Discrete-time models Instructor: Hyungbin Park

Lecture note 1: One-period models

References:
CH 2, 3 in Björk (2004)

1 Introduction

In this section, we consider a simple binomial model. Let

$$\Omega = \{u, d\}$$

and define a probability measure $\mathbb{P}$ on 2^Ω by

$$\mathbb{P}(\varnothing) = 0\,,\ \mathbb{P}(\{u\}) = p_u\,,\ \mathbb{P}(\{d\}) = p_d\,,\ \mathbb{P}(\Omega) = 1$$

where $0 < p_u, p_d < 1$ and $p_u + p_d = 1$. Running time is denoted by t, and we have two points in time, $t = 0$ (today) and $t = 1$ (tomorrow). There are two assets in the market. One is a bank account and the other is a stock.

Definition 1.1. *A bank account is a sequence of deterministic random variables $G_0, G_1 : \Omega \to \mathbb{R}$ given by*

$$\begin{aligned} G_0 &= 1 \\ G_1 &= 1 + R \end{aligned}$$

Here, R is the short interest rate.

Definition 1.2. *The stock price is a sequence of random variables $S_0, S_1 : \Omega \to \mathbb{R}$, and its dynamic behavior is described by*

$$S_0 = s$$

$$\begin{cases} S_1(u) = s_u \\ S_1(d) = s_d \end{cases}$$

where $s, s_d, s_u > 0$ and $s_d < s_u$.

Definition 1.3. *A portfolio is a vector $h = (x, y)$ in $\mathbb{R}^2$. The value process of the portfolio is defined by*

$$V_t^h = xG_t + yS_t\,,\ t = 0, 1\,.$$

Definition 1.4. *An arbitrage is a portfolion h such that*

$$\begin{aligned} &V_0^h = 0 \\ &V_1^h \geq 0 \quad \text{with probability } 1 \\ &V_1^h > 0 \quad \text{with positive probability} \end{aligned}$$

Theorem 1.1. *The binomial model above is free of arbitrage if and only if*

$$\frac{s_d}{s} < 1 + R < \frac{s_u}{s}\,.$$

2 Option pricing

An option is a contract which gives the buyer a specified amount, depending on the value of the underlier, at a specified date. Options are characterized by the payoff and the maturity.

Definition 2.1. *An option payoff is a random variable*

$$X : \Omega \to \mathbb{R}\,.$$

One of the main purposes of this note is to price options.

Definition 2.2. *We say a portfolio h is the* hedging portfolio *or the* replicating portfolio *of an option X if*

$$V_1^h = X\,.$$

Theorem 2.1. *An arbitrage-free price of an option is V_0^h where h is the hedging portfolio of the option.*

3 Risk-neutral measures

Definition 3.1. *A risk-neutral measure is a probability measure $\mathbb{Q}$ on Ω such that*

$$S_0 = \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(S_1)$$

and $\mathbb{Q}(\{u\}) > 0$, $\mathbb{Q}(\{d\}) > 0$.

Theorem 3.1. *The binomial model above is arbitrage-free if and only if a risk-neutral measure exists. In this case,*

$$\mathbb{Q}(\{u\}) = \frac{(1+R)s - s_d}{s_u - s_d}\,,\ \mathbb{Q}(\{d\}) = \frac{s_u - (1+R)s_d}{s_u - s_d}$$

Theorem 3.2. *Consider an option with payoff X with maturity $t = 1$. The arbitrage-free price is*

$$\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X)\,.$$

4 Super-hedging duality

Consider a one-period ($t = 0$ or T) trinomial model. The initial stock price is $S_0 = s$, and there are three possible prices at T: $S_T = s_3$, $S_T = s_2$ and $S_T = s_1$, with probabilities p_u, p_m and p_d, respectively. Assume that $s_1 < s_2 < s_3$ and $p_u, p_m, p_d > 0$. The back account earns zero short interest rate.

In class, we studied that the super-hedging price of an option whose payoff is

$$X = \begin{cases} x_3 & \text{if } S_T = s_3 \\ x_2 & \text{if } S_T = s_2 \\ x_1 & \text{if } S_T = s_1 \end{cases}$$

at maturity T satisfies the super-hedging duality;

$$\inf\{\alpha + \beta s \,|\, X \leq \alpha + \beta S_T\} = \sup\{\mathbb{E}^{\mathbb{Q}}(X) \,|\, \mathbb{Q} \text{ is a risk-neutral measure}\}\,.$$

The proof is as follows

$$\begin{aligned}
\inf_{\substack{x_i \leq \alpha + \beta s_i \\ i=1,2,3}} \alpha + \beta s &= \inf_{\alpha,\beta} \sup_{p_i > 0} \alpha + \beta s + \sum_{i=1}^{3} p_i(x_i - \alpha - \beta s_i) \\
&= \sup_{p_i>0} \inf_{\alpha,\beta} \alpha + \beta s + \sum_{i=1}^{3} p_i(x_i - \alpha - \beta s_i) \\
&= \sup_{p_i>0} \inf_{\alpha,\beta} \alpha(1 - \sum_{i=1}^{3} p_i) + \beta(s - \sum_{i=1}^{3} p_i s_i) + \sum_{i=1}^{3} p_i x_i \\
&= \sup_{\substack{p_i>0 \\ \sum_{i=1}^3 p_i = 1 \\ \sum_{i=1}^3 p_i s_i = s}} \sum_{i=1}^{3} p_i x_i
\end{aligned} \tag{4.1}$$

We have four equalities in these equations. The "$\inf\sup = \sup\inf$" in the second equality is not trivial and can be proven by using the "linear programing".

5 Exercises

Problem 5.1. Consider the binomial model

$$R = 0.2\,,\ s = 110\,,\ s_u = 144\,,\ s_d = 96\,,\ p_u = 0.6\,,\ p_d = 0.4\,.$$

(i) (5 points) Price and hedge a call option with strike price $K = 100$ and maturity $t = 1$.

(ii) (5 points) Find the risk-neutral measure, and evaluate the price of this option by using this risk-neutral measure

Problem 5.2. Consider the one-period trinomial model: $s = 95$, $s_u = 150$, $s_m = 125$, $s_d = 100$, $R = 0.25$, $p_u = 0.2$, $p_m = 0.2$, $p_d = 0.6$.

(i) (5 points) Define $\Omega = \{u, m, d\}$ and let $\mathbb{P}$ be the probability measure on 2^Ω such that $\mathbb{P}(\{u\}) = 0.2$, $\mathbb{P}(\{m\}) = 0.2$, $\mathbb{P}(\{d\}) = 0.6$. Define bank accounts G_0, G_1 and stock prices S_0, S_1 on this space.

(ii) (5 points) Show a risk-neutral measure $\mathbb{Q}$ exists, but is not unique. Give two examples of $\mathbb{Q}$.

(iii) (5 points) Find the super-hedging price of the option with payoff

$$X = \begin{cases} 80 & \text{if } S_1 = 150 \\ 40 & \text{if } S_1 = 125 \\ 0 & \text{if } S_1 = 100 \end{cases}$$

at maturity $t = 1$.

(iv) (10 points) Calculate

$$\sup \left\{ \frac{1}{1+R} \mathbb{E}^{\mathbb{Q}}(X) \,\middle|\, \mathbb{Q} \text{ is a risk-neutral measure} \right\}$$

and confirm that the superhedging duality holds. Can you find a risk-neutral measure $\mathbb{Q}$ which achieves the supremum?

(v) (10 points) Let $\mathcal{P}$ be set of all probability measures $\mathbb{Q}$ on 2^{Ω} such that $S_0 = \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(S_1)$ (not necessarily to satisfy $\mathbb{Q}(\{u\}) > 0$, $\mathbb{Q}(\{m\}) > 0$, $\mathbb{Q}(\{d\}) > 0$). Calculate

$$\sup \left\{ \frac{1}{1+R} \mathbb{E}^{\mathbb{Q}}(X) \,\middle|\, \mathbb{Q} \in \mathcal{P} \right\}$$

and confirm that this is equal to the superhedging price. Find the probability measure $\mathbb{Q} \in \mathcal{P}$ which achieves the supremum.

(vi) (5 points) Let $\mathcal{M}$ be the set of all signed-measures on 2^{Ω} (easy to check that this space $\mathcal{M}$ is a vector space over $\mathbb{R}$). Show that $\mathcal{P}$ is a convex subset of $\mathcal{M}$.

Problem 5.3. (15 points) In class, we merely checked that the superhedging duality holds for a specific example. The proof of the superhedging duality is in Eq.(4.1). Explain why these equalities hold except for the "inf sup = sup inf" in the second equality.

References

Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2004.

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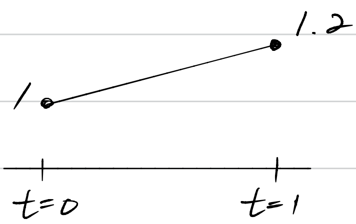
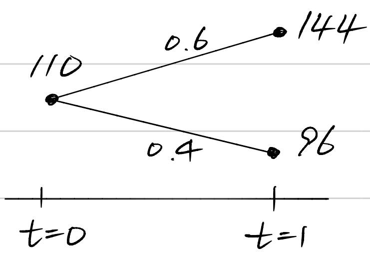
5.1.(i)



Bank

$G_0 = 1$

$G_1 = 1.2$



Stock

$S_0 = 110$

$S_1 = \begin{cases} 144 & \text{with prob. } 0.6 \\ 96 & \text{〃} \quad 0.4 \end{cases}$

Call option payoff $X = (S_1 - 100)_+ = \begin{cases} 44 & \text{if } S_1 = 144 \\ 0 & \text{if } S_1 = 96 \end{cases}$

hedging portfolio $h = (x, y) \in \mathbb{R}^2$ s.t $V_1^h = X$

$$V_1^h = \begin{cases} xG_1 + yS_u = 1.2x + 144y = 44 \\ xG_1 + yS_d = 1.2x + 96y = 0. \end{cases} \Leftrightarrow (x, y) = \left(-\frac{220}{3}, \frac{11}{12}\right)$$

$$V_0^h = -\frac{220}{3}G_0 + \frac{11}{12}S_0 = -\frac{220}{3}\cdot 1 + \frac{11}{12}\cdot 110 = 27.5$$

5.1.(ii). Risk-neutral measure Q s.t $S_0 = \frac{1}{1+R}E^Q(S_1)$

$$\begin{cases} q_u + q_d = 1 \\ 144q_u + 96q_d = 1.2 \times 110 = 132 \end{cases} \Leftrightarrow (q_u, q_d) = \left(\frac{3}{4}, \frac{1}{4}\right)$$

$$\frac{1}{1+R}E^Q(X) = \frac{1}{1.2}\left(\frac{3}{4}\cdot 44 + \frac{1}{4}\cdot 0\right) = 27.5$$

$\therefore$ The risk-neutral measure is given by $Q(\{u\}) = \frac{3}{4}$, $Q(\{d\}) = \frac{1}{4}$, and the arbitrage-free price of option is 27.5

5.2. (i) $\Omega = \{u, m, d\}$

2^{Ω} = the power set of Ω

$P : 2^{\Omega} \to \mathbb{R}$, $P(\{u\}) = 0.2$, $P(\{m\}) = 0.2$, $P(\{d\}) = 0.6$

$\Rightarrow (\Omega, 2^{\Omega}, P)$: probability space.

$G_0, G_1, S_0, S_1 : \Omega \to \mathbb{R}$ random variables.

$$\begin{cases} G_0 = 1 \quad (\text{i.e., } G_0(u) = G_0(m) = G_0(d) = 1). \\ G_1 = 1.25 \\ S_0 = 95 \\ S_1(u) = 150, \ S_1(m) = 125, \ S_1(d) = 100 \end{cases}$$

5.2. (ii). Risk-neutral measure Q should satisfy that

① $Q(\{u\}) > 0$, $Q(\{m\}) > 0$, $Q(\{d\}) > 0$.

② $S = \frac{1}{1+R} \cdot E^{Q}(S_1)$

Let $Q(\{u\}) =: q_u$, $Q(\{m\}) =: q_m$, $Q(\{d\}) =: q_d$.

$$\begin{cases} 0 < q_u, q_m, q_d < 1 \\ q_u + q_m + q_d = 1 \\ 150 q_u + 125 q_m + 100 q_d = 95 \times 1.25 = 118.75 \end{cases}$$

Intersection between two planes with normal vector $(1,1,1)$ and $(150, 125, 100)$ respectively generate line.
It is perpendicular to both of normal vectors.
Thus it is parallel to $(1,1,1) \wedge (150, 125, 100)$.
The line L is $t(1,-2,1) + (\frac{3}{8}, 0, \frac{5}{8})$ and the symmetric equation of L is $8x - 3 = -4y = 8z - 5$
In this case, (q_u, q_m, q_d) is L on first octant.
Hence, $Q = (q_u, q_m, q_d)$ is not unique obviously, and two example of Q is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, $(\frac{1}{8}, \frac{1}{2}, \frac{3}{8})$.
Actually, risk-neutral measure $Q = (q_u, q_m, q_d)$
$= t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8})$ for $t \in (0,1)$.

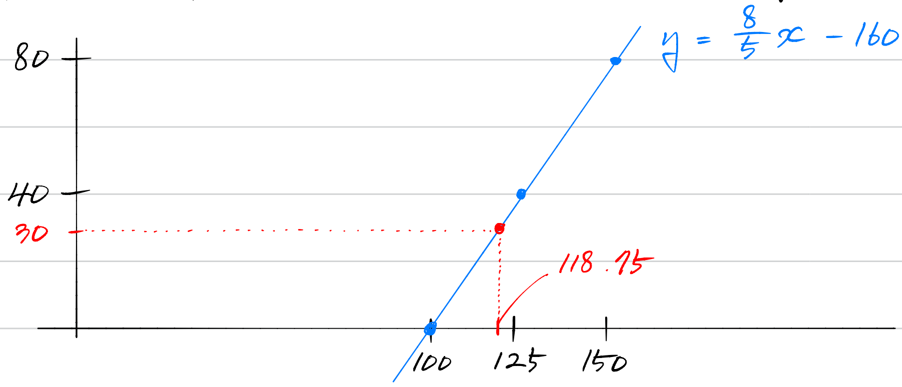
5.2.(iii). super-hedging price $= \inf\{V_0^h \mid X \le V_1^h\}$, $h = (\alpha, \beta)$.

where $X = \begin{cases} 80 & \text{if } S_1 = 150 \\ 40 & \text{if } S_1 = 125 \\ 0 & \text{if } S_1 = 100 \end{cases}$

$$\inf\{V_0^h \mid X \le V_1^h\} = \inf\left\{ \underset{\Downarrow}{\alpha + 95\beta} \;\middle|\; \begin{array}{l} 80 \le 1.25\alpha + 150\beta \\ 40 \le 1.25\alpha + 125\beta \\ 0 \le 1.25\alpha + 100\beta \end{array} \right\}$$

$$\frac{1.25\alpha + 118.75\beta}{1.25}$$

We want to find line $y = 1.25\alpha + \beta x$ above three points, $(150, 80)$, $(125, 40)$, $(100, 0)$ while $1.25\alpha + 118.75\beta$ be minimum.



$\inf\{1.25\alpha + 118.75\beta \mid X \le V_1^h\} = 30.$

$\therefore$ super-hedging price is $\inf\{\alpha + 95\beta \mid X \le V_1^h\} = \frac{30}{1.25} = 24$

Actually, in this case, we can construct hedge portfolio $h = (-128, \frac{8}{5})$, and price is $V_0^h = 24$. $\therefore$ superhedging price is also arbitrage-free price.

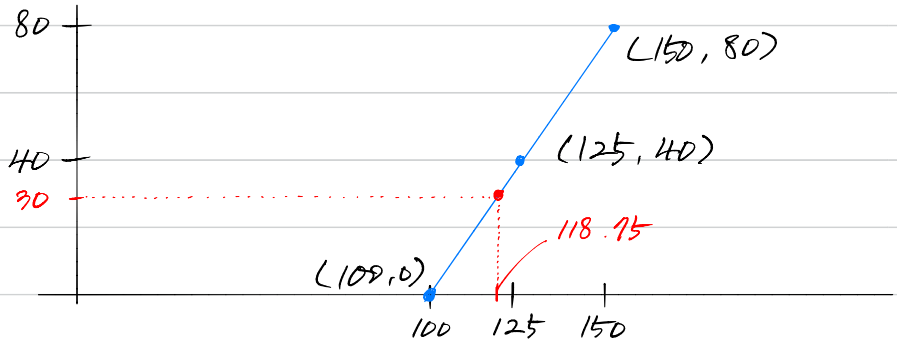
5.2.(iv). $\sup\{\frac{1}{1+R} E^Q(X) \mid Q:$ risk-neutral measure$\}$

$$\begin{cases} 0 < q_u, q_m, q_d < 1 \\ q_u + q_m + q_d = 1 \\ 150 q_u + 125 q_m + 100 q_d = 118.75 \\ 80 q_u + 40 q_m + 0 q_d = E^Q(X) \end{cases}$$

$$\begin{pmatrix} 150 \\ 80 \end{pmatrix} q_u + \begin{pmatrix} 125 \\ 40 \end{pmatrix} q_m + \begin{pmatrix} 100 \\ 0 \end{pmatrix} q_d = \begin{pmatrix} 118.75 \\ E^Q(X) \end{pmatrix}$$

$(150, 80)$, $(125, 40)$, $(100, 0)$ 을 꼭지점으로 하는 삼각형 내부의

$(118.75, E^Q(X))$ 점들 중 $E^Q(X)$의 최대 값.



$(118.75,\ E^Q(X)) = (118.75,\ 30)$.

$\therefore\ \sup\{\frac{1}{1.25} E^Q(X) \mid Q : R.N\} = \frac{30}{1.25} = 24$

$\inf\{ V_0^h \mid X \leq V_1^h\} = \sup\{\frac{1}{1+R} E^Q(X) \mid Q : R.N\}$

$\therefore$ super-hedging duality holds.

Risk-neutral measure Q s.t above supremum is achieved should satisfies that

$$\begin{cases} q_u + q_m + q_d = 1 \\ 150 q_u + 125 q_m + 100 q_d = 118.75 \\ 80 q_u + 40 q_m + 0 q_d = 30 \end{cases}$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 150 & 125 & 100 \\ 80 & 40 & 0 \end{pmatrix} \begin{pmatrix} q_u \\ q_m \\ q_d \end{pmatrix} = \begin{pmatrix} 1 \\ 118.75 \\ 30 \end{pmatrix} \Rightarrow \begin{matrix} Aq = b. \\ \det(A) = 0. \end{matrix}$$

Two examples in 5.2.(ii), $Q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}),\ (\frac{1}{8}, \frac{1}{2}, \frac{3}{8})$ satisfy above three equations. So, we can find risk-neutral measure Q which achives supremum. And it is not unique because of $\det(A) = 0$.

In this case, all risk-neutral measure Q get supremum.

$t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8}) = (0, \frac{3}{4}t, \frac{1}{4}t) + (\frac{3}{8} - \frac{3}{8}t, 0, \frac{5}{8} - \frac{5}{8}t)$

$= (\frac{3}{8} - \frac{3}{8}t, \frac{3}{4}t, \frac{5}{8} - \frac{3}{8}t) =: l_t$ for $t \in (0,1)$.

$l_t \cdot (80, 40, 0) = 80(\frac{3}{8} - \frac{3}{8}t) + 40 \cdot \frac{3}{4}t = 30.\quad \forall t \in (0,1)$.

5.2.(v) $\mathcal{P}$: set of all $Q : 2^{\Omega} \to \mathbb{R}$ s.t $S = \frac{1}{1+R} E^Q(S_1)$

It may be $Q(\{u\}) = 0$ or $Q(\{m\}) = 0$ or $Q(\{d\}) = 0$.

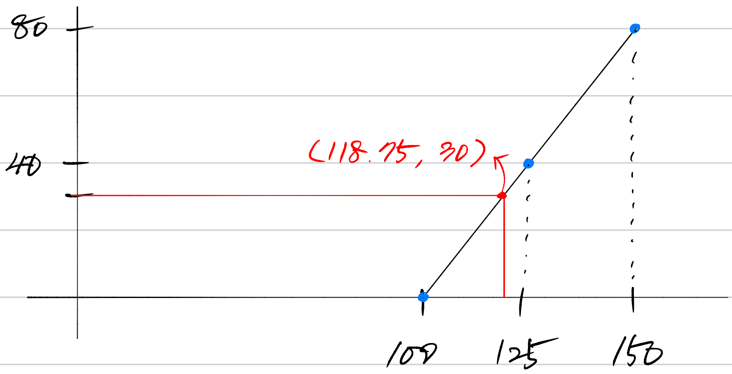
$$\sup\left\{ \frac{1}{1+R} E^Q(X) \,\middle|\, Q \in \mathcal{P} \right\} = \sup\left\{ \frac{1}{1.25}(80 q_u + 40 q_m) \right\}$$

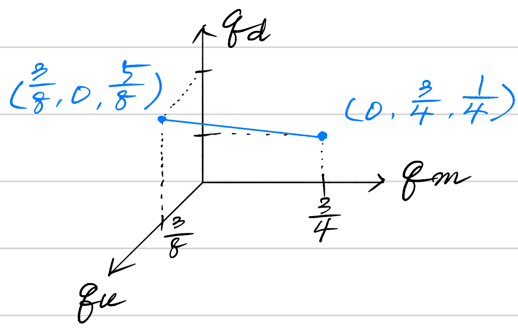
$$= \sup\left\{ \frac{80 q_u + 40 q_m}{1.25} \,\middle|\, \begin{array}{l} 0 \le q_u, q_m, q_d \le 1 \\ q_u + q_m + q_d = 1 \\ 150 q_u + 125 q_m + 100 q_d = 118.75 \end{array} \right\}$$

$$\Rightarrow \begin{cases} 150 q_u + 125 q_m + 100 q_d = 118.75 \\ 80 q_u + 40 q_m + 0 q_d = E^Q(X) \end{cases}$$

$$\Rightarrow \begin{pmatrix} 150 \\ 80 \end{pmatrix} q_u + \begin{pmatrix} 125 \\ 40 \end{pmatrix} q_m + \begin{pmatrix} 100 \\ 0 \end{pmatrix} q_d = \begin{pmatrix} 118.75 \\ E^Q(X) \end{pmatrix}$$

$\Rightarrow$ $(150, 80)$, $(125, 40)$, $(100, 0)$ 세 점을 꼭지점으로 하는 삼각형 위의 $(118.75, E^Q(X))$ 점 들 중 $E^Q(X)$의 최대 값.





$$\therefore \sup\left\{ \frac{E^Q(X)}{1.25} \,\middle|\, Q \in \mathcal{P} \right\} = \frac{30}{1.25} = 24.$$

superhedging price don't change from 5.2.(iii).

Thus it hold $\inf\{ V_0^h \mid X \le V_1^h \} = \sup\left\{ \frac{E^Q(X)}{1+R} \,\middle|\, Q \in \mathcal{P} \right\} = 24.$

$Q \in \mathcal{P}$ that achieves the supremum is not unique because of $\det(A) = 0$. All of Q is line from $(0, \frac{3}{4}, \frac{1}{4})$ to $(\frac{3}{8}, 0, \frac{5}{8})$ in $\mathbb{R}^3$. That is, $Q = (q_u, q_m, q_d)$ is $t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8})$ for $t \in [0, 1]$

5.2.(vi).

$M = \{f \mid f: 2^{\Omega} \to \mathbb{R}\} = \{f = (f_u, f_m, f_d) \mid f_u, f_m, f_d \in \mathbb{R}\}$

$P = \{g \mid g: 2^{\Omega} \to [0,1],\ g(\Omega) = 1,\ \frac{E^g(S_1)}{1+R} = s\}$

$= \{g = (g_u, g_m, g_d) \mid g = t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8})\}$

Note. $f(\{u\}) = f_u$. $\qquad g(\{u\}) = g_u$

$f(\{u, m\}) = f_u + f_m$. $\qquad g(\{u, m\}) = g_u + g_m$

for $f_1, f_2 \in M$, $c \in \mathbb{R}$,

$cf_1 + f_2 = (cf_{1u} + f_{2u}, cf_{1m} + f_{2m}, cf_{1d} + f_{2d}) \in M$

$\therefore$ M is vector space.

P is a convex subset of M if $P \subseteq M$, $\forall g_1, g_2 \in P$, line segment containing g_1, g_2 contained in P

We can consider $M, P \subset \mathbb{R}^3$

$M = \{(f_u, f_m, f_d) \mid f_u, f_m, f_d \in \mathbb{R}\} \cong \mathbb{R}^3$ (isomorphic).

$P = \{(g_u, g_m, g_d) \mid t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8}),\ t \in [0,1]\}$

$P \subset M$ (trivial).

$\forall g_1, g_2 \in P$, $\exists t_1, t_2 \in [0,1]$ s.t $g_1 = t_1(0, \frac{3}{4}, \frac{1}{4}) + (1-t_1)(\frac{3}{8}, 0, \frac{5}{8})$

$g_2 = t_2(0, \frac{3}{4}, \frac{1}{4}) + (1-t_2)(\frac{3}{8}, 0, \frac{5}{8})$

the line segment containing g_1, g_2

$tg_1 + (1-t)g_2$ for $t \in [0,1]$

$= t\{t_1(0, \frac{3}{4}, \frac{1}{4}) + (1-t_1)(\frac{3}{8}, 0, \frac{5}{8})\} + (1-t)\{t_2(0, \frac{3}{4}, \frac{1}{4}) + (1-t_2)(\frac{3}{8}, 0, \frac{5}{8})\}$

$= \underbrace{(tt_1 + t_2 - tt_2)}_{0 \le\ \le 1}(0, \frac{3}{4}, \frac{1}{4}) + \underbrace{(1 - t_2 - tt_1 + tt_2)}_{0 \le\ \le 1}(\frac{3}{8}, 0, \frac{5}{8})$

$(tt_1 + t_2 - tt_2) + (1 - t_2 - tt_1 + tt_2) = 1$.

Thus line segment containing g_1, g_2 is contained in line segment containing $(0, \frac{3}{4}, \frac{1}{4}), (\frac{3}{8}, 0, \frac{5}{8})$, that is, P

$\therefore$ P is convex subset of M.

5.3. $R=0$.

$$\inf\{V_0^h \mid X \le V_1^h\}$$

probability measure.

$\forall P_i > 0,\ i=1,2,3,\ \Rightarrow E^P(X) \le E^P(V_1^h)$

$\Rightarrow E^P(X) - E^P(V_1^h) \le 0.$

$$= \inf\{\alpha + \beta S \mid x_i \le \alpha + \beta S_i,\ i=1,2,3\}$$

$$= \inf_{\substack{x_i \le \alpha+\beta S_i \\ i=1,2,3}} \{\alpha + \beta S\}$$

$x_i \le \alpha + \beta S_i$ 제약조건을 만족하면서 $\alpha + \beta S$ 의 최소.

$$= \inf_{\alpha,\beta}\left\{\sup_{P_i>0}\left\{\alpha+\beta S + \underbrace{(E^P(X) - E^P(V_1^h))}_{\le 0}\right\}\right\}$$

$\hookrightarrow E^P(X) - E^P(V_1^h) \to 0$

삽입된 항이 아닌 항과 sup 으로 인해 $E^P(X) - E^P(V_1^h) \to 0$.

$$= \inf_{\alpha,\beta}\left\{\sup_{P_i>0}\left\{\alpha+\beta S + \underbrace{\sum_{i=1}^{3} P_i (x_i - \alpha - \beta S_i)}_{\le 0}\right\}\right\}$$

$P_i \to > 0$

$$= \sup_{P_i>0}\left\{\inf_{\alpha,\beta}\left\{\alpha+\beta S + \sum_{i=1}^{3} P_i (x_i - \alpha - \beta S_i)\right\}\right\}$$

α, β 로 묶음.

$$= \sup_{P_i>0}\left\{\inf_{\alpha,\beta}\left\{\alpha\underbrace{\left(1-\sum_{i=1}^{3}P_i\right)}_{0} + \beta\underbrace{\left(S-\sum_{i=1}^{3}P_i S_i\right)}_{0} + \underbrace{\sum_{i=1}^{3}P_i x_i}_{+}\right\}\right\}$$

최소가 되기 위해 $\sum_{i=1}^{3} P_i = 1$, $\sum_{i=1}^{3} P_i S_i = S$ 가 되어야 함. $P_i > 0$, $x_i \ge 0$.

$\Rightarrow$ risk-neutral measure 정의.

$$= \sup_{\substack{P_i>0 \\ \sum_{i=1}^{3}P_i = 1 \\ \sum_{i=1}^{3}P_i S_i = S}} \left\{\sum_{i=1}^{3} P_i x_i\right\}.$$

$$= \sup\{E^Q(X) \mid Q : \text{risk-netural measure}\}. \quad \blacksquare$$