Mathematical Statistics 1

Ch.3 Continuous Distributions

Jungsoon Choi

jungsoonchoi@hanyang.ac.kr

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3.1 Normal Distribution, $N(\mu, \sigma^2)$

A random variable X has a normal distribution if its *pdf* is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty,$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$ are called the mean and the variance of X, respectively.

- $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$
- $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1.$
- dnorm(x,mu,sd): $f_X(x)$ in R
- pnorm(x,mu,sd): $P(X \leq x)$ in R

1) $f_X(x) > 0 \quad \forall x \in \mathbb{R}$

2) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$\Rightarrow f_X(x)$: pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

To show : $I \overset{\text{let}}{=} \int_{-\infty}^{\infty} f_X(x)\,dx = 1$ $\quad (I \Leftrightarrow I^2)$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz$$

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{(x^2+y^2)}{2}\right\} dx\,dy$$

$$J = \begin{array}{c|cc} & r & \theta \\ \hline x & \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ y & \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{array}$$

$x = r\cos\theta$

$y = r\sin\theta$.

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2}\right) r\,dr\,d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \left[\exp\left(-\frac{r^2}{2}\right)\right]_0^{\infty} d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta = \frac{1}{2\pi}\cdot 2\pi = 1.$$

$\therefore I^2 = 1$

$\Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

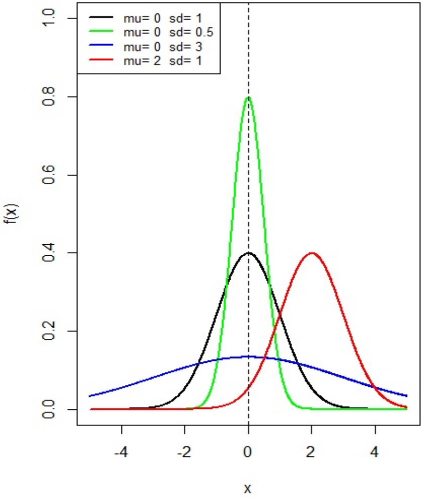
$$X \sim f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

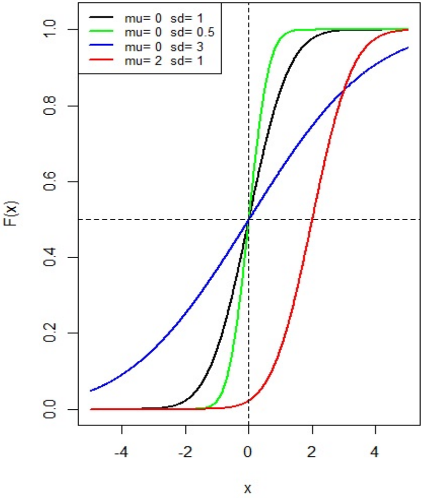
pdf (normal dist.).

$X \sim N(\mu, \sigma^2)$ $\quad \mu \in \mathbb{R}$ $\quad \sigma^2 > 0$.

$E(X)$ $\quad Var(X)$

Normal pdf and cdf





Prove that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Since $f(x) > 0$, $I = \int_{-\infty}^{\infty} f(x)\, dx > 0$ and by letting $z = (x-\mu)/\sigma$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

$$I^2 = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-x^2/2} dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2/2} dy\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) dxdy$$

Let $x = r\cos\theta$, $y = r\sin\theta$. Then,

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r dr d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{1}{2\pi} 2\pi = 1.$$

Thus, $I = \int_{-\infty}^{\infty} f(x)\, dx = 1$.

mgf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty.$$

$$\begin{aligned}
M(t) &= \int_{-\infty}^{\infty} \frac{e^{tx}}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}\left\{x^2 - 2(\mu+\sigma^2 t)x + \mu^2\right\}\right] dx \\
&= \exp\left(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}\left\{x - (\mu+\sigma^2 t)\right\}^2\right] dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right), \quad -\infty < t < \infty.
\end{aligned}$$

The mgf is

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right), \quad -\infty < t < \infty.$$

- The first derivative, $M'(t)$

$$M'(t) = \frac{d}{dt}M(t) = (\mu + \sigma^2 t)\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) = (\mu + \sigma^2 t) \times M(t).$$

- The second derivative, $M''(t)$

$$\begin{aligned} M''(t) = \frac{d^2}{dt^2}M(t) &= \left\{\sigma^2 + (\mu + \sigma^2 t)^2\right\}\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \\ &= \left\{\sigma^2 + (\mu + \sigma^2 t)^2\right\} \times M(t). \end{aligned}$$

- The mean of X, $E(X)$

$$E(X) = M'(0) = \mu \times M(0) = \mu.$$

- The second moment of X, $E(X^2)$

$$M''(0) = (\sigma^2 + \mu^2) \times M(0) = \sigma^2 + \mu^2.$$

- The variance of X, $\text{Var}(X)$

$$\text{Var}(X) = M''(0) - \left\{M'(0)\right\}^2 = \sigma^2.$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$z = \frac{x-\mu}{\sigma}$$

$$= \int_{-\infty}^{\infty} (\sigma z + \mu) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz$$

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x^2 - 2\mu x + \mu^2)}{2\sigma^2} + tx\right\} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\mu^2}{2\sigma^2}\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2 - 2(\mu+\sigma^2 t)x}{2\sigma^2}\right\} dx$$

$$+(\mu+\sigma^2 t)^2 - (\mu+\sigma^2 t)^2$$

$$= \exp\left\{-\frac{\mu^2}{2\sigma^2}\right\} \exp\left\{\frac{(\mu+\sigma^2 t)^2}{2\sigma^2}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\{x-(\mu+\sigma^2 t)\}^2}{2\sigma^2}\right\} dx$$

$$= \exp\left\{-\frac{\mu^2}{2\sigma^2} + \frac{\mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\right\}$$

$$= \exp\left\{\mu t + \frac{\sigma^2}{2} t^2\right\} \qquad t \in \mathbb{R}.$$

$$\begin{tikzcd} & X \sim f_X(x) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ F_X(x) \arrow[rr, leftrightarrow] & & M_X(t) \end{tikzcd}$$

$$M_X(t) = \exp\{\mu t + \frac{\sigma^2}{2}t^2\} \quad t \in \mathbb{R}.$$

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} e^{\mu t + \frac{\sigma^2}{2}t^2}$$

$$= (\mu + \sigma^2 t) \cdot e^{\mu t + \frac{\sigma^2}{2}t^2}$$

$$M_X^{(1)}(0) = \mu$$

$$M_X^{(2)}(t) = \sigma^2 \cdot e^{\mu t + \frac{\sigma^2}{2}t^2} + (\mu + \sigma^2 t)^2 \cdot e^{\mu t + \frac{\sigma^2}{2}t^2}$$

$$= \{\sigma^2 + (\mu + \sigma^2 t)^2\} \cdot e^{\mu t + \frac{\sigma^2}{2}t^2}$$

$$M_X^{(2)}(0) = \sigma^2 + \mu^2$$

$$Var(X) = M_X^{(2)}(0) - \{M_X^{(1)}(0)\}^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

Example 3.3-1

The pdf of X is $\rightarrow X \sim N(-7, 4^2)$.

$$f(x) = \frac{1}{\sqrt{32\pi}} \exp\left\{-\frac{(x+7)^2}{32}\right\}, \quad -\infty < x < \infty.$$

Find the mgf of X. $M_X(t) = \exp\{-7t + 8t^2\} \quad t \in \mathbb{R}$

Example 3.3-2

The mgf of X is

$$M(t) = \exp(5t + 12t^2)$$

Find the pdf of X. $X \sim N(5, 24)$

3.2 Standard Normal Distribution

A random variable Z has a standard normal distribution if its *pdf* is defined by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

- $Z \sim N(0, 1)$.
- $E(Z) = 0$ and $\mathrm{Var}(Z) = 1$
- $\int_{-\infty}^{\infty} f(z)\, dz = 1.$
- $\Phi(z) = Pr(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$

$\Phi(z) = F_Z(z)$

Example 3.3-3

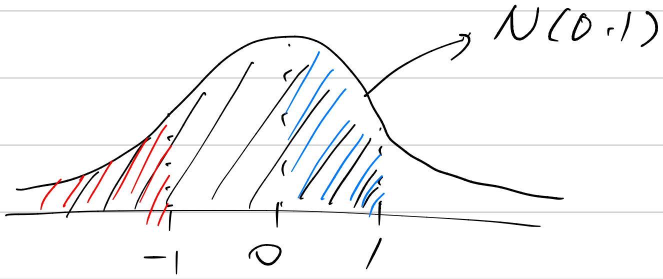
Compute the following probabilities.

- $P(Z \leq 1.24)$
- $P(1.24 \leq Z \leq 2.37)$
- $P(-2.37 \leq Z \leq -1.24)$
- $P(Z \leq -2.14)$
- $P(-2.14 \leq Z \leq 0.77)$

Example 3.3-4

Find a and b values satisfying that $P(Z \leq a) = 0.9147$ and $P(Z \geq b) = 0.0526$.

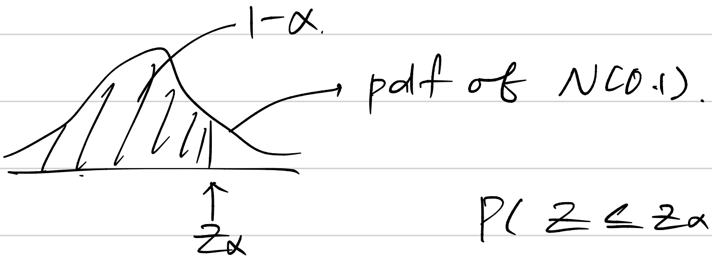
$P(Z \leq 1) = 0.8413$



$$P(Z \leq -1) = 1 - P(Z \leq 1)$$

$$P(0 \leq Z \leq 1) = P(Z \leq 1) - P(Z \leq 0)$$
$$= P(Z \leq 1) - 0.5$$

z_α : $100(1-\alpha)^{th}$ percentile of $N(0,1)$



$$P(Z \leq z_\alpha) = 1-\alpha.$$
$$\Leftrightarrow \quad P(Z \geq z_\alpha) = \alpha.$$

100(1-α) percentile

The z_α is called as the 100(1-α) percentile of $N(0,1)$ if

$$P(Z \geq z_\alpha) = \alpha, \quad Z \sim N(0,1).$$

Due to the symmetric property, $P(Z \leq -z_\alpha) = P(Z \geq z_\alpha) = \alpha$.

Thm 3.3-1

If $X \sim N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma \sim N(0, 1)$.

$$X \sim N(\mu, \sigma^2) \quad \overset{?}{\Rightarrow} \quad P(X \leq \bigcirc) = ?$$

$$Z \sim N(0,1) \quad \Rightarrow \quad P(Z \leq \bigcirc)$$

To show

$$\text{Let} \quad Z = \frac{X-\mu}{\sigma} \quad (\sim N(0,1))$$

$$P(Z \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right) = P(X \leq \mu + \sigma z)$$

$$= \int_{-\infty}^{\mu+\sigma z} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \quad = dw$$

$$\text{Let} \quad w = (x-\mu)/\sigma \qquad dw = \frac{1}{\sigma}\, dx$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{w^2}{2}\right\} dw$$

$$= \Phi(z).$$

$$\underset{\text{cdf of } Z}{\underset{\downarrow}{P(Z \leq z)}} = \underset{\text{cdf of } N(0,1).}{\underset{\downarrow}{\Phi(z)}}$$

$$\therefore \; Z \sim N(0,1)$$

Example 3.3-6

Let $X \sim N(3, 16)$. Compute the following probabilities.

- $P(4 \leq X \leq 8)$

  $\frac{1}{16} \leq z \leq \frac{5}{16}$

- $P(0 \leq X \leq 5)$
- $P(-2 \leq X \leq 1)$

Example 3.3-7

If $X \sim N(25, 36)$, then find the c value such as $P(|X - 25| \leq c) = 0.9544$.

3.3 Relationship between Chi-square and Normal distributions

Thm 3.3-2

If X is $N(\mu, \sigma^2)$, then $V = (X - \mu)^2/\sigma^2 = Z^2$ is $\chi^2(1)$.

$$F_W = P(W \le w) = P(V^2 \le w) = P(-\sqrt{w} \le V \le \sqrt{w})$$
$$= 2P(V \le \sqrt{w})$$

$$= 2\int_0^{\sqrt{w}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv$$

Let $v = \sqrt{y}$ $\quad v^2 = y.$ $\quad 2v\,dv = dy$ $\quad dv = \frac{1}{2\sqrt{y}}\,dy$

$$= 2\int_0^{\sqrt{w}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right) \frac{1}{2\sqrt{y}}\,dy$$

$$= \int_0^{\sqrt{w}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right) y^{-\frac{1}{2}}\,dy.$$

$$\therefore f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w}{2}} \cdot w^{-\frac{1}{2}} \qquad w > 0.$$

$$\int_0^\infty e^{-\frac{w}{2}} \cdot w^{-\frac{1}{2}}\,dw = \sqrt{2\pi}$$

$$\left(\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \cdot e^{-t}\,dt\right).$$

Let $t = \frac{w}{2}$, $\quad dw = 2dt.$

$$= \int_0^\infty e^{-t} \cdot (2t)^{-\frac{1}{2}}\, 2dt = \sqrt{2\pi}.$$

$$= \int_0^\infty e^{-t} \cdot t^{-\frac{1}{2}}\,dt = \underline{\sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right)}_{/\!/}$$

$$f_W = \frac{1}{\sqrt{2\pi}} e^{-\frac{w}{2}} \cdot w^{-\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{1}{2}\right) \cdot 2^{\frac{1}{2}}} \cdot w^{\frac{1}{2}-1} \cdot e^{-\frac{w}{2}} = \chi^2(1).$$

Example 3.3-8

If $Z \sim N(0.1)$, then $P(|Z| < 1.96 = \sqrt{3.841}) =$

Since $Z^2 \sim \chi^2(1)$, $P(Z^2 < 3.841) =$