

Introduction to Differential Geometry I – Homework 10

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Problem 3-3.13 Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the map (a similarity) defined by $F(p) = cp$, $p \in \mathbb{R}^3$, c a positive constant. Let $S \subset \mathbb{R}^3$ be a regular surface and set $F(S) = \bar{S}$. Show that \bar{S} is a regular surface, and find formulas relating the Gaussian and mean curvatures, K and H , of S with the Gaussian and mean curvatures, \bar{K} and \bar{H} , of \bar{S} .

Solution.

□

Let $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow S$ be a local parametrization. Then the map

$$\bar{\mathbf{x}}(u, v) = (F \circ \mathbf{x})(u, v) = c\mathbf{x}(u, v)$$

locally parametrizes the surface \bar{S} . Specially, $\bar{\mathbf{x}}_u = c\mathbf{x}_u$ and $\bar{\mathbf{x}}_v = c\mathbf{x}_v$. Now since S is regular, the vectors \mathbf{x}_u and \mathbf{x}_v are linearly independent, which implies that $\bar{\mathbf{x}}_u$ and $\bar{\mathbf{x}}_v$ are linearly independent. Hence, \bar{S} is regular.

We also have that

$$\bar{E} = \bar{\mathbf{x}}_u \cdot \bar{\mathbf{x}}_v = (c\mathbf{x}_u) \cdot (c\mathbf{x}_u) = c^2(\mathbf{x}_u \cdot \mathbf{x}_u) = c^2E$$

and the same way $\bar{F} = c^2F$ and $G = c^2\bar{G}$. We also have

$$\bar{n} = \frac{\bar{\mathbf{x}}_u \times \bar{\mathbf{x}}_v}{\|\bar{\mathbf{x}}_u \times \bar{\mathbf{x}}_v\|} = \frac{c^2(\mathbf{x}_u \times \mathbf{x}_v)}{c^2\|\mathbf{x}_u \times \mathbf{x}_v\|} = n$$

Hence,

$$\bar{L} = \bar{n} \cdot \bar{\mathbf{x}}_{uu} = n \cdot (c\mathbf{x}_{uu}) = c(n \cdot \mathbf{x}_{uu}) = cL$$

and the same way $\bar{M} = cM$ and $\bar{N} = cN$. Therefore,

$$\begin{aligned}\bar{K} &= \frac{\bar{L}\bar{N} - \bar{M}^2}{\bar{E}\bar{G} - \bar{F}^2} = \frac{cL \cdot cN - (cM)^2}{c^2E \cdot c^2G - (c^2F)^2} = \frac{1}{c^2} \cdot \frac{LN - M^2}{EG - F^2} = \frac{K}{c^2} \\ \bar{H} &= \frac{\bar{E}\bar{N} - 2\bar{F}\bar{M} + \bar{G}\bar{L}}{2(\bar{E}\bar{G} - \bar{F}^2)} = \frac{c^3}{c^4} \cdot \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{H}{c}\end{aligned}$$