

Lecture note 3: Stochastic calculus

1 Exercises

Problem 1.1. (10 points) Let S_0 be the stock price and let C_A and P_A be the prices of American call and put options, respectively, for the same maturity and strike K . Show that

$$S_0 - K \leq C_A - P_A.$$

Problem 1.2. Let C_E and C_A be the prices of European call and American call options, respectively, for the same maturity and strike. Show that

- (i) (10 points) Show that $C_E = C_A$ when the short rate $r \geq 0$.
- (ii) (10 points) The proof in class is not valid when $r = 0$. Explain why.

Problem 1.3. (15 points) Let $f(K)$ be the price of option with payoff

$$(S_T - K)_+^3$$

for $K > 0$. Show that the map $K \mapsto f(K)$ is convex.

Problem 1.4. (10 points) For a time-index set $\mathbb{T} = \{0, 1, \dots, T\}$, let $(\Omega, 2^\Omega, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ be a filtered probability space, and $X : \Omega \rightarrow \mathbb{R}$ be a random variable. Show that a process M defined by

$$M_t = \mathbb{E}(X | \mathcal{F}_t), \quad t = 0, 1, \dots, T$$

is a martingale.

Problem 1.5. (10 points) Let Ω be a set and \mathcal{B} be the Borel σ -algebra of \mathbb{R} . For a map $X : \Omega \rightarrow \mathbb{R}$, show that

$$\sigma(X) := \{X^{-1}(A) : A \in \mathcal{B}\}$$

is a σ -algebra of Ω .

Problem 1.6. (10 points) Let $X = (X_1, X_2, \dots, X_d)^\top$ be jointly normal with the characteristic function

$$\phi_X(t) := \mathbb{E}(e^{it^\top X}) = e^{i\mu^\top t - \frac{1}{2}t^\top \Sigma t}, \quad t \in \mathbb{R}^d$$

for a vector $\mu \in \mathbb{R}^d$ and a nonnegative symmetric matrix Σ . Show that the mean vector is μ and the covariance matrix is Σ .

Problem 1.7. (10 points) Let $X \sim \mathcal{N}(0, 1)$ and let $a > 0$. Define a random variable Y by

$$Y = \begin{cases} X & \text{if } |X| < a \\ -X & \text{if } |X| \geq a \end{cases}$$

Show that $Y \sim \mathcal{N}(0, 1)$. Find the covariance $\text{cov}(X, Y)$ in terms of the density function of the standard normal distribution. Is the random vector (X, Y) a jointly normal?

Problem 1.8. Solve the followings.

- (i) (5 points) Show that if $X_1 \dots, X_n$ are mutually independent, so are $h_1(X_1) \dots, h_n(X_n)$ for any (possibly complex-valued) Borel functions h_1, \dots, h_n .
- (ii) (10 points) Show that if $X_1 \dots, X_n$ are mutually independent and each X_i is normal, then $(X_1 \dots, X_n)$ is jointly normal. Hint: Use (i).

References

Tomas Björk. *Arbitrage theory in continuous time*. Oxford university press, 2004.

1.1 If not exercised before T

$$(t=0) \quad S_0 - K \leq C_A - P_A$$

\uparrow (\because otherwise, \exists arbitrage).

$$(t=T) \quad S_T - K e^{rT} \leq (S_T - K)_+ - (K - S_T)_+ = S_T - K$$

If exercised at $t < T$

$$(t=0) \quad S_0 - K \leq C_A - P_A$$

\uparrow (\because otherwise, \exists arbitrage).

$$(t=t) \quad S_t - K e^{rt} \leq (S_t - K)_+ - (K - S_t)_+ = S_t - K$$

$$\therefore S_0 - K \leq C_A - P_A \quad \square$$

1.2 (i) We know that $C_A \geq C_E$ trivial.

WTS : $C_A = C_E$ when $t=0$. ($\because C_A = C_E$ when $t>0$ proved in class).

Suppose $C_A > C_E$ i.e. $\alpha := C_A - C_E > 0$.

We can make a transaction as following :

Sell American call, Buy European call, Bank α

If C_A is not exercised before T ,

$$(t=0) \quad -C_A \quad C_E \quad \alpha = 0$$

$\downarrow \quad \downarrow \quad \downarrow$

$$(t=T) \quad -(S_T - K)_+ \quad (S_T - K)_+ \quad \alpha = \alpha > 0 \quad (\text{arbitrage}).$$

If C_A is exercised at $t < T$

$$(t=0) \quad -C_A \quad C_E \quad \alpha = 0$$

$$(t=\text{time } t) \quad -(S_t - K)_+ \quad (S_t - K)_+ \quad \alpha = \alpha \quad (\because r=0).$$

$$(t=T) \quad = \alpha > 0 \quad (\text{arbitrage}).$$

This is a contradiction to no-arbitrage.

$$\therefore C_A = C_E$$

1.2. (ii) In $t = 0$ case,

At time $t < T$

$$CA(t) \geq CE(t) \geq St - Ke^{-r(T-t)} = St - K$$

$$\Rightarrow CA(t) \geq CE(t) \geq St - K$$

$$\Rightarrow CA(t) \geq St - K.$$

It means that CA could be exercised at $t < T$.

Thus CA is not exactly same with CE. \square

1.3. WTS : $f(\alpha K + (1-\alpha)K') \leq \alpha f(K) + (1-\alpha)f(K') \quad \alpha \in (0,1), K \neq K' \quad (t=0)$

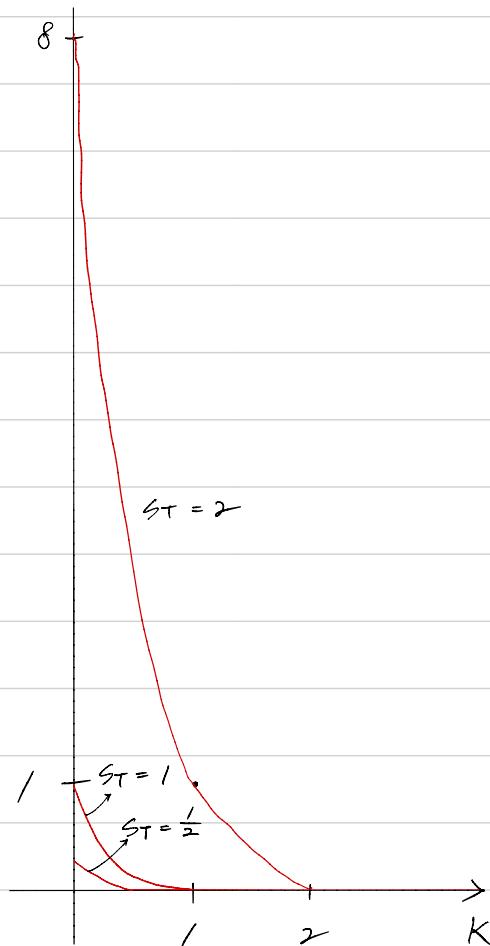
$$\text{i.e., } (St - (\alpha K + (1-\alpha)K'))^+ \leq \alpha(St - K)^+ + (1-\alpha)(St - K')^+ \quad (t=T)$$

i.e., $g(K) = (St - K)^+$ is convex.

The payoff fn $g(K) = (St - K)^+$ is convex. (graph).

$$\therefore f(\alpha K + (1-\alpha)K') \leq \alpha f(K) + (1-\alpha)f(K')$$

$\therefore K \mapsto f(K)$ is convex.



1.4 $(\Omega, \mathcal{F}^{\omega}, (\mathcal{F}_t)_{t \in T}, P)$: filtered prob. space.

$$M_t = E(X|\mathcal{F}_t), t = 0, 1, \dots, T.$$

We need show that $\emptyset M_t$ is adapted (i.e., M_t is \mathcal{F}_t -measurable for $t \in T$)

$$\textcircled{2} E^P(M_{t+1}|\mathcal{F}_t) = M_t \text{ for } t = 0, 1, \dots, T-1$$

$$\textcircled{1} E(M_t|\mathcal{F}_t) = E(E(X|\mathcal{F}_t)|\mathcal{F}_t) = E(X|\mathcal{F}_t) = M_t$$

$\Rightarrow M_t$ is \mathcal{F}_t -measurable for $t \in T$ ($\because \mathcal{F}_t \subseteq \mathcal{F}_T$).

$\Rightarrow M_t$ is adapted.

$$\textcircled{2} E(M_{t+1}|\mathcal{F}_t) = E(E(X|\mathcal{F}_{t+1})|\mathcal{F}_t) = E(X|\mathcal{F}_t) = M_t \text{ for } t = 0, 1, \dots, T-1 \quad \square$$

\uparrow
 $\because \mathcal{F}_t \subseteq \mathcal{F}_{t+1}$.

1.5. \mathcal{B} : Borel σ -alg. of \mathbb{R} .

$$X: \Omega \rightarrow \mathbb{R}.$$

$$\mathcal{G}(X) = \{X^{-1}(A) : A \in \mathcal{B}\}$$

We need to show that $\emptyset \Omega \in \mathcal{G}(X)$.

$$\textcircled{2} A \in \mathcal{G}(X) \Rightarrow A^c \in \mathcal{G}(X).$$

$$\textcircled{3} \bigcup_{x=1}^{\infty} A_x \in \mathcal{G}(X) \text{ for } A_x \in \mathcal{G}(X).$$

$$\textcircled{1} \Omega \in \mathcal{B} \Rightarrow X^{-1}(\Omega) = \Omega \in \mathcal{G}(X).$$

$$\textcircled{2} \text{ For } A \in \mathcal{G}(X), \exists A' \in \mathcal{B} \text{ s.t. } A = X^{-1}(A'). (A')^c \in \mathcal{B}. (\because \mathcal{B} \text{ is } \sigma\text{-alg.}).$$

$$X^{-1}((A')^c) = X^{-1}(\Omega \setminus A') = X^{-1}(\Omega) \setminus X^{-1}(A') = \Omega \setminus A = A^c$$

$$\therefore A \in \mathcal{G}(X) \Rightarrow A^c \in \mathcal{G}(X).$$

$$\textcircled{3} A_1, A_2, \dots \in \mathcal{G}(X) \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{B} \text{ s.t. } X^{-1}(A'_i) = A_1, X^{-1}(A'_2) = A_2, \dots$$

Since \mathcal{B} is σ -alg., $\bigcup_{x=1}^{\infty} A'_x \in \mathcal{B}$ and $X^{-1}(\bigcup_{x=1}^{\infty} A'_x) \in \mathcal{G}(X)$.

$$X^{-1}(\bigcup_{x=1}^{\infty} A'_x) = \bigcup_{x=1}^{\infty} X^{-1}(A'_x) = \bigcup_{x=1}^{\infty} A_x$$

$$\therefore A_1, A_2, \dots \in \mathcal{G}(X) \Rightarrow \text{countable union } \bigcup_{x=1}^{\infty} A_x \in \mathcal{G}(X). \quad \square$$

1.6. \mathbf{X} is jointly normal.

Let $E(\mathbf{X}) = \mu_0$ (mean vector) $\text{Var}(\mathbf{X}) = \Sigma_0$. (covariance matrix)
 $\phi_{\mathbf{X}}(t) = E(e^{it^T \mathbf{X}}) = e^{i\mu_0^T t - \frac{1}{2}t^T \Sigma_0 t}$

Since $\mathbf{Y} := t^T \mathbf{X}$ is normal,

$$\phi_{\mathbf{Y}}(t) = E(e^{it^T \mathbf{Y}}) = e^{i\mu_0^T t - \frac{1}{2}\text{Var}(\mathbf{Y})t^2}$$

$$\phi_{\mathbf{Y}}(1) = E(e^{i\mathbf{Y}}) = e^{i\mu_0^T t - \frac{1}{2}\text{Var}(\mathbf{Y})}$$

$$\therefore E(e^{i\mathbf{Y}}) = E(e^{it^T \mathbf{X}}) = e^{i\mu_0^T t - \frac{1}{2}\text{Var}(t^T \mathbf{X})}$$

$$E(t^T \mathbf{X}) = E(t_1 X_1 + \dots + t_d X_d) = \mu_0^T t$$

$$\text{Var}(t^T \mathbf{X}) = \text{Var}(t_1 X_1 + \dots + t_d X_d)$$

$$\begin{aligned} &= t_1^2 \text{Var}(X_1) + \dots + t_d^2 \text{Var}(X_d) + \sum_{1 \leq i < j \leq d} 2t_i t_j \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^d t_i^2 \text{Cov}(X_i, X_i) + \sum_{1 \leq i < j \leq d} 2t_i t_j \text{Cov}(X_i, X_j) \end{aligned}$$

$$= t^T \Sigma_0 t$$

$$\therefore E(e^{it^T \mathbf{X}}) = e^{i\mu_0^T t - \frac{1}{2}t^T \Sigma_0 t} = e^{i\mu_0^T t - \frac{1}{2}t^T \Sigma_0 t} \quad \text{for } t \in \mathbb{R}^d$$

$$\therefore E(\mathbf{X}) = \mu, \text{Var}(\mathbf{X}) = \Sigma \quad \square$$

$$1.7. X \sim N(0, 1) \Rightarrow \phi_X(t) = e^{-\frac{1}{2}t^2}$$

$$Y = \begin{cases} X & \text{if } |X| < a \\ -X & \text{if } |X| \geq a \end{cases}$$

$$\phi_Y(t) = E(e^{itY}) = \begin{cases} E(e^{itX}) & \text{if } |X| < a \Rightarrow \phi_Y(t) = e^{-\frac{1}{2}t^2} \\ E(e^{-itX}) & \text{if } |X| \geq a \Rightarrow \phi_Y(t) = e^{-\frac{1}{2}(-t)^2} = e^{-\frac{1}{2}t^2} \end{cases}$$

Since characteristic fn of Y is $\phi_Y(t) = e^{-\frac{1}{2}t^2}$,

$$Y \sim N(0, 1).$$

$$\text{Cov}(X, Y) = E(XY) = \begin{cases} E(X^2) & \text{if } |X| < a \\ E(-X^2) & \text{if } |X| \geq a \end{cases}$$

$$\Rightarrow \int_{|x| < a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_{|x| \geq a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\Rightarrow \int_{|x| < a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - (1 - \int_{|x| < a} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx)$$

$$\Rightarrow 2 \int_{-a}^a x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - 1.$$

Assume (X, Y) is jointly normal, then $X+Y$ is normal with density function $f_{X+Y}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Since $X+Y$ is continuous r.v., $P(X+Y=0) = 0$.

But $|X| \geq a \Rightarrow X+Y = X+(-X) = 0$.

$$P(|X| \geq a) = \int_{|x| \geq a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx > 0.$$

$\therefore X+Y$ is not normal. $\neq \square$

1.8.(ii) Since $\{X_i\}$ mutually independent, $\{\mathcal{G}(X_i)\}$ mutually independent.