Introduction to Differential Geometry I – Homework 6

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Problem 2-5.9 Show that a surface of revolution can always be parametrized so that

$$E = E(v),\ F = 0,\ G = 1.$$

Solution. Suppose that the axis of the surface is the z-axis and let $c = (x, 0, z) : I \to \{y = 0, x > 0\}$ be the generating curve parametrized by arc length. The surface itself is parametrized by the map $\mathbf{x} : (0, 2\pi) \times I \to \mathbb{R}^3$ given by the formula $x(u, v) = (x(v) \cos u, x(v) \sin u, z(v))$. Now we have

$$\begin{aligned} \mathbf{x}_u &= (-x(v) \sin u, x(v) \cos u, 0) \\ \mathbf{x}_v &= (x'(v) \cos u, x'(v) \sin u, z'(v)). \end{aligned}$$

Thus,

$$\begin{aligned} E &= x(v)^2 \sin^2 u + x(v)^2 \cos^2 u = x(v)^2 (\sin^2 u + \cos^2 u) = x(v)^2 \\ F &= -x(v)x'(v) \sin u \cos u + x(v)x'(v) \sin u \cos u + 0 \cdot z'(u) = 0 \\ G &= x'(v)^2 \cos^2 u + x'(v)^2 \sin^2 u + z'(v)^2 = x'(v)^2 (\cos^2 u + \sin^2 u) + z'(v) \\ &= x'(v)^2 + z'(v)^2 = c'(v) \cdot c'(v) = 1 \end{aligned}$$

Hence, we found one such parametrization. □

Problem 2-6.7 Show that if a regular surface S contains an open set diffeomorphic to a Mobius strip, then S is nonorientable.

Solution. Suppose that the surface S is orientable and that it contains an open subset $V \subseteq S$ diffeomorphic to the Mobius strip. Since S is orientable, by proposition 1 there exists a differentiable field of unit normal vectors $N : S \to \mathbb{R}^3$ on S. Then, because differentiability is a local property, $N|_V : V \to \mathbb{R}^3$ is a differentiable field of unit normal vectors on V. This, again by proposition 1, implies that V is orientable, which is a contradiction because V is diffeomorphic to the nonorientable Mobius strip, so S is nonorientable. □