Homework 7

Due date: 2019. 6. 3.

1. Let G be a group of permutations. For each σ in G, define

$$sgn(\sigma) = \begin{cases} +1 & if\ \sigma\ is\ an\ even\ permutation, \\ -1 & \text{if}\ \sigma\ is\ an\ odd\ permutation. \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel?

2. Prove that the mapping from $G \oplus H$ to G given by $(g,h) \to g$ is a homomorphism. What is the kernel? This mapping is called the *projection* of $G \oplus H$ onto G.

3. How many homomorphsims are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$?

4. Prove that $(\mathbb{Z} \oplus \mathbb{Z})/(\langle (a,0) \rangle \times \langle (0,b) \rangle) \approx \mathbb{Z}_a \oplus \mathbb{Z}_b$.

5. (Second Isomorphism Theorem) If K is a subgroup of G and N is a normal subgroup of G, prove $K/(K \cap N) \approx (KN)/N$.

6. (Third Isomorphism Theorem) If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$.

7. Find all Abelian groups (up to isomorphism) of order 360.

현대대수1 - HW7

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#1. (well defined)

suppose that $\sigma_1 = \sigma_2$

Since every permutation can be into transposition will be either always odd over even, eighter both σ_1 and σ_2 are even or both odd.

$\therefore sgn(\sigma_1) = sgn(\sigma_2) \Rightarrow sgn$ is a well defined function.

(homomorphism)

$$sgn(\alpha\beta) = \begin{cases} +1, & \text{if } \alpha\beta \text{ is even} \Leftrightarrow \alpha,\beta \text{ both even or both odd.} \\ -1, & \text{if } \alpha\beta \text{ is odd} \Leftrightarrow \text{one is even, other odd.} \end{cases}$$

$$= \begin{cases} +1, & \text{if } sgn(\alpha) = sgn(\beta) \\ -1, & \text{if } sgn(\alpha) \neq sgn(\beta) \end{cases}$$

$$= \begin{cases} +1, & \text{if } sgn(\alpha)\, sgn(\beta) = 1 \\ -1, & \text{if } sgn(\alpha)\, sgn(\beta) = -1 \end{cases}$$

$$= sgn(\alpha)\, sgn(\beta)$$

$\therefore sgn$ is indeed a homomorphism.

$$\ker(sgn) = \{ g \in G \mid sgn(g) = 1 \}$$
$$= \{ g \in G \mid g \text{ is even} \} = A_n$$

#2. Let $\phi : G \oplus H \to G$ be a map defined as $\phi(g,h) = g$

For $(g_1, h_1), (g_2, h_2) \in G \oplus H$

$$\phi((g_1,h_1)(g_2,h_2)) = \phi(g_1g_2, h_1h_2)$$
$$= g_1 g_2$$
$$= \phi(g_1,h_1) \cdot \phi(g_2,h_2)$$

$\therefore$ ϕ is a homomorphism

$\text{Ker}\,\phi = \{(g,h) \mid \phi(g,h) = e\}$

since $\phi(g,h) = g$, $\quad \therefore g = e$

$\therefore \ker\phi = \{(e,h) \mid h \in H\}$

#3. If $\phi: \mathbb{Z}_{20} \to \mathbb{Z}_8$ is onto, then there is $g \in \mathbb{Z}_{20}$

s.t $\phi(g) = 1 \in \mathbb{Z}_8$

$\Rightarrow |\phi(g)| = 8$, $|\phi(g)| \mid |g|$

$\Rightarrow 8 \mid 20 \quad (\rightarrow\leftarrow)$.

$\therefore$ There is no homomorphism then,

$|\phi(1)| \mid \gcd(8, 20) = 4$

$\therefore$ $\phi(1)$ is in a unique subgroup of order 4

which is $2\mathbb{Z}_8$

$\phi: 1 \mapsto 0$, $\quad \phi: 1 \mapsto 2$, $\quad \phi: 1 \mapsto 4$. $\quad \phi: 1 \mapsto 6$

There are 4 homomorphisms.

#4. Prove that $(\mathbb{Z} \oplus \mathbb{Z}) / (\langle(1,0)\rangle \times \langle 0,b\rangle) \approx \mathbb{Z}_a \oplus \mathbb{Z}_b$

Define mapping $\phi: \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_a \oplus \mathbb{Z}_b$,

with $\phi((m,n)) = (m,n) \pmod a, \bmod b)$.

kernel of $\phi \Rightarrow \phi((m,n)) = (0,0)$.

$\Rightarrow m \equiv 0 \pmod a$, $\quad n \equiv 0 \pmod b$.

$\quad m \in \langle a \rangle$, $\quad n \in \langle b \rangle$

$\therefore \ker(\phi) = \{(m,n) \mid m \in \langle a \rangle, n \in \langle b \rangle\} = \langle(a,0)\rangle \times \langle(0,b)\rangle$

By First Isomorphism Theorem,

$(\mathbb{Z} \oplus \mathbb{Z}) / (\langle(1,0)\rangle \times \langle(0,b)\rangle) \approx \mathbb{Z}_a \oplus \mathbb{Z}_b$

#5. (Second Isomorphism Theorem).

Define $\phi : K \to \frac{KN}{N}$ $(K \leq G,\ N \triangleleft G)$

$k \mapsto kN$

$\phi(k_1 k_2) = k_1 k_2 N = (k_1 N)(k_2 N) = \phi(k_1)\phi(k_2)$

Let $x \in \frac{KN}{N}$, then $x = kN$ for some $k \in K$

$\phi(k) = kN$, i.e ϕ is onto

$\ker\phi = \{k \in K \mid kN = N\} = K \cap N$

$\Rightarrow \frac{K}{K \cap N} \approx \frac{KN}{N}$ ▨.

#6. (Third Isomorphism theorem).

Define $\phi : G/N \to G/M$

$gN \mapsto gM$

well-define) Image is exactly G/M

kernel of ϕ

$\Rightarrow \phi(gN) = M \Rightarrow \underbrace{gM = M}_{g \in M} \Rightarrow \ker(\phi) = \{gN \mid g \in M\} = M/N$

By first isomorphism theorem.

$(G/N)/(M/N) \approx G/M$

#7. $360 = 2^3 \times 3^2 \times 5$

Fundamental theorem of finite abelian groups,

up to isomorphism :

- $\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5$
- $\mathbb{Z}_{2^3} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$