수리통계학2 - HW1

파이낸스경영학과
2017004093
양 준 우

5.3-3.

X_1, X_2 : indep.

$f_1(x_1) = 2x_1,\ 0 < x_1 < 1$

$f_2(x_2) = 4x_2^3,\ 0 < x_2 < 1$

(a) $P(0.5 < X_1 < 1 \text{ and } 0.4 < X_2 < 0.8)$.

$= P(0.5 < X_1 < 1) \cdot P(0.4 < X_2 < 0.8)$ ($\because$ indep).

$F_1(x_1) = \int_0^{x_1} 2t\,dt = [t^2]_0^{x_1} = x_1^2$

$F_2(x_2) = \int_0^{x_2} 4t^3\,dt = [t^4]_0^{x_2} = x_2^4$

$P(0.5 < X_1 < 1) = F_1(1) - F_1(0.5) = 1^2 - 0.5^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$P(0.4 < X_2 < 0.8) = F_2(0.8) - F_2(0.4) = \frac{8^4 - 4^4}{10000} = \frac{384}{1000}$

$\therefore \frac{3}{4} \cdot \frac{384}{1000} = 0.288$

(b) $E[X_1^2 X_2^3] = \int_0^1\int_0^1 x_1^2 x_2^3 \cdot f_{12}(x_1, x_2)\,dx_1\,dx_2$

$= \int_0^1\int_0^1 x_1^2 x_2^3 f_1(x_1) f_2(x_2)\,dx_1\,dx_2$ ($\because$ indep).

$= \int_0^1\int_0^1 x_1^2 x_2^3\, 2x_1\, 4x_2^3\,dx_1\,dx_2$

$= 8\int_0^1\int_0^1 x_1^3 x_2^6\,dx_1\,dx_2$

$= 8\int_0^1 x_1^3\,dx_1 \cdot \int_0^1 x_2^6\,dx_2$

$= 8\left[\frac{1}{4}x_1^4\right]_0^1 \cdot \left[\frac{1}{7}x_2^7\right]_0^1 = 8 \cdot \frac{1}{4} \cdot \frac{1}{7} = \frac{2}{7}$

5.3-4.

$X_1, X_2 \sim 2 \cdot e^{-2x},\ 0 < x < \infty$

$\exp(\frac{1}{2}) = \text{Gamma}(1, \frac{1}{2})$.

(a) $F_X(x) = 1 - e^{-2x} \quad 0 < x < \infty$

$P(0.5 < X_1 < 1.0,\ 0.7 < X_2 < 1.2)$

$= \{(1 - e^{-2\cdot 1}) - (1 - e^{-2\cdot 0.5})\} \cdot \{(1 - e^{-2\cdot 1.2}) - (1 - e^{-2\cdot 0.7})\}$

$= (e^{-1} - e^{-2})(e^{-1.4} - e^{-2.4})$
$= e^{-2.4} + e^{-4.4} - 2e^{-3.4}$

(b) $E[X_1(X_2 - 0.5)^2]$
$= E[X_1] \cdot E[(X_2 - 0.5)^2]$ ($\because$ indep).
Since $E[X] = \beta = 0.5$, $Var[X] = \beta^2 = 0.5^2$
$E[X_1] \cdot E[(X_2 - 0.5)^2]$
$= 0.5 \cdot Var[X_2] = 0.5 \cdot 0.5^2 = 0.125$

5.3 - 10.

X_1, X_2, X_3 : iid, $f(x) = (\frac{3}{4})(\frac{1}{4})^{x-1}$ $x = 1, 2, 3, \cdots$ $Geo(\frac{3}{4})$.

(a) $P(X_1 = 1, X_2 = 3, X_3 = 1)$.
$= (\frac{3}{4}) \cdot (\frac{3}{4}) (\frac{1}{4})^2 \cdot (\frac{3}{4})$ ($\because$ indep).
$= \frac{3^3}{4^5} = \frac{27}{1024}$

(b) Let $Y = X_1 + X_2 + X_3$
$M_Y(t) = [M_X(t)]^3$
$M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} (\frac{3}{4})(\frac{1}{4})^{x-1} = \frac{\frac{1}{4} \cdot e^t}{1 - (\frac{3}{4})e^t}$
$M_Y(t) = \left| \frac{\frac{1}{4} \cdot e^t}{1 - (\frac{3}{4})e^t} \right|^3$ $\Rightarrow$ mgf technique X.
$P(X_1 + X_2 + X_3 = 5) = 3 \cdot P(X_1 = 1, X_2 = 1, X_3 = 3)$
$+ 3 \cdot P(X_1 = 1, X_2 = 2, X_3 = 2)$
$P(X_1 = 1, X_2 = 2, X_3 = 2) = (\frac{3}{4})^3 \cdot (\frac{1}{4})^2 = \frac{27}{1024}$
$\therefore P(X_1 + X_2 + X_3 = 5) = 3 \cdot \frac{27}{1024} + 3 \cdot \frac{27}{1024} = \frac{162}{1024}$

(c). $Y = \max(X_1, X_2, X_3)$
$P(Y \leq 2) = P(X_1 \leq 2) P(X_2 \leq 2) P(X_3 \leq 2)$
$= (\frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4})^3 = (\frac{15}{16})^3$

5.3 - 12.

$X_1, X_2, X_3 \sim exp(1) = Gamma(1, 1)$. iid.
$P(1 < \min X_i) = P(1 < X_1, 1 < X_2, 1 < X_3) = P(1 < X_1)^3$

$F_X(x) = \int_0^x e^{-t}\,dt = [-e^{-t}]_0^x = 1-e^{-x}$
$P(1 < X_1) = 1 - P(X_1 < 1) = 1-(1-e^{-1}) = \frac{1}{e}$
$\therefore P(1 < \min X_i) = \frac{1}{e^3}$

5.4 - 3

X_1, X_2, X_3 : indep.
$X_1 \sim Poi(2)$, $X_2 \sim Poi(1)$, $X_3 \sim Poi(4)$
(a) $Y = X_1 + X_2 + X_3$.
$M_{X_1}(t) = \exp\{2(e^t - 1)\}$
$M_Y(t) = \prod_{i=1}^{3} M_{X_i}(t) = \exp\{(e^t-1)(2+1+4)\}$
$= \exp\{7(e^t-1)\}$: mgf of $Poi(7)$.
(b) $\therefore Y \sim Poi(7)$.
(c) $f_Y(y) = \frac{e^{-7} \cdot 7^y}{y!}$, $y = 0, 1, 2, \dots$
$P(3 \leq Y \leq 9) = \sum_{y=3}^{9} f_Y(y) = \sum_{y=3}^{9} \frac{e^{-7} \cdot 7^y}{y!}$

5.4 - 5

$Z_1, \cdots, Z_7 \sim N(0,1)$
$W = Z_1^2 + \cdots + Z_7^2$
$Z^2 \sim \chi^2(1)$. $W \sim \chi^2(7)$.
$P(1.69 < W < 14.07) = P(W < 14.07) - P(W < 1.69)$
$= 0.95 - 0.025 = 0.925$

5.4 - 7.

$X_1, X_2, X_3 \sim Gamma(7, 5)$.
(a) $Y = X_1 + X_2 + X_3$. $(1-5t)^{-7}$
$M_X(t) = (1-5t)^{-7}$
$M_Y(t) = \{(1-5t)^{-7}\}^3 = (1-5t)^{-21}$
: mgf of $Gamma(21, 5)$.
(b) $\therefore Y \sim Gamma(21, 5)$.

5.4 - 22.

X_1, X_2 : indep.

$X_1 \sim \chi^2(r_1)$, $Y = X_1 + X_2 \sim \chi^2(r)$. $r_1 < r$

(a). $M_{X_1}(t) = (1-2t)^{-\frac{r_1}{2}}$

$M_Y(t) = (1-2t)^{-\frac{r}{2}}$

$= M_{X_1}(t) \cdot M_{X_2}(t) = (1-2t)^{-\frac{r_1}{2}} \cdot M_{X_2}(t)$.

$\therefore M_{X_2}(t) = (1-2t)^{-\frac{(r-r_1)}{2}}$

(b) : mgf of $\chi^2(r-r_1)$. = Gamma$(\frac{r-r_1}{2}, 2)$

$\therefore X_2 \sim \chi^2(r-r_1)$.

5.5 - 8.

$X \sim N(162.05, 35.24)$.
$Y \sim N(145.93, 49.39)$.) indep.

$P(X > Y)$.

Let $W = X - Y$. $E[W] = 162.05 - 145.93 = 16.12$

$Var[W] = Var[X-Y] = Var[X] + Var[Y] = 84.63$

WTS : $X, Y \sim N. \Rightarrow W = X - Y \sim N(16.12, 84.63)$.

mgf of Normal distribution : $\exp\{\mu t + \frac{\sigma^2}{2}t^2\}$

$M_W(t) = E[e^{tW}] = E[e^{t(X-Y)}] = E[e^{tX}] \cdot E[e^{-tY}]$

$= \exp\{162.05t + \frac{35.24}{2}t^2\} \cdot \exp\{-145.93t + \frac{49.39}{2}t^2\}$

$= \exp\{16.12t + \frac{84.63}{2}t^2\}$

: mgf of $N(16.12, 84.63)$.

$\therefore W = X - Y \sim N(16.02, 84.63)$.

$P(X > Y) = P(X-Y > 0) = P(W > 0)$

$= P(Z > \frac{-16.02}{\sqrt{84.63}})$ $(Z \sim N(0,1))$.

$\doteqdot P(Z > -1.7414) = 0.9592$

5.5 - 14.

$T \sim t(r)$

$E(Z)$, $E(1/\sqrt{U})$, $E(Z^2)$, $E(1/U)$. $=$ Gamma$(\frac{r}{2}, 2)$

$T = Z/\sqrt{U/r}$, $Z \sim N(0,1)$, $U \sim \chi^2(r)$, $Z^2 \sim \chi^2(1)$

$E[Z] = 0$, $E[Z^2] = 1$

$$E[1/\sqrt{U}) = \int_0^\infty \frac{1}{\sqrt{u}} \frac{1}{\Gamma(\frac{r}{2}) \cdot 2^{\frac{r}{2}}} \cdot u^{\frac{r}{2}-1} \cdot \exp(-\frac{u}{2})\, du.$$

$$= \int_0^\infty \frac{1}{\Gamma(\frac{r}{2}) \cdot 2^{\frac{r}{2}}} \cdot u^{\frac{(r-1)}{2}-1} \cdot \exp(-\frac{u}{2})\, du$$

$$= \frac{\Gamma(\frac{r-1}{2})}{2^{\frac{1}{2}}\Gamma(\frac{r}{2})} \int_0^\infty \underline{\frac{1}{\Gamma(\frac{r-1}{2}) \cdot 2^{\frac{r-1}{2}}} \cdot u^{\frac{(r-1)}{2}-1} \cdot \exp(-\frac{u}{2})\, du}$$

pdf of $\chi^2(r-1) =$ Gamma$(\frac{r-1}{2}, 2)$.

$$= \frac{\Gamma(\frac{r-1}{2})}{\sqrt{2}\,\Gamma(\frac{r}{2})}$$

$$E[1/U] = \int_0^\infty \frac{1}{u} \frac{1}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} u^{\frac{r}{2}-1} \cdot e^{-\frac{u}{2}}\, du$$

$$= \int_0^\infty \frac{1}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} u^{(\frac{r}{2}-1)-1} \cdot e^{-\frac{u}{2}}\, du$$

$$= \frac{1}{(\frac{r}{2}-1) \cdot 2} \int_0^\infty \underline{\frac{1}{\Gamma(\frac{r}{2}-1) 2^{\frac{r}{2}-1}} u^{(\frac{r}{2}-1)-1} \cdot e^{-\frac{u}{2}}}\, du$$

pmf of Gamma$(\frac{r}{2}-1, 2) = \chi^2(r-2)$.

$$= \frac{1}{r-2}$$

$$\therefore E[T] = E[Z/\sqrt{U/r}] = E[Z] \cdot E[1/\sqrt{U/r}]$$

$$= E[Z] \cdot E[1/\sqrt{U}] \cdot \sqrt{r}$$

$$= 0 \cdot \frac{\sqrt{r}\,\Gamma((r-1)/2)}{\sqrt{2}\,\Gamma(r/2)} = 0 \quad \text{provided } r \geq 2$$

$$\mathrm{Var}[T] = \mathrm{Var}[Z/\sqrt{U/r}] = E[Z^2/(U/r)] - \{E[T]\}^2$$

$$= E[Z^2] \cdot E[r/U] = 1 \cdot r \cdot E[1/U]$$

$$= r \cdot \frac{1}{r-2} = \frac{r}{r-2} \quad \text{provided } r \geq 3.$$

5.5 - 15.

$T \sim t(17)$.

(a) $t_{0.01}(17) = 2.56693$ (b) $t_{0.95}(17) = -1.7396$

(c) $P(-1.740 \leq T \leq 1.740) \doteqdot 0.95 - 0.0499 \doteqdot 0.9$

5.6 - 2.

$X_1, \cdots, X_{15}$: iid. $f_X(x) = \frac{3}{2}x^2 \quad -1 < x < 1$

$Y = X_1 + \cdots + X_{15}$

$E[X] = \int_{-1}^{1} x \cdot \frac{3}{2}x^2 dx = \frac{3}{2} \cdot \frac{1}{4}[x^4]_{-1}^{1} = 0.$

$Var[X] = E[X^2] = \int_{-1}^{1} \frac{3}{2}x^4 dx = \frac{3}{2} \cdot \frac{1}{5}[x^5]_{-1}^{1} = \frac{3}{2} \cdot \frac{1}{5} \cdot 2 = \frac{3}{5}$

$E[Y] = \sum_{i=1}^{15} \mu_i = 0, \quad Var[Y] = \sum_{i=1}^{15} \sigma_i^2 = 15 \cdot \frac{3}{5} = 9$

$\therefore Y \sim (0, 3^2).$

$P(-0.3 \leq Y \leq 1.5) = 0.22788.$

By using CLT, $\frac{Y-0}{3} =: Z \sim N(0,1).$

$P(\frac{-0.3}{3} \leq Z \leq \frac{1.5}{3}) = P(-0.1 \leq Z \leq 0.5)$ where $Z \sim N(0,1)$

$\doteqdot 0.69146 - 0.46017 \doteqdot 0.23129$

5.6 - 5.

$X_1, \cdots, X_{18} \sim \chi^2(1), \quad E[X] = 1, \quad Var[X] = 2$

(a) $Y = X_1 + \cdots + X_{18}$

$M_Y(t) = \prod_{i=1}^{18} M_X(t) = [M_X(t)]^{18} = (1-2t)^{-18}$

: mgf of $\chi^2(18).$

$\therefore Y \sim \chi^2(18).$

(b). $P(Y \leq 9.390) = 0.05 \quad P(Y \leq 34.80) = 0.99.$

$E[Y] = 18, \quad Var[Y] = 36$

By using CLT, $\frac{Y-18}{6} \sim N(0,1).$

$P(Y \leq 9.39) \doteqdot P(Z \leq \frac{9.39-18}{6}) = 0.07564$

$P(Y \leq 34.8) \doteqdot P(Z \leq \frac{34.8-18}{6}) = 0.99744$

5.6 - 8.

$X \sim (24.43, 2.20)$

(a) $E[\overline{X}] = \sum_{i=1}^{30} \frac{1}{30}\mu_i = \frac{1}{30} \cdot 30 \cdot 24.43 = 24.43$

(b) $Var[\overline{X}] = \sum_{i=1}^{30} \frac{1}{30^2} \cdot \sigma_i^2 = 30 \cdot \frac{1}{30^2} \cdot 2.2 = \frac{2.2}{30}$

$\overline{X} \sim (24.43, \frac{2.2}{30})$

(c) $P(24.17 \leq \overline{X} \leq 24.82) \approx P\left(\frac{24.17-24.43}{\sqrt{2.2/30}} \leq Z \leq \frac{24.82-24.43}{\sqrt{2.2/30}}\right)$

$= 0.92509 - 0.16849 = 0.75659$

5.7-4.

$X \sim Bin(48, 0.75)$.

$E[X] = 48 \cdot 0.75 = 36$, $Var[X] = 48 \cdot 0.75 \cdot 0.25 = 9$

$P(35 \leq X \leq 40) = P(34.5 \leq X \leq 40.5)$

Apply adjust CLT, $\approx P\left(\frac{34.5-36}{3} \leq Z \leq \frac{40.5-36}{3}\right)$

$\fallingdotseq 0.93319 - 0.30853 \fallingdotseq 0.62465$

5.7-7.

$X \sim Poi(49)$. $E[X] = 49$, $Var[X] = 49$.

$P(45 < X < 60)$ Apply adjust CLT,

$\approx P\left(\frac{45.5-49}{7} < Z < \frac{59.5-49}{7}\right) \fallingdotseq 0.93319 - 0.30853 \fallingdotseq 0.62465$

5.7-13

$X_1, \cdots, X_{36} \sim Geo(\frac{3}{4})$ iid $f_X(x) = \frac{3}{4} \cdot \left(\frac{1}{4}\right)^{x-1}$, $x = 1, 2, \cdots$

(a) Let $Y := \sum_{i=1}^{36} X_i$

$E[X] = \sum_{x=1}^{\infty} x \cdot \frac{3}{4}\left(\frac{1}{4}\right)^{x-1} = \frac{3}{4}\left(1 + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right)^2 + \cdots\right) = \frac{3}{4} \cdot \frac{1}{(1-\frac{1}{4})^2} = \frac{4}{3}$

$E[X(X+1)] = \sum_{x=1}^{\infty} x(x+1)\frac{3}{4} \cdot \left(\frac{1}{4}\right)^{x-1} = \frac{3}{4}\left(1 \cdot 2 + 2 \cdot 3 \cdot \left(\frac{1}{4}\right) + 3 \cdot 4 \cdot \left(\frac{1}{4}\right)^2 + \cdots\right)$

$= \frac{3}{4} \cdot \frac{2}{(1-\frac{1}{4})^3} = \frac{3}{2} \cdot \left(\frac{4}{3}\right)^3 = \frac{32}{9}$

$Var[X] = E[X(X+1)] - E[X] - \{E[X]\}^2 = \frac{32}{9} - \frac{4}{3} - \left(\frac{4}{3}\right)^2 = \frac{4}{9}$

$E[Y] = 36 \cdot E[X] = 48$

$Var[Y] = 36 \cdot Var[X] = 16$

$P(46 \leq \sum_{i=1}^{36} X_i \leq 49) \approx P\left(\frac{45.5-48}{4} \leq Z \leq \frac{49.5-48}{4}\right)$

$\fallingdotseq 0.64616 - 0.26598 \fallingdotseq 0.38018$

(b) $\overline{X} \sim \left(\frac{4}{3}, \frac{4}{9} \cdot \frac{1}{36} = \frac{1}{9^2}\right)$

$P(1.25 \leq \overline{X} \leq 1.50) \approx P\left(\frac{1.25 - 0.5 \cdot \frac{1}{36} - \frac{4}{3}}{1/9} \leq Z \leq \frac{1.5 + 0.5 \cdot \frac{1}{36} - \frac{4}{3}}{1/9}\right)$

$\fallingdotseq 0.94791 - 0.19078 \fallingdotseq 0.75713$

5.8 - 3.

$X \sim \left(\frac{7}{2}, \frac{35}{12}\right)$.

Chebyshev's inequality : $P(|X-\mu| > k\sigma) < \frac{1}{k^2}$

$\Rightarrow P\left(\left|X-\frac{7}{2}\right| > k\cdot\sqrt{\frac{35}{12}}\right) < \frac{1}{k^2}$

$\Rightarrow P\left(\left|X-\frac{7}{2}\right| < k\cdot\sqrt{\frac{35}{12}}\right) > 1-\frac{1}{k^2}$

$k\cdot\sqrt{\frac{35}{12}} = \frac{5}{2}$. $\quad k = \frac{5}{2}\cdot\sqrt{\frac{12}{35}}$. $\quad k^2 = \frac{25}{4}\cdot\frac{12}{35} = \frac{15}{7}$

$\therefore P\left(\left|X-\frac{7}{2}\right| < \frac{5}{2}\right) > 1-\frac{7}{15} = \frac{8}{15}$

$\therefore$ lower bound : $\frac{8}{15}$

5.8 - 6.

$X \sim (98, 72)$. $\bar{X} = \frac{1}{19}\sum_{i=1}^{19} X_i \quad \bar{X} \sim \left(98, \frac{72}{19}\right)$

$P(93 < \bar{X} < 103) = P(93-98 < \bar{X}-98 < 103-98)$

$= P(|\bar{X}-\mu_{\bar{x}}| < 5)$

By Chebyshev's inequality,

$P(|\bar{X}-\mu_{\bar{x}}| < k\sigma_{\bar{x}}) > 1-\frac{1}{k^2}$

$k\cdot\sigma_{\bar{x}} = k\cdot\sqrt{\frac{72}{19}} = 5$, $\quad k = 5\cdot\sqrt{\frac{19}{72}}$, $\quad k^2 = \frac{475}{72}$

$\therefore P(|\bar{X}-98| < 5) > 1-\frac{72}{475} = \frac{403}{475} \doteqdot 0.84842$

$\therefore$ lower bound : 0.84842