미분방정식 HW 4-2 양준우 (2020-91659)

6.2-15.

$y'' - 2y' + 2y = \cos t.$ $y(0) = 1,$ $y'(0) = 1.$

$$\overset{\mathcal{L}}{\Rightarrow} (s^2 - 2s + 2)Y = s - 1 + \frac{s}{s^2+1}$$

$$\Rightarrow Y = \frac{s-1}{(s^2-2s+2)} + \underline{\frac{s}{(s^2+1)(s^2-2s+2)}} \quad (*)$$

$$= \frac{s-1}{(s-1)^2+1} + \underline{\frac{1}{5}\left(\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right)}$$

$$= \frac{s-1}{(s-1)^2+1} + \frac{1}{5}\left(\frac{s}{s^2+1} - 2 \cdot \frac{1}{s^2+1} - \frac{s-1}{(s-1)^2+1} + 3 \cdot \frac{1}{(s-1)^2+1}\right)$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} y = e^t \cos t + \frac{1}{5}\left(\cos t - 2\sin t - e^t \cos t + 3e^t \sin t\right)$$

$$= \frac{1}{5}\cos t - \frac{2}{5}\sin t + \frac{4}{5}e^t \cos t + \frac{3}{5}e^t \sin t \quad \square.$$

(*). $(s^2-2s+2)(As+B) + (Cs+D)(s^2+1) = s.$

$As^3 + (B-2A)s^2 + (2A-2B)s + 2B$

$Cs^3 + Ds^2 + Cs + D$

$A + C = 0. \Rightarrow C = -A.$

$$\left.\begin{array}{l} -2A + B + D = 0 \Rightarrow -2A + B - 2B = -2A - B = 0. \\ 2A - 2B + C = 1 \Rightarrow 2A - 2B - A = A - 2B = 1. \end{array}\right\} \Rightarrow \begin{array}{ll} A = \frac{1}{5} & C = -\frac{1}{5} \\ B = -\frac{2}{5} & D = \frac{4}{5} \end{array}$$

$2B + D = 0. \Rightarrow D = -2B.$

6.2 − 16.

$y'' + 2y' + y = 4e^{-t}$; $y(0) = 2$, $y'(0) = -1$.

$$\overset{\mathcal{L}}{\Rightarrow} (s^2+2s+1)Y = 2s+3+\frac{4}{s+1}$$

$$\Rightarrow Y = 2\cdot\frac{s+1}{(s+1)^2} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

$$= 2\cdot\frac{1}{s+1} + \frac{1}{(s+1)^2} + 2\frac{2}{(s+1)^3}$$

① $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

② $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} y = 2e^{-t} + e^{-t}\cdot t + 2\cdot e^{-t}\cdot t^2 \quad \square.$$

6.4 − 3.(b).

$y'' + 4y = \sin t + u_\pi(t)\sin(t-\pi)$. $y(0) = 1$, $y'(0) = 0$.

$$\overset{\mathcal{L}}{\Rightarrow} (s^2+4)Y = s+4+\frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}$$

$$\Rightarrow Y = \frac{s+4}{s^2+4} + \frac{1}{(s^2+4)(s^2+1)} + e^{-\pi s}\frac{1}{(s^2+4)(s^2+1)}$$

$$= \frac{s}{s^2+4} + 2\frac{2}{s^2+4} + \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) + \frac{1}{3}e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$$

$$= \frac{s}{s^2+4} + 2\cdot\frac{2}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\cdot\frac{2}{s^2+4} + \frac{1}{3}e^{-\pi s}\cdot\frac{1}{s^2+1} - \frac{1}{6}e^{-\pi s}\frac{2}{s^2+4}$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} y = \cos 2t + 2\sin 2t + \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \frac{1}{3}u_\pi(t)\sin(t-\pi) - \frac{1}{6}u_\pi(t)\sin(2t-\pi)$$

$$= \cos 2t + \frac{11}{6}\sin 2t + \frac{1}{3}\sin t - \frac{1}{3}u_\pi(t)\sin t - \frac{1}{6}u_\pi(t)\sin 2t$$

$$= \cos 2t + \left(\frac{11}{6} - \frac{1}{6}u_\pi(t)\right)\sin 2t + \left(\frac{1}{3} - \frac{1}{3}u_\pi(t)\right)\sin t. \quad \square.$$

6.4 − 4.(b).

$$y'' + 3y' + 2y = 1 - u_{10}(t). \quad y(0) = 1, \quad y'(0) = 0$$

$$\overset{\mathcal{L}}{\Rightarrow} (s^2 + 3s + 2)Y = s + 3 + \frac{1}{s} - \frac{1}{s}e^{-10s}$$

$$\Rightarrow Y = \frac{s+3}{(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)} - \frac{e^{-10s}}{s(s+1)(s+2)}$$

$$= \frac{2}{s+1} - \frac{1}{s+2} + \left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}\right)(1 - e^{-10s})$$

Let $\mathcal{L}^{-1}\{\frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} =: g(t).$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} y = 2e^{-t} - e^{-2t} + g(t) - u_{10}(t)\,g(t-10).$$

$$= \frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t} - u_{10}(t)\,g(t-10).$$

$$= \frac{1}{2}(1 + 2e^{-t} - e^{-2t}) - \frac{1}{2}u_{10}(t)\left(1 - 2e^{-(t-10)} + e^{-2(t-10)}\right) \quad \square.$$

6.5 - 7.(a).

$y'' + 2y' + 2y = \cos t + \delta(t-\pi)$. $y(0) = y'(0) = 0$.

$\overset{\mathcal{L}}{\Rightarrow} (s^2+2s+2)Y = \dfrac{s}{s^2+1} + e^{-\pi s}$ $\quad (\because \mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s})$.

$$\Rightarrow Y = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{1}{(s+1)^2+1}e^{-\pi s}$$

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2} + \frac{1}{(s+1)^2+1}e^{-\pi s}$$

(*).

$$= \frac{1}{5}\left(\frac{s+2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right) + \frac{1}{(s+1)^2+1}e^{-\pi s}$$

$$= \frac{1}{5}\left(\frac{s}{s^2+1} + 2\cdot\frac{1}{s^2+1} - \frac{s+1}{(s+1)^2+1} - 3\cdot\frac{1}{(s+1)^2+1}\right) + \frac{1}{(s+1)^2+1}e^{-\pi s}$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} y = \frac{1}{5}\left(\cos t + 2\sin t - e^{-t}\cos t - 3e^{-t}\sin t\right) + u_\pi(t)\cdot e^{-(t-\pi)}\cdot \sin(t-\pi).$$

$$= \frac{1}{5}\left(\cos t + 2\sin t - e^{-t}\cos t - 3e^{-t}\sin t\right) - u_\pi(t)\cdot e^{-(t-\pi)}\cdot \sin t. \quad \square.$$

(*). $(As+B)(s^2+2s+2) + (Cs+D)(s^2+1) = s$.

$A + C = 0. \Rightarrow C = -A.$

$2A + B + D = 0 \Rightarrow 2A - B = 0.$

$2A + 2B + C = 1 \Rightarrow A + 2B = 1.$

$\Rightarrow A = \frac{1}{5}, \; C = -\frac{1}{5}, \; B = \frac{2}{5}, \; D = -\frac{4}{5}$

$2B + D = 0. \Rightarrow D = -2B.$