $A \cap E_\beta$ is an open and closed subset of E_β . Note that $A \cap E_\beta \supset E_{\alpha_0} \cap E_\beta \neq \emptyset$. Thus, $A \cap E_\beta = E_\beta$, $A = \bigcup_{\alpha \in A} E_\alpha$, $B = \emptyset$. \square

Definition 31. A **connected component** of X is a maximal connected subset of X .^a

^a A is a connected component of X if there is no connected subset of X which contain A .

Assume that C is a connected component of X and U is a connected subset. If $C \cap U \neq \emptyset$, $C \cup U$ is a connected (by Theorem 34) subset which contains C .
 \nmid . Thus we have only two possible cases: $C \cap U = \emptyset$ or $C \supset U$. This implies, two connected components are disjoint and X can be partitioned into a disjoint union of connected components.

Definition 32. A space X is **path-connected** if for every $x, y \in X$, there is a path from x to y .

Note. The existence of a path between two points is an equivalence relation on the points of X . The equivalence classes of such a relation are called **path-components**.

A path-connected space is connected. Thus a connected component is split into path-components.

Theorem 35 (25.5). If a space X is locally path-connected, then connected components and path-components of X are the same.

Proof. Let C be a connected component and P be a path-component s.t. $C \cap P \neq \emptyset$. Since P is connected, $C \supset P$. Suppose that $C \neq P$. Let Q be the union of path-components other than P . $C = P \sqcup Q$. Since X is locally path-connected, each path-component is open. Thus P, Q are open, hence $P \sqcup Q$ is a separation of C . \nmid \square

10.61 Jordan separation theorem

Definition 33. Let A be a subspace of a connected space X . We say A **separates** X if $X - A$ is not connected. A is an **arc** if $A \cong [0, 1]$, that is, there is a continuous map $\alpha: [0, 1] \rightarrow X$ s.t. α is injective, $\alpha(I) = A$. A is a **simple closed curve** if $A \cong S^1$, i.e. there is a continuous map $\alpha: [0, 1] \rightarrow X$ s.t. $\alpha(I) = A$, $\alpha(0) = \alpha(1)$, α is injective on $(0, 1)$.

The main content of this section is the proof of the following theorem.

Theorem 36 (61.3, Jordan separation theorem). Any simple closed curve in S^2 separates S^2 .

Lemma 16 (61.1). Let C be a compact subspace of S^2 , $b \in S^2 - C$, $h: S^2 - \{b\} \rightarrow \mathbb{R}^2$ be a homeomorphism, and U be a component of $S^2 - C$. If $b \notin U$, then $h(U)$ is a bounded component of $\mathbb{R}^2 - h(C)$. If $b \in U$, then $h(U - \{b\})$ is an unbounded component of $\mathbb{R}^2 - h(C)$.

Proof. (i) $U - \{b\}$ is connected. If $b \notin U$, $U - \{b\} = U$ is connected. Assume $b \in U$. Let $U - \{b\} = A \sqcup B$ be a separation. Choose an open nbh W of b in S^2 so that $W \cong$ open disk in \mathbb{R}^2 , $W \cap C = \emptyset$. Then $W - \{b\}$ is connected, hence we may say $W - \{b\} \subset A$. Thus $(A \cup \{b\}) \sqcup B$ is a separation of U . \nmid

(ii) Let $\{U_\alpha\}$ be the collection of all components of $S^2 - C$. $S^2 - C$ (\cong open subset of \mathbb{R}^2 is locally connected, hence each U_α is open in $S^2 - C$. By (i), $\{U_\alpha - \{b\}\}$ is the collection of open, disjoint, and connected subsets. The homeomorphism h preserves such properties.

Thus $\{h(U_\alpha - \{b\})\}$ is the collection of all components of $\mathbb{R}^2 - h(C)$.

- (iii) If $b \in U_{\alpha_0}$, then $h(U_{\alpha_0} - \{b\})$ is an unbounded component of $\mathbb{R}^2 - h(C)$. $(S^2 - C) - U_{\alpha_0}$ is bounded and closed, hence compact in $S^2 - C$. $h((S^2 - C) - U_{\alpha_0}) = \bigsqcup_{\alpha \neq \alpha_0} h(U_\alpha - \{b\})$. Each $h(U_\alpha - \{b\})$ ($\alpha \neq \alpha_0$) is bounded. □

Lemma 17 (61.2). Let $a, b \in S^2$, A be a compact space, $f: A \rightarrow S^2 - \{a, b\}$ be a continuous map. If a, b are in the same component of $S^2 - f(A)$, then f is null-homotopic.

Proof. Note that there is a homeomorphism $h: S^2 - \{a, b\} \rightarrow \mathbb{R}^2 - \{O\}$. If $h \circ f: A \xrightarrow{f} S^2 - \{a, b\} \xrightarrow{h} \mathbb{R}^2 - \{O\}$ is null-homotopic, then $h \circ f \simeq$ constant map, hence $f \simeq h^{-1} \circ$ constant map.

Therefore, it's enough to show: For a continuous map $g: A \rightarrow \mathbb{R}^2 - \{O\}$, if O lies in the unbounded component of $\mathbb{R}^2 - g(A)$, then g is null-homotopic. (If a, b are in the same component of $S^2 - f(A)$, then O is in the unbounded component of $\mathbb{R}^2 - (h \circ f)(A)$ by Lemma 16.)

Choose a disk B centered at O in \mathbb{R}^2 so that $g(A) \subset B$. And choose a point $p \in B$. Then O and p are in the unbounded component of $\mathbb{R}^2 - g(A)$. \mathbb{R}^2 is locally path-connected, hence so is the open subset $\mathbb{R}^2 - g(A)$. (Thus the components and the path-components of $\mathbb{R}^2 - g(A)$ are the same.) O and p are in the same path-component of $\mathbb{R}^2 - g(A)$. So, there is a path α in $\mathbb{R}^2 - g(A)$ from O to p .

Define a homotopy $G: A \times I \rightarrow \mathbb{R}^2 - \{O\}$ by $G(x, t) = g(x) - \alpha(t)$. $G(x, t) \neq O$ because of $g(A) \cap \alpha(I) = \emptyset$. ($G(x, 0) = g(x)$, $G(x, 1) = g(x) - p$) Also, define a homotopy $H: A \times I \rightarrow \mathbb{R}^2 - \{O\}$ by $H(x, t) = tg(x) - p$. ($H(x, 0) = -p$, $H(x, 1) = g(x) - p = G(x, 1)$) Therefore, by G and H , $g(x)$ is null-homotopic. □

Proof (of Theorem 36). $S^2 - C$ is open, hence it is locally path-connected, $\{\text{path-components}\} = \{\text{connected components}\}$. Suppose that $S^2 - C$ is path-connected. Let $C = A_1 \cup A_2$ and $X = S^2 - \{a, b\}$. And let $U = S^2 - A_1$, $V = S^2 - A_2$, $x_0 \in U \cap V$, $i: U \hookrightarrow X$, and $j: V \hookrightarrow X$. (Then, $X = U \cup V$, $U \cap V = S^2 - (A_1 \cup A_2) = S^2 - C$ (path-connected).) By special van-Kampen theorem, $i_*(\pi_1(U, x_0))$ and $j_*(\pi_1(V, x_0))$ generate $\pi_1(X, x_0)$.

Claim: i_* and j_* is trivial homomorphisms. The claim implies, $\pi_1(X, x_0)$ should be trivial, but, $\pi_1(X) \cong \pi_1(\mathbb{R}^2 - \{\text{a point}\}) \cong \mathbb{Z}$ is not trivial. \nmid

Proof of claim: Let $p: I \rightarrow S^1$ be the loop $p(t) = e^{2\pi it}$. Then $[p]$ generates $\pi_1(S^1, b_0)$. For a loop $f \in \mathcal{L}(U, x_0)$, Let $h: S^1 \rightarrow U$ be the loop s.t. $h \circ p = f$. Consider the map $i \circ h: S^1 \xrightarrow{h} U \xrightarrow{i} X = S^2 - \{a, b\}$. $i(h(S^1)) = h(S^1) \cap A_1 = \emptyset$, hence a and b are in the same path-component (= conn. comp.) of $S^2 - i(h(S^1))$. Applying Lemma 17 to $i \circ h: S^1 \rightarrow X$, we know that $i \circ h$ is null-homotopic. By Lemma 12, $(i \circ h)_*$ is the trivial

homomorphism. Thus,

$$(i \circ h)_*([p]) = [i \circ h \circ p] = [i \circ f] = i_*([f]) = i_*([e_{x_0}]),$$

hence, i_* is trivial. Similarly, so is j_* . \square

Theorem 37 (61.4, A general separation theorem). If A_1 and A_2 are closed connected subsets of S^2 s.t. $A_1 \cap A_2 = \{\text{two points}\}$, then $A_1 \cap A_2$ separates S^2 .

Proof. $A_1 \cup A_2 \neq S^2$. Because $S^2 - \{a, b\}$ is connected. $(A_1 \cup A_2) - \{a, b\} = (A_1 - \{a, b\}) \sqcup (A_2 - \{a, b\})$ (both are open). Thus $(A_1 \cup A_2) - \{a, b\}$ is disconnected. The remainder of proof is same with that of Theorem 36. \square

Note. HW7: Exercise §61 – #1, #2.

10.62 Invariance of domain

Theorem 38 (62.3). If U is an open subset of \mathbb{R}^n and $f: U \rightarrow S^n$ is continuous and injective, then $f(U)$ is open in S^n and the inverse function $f^{-1}: f(U) \rightarrow U$ is continuous. ($\therefore u \cong f(U)$ by f)

In this section, we prove this theorem for $n = 2$.

Lemma 18 (62.1, Homotopy extension lemma). Let X be a space s.t. $X \times I$ is normal, A be a closed subset of X , Y be a open subset of \mathbb{R}^n , and $f: A \rightarrow Y$ be a continuous map. If f is null-homotopic, then f can be extended to a continuous map $g: X \rightarrow Y$ that is null-homotopic.

Proof. Let $F: A \times I \rightarrow Y$ be a homotopy between f and c_{y_0} . Define $\bar{F}: (A \times I) \cup (X \times \{1\}) \rightarrow \mathbb{R}^n$ by $\bar{F}|_{A \times I} = F$ and $\bar{F}(x, 1) = y_0$. Applying the Tietze extension theorem n -times, \bar{F} can be extended to a continuous map $G: X \times I \rightarrow \mathbb{R}^n$. The map $x \mapsto G(x, 0)$ is an extension of f , but it may map X into \mathbb{R}^n , rather than Y . So, we need to do something more.

Let $U = G^{-1}(Y)$. Then $U \supset (A \times I) \cup (X \times \{1\})$. By the Tube lemma (26.8), there is an open subset W of X s.t. $W \times I \subset U$, $A \subset W$. Apply the Urysohn lemma to (X, A, W^c) , we have a continuous map $\phi: X \rightarrow [0, 1]$ s.t. $\phi(x) = 0 \ \forall x \in A$, $1 \ \forall x \in X - W$. Then the map $x \mapsto (x, \phi(x))$ carries X into $(W \times I) \cup (X \times \{1\}) \subset U$. $g(x) = G(x, \phi(x))$ is a continuous map from X to Y . For $x \in A$, $g(x) = G(x, \phi(x)) = G(x, 0) = f(x)$ (extension of f). Define $H: X \times I \rightarrow Y$ by $H(x, t) = G(x, (1 - t)\phi(x) + t)$. Then,

- $H(x, 0) = G(x, \phi(x)) = g(x)$ (extension of f)
- $H(x, 1) = G(x, 1) = c_{y_0}$

Thus, H is a homotopy between g and c_{y_0} . \square

Lecture 19
Wed, Nov 3

Lemma 19 (62.2, Borsuk lemma). Let $a, b \in S^2$, A be a compact space, and $f: A \rightarrow S^2 - \{a, b\}$ be a continuous and injective map. If f is null-homotopic, then a and b are in the same component of $S^2 - f(A)$.

Proof. Because A is compact, $S^2 - \{a, b\}$ is Hausdorff, and f is injective, by Theorem 26.6, $A \cong f(A)$ by f . Let $h: S^2 - \{b\} \rightarrow \mathbb{R}^2$ be a homeomorphism s.t. $h(a) = O$ ($\because f^{-1}: f(A) \rightarrow A$ is continuous). By Lemma 17, if O is in unbounded component of $\mathbb{R}^2 - h(f(A))$, then a, b are in the same component of $S^2 - f(A)$.

Consider the map $A \xrightarrow{h \circ f} h(f(A)) \xrightarrow{j} \mathbb{R}^2 - \{O\}$. Let $H: A \times I \rightarrow S^2 - \{a, b\}$ be a homotopy between f and c_{y_0} ($y_0 \neq a, b$). Define $J: h(f(A)) \times I \rightarrow S^2 - \{O\}$ by $J(z, t) = (h \circ H)((h \circ f)^{-1}(z), t)$. Then,

- $J(z, 0) = h \circ f \circ (h \circ f)^{-1}(z) = z = j(z)$
- $J(z, 1) = (h \circ c_{y_0})((h \circ f)^{-1}(z)) = h(y_0) = c_{h(y_0)}$

Thus, j is null-homotopic. Now, it's enough to show: If X is compact subspace of $\mathbb{R}^2 - \{O\}$ and $j: X \hookrightarrow \mathbb{R}^2 - \{O\}$ is null-homotopic, then O is in the unbounded component of $\mathbb{R}^2 - X$.

Let C be the component of $\mathbb{R}^2 - X$ containing O . Suppose C is bounded. Let D be the union of the other components ($\mathbb{R}^2 - X = C \sqcup D$). $\mathbb{R}^2 - X$ is open in X , and C, D are open in $\mathbb{R}^2 - X$, hence C, D are open in \mathbb{R}^2 . Thus $X \cup C$ is closed and normal in \mathbb{R}^2 .

Apply Lemma 18 to $(X \cup C, X, j: X \hookrightarrow \mathbb{R}^2 - \{O\})$. Then j can be extended to a map $k: X \cup C \rightarrow \mathbb{R}^2 - \{O\}$. And extend k to a map $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2 - \{O\}$ by $g(x) = x$ for all $x \in D$. Let B be a closed ball in \mathbb{R}^2 centered at O s.t. $\text{Int } B \supset X \cup C$. Note that $g(x) = x$ for all $x \in \partial B$. Define $g_1: B \rightarrow \partial B$ by $g_1(x) = (\text{Radius of } B) \times \frac{g(x)}{\|g(x)\|}$. Then g_1 is a retraction of B onto ∂B , which is a contradiction to Theorem 20. \square

Proof (of Theorem 38 for $n = 2$). (i) For a closed ball B in \mathbb{R}^2 containing U , $f(B)$ does not separate S^2 . Let $a, b \in S^2 - f(B)$. The identity map $i: B \rightarrow B$ is null-homotopic ($i \simeq^H c_{x_0}$). Consider the map $f \circ H: B \times I \xrightarrow{H} B \xrightarrow{f} S^2 - \{a, b\}$. Then,

- $(f \circ H)(x, 0) = (f \circ i)(x) = f(x)$
- $(f \circ H)(x, 1) = f(x_0)$

for all $x \in B$. Thus, $f|_B: B \rightarrow S^2 - \{a, b\}$ is null-homotopic, hence by Lemma 19, a, b are in the same component of $S^2 - f(B)$.

(ii) If B is a closed disk of \mathbb{R}^2 s.t. $B \subset U$, then $f(\text{Int } B)$ is open in S^2 . Let $C = f(\partial B)$. Since f is continuous and injective, C is a simple closed curve on S^2 , hence it separates S^2 . Let V be the component of $S^2 - C$ s.t. $V \supset f(\text{Int } B)$ and W be the union of the other components. Because S^2 is locally connected, V and W are open in S^2 . In fact, $V = f(\text{Int } B)$ (open). Otherwise, select a point

$a \in V$ s.t. $a \notin f(\text{Int } B)$ and another point $b \in W$. By (i), $S^2 - f(B)$ is a connected and contains both a and b . But $S^2 - f(B) \subset S^2 - C$, $S^2 - f(B)$ is contained in a component of $S^2 - C$. Both a and b are also in the component. \nmid

(iii) Since U is open, for any $x \in U$, we can select a closed ball B_x so that $B_x \subset U$. By (ii), $f(\text{Int } B_x)$ is open in S^2 . $U = \bigcup_{x \in U} \text{Int } B_x$. $f(U) = \bigcup_{x \in U} f(\text{Int } B_x)$ is open. \square

Note. HW7: Exercise §62 – #6.

10.63 Jordan curve theorem

Theorem 39 (63.4, Jordan curve theorem). Let C be a simple closed curve in S^2 . Then,

- (i) C separates S^2 into precisely two components W_1 and W_2 .
- (ii) $\partial W_1 = C = \partial W_2$.

Theorem 40 (Schöflies theorem). $\overline{W}_1 \cong B^2 \cong \overline{W}_2$.

Theorem 41 (General version). For a subspace C of S^n , if $C \cong S^{n-1}$, then C separates S^n into precisely two components W_1, W_2 , and $\partial W_1 = C = \partial W_2$.

Remark. For S^3 , the Schöflies theorem is true if C is a smooth manifold. Otherwise there exists counterexamples.

Example. Alexander's horned sphere which is homeomorphic to S^2 separates S^3 into W_1 and W_2 s.t. $\overline{W}_1 \cong B^3$ but W_2 is not simply connected.

In this section, we prove Theorem 39.

Lecture 20
Mon, Nov 8

Theorem 42 (63.1). Assume that

- $X = U \cup V$ s.t. U, V are open, $U \cap V = A \sqcup B$, A, B are open.
- There is a path α in U from a point $a \in A$ to a point $b \in B$.
- There is a path β in V from b to a .

Then,

- (i) The path-homotopy class $[f]$ generates an infinite cyclic subgroup of $\pi_1(X, a)$.
- (ii) If $\pi_1(X, a)$ is itself infinite cyclic, it is generated by $[f]$.
- (iii) Assume there is a path γ in U from a to the point a' of A , and that there is a path δ in V from a' to a . Let g be the loop $g = \gamma * \delta$. Then the subgroups of $\pi_1(X, a)$ generated by $[f]$ and $[g]$ intersect in the identity element alone.

Chapter 11

Seifert–van Kampen theorem

Note. This doesn't follow the book very well.

Definition 34. A **free group** on a set X consists of a group F_X and a map $i: X \rightarrow F_X$ such that the following holds: For any group G and any map $f: X \rightarrow G$, there exists a unique morphism of groups $\phi: F_X \rightarrow G$ such that

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow f & \downarrow \exists! \phi \\ & & G \end{array}$$

Note. The free group of a set is unique. Suppose $i: X \rightarrow F_X$ and $j: X \rightarrow F'_X$ are free groups.

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow j & \downarrow \exists \phi \\ & & F'_X \end{array} \quad \begin{array}{ccc} X & \xrightarrow{j} & F'_X \\ & \searrow i & \downarrow \exists \psi \\ & & F_X \end{array}$$

Then

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow i & \downarrow \psi \circ \phi \\ & & F_X \end{array}$$

Then by uniqueness, $\psi \circ \phi$ is 1_{F_X} , and likewise for $\phi \circ \psi$.

Note. The free group on a set always exists. You can construct it using “irreducible words”.

Example. Consider $X = \{a, b\}$. An example of a word is $aaba^{-1}baa^{-1}bbb^{-1}a$. This is not a irreducible word. The reduced form is $aaba^{-1}bba = a^2ba^{-1}b^2a$. Then F_X is the set of irreducible words.

Example. If $X = \{a\}$, then $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. Exercise: check that \mathbb{Z} satisfies the universal property.

Example. If $X = \emptyset$, then $F_X = 1$.

Definition 35. Let G_i with $i \in I$, be a set of groups. Then the **free product** of these groups denoted by $*_{i \in I} G_i$ is a group G together with morphisms $j_i: G_i \rightarrow G$ such that the following universal property holds: Given any group H and a collection of morphisms $f_i: G_i \rightarrow H$, then there exists a unique morphism $f: G \rightarrow H$, such that for all $i \in I$, the following diagram commutes:

$$\begin{array}{ccc} G_i & \xrightarrow{j_i} & G \\ & \searrow f_i & \downarrow \exists! f \\ & & H \end{array}$$

Note. Again, $*_{i \in I} G_i$ is unique.

Example. Construction is similar to the construction of a free group. Let $I = \{1, 2\}$ and $G_1 = G$, $G_2 = H$. Then $G * H$. Elements of $G * H$ are “words” of the form $g_1 h_1 g_2 h_2 g_3$, $g_1 h_1 g_2 h_2$, or $h_1 g_1 h_2 g_2 h_3 g_3$ or $h_1 g_1 h_2$, ... with $g_j \in G$, $h_j \in H$.

Note. $G * H$ is always infinite and nonabelian if $G \neq 1 \neq H$. Even if G, H are very small, for example $\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, t\} * \{1, s\}$. Then $ts \neq st$ and the order of ts is infinite.

Note. $\mathbb{Z} * \mathbb{Z} = F_{a,b}$. In general: $F_X = *_{x \in X} \mathbb{Z}$.

11.70 The Seifert–van Kampen theorem

Theorem 43 (70.1, Seifert–van Kampen theorem). Let $X = U \cup V$ where $U, V, U \cap V$ are open and path connected.^a Let $x_0 \in U \cap V$. For any group H and 2 morphisms $\Phi_1: \pi(U, x_0) \rightarrow H$ and $\Phi_2: \pi(V, x_0) \rightarrow H$ such that $\Phi_1 \circ i_1$ and $\Phi_2 \circ i_2$, there exists exactly one $\Phi: \pi(X, x_0) \rightarrow H$ making the diagram commute

$$\begin{array}{ccccc}
 & & \pi(U, x_0) & & \\
 & \nearrow i_1 & \downarrow j_1 & \searrow \Phi_1 & \\
 \pi(U \cap V, x_0) & \xrightarrow{i} & \pi(x, x_0) & \dashrightarrow^{\Phi} & H \\
 & \searrow i_2 & \uparrow j_2 & \nearrow \Phi_2 & \\
 & & \pi(V, x_0) & &
 \end{array}$$

i_1, i_2, i, j_1, j_2 are induced by inclusions.

^aNote that U, V should also be path connected!

Theorem 44 (70.2, Seifert–van Kampen theorem (classical version)). Assume the hypotheses of the Theorem 43. Let $j: \pi(U, x_0) * \pi(V, x_0) \rightarrow \pi(X, x_0)$ (induced by j_1 and j_2). On elements of $\pi(U, x_0)$ it acts like j_1 , on elements of $\pi(V, x_0)$ it acts like j_2 .

$$\begin{array}{ccc}
 G_1 & & \\
 \downarrow & \searrow f_1 & \\
 G_1 * G_2 & \dashrightarrow^f & H \\
 \uparrow & \nearrow f_2 & \\
 G_2 & &
 \end{array}$$

Then j is surjective^a and the kernel of j is the normal subgroup of $\pi(U, x_0) * \pi(V, x_0)$ generated by all elements of the form $i_1(g)^{-1}i_2(g)$, where $g \in \pi(U \cap V, x_0)$.

^aThis is the only place where algebraic topology is used. We've proved this last week. The groups U and V generate the whole group. The rest of this theorem follows from the previous theorem.

Proof. • j is surjective. (later)

- Let N be the normal subgroup generated by $i_1(g)^{-1}i_2(g)$. Then we claim that $N \subset \ker(j)$. This means we have to show that $i_1(g)^{-1}i_2(g) \in \ker(j)$. $j(i_1(g)) = j_1(i_1(g))$ by definition of j . Looking at the diagram, we find that $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$. Therefore $j(i_1(g)^{-1}i_2(g)) = 1$, which proves that elements of the form $i_1(g)^{-1}i_2(g)$ are in the kernel.

- Since $N \subset \ker j$, there is an induced morphism

$$\begin{aligned} k: (\pi_1(U, x_0) * \pi_1((V, x_0)))/N &\longrightarrow \pi_1(X, x_0) \\ gN &\longmapsto j(g). \end{aligned}$$

To prove that $N = \ker j$, we have to show that k is injective. Because this would mean that we've divided out the whole kernel of j .

Now we're ready to use the previous theorem. Let $H = (\pi(U) * \pi(V))/N$. Repeating the diagram:

$$\begin{array}{ccccc}
& & \pi(U, x_0) & & \\
& \nearrow i_1 & \downarrow j_1 & \searrow \Phi_1 & \\
\pi(U \cap V, x_0) & \xrightarrow{i} & \pi(x, x_0) & \xrightleftharpoons[\leftarrow k]{\rightarrow \Phi} & H \\
& \searrow i_2 & \uparrow j_2 & \nearrow \Phi_2 & \\
& & \pi(V, x_0) & &
\end{array}$$

Now, we define $\Phi_1: \pi(U, x_0) \rightarrow H$ which is given by $g \mapsto gN$, and $\Phi_2: \pi(V, x_0) \rightarrow H$ given by $g \mapsto gN$. For the theorem to work, we needed that $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$. This is indeed the case: let $g \in \pi(U \cap V)$. Then $\Phi_1(i_1(g)) = i_1(g)N$ and $\Phi_2(i_2(g)) = i_2(g)N$ and $i_1(g)N = i_2(g)N$ because $i_1(g)^{-1}i_2(g) \in N$.

The conditions of the previous theorem are satisfied, so there exists a Φ such that the diagram commutes.

Note that we also have $k: H \rightarrow \pi(X)$. We claim that $\Phi \circ k = 1_H$, which would mean that k is injective, concluding the proof. It's enough to prove that $\Phi \circ k(gN) = gN$ for all $g \in \pi(U)$ and $\forall g \in \pi(V)$, as these g 's generate the product of the groups. If a map is the identity on the generators, it is the identity on the whole group.

Let $g \in \pi(U)$. Then $(\Phi \circ k)(gN) = \Phi(k(gN)) = \Phi(j(g))$, per definition of k . On elements of $\pi(U)$, $j \equiv j_1$, so $\Phi(j(g)) = \Phi(j_1(g)) = \Phi_1(g)$ by looking at the diagram, and per definition of Φ_1 , we find that $\Phi(g) = gN$. So we've proven that $(\Phi \circ k)(gN) = gN$. This means that N is the kernel, so we've proved that k is an isomorphism.

Corollary 7. Suppose $U \cap V$ is simply connected, so $\pi_1(U \cap V, x_0)$ is the trivial group. In this case $N = \ker j = 1$, hence $\pi(U, x_0) * \pi(V, x_0) \rightarrow \pi(X, x_0)$ is an isomorphism.

Corollary 8. Suppose U is simply connected. Then $\pi(X, x_0) \cong \pi(V, x_0)/N$ where N is the normal subgroup generated by the image of $i_2: \pi(U \cap V) \rightarrow \pi(V, x_0)$.

Chapter 13

Classification of covering spaces

13.79 Equivalence of covering spaces

Definition 36. Let (E, p) and (E', p') be two coverings of a space B . An *equivalence* between (E, p) and (E', p') is a homeomorphism $h: E \rightarrow E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ & \searrow p & \swarrow p' \\ & B & \end{array}$$

is commutative. $p' \circ h = p$.

Lemma 20 (79.1, General lifting lemma). Let $p: E \rightarrow B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.^a Let $f: Y \rightarrow B$, $y_0 \in Y$, $b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f}: Y \rightarrow E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{array}{ccc} & & (E, e_0) \\ & \nearrow \tilde{f} & \downarrow p \\ (Y, y_0) & \xrightarrow{f} & (B, b_0) \end{array}$$

iff $f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$. If \tilde{f} exists then it is unique.

^aFrom now on, all spaces are locally path connected: Every neighborhood contains an open that is path connected.

Proof. Suppose \tilde{f} exists. Then $p \circ \tilde{f} = f$, so $(p \circ \tilde{f})_*\pi(Y, y_0) = \pi(Y, y_0)$. The left hand side is of course $p_*(\tilde{f}_*(\pi(Y, y_0))) \subset p_*(\pi(E, e_0))$, so $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$.

Conversely, we'll show the uniqueness first. Suppose \tilde{f} exists.
 $p \circ (\tilde{f} \circ \alpha) = f \circ \alpha$, so $\tilde{f} \circ \alpha$ is the unique lift of $f \circ \alpha$ starting at e_0 .
Hence $\tilde{f}(y)$ the endpoint of the unique lift of $f \circ \alpha$ to E starting at e_0 .

This also shows how you can define \tilde{f} : choose a path α from y_0 to y .
Lift $f \circ \alpha$ to a path starting at e_0 . Define $\tilde{f}(y)$ = the end point of this lift.
Is this well defined? Is \tilde{f} continuous?

Well defined As $[\alpha] * [\bar{\beta}] \in \pi(Y, y_0)$,

$$f_*([\alpha] * [\bar{\beta}]) = ([f \circ \alpha] * [f \circ \bar{\beta}]) \in f_*(\pi_1(Y, y_0))$$

which is by assumption a subgroup of $p_*(\pi(E, e_0)) = H$.

And now, by Lemma 10.(iii), a loop in the base space lifts to a loop in E if the loop is in H . This lift is of course just $\gamma * \delta$, so the end points in the drawing should be connected! this means that $\bar{\delta}$ is the lift of $f \circ \beta$ starting at e_0 , so the endpoint of the lift of $f \circ \beta$ is the endpoint of the lift of $f \circ \alpha$. Therefore $\tilde{f}(y)$ is well defined.

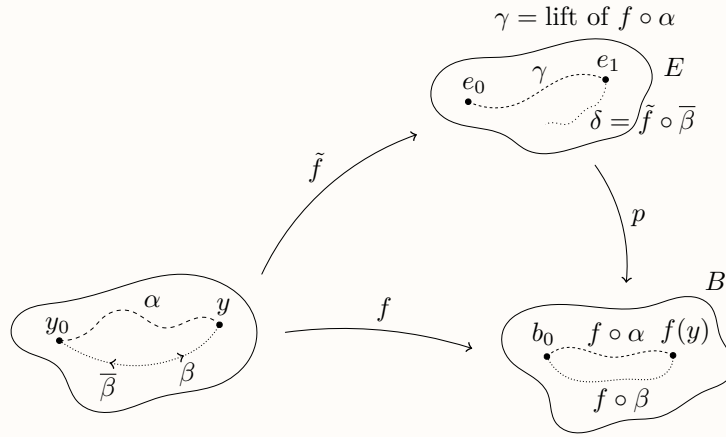


Figure 13.1: well defined general lifting lemma

Continuity We prove that \tilde{f} is continuous.

- Choose a neighborhood of $\tilde{f}(y_1)$, say N .
- Take U , a path connected open neighborhood of $f(y_1)$ which is evenly covered, such that the slice $p^{-1}(U)$ containing $\tilde{f}(y_1)$ is completely contained in N .

Can we do this? The inverse image of U is a pile of pancakes. One of these pancakes contains $\tilde{f}(y_1)$. Then, because N is a neighborhood of $\tilde{f}(y_1)$, we can shrink the pancake such that it is contained in N .

- Choose a path connected open which contains y_1 such that $f(W) \subset U$. We can do this because of continuity of f .

- Take $y \in W$. Take a path β in W from y_1 to y . (Here we use that W is path connected.) Now consider $p|_V$ and defined Then $\alpha * \beta$ is path fro y_0 to y , $f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$. Then $\tilde{f} \circ \alpha * (p^{-1}|_V \circ f \circ \beta)$ is the lift of $f \circ (\alpha * \beta)$ starting at y_0 . So by definition of \tilde{f} , we have that $\tilde{f}(y)$ is the endpoint of that lift, which belongs to $V \subset N$. This means that $\tilde{f}(W) \subset N$, which proves continuity.

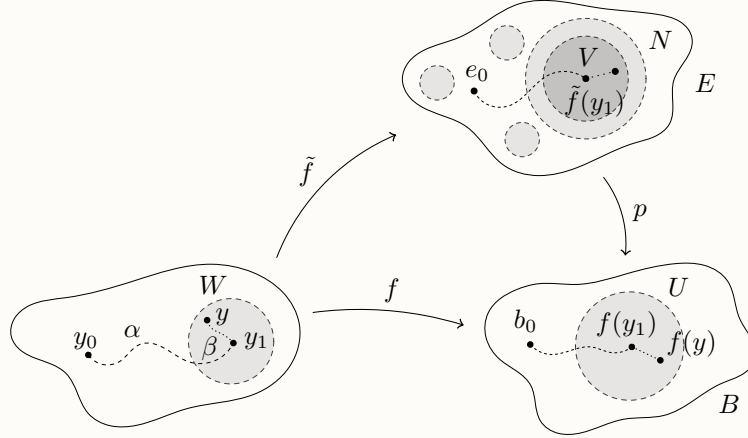


Figure 13.2: Proof of the continuity of the general lifting lemma

□

Example. Take $Y = [0, 1]$. Then f is a path, then we showed that every map can be lifted. And indeed, the condition holds: $f_*(\pi(Y, y_0)) = 1$, the trivial group, which is a subgroup of all groups.

Lemma 21 (General lifting lemma, short statement). Short statement:

$$\begin{array}{ccc} & (E, e_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (Y, y_0) & \xrightarrow{f} & (B, b_0) \end{array}$$

$$\exists! \tilde{f} \iff f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0).$$

Theorem 45 (79.2). Let $p: (E, e_0) \rightarrow (B, b_0)$ and $p': (E', e'_0) \rightarrow (B, b_0)$ be coverings, and $H_0 = p_*\pi(E, e_0)$ and $H'_0 = p'_*\pi(E', e'_0) \leq \pi(B, b_0)$. Then there exists an equivalence $h: (E, p) \rightarrow (E', p')$ with $h(e_0) = e'_0$ iff $H_0 = H'_0$. Not isomorphic, but really the same as a subgroup of $\pi(B, b_0)$. In that case, h is unique.

Proof. \Rightarrow Suppose h exists. Then

$$\begin{array}{ccc} (E, e_0) & \xrightarrow{h} & (E', e'_0) \\ & \searrow p & \downarrow p' \\ & & (B, b_0) \end{array} .$$

Therefore $p_*\pi(E, e_0) = p'_*(h_*\pi(E, e_0))$. Since h is a homeomorphism, it induces an isomorphism, so $p'_*(h_*\pi(E, e_0)) = p'_*(\pi(E', e'_0))$.

\Leftarrow

$$\begin{array}{ccc} & & (E', e'_0) \\ & \nearrow k & \downarrow p' \\ (E, e_0) & \xrightarrow{p} & (B, b_0) \end{array} .$$

By the previous lemma, there exists a unique k iff $p_*\pi(E, e_0) \subset p'_*\pi(E', e'_0)$ or equivalently $H_0 \subset H'_0$, which is the case. Reversing the roles, we get

$$\begin{array}{ccc} & & (E, e_0) \\ & \nearrow l & \downarrow p \\ (E', e'_0) & \xrightarrow{p'} & (B, b_0) \end{array}$$

for the same reasoning, l exists. Now, composing the diagrams

$$\begin{array}{ccc} & & (E, e_0) \\ & \nearrow l \circ k & \downarrow p \\ (E, e_0) & \xrightarrow{p} & (B, b_0) \end{array} \quad \begin{array}{ccc} & & (E', e'_0) \\ & \nearrow k \circ l & \downarrow p' \\ (E', e'_0) & \xrightarrow{p'} & (B, b_0) \end{array} .$$

But placing the identity in place of $l \circ k$ or $k \circ l$, this diagram also commutes! By unicity, we have that $l \circ k = 1_E$ and $k \circ l = 1_{E'}$. Therefore, k and l are homeomorphism $k(e_0) = e'_0$.

Uniqueness is trivial, because of the general lifting theorem. \square

Note that this doesn't answer the question 'is there a equivalence between two coverings', it only answers the question 'is there an equivalence between two coverings mapping $e_0 \rightarrow e'_0$ '. So now, we seek to understand the dependence of H_0 on the base point.

Lemma 22 (79.3). Let (E, p) be a covering of B . Let $e_0, e_1 \in p^{-1}(b_0)$. Let $H_0 = p_*\pi(E, e_0)$, $H_1 = p_*\pi(E, e_1)$.

- Let γ be a path from e_0 to e_1 and let $p \circ \gamma = \alpha$ be the induced *loop* at b_0 . Then $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$, so H_0 and H_1 are conjugate inside $\pi(B, b_0)$.
- Let H be a subgroup of $\pi(B, b_0)$ which is conjugate to H_0 , then there is a point $e \in p^{-1}(b_0)$ such that $H = p_*\pi(E, e)$.

So a covering space induces a conjugacy class of a subgroup of $\pi(B, b_0)$.

Proof. • Let $[h] \in H_1$, so this means that $h = p \circ \tilde{h}$, where \tilde{h} is a loop based at e_1 . Then $(\gamma * \tilde{h}) * \bar{\gamma}$ is a loop based at e_0 . This means that the path class $[p((\gamma * \tilde{h}) * \bar{\gamma})] \in H_0$. This means that $[p \circ \gamma] * [h] * [p \circ \bar{\gamma}] \in H_0$, or $[\alpha] * [h] * [\alpha]^{-1} \in H_0$. So we showed that if we take any element of H_1 and we conjugate it with α , we end up in H_0 , so $[\alpha] * H_1 * [\alpha]^{-1} \subset H_0$.

For the other inclusion, consider $\bar{\gamma}$ going from $e_1 \rightarrow e_0$. The same argument shows that $[\alpha]^{-1} * H_0 * [\alpha] \subset H_1$, or $H_0 \subset [\alpha] * H_1 * [\alpha]^{-1}$. This proves that $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$.

- Take $H = [\beta] * H_0 * [\beta]^{-1}$ for some $[\beta] \in \pi(B, b_0)$. So $H_0 = [\beta]^{-1} * H * [\beta]$. Take $\alpha = \bar{\beta}$. Then $H_0 = [\alpha] * H * [\alpha]^{-1}$, where α, β are loops based at b_0 . Let γ be the unique lift of α starting at e_0 . Take $e = \gamma(1)$, the end point of γ . (So $p(e) = b_0$) From the first bullet point, it follows that $p_*\pi(E, e_0) = H'$ satisfies $H_0 = [\alpha] * H' * [\alpha]^{-1}$. So we have both $H_0 = [\alpha] * H * [\alpha]^{-1} = H_0 = [\alpha] * H' * [\alpha]^{-1}$. This implies that $H' = H$.

□

This completely answers the question: when are two covering spaces equivalent?

Theorem 46 (79.4). Let (E, p) and (E', p') be two coverings, $e_0 \in E$, $e'_0 \in E'$ with $p(e_0) = p'(e'_0) = b_0$. Let $H_0 = p_*\pi(E, e_0)$, $H'_0 = p'_*\pi(E', e'_0)$. Then (E, p) and (E', p') are equivalent iff H_0 and H'_0 are conjugate inside $\pi(B, b_0)$.

Question: can we reach every possible subgroup? Answer: yes, in some conditions.

13.80 The universal covering space

Definition 37. Let B be a path connected and locally path connected space. A covering space (E, p) of B is called a **universal covering space** if E is simply connected, so $\pi(E, e_0) = 1$.

Remark. Any two universal coverings are equivalent. Even more, we can choose any base point we want.

$$\begin{array}{ccc} (E, e_0) & \xrightarrow{h(e_0)=e'_0} & (E', e'_0) \\ & \searrow p & \downarrow p' \\ & & (B, b_0) \end{array}$$

h exists because the groups of (E, e_0) and (E', e'_0) are trivial.

Lemma 23 (80.2). Suppose

$$\begin{array}{ccc} X & & \\ \downarrow p & \searrow q & \\ & Y & \\ & \swarrow r & \\ Z & & \end{array}$$

If p and r are covering maps, then also q is a covering map. (Also: if q and p are covering maps, then so is r . Not the case for $q, r \Rightarrow p$!)

Proof. • q is a surjective map. Choose a base point in x_0 , and call $y_0 = q(x_0)$, $z_0 = r(y_0)$. Certainly, y_0 lies in the image of q . Now, take $y \in Y$, and choose a path $\tilde{\alpha}$ from y_0 to y . Now, denote by α the projection of $\tilde{\alpha}$, a path from z_0 to $r(y)$. Let $\tilde{\alpha}$ be the unique lift of α to X starting at x_0 . This is defined as we assume that p is a covering map. Then $q \circ \tilde{\alpha}$ is a path starting at $q(\tilde{\alpha}(0)) = q(x_0) = y_0$. Moreover, $q \circ \tilde{\alpha}$ is a lift of $\alpha = r \circ \tilde{\alpha}$ to Y . Indeed consider the projection, $r \circ q \circ \tilde{\alpha} = p \circ \tilde{\alpha} = \alpha$. Of course, $\tilde{\alpha}$ is also a lift from α starting at y_0 . Since r is assumed to be a covering, and lifts of paths are unique, we get that $q \circ \tilde{\alpha} = \tilde{\alpha}$, so the end points are the same: $q(\tilde{\alpha}(1)) = \tilde{\alpha}(1) = y$, so y lies in the image of q .

The only fact we've used is that q is a continuous map, so that $q \circ \tilde{\alpha}$ is again a path.

- Now we show that every point of y has a neighborhood that is evenly covered. Choose $y \in Y$ and project it down to Z . $r(y)$ has a neighbourhood U that is evenly covered by p , and also by r . Now we can shrink it so that is evenly covered by both covering maps. We can also choose it to be path connected.

So $p^{-1}(U) = \bigcup_{\alpha \in I} U_\alpha$, and $r^{-1}(U) = \bigcup_{\beta \in J} V_\beta$. Let V be the slice containing Y . Then we claim that V will be evenly covered by U .

Consider a U_α . Then $q(U_\alpha)$ is connected and contained in $\bigcup_{\beta \in J} V_\beta$, but all these V_β are disjoint, so there is exactly one V_β such that $q(U_\alpha) \subset V_\beta$.

Now, let $I' = \{\alpha \mid q(U_\alpha) \subset V\}$. For any $\alpha \in I'$, we have the diagram

$$\begin{array}{ccc}
 U_\alpha & & \\
 \downarrow p & \searrow q & \\
 & & V \\
 & \swarrow r & \\
 U & &
 \end{array}$$

As r and p is a homeomorphism, q is also a homeomorphism. Hence $q^{-1}(V) = \bigcup_{\alpha \in I'} U_\alpha$, and $q|_{U_\alpha}: U_\alpha \rightarrow V$ is a homeomorphism.

This means that q is a covering projection. \square

Why is this useful? Because now we can say why the universal covering space is a universal covering space.

Theorem 47 (80.3). Let (E, p) be a universal covering of B . Let (X, r) be a another covering of B . Then there exists a map $q: E \rightarrow X$ such that $r \circ q = p$ and q is a covering map.

$$\begin{array}{ccc}
 E & & \\
 \downarrow p & \searrow q & \\
 & & X \\
 & \swarrow r & \\
 B & &
 \end{array}$$

Every covering space is itself covered by the universal covering space, if it exists.

Proof. Drawing the diagram differently,

$$\begin{array}{ccc}
 & & X \\
 & \nearrow q & \downarrow r \\
 E & \xrightarrow{p} & B
 \end{array}$$

Choose e_0, x_0 mapped to $b_0 \in B$. Then $\pi(E, e_0) = 1 \subset r_*\pi(X, x_0)$. Then there exists a map q by the general lifting lemma. So q makes the diagram commutative. By the previous result, q is a covering map. \square

13.81 Covering transformations

Definition 38. Let (E, p) be a covering of B . We define

$$C(E, p, B) = \{h: E \rightarrow E \mid h \text{ is an equivalence of covering spaces}\}.$$

Elements of this set are homeomorphism h such that $p \circ h = p$. The composition of two such elements is again such an elements, same for inverse. This means that C is a group, the **group of covering transformations**, also called Deck-transformations.

Example. Consider the covering space $\mathbb{R} \rightarrow S^1$ defined by $t \mapsto e^{2\pi i t}$. For

any $z \in \mathbb{Z}$, there is a map $h_z: \mathbb{R} \rightarrow \mathbb{R}$ given by $r \mapsto r + z$, which is a covering transformation. Indeed $e^{2\pi i t} = e^{2\pi i(t+z)}$. Claim: these are the only covering transformations. Conclusion: $C(\mathbb{R}, p, S^1) = (\mathbb{Z}, +)$.

Proof. Suppose $h: \mathbb{R} \rightarrow \mathbb{R}$ is another covering transformation. We certainly have $h(0) = z$ for some $z \in \mathbb{Z}$. Therefore, $h(0) = h_z(0)$, from this follows immediately that $h \equiv h_z$.

Why? ‘If two covering transformations agree in one point, they agree everywhere.’ Indeed, $h_1, h_2 \in C(E, p, B)$ and $h_1(e) = h_2(e) \Rightarrow h_1 \equiv h_2$, because

$$\begin{array}{ccc} & & E \\ & \nearrow h_1 \text{ and } h_2 & \downarrow \\ E & \xrightarrow{p} & B \end{array}$$

and, h_1 and h_2 are both lifts of p and there is a unique lift when fixing the base point, so h_1 and h_2 agree. \diamond

Goal: what is the structure of the group $C(E, p, B)$ in terms of fundamental groups? Let (E, p) be a covering of B . $p(e_0) = b_0$, $H_0 = p_*\pi(E, e_0)$. Remember:

$$\begin{aligned} \Phi: \pi(B, b_0)/H_0 &\longrightarrow p^{-1}(b_0) \\ H_0 * [\alpha] &\longmapsto \tilde{\alpha}(t) \end{aligned}$$

is a bijection, where $\tilde{\alpha}$ is the unique lift of α starting at e_0 .

Now, consider $\psi: C(E, p, B) \rightarrow p^{-1}(b_0)$ given by $h \mapsto h(e_0)$. ψ is injective. Reason: same as before, if they agree on one point, these are the same. In general ψ will not be surjective.

Lemma 24 (81.1). $\text{Im } \Phi(N_{\pi(B, b_0)}(H_0)/H_0)$, where

$$N_{\pi(B, b_0)}(H_0) = \{[\alpha] \in \pi(B, b_0) \mid [\alpha] * H_0 * [\alpha]^{-1} = H_0\},$$

which is the largest subgroup of $\pi(B, b_0)$ in which H_0 is normal.

Proof. Consider $H_0 * [\alpha]$.^a Then $\Phi(H_0 * [\alpha]) = \tilde{\alpha}(1)$, where $\tilde{\alpha}$ is the lift of α starting at e_0 . Let's denote $\tilde{\alpha}(1) = e_1$. Question: which of these elements lie in the image of ψ .

e_1 lies in the image of ψ iff there exists a covering transformation h sending e_0 to e_1 , which is equivalent to $H_0 = H_1 = p_*\pi(E, e_1)$. On the other hand, we also know that $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$. Conclusion: e_1 lies in the image of ψ iff $H_0 = [\alpha] * H_0 * [\alpha]^{-1}$, iff $[\alpha] \in N_{\pi(B, b_0)}(H_0)$. \square

^aIn the book, they use $[\alpha] * H_0$. This is not wrong, as for elements in the normalizer, left and right cosets are the same, so writing $[\alpha] * H_0$ is allowed. But in general, we write $H_0 * [\alpha]$.

This means we have the following situation:

$$C(E, p, B) \xrightarrow{\psi} \text{Im } \psi \subset p^{-1}(b_0) \xrightarrow{\Psi^{-1}} \frac{N_{\pi(B, b_0)}(H_0)}{H_0}$$

$$h \longmapsto h(e_0) = e_1 \longmapsto H_0 * [\alpha], \alpha = p \circ \gamma$$

Theorem 48 (81.2). The map $\Phi^{-1} \circ \psi: C(E, p, B) \rightarrow N_{\pi(B, b_0)}(H_0)/H_0$ is an isomorphism of groups.

Proof. Let $h, k \in C(E, p, B)$ with $h(e_0) = e_1$ and $k(e_0) = e_2$. Then

$$\begin{aligned} (\Phi^{-1} \circ \psi)(h) &= H_0 * [\alpha] & \alpha &= p \circ \gamma \\ (\Phi^{-1} \circ \psi)(k) &= H_0 * [\beta] & \beta &= p \circ \delta. \end{aligned}$$

Then call $h(k(e_0)) = h(e_2) = e_3$. Claim: $(h \circ \delta)(0) = h(e_1) = e_1$ and $(h \circ \delta)(1) = h(e_2) = e_3$. Then $\varepsilon := \gamma * (h \circ \delta)$ is a path from e_0 to e_3 . This implies that $(\Phi^{-1} \circ \psi)(h \circ k) = H_0 * [p \circ \varepsilon]$. Then $p \circ \varepsilon = (p \circ \gamma) * (p \circ h \circ \delta) = \alpha * (p \circ \delta) = \alpha * \beta$. This means that $(\Phi^{-1} \circ \psi)(h \circ k) = H_0 * [\alpha * \beta]$.

This shows that this is indeed a morphism. \square

What can we do with this? Some covering spaces are nice: e.g. when the normalizer is the entire group. (Which is the case when Abelian)

Definition 39. A covering space (E, p) of B is called **regular** if H_0 is normal in $\pi(B, b_0)$, where $b_0 \in B$, $p(e_0) = b_0$, $H_0 = p_*\pi(E, e_0)$.

This is the case iff for every $e_1, e_2 \in p^{-1}(b_0)$, there exists an $h \in C(E, p, B)$ such that $h(e_1) = e_2$.

$$\begin{array}{ccc} (E, e_1) & \xrightarrow{h} & (E, e_2) \\ & \searrow p & \downarrow p \\ & & B \end{array}$$

This h exists iff $H_1 = H_2$. $H_1 = p_*\pi(E, e_1)$ and $H_2 = p_*\pi(E, e_2)$. If H_0 is normal, then H_1 and H_2 are the same as H_0 , because they are conjugate to H_0 . So for H_1, H_2 the h exists. Conversely: exercise.

In this case, $(\Phi^{-1} \circ \psi): C(E, p, B) \rightarrow \pi(B, b_0)/H_0$ is an isomorphism. Special case: Let (E, p) be the universal covering space, so $\pi(E, e_0) = 1$, or $H_0 = 1$. In this case, $\Phi^{-1} \circ \psi: C(E, p, B) \rightarrow \pi(B, b_0)$ is an isomorphism.

Example. $C(\mathbb{R}, p, S^1) \cong \mathbb{Z} \cong \pi(S^1, b_0)$.

Chapter 14

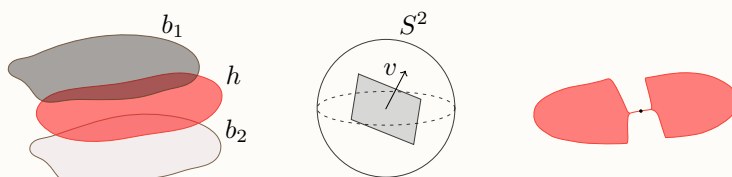
Singular homology

Theorem 49 (Ham sandwich theorem). Suppose you give me two pieces of bread and 1 slice of ham. Then it is possible to divide both the pieces of bread and the slice of ham in equal pieces by 1 straight cut of knife.

Proof. Consider for each $v \in S^2$ a plane $P_v \subset \mathbb{R}^3$. $P_v \perp v$ and P_v cuts the slice of ham exactly in two. We defined the *upper side* of the plane to be the half to which v is pointing to.

If you have some weird ham which you can cut in multiple places in half, then you take the middle of the line segment. This makes it unique.

Note that $P_v = P_{-v}$.



Now, consider

$$f: S^2 \rightarrow \mathbb{R}^2 \text{ given by } v \mapsto (f_1(v), f_2(v)).$$

Then $f_1(v)$ is the volume of bread b_1 above P_v . Then $f_2(v)$ is the volume of bread b_2 above P_v .

Now, you should believe that f_1 and f_2 are continuous. (Proving this precisely needs measure theory etc.) So, now, we can use the Borsak Ulam theorem. So there exists a $v \in S^2$ such that $f(v) = f(-v)$. So $f_1(v) = f_1(-v)$, so volume of bread b_1 above P_v is the volume of bread b_1 below P_v , and similar for f_2 . This proves the Ham sandwich theorem. \square