Homework for Chapter 5 and 6

Junwoo Yang

June 18, 2020

Problem 1. Find a compact topological space which does not satisfy the first countability axiom and justify your answer.

Proof. Let topological space X be $\mathbb{R}$ with finite complement topology. We want to show that X is compact but does not satisfy the first countability axiom.

First, let's check compactness. Let $\{U_i | i \in I\}$ be open cover of X. Take any $U_0 \in \{U_i\}$. Then $X \setminus U_0$ contains only finite number of points $\{x_1, \cdots, x_n\}$. For any $1 \leq k \leq n$, choose $U_k \in \{U_i\}$ containing k. Such elements exist since $\{U_i\}$ covers X. Then $\{u_0, u_1, \cdots, u_n\}$ is a finite subcover of X. Thus, X is compact. Now, if we show X does not satisfy first countability axiom, we are done. Pick arbitrarily $x \in \mathbb{R}$, and suppose $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ is a countable family of open neighborhoods of x. For each $n \in \mathbb{N}$, there is by definition a finite $F_n \subset \mathbb{R}$ such that $B_n = \mathbb{R} \setminus F_n$. Let $C \coloneqq \{x\} \cup \bigcup_{n\in\mathbb{N}} F_n$. Since C is the union of countably many finite sets, so C is countable.

Now pick $y \in \mathbb{R} \setminus C$. Note that $y \in B_n$ for any n. Let $U \coloneqq \mathbb{R} \setminus \{y\}$. By definition, U is open, and clearly $x \in U$. However, for any $n \in \mathbb{N}$, $y \in B_n \setminus U$, so $B_n \not\subset U$. This implies that $\mathcal{B}$ is not a local base at x. Since $x \in \mathbb{R}$ and $\mathcal{B}$ was arbitrary, X is not first countable. □

Problem 2. Show that the real line equipped with the lower limit topology $\mathbb{R}_{ll}$ is not metrizable. (Hint: First check that $\mathbb{R}_{ll}$ is separable and use **Propositon 5.3**.)

Proof. We first check $\mathbb{R}_{ll}$ is separable. Recall that X is separable if there exists countable subset whose closure is X. Construct subset $\{[z, z+1) : z \in \mathbb{Z}\}$. Then their closure $\overline{\{[z, z+1)\}} = \mathbb{R}$. Thus, $\mathbb{R}_{ll}$ is separable.

Now, we claim that $\mathbb{R}_{ll}$ is not second countable. Let B be basis of $\mathbb{R}_{ll}$. By definition of basis, for any open set U and $x \in U$, there exists $B_\alpha \in B$ such that $x \in B_\alpha \subset U$. For all $x \in \mathbb{R}$, pick $U = [x, x+\varepsilon) \in \mathbb{R}_{ll}$ for $\varepsilon > 0$. Then, for each $x \in \mathbb{R}$, there exists $B_x \in B$ such that $x \in B_x \subset [x, x+\varepsilon)$. This B_x has an infimum equal to x, so for different x, the corresponding B_x is different. Thus, the cardinality of B is at least $|\mathbb{R}|$, which is uncountable. So, our claim is proved.

Proposition 5.3. *A separable metric space satisfies the second axiom of countability.*

If $\mathbb{R}_{ll}$ is metric space (or metrizable), $\mathbb{R}_{ll}$ is second countable from above proposition. This is a contradiction. Hence, $\mathbb{R}_{ll}$ is not metrizable. □

Problem 3. Let $f : X \to Y$ be a closed, continuous surjective map between topological spaces. Suppose further that inverse image of any singleton set in Y is compact in X.

(a) Show that if X is Hausdorff, then so is Y.

(b) Show that if X is normal, then so is Y.

(c) Show that if X is locally compact, then so is Y.

(d) Show that if X satisfy the second countability, then so is Y.

Proof. (a) Suppose Y is not Hausdorff. There exists distinct y_1, y_2 such that there doesn't exist open V_1, V_2 with $V_1 \ni y_1$, $V_2 \ni y_2$, $V_1 \cap V_2 = \emptyset$. Namely, every open sets containg y_1, y_2 respectively have non-empty intersection and so are their inverse images, i.e. $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) \neq \emptyset$. Let $C_1 := f^{-1}(y_1)$, $C_2 := f^{-1}(y_2)$. Since C_1 and C_2 are compact and disjoint, by Proposition 5.5, there exists disjoint open sets $U_1 \supset C_1$, $U_2 \supset C_2$. Now if there exists open sets $W_1 \ni y_1$, $W_2 \ni y_2$ such that $f^{-1}(W_1) \subset U_1$, $f^{-1}(W_2) \subset U_2$, we are done. Because if it is, there is contradiction between $W_1 \cap W_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$. Let $W_1 := Y \setminus f(X \setminus U_1)$. Note that $X \setminus U_1$ is closed, $f(X \setminus U_1)$ is also closed, and $Y \setminus f(X \setminus U_1)$ is open in Y. Then, the inverse image of W_1,

$$f^{-1}(W_1) = f^{-1}(Y \setminus f(X \setminus U_1)) = f^{-1}(Y) \setminus f^{-1}(f(X \setminus U_1)) = X \setminus f^{-1}(f(X \setminus U_1))$$

where the last equality is due to surjection. Since $f^{-1}(f(X \setminus U_1)) \supset X \setminus U_1$, taking complements gives that $X \setminus f^{-1}(f(X \setminus U_1)) \subset U_1$. We proved $f^{-1}(W_1) \subset U_1$. It is exactly the same with W_2 as well.

(b) Since X is Hausdorff, so is Y by (a). What we need to show is that for any disjoint closed sets $V_1 \ni y_1$, $V_2 \ni y_2$, there exists open $U_1 \supset V_1$, $U_2 \supset V_2$ with $U_1 \cap U_2 = \emptyset$. Since f is continuous, $f^{-1}(V_1)$, $f^{-1}(V_2)$ are disjoint closed sets in X. By normality of X, there are open O_1, O_2 with $f^{-1}(V_1) \subset O_1$, $f^{-1}(V_2) \subset O_2$, $O_1 \cap O_2 = \emptyset$. Note that $f^{-1}(y_1) \subset f^{-1}(V_1) \subset O_1$. By (a), there are open sets $W^1_{y_1}, W^2_{y_2} \subset Y$ with $f^{-1}(W^1_{y_1}) \subset O_1$, $f^{-1}(W^2_{y_2}) \subset O_2$. Let $W^1 := \bigcup_{y_1 \in V_1} W^1_{y_1}$, $W^2 := \bigcup_{y_2 \in V_2} W^2_{y_2}$. Note that $W^1 \supset V_1$, $W^2 \supset V_2$, and they are open. Now we claim that $W^1 \cap W^2 = \emptyset$. Suppose that there is $z \in W^1 \cap W^2$. Since $z \in W^1_{y_1}$ for some $y_1 \in V_1$, $f^{-1}(z) \subset f^{-1}(W^1_{y_1}) \subset O_1$. In same way, since $z \in W^1_{y_2}$ for some $y_2 \in V_2$, $f^{-1}(z) \subset f^{-1}(W^2_{y_2}) \subset O_2$. This is a contradiction to $O_1 \cap O_2 = \emptyset$ and f is onto map, so $f^{-1}(z) \neq \emptyset$.

(c) What we need to show is that for any $y \in Y$, there is open set $O_y \ni y$ whose closure $\overline{O_y}$ is compact. For arbitrary $y \in Y$, consider $f^{-1}(y)$ which is compact. Since X is locally compact, for $x \in f^{-1}(y)$, there is open U_x such that $\overline{U_x}$ is compact. $\{U_x : x \in f^{-1}(y)\}$ is open covering of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there exists finite subcovering $U = U_{x_1} \cup \cdots \cup U_{x_n}$. Note U is open. Let $\overline{U} := \bigcup_{i=1}^n \overline{U_{x_i}}$. $\overline{U}$ is finite union of compact sets, so $\overline{U}$ is compact. By (a) there is $V \subset Y$ such that $y \in V$, $f^{-1}(V) \subset U$. That is, $y \in V \subset f(U) \subset f(\overline{U})$. Since $\overline{U}$ is compact and f is continuous, $f(\overline{U})$ is compact. Hence, for any $y \in Y$, there is compact subspace $f(\overline{U})$ that contains a neighborhood $f(U)$ of y.

(d) Let $V \subset Y$ open, $y \in V$. Then $f^{-1}(y) \subset f^{-1}(V)$, $f^{-1}(y)$ is compact, and $f^{-1}(V)$ is open. Since X is second countable, there exists countable basis B of X. For every $x \in f^{-1}(y)$, there is $B_x \in B$ with $x \in B_x \subset f^{-1}(V)$. $\{B_x : x \in f^{-1}(y)\}$ is open covering of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there is finite subcovering $B_y = \bigcup_{i=1}^n B_i$. By (a), there is open set $W \subset Y$ such that $y \in W$, $f^{-1}(W) \subset B_i \subset f^{-1}(V)$. Thus, $y \in W \subset V$.

Lemma. *Let C be collection of open sets of X. For any open set U of X, $x \in U$, if there is $x \in C_\alpha \subset U$ where $C_\alpha \in C$, then C is basis.*

Let W^J be union of all open sets $W \subset Y$ such that $f^{-1}(W) \subset \bigcup B_i$. $\{W_I : I \text{ finite}\}$ is basis of Y. This is finite. Hence, Y is second countable.

□