# Mathematical Statistics 1

## Ch.2 Discrete Distributions

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# Ch.2.1 Random Variables of the Discrete Type

# 1.1 Review of Random Variable

r.v [ disc r.v / cont r.v ]

## Definition

A *random variable* is a function from a sample space $\mathcal{S}$ into the real numbers.

- discrete random variable: countable numbers of points
- continuous random variable: numerical values in an interval or collection of intervals

$S$: Height $\longrightarrow$ $\mathbb{R}$: $\mathbb{R}^+$

cont

$X$ [ h+ 크기 0 . / other ]

H $\longrightarrow$ 0, 1

disc r.v ✓

같은 sample space 여도 discrete, continuous 로 나뉠 수 있음.

## Examples

Let X be the random variable.

(a) Toss two dice. X=sum of the numbers

(b) Toss a coin 5 times. X=number of heads in 5 tosses.

# 1.2 Probability mass function (pmf)

## Definition

The *probability mass function (pmf)* $f_X(x)$ of a discrete random variable $X$ is is a function defined by

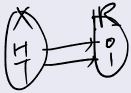
$$f_X(x) = P(X = x)$$

and it satisfies the following properties:

(a) $f_X(x) > 0, \quad x \in \overset{\text{가이 set}}{\mathcal{X}},$

(b) $\sum_{x \in \mathcal{X}} f_X(x) = 1,$

(c) $P(X \in \mathbf{A}) = \sum_{x \in \mathbf{A}} f_X(x)$, where $\mathbf{A} \subset \mathcal{X}$.




$x = \{0, 1\}$

$4 \times 4 = 16.$

1 2 3 4

1 2 3 4.

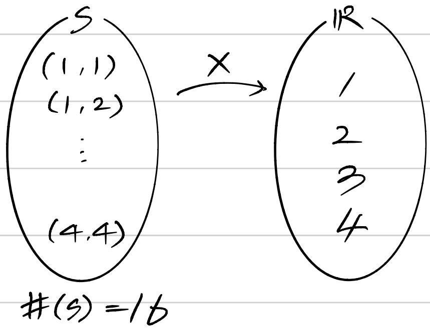
$X = 1 \quad 2 \quad 3 \quad 4.$

$\frac{1}{16} \quad \frac{3}{16} \quad \frac{5}{16} \quad \frac{7}{16}$

### Example 2.1-3

Roll a four-sided die twice, and let $X$ equal the larger of the two outcomes if they are different and the common value if they are the same. Find the pmf of $X$ and provide its bar graph and probability histogram.

$X \sim f_X(x) = P(X=x)$




$\#(S) = 16$

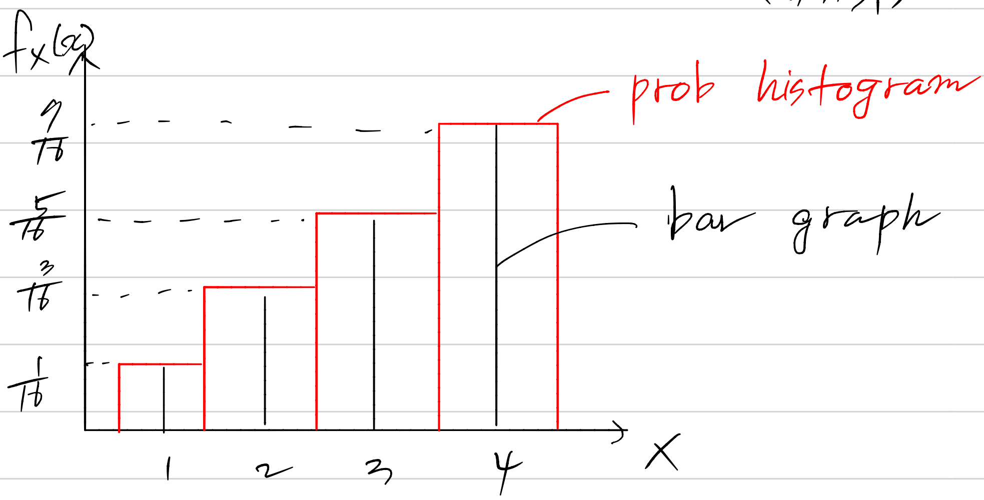
$f_X(x) = P(X=x)$

$P(X=1) = P(\{(1,1)\}) = \frac{1}{16}$

$P(X=2) = P(\{(1,2),(2,1),(2,2)\}) = \frac{3}{16}$

$P(X=3) = P(\{(1,3),(2,3),(3,3),(3,2),(3,1)\}) = \frac{5}{16}$

$P(X=4) = P(\{(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1)\}) = \frac{7}{16}$




$$f_X(x) = \begin{cases} \frac{1}{16}, & x=1 \\ \frac{3}{16}, & x=2 \\ \frac{5}{16}, & x=3 \\ \frac{7}{16}, & x=4 \end{cases}$$

OR

$f_X(x) = \frac{2x-1}{16}$, $x = 1, 2, 3, 4$.

☆ 쓰는거 잊지 말기.

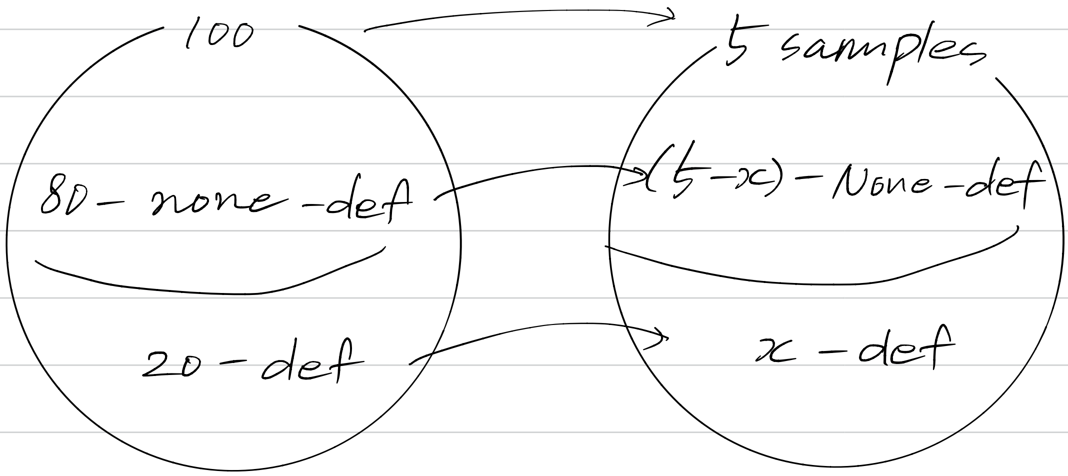
안쓰면 pmf 인지 모름.

can not be pmf

## Example 2.1-6

A lot (collection) consisting of 100 fuses is inspected by the following procedure: Five fuses are chosen at random and tested; if all 5 blow at the correct amperage, the lot is accepted. Suppose that the lot contains 20 defective fuses. If $X$ is a random variable equal to the number of defective fuses in the sample of 5, find the pmf of $X$ and the probability of accepting the lot.

$X \sim f_X(x) = P(X=x)$




$$f_X(x) = P(X=x)$$

$$= \frac{{}_{80}C_{5-x} \times {}_{20}C_x}{{}_{100}C_5} \qquad x = 0, 1, \dots, 5$$

$$f_X(0) = P(X=0)$$

## 1.3 (Cumulative) Distribution Function

**Definition**

The *cumulative distribution function (cdf)* of a random variable $X$, denoted by $F_X(x)$, is defined by

$$F_X(x) = P(X \le x), \text{ for } x \in \mathbb{R}.$$

big difference with pmf
pmf : discreate r.v
cdf : discreate or continuous r.v

**Example**

In tossing three coins, $X$=number of heads observed. Compute the cdf of $X$.

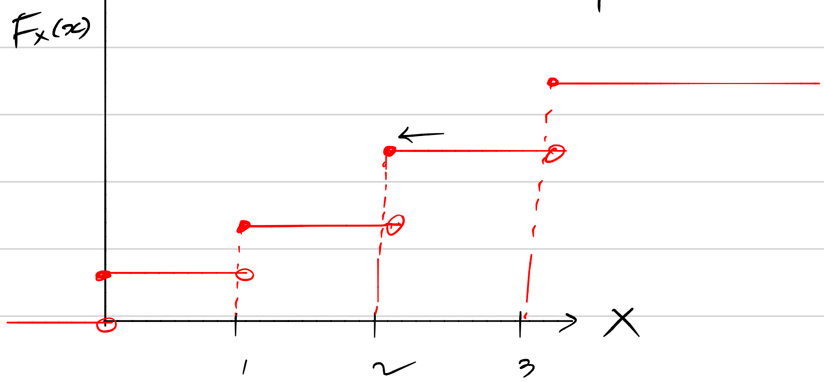
* properties of cdf $F_X(x)$

1) $\lim_{x\to-\infty} F_X(x) = 0$ & $\lim_{x\to+\infty} F_X(x) = 1$

2) non-decreasing function
(i.e. $a<b$ implies $F_X(a) \le F_X(b)$)

3) right-continuous function
$\lim_{x\downarrow x_0} F_X(x) = F_X(x_0)$
$\lim_{x\to x_0^+}$

* For a discreate r.v,
cdf is step-function




$F_X(x) = P(X \le x)$ $x \in \mathbb{R}$

$$F_X(b) = P(X \le b) = P((X \le a) \cup (a < X \le b))$$
$$= P(X \le a) + \underbrace{P(a < X \le b)}_{\ge 0} \ge \underset{F_X(a)}{\underset{\shortparallel}{P(X \le a)}}$$

| $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f_X(x) = P(X=x)$ | $\frac{1}{8}$ | $3/8$ | $3/8$ | $1/8$ |

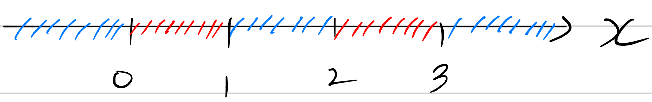
$F_X(x) = P(X \le x) =$

$0$

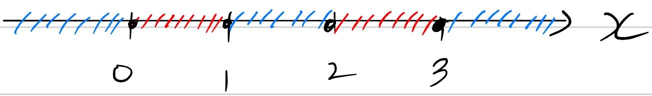
$P(X<0) + P(X<0) + P(X=0) = \frac{1}{8}$

$P(X \le 0) + P(0 < X < x) = \frac{1}{8}$

$P(X$



| $X$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|
| $f_X(x)=P(X=x)$ | $\frac{1}{8}$ | $3/8$ | $3/8$ | $1/8$ |



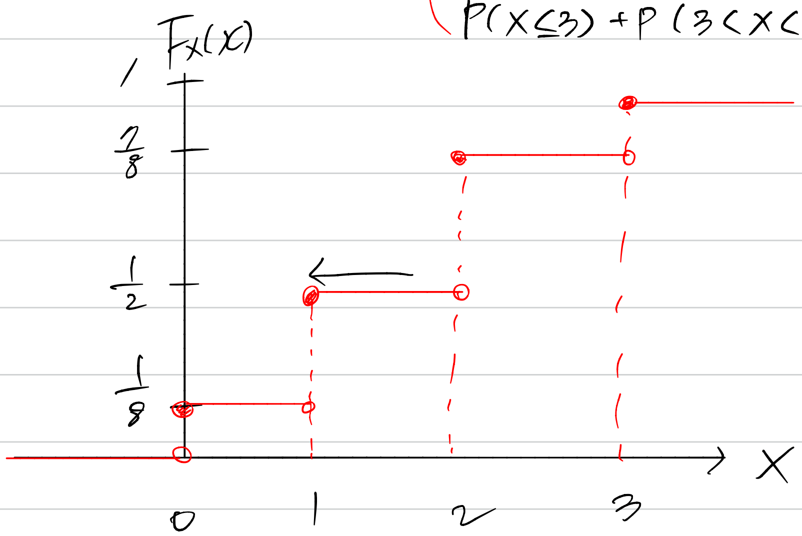

$$F_X(x)=P(X\le x)=\begin{cases} 0 & x<0 \\ P(X\le 0)+\overset{0}{P(X<0)}+P(X=0)=\frac{1}{8} & x=0 \\ P(X\le x)=\underset{\frac{1}{8}}{P(X\le 0)}+\underset{0}{\underline{P(0<X<x)}}=\frac{1}{8} & 0<x<1 \\ P(X\le 1)=\underline{P(X<1)}+P(X=1)=\frac{1}{8}+\frac{3}{8}=\frac{4}{8} & x=1 \\ P(X\le 1)+\underset{0}{P(1<X<x)}=\frac{4}{8} & 1<x<2 \\ P(X\le 2)=P(X<2)+P(X=2)=\frac{4}{8}+\frac{3}{8}=\frac{7}{8} & x=2 \\ P(X\le 2)+P(2<X<x)=\frac{7}{8} & 2<x<3 \\ P(X\le 3)=P(X<3)+P(X=3)=1 & x=3 \\ P(X\le 3)+P(3<X<x)=1 & x>3 \end{cases}$$

$\Rightarrow 0\le x<1$

$\Rightarrow 1\le x<2$

$\Rightarrow 2\le x<3$

$x>3$



# Ch.2.2 Mathematical Expectation

# 2.1 Expectation

**Definition for the discrete r.v**

Let $f_X(x)$ be the pmf of the random variable $X$ of the discrete type with space $\mathcal{X}$. The expected value, or expectation of the function $g(X)$, $E[g(X)]$, is defined by

$$E[g(X)] = \sum_{x \in \mathcal{X}} g(x) f_X(x)$$

provided that the sum exists. If $E|g(X)| < \infty$, then $E[g(X)]$ exists.

1) $X \sim f_X(x) = P(X=x)$ : pmf

2) $F_X(x) = P(X \le x)$ : cdf

3) Expectation of $g(x)$ → $X, X^2, X(X-3)$

$$E[g(X)] = \sum_{x \in X} g(x) \underline{f_X(x)}$$

pmf of $x$

## Example 2.2-2

Let the random variable $X$ have the pmf $f_X(x) = 1/3$, $x \in \mathcal{X} = \{-1, 0, 1\}$. Let $g(X) = X^2$. Compute the expectation of $g(X) = X^2$.

**Thm 2.2-1**

Let $X$ be a r.v. and a, b, and c be constants. For any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

- $E(c) = c$
- $E[ag_1(X)] = aE[g_1(X)]$
- $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$

Assume that $X \sim f_X(x)$

1) $E[c] = \sum_{x \in X} c f_X(x) = c \underbrace{\sum_{x \in X} f_X(x)}_{=1} = c$

2) $E[a \cdot g_1(X)] = \sum_{x \in X} a g_1(x) f_X(x) = a \sum_{x \in X} g_1(x) f_X(x)$

$= a \cdot E[g_1(X)]$

3) $E[a g_1(X) + b g_2(X)] = \sum_{x \in X} \{a g_1(x) + b g_2(x)\} f_X(x)$

$= \sum_{x \in X} \{a g_1(x) f_X(x) + b g_2(x) f_X(x)\}$

$= \sum_{x \in X} a g_1(x) f_X(x) + \sum_{x \in X} b g_2(x) f_X(x)$

$= a \cdot E[g_1(X)] + b E[g_2(X)]$

4) mean $\mu_X = E(X)$

variance $Var(X) = \sigma_X^2 = E[(X - \mu_X)^2]$

st.d $\sigma_X$

$1.0$
$\overline{2}$

## Example 2.2-3

Let the random variable $X$ have the pmf

$E[X] = 4$

$$f_X(x) = \frac{x}{10}, \quad x = 1, 2, 3, 4.$$

Compute $E[X(5-X)]$. $= E[5X - X^2] = 5E[X] - E[X^2]$

$g(x) = 5x - x^2$

$4 \quad 12 \quad 18 \quad 16$

$4 \cdot \frac{1}{10} \quad 6 \cdot \frac{2}{10} \quad 6 \cdot \frac{3}{10} \quad 4 \cdot \frac{4}{10}$

## Example 2.2-4

Let $u(x) = (x - b)^2$, where $b$ is not a function of $X$, and suppose $E[(X - b)^2]$ exists. Find the value of $b$ for which $E[(X - b)^2]$ is a minimum.

$$E[(X-b)^2] = E[X^2-2bX+b^2]$$

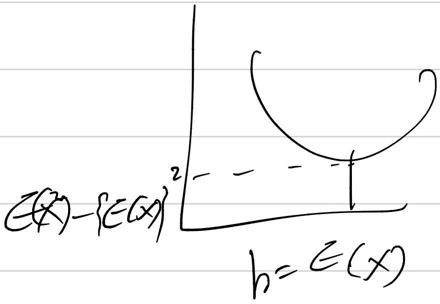
$$E[(X-b)^2] = E[X^2-2bX+b^2]$$

Since $b$ is not a f of $X$

$$E[(X-b)^2] = E[X^2]-2bE[X]+b^2$$

$$= \{b-E(X)\}^2 + E(X^2) - \{E(X)\}^2$$

$$b = E(X).$$

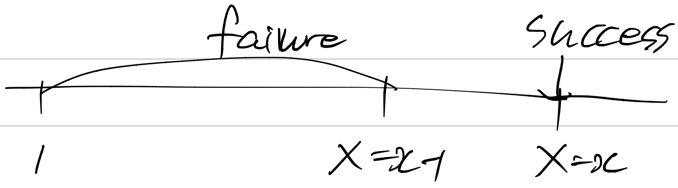


### Example 2.2-6 (geometric distribution)

An experiment has probability of success $p$, where $0 < p < 1$, and probability of failure $1 - p = q$. This experiment is repeated independently until the first success occurs; say this happens on the $X$ trial. Find the expectation of $X$.

$X$ = # of trials until the first success happens

$f_X(x) = P(X=x) = (1-p)^{x-1}p \quad x = 1, 2, 3$




total trials $= x$

$$\frac{1 \text{ success}}{(x-1) \text{ trials} \to \text{failure}}$$

$$E[X] = \sum_{x=1}^{\infty} x f_X(x) = \sum_{x=1}^{\infty} x(1-p)^{x-1}p$$

$$E(X) = p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots$$

$$-\ (1-p)E(X) = (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots$$

$$pE(X) = p + (1-p)p + (1-p)^2 + \cdots$$

$$= p\,[1 + (1-p) + (1-p)^2 + \cdots] = p \cdot \frac{1}{1-(1-p)} = 1$$

$$\therefore E(X) = \frac{1}{p}$$

eg) $p = \frac{1}{10} \quad E(X) = 10$

# Ch.2.3 Special Mathematical Expectation

# 3.1 Mean

## Definition

The mean of the random variable $X$ with the pmf $f_X(x)$ is

$$\mu_X = E[X] = \sum_{x \in \mathcal{X}} x f_X(x)$$

## Property

$$\sum_{x \in \mathcal{X}} (x - \mu_X) f_X(x) = 0$$

$$\sum_{x \in X} (x - \mu_X) f_X(x) \quad = E[X - \mu_X]$$

$$= \sum_{x \in X} \{ x f_X(x) - \mu_X f_X(x) \} \quad = E[X] - \mu_X = 0$$

$$= \sum_{x \in X} x f_X(x) - \mu_X \sum_{x \in X} f_X(x)$$

$$= \mu_X - \mu_X = 0$$

# 3.2 Variance and Standard Deviation

## Definition

The variance of the random variable $X$ with the pmf $f_X(x)$ is

$$\sigma_X^2 = Var(X) = E[(X-\mu_X)^2] = \sum_{x \in \mathcal{X}} (x-\mu_X)^2 f_X(x).$$

The standard deviation of the random variable $X$ is

$$\sigma_X = \sqrt{\sigma_X^2}$$

## Property

$$\sigma_X^2 = E(X^2) - \mu_X^2$$

$$
\begin{aligned}
Var(X) &= E[(X-\mu_X)^2] \\
&= E[X^2 - 2\mu_X X + \mu_X^2] \\
&= E(X^2) - 2\mu_X \underbrace{E(X)}_{=\mu_X} + \mu_X^2 \\
&= E(X^2) - \mu_X^2 \\
&= E(X^2) - \{E(X)\}^2
\end{aligned}
$$

## Example 2.3-1

Let $X$ be the number of spots on the side facing upwards after a six-sided die is cast at random. If everything is fair about this experiment,

$X = \{1, 2, 3, 4, 5, 6\}$.

- Find the pmf of $X$. $\frac{1}{6}$ . . . . .
- Compute the mean of $X$. $E(X) = \frac{7}{2}$
- Compute the variance and standard deviation of $X$.

2.5 1.5 0.5 0.5 1.5 2.5

$$X \sim f_X(x) = \frac{1}{6} \quad x = 1, 2, \cdots, 6$$

$$E(X) = \sum_{x=1}^{6} x f_X(x) = \frac{1}{6}(1+2+\cdots+6) = \frac{21}{6} = \frac{7}{2}$$

$$E(X^2) = \sum_{x=1}^{6} x^2 f_X(x) = \frac{1}{6}(1+2^2+\cdots+6^2) = \frac{91}{6}$$

$$\sigma_X^2 = Var(X) = E(X^2) - \{E(X)\}^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\sigma_X = \sqrt{\frac{35}{12}}$$

### Example 2.3-2

Let $X$ have the pmf $f_X(x) = 1/3$, $x = -1, 0, 1$. Let $Y$ have the pmf $f_Y(y) = 1/3$, $y = -2, 0, 2$.

- Compute the means of $X$ and $Y$.
- Compute the variances and standard deviations of $X$ and $Y$.

$X \sim f_X(x) = \frac{1}{3}, \quad x = -1, 0, 1$

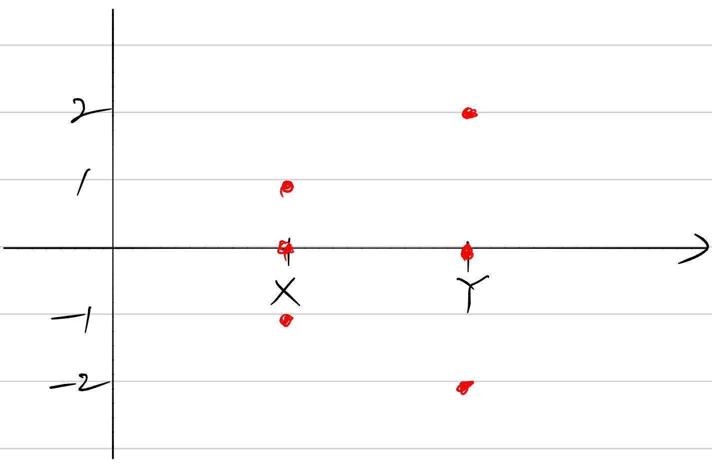
$Y \sim f_Y(y) = \frac{1}{3}, \quad y = -2, 0, 2$

$E(X) = \sum_{x=-1}^{0} x f_X(x) = \frac{1}{3}(-1+0+1) = 0$

$E(Y) = \frac{1}{3}(-2+0+2) = 0$

$\sigma_X^2 = Var(X) = E(X^2) = \frac{1}{3}(1+0+1) = \frac{2}{3}$

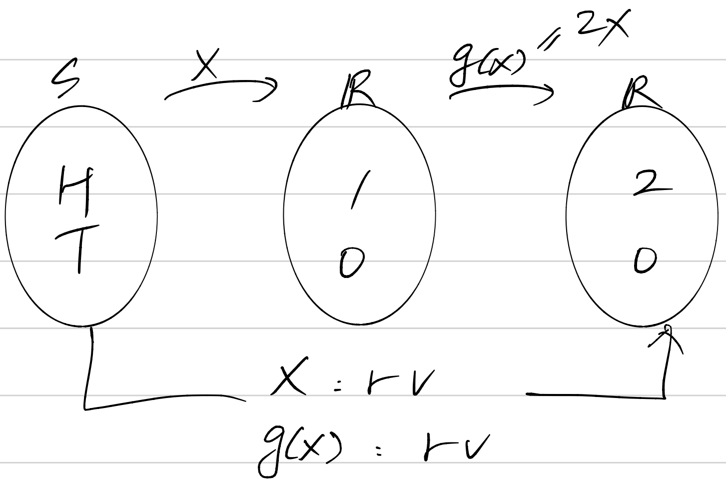
$\sigma_Y^2 = Var(Y) = E(Y^2) = \frac{1}{3}(4+0+4) = \frac{8}{3}$



**Thm.**

Let $X$ be a random variable with mean $\mu_X$ and variance $\sigma_X^2$. Then, $Y = aX + b$, where $a$ and $b$ are constants, is also a random variable.

- $\mu_Y = E(Y) = a\mu_X + b$
- $\sigma_Y^2 = Var(Y) = a^2\sigma_X^2$
- $\sigma_Y = |a|\sigma_X$




$X \quad E(X) = \mu_X \quad \cdot \ Var(X) = \sigma_X^2$

$Y = aX + b \Rightarrow E(Y)$ & $Var(Y)$

1) $\mu_Y = E(Y) = E(aX+b) = aE(X) + b$

2) $$\begin{aligned} \sigma_Y^2 = Var(Y) &= E[(Y-\mu_Y)^2] \\ &= E[\{(aX+b)-(a\mu_X+b)\}^2] \\ &= E[a^2(X-\mu_X)^2] \\ &= a^2 E[(X-\mu_X)^2] \\ &= a^2 Var(X) = a^2\sigma_X^2 \end{aligned}$$

3) $\sigma_Y = \sqrt{\sigma_Y^2} = |a|\sigma_X$

## 3.3 Moment

### Definition

For each positive integer $r$, the $r$th *moment* of $X$, $\mu_r'$ is

$$\mu_r' = E(X^r).$$

The $r$th *central moment* of $X$, $\mu_r$, is

$$\mu_r = E(X - \mu_X)^r,$$

where $\mu_X = \mu_1' = EX$.

$r^{th}$ moment of $X = E[X^r]$

$1^{st}$ moment of $X = E(X) = \mu_X$

$2^{nd}$ " $= E(X^2) = \mu_2'$

$\Rightarrow Var(X)$

$r^{th}$ central moment of $X = E[(X-\mu_X)^r]$

$1^{st}$ " $= E(X-\mu_X) = 0$

$2^{nd}$ $= E[(X-\mu_X)^2] = Var(X)$

# 3.4 Moment-Generating Function (mgf)

## Definition of mgf

Let $X$ be a discrete random variable with pmf $f_X(x)$ and space $\mathcal{X}$. The *moment generating function (mgf)* of $X$ (or $F_X$), denoted by $M_X(t)$, is

always positive

$$M_X(t) = E_X[e^{tX}] = \sum_{x \in \mathcal{X}} e^{tx} f_X(x),$$

provided that the expectation exists for $t$ in some neighborhood of 0. That is, there is a $h > 0$ such that, for all $t$ in $-h < t < h$, $E[e^{tX}]$ exists. Note that $M_X(0) = E(1) = 1$. $\sum_{x \in \mathcal{X}} f_X(x) = 1$

## Example 2.3-5

The mgf of $X$ is given by

$$M_X(t) = e^t \frac{3}{6} + e^{2t} \frac{2}{6} + e^{3t} \frac{1}{6}, \quad -\infty < t < \infty.$$

Find the pmf of $X$.

| 1 | 2 | 3 |
|---|---|---|
| $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |

$$
\begin{tikzcd}
 & X \sim f_X(x) = P(X=x) : \text{pmf} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\text{mgf} : M_X(t) = E[e^{tX}] \arrow[ur, shift left] \arrow[rr, shift left] & & F_X(x) = P(X \le x) : \text{cdf} \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] = \sum_{x \in \mathcal{X}} e^{tx} f_X(x) \\
&= e^{t}\frac{3}{6} + e^{2t}\frac{2}{6} + e^{3t}\frac{1}{6} \qquad t \in \mathbb{R} \\
&= e^{t\times 1} f_X(1) + e^{t\times 2} f_X(2) + e^{t\times 3} f_X(3)
\end{aligned}
$$

$$
\begin{cases}
f_X(1) = \frac{3}{6} \\
f_X(2) = \frac{2}{6} \\
f_X(3) = \frac{1}{6}
\end{cases}
\qquad f_X(x) = \frac{4-x}{6}, \quad x = 1, 2, 3
$$

## Relationship between mgf and moment

If $X$ has mgf $M_X(t)$, then for each positive integer $r$,

$$E(X^r) = M_X^{(r)}(0),$$

$$\text{where} \quad M_X^{(r)}(0) = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0}.$$

$E(X) \leftarrow M_X(t)$

1) $M_X(t)$

2) $M_X^{(1)}(t) = \frac{d}{dt} M_X(t)$

3) $M_X^{(1)}(0) = E(X)$

function of $t$

$$M_X(t) = E[e^{tX}] = \sum_{x \in X} e^{tx} f_X(x)$$

상수와 마찬가지

$$\text{Let } r=1, \quad M_X^{(1)}(t) = \frac{d}{dt} M_X(t) = \frac{d}{dt}\left\{ \sum_{x \in X} e^{tx} f_X(x) \right\}$$

$$= \sum_{x \in X} e^{tx}\, x f_X(x)$$

$$\Rightarrow M_X^{(1)}(0) = \sum_{x \in X} e^{0} x f_X(x) = \sum x f_X(x) = E(X)$$

$$\text{Let } r=2, \quad M_X^{(2)}(t) = \frac{d}{dt^2} M_X(t) = \sum_{x \in X} e^{tx} x^2 f_X(x)$$

$$\Rightarrow M_X^{(2)}(0) = \sum_{x \in X} x^2 f_X(x) = E(X^2)$$

In general, $M_X^{(r)}(0) = E(X^r)$

$X \sim f_X(x) \Rightarrow E(X),\ Var(X)$

$\sim M_X(t) \Rightarrow E(X),\ Var(X)$

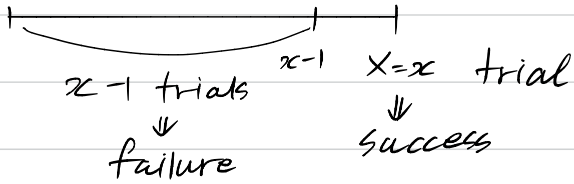
$$\overset{\shortparallel}{M_X^{(1)}(0)} \qquad \overset{\shortparallel}{E(X^2)} - \{E(X)\}^2 = M_X^{(2)}(0) - \{M_X^{(1)}(0)\}^2$$

### Example 2.3-7

Suppose $X$ has the geometric distribution of Example 2.2-6. Find the mgf of $X$. Using the mgf, find the mean and variance of $X$.

X = # of total trials until the 1st success happens

$f_X(x) = P(X=x) = q^{x-1}\cdot p \qquad x = 1, 2, 3, \cdots \qquad q = 1-p$




$$M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx}\, p\, q^{x-1} = \frac{p}{q}\sum_{x=1}^{\infty}[e^t\cdot q]^x$$

$$= \frac{p}{q}\left[(qe^t)^1 + (qe^t)^2 + \cdots\right]$$

$$= \frac{p}{q}\,\frac{qe^t}{1-(qe^t)} = \frac{pe^t}{1-qe^t}, \qquad e^t < \frac{1}{q} \quad (t < -\ln q)$$

$(0 < qe^t < 1)$

$$M_X^{(1)}(t) = \frac{p\cdot e^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$\Rightarrow M_X^{(1)}(0) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} = \mu_X$$

$$M_X^{(2)}(t) = \frac{pe^t(1-qe^t)^2 - pe^t\,2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{(1-qe^t)pe^t + 2pqe^{2t}}{(1-qe^t)^3} = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

$$\Rightarrow M_X^{(2)}(0) = \frac{(1-q)p + 2pq}{(1-q)^3} = \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2} = E(X^2)$$

· $Var(X) = E(X^2) - \{E(X)\}^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$