Mathematical Statistics II

Ch.5 Distributions of Functions of Random Variables

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Ch5.3 Several Independent Random Variables

5.3.1 Random Sample

Random Sample

The random variables $X_1, \ldots, X_n$ are called a random sample of size n from the common distribution, $f(x)$ if $X_1, \ldots, X_n$ are mutually independent random variables and their distribution is the same, $f(x)$. Alternatively, $X_1, \ldots, X_n$ are called independent and identically distributed (iid) random variable with $f(x)$. The joint pdf or pmf of $X_1, \ldots, X_n$ is given by

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\cdots f(x_n) = \prod_{i=1}^{n} f(x_i)$$

In Ch4. X_1 & X_2 are indep.

$\Rightarrow f(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$

$\Rightarrow$ Extending with the same dist^n

(eg. 5.3-1) Toss 2 dice indep.

Let X_1 & X_2 be a value appearing in ith die

$X_1, X_2 \sim f(x)$.

$f(x) = \frac{1}{6} \quad x = 1, \cdots, 6.$

$\Rightarrow$ Joint pmf of X_1 & X_2

$f(x_1, x_2) = \prod_{i=1}^{2} f_{X_i}(x_i) = \prod_{i=1}^{2} \left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)^2 \quad x_1, x_2 = 1, \cdots, 6.$

cf). mutually independent

pairwise 〃 $\prod f_{X_i}(x_i)$

Def) $X_1, \cdots, X_n$: random sample

(or independent and identically distributed, iid)

if ① mutually indep. $f(x_1, \cdots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$

② same distribution. $f_{X_i}(x_i) = f_X(x) \quad i = 1, \cdots, n$

$(X_i \sim f(x) \quad i = 1, \cdots, n)$

iid = random sample

expression).

- $X_i \overset{iid}{\sim} f(x), \quad i = 1, \cdots, n, \quad$ iid
- $X_i, \quad i = 1, \cdots, n$ be a random sample.

indep, same dist^n

$\Rightarrow f(x_1, \cdots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$

Example 5.3-2

Let $X_1,\ X_2,\ X_3$ be a random sample from a distribution with pdf

$$f_X(x) = e^{-x}, \quad x > 0$$

Find $P(0 < X_1 < 1,\ 2 < X_2 < 4,\ 3 < X_3 < 7)$.

$f_X(x) = e^{-x}, \quad x > 0$

Joint pdf

$$f(x_1, x_2, x_3) = e^{-x_1} e^{-x_2} e^{-x_3}$$
$$= e^{-(x_1+x_2+x_3)}, \quad x_i > 0 \quad i = 1, 2, 3$$

$P(0 < X_1 < 1,\ 2 < X_2 < 4,\ 3 < X_3 < 7)$

$$= \int_0^1 e^{-x_1} dx_1 \cdot \int_2^4 e^{-x_2} dx_2 \cdot \int_3^7 e^{-x_3} dx_3$$

$$= \left[-e^{-x_1}\right]_0^1 \cdot \left[-e^{-x_2}\right]_2^4 \cdot \left[-e^{-x_3}\right]_3^7$$

$$= (1 - e^{-1})(e^{-2} - e^{-4})(e^{-3} - e^{-7})$$

($\because$ X_1, X_2, X_3 are independent).

Let $X_1, \ldots, X_n$ be n independent random variables each of which has own pmf $f_i(x_i),\ i = 1, \ldots, n$. Let $Y = u(X_1, X_2, \ldots, X_n)$ with the pmf $f_Y(y)$. Then

$$E(Y) = \sum_y y f_Y(y) = \sum_{x_1} \cdots \sum_{x_n} u(x_1, \ldots, x_n) \underset{f_{X_1}}{f_1(x_1)} \cdots \underset{f_{X_n}}{f_n(x_n)},$$

provided that these summations exist.

$X_i \sim f_{X_i}(x_i) \quad i = 1, \cdots, n \quad \leftarrow$ indep.

$Y = u(X_1, \cdots, X_n) \sim ? \quad f_Y(y)$

$$E[Y] = \sum_{\forall y} y f_Y(y)$$

$$\parallel$$

$$E[u(X_1, \cdots, X_n)] = \sum_{\forall x_1} \cdots \sum_{\forall x_n} u(x_1, \cdots, x_n) \cdot f(x_1, \cdots, x_n)$$

$$= \sum_{\forall x_1} \cdots \sum_{\forall x_n} u(x_1, \cdots, x_n) \cdot f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

X_1, X_2

$$E[g(X_1, X_2)] = \sum_{\forall x_1} \sum_{\forall x_2} g(x_1, x_2) \cdot f(x_1, x_2)$$

Thm 5.3-1

Let $X_1, \ldots, X_n$ be n independent random variables. Let $Y = u_1(X_1) \cdots u_n(X_n)$. If $E[u_i(X_i)],\ i = 1, \ldots, n$ exist,

$$E(Y) = E[u_1(X_1) \cdots u_n(X_n)]$$

$$= E[u_1(X_1)] \cdots E[u_n(X_n)] = \prod_{i=1}^{n} E[u_i(X_i)].$$

X_1, X_2.

$Y = \underset{u_1(X_1)}{(2X_1+1)} \cdot \underset{u_2(X_2)}{e^{-X_2}}$

$X_i \sim f_{X_i}(x_i) \quad i = 1, \cdots, n$ independent.

$Y = u_1(X_1) \cdots u_n(X_n)$

$E[Y] = E[u_1(X_1) \cdots u_n(X_n)]$

$$= \underbrace{\int \cdots \int}_{n} u_1(x_1) \cdots u_n(x_n) \cdot f(x_1 \cdots x_n)\, dx_1 \cdots dx_n$$

$$= \underbrace{\int \quad \int}_{n} u_1(x_1) \cdots u_n(x_n) \cdot f_1(x_1) \cdots f_n(x_n)\, dx_1 \cdots dx_n \quad (\because \text{indep}).$$

$$= \left[\int u_1(x_1) f_1(x_1)\, dx_1\right] \times \cdots \times \left[\int u_n(x_n) f_n(x_n)\, dx_n\right]$$

$$= E[u_1(X_1)] \cdots E[u_n(X_n)]$$

$$= \prod_{i=1}^{n} E[u_i(X_i)]$$

eg). $X_1 \sim Unif(0,1)$, $X_2 \sim Unif(0,1)$ indep.

$Y = (2X_1+3)(5X_2+1)$

$\Rightarrow E[Y] = E[(2X_1+3)(5X_2+1)]$

$= E[2X_1+3] \cdot E[5X_2+1]$ $(\because$ indep$)$.

$= 2\underbrace{E[X_1]}_{\frac{1}{2}} + 3 + 5\underbrace{E[X_2]}_{\frac{1}{2}} + 1$

$$\int_0^1 \int_0^1 (2x_1+3)(5x_2+1) \underset{f_{X_1}(x_1) f_{X_2}(x_2)}{f(x_1, x_2)}\, dx_1 dx_2$$

$$= \int_0^1 (2x_1+3) f_{X_1}(x_1)\, dx_1 \quad ''$$

$$= E[2X_1+3] \cdot E[5X_2+1].$$

Thm 5.3-2

Let $X_1, \ldots, X_n$ be n independent random variables, each of which has own mean μ_i and variance σ_i^2, $i = 1, \ldots, n$. Suppose that $Y = \sum_{i=1}^{n} a_i X_i$, where $a_1, \ldots, a_n$ are real constants.

$$E[Y] = \mu_Y = \sum_{i=1}^{n} a_i \mu_i, \quad Var(Y) = \sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

$X_i \sim ?\ (\mu_i, \sigma_i^2)$. $i = 1, \cdots, n$, indep.

$Y = \sum_{i=1}^{n} a_i X_i \sim ?\ (\mu_Y, \sigma_Y^2)$.

$$\mu_Y = E[Y] = E[\sum_{i=1}^{n} a_i X_i] = E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n]$$

$$= a_1 E(X_1) + a_2 E(X_2) + \cdots + a_n E(X_n) = \sum_{i=1}^{n} a_i \mu_i$$

$$\sigma_Y^2 = Var(Y) = E[(Y - \mu_Y)^2] = E[(\sum_{i=1}^{n} a_i X_i - \sum a_i \mu_i)^2]$$

$$= E[(\sum_{i=1}^{n} a_i (X_i - \mu_i))^2]$$

$$= E[\{a_1(X_1 - \mu_1) + a_2(X_2 - \mu_2) + \cdots + a_n(X_n - \mu_n)\}^2]$$

$$= E[a_1^2(X_1 - \mu_1)^2 + a_2^2(X_2 - \mu_2)^2 + \cdots + a_n^2(X_n - \mu_n)^2 + \underline{2\sum_{i<j}\sum a_i a_j (X_i - \mu_i)(X_j - \mu_j)}]$$

$$= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2 + \underline{\qquad\qquad}$$

For $i \neq j$. indep.

$\underline{E[a_i a_j (X_i - \mu_i)(X_j - \mu_j)]}$

$= a_i a_j \underline{E(X_i - \mu_i)} \cdot E(X_j - \mu_j)$ ($\because$ independent).

$= E(X_i) - \mu_i = \mu_i - \mu_i = 0$.

$= 0$.

$\therefore Y \sim (\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2)$.

$X_1, \cdots, X_n$ independent R.V.

$Y = \sum_{i=1}^{n} a_i X_i \qquad a_i \in \mathbb{R}. \qquad \exists\, \mu_i,\ \sigma_i^2$

$\mu_Y,\ \sigma_Y^2$ //.

$$\mu_Y = E[Y] = E[a_1 X_1 + \cdots + a_n X_n]$$

$$= a_1 E[X_1] + \cdots + a_n E[X_n]$$

$$= a_1 \mu_1 + \cdots + a_n \mu_n = \sum_{i=1}^{n} a_i \mu_i$$

$$\sigma_Y^2 = E[(Y - \mu_Y)^2]$$

$$= E\left[\left(\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i \mu_i\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} a_i (X_i - \mu_i)\right)^2\right]$$

$$= E[(a_1(X_1 - \mu_1) + \cdots + a_n(X_n - \mu_n))^2]$$

$$= E[a_1^2 (X_1 - \mu_1)^2 + \cdots + a_n^2 (X_n - \mu_n)^2 + \sim\sim]$$

$$= a_1^2 E[(X_1 - \mu_1)^2] + \cdots + a_n^2 E[(X_n - \mu_n)^2] + E[\sim\sim]$$

$$= \sum_{i=1}^{n} a_i^2 \sigma_i^2 + E[\sim\sim].$$

$\sim\sim = 2 \sum_{i<j} \sum a_i a_j (X_i - \mu_i)(X_j - \mu_j) \qquad i, j = 1, \cdots, n.$

For $i \neq j$

$E[a_i a_j (X_i - \mu_i)(X_j - \mu_j)]$

$= a_i a_j E[X_i - \mu_i] E[X_j - \mu_j]$

$= a_i a_j (\underbrace{E[X_i] - \mu_i}_{0})(\underbrace{E[X_j] - \mu_j}_{0}).$

Example 5.3-4

Let X_1 and X_2 be independent random variables with $\mu_1 = -4$ and $\mu_2 = 3$ and variances $\sigma_1^2 = 4$ and $\sigma_2^2 = 9$. Find the mean and variance of $Y = 3X_1 - 2X_2$.

Example 5.3-5

Let X_1 and X_2 be a random sample from a distribution with mean μ and variances σ^2. Find the mean and variance of $Y = X_1 - X_2$.

ex. 5.3.4).

X_1, X_2 indep. $\mu_1 = -4$ $\mu_2 = 3$, $\sigma_1^2 = 4$, $\sigma_2^2 = 9$.

$Y = 3X_1 - 2X_2$.

$\mu_Y = \sum_{i=1}^{2} a_i \mu_i = 3 \cdot -4 - 2 \cdot 3 = -12 - 6 = -18.$

$\sigma_Y^2 = \sum_{i=1}^{2} a_i^2 \sigma_i^2 = 3^2 \cdot 4 + (-2)^2 \cdot 9 = 36 + 36 = 72.$

ex. 5.3.5).

X_1, X_2 : random sample. μ, σ^2.

$Y = X_1 - X_2$

$\mu_Y = 1 \cdot \mu - 1 \cdot \mu = \mu - \mu = 0.$

$\sigma_Y^2 = 1^2 \sigma^2 + (-1)^2 \sigma^2 = 2\sigma^2$

5.3.2 Statistic

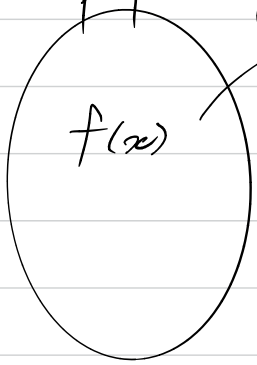
Statistic

Let $X_1, \ldots, X_n$ be a random sample. Any function of the random sample, $g(X_1, \ldots, X_n)$, is called a statistic.

Examples

i) $\overline{X} = \sum_{i=1}^{n} X_i/n$: mean of random sample (sample mean)

ii) $S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$: sample variance pop



indep. sample.

$X_1 \cdots X_n \sim f(x)$.

$X_i \sim f(x)$, iid.

$\bar{X} = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i \sim \boxed{?}$ //.

$\min\{X_i\}$, $\max\{X_i\}$: statistic

$X_i \sim f(x)$.

$M_{X_i}(t)$: mgf.

$Y = g(X_1, \cdots, X_n) \sim \boxed{f_Y(y)}$ $\quad M_Y(t)$.

Ch5.4 The Moment-Generating Function Technique

Example 5.4-1

Let X_1 and X_2 be independent random variables with uniform distributions on $\{1, 2, 3, 4\}$. Find the mgf of $Y = X_1 + X_2$.

$$// \ \frac{\partial}{\partial x} \cdot x \cdot e^{tx}$$

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}]$$

$$= E[\underbrace{e^{tX_1}}_{u_1(X_1)} \cdot \underbrace{e^{tX_2}}_{u_2(X_2)}] = E[e^{tX_1}] \cdot E[e^{tX_2}] \quad (\because \text{ indep}).$$

$$= M_{X_1}(t) \cdot M_{X_2}(t) = [M_{X_1}(t)]^2$$

$$= \frac{1}{16}(e^t + e^{2t} + e^{3t} + e^{4t})^2, \quad t \in \mathbb{R}$$

where $X \sim$ unif on $\{1, 2, 3, 4\}$.

$$f_X(x) = \frac{1}{4}, \quad x = 1, 2, 3, 4.$$

$$M_{X_1}(t) = E[e^{tX_1}] = \sum_{x \in \{1,2,3,4\}} e^{tx} f_1(x)$$

$$= \frac{1}{4}(e^t + e^{2t} + e^{3t} + e^{4t}), \quad t \in \mathbb{R}$$

just indep.
not random sample.

Thm 5.4-1

Let $X_1, \ldots, X_n$ be n independent random variables with respective moment generating functions $M_{X_i}(t),\ i = 1, \ldots, n$, where $-h_i < t < h_i$, for positive number h_i. The mgf of $Y = \sum_{i=1}^{n} a_i X_i$ is

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t), \quad \text{where} \quad -h_i < a_i t < h_i.$$

$X_i \overset{\text{indep}}{\sim} \bigcirc\,(\mu_i, \sigma_i^2)$. $M_{X_i}(t)$: mgf.

$Y = \sum_{i=1}^{n} a_i X_i \sim \square\,(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2)$. $M_Y(t)$.

$X_i \sim M_{X_i}(t)$ indep. ; $i=1, \cdots, n$. , $-h_i < t < h_i$

$Y = \sum_{i=1}^{n} a_i X_i \sim M_Y(t) \Leftarrow$ how to find ?

$$M_Y(t) = E[e^{tY}] = E[e^{t\sum_{i=1}^{n} a_i X_i}]$$

$$= E[e^{t(a_1 X_1)} \cdots e^{t(a_n X_n)}]$$

$$= E[e^{t a_1 X_1}] \cdots E[e^{t a_n X_n}] \quad (\because \text{indep}).$$

$$= \prod_{i=1}^{n} E[e^{t a_i X_i}]$$

$$= \prod_{i=1}^{n} M_{X_i}(a_i t) \quad \text{where} \quad -h_i < a_i t < h_i$$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t).$$

If $X_i \sim M_X(t)$, then $M_Y(t) = \prod_{i=1}^{n} M_X(a_i t)$.

Corollary 5.4-1

Let $X_1, \ldots, X_n$ be n independent random sample from a distribution with mgf $M_X(t)$, where $-h < t < h$.

- mgf of $Y = \sum_{i=1}^{n} X_i$ is 1) $X_1 + \cdots + X_n$

$$M_Y(t) = \prod_{i=1}^{n} M_X(t) = [M_X(t)]^n, \qquad -h < t < h.$$

- mgf of $\overline{X} = \sum_{i=1}^{n} X_i/n$ is 2) $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$ $\left(a_i = \frac{1}{n}\right)$

$$M_{\overline{X}}(t) = \prod_{i=1}^{n} M_X\left(\frac{t}{n}\right) = \left[M_X\left(\frac{t}{n}\right)\right]^n \qquad -h < \frac{t}{n} < h$$

Example 5.4-2

Let $X_1, \ldots, X_n$ be the independent random variable following Bernoulli trial with success probability p. Let $Y = \sum_{i=1}^{n} X_i$. Find the mgf of Y.

Example 5.4-3

Let $X_1,\ X_2,\ X_3$ be a random sample from the exponential distribution with mean θ. Find the mgf of $Y = \sum_{i=1}^{3} X_i$ and $\overline{X}$.

☆

ex. 5.3.2. $X_i \sim Ber(p)$. iid, $i = 1, \cdots, n$.

$f(x) = (1-p)^{1-x} \cdot p^x$, $x = 0, 1$

$M_{X_1}(t) = E[e^{tX_1}] = \sum_{x_1=0}^{1} e^{tx_1} f(x_1) = (1-p) + pe^t$ $\quad t \in \mathbb{R}$.

$Y = \sum_{i=1}^{n} X_i$

$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = [M_X(t)]^n = [(1-p) + pe^t]^n$ $\quad t \in \mathbb{R}$.

$\therefore Y \sim Bin(n, p)$.

pop

$X_1 \cdots X_n \sim Ber(p)$

$\Sigma X_i \sim Bin(n, p)$.

ex. 5.4.3. $X_1, X_2, X_3 \sim Exp(\theta) = Gamma(1, \theta)$.

$X_i \sim M_X(t) = \frac{1}{1-\theta t}$, $t < \frac{1}{\theta}$

cf).

$X \sim Gamma(\alpha, \beta)$.

$M_X(t) = (1-\beta t)^{-\alpha}$

$t < \frac{1}{\beta}$

1) $Y = X_1 + X_2 + X_3$.

$M_Y(t) = \prod_{i=1}^{3} [\frac{1}{1-\theta t}] = [\frac{1}{1-\theta t}]^3$, $\quad t < \frac{1}{\theta}$

: mgf of $Gamma(3, \theta)$

$\therefore Y \sim Gamma(3, \theta)$.

2) $\bar{X} = \sum_{i=1}^{3} \frac{1}{3} X_i \sim Gamma(3, \frac{\theta}{3})$.

$M_{\bar{X}}(t) = \prod_{i=1}^{3} M_X(\frac{t}{3}) = \prod_{i=1}^{3} [\frac{1}{1-\frac{\theta t}{3}}]$

$= [1 - \frac{\theta}{3} t]^{-3}$, $\quad t < \frac{3}{\theta}$

: mgf of $Gamma(3, \frac{\theta}{3})$.

Thm 5.4-2

Let $X_1, \ldots, X_n$ be independent chi-square random variables with $r_1, r_2, \ldots, r_n$ degrees of freedom, respectively. Then, the distribution of $Y = \sum_{i=1}^{n} X_i$ is $\chi^2(r_1 + \cdots + r_n)$.

$Exp(\beta) \equiv Gamma(1, \beta)$.

$X \sim N(\mu, \sigma^2)$.

$\frac{1}{1-\beta t}$ $\left(\frac{1}{1-\beta t}\right)^{\alpha}$

$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$t < \frac{1}{\beta}$

$Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2(1)$

$\chi^2(r) \equiv Gamma\left(\frac{r}{2}, 2\right)$.

$X_i \sim \chi^2(r_i)$

$\left(\frac{1}{1-2t}\right)^{\frac{r}{2}}$

$Y = \Sigma X_i \sim \chi^2(\Sigma r_i)$

$$\left.\begin{array}{l} Z_1^2 \sim \chi^2(1) \\ \vdots \\ Z_n^2 \sim \chi^2(1) \end{array}\right) \Leftrightarrow \begin{array}{l} Z_i^2 \sim \chi^2(1) \text{ indep.} \\ \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n). \end{array}$$

$t < \frac{1}{2}$

$Y = \sum_{i=1}^{n} X_i$ $\quad X_i \sim \chi^2(r_i). \equiv Gamma\left(\frac{r_i}{2}, 2\right)$

$X_i \sim M_i(t) = \left(\frac{1}{1-2t}\right)^{\frac{r_i}{2}}$

$M_Y(t) = E[e^{tY}]$

$\prod_{i=1}^{n} M_i(t). = \prod_{i=1}^{n} \left(\frac{1}{1-2t}\right)^{\frac{r_i}{2}}$

$\left(\frac{1}{1-2t}\right)^{\frac{1}{2}(r_1 + \cdots + r_n)}$.

: mgf of $\chi^2(r_1 + \cdots + r_n)$.

cf). X : Gamma $(\alpha, \beta) \Rightarrow M_X(t) = (1-\beta t)^{-\alpha}$, $t < \frac{1}{\beta}$

Thm $X_i \sim \chi^2(r_i)$ indep. $i = 1, \cdots, n$.

$Y = \sum_{i=1}^{n} X_i$

Since $X_i \sim \chi^2(r_i) = \text{Gamma}\left(\frac{r_i}{2}, 2\right)$,

$M_{X_i}(t) = (1-2t)^{-\frac{r_i}{2}}$, $t < \frac{1}{2}$

$$M_Y(t) = E[e^{tY}] = E[e^{t\sum X_i}] = \prod_{i=1}^{n} E[e^{tX_i}]$$

$$= \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} (1-2t)^{-\frac{r_i}{2}} = (1-2t)^{-\frac{\sum r_i}{2}}, \quad t < \frac{1}{2}$$

: mgf of $\text{Gamma}\left(\frac{\sum r_i}{2}, 2\right)$

$\therefore Y \sim \text{Gamma}\left(\frac{\sum r_i}{2}, 2\right) = \chi^2\left(\sum_{i=1}^{n} r_i\right)$

Cor 1). $Z_1 \cdots Z_n \sim N(0,1) \Rightarrow Z_i^2 \sim \chi^2(1)$ $i = 1, \cdots, n$

$W = Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi^2(n)$.

Cor 2). $X_i \sim N(\mu_i, \sigma_i^2)$ indep.

$\Rightarrow Z_i = \frac{X_i - \mu_i}{\sigma_i} \sim N(0,1)$.

$Z_i^2 = \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi^2(1)$. $i = 1, \cdots, n$.

$\sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$

$\sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi^2(n)$.

independent

Corollary 5.4-2

Let $Z_1, \ldots, Z_n$ have standard normal distributions, $N(0,1)$. Then, $W = Z_1^2 + Z_2^2 + \cdots + Z_n^2$ has a distribution, $\chi^2(n)$.

Corollary 5.4-3

Let $X_1, \ldots, X_n$ have independent normal distributions, $N(\mu_i, \sigma_i^2)$.

$$W = \sum_{i=1}^{n} \frac{(X_i - \mu_i)^2}{\sigma_i^2} \sim \chi^2(n)$$