Modern Algebra I – Homework 3

Junwoo Yang

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1. Find all generators of $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_{20}$.

   Proof. If n is generator of $\mathbb{Z}_k$, then $\gcd(n,k)=1$ for $n \in \mathbb{Z}_k$.

   (a) $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$. $\langle 1\rangle = \langle 5\rangle = \mathbb{Z}_6$. $\therefore$ generators of $\mathbb{Z}_6 = \{1,5\}$.

   (b) $\mathbb{Z}_8 = \{0,1,2,3,44,5,6,7\}$. $\langle 1\rangle = \langle 3\rangle = \langle 5\rangle = \langle 7\rangle = \mathbb{Z}_8$. $\therefore$ generators of $\mathbb{Z}_8 = \{1,3,5,7\}$.

   (c) $\mathbb{Z}_{20} = \{0,1,2,\cdots,19\}$. generators of $\mathbb{Z}_{20} = \{1,3,7,9,11,13,17,19\}$. □

2. List the elements of the subgroups $\langle 3\rangle$, $\langle 7\rangle$ in $U(20)$.

   Proof. $U(20) = \{1,3,7,9,11,13,17,19\}$, $\langle 3\rangle = \{1,3,9,7\}$, $\langle 7\rangle = \{1,7,9,3\}$. □

3. Let a be an element of a group and let $|a| = 15$. Compute the orders of a^3, a^5, a^6, a^{10} of G.

   Proof. $|a| = 15 \Rightarrow a^{15} = e$. $(a^3)^5 = (a^5)^3 = (a^6)^5 = (a^{10})^3 = e$. $\therefore |a^3| = 5$, $|a^5| = 3$, $|a^6| = 5$, $|a^{10}| = 3$. □

4. In $\mathbb{Z}_{24}$ list all generators for the subgroup of order 8. Let $G = \langle a\rangle$ and let $|a| = 24$. List all generators for the subgroup of order 8.

   Proof. $\mathbb{Z}_{24} = \langle 1\rangle$ is cyclic group of order 24. Subgroup of order 8 is $|\langle 1^k\rangle| = \frac{24}{\gcd(24,k)} = 8$. $\gcd(24,k) = 3$. Therefore $k = 3,9,15,21$. $\langle 3\rangle = \langle 9\rangle = \langle 15\rangle = \langle 21\rangle$. All generators of $\langle 3\rangle$ are of the form $k \cdot 3$ where $\gcd(8,k) = 1$. Thus, $k = 1,3,5,7$ and the generators of $\langle 3\rangle$ are $3,9,15,21$. In $\langle a\rangle$, there is a unique subgroup of order 8 which is $\langle a^3\rangle = \langle a^9\rangle = \langle a^{15}\rangle = \langle a^{21}\rangle$. All generators of $\langle a^3\rangle$ are of the form $(a^3)^k$ where $\gcd(8,k) = 1$. Therefore, $k = 1,3,5,7$ and the generators of $\langle a^3\rangle$ are a^3, a^9, a^{15}, and a^{21}. □

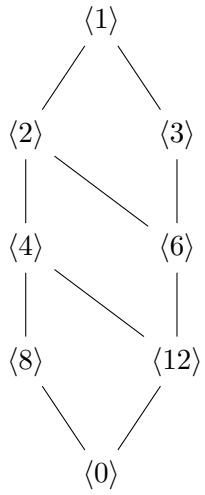
5. Determine the subgroup lattices for $\mathbb{Z}_{24}$.

   Proof. Subgroup of $\mathbb{Z}_{24}$ are

   $$\{0,1,2,\cdots,23\},\ \{0,2,4,\cdots,22\},\ \{0,3,6,\cdots,21\},\ \{0,4,8,\cdots,22\},$$

   $$\{0,6,12,18\},\ \{0,8,16\},\ \{0,12\},\ \{0\}.$$

The lattices is



□

6. Let p be a prime number. Find the number of generators of the cyclic group $\mathbb{Z}_{p^r}$, where r is an positive integer.

Proof. For any x in the set of generators of $\mathbb{Z}_{p^r}$, $\gcd(x, p^r) = 1$. Since p is prime, $x \in \mathbb{Z}_{p^r}$. If $\gcd(x, p^r) \neq 1$, then $x = np$ for $n = 0, 1, \cdots$.

$$\therefore \underbrace{0, p, 2p, \cdots, p^r - p}_{p^{r-1} - 1} \text{ are not generator.}$$

Therefore, the number of generators of $\mathbb{Z}_{p^r}$ is $|\mathbb{Z}_{p^r}| - (p^{r-1} - 1 + 1) = p^r - p^{r-1}$. □

7. Show that $\mathbb{Z}_p$ has no proper nontrivial subgroups if p is a prime number.

Proof. $\mathbb{Z}_p = \{0, 1, 2, \cdots, p-1\}$. For all $x \in \mathbb{Z}_p \setminus \{0\}$, $\gcd(x, p) = 1$. Thus, all x are generator,

$$|\langle x \rangle| = |x| = \frac{p}{\gcd(x, p)} = \frac{p}{1} = p.$$

Since $\mathbb{Z}_p$ is cyclic, H must be cyclic. Thus, $H = \langle a \rangle$ for $a \in \mathbb{Z}_p$.

(i) $a = 0$, H is trivial subgroup.

(ii) $a \neq 0$, $a \in \mathbb{Z}_p^*$, $|H| = |\langle a \rangle| = |a| = p = |\mathbb{Z}_p| \Rightarrow H = \mathbb{Z}_p$.

Therefore, subgroup of $\mathbb{Z}_p$ is always either $\{0\}$ or $\mathbb{Z}_p$. □

8. Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Proof. $\langle a \rangle \cap \langle b \rangle$ is a group. Let $c \in \langle a \rangle \cap \langle b \rangle$. Then, $c \in \langle a \rangle$, $c \in \langle b \rangle$. Thus, $|c| \Big| \gcd(10, 21) = 1 \Rightarrow |c| = 1$. So, $c' = c = e$. Therefore, only elements of $\langle a \rangle \cap \langle b \rangle$ is e. □

9. Suppose that $|x| = n$. Find a necessary and sufficient condition on r and s such that $\langle x^r \rangle \subseteq \langle x^s \rangle$.

Proof. $\langle x^r \rangle \subseteq \langle x^s \rangle$ if and only if $x^r \in \langle x^s \rangle$. $|x^s| = \frac{n}{(n,s)} \Big| n$. $\langle x^s \rangle = \langle x^{(n,s)} \rangle = \{1, x^{(n,s)}, x^{2(n,s)}, \cdots\}$.

$$\because \forall k | n, \text{ subgroup of } \langle a \rangle : \left\{1, a^{\frac{n}{k}}, a^{\frac{2n}{k}} \cdots a^{\frac{(k-1)n}{k}}\right\} = \langle a^{\frac{n}{k}} \rangle.$$

$\therefore x^r \in \langle x^s \rangle = \langle x^{(n,s)} \rangle \Leftrightarrow (n,s) \Big| r.$ □