ADVANCED CALCULUS 1
ASSIGNMENT # 3 : 2019 SPRING

§4.1. # 1.
(a) Let $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$. Prove that f is continuous.
(b) Let $f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x$. Prove that f is continuous.

§4.5. # 3. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point.

§4.6. # 3. Must a bounded continuous function on $\mathbb{R}$ be uniformly continuous?

§4.6. # 6.
(a) Show that $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous iff there exist an $\varepsilon > 0$ and sequences x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Generalize this statement to metric spaces.
(b) Use (a) on $\mathbb{R}$ to prove that $f(x) = x^2$ is not uniformly continuous.

§4.7. # 5. Let f be continuous on $[3, 5]$ and differentiable on $(3, 5)$, and suppose that $f(3) = 6$ and $f(5) = 10$. Prove that, for some point x_0 in the open interval $(3, 5)$, the tangent line to the graph of f at x_0 passes through the origin. Illustrate your result with a sketch.

§4.8. # 7. Let $f : [0, 1] \to \mathbb{R},\ f(x) = 1$ if $x = \frac{1}{n}$, n an integer, and $f(x) = 0$ otherwise.
(a) Prove that f is integrable.
(b) Show that $\int_0^1 f(x)\, dx = 0$.

(Exercises for Chapter 4)

12.
(a) A map $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz on A if there is a constant $L \geq 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$, for all $x, y \in A$. Show that a Lipschitz map is uniformly continuous.
(b) Find a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous and hence is not Lipschitz.
(c) Is the sum (product) of two Lipschitz functions again a Lipschitz function?
(d) Is the sum (product) of two uniformly continuous functions again uniformly continuous?
(e) Let f be defined and have a continuous derivative on $(a - \varepsilon, b + \varepsilon)$ for some $\varepsilon > 0$. Show that f is a Lipschitz function $[a, b]$.