Homework 6

Due date: 2019. 5. 27.

1. Prove or disprove that $\mathbb{Z} \oplus \mathbb{Z}$ is a cyclic group.

2. Find the order of the given element of the external direct product.
 (1) $(2,3)$ in $\mathbb{Z}_6 \oplus \mathbb{Z}_{15}$
 (2) $(3,6,12,16)$ in $\mathbb{Z}_4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{24}$

3. How many elements of order 2 are in $\mathbb{Z}_{2000000} \oplus \mathbb{Z}_{4000000}$?

4 Prove or disprove.
 (1) $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \approx \mathbb{Z}_{60} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$.
 (2) $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \approx \mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$.

5. How many isomorphisms are there from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_3$?

6. (1) Prove $\mathbb{R} \oplus \mathbb{R} \approx \mathbb{C}$.
 (2) Prove $\mathbb{R}^* \oplus \mathbb{R}^* \not\approx \mathbb{C}^*$.

7. Show that if H and N are subgroups of a group G, and N is normal subgroup in G, then $H \cap N$ is normal subgroup in H.

8. Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a,b,d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2,\mathbb{R})$?

9. Find the order of the given factor group.
 (1) $(\mathbb{Z}_4 \oplus \mathbb{Z}_{12})/(<2> \oplus <2>)$.
 (2) $(\mathbb{Z}_{10} \oplus U(10))/\langle (2,9) \rangle$.

10. Let $G = U(32)$ and $H = \{1,31\}$. The group G/H is isomorphic to one of $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one by elimination.

현대대수1 - HW6

파이낸스경영학과

2017004093

양 준 우

#1. Suppose $\mathbb{Z} \oplus \mathbb{Z}$ is a cyclic group.

Then all elements are multiple of a generator $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$. $\exists m \in \mathbb{Z}$ s.t $(1,0) = m(a,b)$

$\therefore b = 0$

Also, there is $n \in \mathbb{Z}$ s.t $(0,1) = n(a,b) \Rightarrow a = 0$

Hence, $(a,b) = (0,0)$, which is not a generator of $\mathbb{Z} \oplus \mathbb{Z}$.

$\therefore \mathbb{Z} \oplus \mathbb{Z}$ is not a cyclic group.

#2. (1) $(2,3)$ in $\mathbb{Z}_6 \oplus \mathbb{Z}_{15}$.

In $\mathbb{Z}_6$, $|2| = 3$, In $\mathbb{Z}_{15}$, $|3| = 5$

$\therefore |(2,3)| = \text{lcm}(3,5) = 15$.

(2) $(3,6,12,16)$ in $\mathbb{Z}_4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{24}$

In $\mathbb{Z}_4$, $|3| = 4$, In $\mathbb{Z}_{12}$, $|6| = 2$

In $\mathbb{Z}_{20}$, $|12| = \dfrac{20}{\gcd(12,20)} = \dfrac{20}{4} = 5$

In $\mathbb{Z}_{24}$, $|16| = \dfrac{24}{\gcd(16,24)} = \dfrac{24}{8} = 3$

$\therefore |(3,6,12,16)| = \text{lcm}(4,2,5,3) = 60$.

#3. Let $(a,b) \in \mathbb{Z}_{2000000} \oplus \mathbb{Z}_{4000000}$ with $|(a,b)| = 2$.

Case1) $|a| = 2$ then $|b| = 1$ or $|b| = 2$

$$\frac{2000000}{\gcd(k,2000000)} = 2 \Rightarrow \gcd(k,2000000) = 1000000$$

$$\Rightarrow k = 1000000$$

$$\therefore a \in \{1000000\}$$

If $|b|=1$, then $b \in \{0\}$ $\Rightarrow (1000000, 0)$,
If $|b|=2$, then $b \in \{2000000\}$ $(1000000, 2000000)$
case 2) $|a|=1$, then $|b|=2$
Since $|a|=1$, $a \in \{0\}$ $\Rightarrow (0, 2000000)$
Since $|b|=2$, $b \in \{2000000\}$
There are 3 elements of order 2.

#4. (1) $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \approx \mathbb{Z}_{60} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$

proof). $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6$

$$\approx (\mathbb{Z}_5 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_4) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3)$$
$$\approx (\mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2$$
$$\approx \mathbb{Z}_{60} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2 \ \blacksquare$$

(2) $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \approx \mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$

disprove). $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \approx \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$\mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12} \approx \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$

Also, $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6$ has more than three elements of order 2, such as $\{(5,6,3), (0,6,3), (5,6,0), (5,0,3)\}$, while $\mathbb{Z}_{15} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ has only three elements $\{(0.0.6), (0,2,0), (0,2,6)\}$

#5. $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_3$

Isomorphisms map generators to generators.
The elements of order 12 in $\mathbb{Z}_4 \oplus \mathbb{Z}_3$
$\Rightarrow$ since $\mathrm{lcm}(4,3) = 12$,
In $\mathbb{Z}_4$, elements of order 4 is 1, 3
In $\mathbb{Z}_3$, elements of order 3 is 1, 2
$\therefore$ elements of order 12 in $\mathbb{Z}_4 \oplus \mathbb{Z}_3$
: $(1,1)$, $(1,2)$, $(3,1)$, $(3,2)$

Since the isomorphisms are completely determined by where 1 maps, there are four isomorphisms.

#6. (1) prove $\mathbb{R} \oplus \mathbb{R} \approx \mathbb{C}$.

$\phi : \mathbb{R} \oplus \mathbb{R} \to \mathbb{C}$

$(a, b) \longmapsto a + bi$

① homomorphism

$\forall (a_1, b_1), (a_2, b_2) \in \mathbb{R} \oplus \mathbb{R}$

$$\begin{aligned} \phi((a_1, b_1), (a_2, b_2)) &= \phi((a_1 + a_2, b_1 + b_2)) \\ &= (a_1 + b_1 i) + (a_2 + b_2 i) \\ &= \phi(a_1, b_1) + \phi(a_2, b_2) \end{aligned}$$

② one-to-one.

$\phi(a_1, a_2) = \phi(a_2, b_2)$

$\Rightarrow a_1 + b_1 i = a_2 + b_2 i \Rightarrow a_1 = a_2,\ b_1 = b_2$

③ onto.

$\forall a + bi \in \mathbb{C},\ \phi(a, b) = a + bi$

$\therefore \phi$ is isomorphism, $\mathbb{R} \oplus \mathbb{R} \approx \mathbb{C}$

(2) prove $\mathbb{R}^* \oplus \mathbb{R}^* \not\approx \mathbb{C}^*$

Identity of $\mathbb{R}^* \oplus \mathbb{R}^* = (1, 1)$

Identity of $\mathbb{C}^* = 1$

$\mathbb{R}^* \oplus \mathbb{R}^*$에서 제곱하여 $(1, 1)$을 만들 수 있는 원소 :

$(1,1), (1,-1), (-1,1), (-1,-1)$ 4개.

$\mathbb{C}^*$에서 제곱하여 1을 만들 수 있는 원소 : $\{1, -1\}$ 2개.

$\therefore \mathbb{R}^* \oplus \mathbb{R}^* \not\approx \mathbb{C}^*$

#7. $H \leq G$, $N \triangleleft G$

WTS $\forall h \in H$, $x \in N \cap H \Rightarrow hxh^{-1} \in N \cap H$

$N \cap H$ is non-empty set ($e \in N \cap H$.)

Since $h, h^{-1}, x \in H$, $hxh^{-1} \in H$

Since N is a normal subgroup. $h \in H \subset G$

$hxh^{-1} \in G$.

$\therefore hxh^{-1} \in N \cap H$

$\therefore H \cap N$ is normal subgroup in H. ▨

#8. $H = \{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R}, ad \neq 0 \}$

If $H \triangleleft G$, then for $\forall x \in GL(2, \mathbb{R})$, $xHx^{-1} \in H$

Let $x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in GL(2, \mathbb{R})$

Then $x^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Let $y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H$

Then $x^{-1}yx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \notin H$.

$\therefore H$ is not a normal subgroup of $GL(2, \mathbb{R})$

#9. (1) $(\mathbb{Z}_4 \oplus \mathbb{Z}_{12}) / (\langle 2 \rangle \oplus \langle 2 \rangle)$

Let $H = \langle 2 \rangle \oplus \langle 2 \rangle$, $G = \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$

$H = \{(0,0), (2,2), (0,4), (2,6), (0,8), (2,10)\}$

$|H| = 6$.

$|\mathbb{Z}_4 \oplus \mathbb{Z}_{12}| = 4 \times 12 = 48$.

$\therefore |(\mathbb{Z}_4 \oplus \mathbb{Z}_{12}) / \langle 2 \rangle \oplus \langle 2 \rangle| = |\mathbb{Z}_4 \oplus \mathbb{Z}_{12}| / |\langle 2 \rangle \oplus \langle 2 \rangle| = \frac{48}{6} = 8$

(2) $(\mathbb{Z}_{10} \oplus U(10)) / \langle (2, 9) \rangle$

Let $H = \langle (2,9) \rangle$, $G = \mathbb{Z}_{10} \oplus U(10)$

$|U(10)| = 10 \times (1 - \frac{1}{2}) \times (1 - \frac{1}{5}) = 10 \times \frac{1}{2} \cdot \frac{4}{5} = 4$

$|\mathbb{Z}_{10}| = 10$.

$\therefore |G| = 10 \cdot 4 = 40$.

$H = \{(2,9), (4,1), (6,9), (8,1), (0,9), (2,1), (4,9),$
$(6,1), (8,9), (0,1)\}$

$\therefore |H| = 10.$

$\therefore |G/H| = \dfrac{|G|}{|H|} = \dfrac{40}{10} = 4.$

#10. $G = U(32)$, $H = \{1, 31\}$

$\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$G = U(32) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, \cdots, 31\}$

$H = \{1, 31\}$ $\quad 9H = \{9, 23\}$

$3H = \{3, 29\}$ $\quad 11H = \{11, 21\}$

$5H = \{5, 27\}$ $\quad 13H = \{13, 19\}$

$7H = \{7, 25\}$ $\quad 15H = \{15, 17\}$

$(3H)^2 = 9H \neq H \Rightarrow 3H$ has order at least 4.

$\Rightarrow G/H$ is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$(3H)^4 = 17H \neq H \Rightarrow |3H| \neq 4 \Rightarrow G/H$ is not isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$

$\Rightarrow |3H| = 8.$

$\therefore G/H \approx \mathbb{Z}_8.$