Probability Theory – Exercise 3

Junwoo Yang

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Problem 1

Let $\mathrm{Var}(X)$ be the variance of a random variable X.

(a) Find $\mathrm{Var}(aX)$ in terms of $\mathrm{Var}(X)$.

Proof.

$$\mathrm{Var}(aX) = \mathbb{E}(a^2X^2) - \mathbb{E}(aX)^2 = a^2\mathbb{E}(X^2) - a^2\mathbb{E}(X)^2 = a^2(\mathbb{E}(X^2) - \mathbb{E}(X)^2) = a^2\mathrm{Var}(X). \quad \square$$

(b) Find $\mathrm{Var}(X)$ of $X : [0,1] \to \mathbb{R}$ given by $X(\omega) = \min\{\omega, 1-\omega\}$.

Proof.

$$F_X(y) = P(\{\omega \in [0,1] : \min\{\omega, 1-\omega\} \le y\}) = 2\int_{-\infty}^{y} \mathbf{1}_{[0,\frac{1}{2}]}(\omega)\,\mathrm{d}\omega = \begin{cases} 0 & \text{if } y < 0 \\ 2y & \text{if } 0 \le y \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < y \end{cases}$$

$$\begin{aligned}
f_X(y) &= \frac{\mathrm{d}F_X(y)}{\mathrm{d}y} = 2\mathbf{1}_{[0,\frac{1}{2}]}(y) \\
\mathbb{E}(X) &= \int_0^1 X\,\mathrm{d}P = \int_{\mathbb{R}} x\,\mathrm{d}P_X(x) = \int x f_X(x)\,\mathrm{d}x = \int x2\mathbf{1}_{[0,\frac{1}{2}]}(x)\,\mathrm{d}x = \int_0^{\frac{1}{2}} 2x\,\mathrm{d}x = x^2\Big|_0^{\frac{1}{2}} = \frac{1}{4} \\
\mathbb{E}(X^2) &= \int x^2 f_X(x)\,\mathrm{d}x = \int x^2 2\mathbf{1}_{[0,\frac{1}{2}]}(x)\,\mathrm{d}x = \int_0^{\frac{1}{2}} 2x^2\,\mathrm{d}x = \frac{2}{3}x^3\Big|_0^{\frac{1}{2}} = \frac{1}{12} \\
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{12} - \frac{1}{16} = \frac{1}{48}
\end{aligned}$$

$\square$

(c) If $a_1, \ldots, a_n, b$ are arbitrary real numbers and $X_1, \ldots, X_n$ are random variables, show

$$\mathrm{Var}(a_1X_1 + \cdots + a_nX_n + b) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) + 2\sum_{1 \le i < j \le n} a_i a_j \mathrm{Cov}(X_i, X_j).$$

Proof. Let $Z = a_1 X_1 + \cdots + a_n X_n + b = \sum_{i=1}^{n} a_i X_i + b$. Then, we get that

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{i=1}^{n} a_i \mathbb{E}(X_i) + b \\
Z^2 &= \sum_{i=1}^{n} a_i^2 X_i^2 + 2 \sum_{1 \le i < j \le n} a_i a_j X_i X_j + 2b \sum_{i=1}^{n} a_i X_i + b^2 \\
\mathbb{E}(Z^2) &= \sum_{i=1}^{n} a_i^2 \mathbb{E}(X_i^2) + 2 \sum_{1 \le i < j \le n} a_i a_j \mathbb{E}(X_i X_j) + 2b \sum_{i=1}^{n} a_i \mathbb{E}(X_i) + b^2 \\
\mathbb{E}(Z)^2 &= \sum_{i=1}^{n} a_i^2 \mathbb{E}(X_i)^2 + 2 \sum_{1 \le i < j \le n} a_i a_j \mathbb{E}(X_i)\mathbb{E}(X_j) + 2b \sum_{i=1}^{n} a_i \mathbb{E}(X_i) + b^2 \\
\mathrm{Var}(Z) &= \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 \\
&= \sum_{i=1}^{n} a_i^2 (\mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2) + 2 \sum_{1 \le i < j \le n} a_i a_j (\mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)) \\
&= \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i) + 2 \sum_{1 \le i < j \le n} a_i a_j \mathrm{Cov}(X_i, X_j).
\end{aligned}$$

□

Problem 2

Find the correlation $\rho_{X,Y}$ if $X = 2Y + 1$.

Proof.

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(2Y + 1) = 2\mathbb{E}(Y) + 1 \\
X - \mathbb{E}(X) &= 2(Y - \mathbb{E}(Y)) \\
\mathrm{Var}(X) &= \mathbb{E}((X - \mathbb{E}(X))^2) = 4(\mathbb{E}((Y - \mathbb{E}(Y))^2)) = 4\mathrm{Var}(Y) \\
\mathrm{Cov}(X, Y) &= \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) = 2\mathbb{E}((Y - \mathbb{E}(Y))^2) = 2\mathrm{Var}(Y) \\
\rho_{X,Y} &= \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)^{\frac{1}{2}} \mathrm{Var}(Y)^{\frac{1}{2}}} = \frac{2\mathrm{Var}(Y)}{2\mathrm{Var}(Y)} = 1
\end{aligned}$$

□

Problem 3

Find F_X the distribution function of a random variable $X : [0, 1] \to \mathbb{R}$ ginven by $X(\omega) = \min\{\omega, 1 - \omega\}$.

Proof.

$$F_X(y) = P(\{\omega \in [0, 1] : \min\{\omega, 1 - \omega\} \le y\}) = 2 \int_{-\infty}^{y} \mathbf{1}_{[0, \frac{1}{2}]}(\omega) \,\mathrm{d}\omega = \begin{cases} 0 & \text{if } y < 0 \\ 2y & \text{if } 0 \le y \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < y \end{cases}.$$

□

Problem 4

Suppose that X, Y are independent random variables and that f, g are Borel measurable functions on $\mathbb{R}$. Show that the random variables $f(X)$, $g(Y)$ are independent.

Proof. Let $(\Omega, \mathcal{F}, P)$ be probability space. We want to show that

$$P(\{\omega \in \Omega : f(X(\omega)) \in B,\ g(Y(\omega)) \in C\}) = P(\{\omega : f(X(\omega)) \in B\})P(\{\omega : g(Y(\omega)) \in C\})$$

for all Borel sets B, C. Note that $f^{-1}(B)$ and $g^{-1}(C)$ are Borel sets because f, g are Borel measurable functions. Then, by definition of independence of random variables, we get that

$$\begin{aligned} P(f(X) \in B \cap g(Y) \in C) &= P(X \in f^{-1}(B) \cap Y \in g^{-1}(C)) \\ &= P(X \in f^{-1}(B))P(Y \in g^{-1}(C)) \\ &= P(f(X) \in B)P(g(Y) \in C). \end{aligned}$$

Therefore, random variables $f(X)$, $g(Y)$ are independent. $\square$

Problem 5

Show that $|\rho_{X,Y}| = 1$ if and only if $X_c = X - \mathbb{E}(X)$ and $Y_c = Y - \mathbb{E}(Y)$ are linearly dependent, that is, $P(\{\omega \in \Omega : aX_c(\omega) + bY_c(\omega) = 0\}) = 1$ for some real numbers a and b, not both 0.

Proof. Without loss of generality, assume that $\|X_c\|_2$ and $\|Y_c\|_2$ are non-zero.
($\Rightarrow$) The correlation $\rho_{X,Y}$ is cosine of angle between X_c and Y_c. $\rho_{X,Y} = \cos\theta = 1$ means that θ is an even multiple of π, i.e. X_c and Y_c have same direction. In this case, Y_c is just a positive scalar multiple of X_c, i.e. $Y_c = tX_c$ for $t \in \mathbb{R}$, $t > 0$. If $\rho_{X,Y} = \cos\theta = -1$, θ is an odd multiple of π. This means X_c and Y_c have opposite direction and Y_c can be written as tX_c for $t \in \mathbb{R}$, $t < 0$. Thus X_c and Y_c are linearly dependent.
($\Leftarrow$) TFAE

$$\begin{aligned} & P(\{\omega \in \Omega : aX_c(\omega) + bY_c(\omega) = 0\}) = 1 \ \text{ for some } a, b \in \mathbb{R} \setminus \{0\} \\ \iff & \forall \omega \in \Omega,\ Y_c(\omega) = tX_c(\omega) \ \text{ for some } t \in \mathbb{R} \setminus \{0\} \\ \iff & Y - \mathbb{E}(Y) = t(X - \mathbb{E}(X)) \\ \iff & Y = tX - t\,\mathbb{E}(X) + \mathbb{E}(Y) = tX + c \ \text{ where } c = -t\,\mathbb{E}(X) + \mathbb{E}(Y) \text{ ; constant.} \end{aligned}$$

We want to show that $|\mathrm{Cov}(X,Y)| = \mathrm{Var}(X)^{\frac{1}{2}}\mathrm{Var}(Y)^{\frac{1}{2}}$.

$$\because\ |\rho_{X,Y}| = \left| \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)^{\frac{1}{2}}\mathrm{Var}(Y)^{\frac{1}{2}}} \right| = \frac{|\mathrm{Cov}(X,Y)|}{\mathrm{Var}(X)^{\frac{1}{2}}\mathrm{Var}(Y)^{\frac{1}{2}}}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$
\begin{aligned}
&= \mathbb{E}(tX^2 + cX) - \mathbb{E}(X)\mathbb{E}(tX + c) \\
&= t\,\mathbb{E}(X^2) + c\,\mathbb{E}(X) - t\,\mathbb{E}(X)^2 - c\,\mathbb{E}(X) \\
&= t\left(\mathbb{E}(X^2) - \mathbb{E}(X)^2\right) \\
&= t\,\mathrm{Var}(X).
\end{aligned}
$$

$$
|\mathrm{Cov}(X,Y)| = \begin{cases} t\,\mathrm{Var}(X) & \text{if } t > 0 \\ -t\,\mathrm{Var}(X) & \text{if } t < 0 \end{cases}
$$

$$
\begin{aligned}
\mathrm{Var}(X)^{\frac{1}{2}}\,\mathrm{Var}(Y)^{\frac{1}{2}} &= \mathrm{Var}(X)^{\frac{1}{2}}\,\mathrm{Var}(tX + c)^{\frac{1}{2}} \\
&= \mathrm{Var}(X)^{\frac{1}{2}}\,\mathrm{Var}(tX)^{\frac{1}{2}} \\
&= \sqrt{t^2}\,\mathrm{Var}(X) \\
&= \begin{cases} t\,\mathrm{Var}(X) & \text{if } t > 0 \\ -t\,\mathrm{Var}(X) & \text{if } t < 0 \end{cases}
\end{aligned}
$$

$\therefore\ |\mathrm{Cov}(X,Y)| = \mathrm{Var}(X)^{\frac{1}{2}}\,\mathrm{Var}(Y)^{\frac{1}{2}}.$ $\square$