Discrete-time models Instructor: Hyungbin Park

Lecture note 2: Multi-period models

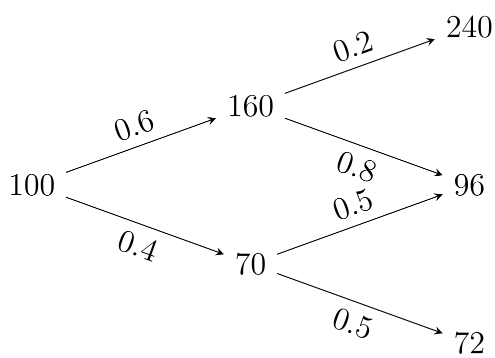
References:
CH 2, 3 in Björk (2004)

1 Exercises

Problem 1.1. (30 points) Consider a two-period binomial model with the time index $t = 0, 1, 2$. This model has two underlying assets; a bank account process $(G_t)_{t=0,1,2}$ given by

$$G_0 = G_1 = 1\,,\ G_2 = 1.2$$

and stock with price process given by the following tree.



(i) (5 points) Construct a sample space Ω, a probability $\mathbb{P}$, a filtration $(\mathcal{F}_t)_{t=0,1,2}$, and a stochastic process $(S_t)_{t=0,1,2}$ representing this stock price tree.

(ii) (5 points) Evaluate the conditional expectation $\mathbb{E}^{\mathbb{P}}(S_2/G_2|\mathcal{F}_1)$.

(iii) (5 points) Consider an option with payoff

$$X_2 = \begin{cases} 120 & \text{if } S_2 = 240 \\ 0 & \text{if } S_2 = 96 \\ -12 & \text{if } S_2 = 72 \end{cases}$$

at maturity $t = 2$. Find the price process $(X_t)_{t=0,1,2}$ and the hedging portfolio $h = (\pi_t, \phi_t)_{t=0,1,2}$ of this option.

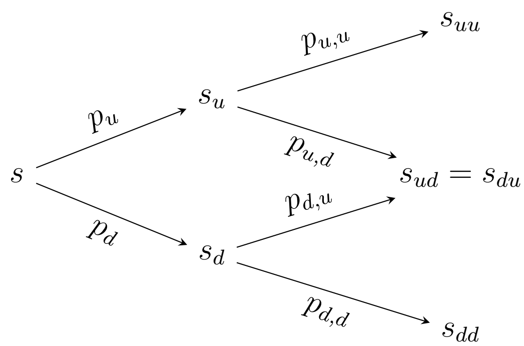
(iv) (5 points) Show that h is self-financing (Recall that we defined $(\pi_2, \phi_2) := (\pi_1, \phi_1)$ in class).

(v) (5 points) Show that the process $\phi = (\phi_t)_{t=0,1,2}$ is adapted .

(vi) (5 points) Find the risk-neutral measure $\mathbb{Q}$.

(vii) (5 points) Evaluate the conditional expectations $\mathbb{E}^{\mathbb{Q}}(S_2/G_2|\mathcal{F}_1)$ and $\mathbb{E}^{\mathbb{Q}}(X_2/G_2|\mathcal{F}_1)$.

Problem 1.2. (30 points) Consider the binomial model: $(G_t)_{t=0,1,2} = ((1+R)^t)_{t=0,1,2}$ and $(S_t)_{t=0,1,2}$ is given as



for $R \geq 0$, $p_u, p_d, p_{u,u}, p_{u,d}, p_{d,u}, p_{d,d} > 0$, $p_u + p_d = p_{u,u} + p_{u,d} = p_{d,u} + p_{d,d} = 1$, $s_u > s_d$, $s_{uu} > s_{ud} = s_{du} > s_{dd}$. Show that the followings are equivalent.

(i) This market satisfies the no-arbitrage condition.

(ii) $\frac{s_d}{s} < 1 + R < \frac{s_u}{s}$, $\frac{s_{ud}}{s_u} < 1 + R < \frac{s_{uu}}{s_u}$, $\frac{s_{dd}}{s_d} < 1 + R < \frac{s_{du}}{s_d}$.

(iii) A risk-neutral measure exists.

Problem 1.3. (5 points) For a finite set Ω and a time-index set $\mathbb{T} = \{0, 1, \cdots, T\}$, consider a filtered probability space $(\Omega, 2^\Omega, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable. Show that a process M defined by

$$M_t = \mathbb{E}(X|\mathcal{F}_t)\,,\ t = 0, 1, \cdots, T$$

is a martingale.

Problem 1.4. (10 points) Consider the two-period ($t = 0, 1, 2$) binomial model satisfying the no-arbitrage condition (thus, a risk-neutral measure exists). Let X_2 be an option payoff with maturity $T = 2$, and let $(X_t)_{t=0,1,2}$ be the arbitrage-free price process of this option. Show that a market with three assets $(G_t)_{t=0,1,2}$, $(S_t)_{t=0,1,2}$, $(X_t)_{t=0,1,2}$ is free of arbitrage. (Here, a portfolio is a three-dimensional adapted process.) Hint: Get an idea from the proof of “The existence of a RN measure implies the no-arbitrage condition” we studied in class.

References

Tomas Björk. *Arbitrage theory in continuous time*. Oxford university press, 2004.

금융수학1 - HW2

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1.1.(i).

$\Omega = \{uu, ud, du, dd\}$.

2^Ω : the power set of Ω

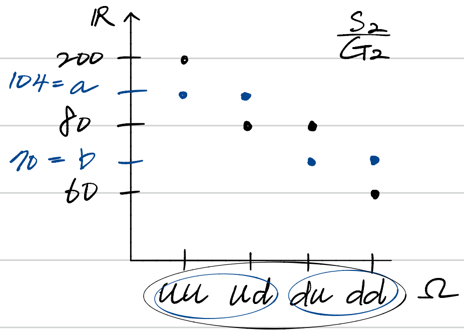
$\mathbb{P} : 2^\Omega \to \mathbb{R}$ probability measure with $\mathbb{P}(\{uu\}) = 0.12$,
$\mathbb{P}(\{ud\}) = 0.48$, $\mathbb{P}(\{du\}) = 0.2$, $\mathbb{P}(\{dd\}) = 0.2$

$(\mathcal{F}_t)_{t=0,1,2}$: filteration. $\mathcal{F}_0 = \{\phi, \Omega\}$,
$\mathcal{F}_1 = \{\phi, \{uu, ud\}, \{du, dd\}, \Omega\}$
$(\Omega, 2^\Omega, (\mathcal{F}_t), \mathbb{P})$: filtered prob. space $\quad \mathcal{F}_2 = 2^\Omega$

$(S_t)_{t=0,1,2}$: $\Omega \times \mathbb{T} \to \mathbb{R}$. stochastic process.
$S_0 = 100$.
$S_1(uu) = S_1(ud) = 160$, $S_1(du) = S_1(dd) = 70$
$S_2(uu) = 240$, $S_2(ud) = S_2(du) = 96$, $S_2(dd) = 72$.

1.1.(ii)

$E^{\mathbb{P}}\left(\frac{S_2}{G_2} \middle| \mathcal{F}_1\right)$



$\frac{S_2}{G_2}(uu) = \frac{240}{1.2} = 200$
$\frac{S_2}{G_2}(ud) = \frac{S_2}{G_2}(du) = \frac{96}{1.2} = 80$
$\frac{S_2}{G_2}(dd) = \frac{72}{1.2} = 60$

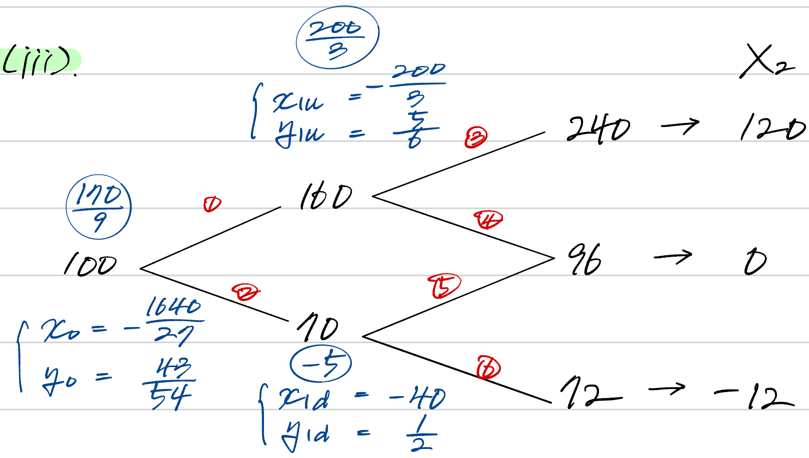
$200 \times P(\{uu\}) + 80 \times P(\{ud\}) = a \times P(\{uu\}) + a \times P(\{ud\})$.

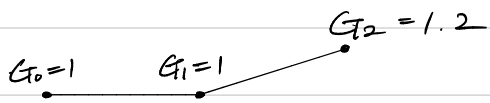
$200 \times 0.12 + 80 \times 0.48 = 0.6a. \Rightarrow a = 104.$

$80 \times 0.2 + 60 \times 0.2 = 0.4b \Rightarrow b = 70.$

$\therefore \begin{cases} E^{\mathbb{P}}\left(\frac{S_2}{G_2} \middle| \mathcal{F}_1\right)(uu) = E^{\mathbb{P}}\left(\frac{S_2}{G_2} \middle| \mathcal{F}_1\right)(ud) = 104 \\ E^{\mathbb{P}}\left(\frac{S_2}{G_2} \middle| \mathcal{F}_1\right)(du) = E^{\mathbb{P}}\left(\frac{S_2}{G_2} \middle| \mathcal{F}_1\right)(dd) = 70. \end{cases}$

1.1.(iii).





$$\begin{cases} 1.2x_{1u} + 240y_{1u} = 120 \\ 1.2x_{1u} + 96y_{1u} = 0 \end{cases} \Leftrightarrow \begin{cases} x_{1u} = -\frac{200}{3} \\ y_{1u} = \frac{5}{6} \end{cases}$$

$\Rightarrow$ price $= x_{1u} + 160y_{1u} = \frac{200}{3}$

$$\begin{cases} 1.2x_{1d} + 96y_{1d} = 0 \\ 1.2x_{1d} + 72y_{1d} = -12 \end{cases} \Leftrightarrow \begin{cases} x_{1d} = -40. \\ y_{1d} = \frac{1}{2} \end{cases}$$

$\Rightarrow$ price $= x_{1d} + 70y_{1d} = -5$

$$\begin{cases} x_0 + 160y_0 = \frac{200}{3} \\ x_0 + 70y_0 = -5 \end{cases} \Leftrightarrow \begin{cases} x_0 = -\frac{1640}{27} \\ y_0 = \frac{43}{54} \end{cases}$$

price $= x_0 + 100y_0 = \frac{170}{9}$

$(X_t)_{t=0,1,2} : \Omega \times \mathbb{T} \to \mathbb{R}$ stochastic process.

$X_0 = \frac{170}{9}$

$X_1(uu) = X_1(ud) = \frac{200}{3}$, $X_1(du) = X_1(dd) = -5$

$X_2(uu) = 120$, $X_2(ud) = X_2(du) = 0$, $X_2(dd) = -12$.

$h = (\pi_t, \phi_t)_{t=0,1,2} : \Omega \times \mathbb{T} \to \mathbb{R}$ stochastic process.

$(\pi_0, \phi_0) = (-\frac{1640}{27}, \frac{43}{54})$

$(\pi_1, \phi_1)(uu) = (\pi_1, \phi_1)(ud) = (-\frac{200}{3}, \frac{5}{6})$.

$(\pi_1, \phi_1)(du) = (\pi_1, \phi_1)(dd) = (-40, \frac{1}{2})$.

$(\pi_2, \phi_2) = (\pi_1, \phi_1)$.

1.1.(IV)

h is self-financing

if $V_{t+1}^h - V_t^h = x_t(G_{t+1} - G_t) + y_t(S_{t+1} - S_t)$.

1). $V_1^h - V_0^h = x_0(G_1 - G_0) + y_0(S_1 + S_0)$.

① $\frac{200}{3} - \frac{170}{9} = \frac{430}{9} = -\frac{1640}{27}(1-1) + \frac{43}{54}(160-100) = \frac{1290}{27} = \frac{430}{9}$

② $-5 - \frac{170}{9} = -\frac{215}{9} = -\frac{1640}{27}(1-1) + \frac{43}{54}(70-100) = -\frac{645}{27} = -\frac{215}{9}$

2) $V_2^h - V_1^h = x_1(G_2 - G_1) + y_1(S_2 - S_1)$

③ $120 - \frac{200}{3} = \frac{160}{3} = -\frac{200}{3}(1.2-1) + \frac{5}{6}(240-160) = -\frac{40}{3} + \frac{200}{3} = \frac{160}{3}$

④ $0 - \frac{200}{3} = -\frac{200}{3} = -\frac{200}{3}(1.2-1) + \frac{5}{6}(96-160) = -\frac{40}{3} - \frac{160}{3} = -\frac{200}{3}$

⑤ $0 - (-5) = 5 = -40(1.2-1) + \frac{1}{2}(96-70) = -8 + 13 = 5$

⑥ $-12 - (-5) = -7 = -40(1.2-1) + \frac{1}{2}(72-70) = -8 + 1 = -7$.

$\therefore$ h is self-financing.

1.1.(V).

$$\phi = (\phi_t)_{t=0,1,2} \quad \begin{cases} \phi_0 = \frac{43}{54} \\ \phi_1(uu) = \phi_1(ud) = \frac{5}{6}, \quad \phi_1(du) = \phi_1(dd) = \frac{1}{2} \\ \phi_2 = \phi_1 \end{cases}$$

ϕ is adapted $\Leftrightarrow$ $\phi_t^{-1}(a) \in \mathcal{F}_t$ for $\forall a \in \mathbb{R}$, $t \in \{0,1,2\}$.

case 1). ϕ_0 is $\mathcal{F}_0$-measurable. $\mathcal{F}_0 = \{\phi, \Omega\}$

$\phi_0^{-1}(\frac{43}{54}) = \Omega \in \mathcal{F}_0$, for $\forall a \in \mathbb{R}$ s.t $a \neq \frac{43}{54}$ $\phi_0^{-1}(a) = \phi \in \mathcal{F}_0$.

Case 2). ϕ_1 is $\mathcal{F}_1$-measurable. $\mathcal{F}_1 = \{\phi, \{uu, ud\}, \{du, dd\}, \Omega\}$.

$\phi_1^{-1}(\frac{5}{6}) = \{uu, ud\} \in \mathcal{F}_1$, $\phi_1^{-1}(\frac{1}{2}) = \{du, dd\} \in \mathcal{F}_1$,

for $\forall a \in \mathbb{R}$ s.t $a \neq \frac{5}{6}$, $a \neq \frac{1}{2}$, $\phi_1^{-1}(a) = \phi \in \mathcal{F}_1$.

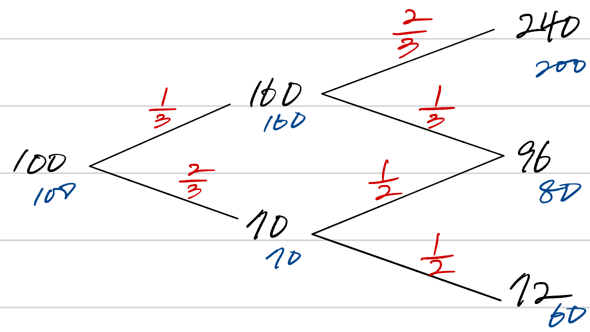
Case 3). ϕ_2 is $\mathcal{F}_2$-measurable. $\mathcal{F}_2 = 2^{\Omega}$

$\phi_2^{-1}(\frac{5}{6}) = \{uu, ud\} \in \mathcal{F}_2$, $\phi_2^{-1}(\frac{1}{2}) = \{du, dd\} \in \mathcal{F}_2$

for $\forall a \in \mathbb{R}$ s.t $a \neq \frac{5}{6}$, $a \neq \frac{1}{2}$ $\phi_2^{-1}(a) = \phi \in \mathcal{F}_2$.

$\therefore$ ϕ is adapted.

1.1.(vi). Q. $\left(\frac{S_t}{G_t}\right)$.



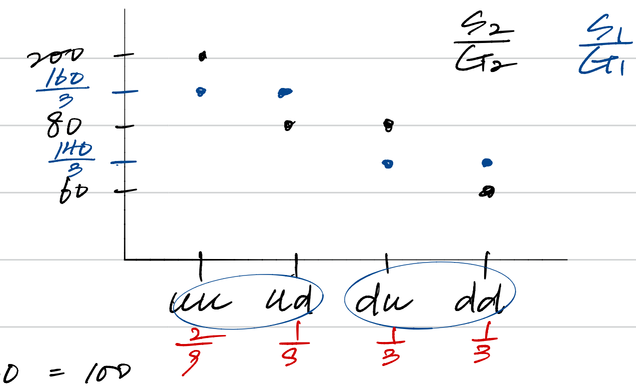
$$\begin{cases} Q(\{uu\}) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9} \\ Q(\{ud\}) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \\ Q(\{du\}) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \\ Q(\{dd\}) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \end{cases}$$

Q is risk-neutral measure.

$\Leftrightarrow$ $P \simeq Q$: trivial.

$\left(\frac{S_t}{G_t}\right)_{t=0,1,2}$: martingale on Q. $\Rightarrow E^Q\left(\frac{S_{t+1}}{G_{t+1}} \middle| \mathcal{F}_t\right) = \frac{S_t}{G_t}$

i.e ETS $\begin{cases} E^Q\left(\frac{S_2}{G_2} \middle| \mathcal{F}_1\right) = \frac{S_1}{G_1} \\ E^Q\left(\frac{S_1}{G_1} \middle| \mathcal{F}_0\right) = \frac{S_0}{G_0} \end{cases}$



$$\Rightarrow E^Q\left(\frac{S_2}{G_2}\right) = E^Q\left(\frac{S_1}{G_1}\right) = E^Q\left(\frac{S_0}{G_0}\right).$$

$$E^Q\left(\frac{S_2}{G_2}\right) = \frac{2}{9} \cdot 200 + \frac{1}{9} \cdot 80 + \frac{1}{3} 80 + \frac{1}{3} 60 = 100$$

$$E^Q\left(\frac{S_1}{G_1}\right) = \frac{2}{9} 160 + \frac{1}{9} 160 + \frac{1}{3} 70 + \frac{1}{3} 70 = 100$$

$$E^Q\left(\frac{S_0}{G_0}\right) = 100.$$

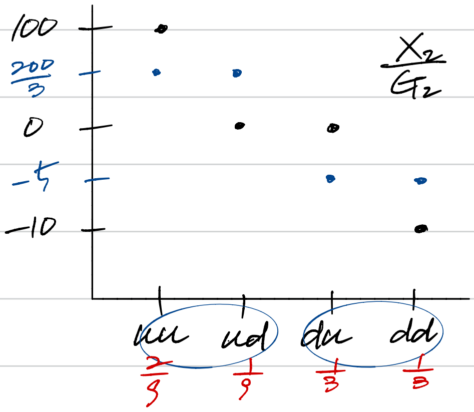
$\therefore$ Q is RN.

1.1. (vii)

$E^Q(\frac{S_2}{G_2}|\mathcal{F}_1) = \frac{S_1}{G_1}$ ($\because$ Q is RN, $(\frac{S_t}{G_t})_{t=0,1,2}$ is martingale.)

$$\Rightarrow \begin{cases} E^Q(\frac{S_2}{G_2}|\mathcal{F}_1)(uu) = E^Q(\frac{S_2}{G_2}|\mathcal{F}_1)(ud) = 160. \\ E^Q(\frac{S_2}{G_2}|\mathcal{F}_1)(du) = E^Q(\frac{S_2}{G_2}|\mathcal{F}_1)(dd) = 10. \end{cases}$$

$$E^Q(\frac{X_2}{G_2}|\mathcal{F}_1) = \begin{cases} E^Q(\frac{X_2}{G_2}|\mathcal{F}_1)(uu) = E^Q(\frac{X_2}{G_2}|\mathcal{F}_1)(ud) = \frac{200}{3} \\ E^Q(\frac{X_2}{G_2}|\mathcal{F}_1)(du) = E^Q(\frac{X_2}{G_2}|\mathcal{F}_1)(dd) = -5. \end{cases}$$



$\frac{2}{9}\cdot 100 + \frac{1}{9}\cdot 0 = \frac{a}{3} \Rightarrow a = \frac{200}{3}$

$\frac{1}{3} 0 + \frac{1}{3}\cdot(-10) = \frac{2}{3}b \Rightarrow b = -5$

1.2. (i) $\Leftrightarrow$ (ii).

($\Rightarrow$). a). Suppose $\frac{S_d}{S} \geq 1+R$. $\Rightarrow S_d \geq (1+R)S$

Let $h = (-S, 1)$. $\Rightarrow V_0^h = -S\cdot 1 + 1\cdot S = 0$.

$$V_1^h = \begin{cases} -S(1+R) + S_u > 0 & (\because S_u > S_d). \\ -S(1+R) + S_d \geq 0. & (\because S_d \geq (1+R)S). \end{cases}$$

$\Rightarrow h$ is arbitrage.

b). Suppose $1+R \geq \frac{S_u}{S} \Rightarrow (1+R)S \geq S_u$

Let $h = (S, -1)$.

$\Rightarrow V_0^h = S\cdot 1 - 1\cdot S = 0$.

$$V_1^h = \begin{cases} S(1+R) - S_u \geq 0. & (\because (1+R)S \geq S_u). \\ S(1+R) - S_d > 0 & (\because S_u > S_d) \end{cases}$$

$\Rightarrow h$ is arbitrage.

We can show above for $\frac{S_{ud}}{S_u} < 1+R < \frac{S_{uu}}{S_u}$ and $\frac{S_{dd}}{S_d} < 1+R < \frac{S_{du}}{S_d}$

Similarly. $\square$

$(\Leftarrow)$. Suppose h is arbitrage. s.t $P(V_0^h = 0) = 1$ $\quad h = (x, y)$.

$P(V_2^h \geq 0) = 1$.

$V_0^h = x + Sy = 0$. $\quad P(V_2^h > 0) > 0$

$$V_2^h = \begin{cases} (1+R)^2 x + S_{uu} y > 0 \\ (1+R)^2 x + S_{ud} y \geq 0 \\ (1+R)^2 x + S_{dd} y \geq 0. \end{cases}$$

$(\because S_{uu} > S_{ud} > S_{dd})$.

$$\Rightarrow \begin{cases} x + \frac{S_{uu}}{(1+R)^2} y > 0 \\ x + \frac{S_{ud}}{(1+R)^2} y \geq 0 \\ x + \frac{S_{dd}}{(1+R)^2} y \geq 0 \end{cases}$$

$\leftarrow x + \frac{S_u}{1+R} y$ $\left(\because \frac{S_{ud}}{S_u} < 1+R < \frac{S_{uu}}{S_u}\right)$.

$\leftarrow x + \frac{S_d}{1+R} y$. $\left(\because \frac{S_{dd}}{S_d} < 1+R < \frac{S_{du}}{S_d}\right)$.

$x + Sy = 0$ $\left(\because \frac{S_d}{S} < 1+R < \frac{S_u}{S}\right)$.

$\Rightarrow\!\Leftarrow$ $\square$.

1.2. (ii) $\Leftrightarrow$ (iii).