Introduction to Differential Geometry I – Homework 7

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Problem 3-2.8 Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:
a. Paraboloid of revolution $z = x^2 + y^2$.
b. Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.
c. Catenoid $x^2 + y^2 = \cosh^2 z$.

Solution. **a.** We can parametrize the surface as $\mathbf{x}(u, v) = (u^2, v^2, u^2 + v^2)$.
Then, $\mathbf{x}_u = (1, 0, 2u)$, $\mathbf{x}_v = (0, 1, 2v)$ and the normal N is

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{(1, 0, 2u) \wedge (0, 1, 2v)}{|(1, 0, 2u) \wedge (0, 1, 2v)|} = \frac{(-2u, -2v, 1)}{|(-2u, -2v, 1)|}$$

Since coordinate z of N is constantly posivive, Gauss map covers only the positive half of the unit sphere.

b. We can parametrize it as $\mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, \sinh v)$.
Then $\mathbf{x}_u = (-\cosh v \sin u, \cosh v \cos u, \sinh v)$, $\mathbf{x}_v = (\sinh v \cos u, \sinh v \sin u, \cosh v)$ and normal is

$$\begin{aligned}
N &= \frac{(-\cosh v \sin u, \cosh v \cos u, \sinh v) \wedge (\sinh v \cos u, \sinh v \sin u, \cosh v)}{|(-\cosh v \sin u, \cosh v \cos u, \sinh v) \wedge (\sinh v \cos u, \sinh v \sin u, \cosh v)|} \\
N_x &= \cos u \cosh^2 v - \sin u \sinh^2 v \\
N_y &= \cosh^2 v \sin u + \cos u \sinh^2 v \\
N_z &= -\cos^2 u \cosh v \sinh v - \cosh v \sin^2 u \sinh v
\end{aligned}$$

From here we can observec that Gauss map covers the whole unit sphere.

c. Let $f(x, y, z) = x^2 + y^2 - \cosh^2 z$. Then the Catenoid is $f^{-1}(0)$ and the normal is

$$N = \frac{(f_x, f_y, f_z)}{|(f_x, f_y, f_z)|} = \frac{(2x, 2y, 2\cosh z \sinh z)}{|(2x, 2y, 2\cosh z \sinh z)|}$$

From here we can observe that x, y, z can be any value of $\mathbb{R}$. Thus the Gauss map covers the whole unit sphere. $\square$