Topology I – Homework 1

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Problem 1.1 Show that the Euclidean metric on $\mathbb{R}^n$ is a metric on $\mathbb{R}^n$.

Proof. Let $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

1. $(x_i - y_i)^2 \geq 0 \ ^\forall i \ \Rightarrow \ d \geq 0$.

2. $d = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$ $(\Rightarrow)$ $(x_i - y_i)^2 = 0 \ ^\forall i \Rightarrow x_i = y_i \ ^\forall i \Rightarrow \mathbf{x} = \mathbf{y}$
   $(\Leftarrow)$ trivial

3. $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = d(\mathbf{y}, \mathbf{x})$

4. $\sqrt{\sum_{i=1}^n (x_i - y_i)^2} \leq \sqrt{\sum_{i=1}^n (x_i - z_i)^2} + \sqrt{\sum_{i=1}^n (z_i - y_i)^2}$
   Recall that $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ and $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. So, $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.
   First let's establish the Cauchy-Schwarz inequality: $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.
   Consider $|\mathbf{x} - c\mathbf{y}|^2 = (\mathbf{x} - c\mathbf{y})(\mathbf{x} - c\mathbf{y}) = c^2\mathbf{y} \cdot \mathbf{y} - 2c\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{x} = |\mathbf{y}|^2c^2 - 2(\mathbf{x} \cdot \mathbf{y})c + |\mathbf{x}|^2$.
   This is a quadratic in c and since $|\mathbf{x} - c\mathbf{y}|^2 \geq 0$, we have $|\mathbf{y}|^2c^2 - 2(\mathbf{x} \cdot \mathbf{y})c + |\mathbf{x}|^2 \geq 0$.
   Thus this quadratic either has a repeated real root or complex roots. Thus its discriminant is non-positive. So $(-2(\mathbf{x} \cdot \mathbf{y}))^2 - 4|\mathbf{y}|^2|\mathbf{x}|^2 \leq 0$. This means $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ as we required. By using this,

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y})(\mathbf{x} + \mathbf{y}) = |\mathbf{x}|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x} \cdot \mathbf{y}| + |\mathbf{y}|^2 \leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2 = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

   Thus, $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}|+|\mathbf{y}|$. $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x}-\mathbf{y}| = |\mathbf{x}-\mathbf{z}+\mathbf{z}-\mathbf{y}| \leq |\mathbf{x}-\mathbf{z}|+|\mathbf{z}-\mathbf{y}| = d(\mathbf{x}, \mathbf{z})+d(\mathbf{z}, \mathbf{y})$.

Thus, the Euclidean metric on $\mathbb{R}^n$ is a metric on $\mathbb{R}^n$. □

Problem 1.2 On $\mathbb{R}^2$, let $d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Show that d is a metric on $\mathbb{R}^2$

Proof.

1. $|x_1 - y_1| \geq 0$, $|x_2 - y_2| \geq 0 \Rightarrow d \geq 0$.

2. $d = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$
   $(\Rightarrow)$ $|x_1 - y_1| = |x_2 - y_2| = 0 \Rightarrow x_1 = y_1$, $x_2 = y_2 \Rightarrow \mathbf{x} = \mathbf{y}$
   $(\Leftarrow)$ trivial.

3. $d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{|y_1 - x_1|, |y_2 - x_2|\} = d(\mathbf{y}, \mathbf{x})$

4.

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} &\leq \max\{|x_1 - z_1| + |z_1 - y_1|, |x_2 - z_2| + |z_2 - y_2|\} \\ &\leq \max\{|x_1 - z_1|, |x_2 - z_2|\} + \max\{|z_1 - y_1|, |z_2 - y_2|\} \\ &= d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \end{aligned}$$

Thus, d is a metric on $\mathbb{R}^2$. $\square$