Stochastic calculus Instructor: Hyungbin Park

Lecture note 3: Stochastic calculus

# 1 Exercise

**Problem 1.1.** (45 points) Consider a Brownian motion $B = (B_t)_{t\geq 0}$. For $0 < s < t$, evaluate the followings. Use the cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$$

of the standard normal density if needed.

(i) $\mathbb{P}(B_1 > 1, B_3 - B_2 > 1)$

(ii) $\mathbb{P}(2B_1 - B_2 > -1)$

(iii) $\mathbb{P}(B_1 < 0, B_2 > (1-\sqrt{3})B_1)$

(iv) $\mathbb{P}(B_3 < 1 \,|\, B_1)$

(v) $\mathbb{P}(B_1 < 2 \,|\, B_2)$

(vi) $\mathbb{E}(B_s^2 e^{2B_t})$

(vii) $\mathrm{Var}(2B_3 - B_2)$

(viii) $\mathrm{Cov}(e^{B_t}, e^{-2B_s})$

(ix) $\mathbb{E}(B_1 + B_3 | B_1 - 2B_2)$.

For random variable $X, Y$ and a Borel set $A$, the notation $\mathbb{P}(X \in A|Y)$ means $\mathbb{E}(\mathbb{I}_{\{X\in A\}}|\sigma(Y))$.

**Problem 1.2.** (10 points) Let $0 \leq s < t$. Show that $B_t - B_s$ is independent of $\sigma(B_u | 0 \leq u \leq s)$. Read Problem 1.4 on page 49 in (Karatzas and Shreve, Brownian Motion and Stochastic Calculus, 1991). You can find the solution from the book.

**Problem 1.3.** Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(i) (10 points) Show that $(X_t)_{t\geq 0} = (\frac{1}{\sqrt{c}} B_{ct})_{t\geq 0}$ is a Brownian motion for any $c > 0$,

(ii) (5 points) Use the time inversion formula and the law of iterated logarithm of Brownian motion to show that for $s \geq 0$

$$\mathbb{P}\left( \liminf_{t\to 0+} \frac{B_{t+s} - B_s}{\sqrt{2t \ln\ln \frac{1}{t}}} = -1\,,\ \limsup_{t\to 0+} \frac{B_{t+s} - B_s}{\sqrt{2t \ln\ln \frac{1}{t}}} = 1 \right) = 1\,.$$

**Problem 1.4.** Solve the following problems.

(i) (5 points) Let $f : [0,T] \to \mathbb{R}$ be a RCLL function. Show that $f$ is bounded.

(ii) (10 points) Let $(X_t)_{t\geq 0}$ be a RCLL Gaussian process. Show that

$$\Big(\int_0^t X_s\,ds\Big)_{t\geq 0}$$

is a continuous Gaussian process. You may use, without proof, the fact that the limit (in the sense of convergence in distribution) of a sequence of normal random variables is normal.

(iii) (5 points) For $T > 0$, find the distribution of

$$\int_0^T uB_u\,du\,.$$

(iv) (5 points) Calculate

$$\mathbb{E}(e^{\int_0^T uB_u\,du})$$

**Problem 1.5.** (Fractional Brownian motion) (25 points) Let $0 < H < 1$. A continuous Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with mean zero and covariance

$$\mathrm{cov}(B_s^H, B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$$

is called a fractional Brownian motion with parameter $H$.

(i) Show that if $H = 1/2$, then $B^H$ is the standard Brownian motion.

(ii) Let $B^H$ be a fractional Brownian motion with parameter $H$. Show that for any $h > 0$, the process $X$ given by

$$X_t = B_{t+h}^H - B_h^H$$

is a fractional Brownian motion with parameter $H$.

(iii) Deduce that a fractional Brownian motion has stationary increments, that is, $B_t^H - B_s^H$ has the same distribution with $B_{t-s}^H$ for $0 < s < t$.

(iv) Let $0 \leq u \leq s \leq t$. Evaluate $\mathbb{E}(B_u^H(B_t^H - B_s^H))$. For which $H$ the increment $B_t^H - B_s^H$ is independent of the past $\sigma(B_u^H : 0 \leq B_u^H \leq s)$?

(v) Show that

$$\Big(\int_0^t B_u^H \mathbb{I}_{[1,2)}(u) - 2B_u^H \mathbb{I}_{[3,\infty)}(u)\,du\Big)_{t\geq 0}$$

is a Gaussian process.

Hint: The proofs of the above problems are similar with the Brownian motion case we did in class.

# References