Topology 2

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Based on lectures by Youngsik Huh in fall 2021

Contents

Chapter 0

Introduction

0.1 Topological spaces

Definition 1. A ***topological space*** is $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ a family of subsets of X, called open sets, such that

- $\emptyset, X \in \mathcal{T}$
- $\bigcup_{i \in I} U_i \in \mathcal{T}$ whenever $U_i \in \mathcal{T}$ for all i
- $\bigcap_{i<n} U_i \in \mathcal{T}$ whenever $U_i \in \mathcal{T}$ for all i.

Let (X, σ) be a topological space.

Definition 2. An open subset that contains $p \in X$ is called a ***(open) neighborhood*** of p.

Definition 3. If $Y \subset X$ then $(Y, \mathcal{T}_Y)$ is a topological space, where

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}.$$

We call $\mathcal{T}_Y$ the ***subspace topology***.

Example. Endowing $\mathbb{R}^2$ with the Euclidean topology, the subspace topology on $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is also Euclidean topology.

Definition 4. An ***equivalence relation*** is a relation $x \sim y$ so that $x \sim x$; if $x \sim y$ then $y \sim x$; and if $x \sim y$ and $y \sim z$, then $x \sim z$. Given an equivalence relation defined on X, $X/_\sim$ is the set of ***equivalence classes***.

Definition 5. Let $\sim$ be an equivalence relation on X. Consider a surjective map $\pi \colon X \to X/_\sim$ given by $x \mapsto [x]$. Then $X/_\sim$ equipped with ***quotient topology*** is a topological space, where the open sets are the subsets $U \subset X/_\sim$ such that $\pi^{-1}(U)$ is open in X.

Example. Let X be the closed unit ball, $\{(x, y) : x^2 + y^2 \leq 1\}$, in $\mathbb{R}^2$ and X^* be the partition of X consisting of all the one-point sets $\{(x, y)\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Then X^* is homeomorphic to $S^2(r)$.

Definition 6. A function $f\colon X_1 \to X_2$ is ***continuous*** if $f^{-1}(U)$ is open in X_1 for every open set $U \subset X_2$.

Definition 7. A topological space X is ***Hausdorff*** if $\forall x, y \in X$, there exists neighborhoods U of x, V of y such that $U \cap V = \emptyset$.

Definition 8. Let $(X, \mathcal{T})$ be a topological space. A ***basis*** for $\mathcal{T}$ is a subset $\mathcal{B} \subset \mathcal{T}$ such that every open set of X is a union of elements of $\mathcal{B}$.

Definition 9. A topological space $(X, \mathcal{T})$ is ***second countable*** if there exists a countable basis.

Example. $\mathbb{R}^n$ is second countable. Indeed $\{B_{\frac{1}{m}}(x) \mid x \in \mathbb{Q}^n,\, m \in \mathbb{N}\}$ is a countable basis for the topology. Here $B_r(x)$ is the open ball with radius r around x.

Definition 10. A ***topological manifold*** M of dimension of m is a second countable, Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^m$.

Remark. 'Locally homeomorphic to $\mathbb{R}^m$' means that $\forall p \in M$, there exists a neighborhood U of p and a homeomorphism $\phi\colon U \to V \subset \mathbb{R}^m$. Recall that homeomorphism means: bijective map that is continuous in both directions.

0.2 What is algebraic topology?

Functor from category of topological spaces to the category of groups.

- Category: set of spaces and morphisms.
- Functor: $X \rightsquigarrow G_X$ and $f\colon X \to Y \rightsquigarrow f_*\colon G_X \to G_Y$ such that
  - $(f \circ g)_* = f_* \circ g_*$
  - $(1_X)_* = 1_{G_X}$

Two systems we'll discuss:

- fundamental groups
- homology groups

Example. Suppose we have a functor. If $G_X \ncong G_Y$, then X and Y are not homeomorphic. If 'shadows' are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f \colon X \to Y$ and $g \colon Y \to X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* \colon G_X \to G_Y$ and $g_* \colon G_Y \to G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* \colon G_X \to G_Y$ is an isomorphism. ◇

0.3 Fundamental group

Pick a base point x_0 and consider it fixed. (The fundamental group will not depend on it. We assume all spaces are path connected) $X \rightsquigarrow \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f \colon I = [0,1] \to X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

Example. Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the 'constant' loop.

Example. Let $X = S^1$ and pick x_0 on the circle. Two options:

- The loop is trivial equivalent to the constant loop
- The loop goes around the circle.
- The loop goes around the circle, twice.
- The loop goes around the circle, clockwise, once
- ...

$\pi(S^1) \cong \mathbb{Z}$ (proof will follow)

The composition of loops is simply pasting them. In the case of the circle, the loop $-1\circ$ the loop 2 is the loop 1.

Suppose $\alpha \colon I \to X$ and $f \colon X \to Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

Theorem 1 (Fixed point theorem of Brouwer)**.** Any continuous map from a rectangle to itself has at least one fixed point.

Proof. Suppose there is no fixed point, so $f(x) \neq x$ for all $x \in B^2$. Then we can construct map $r\colon B^2 \to S_1$ as follows: take the intersection of the boundary and half ray between $f(x)$ and x. If x lies on the boundary, we have the identity map. This map is continuous. Then we have $S^1 \to B^2 \to S^1$, via the inclusion and r. Looking at the fundamental groups:

$$\pi(S^1) = \mathbb{Z} \to \pi(B^2) = 1 \to \pi(S^1) = \mathbb{Z}.$$

The map from $\pi(S^1) \to \pi(S^1)$ is the identity map, but the first map maps everything on 1. $\square$

Chapter 9

Fundamental group

9.51 Homotopy of paths

Definition 11. Let $f, g\colon X \to Y$ be continuous maps. Then a ***homotopy*** between f and g is a continuous map $H\colon X \times I \to Y$ such that

- $H(x,0) = f(x)$, $H(x,1) = g(x)$
- For all $t \in I$, define $f_t\colon X \to Y$ given by $x \mapsto H(x,t)$

We say that f is ***homotopic*** to g and write $f \simeq g$. If g is a constant map, we say that f is ***null homotopic***.

Definition 12. Let $f, g\colon I \to X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H\colon I \times I \to X$ is a ***path homotopy*** between f and g if

- $H(s,0) = f(s)$ and $H(s,1) = g(s)$ (homotopy between maps)
- $H(0,t) = x_0$ and $H(1,t) = x_1$ (start and end points fixed)

We say that f is ***path homotopic*** to g and write $f \simeq_p g$.

Lemma 1. $\simeq$ and $\simeq_p$ are equivalence relations.

Proof.
- Reflective: $F(x,t) = f(x)$
- Symmetric: $G(x,t) = H(x, 1-t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x,t) = \begin{cases} H_1(x,2t) & 0 \le t \le \frac{1}{2} \\ H_2(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

□

Example (Trivial, but important). Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g \colon X \to C$ are homotopic.
- Any two paths $f, g \colon I \to C$ with $f(0) = g(0)$ and $g(1) = f(1)$ are path homopotic.

Choose $H \colon X \times I \to C$ defined by $(x, t) \mapsto H(x, t) = (1 - t)f(x) + tg(x)$.

Product of paths

Let $f \colon I \to X$, $g \colon I \to X$ be paths, $f(1) = g(0)$. Define

$$f * g \colon I \to X \text{ given by } s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g \colon I \to X \,|\, g \simeq_p f\}$.

Theorem 2.

1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
2. Let e_x denote the constant path $e_x \colon I \to X$ given by $s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Let $\overline{f} \colon I \to X$ given by $s \mapsto f(1 - s)$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Proof. First two observations

- Suppose $f \simeq_p g$ via homotopy H, $f, g \colon I \to X$. Let $k \colon X \to Y$. Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.
- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

2. Take $e_0 \colon I \to I$ given by $s \mapsto 0$. Take $i \colon I \to I$ given by $s \mapsto s$. Then $e_0 * i$ is a path from 0 to $1 \in I$. The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 * \simeq i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

3. Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

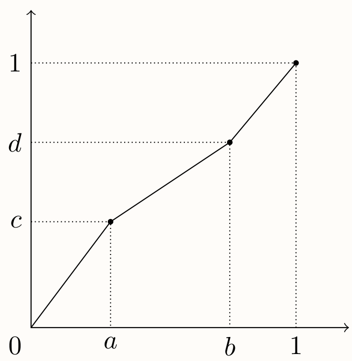
$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \overline{f} &\simeq_p e_{x_0} \\ [f] * [\overline{f}] &= [e_{x_0}]. \end{aligned}$$

1. Remark: Only defined if $f(1) = g(0)$, $g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

   Assume the product is defined. Suppose $[a, b]$, $[c, d]$ are intervals in $\mathbb{R}$. Then there is a unique positive (positive slope), linear map from $[a, b] \to [c, d]$. For any $a, b \in [0, 1)$ with $0 < a < b < 1$, we define a path

$$\begin{aligned} k_{a,b} \colon [0, 1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 0] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

   Then $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$.
   Let γ be that path $\gamma \colon I \to I$ with the following graphs:



   Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

   which is what we wanted to show.

□

9.52 Fundamental group

Definition 13. Let X be a space and $x_0 \in X$, then the ***fundamental group*** of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f\colon I \to X,\ f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\overline{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

Theorem 3 (52.1)**.** Let X be a space, $x_0, x_1 \in X$ and $\alpha\colon I \to X$ a path from x_0 to x_1. Then

$$\begin{aligned} \hat{\alpha}\colon \pi(X, x_0) &\longrightarrow \pi(x, x_1) \\ [f] &\longmapsto [\overline{\alpha}] * [f] * [\alpha]. \end{aligned}$$

is an isomorphisms of groups. Note however that this isomorphism depends on α.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned} \hat{\alpha}([f] * [g]) &= [\overline{\alpha}] * [f] * [g] * [\alpha] \\ &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}[f] * \hat{\alpha}[g]. \end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic.
□

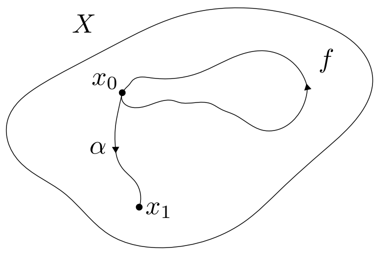


Figure 9.1: construction of the group homeomorphism

Remark. If : $(x, x_0) \to (Y, y_0)$ is a map of pointed topology spaces ($f\colon X \to$

Y continuous and $f(x_0) = y_0$). Then

$$f_* \colon \pi(X, x_0) \to \pi(Y, y_0) \text{ given by } [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two 'rules' discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \quad (1_X)_* = 1_{\pi(X,x_0)}.$$

Definition 14. Let X be a topological space, then X is ***simply connected*** if X is path connected and $\pi_1(X, x_0) = 0$ for some $x_0 \in X$.

Remark. If trivial for one base point, it's trivial for all base points.

Example. Any convex subset $C \subset \mathbb{R}^n$ is simply connected.

Example (Wrong proof of $\pi(S^2)$ being trivial)**.** Let f be a path from $[0, 1] \to S^2$. Then pick $y_0 \in \mathrm{Im}(f)$. Then $S^2 \setminus \{y_0\} \approx \mathbb{R}^2$. Then use $\mathbb{R}^2$.

This is wrong because we cannot always find $y_0 \in \mathrm{Im}(f)$. Space filling loops! We'll see the correct proof later on.

Lemma 2 (52.3)**.** Suppose X is simply connected and $\alpha, \beta \colon I \to X$ two paths with same start and end points. Then $\alpha \simeq_p \beta$.

Proof. Simply connected implies loops are homotopic? Consider $\alpha * \overline{\beta} \simeq_p e_{x_0}$, since the space is imply connected.

$$([\alpha] * [\overline{\beta}]) * [\beta] = [e_{x_0}] * [\beta] = [\beta]$$
$$[\alpha] * ([\overline{\beta}] * [\beta]) = [\alpha] * [e_{x_0}] = [\alpha].$$

And therefore $\alpha \simeq_p \beta$. (Note: make sure end and start point matchs when using $*$) $\square$

9.53 Covering spaces

Definition 15. Let $p \colon E \to B$ be continuous surjective map. The open set $U \subset B$ is ***evenly covered*** if $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$ with

- V_α open in E
- $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$
- $p|_{V_\alpha} \colon V_\alpha \to U$ is a homeomorphism.

Remark. If $U' \subset U$, also open and U is evenly covered, then also U'.

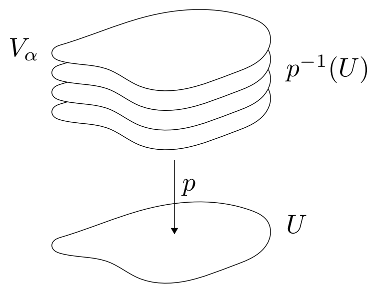


Figure 9.2: evenly covered

Definition 16. Let $p\colon E \to B$ be continuous and surjective. Then p is a ***covering map*** if $\forall b \in B$, $\exists U \subset B$ open, containing b such that U is evenly covered by p. Then (E, p) is called a ***covering space***.

Example. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Take $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. Note that $\mathbb{R}$ is an easier space than S^1, and so will be π_1 (1 vs $\mathbb{Z}$).

Proposition 1. A covering map is always a open map.

Proof. Exercise. □

Proposition 2. For any $b \in B$, $p^{-1}(b)$ is a discrete subset of E. (No accumulation point)

Proof. Indeed for any $\alpha \in I$, $V_\alpha \cap p^{-1}(b)$ is exactly one point. □

Remark. A covering is always local homeomorphism. But there are surjective local homeomorphism which are not covering maps. A covering map is more than a surjective local homeomorphism.

For example, $p\colon \mathbb{R}_0^+ \to S^1$ given by $t \mapsto e^{2\pi i t}$. Consider the inverse image of a neighborhood around 1. When we restrict p to the part around 0, it is no longer a homeomorphism (we don't get the whole neighborhood around one).

Creating new covering spaces out of old ones

- Suppose $p\colon E \to B$ is a covering and $B_0 \subset B$ is a subspace with the subspace topology. Let $E_0 = p^{-1}(B_0)$ and $p_0 = p|_{E_0}$. Then (E_0, p_0) is a covering of B_0.
- Suppose that (E, p) is a covering of B and (E', p') is a covering of B', then $(E \times E', p \times p')$ is a covering of $B \times B'$.

Example. Let $T^2 = S^1 \times S^1$.

- $p\colon \mathbb{R}^2 \to S^1 \times S^1$ given by $(t, s) \mapsto (e^{ait}, e^{bis})$.
- $p'\colon \mathbb{R} \times S^1 \to T^2$ given by $(t, z) \mapsto (e^{ait}, z^n)$.
- $p\colon S^1 \times S^1 \to T^2$ given by $(z_1, z_2) \mapsto (z_1^n, z_2^m)$.

These are the only types of coverings of the torus. We'll prove this later on.

9.54 Fundamental group of the circle

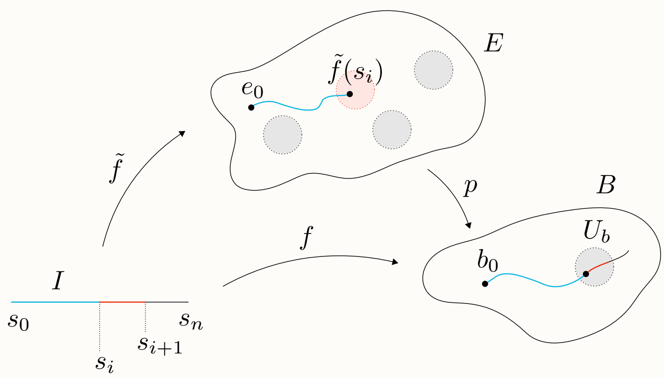
Definition 17. Let $p\colon E \to B$ be a map. If f is a continuous mapping of some space X into B, a ***lifting*** of f is a map $\tilde{f}\colon X \to E$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Given f, when can f be lifted to E? In this section, we'll only consider $X = [0, 1]$, $X = [0, 1]^2$.

Lemma 3 (54.1, Important result)**.** Suppose (E, p) is a covering of B, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$. Suppose that $f\colon I \to B$ is a path starting at b_0. Then there exists a unique lift $\tilde{f}\colon I \to E$ of f with $\tilde{f}(0) = e_0$.

Proof. For any b of B, we choose an open U_b such that U_b is evenly covered by p. Then $\{f^{-1}(U_b) \mid b \in B\}$ is an open cover of I, which is compact. There is a number $\delta > 0$ such that any subset of I of diameter $\leq \delta$ is contained entirely in one of these opens $f^{-1}(U_b)$. (Lebesgue number lemma). Now, we divide the interval into pieces $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $|s_{i+1} - s_i| \leq \delta$. For nay i, we have that $f([s_i, s_{i+1}]) \subset U_b$ for some b.



We now construct $\tilde{f}$ by induction on $[0, s_i]$.

- $\tilde{f}(0) = e_0$
- Assume $\tilde{f}$ has been defined on $[0, s_i]$. Let U be an open such that $f[s_i, s_{i+1}] \subset U_b$.

  There is exactly one slice V_α in $p^{-1}(U_b)$ containing $\tilde{f}(s_i)$. We define $\forall s \in [s_i, s_{i+1}]\colon \tilde{f}(s) = (p|_{V_\alpha})^{-1} \circ f(s)$. By the pasting lemma, $\tilde{f}$ is continuous.
- In this way, we can construct $\tilde{f}$ on the whole of I.

Uniqueness works in exactly the same way, by induction. □

Lemma 4 (54.2)**.** (E, p) is a covering of B, $b_0 \in B$, $e_0 \in E$, with $p(e_0) = b_0$. Suppose $F\colon I \times I \to B$ is a continuous map with $f(0,0) = b_0$, then there is a unique $\tilde{F}\colon I \times I \to E$. Moreover, if F is a path homotopy, then also $\tilde{F}$ is a path homotopy.

Proof. Same as in the one dimensional case. □

Theorem 4 (54.3)**.** Let (E, p) be a covering of B, $b_0 \in B$, $e_0 \in E$ with $p(e_0) = b_0$. Let f, g be two paths in B starting in b_0 s.t. $f \simeq_p g$ (so f and g end at the same point). Let $\tilde{f}, \tilde{g}$ be the unique lifts of f, g starting at e_0. Then $\tilde{f} \simeq_p \tilde{g}$, and so $\tilde{f}(1) = \tilde{g}(1)$.

Proof. $F\colon I \times I \to B$ is a path homotopy between f and g. Then $\tilde{F}\colon I \times I \to E$ with $\tilde{F}(0,0) = e_0$. Then $\tilde{F}$ is a path homotopy, by the previous result, between $\tilde{F}(\cdot, 0)$ and $\tilde{F}(\cdot, 1)$. Note that $p \circ \tilde{F}(t, 0) = F(t, 0) = f(t)$ and $p \circ \tilde{F}(t, 1) = F(t, 1) = g(t)$. By uniqueness $\tilde{F}(\cdot, 0) = \tilde{f}$, $\tilde{F}(\cdot, 1) = \tilde{g}$. □

We've shown that homotopy from below lifts to above. The converse is easy. Now we're ready to discuss the relation between groups and covering spaces.

Definition 18. Let (E, p) be a covering of B. $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then the ***lifting correspondence*** is the map

$$\begin{aligned} \phi\colon \pi(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1) \end{aligned}$$

where $\tilde{f}$ is the unique lift of f, starting at e_0. This is well-defined because $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$. This ϕ depends on the choice of e_0.

Theorem 5 (54.4)**.** It E is path connected, then ϕ is a surjective map. If E is simply connected, then ϕ is a bijective map.

Proof. Suppose E is path connected, and let $e_0, e_1 \in p^{-1}(b_0)$. Consider a path $\tilde{f}\colon I \to E$ with $\tilde{f}(0) = e_0$ and $\tilde{f}(1) = e_1$. This is possible because E is path connected. Let $f = p \circ \tilde{f}\colon I \to B$ with $f(0) = p(e_0) = b_0$ and

$f(1) = p(e_1) = b_0$, so f is a loop based at b_0. So f is a loop at b_0 and its unique lift to E starting at e_0 is $\tilde{f}$. Hence $\phi[f] = \tilde{f}(1) = e_1$, which shows that ϕ is surjective.

Now assume that E is simply connected (group is trivial). Consider $[f], [g] \in \pi(B_0)$ with $\phi[f] = \phi[g]$. This implies $\tilde{f}(1) = \tilde{g}(1)$. These start at e_0. It follows from Lemma 2 that $\tilde{f} \simeq_p \tilde{g}$. □

Example. Take the circle and the real line as covering space. Then $p^{-1}(1) = \mathbb{Z}$. So we know that as a set $\pi(S^1)$ is countable. Therefore, $p \circ \tilde{f} \simeq_p p \circ \tilde{g}$. This implies that $f \simeq_p g$, and therefore $[f] = [g]$.

Theorem 6 (54.5). $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$.

Proof. Take $b_0 = 1$ and $e_0 = 0$ and $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. Then $p^{-1}(b_0) = \mathbb{Z}$. And since, $\mathbb{R}$ is simply connected, we have that $\pi\colon \pi(S, 1) \to \mathbb{Z}$ given by $[f] \mapsto \tilde{f}(1)$ is a bijection.

Now we'll show that it's a morphism. Let $[f]$ and $[g]$ elements of the fundamental group of S^1 and assume that $\phi[f] = \tilde{f}(1) = m$ and $\phi[g] = \tilde{g}(1) = n$.

We're going to prove that $\phi([f] * [g]) = \phi([f]) + \phi([g]) = n + n$. Define $\tilde{\tilde{g}}\colon I \to \mathbb{R}$ given by $t \mapsto \tilde{g}(t) + m$. Then $p \circ \tilde{\tilde{g}} = g$, as $p(s + m) = p(s)$ for all m. Now, look at $\tilde{f} * \tilde{\tilde{g}}$. This is a lift of $p \circ (\tilde{f} * \tilde{\tilde{g}}) = (p \circ \tilde{f}) * (p \circ \tilde{\tilde{g}}) = f * g$, which starts at 0. Hence, $\phi([f] * [g]) = \phi([f * g]) =$ the end point of $\tilde{f} * \tilde{\tilde{g}}$, so $\tilde{\tilde{g}}(1) = \tilde{g}(1) + m = n + m$. □

The following lema makes the fact that the covering space is simpler than the space itself exact.

Lemma 5 (54.6). Let (E, p) be a covering of B, $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then

1. $p_*\colon \pi(E, e_0) \to \pi(B, b_0)$ is a monomorphism (injective).

2. Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence induces a well defined map
$$\begin{aligned} \Phi\colon \pi_1(B, b_0)/H &\longrightarrow p^{-1}(b_0) \\ H * [f] &\longmapsto \phi[f], \end{aligned}$$
so ϕ is constant on right cosets. Dividing by H makes ϕ always bijective, even when E is not simply connected.

3. Let f be a loop based at b_0, then $\tilde{f}$ is a loop at e_0 iff $[f] \in H$.

Proof.

1. Let $\tilde{f}\colon I \to E$ be a loop at e_0 and assume that $p_*[\tilde{f}] = 1$. (Then we'd like to show that f itself is trivial.) This implies $p \circ \tilde{f} \simeq_p e_{b_0}$. This implies that $\tilde{f} \simeq_p \tilde{e}_{b_0} = e_{e_0}$, or $[\tilde{f}] = 1$.

2. We have to prove two things:

Well defined $H * [f] = H[g] \Rightarrow \phi(f) = \phi(g)$.

Assume $[f] \in H * [g]$, or $H * [f] = H * [g]$. This means that $[f] = [h] * [g]$, were $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ at e_0. In other words $[f] = [h * g]$, or $f \simeq_p h * g$. Let $\tilde{f}$ be the unique lift of f starting at e_0. Let $\tilde{g}$ be the unique lift of g starting at e_0. Then $\tilde{h} * \tilde{g}$ (which is allowed, $\tilde{h}$ is a loop) the unique lift of $h * g$ starting at e_0.

$\tilde{f}(1) = \phi(f) = \phi(h * g) = (\tilde{h} * \tilde{g})(1) = \tilde{g}(1) = \phi(g)$. If the cosets are the same, then the end points of the lifts are also the same.

Injective $H * [f] = H * [g] \Leftarrow \phi(f) = \phi(g)$.

The end points of f and g are the same. Now consider $\tilde{h} = \tilde{f} * \overline{\tilde{g}}$. Then $[\tilde{h}] * [\tilde{g}] = [\tilde{f}] * [\overline{\tilde{g}}] * [\tilde{g}] = [\tilde{f}]$. By applying p_*, $[h] * [g] = [f]$.

3. Trivial. Exercise. Apply 2 with the constant path.

□

Remark. $k \colon X \to Y$ induces a morphism k_*, we've proved that earlier. Here we only showed injectiveness.

9.55 Retractions and fixed points

Definition 19. Let $A \subset X$, then A is a ***retract*** of X iff there exists a map $r \colon X \to A$ such that $r|_A = 1|_A$, i.e. $r(a) = a$ for all $a \in A$. The map r is called a ***retraction***.

Chapter 10

Separation theorems in the plane

10.63 Jordan curve theorem

https://en.wikipedia.org/wiki/Jordan_curve_theorem

Chapter 11

Seifert–Van Kampen theorem

https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem

Note. This doesn't follow the book very well.

Definition 20. A ***free group*** on a set X consists of a group F_X and a map $i\colon X \to F_X$ such that the following holds: For any group G and any map $f\colon X \to G$, there exists a unique morphism of groups $\phi\colon F_X \to G$ such that

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"] & F_X \arrow[d, "\exists! \phi"] \\ & G \end{tikzcd}.$$

Note. The free group of a set is unique. Suppose $i\colon X \to F_X$ and $j\colon X \to F'_X$ are free groups.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"] & F_X \arrow[d, "\exists \phi"] \\ & F'_X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "i"] & F'_X \arrow[d, "\exists \psi"] \\ & F_X \end{tikzcd}.$$

Then

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i"] & F_X \arrow[d, "\psi \circ \phi"] \\ & F_X \end{tikzcd}.$$

Then by uniqueness, $\psi \circ \phi$ is 1_{F_X}, and likewise for $\phi \circ \psi$.

Note. The free group on a set always exists. You can construct it using "irreducible words".

Example. Consider $X = \{a, b\}$. An example of a word is $aaba^{-1}baa^{-1}bbb^{-1}a$. This is not a irreducible word. The reduced form is $aaba^{-1}bba = a^2ba^{-1}b^2a$. Then F_X is the set of irreducible words.

Example. If $X = \{a\}$, then $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. Exercise: check that $\mathbb{Z}$ satisfies the universal property.

Example. If $X = \emptyset$, then $F_X = 1$.

Definition 21. Let G_i with $i \in I$, be a set of groups. Then the ***free product*** of these groups denoted by $*_{i\in I} G_i$ is a group G together with morphisms $j_i \colon G_i \to G$ such that the following universal property holds: Given any group H and a collection of morphisms $f_i \colon G_i \to H$, then there exists a unique morphism $f \colon G \to H$, such that for all $i \in I$, the following diagram commutes:

$$\begin{tikzcd} G_i \arrow[r, "j_i"] \arrow[dr, "f_i"'] & G \arrow[d, "\exists! f"] \\ & H \end{tikzcd} .$$

Note. Again, $*_{i\in I} G_i$ is unique.

Example. Construction is similar to the construction of a free group. Let $I = \{1, 2\}$ and $G_1 = G$, $G_2 = H$. Then $G * H$. Elements of $G * H$ are "words" of the form $g_1h_1g_2h_2g_3$, $g_1h_1g_2h_2$, or $h_1g_1h_2g_2h_3g_3$ or $h_1g_1h_2$, $\cdots$ with $g_j \in G$, $h_j \in H$.

Note. $G * H$ is always infinite and nonabelian if $G \neq 1 \neq H$. Even if G, H are very small, for example $\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, t\} * \{1, s\}$. Then $ts \neq st$ and the order of ts is infinite.

Note. $\mathbb{Z} * \mathbb{Z} = F_{a,b}$. In general: $F_X = *_{x\in X}\mathbb{Z}$.

11.70 The Seifert–Van Kampen theorem

Theorem 7 (70.1, Seifert–Van Kampen theorem)**.** Let $X = U \cup V$ where $U, V, U \cap V$ are open and path connected.[a] Let $x_0 \in U \cap V$. For any group H and 2 morphisms $\Phi_1 : \pi(U, x_0) \to H$ and $\Phi_2 : \pi(V, x_0) \to H$ such that $\Phi_1 \circ i_1$ and $\Phi_2 \circ i_2$, there exists exactly one $\Phi : \pi(X, x_0) \to H$ making the diagram commute

$$\begin{tikzcd}
 & \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, "\Phi"] & H \\
 & \pi(V, x_0) \arrow[u, "j_2"] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}.$$

i_1, i_2, i, j_1, j_2 are induced by inclusions.

[a]Note that U, V should also be path connected!

Theorem 8 (70.2, Seifert–Van Kampen theorem (classical version))**.** Assume the hypotheses of the Theorem 7. Let $j : \pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ (induced by j_1 and j_2). On elements of $\pi(U, x_0)$ it acts like j_1, on elements of $\pi(V, x_0)$ it acts like j_2.

$$\begin{tikzcd}
G_1 \arrow[d] \arrow[dr, "f_1"] & \\
G_1 * G_2 \arrow[r, dashed, "f"] & H \\
G_2 \arrow[u] \arrow[ur, "f_2"'] &
\end{tikzcd}.$$

Then j is surjective[a] and the kernel of j is the normal subgroup of $\pi(U, x_0) * \pi(U, x_0)$ generated by all elements of the form $i_1(g)^{-1} i_2(g)$, were $g \in \pi(U \cap V, x_0)$.

[a]This is the only place where algebraic topology is used. We've proved this last week. The groups U and V generate the whole group. The rest of this theorem follows from the previous theorem.

Proof.
- j is surjective. (later)
- Let N be the normal subgroup generated by $i_1(g)^{-1} i_2(g)$. Then we claim that $N \subset \ker(j)$. This means we have to show that $i_1(g)^{-1} i_2(g) \in \ker j$. $j(i_1(g)) = j_1(i_1(g))$ by definition of j. Looking at the diagram, we find that $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$. Therefore $j(i_1(g)^{-1} i_2(g)) = 1$, which proves that elements of the form $i_1(g)^{-1} i_2(g)$ are in the kernel.
- Since $N \subset \ker j$, there is an induced morphism

$$\begin{aligned} k \colon (\pi_1(U, x_0) * \pi_1((V, x_0))/N &\longrightarrow \pi_1(X, x_0) \\ gN &\longmapsto j(g). \end{aligned}$$

To prove that $N = \ker j$, we have to show that k is injective. Because this would mean that we've divided out the whole kernel of j.

Now we're ready to use the previous theorem. Let $H = (\pi(U) * \pi(V))/N$. Repeating the diagram:

$$\begin{tikzcd}
 & \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, shift left, "\Phi"] & H \arrow[l, shift left, "k"] \\
 & \pi(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}.$$

Now, we define $\Phi_1 \colon \pi(U, x_0) \to H$ given by $g \mapsto gN$, and $\Phi_2 \colon \pi(V, x_0) \to H$ given by $g \mapsto gN$. For the theorem to work, we needed that $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$. This is indeed the case: let $g \in \pi(U \cap V)$. Then $\Phi_1(i_1(g)) = i_1(g)N$ and $\Phi_2(i_2(g)) = i_2(g)N$ and $i_1(g)N = i_2(g)N$ because $i_1(g)^{-1}i_2(g) \in N$.

The conditions of the previous theorem are satisfied, so there exists a Φ such that the diagram commutes.

Note that we also have $k \colon H \to \pi(X)$. We claim that $\Phi \circ k = 1_H$, which would mean that k is injective, concluding the proof. It's enough to prove that

□

Corollary 8.1. Suppose $U \cap V$ is simply connected, so $\pi_1(U \cap V, x_0)$ is the trivial group. In this case $N = \ker j = 1$, hence $\pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ is an isomorphism.

Corollary 8.2. Suppose U is simply connected. Then $\pi(X, x_0) \cong \pi(V, x_0)/N$ where N is the normal subgroup generated by the image of $i_2 \colon \pi(U \cap V) \to \pi(V, x_0)$.

Example. Let X be the figure 8 space.

Chapter 12

Classification of surfaces

Chapter 13

Classification of covering spaces

Lemma 6 (79.1, General lifting lemma). Let $p\colon E \to B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.[a] Let $f\colon Y \to B$, $y_0 \in Y$, $b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f}\colon Y \to E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

iff $f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$. If $\tilde{f}$ exists then it is unique.

[a]From now on, all spaces are locally path connected: Every neighborhood contains an open that is path connected.

Proof. Suppose $\tilde{f}$ exists. Then $p \circ \tilde{f} = f$, so $(p \circ \tilde{f})_*\pi(Y, y_0) = \pi(Y, y_0)$. The left hand side is of course $p_*(\tilde{f}_*(\pi(Y, y_0)) \subset p_*(\pi(E, e_0))$, so $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$.

Conversely, we'll show the uniqueness first. Suppose $\tilde{f}$ exists.

$p \circ (\tilde{f} \circ \alpha) = f \circ \alpha$, so $\tilde{f} \circ \alpha$ is the unique lift of $f \circ \alpha$ starting at e_0. Hence $f(y)$ the endpoint of the unique lift of $f \circ \alpha$ to E starting at e_0.

This also shows how you can define $\tilde{f}$: choose a path α from y_0 to y. Lift $f \circ \alpha$ to a path starting at e_0. Define $\tilde{f}(y) =$ the end point of this lift. Is this well defined? Is $\tilde{f}$ continuous?

Well defined As $[\alpha] * [\overline{\beta}] \in \pi(Y, y_0)$,

$$f_*([\alpha] * [\overline{\beta}]) = ([f \circ \alpha] * [f \circ \overline{\beta}) \in f_*(\pi_1(Y, y_0))$$

which is by assumption a subgroup of $p_*(\pi(E, e_0)) = H$.

And now, by Lemma 3, a loop in the base space lifts to a loop in E if the loop is in H. This lift is of course just $\gamma * \delta$, so the end points in the drawing should be connected! this means that $\overline{\delta}$ is the lift of $f \circ \beta$ starting at e_0, so the endpoint of the lift of $f \circ \beta$ is the endpoint of the lift of $f \circ \alpha$. Therefore $\tilde{f}(y)$ is well defined.

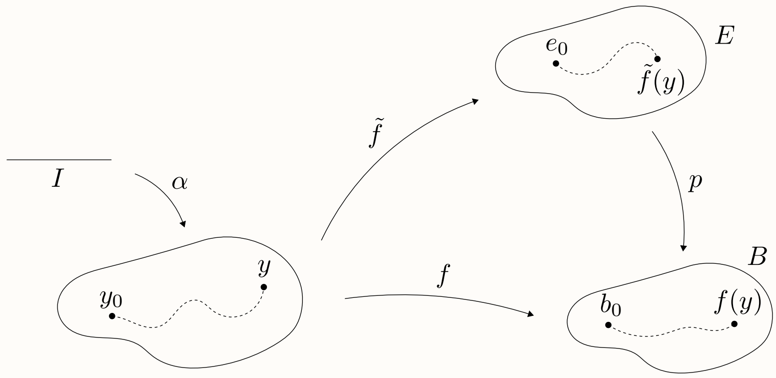


Figure 13.1: General lifting lemma

Continuity We prove that $\tilde{f}$ is continuous.

- Choose a neighborhood of $\tilde{f}(y_1)$, say N.
- Take U, a path connected open neighborhood of $f(y_1)$ which is evenly covered, such that the slice $p^{-1}(U)$ containing $\tilde{f}(y_1)$ is completely contained in N.

  Can we do this? The inverse image of U is a pile of pancakes. One of these pancakes contains $\tilde{f}(y_1)$. Then, because N is a neighborhood of $\tilde{f}(y_1)$, we can shrink the pancake such that it is contained in N.
- Choose a path connected open which contains y_1 such that $f(W) \subset U$. We can do this because of continuity of f.
- Take $y \in W$. Take a path β in W from y_1 to y. (Here we use that W is path connected.) Now consider $p|_V$ and defined

□

Example. Take $Y = [0, 1]$. Then f is a path, then we showed that every map can be lifted. And indeed, the condition holds: $f_*(\pi(Y, y_0)) = 1$, the trivial group, which is a subgroup of all groups.

Lemma 7 (General lifting lemma, short statement)**.** Short statement:

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

$\exists! \tilde{f} \iff f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$.

Definition 22. Let (E, p) and (E', p') be two coverings of a space B. An ***equivalence*** between (E, p) and (E', p') is a homeomorphism $h\colon E \to E'$ such that

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & B \end{tikzcd}$$

is commutative. $p' \circ h = p$.

Theorem 9 (79.2)**.** Let $p\colon (E, e_0) \to (B, b_0)$ and $p'\colon (E', e_0') \to (B, b_0)$ be coverings, and $H_0 = p_*\pi(E, e_0)$ and $H_0' = p'_*\pi(E', e_0') \leq \pi(B, b_0)$. Then there exists and equivalence $h\colon (E, p) \to (E', p')$ with $h(e_0) = e_0'$ iff $H_0 - H_0'$. Not isomorphic, but really the same as a subgroup of $\pi(B, b_0)$. In that case, h is unique.

Proof. $\boxed{\Rightarrow}$ Suppose h exists. Then

$$\begin{tikzcd} (E, e_0) \arrow[r, "h"] \arrow[dr, "p"'] & (E', e_0') \arrow[d, "p'"] \\ & (B, b_0) \end{tikzcd}.$$

Therefore $p_*\pi(E, e_0) = p'_*(h_*\pi(E, e_0))$. Since h is a homeomorphism, it induces an isomorphism, so $p'_*(h_*\pi(E, e_0)) = p'_*(\pi(E', e_0')$.

$\boxed{\Leftarrow}$

$$\begin{tikzcd} & (E', e_0') \arrow[d, "p'"] \\ (E, e_0) \arrow[ur, dashed, "k"] \arrow[r, "p"] & (B, b_0) \end{tikzcd}$$

By the previous lemma, there exists a unique k iff $p_*\pi(E, e_0) \subset p'_*\pi(E', e_0')$ or equivalently $H_0 \subset H_0'$, which is the case. Reversing the roles, we get

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E', e_0') \arrow[ur, "l"] \arrow[r, "p'"] & (B, b_0) \end{tikzcd}$$

for the same reasoning, l exists. Now, composing the diagrams

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E, e_0) \arrow[ur, dashed, "l \circ k"] \arrow[r, "p"] & (B, b_0) \end{tikzcd} \qquad \begin{tikzcd} & (E', e_0') \arrow[d, "p'"] \\ (E', e_0') \arrow[ur, dashed, "k \circ l"] \arrow[r, "p'"] & (B, b_0) \end{tikzcd}$$

But placing the identity in place of $l \circ k$ or $k \circ l$, this diagram also commutes! By unicity, we have that $l \circ k = 1_E$ and $k \circ l = 1_{E'}$. Therefore, k and l are homeomorphism $k(e_0) = e_0'$.

Uniqueness is trivial, because of the general lifting theorem.

□

Note that this doesn't answer the question 'is there a equivalence between two coverings', it only answers the question 'is there an equivalence between two coverings mapping $e_0 \to e_0'$'. So now, we seek to understand the dependence of H_0 on the base point.

Lemma 8 (79.3)**.** Let (E, p) be a covering of B. Let $e_0, e_1 \in p^{-1}(b_0)$. Let $H_0 = p_*\pi(E, e_0)$, $H_1 = p_*\pi(E, e_1)$.

- Let γ be a path from e_0 to e_1 and let $p \circ \gamma = \alpha$ be the induced *loop* at b_0. Then $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$, so H_0 and H_1 are conjugate inside $\pi(B, b_0)$.
- Let H be a subgroup of $\pi(B, b_0)$ which is conjugate to H_0, then there is a point $e \in p^{-1}(b_0)$ such that $H = p_*\pi(E, e)$.

So a covering space induces a conjugacy class of a subgroup of $\pi(B, b_0)$.

This completely answers the question: when are two covering spaces equivalent?

Corollary 9.1. Let (E, p) and (E', p') be two coverings, $e_0 \in E$, $e_0' \in E'$ with $p(e_0) = p(e_0') = b_0$. Let $H_0 = p_*\pi(E, e_0)$, $H_0' = p_*'\pi(E', e_0')$. Then (E, p) and (E', p') are equivalent iff H_0 and H_0' are conjugate inside $\pi(B, b_0)$.

Question: can we reach every possible subgroup? Answer: yes, in some conditions.

13.80 Universal covering space

Definition 23. Let B be a path connected and locally path connected space. A covering space (E, p) of B is called a ***universal covering space*** if E is simply connected, so $\pi(E, e_0) = 1$.