Topology I – Homework 4

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May 8, 2020

Problem 4.1 Let $(X_1, d_1), \cdots, (X_n, d_n)$ be metric spaces and $X = X_1 \times \cdots \times X_n$. Show that the three product metrics

(i) $d(x, y) = \sqrt{\sum_{i=1}^n d_i(x_i, y_i)^2}$

(ii) $d(x, y) = \max\{d_1(x_1, y_1), \cdots, d_n(x_n, y_n)\}$

(iii) $d(x, y) = \sum_{i=1}^n d_i(x_i, y_i)$

where $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, satisfy the conditions

($\star$) a sequence $\{x^j = (x_1^j, \cdots, x_n^j)\}_{j=1}^\infty$ converges to $x = (x_1, \cdots, x_n)$ in X if and only if for each k, $\{x_k^j\}$ converges to x_k in X_k,

($\star\star$) $d_k(x_k, y_k) \leq d(x, y)$ for all k.

Proof. ($\star$) Suppose a sequence $\{x^j\}$ converges to x. Then, $\lim_{j\to\infty} d(x, x^j) = 0$. Thus, all three product metrics

$$\sqrt{\sum_{i=1}^n d_i(x_i, x_i^j)^2}, \quad \max\{d_1(x_1, x_1^j), \cdots, d_n(x_n, x_n^j)\}, \quad \sum_{i=1}^n d_i(x_i, x_i^j)$$

tend to 0 as j goes to infinity. Therefore, for each k, $\{x_k^j\}$ converges to x_k in X_k.
Conversely, since all $\{x_k^j\}$ converges to x_k in X_k, $d(x, x^j)$ tends to 0 as j goes to infinity. Thus, $\{x^j\}$ converges to x.
($\star\star$) For all $k \in [1, n]$, inequalities hold obviously

(i) $d_k(x_k, y_k) = \sqrt{d_k(x_k, y_k)^2} \leq d(x, y) = \sqrt{\sum_{i=1}^n d_i(x_i, y_i)^2}$

(ii) $d_k(x_k, y_k) \leq d(x, y) = \max\{d_1(x_1, y_1), \cdots, d_n(x_n, y_n)\}$

(iii) $d_k(x_k, y_k) \leq d(x, y) = \sum_{i=1}^n d_i(x_i, y_i)$.

$\square$