금융수학 1 - HW7 양준우 (2020-91659)

Prob. 1.1.

$X_t = tB_t^3 + \sin(B_t^2)$.

To show $Y_t = \int_0^t sB_s \, dX_s$ is well-defined

need to check

(i) X_t is Ito process.

(ii) $tB_t \in \mathcal{H}(X) = \{ g : \text{prog. m'ble} \mid P(\int_0^T |g_t b_t| + |g_t \sigma_t|^2 dt < \infty) = 1 \}$

(i) $X_t = tB_t^3 + \sin(B_t^2)$

$dX_t = df(t, B_t) = f_t dt + f_x dB_t + \frac{1}{2} f_{xx} dt$.

$= B_t^3 dt + (3tB_t^2 + 2B_t \cos(B_t^2)) dB_t + (3tB_t - 2B_t^2 \sin(B_t^2)) dt$

$= (B_t^3 + 3tB_t - 2B_t^2 \sin(B_t^2)) dt + (3tB_t^2 + 2B_t \cos(B_t^2)) dB_t$

$b_t := B_t^3 + 3tB_t - 2B_t^2 \sin(B_t^2)$ are conti. & adapted.

$\sigma_t := 3tB_t^2 + 2B_t \cos(B_t^2)$ $\Rightarrow$ prog. m'ble. & $b, \sigma \in \mathcal{H}^2_{loc}$

Since $b, \sigma \in \mathcal{H}^2_{loc}$, $\int_0^T b_u^2 du < \infty$, $\int_0^T \sigma_u^2 < \infty$.

$\int_0^T b_u^2 < \infty \Rightarrow \int_0^T b_u du < \infty$.

$\therefore P(\int_0^T |b_u| + |\sigma_u|^2 du < \infty) = 1$.

Thus, X_t is Ito process.

(ii). $g_t = tB_t$ is conti. and adapted $\Rightarrow$ prog. m'ble & $g_t \in \mathcal{H}^2_{loc}$

Since $g_t b_t$, $g_t \sigma_t$ are conti. and adapted $\Rightarrow$ $g_t b_t, g_t \sigma_t \in \mathcal{H}^2_{loc}$

$\therefore P(\int_0^T |tB_t b_t| + |tB_t \sigma_t|^2 dt < \infty) = 1$.

$\therefore tB_t \in \mathcal{H}(X)$.

Hence, $Y_t = \int_0^t sB_s \, dX_s$ is well-defined.

$Y_t = \int_0^t sB_s \, dX_s = \int_0^t sB_s (b_t dt + \sigma_t dB_t) = \int_0^t sB_s b_s dt + \int_0^t sB_s \sigma_s dB_s$

i.e., dynamics of Y_t is

$dY_t = tB_t b_t dt + tB_t \sigma_t dB_t)$

$= tB_t (B_t^3 + 3tB_t - 2B_t^2 \sin(B_t^2)) dt + tB_t (3tB_t^2 + 2B_t \cos(B_t^2)) dB_t$

$= (tB_t^4 + 3t^2 B_t^2 - 2tB_t^3 \sin(B_t^2)) dt + (3t^2 B_t^3 + 2tB_t^2 \cos(B_t^2)) dB_t$

Since Y_t is also Ito process,

$$\langle Y \rangle_T = \int_0^T \left(3t^2 B_t^3 + 2tB_t^2 \cos(B_t^2)\right)^2 dt$$

$$\langle X, Y \rangle_T = \int_0^T tB_t G_t^2 \, dt = \int_0^T tB_t \left(3tB_t^2 + 2B_t \cos(B_t^2)\right)^2 dt$$

Prob. 1.2.(i).

$$\left(\arctan B_t + \int_0^t \frac{B_s}{(1+B_s^2)^2} ds\right)_{t \geq 0}$$

$f(t) = \arctan t.$ $\quad f \in C^2$

$$df(B_t) = d(\arctan B_t) = \frac{1}{1+B_t^2} dB_t + \frac{1}{2} \cdot \frac{-2B_t}{(1+B_t^2)^2} d\langle B \rangle_t$$

$$= -\frac{B_t}{(1+B_t^2)^2} dt + \frac{1}{1+B_t^2} dB_t$$

Prob. 1.2.(ii).

Prob. 1.3.

$X_t = \int_0^t B_s^2\, dB_s$, $Y_t = B_t = \int_0^t 1\, dB_s$

$dX_t = B_t^2\, dB_t$ $\quad dY_t = dB_t$

B_s^2 and 1 : continuous and adapted $\Rightarrow$ prog. m'ble. & $\in \mathcal{H}^2_{loc}$

$\therefore X_t, Y_t$: Ito processes.

Let $f(t,x,y) = t^2 e^x y^2 \in C^{1,2,2}$

$$
\begin{aligned}
d(t^2 e^{X_t} Y_t^2) &= 2t e^{X_t} Y_t^2\, dt + t^2 e^{X_t} Y_t^2\, dX_t + 2t^2 e^{X_t} Y_t\, dY_t \\
&\quad + \frac{1}{2}\cdot t^2 e^{X_t} Y_t^2\, d\langle X\rangle_t + 2t^2 e^{X_t} Y_t\, d\langle X,Y\rangle_t + \frac{1}{2}\cdot 2t^2 e^{X_t}\, d\langle Y\rangle_t \\
&= 2t e^{X_t} Y_t^2\, dt + t^2 e^{X_t} Y_t^2 B_t^2\, dB_t + 2t^2 e^{X_t} Y_t\, dB_t \\
&\quad + \frac{1}{2}\cdot t^2 e^{X_t} Y_t^2 B_t^4\, dt + 2t^2 e^{X_t} Y_t B_t^2\, dt + t^2 e^{X_t}\, dt \\
&= \left(2t e^{X_t} Y_t^2 + \frac{1}{2}\cdot t^2 e^{X_t} Y_t^2 B_t^4 + 2t^2 e^{X_t} Y_t B_t^2 + t^2 e^{X_t}\right) dt \\
&\quad + \left(t^2 e^{X_t} Y_t^2 B_t^2 + 2t^2 e^{X_t} Y_t\right) dB_t \\
&= \left(2t e^{X_t} B_t^2 + \frac{1}{2}\cdot t^2 e^{X_t} B_t^6 + 2t^2 e^{X_t} B_t^3 + t^2 e^{X_t}\right) dt \\
&\quad + \left(t^2 e^{X_t} B_t^4 + 2t^2 e^{X_t} B_t\right) dB_t \quad \square. \\
&= e^{\int_0^t B_s^2 dB_s}\left[\left(2t B_t^2 + \frac{1}{2}\cdot t^2 B_t^6 + 2t^2 B_t^3 + t^2\right) dt + \left(t^2 B_t^4 + 2t^2 B_t\right) dB_t\right]
\end{aligned}
$$

Prob. 1.4.

$dX_t = aX_t\, dt + bX_t\, dB_t$, $\quad X_0 = 1$.

$dY_t = cY_t\, dt + dY_t\, dB_t$, $\quad Y_0 = 1$.

$$
\begin{aligned}
dU &= d(XY) = Y_t\, dX_t + X_t\, dY_t + d\langle X,Y\rangle_t \\
&= aX_tY_t\, dt + bX_tY_t\, dB_t + cX_tY_t\, dt + dX_tY_t\, dB_t + bdX_tY_t\, dt \\
&= (a+c+bd)U_t\, dt + (b+d)U_t\, dB_t
\end{aligned}
$$

$$
\begin{aligned}
dV &= d(X/Y) = \frac{1}{Y_t}\, dX_t - \frac{X_t}{Y_t^2}\, dY_t - \frac{1}{Y_t^2}\, d\langle X,Y\rangle_t + \frac{1}{2}\frac{2X_t}{Y_t^3}\, d\langle Y\rangle_t \\
&= \frac{1}{Y_t}(aX_t\, dt + bX_t\, dB_t) - \frac{X_t}{Y_t^2}(cY_t\, dt + dY_t\, dB_t) - \frac{1}{Y_t^2}(bdX_tY_t\, dt) \\
&\quad + \frac{X_t}{Y_t^3}(d^2 Y_t^2\, dt) \\
&= (a-c-bd+d^2)V_t\, dt + (b-d)V_t\, dB_t
\end{aligned}
$$

Prob. 1.5.(i).

$B = (B_t^{(1)}, \cdots, B_t^{(d)})_{t\ge 0}$.

$f \in C^{1,2}([0,\infty)\times\mathbb{R}^d)$

$df(t,B_t) = f_t(t,B_t)dt + \nabla f(t,B_t)dB_t + \frac{1}{2}\Delta f(t,B_t)dt$

By Taylor expansion for multivariate function,

$$f(t,X_t) = f(t, X^{(1)},\cdots,X^{(d)})$$
$$= f(0,X_0) + f_t(t_0,X_0)(t-t_0) + f_{x_1}(t_0,X_0)(X^{(1)}-X_0^{(1)})$$
$$+\cdots + f_{x_d}(t_0,X_0)(X^{(d)}-X_0^{(d)}) + \frac{1}{2}\sum_{i=1}^d f_{x_ix_i}(t_0,X_0)(X^{(i)}-X_0^{(i)})^2$$
$$+ \sum_{1\le i<j\le d} f_{x_ix_j}(t_0,X_0)(X^{(i)}-X_0^{(i)})(X^{(j)}-X_0^{(j)}) + \cdots$$

$$f(t,B_t)-f(t_0,B_0) = \sum_{\pi_n}(f(t_{i+1},B_{t_{i+1}}) - f(t_i,B_{t_i}))$$
$$\simeq \sum_{\pi_n}\{f_t(t_i,B_{t_i})(t_{i+1}-t_i) + \sum_{j=1}^d f_{x_j}(t_i,B_{t_i})(B^{(j)}_{t_{i+1}}-B^{(j)}_{t_i})$$
$$+\frac{1}{2}\sum_{j=1}^d f_{x_jx_j}(t_i,B_{t_i})\underbrace{(B^{(j)}_{t_{i+1}}-B^{(j)}_{t_i})^2}_{dt}$$
$$+\sum_{1\le j<k\le d} f_{x_ix_j}(t_i,B_{t_i})\underbrace{(B^{(j)}_{t_{i+1}}-B^{(j)}_{t_i})(B^{(k)}_{t_{i+1}}-B^{(k)}_{t_i})}_{0} + \cdots\}$$

(as $\|\pi_n\|\to 0$).

$(dt)^2 = dt\,dB = 0$, $(dB)^2 = dt$

$$\to \int_0^t f_t(u,B_u)du + \sum_{j=1}^d\int_0^t f_{x_j}(u,B_u)dB_u^{(j)} + \frac{1}{2}\sum_{j=1}^d\int_0^t f_{x_jx_j}(u,B_u)du$$
$$= \int_0^t f_t(u,B_u)du + \int_0^t\nabla f(u,B_u)dB_u + \frac{1}{2}\int_0^t\Delta f(u,B_u)du$$

i.e., $df(t,B_t) = f_t dt + \nabla f dB_t + \frac{1}{2}\Delta f dt$; dynamics).

Prob. 1.5.(ii).

Similarly to (i),

$$f(t,X_t)-f(0,X_0) = \sum_{\pi_n}(f(t_{i+1},X_{t_{i+1}}) - f(t_i,B_{t_i}))$$
$$\simeq \sum_{\pi_n}\{f_t(t_i,X_{t_i})(t_{i+1}-t_i) + \sum_{j=1}^d f_{x_j}(t_i,X_{t_i})(X^{(j)}_{t_{i+1}}-X^{(j)}_{t_i})$$
$$+\frac{1}{2}\sum_{j=1}^d f_{x_jx_j}(t_i,X_{t_i})(X^{(j)}_{t_{i+1}}-X^{(j)}_{t_i})^2$$
$$+\sum_{1\le j<k\le d} f_{x_ix_j}(t_i,X_{t_i})(X^{(j)}_{t_{i+1}}-X^{(j)}_{t_i})(X^{(k)}_{t_{i+1}}-X^{(k)}_{t_i}) + \cdots\}$$

(as $\|\pi_n\|\to 0$).

$$f(t,X_t)-f(0,X_0) = \int_0^t f_t\,dt + \sum_{i=1}^d\int_0^t f_{x_i}\,dX_t^{(i)}$$
$$+\frac{1}{2}\sum_{i=1}^d f_{x_ix_i}\,d\langle X^{(i)}\rangle_t + \sum_{1\le i<j\le d}\int_0^t f_{x_ix_j}\,d\langle X^{(i)},X^{(j)}\rangle_t$$
$$= \int_0^t f_t\,dt + \sum_{i=1}^d\int_0^t f_{x_i}\,dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^d\int_0^t f_{x_ix_j}\,d\langle X^{(i)},X^{(j)}\rangle_t$$

$$\Rightarrow df(t,X_t) = f_t(t,X_t)dt + \sum_{i=1}^d f_{x_i}(t,X_t)dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^d f_{x_ix_j}(t,X_t)d\langle X^{(i)},X^{(j)}\rangle_t$$

Prob. 1.6.

$$X_t = \cos(B_t^{(1)})\sin(B_t^{(1)}B_t^{(2)})$$

$$dX_t = \left(-\sin(B_t^{(1)})\sin(B_t^{(1)}B_t^{(2)}) + \cos(B_t^{(1)})\cos(B_t^{(1)}B_t^{(2)})B_t^{(2)}\right)dB_t^{(1)}$$

$$+\left(\cos(B_t^{(1)})\cos(B_t^{(1)}B_t^{(2)})B_t^{(1)}\right)dB_t^{(2)}$$

$$+\frac{1}{2}\left(-\cos(B_t^{(1)})\sin(B_t^{(1)}B_t^{(2)}) - \sin(B_t^{(1)})\cos(B_t^{(1)}B_t^{(2)})B_t^{(2)} - \sin(B_t^{(1)})\cos(B_t^{(1)}B_t^{(2)})B_t^{(2)}\right.$$

$$\left.-\cos(B_t^{(1)})\sin(B_t^{(1)}B_t^{(2)}){B_t^{(2)}}^2\right)dt$$

$$+\frac{1}{2}\left(-\cos(B_t^{(1)})\sin(B_t^{(1)}B_t^{(2)}){B_t^{(1)}}^2\right)dt$$

$$= \underbrace{\left(-\sin(B_t^{(1)})\sin(B_t^{(1)}B_t^{(2)}) + \cos(B_t^{(1)})\cos(B_t^{(1)}B_t^{(2)})B_t^{(2)}\right)}_{\alpha}dB_t^{(1)}$$

$$+\underbrace{\left(\cos(B_t^{(1)})\cos(B_t^{(1)}B_t^{(2)})B_t^{(1)}\right)}_{\beta}dB_t^{(2)}$$

$$+\frac{1}{2}\left(-\cos(B_t^{(1)})\sin(B_t^{(1)}B_t^{(2)})({B_t^{(1)}}^2+{B_t^{(2)}}^2+1) - 2\sin(B_t^{(1)})\cos(B_t^{(1)}B_t^{(2)})B_t^{(2)}\right)dt$$

$$Y_t = t^3\sin B_t^{(2)} + \int_0^t s^2(B_s^{(1)})^2\,dB_s^{(2)}$$

$$dY_t = d(t^3\sin B_t^{(2)}) + t^2{B_t^{(1)}}^2\,dB_t^{(2)}.$$

$$d(t^3\sin B_t^{(2)}) = 3t^2\sin B_t^{(2)}\,dt + t^3\cos B_t^{(2)}\,dB_t^{(2)} - \frac{1}{2}\cdot t^3\sin B_t^{(2)}\,dt.$$

$$= \left(-\frac{1}{2}t^3+3t^2\right)\sin B_t^{(2)}\,dt + t^3\cos B_t^{(2)}\,dB_t^{(2)}$$

$$dY_t = \left(-\frac{1}{2}t^3+3t^2\right)\sin B_t^{(2)}\,dt + \underbrace{t^3\cos B_t^{(2)}}_{\gamma}dB_t^{(2)} + \underbrace{t^2{B_t^{(1)}}^2}_{\delta}dB_t^{(2)}.$$

$$\langle X\rangle_t = \int_0^t(\alpha^2+\beta^2)\,du$$

$$\langle Y\rangle_t = \int_0^t(\gamma^2+\delta^2)\,du$$

$$\langle X, Y\rangle_t = \int_0^t \beta\gamma\,du.$$

Prob. 1.7.

$$d(B^{(1)}B^{(2)}) = B_t^{(2)}\,dB_t^{(1)} + B_t^{(1)}\,dB_t^{(2)}$$

$$\begin{aligned}
d(B_t^{(1)}B_t^{(2)}B_t^{(3)}) &= d(B_t^{(1)}B_t^{(2)}\cdot B_t^{(3)}) \\
&= B_t^{(3)}\,d(B_t^{(1)}B_t^{(2)}) + (B_t^{(1)}B_t^{(2)})\,dB_t^{(3)} \\
&= B_t^{(3)}(B_t^{(2)}\,dB_t^{(1)} + B_t^{(1)}\,dB_t^{(2)}) + B_t^{(1)}B_t^{(2)}\,dB_t^{(3)} \\
&= B_t^{(2)}B_t^{(3)}\,dB_t^{(1)} + B_t^{(3)}B_t^{(1)}\,dB_t^{(2)} + B_t^{(1)}B_t^{(2)}\,dB_t^{(3)}
\end{aligned}$$

$\Rightarrow B_t^{(1)}B_t^{(2)}B_t^{(3)} = 0 + \int_0^t (B_u^{(2)}B_u^{(3)},\ B_u^{(3)}B_u^{(1)},\ B_u^{(1)}B_u^{(2)})\,dB_u$

$B_t^{(2)}B_t^{(3)},\ B_t^{(3)}B_t^{(1)},\ B_t^{(1)}B_t^{(2)}$: conti. + adapted $\Rightarrow$ prog. m'ble.

$E\int_0^T (B_t^{(i)}B_t^{(j)})^2\,dt = \int_0^T t^2\,dt = \frac{T^3}{3} < \infty \quad \forall\, i,j \in \{1,2,3\}$

$\therefore (B_t^{(2)}B_t^{(3)},\ B_t^{(3)}B_t^{(1)},\ B_t^{(1)}B_t^{(2)})_{0\le t\le T} \in \mathcal{H}^2(\Omega\times[0,T],\ \mathbb{R}^3,\ \mathcal{F},\ (\mathcal{F}_t),\ P)$.

$B_t^{(1)}B_t^{(2)}B_t^{(3)} = \int_0^t (B_t^{(2)}B_t^{(3)},\ B_t^{(3)}B_t^{(1)},\ B_t^{(1)}B_t^{(2)})\,dB_u$ is mart. each $T>0$

$\therefore (B_t^{(1)}B_t^{(2)}B_t^{(3)})_{t\ge 0}$ is a martingale. $\square$

Prob. 1.8.(i).

$dX_t = a(\theta - X_t)dt + \sigma dB_t$, $X_0 = x$.

Let $Y_t = e^{at}X_t$

$$dY_t = d(e^{at}X_t) = ae^{at}X_t dt + e^{at}dX_t$$
$$= ae^{at}X_t dt + e^{at}(a(\theta - X_t)dt + \sigma dB_t)$$
$$= a\theta e^{at}dt + \sigma e^{at}dB_t$$

$$Y_t = Y_0 + a\theta\int_0^t e^{au}du + \sigma\int_0^t e^{au}dB_u$$
$$= x + a\theta\cdot\frac{1}{a}(e^{at}-1) + \sigma\int_0^t e^{au}dB_u$$

$$X_t = e^{-at}Y_t = xe^{-at} + \theta(1-e^{-at}) + \sigma\int_0^t e^{-a(t-u)}dB_u$$

Prob. 1.8.(ii).

$(X_t)_{t\geq 0}$ is Gaussian process

$\Leftrightarrow \sum_{i=1}^n \lambda_i X_{t_i}$ is normal for any $\lambda_i \in \mathbb{R}$, $\{0\leq t_1\leq t_2\leq\cdots\leq t_n\leq T\}$.

Since $xe^{-at}+\theta(1-e^{-at})$, σ e^{-at} are constants,

ETS $\sum_{i=1}^n \lambda_i\int_0^{t_i} e^{au}dB_u$ is normal. $\Leftrightarrow$ $(\int_0^t e^{au}dB_u)_{0\leq t\leq T}$ is Gaussian.

$(e^{ax})^{-1}(A)\in B(\mathbb{R})$ for $^\forall A\in B(\mathbb{R})$ $\Rightarrow$ e^{ax} is Borel function.

$\int_0^T e^{2au}du = \frac{1}{2a}(e^{2aT}-1) < \infty$. for each $T>0$.

By HW6. Prob. 1.2.(ii), $(\int_0^t e^{au}dB_u)_{0\leq t\leq T}$ is Gaussian process.

$\therefore$ $(X_t)_{t\geq 0}$ is Gaussian process. $\square$.

Prob. 1.8.(iii).

Since $(X_t)_{t\geq 0}$ is Gaussian process, X_T is normal.

Then $E(e^{X_T}) = e^{E(X_T)+\frac{1}{2}Var(X_T)}$

$$E(X_T) = E(xe^{-aT}+\theta(1-e^{-aT})+\sigma\int_0^T e^{-a(T-u)}dB_u)$$
$$= xe^{-aT}+\theta(1-e^{-aT})+\sigma e^{-aT}E(\int_0^T e^{au}dB_u) \quad *$$
$$= xe^{-aT}+\theta(1-e^{-aT})+\sigma e^{-aT}E(\int_0^0 e^{au}dB_u)$$
$$= xe^{-aT}+\theta(1-e^{-aT}) = \theta + (x-\theta)e^{-aT}$$

$*$ e^{au} : conti. + adapted $\Rightarrow$ prog. m'ble. & $E\int_0^T e^{2au}du = \frac{1}{2a}(e^{2aT}-1) < \infty$

$\Rightarrow e^{au}\in\mathcal{H}^2 \Rightarrow (\int_0^t e^{au}dB_u)_{0\leq t\leq T}$: martingale.

$$Var(X_T) = Var(xe^{-aT} + \theta(1-e^{-aT}) + \sigma\int_0^T e^{-a(T-u)}dB_u)$$
$$= \sigma^2 e^{-2aT} Var(\int_0^T e^{au} dB_u)$$
$$= \sigma^2 e^{-2aT}(E((\int_0^T e^{au} dB_u)^2) - (E(\int_0^T e^{au} dB_u))^2)$$
$$= \sigma^2 e^{-2aT} E(\int_0^T e^{2au} du) \quad (\because E(\int_0^T e^{au} dB_u) = 0,\ e^{au} \in \mathcal{H}^2, \text{ by Ito isometry}).$$
$$= \sigma^2 e^{-2aT} \cdot \frac{1}{2a}(e^{2aT}-1)$$
$$= \frac{\sigma^2}{2a}(1-e^{-2aT})$$
$$\therefore E(e^{X_T}) = e^{E(X_T) + \frac{1}{2}Var(X_T)} = e^{\theta + (x-\theta)e^{-aT} + \frac{\sigma^2}{4a}(1-e^{-2aT})}$$

Prob. 1.9.

$$dX_t = r_t dt + \sigma X_t dB_t$$
Let $Y_t = e^{-\sigma B_t + \frac{1}{2}\sigma^2 t}$, $f(t,x) = e^{-\sigma x + \frac{1}{2}\sigma^2 t} \in C^{1,2}$
$$dY_t = \frac{1}{2}\sigma^2 Y_t dt - \sigma Y_t dB_t + \frac{1}{2}\sigma^2 Y_t dt$$
$$= \sigma^2 Y_t dt - \sigma Y_t dB_t \quad : \text{Ito process.}$$
$$d(X_tY_t) = Y_t dX_t + X_t dY_t + d\langle X,Y\rangle_t$$
$$= r_tY_t dt + \sigma X_tY_t dB_t + \sigma^2 X_tY_t dt - \sigma X_tY_t dB_t - \sigma^2 X_tY_t dt$$
$$= r_tY_t dt$$
$$\Rightarrow X_tY_t = X_0Y_0 + \int_0^t r_uY_u du$$
$$X_t = \frac{X_0}{Y_t} + \frac{1}{Y_t}\int_0^t r_uY_u du. \quad (\because Y_0 = 1,\ Y_t \neq 0).$$
$$= e^{\sigma B_t - \frac{1}{2}\sigma^2 t}(X_0 + r\int_0^t u e^{-\sigma B_u + \frac{1}{2}\sigma^2 u} du).$$