Advanced Calculus I – Assignment 3

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May 27, 2019

§4.1 #1

(a) Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Prove that f is continuous.

Proof. For any $\varepsilon > 0$, $x_0 \in \mathbb{R}$, pick $\delta < \sqrt{\varepsilon + x_0^2} - |x_0|$. Then, $|x - x_0| < \delta$ implies

$$\begin{aligned} |x^2 - x_0^2| &= |x - x_0 + 2x_0||x - x_0| < (|x - x_0| + 2|x_0|)|x - x_0| < (\delta + 2|x_0|)\delta \\ &< (\sqrt{\varepsilon + x_0^2} + |x_0|)(\sqrt{\varepsilon + x_0^2} - |x_0|) = \varepsilon + x_0^2 - x_0^2 = \varepsilon. \end{aligned}$$ □

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$. Prove that f is continuous.

Proof. For any $\varepsilon > 0$, $(x_0, y_0) \in \mathbb{R}^2$, choose $\delta = \varepsilon$. Then, $\|(x, y) - (x_0, y_0)\| < \delta$ implies

$$|x - x_0| = \sqrt{(x - x_0)^2} < \sqrt{(x - x_0)^2 + (y - y_0)^2} = \|(x, y) - (x_0, y_0)\| < \delta = \varepsilon.$$ □

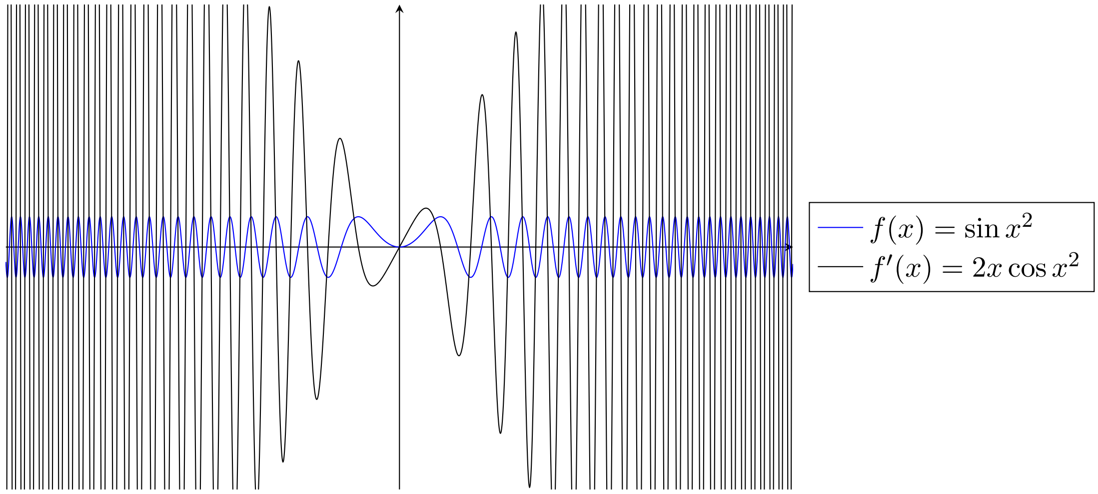
§4.5 #3 Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point.

Proof. If $f(0) = 0$ or $f(1) = 1$, it is done. Suppose $f(0) \neq 0$ and $f(1) \neq 1$, that is, $f(0) \in (0, 1]$, $f(1) \in [0, 1)$. Let $g(x) = x - f(x)$. Then, $g(0) = 0 - f(0) = -f(0) < 0$, $g(1) = 1 - f(1) > 0$. Since g is continuous and $[0, 1]$ is connected, by intermediate value theorem, there is $x_0 \in [0, 1]$ such that $g(x_0) = 0$. Hence, f has fixed point x_0. □

§4.6 #3 Must a bounded continuous function on $\mathbb{R}$ be uniformly continuous?

Proof. Consider $f(x) = \sin x^2$. Note that f is bounded continuous and the farther away x is from the origin, the shorter the periodicity of f. As $|x|$ goes to infinity, furthermore, oscillation range of f is constant, while $f' = 2x \cos x^2$ oscillate largely as shown in the figure.
Once we choose $\delta > 0$ as possible as small for given $\varepsilon > 0$, there must exist infinitely x_0 whose δ-ball $B(x_0, \delta)$ contains point x such that $|f(x) - f(x_0)| > \varepsilon$. For example, fix $\varepsilon = \frac{1}{2}$. We want to show that for any $\delta > 0$, there are points x and y such that $|x - y| < \delta$ and $|f(x) - f(y)| > \frac{1}{2}$. Let $x = \sqrt{2n\pi}$, $y = \sqrt{2n\pi + \pi/2}$, $n \in \mathbb{N}$. Then,

$$y - x = \sqrt{2n\pi + \pi/2} - \sqrt{2n\pi} = \frac{\pi/2}{\sqrt{2n\pi + \pi/2} + \sqrt{2n\pi}} < \frac{\pi/2}{2\sqrt{2n\pi}}$$



tends to 0 as n goes to infinity. We can choose n such that

$$|y - x| < \frac{\pi/2}{2\sqrt{2n\pi}} < \delta.$$

But $|f(x) - f(y)| = |\sin 2n\pi - \sin(2n\pi + \pi/2)| = |0 - 1| = 1 > \frac{1}{2}$. So, we can pick n for any $\delta > 0$. Thus, there is no δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Hence f is not uniformly continuous. □

§4.6 #6

(a) Show that $f : \mathbb{R} \to \mathbb{R}$ is *not* uniformly continuous iff there exist an $\varepsilon > 0$ and sequences x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Generalize this statement to metric spaces.

Proof. □

(b) Use (a) on $\mathbb{R}$ to prove that $f(x) = x^2$ is not uniformly continuous.

Proof. □

§4.7 #5 Let f be continuous on $[3, 5]$ and differentiable on $(3, 5)$, and suppose that $f(3) = 6$ and $f(5) = 10$. Prove that, for some point x_0 in the open interval $(3, 5)$, the tangent line to the graph of f at x_0 passes through the origin. Illustrate your result with a sketch.

Proof. □

§4.8 #7 Let $f : [0, 1] \to \mathbb{R}$, $f(x) = 1$ if $x = \frac{1}{n}$, n an integer, and $f(x) = 0$ otherwise.

(a) Prove that f is integrable.

Proof. □

(b) Show that $\int_0^1 f(x)\,\mathrm{d}x = 0$.

Proof. □

Chapter 4 #12

(a) A map $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz on A if there is a constant $L \geq 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$, for all $x, y \in A$. Show that a Lipschitz map is uniformly continuous.

Proof. □

(b) Find a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous and hence is not Lipschitz.

Proof. □

(c) Is the sum (product) of two Lipschitz functions again a Lipschitz function?

Proof. □

(d) Is the sum (product) of two uniformly continuous functions again uniformly continuous?

Proof. □

(e) Let f be defined and have a continuous derivative on $(a - \varepsilon, b + \varepsilon)$ for some $\varepsilon > 0$. Show that f is a Lipschitz function $[a, b]$.

Proof. □