Homework 9

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Exercise 3-3.5 Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\, v - \frac{v^3}{3} + vu^2,\, u^2 - v^2\right)$$

and show that

(a) The coefficients of the first fundamental form are

$$E = G = (1+u^2+v^2)^2, \quad F = 0.$$

(b) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

(c) The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

(d) The lines of curvature are the coordinate curves.

(e) The asymptotic curves are $u + v = \text{const.}$, $u - v = \text{const.}$

Proof. (a) $\mathbf{x}_u = (1-u^2+v^2,\, 2uv,\, 2u)$ and $\mathbf{x}_v = (2uv,\, 1-v^2+u^2,\, -2v)$. Therefore,

$$\begin{aligned}
E &= \mathbf{x}_u \cdot \mathbf{x}_u = (1-u^2+v^2)^2 + (2uv)^2 + (2u)^2 \\
&= 1 + u^4 + v^4 - 2u^2 + 2v^2 - 2u^2v^2 + 4u^2v^2 + 4u^2 \\
&= 1 + u^4 + v^4 + 2u^2 + 2v^2 + 2u^2v^2 = (1+u^2+v^2)^2 \\
F &= \mathbf{x}_u \cdot \mathbf{x}_v = (1-u^2+v^2)\cdot 2uv + 2uv(1-v^2+u^2) + 2u\cdot(-2v) \\
&= 4uv - 4uv = 0 \\
G &= \mathbf{x}_v \cdot \mathbf{x}_v = (2uv)^2 + (1-v^2+u^2)^2 + (-2v)^2 = \cdots = E
\end{aligned}$$

(b) We calculate

$$\begin{aligned}
\mathbf{x}_u \wedge \mathbf{x}_v &= \begin{vmatrix} i & j & k \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \end{vmatrix} \\
&= (-4uv^2 - 2u + 2uv^2 - 2u^3)i + (4u^2v + 2v - 2u^2v + 2v^3)j \\
&\quad + (1 - v^2 + u^2 - u^2 + u^2v^2 - u^4 + v^2 - v^4 + v^2u^2 - 4u^2v^2)k \\
&= (-2u(u^2+v^2+1),\, 2v(u^2+v^2+1),\, 1-(u^2+v^2)^2) \\
|\mathbf{x}_u \wedge \mathbf{x}_v|^2 &= 4u^2(u^2+v^2+1)^2 + 4v^2(u^2+v^2+1)^2 + (1-(u^2+v^2)^2)^2 \\
&= (u^2+v^2+1)^4.
\end{aligned}$$

Thus,

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{(-2u,\, 2v,\, 1-u^2-v^2)}{u^2+v^2+1}.$$

Since $\mathbf{x}_{uu} = (-2u,\, 2v,\, 2)$, $\mathbf{x}_{uv} = (2v,\, 2u,\, 0)$, and $\mathbf{x}_{vv} = (2u,\, -2v,\, -2)$,

$$\begin{aligned}
e = \langle N, \mathbf{x}_{uu}\rangle &= \frac{4u^2+4v^2+2(1-u^2-v^2)}{u^2+v^2+1} = \frac{2(1+u^2+v^2)}{u^2+v^2+1} = 2 \\
f = \langle N, \mathbf{x}_{uv}\rangle &= \frac{-4uv+4uv+0\cdot(1-u^2-v^2)}{u^2+v^2+1} = 0 \\
g = \langle N, \mathbf{x}_{vv}\rangle &= \frac{-4u^2-4v^2-2(1-u^2-v^2)}{u^2+v^2+1} = \frac{-2(1+u^2+v^2)}{u^2+v^2+1} = -2
\end{aligned}$$

(c) We first calculate

$$\begin{aligned}
K &= \frac{eg-f^2}{EG-F^2} = \frac{2\cdot(-2)-0^2}{(1+u^2+v^2)^4-0^2} = \frac{-4}{(1+u^2+v^2)^4} \\
H &= \frac{gE-2fF+eG}{2(EG-F^2)} = \frac{(1+u^2+v^2)^2\cdot 2 - 2\cdot 0\cdot 0 + (1+u^2+v^2)^2(-2)}{2(1+u^2+v^2)^4} = 0
\end{aligned}$$

Specially, the Enneper surface is minimal. Now from the relations

$$K = k_1k_2, \quad H = \frac{k_1+k_2}{2},$$

we get a system of equations for k_1 and k_2. From $H=0$ we get $k_2 = -k_1$. Now

$$K = -k_1^2 \Rightarrow \frac{-4}{(1+u^2+v^2)^4} = -k_1^2 = k_1 = \frac{\pm 2}{(1+u^2+v^2)^2}$$

and therefore

$$k_2 = -k_1 = \frac{\mp 2}{(1+u^2+v^2)^2}$$

Since the principal curvatures k_1 and k_2 can be switched (the order doesn't matter), we actually have that

$$k_1 = \frac{2}{(1+u^2+v^2)}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

(d) The curve $c(t) = \mathbf{x}(u(t),\, v(t))$, which lies on S, is a line of curvature if and only if u and v satisfy the differential equation

$$\begin{vmatrix} v'^2 & -u'v & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

Substituting the values for the Enneper surface we get

$$\begin{aligned}
0 &= \begin{vmatrix} v'^2 & -u'v' & u'^2 \\ (1+u^2+v^2)^2 & 0 & (1+u^2+v^2)^2 \\ 2 & 0 & -2 \end{vmatrix} = \text{(Lapl. exp. 2nd col.)} \\
&= (-1)^4(-u'v')\begin{vmatrix} v'^2 & u'^2 \\ 2 & -2 \end{vmatrix} = -u'v'(-2v'^2-2u'^2) = 2u'v'(u'^2+v'^2)
\end{aligned}$$

which is equivalent to $u'=0$ or $v'=0$ or $u'^2+v'^2=0$. The last case implies $u'=v'=0$, which is covered by first two cases. Now we have that $j'=0 \Rightarrow u = \text{const.}$ or $v'=0 \Rightarrow$

$v = \text{const.}$, which are exactly the coordinate curves.

(e) The curve c is an asymproric curve if and only if u and v satisfy the differential equation

$$eu'^2 + 2fu'v' + gv'^2 = 0$$

Substituting the values for the Enneper surface we get

$$2u'^2 - 2v'^2 = 0 \Rightarrow 2(u' - v')(u' + v') = 0 \Rightarrow u' \pm v' = 0 \Rightarrow u \pm v = \text{const.}$$

□

Claim 1. Fix $p \in S$. Choose a unit normal vector n at p. Let $h\colon S \to \mathbb{R}$ s.t. $q \mapsto \langle q - p, n\rangle$ be the height function relative to p with n. Show that $q \in S$ is a critical point of h if and only if $n \perp T_qS$ (i.e., $n = \pm N(q)$).

Proof. For $q \in S$ and for any $v \in T_qS$, choose $\alpha\colon (-\varepsilon, \varepsilon) \to S$ s.t. $\alpha(0) = q$ and $\alpha'(0) = v$. Then,

$$dh_q(v) = \left.\frac{d}{dt}\right|_{t=0} h(\alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} \langle \alpha(t) - p, n\rangle = \langle \alpha'(0), n\rangle = \langle v, n\rangle,$$

and TFAE:

$$\begin{aligned} q \text{ is a critical point} &\Leftrightarrow dh_q \text{ is not surjective} \\ &\Leftrightarrow dh_q \equiv 0 \\ &\Leftrightarrow dh_q(v) = \langle v, n\rangle = 0 \text{ for all } v \in T_qS \\ &\Leftrightarrow v \perp n \text{ for all } v \in T_qS \\ &\Leftrightarrow n \perp T_qS \end{aligned}$$

□

Claim 2. There is no compact regular surface with negative Gauss curvature everywhere.

Proof. Suppose not. Let S be a compact regular surface with negative Gauss curvature everywhere. Choose any point $p \in S$. Then, $h\colon S \to \mathbb{R}$ s.t. $q \mapsto \langle q - p, n\rangle$ is continuous. Since S is compact, $h(S)$ is also compact, i.e. $h(S)$ is closed and bounded. Then, $h(S)$ has a maximum with some point $q \in S$ s.t. $h(q) = \max_{q' \in S}\langle q' - p, n\rangle$. Hence, for any $\alpha\colon (-\varepsilon, \varepsilon)$ s.t. $\alpha(0) = q$ and $\alpha'(0) = v \in T_qS$, $dh_q(v) = \left.\frac{d}{dt}\right|_{t=0} h(\alpha(t)) = 0$. Thus, $dh_q \equiv 0$, so q is a critical point of h. By Claim 1, $n \perp T_qS$. Since q is a hyperbolic point, by Proposition 1, in each neighborhood of q there exist points $q_1, q_2 \in S$ in both sides of $T_p(S)$. Then,

$$h(q_i) = \langle q_i - p, n\rangle = \langle q_i - q, n\rangle + \langle q - p, n\rangle = h(q) + \langle q_i - q, n\rangle \quad \text{for } i = 1, 2.$$

Since $n \perp T_pS$ and q_1, q_2 are in both sides of $T_p(S)$, signs of $\langle q_1 - q, n\rangle$ and $\langle q_2 - q, n\rangle$ are different. WLOG, assume $\langle q_1 - q, n\rangle > 0$ and $\langle q_2 - q, n\rangle < 0$. Then, $h(q_1) = h(q) + \langle q_1 - q, n\rangle > h(q)$ and it is a contradiction to the maximality of $h(q)$. Therefore, there is no compact regular surface with negative Gauss curvature everywhere. □