

Probability Theory – Final Exam

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1. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & \text{if } 0 < x, y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Compute $P(X > 1, Y < 1)$ and $P(X < Y)$.

2. If X, Y have joint density $f_{X,Y}$ given by

$$f_{X,Y}(x, y) = \begin{cases} e^{-y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise,} \end{cases}$$

compute the density f_{X+Y} of $X + Y$.

3. Let

$$A_n = \begin{cases} (-\frac{1}{n}, 1] & \text{if } n \text{ is odd} \\ (-1, \frac{1}{n}] & \text{if } n \text{ is even.} \end{cases}$$

Find $\limsup_n A_n$ and $\liminf_n A_n$.

4. Find the characteristic functions of random variable X with

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, 3, \dots$$

5. Let X_n ($n = 1, 2, \dots$) be independent random variables, each uniformly distributed over $[0, 1]$. Calculate $\mathbb{E}(S_n)$ and $\text{Var}(S_n)$ where $S_n = X_1 + \dots + X_n$.

6. Let X and Y be independent Gaussian random variables with expectation 0 and variance 1. Find the value T with $P(3X + 2Y > 5) = 1 - \Phi(T)$ where

$$\Phi(t) = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

7. Suppose that X_1, X_2, \dots are i.i.d. If $Y_n = X_n \mathbb{I}_{\{\omega: |X_n(\omega)| \leq n\}}$ is a truncated random variable, show that

$$\sum_{n=1}^{\infty} P(Y_n \neq X_n) \leq \mathbb{E}(|X_1|).$$

8. Suppose that X_1, \dots, X_n are independent random variables. Show that for any $a \geq 0$,

$$P\left(\max_{1 \leq k \leq n} |S_k| \geq 3a\right) \leq 3 \max_{1 \leq k \leq n} P(|S_k| \geq a)$$

where $S_k = X_1 + \dots + X_k$.