Stochastic Calculus Instructor: Hyungbin Park

Lecture note 3: Stochastic calculus

1 Exercises

Problem 1.1. (20 points) Define

$$X_t = tB_t^3 + \sin(B_t^2)\,.$$

Show that the Ito integration

$$Y_t = \int_0^t sB_s\,dX_s$$

is well-defined. Find the dynamics of Y. What are $\langle Y\rangle_T$ and $\langle X, Y\rangle_T$ for $T \geq 0$?

Problem 1.2. (20 points) Solve the following problems

(i) Show that

$$\arctan B_t + \int_0^t \frac{B_t}{(1+B_t^2)^2}\,ds\,,\ t \geq 0$$

is a martingale.

(ii) Show that $(e^{t/2}\cos B_t)_{t\geq 0}$ is a martingale. Find $\mathbb{E}(\cos B_T)$.

Problem 1.3. (10 points) Define $X_t = \int_0^t B_s^2\,dB_s$ and $Y_t = B_t$. Evaluate $d(t^2 e^{X_t} Y_t^2)$.

Problem 1.4. (10 points) Let X and Y be two Ito processes given as

$$\begin{aligned} dX_t &= aX_t\,dt + bX_t\,dB_t\,, \quad X_0 = 1 \\ dY_t &= cY_t\,dt + dY_t\,dB_t\,, \quad Y_0 = 1 \end{aligned}$$

for $a, b, c, d \in \mathbb{R}$. Find the dynamics of $U := XY$ and $V := X/Y$.

Problem 1.5. Let $B = (B_t^{(1)}, \cdots, B_t^{(d)})_{t\geq 0}$ be a d-dimensional Brownian motion. Prove the following Ito formulas. Write down the main idea as we did in class. Do not provide any rigorous proof.

(i) (10 points) For $f \in C^{1,2}([0,\infty)\times\mathbb{R}^d)$, show that

$$df(t, B_t) = f_t(t, B_t)\,dt + \nabla f(t, B_t)\,dB_t + \frac{1}{2}\Delta f(t, B_t)\,dt$$

where ∇ is the gradient operator and Δ is the Laplacian operator.

(ii) (10 points) For $f \in C^{1,2}([0,\infty)\times\mathbb{R}^d)$ and n-dimensional Ito process $X = (X_t^{(1)}, \cdots, X_t^{(n)})_{t\geq 0}$, show that

$$df(t, X_t) = f_t(t, X_t)\,dt + \sum_{i=1}^n f_{x_i}(t, X_t)\,dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^n f_{x_i x_j}(t, X_t)\,d\langle X^{(i)}, X^{(j)}\rangle_t$$

Problem 1.6. (20 points) Let $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t\geq 0}$ be a three-dimensional Brownian motion. Define

(i) $X_t = \cos(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)})$

(ii) $Y_t = t^3 \sin B_t^{(2)} + \int_0^t s^2 (B_s^{(1)})^2 \, dB_s^{(3)}$.

Find the dynamics of these processes. Evaluate the quadratic variations $\langle X \rangle_t$, $\langle Y \rangle_t$, and $\langle X, Y \rangle_t$ for $t \geq 0$.

Problem 1.7. (10 points) Let $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t\geq 0}$ be a three-dimensional Brownian motion. Show that

$$(B_t^{(1)} B_t^{(2)} B_t^{(3)})_{t\geq 0}$$

is a martingale.

Problem 1.8. (Ornstein-Uhlenbeck process) Consider the stochastic differential equation (SDE)

$$dX_t = a(\theta - X_t)\, dt + \sigma \, dB_t \,, \; X_0 = x$$

for $a \neq 0$ and $\sigma > 0$.

(i) (5 points) Solve this SDE. Hint: apply Ito's formula to $d(e^{at} X_t)$.

(ii) (10 points) Show that the solution $(X_t)_{t\geq 0}$ is a Gaussian process.

(iii) (5 points) Find $\mathbb{E}(e^{X_T})$

Problem 1.9. (15 points) Solve

$$dX_t = rt\, dt + \sigma X_t \, dB_t$$

for $r \in \mathbb{R}$ and $\sigma > 0$. Hint: Calculate $d(XY)$ for $Y_t = e^{-\sigma B_t + \frac{1}{2}\sigma^2 t}$.

References