Mathematical Statistics II – Final Exam

December 17, 2019

1. Present and prove the Cramer-Rao inequality, clearly.
2. Let $X_1, \ldots, X_{10}$ be a random sample from $N(0, \theta = \sigma^2)$.
   (1) Find the MLE of θ, $\hat{\theta}_{\text{MLE}}$ using the S.S's pdf for θ.
   (2) Is $\hat{\theta}_{\text{MLE}}$ is a UMVUE? Provide the evidence.
   (3) Provide the 90% confidence interval for θ. Here, we consider the equal probability for the tail part.

   Now, we are interested in testing

   $$H_0 : \sigma^2 = 1 \quad \text{vs} \quad H_1 : \sigma^2 = 2,$$

   where Type I error is a value of $\alpha = 0.1$.
   (4) Find a most powerful critical region using the percentile.
   (5) Provide Type II error.
3. Suppose $Y_1, \ldots, Y_n$ forms a random sample from the uniform distribution on the interval $[0, \theta]$. Let $Y_{(n)}$ be the maximum order statistic from the sample. We know that $Y_{(n)}$ is a S.S for θ and the Beta distribution, $X \sim Beta(\alpha, \beta)$, is expressed as

   $$f(x) = \frac{(\alpha + \beta - 1)!}{(\alpha - 1)!(\beta - 1)!} x^{\alpha - 1}(1 - x)^{\beta - 1}, \quad 0 < x < 1.$$

   (1) Find the MME of θ, $\hat{\theta}_1$, and show that $\hat{\theta}_1$ is UE of θ. Also, calculate MSE of $\hat{\theta}_1$.
   (2) Use (1) results and Rao-Blackwell Thm., find a better UE of θ, $\hat{\theta}_2$, rather than $\hat{\theta}_1$.
   (3) Compute MSE of $\hat{\theta}_2$ and compare the MSEs of $\hat{\theta}_1$ and $\hat{\theta}_2$.

   We wish to test the null hypothesis $H_0 : \theta = 1$ vs $H_1 : \theta \neq 1$, using rejection region $C = \{Y_{(n)} \leq k \text{ or } Y_{(n)} > 1\}$ and $0 \leq k < 1$.
   (4) Find the UE of θ using the pdf of $Y_{(n)}$, $\hat{\theta}_3$. Calculate MSE of $\hat{\theta}_3$.
   (5) If $\alpha = 0.05$ is the significance level, find k in terms of n.
   (6) Find thee power function of the test. Use the value of k found in (5).
4. To investigate the impact of the city's new policy to reduce ambient air pollution, the amount of particulate matter (in $\mu g/m^3$) measured at the stations before and after the new policy. Let X_i and Y_i be the amount of particulate matter at $i(= 1, \ldots, 16)$ station before and after the policy, respectively. We assume that $D_i = Y_i - X_i$ follows a normal distribution, $N(\mu_D, \sigma_D^2)$.
   (1) Construct 95% confidence interval for μ_D.
   (2) To argue that the new policy is working well, write down the null and alternative hypotheses.
   (3) Find a uniformly most powerful critical region of size $\alpha = 0.05$ under (2).