Probability Theory – Exercise 5

Junwoo Yang

November 15, 2020

Problem 1

For each of the following decide whether $f_n \to 0$ (i) in L^p, (ii) uniformly, (iii) pointwise, (iv) a.e.

(a) $f_n = \mathbf{1}_{[n,n+\frac{1}{n}]}$,

Proof. Note that $f_n(x) = 0$ for $x < 1$ and any $n \in \mathbb{N}$. For any given $x \geq 1$, take $N \in \mathbb{N}$ such that $N \leq x < N+1$. Then $f_n(x) = 0$ for $n > N$. Thus, f_n converges to 0 pointwise (also a.e.). But, f_n does not converge uniformly since $\sup_{x\in\mathbb{R}} |f_n - 0| = 1$ for all n. However, f_n converges to 0 in L^p.

$$\because \|f_n - 0\|_p = \left(\int |f_n - 0|^p \,\mathrm{d}m\right)^{\frac{1}{p}} = \left(\int f_n^p \,\mathrm{d}m\right)^{\frac{1}{p}} = \left(\int f_n \,\mathrm{d}m\right)^{\frac{1}{p}} = \left(\int_n^{n+\frac{1}{n}} 1 \,\mathrm{d}m\right)^{\frac{1}{p}} = \frac{1}{n^{\frac{1}{p}}}$$

tends to 0 as n goes ∞. □

(b) $f_n = n\mathbf{1}_{[0,\frac{1}{n}]} - n\mathbf{1}_{[-\frac{1}{n},0]}$.

Proof. Note that $f_n(0) = n - n = 0$ for all $n \in \mathbb{N}$. For any given non-zero $x \in \mathbb{R}$, take $N \in \mathbb{N}$ such that $\frac{1}{N} < |x|$. Then $f_n(x) = 0$ for $n > N$. Thus, f_n converges to 0 pointwise (also a.e.). But f_n does not converge uniformly since $\sup_{x\in\mathbb{R}} |f_n(x) - 0| = n \to \infty$ as $n \to \infty$. f_n also does not converge in L^p.

$$\begin{aligned}\|f_n - 0\|_p &= \left(\int |f_n|^p \,\mathrm{d}m\right)^{\frac{1}{p}} = \left(\int |n\mathbf{1}_{[0,\frac{1}{n}]} - n\mathbf{1}_{[-\frac{1}{n},0]}|^p \,\mathrm{d}m\right)^{\frac{1}{p}} = \left(\int n^p\mathbf{1}_{[-\frac{1}{n},\frac{1}{n}]} \,\mathrm{d}m\right)^{\frac{1}{p}} \\ &= \left(\int_{-\frac{1}{n}}^{\frac{1}{n}} n^p \,\mathrm{d}m\right)^{\frac{1}{p}} = \left(n^p\frac{2}{n}\right)^{\frac{1}{p}} = (2n^{p-1})^{\frac{1}{p}} \to \infty \text{ as } n \to \infty.\end{aligned}$$

Note that null set $\{0\}$ does not affect integral. □

Problem 2

If $X_n \to X$ and $Y_n \to Y$ in probability, show that $X_n + Y_n \to X + Y$ in probability and $X_nY_n \to XY$ in probability.

Proof. Since X_n, Y_n converge to X, Y in probability, respectively, $P(|X_n - X| > \frac{\epsilon}{2})$, $P(|Y_n - Y| > \frac{\epsilon}{2})$ tend to 0 as n goes ∞. At first, we prove $X_n + Y_n$ converges to $X + Y$ in probability.

$$\begin{aligned} P(|(X_n + Y_n) - (X + Y)| > \epsilon) &= P(|(X_n - X) + (Y_n - Y)| > \epsilon) \\ &\leq P(|X_n - X| + |Y_n - Y| > \epsilon) \\ &\leq P\left(|X_n - X| \geq \frac{\epsilon}{2} \cup |Y_n - Y| > \frac{\epsilon}{2}\right) \\ &\leq P\left(|X_n - X| \geq \frac{\epsilon}{2}\right) + P\left(|Y_n - Y| > \frac{\epsilon}{2}\right) \to 0 \end{aligned}$$

□

Problem 3

Show that if (X_n) is a Cauchy sequence in probability (i.e. $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. $m, n \geq N$ implies $P\{\omega : |X_m(\omega) - X_n(\omega)| \geq \epsilon\} < \epsilon$), then there is a random variable X such that $X_n \to X$ in probability.

Proof. For each $k \geq 1$, there exist n_k such that for n, $m \geq n_k$, $P(|X_n - X_m| > 2^{-k}) < 2^{-k}$. We are going to show subsequence (X_{n_k}) is a Cauchy sequence in $\mathbb{R}$ almost surely. Borel-Cantelli lemma says that

$$\sum_{k=1}^{\infty} P(|X_{n_k} - X_{n_{k+1}}| > 2^{-k}) < \sum_{k=1}^{\infty} 2^{-k} < \infty \quad \text{implies} \quad P(\limsup_{k\to\infty}\{|X_{n_k} - X_{n_{k+1}}| > 2^{-k}\}) = 0.$$

So, for all ω, except for those belonging to an event of probability 0, the subsequnce $X_{n_k}(\omega)$ is a Cauchy sequence of real numbers, which in turn must converge to a finite limit, that can be denoted $X(\omega)$. So X_{n_k} converges to X almost surely. Then, by X_n's Cauchy convergence in probability and $X_{n_k} \to X$ almost surely, we get for any $\varepsilon > 0$,

$$P(|X_n - X| > \varepsilon) \leq P\left(|X_n - X_{n_k}| > \frac{\varepsilon}{2}\right) + P\left(|X_{n_k} - X| > \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for sufficiently large n and n_k. Hence, X_n converges to X in probability. □