

# Topology II – Homework 2

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October 3, 2021

## 1 Proof of Theorem 51.2

**Theorem 51.2.** The product  $*$  has the following properties:

- (i) Associative:  $([f] * [g]) * [h] = [f] * ([g] * [h])$
- (ii) Let  $e_x$  denote the constant path  $e_x: I \rightarrow X$  given by  $s \mapsto x$ ,  $x \in X$ . If  $f(0) = x_0$  and  $f(1) = x_1$  then  $[e_{x_0}] * [f] = [f]$  and  $[f] * [e_{x_1}] = [f]$ .
- (iii) Let  $\bar{f}: I \rightarrow X$  given by  $s \mapsto f(1 - s)$ . Then  $[f] * [\bar{f}] = [e_{x_0}]$  and  $[\bar{f}] * [f] = [e_{x_1}]$ .

**Proof.** First two observations

- Suppose  $f \simeq_p g$  via homotopy  $H$ ,  $f, g: I \rightarrow X$ . Let  $k: X \rightarrow Y$ . Then  $k \circ f \simeq_p k \circ g$  using  $k \circ H$ .
- If  $f * g$  (not necessarily path homotopic). Then  $k \circ (f * g) = (k \circ f) * (k \circ g)$ .

Now, the proof

- (ii) Take  $e_0: I \rightarrow I$  given by  $s \mapsto 0$ . Take  $i: I \rightarrow I$  given by  $s \mapsto s$ . Then  $e_0 * i$  is a path from 0 to 1  $\in I$ . The path  $i$  is also such a path. Because  $I$  is a convex subset,  $e_0 * i$  and  $i$  are path homotopic,  $e_0 * i \simeq i$ . Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

- (iii) Note that  $i * \bar{i} \simeq_p e_0$ . Now, applying the same rules, we get

$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \bar{f} &\simeq_p e_{x_0} \\ [f] * [\bar{f}] &= [e_{x_0}]. \end{aligned}$$

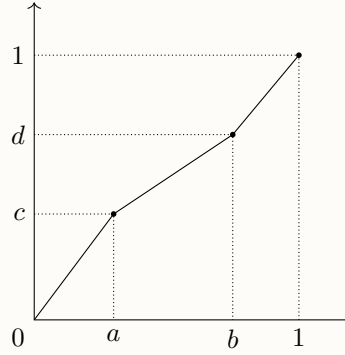
- (i) Remark: Only defined if  $f(1) = g(0)$ ,  $g(1) = h(0)$ . Note that  $f * (g * h) \neq (f * g) * h$ . The trajectory is the same, but the speed is not.

Assume the product is defined. Suppose  $[a, b]$ ,  $[c, d]$  are intervals in  $\mathbb{R}$ . Then there is a unique positive (positive slope), linear map from  $[a, b] \rightarrow [c, d]$ . For any  $a, b \in [0, 1]$  with  $0 < a < b < 1$ , we define a path

$$\begin{aligned} k_{a,b}: [0, 1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\ [b, 1] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X \end{aligned}$$

Then  $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$  and  $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$ .

Let  $\gamma$  be that path  $\gamma: I \rightarrow I$  with the following graphs:



Note that  $\gamma \simeq_p i$ . Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

which is what we wanted to show.

□

## 2 Exercises

**Exercise 51.1.** Show that if  $h, h': X \rightarrow Y$  are homotopic and  $k, k': Y \rightarrow Z$  are homotopic, then  $k \circ h$  and  $k' \circ h'$  are homotopic.

*Proof.* Let  $H: X \times I \rightarrow Y$  and  $K: Y \times I \rightarrow Z$  be homotopies between  $h, h'$  and  $k, k'$  respectively, i.e.  $H(x, 0) = h(x)$ ,  $H(x, 1) = h'(x)$ ,  $K(y, 0) = k(y)$ , and  $K(y, 1) = k'(y)$ . Then, define the map  $F: X \times I \rightarrow Z$  by  $F(x, t) = K(H(x, t), t)$ . This is continuous and defines a homotopy between  $F(x, 0) = K(H(x, 0), 0) = K(h(x), 0) = k(h(x)) = k \circ h$  and  $F(x, 1) = K(H(x, 1), 1) = K(h'(x), 1) = k'(h'(x)) = k' \circ h'$ . □

**Exercise 51.2.** Given spaces  $X$  and  $Y$ , let  $[X, Y]$  denote the set of homotopy classes of maps of  $X$  into  $Y$ .

- (a) Let  $I = [0, 1]$ . Show that for any  $X$ , the set  $[X, I]$  has a single element.
- (b) Show that if  $Y$  is path connected, the set  $[I, Y]$  has a single element.

*Proof.* To explain more about homotopy class, given two topological spaces  $X$  and  $Y$ , place an equivalence relation on the continuous maps  $f: X \rightarrow Y$  using homotopies, and write  $f_1 \sim f_2$  if  $f_1$  is homotopic to  $f_2$ .

- (a) We need to show that all continuous maps of  $X$  into  $I$  are homotopic to each other; we do this by showing that every continuous map  $f: X \rightarrow I$  is homotopic to the constant map  $f_0: X \rightarrow I$  defined by  $f_0(x) = 0$  for all  $x \in X$ . This is indeed the case, and an explicit homotopy is given by  $F: X \times I \rightarrow I$  defined by  $F(x, t) = tf(x)$ , which is clearly continuous, and satisfies  $F(x, 0) = 0 = f_0(x)$  and  $F(x, 1) = f(x)$ .
- (b) Assuming  $Y$  is path-connected, we need to show that any two continuous maps from  $I$  to  $Y$  are homotopic. First we show that every continuous map  $f: I \rightarrow Y$  is homotopic to the constant map  $I \rightarrow Y$  which maps every element of  $I$  to  $f(0)$ . Indeed, consider  $F: I \times I \rightarrow Y$  given by  $F(s, t) = f(st)$ , which is continuous. This is a homotopy between the constant map  $F(s, 0) = f(0)$  and  $F(s, 1) = f(s)$ . (In other terms: we have shown that every path in  $Y$  can be homotoped (not fixing the end points) to the constant path at its starting point).

Next, given two points  $y, y' \in Y$ , let  $f, f': I \rightarrow Y$  be the constant maps taking the values  $f(s) = y$  and  $f'(s) = y' \forall s \in I$ . Since  $Y$  is path-connected, there exists a path  $g: I \rightarrow Y$  such that  $g(0) = y$  and  $g(1) = y'$ . We then consider the map  $F: I \times I \rightarrow Y$  defined by  $F(s, t) = g(t)$ , which gives a homotopy between  $F(s, 0) = g(0) = y = f(s)$  and  $F(s, 1) = g(1) = y' = f'(s)$ . Thus, any path is homotopic to a constant path, and any two constant paths are homotopic to each other (again, not fixing the end points); it follows that any two maps  $I \rightarrow Y$  are homotopic.

□

**Exercise 51.3.** A space  $X$  is said to be contractible if the identity map  $i_X: X \rightarrow X$  is null-homotopic.

- (a) Show that  $I$  and  $\mathbb{R}$  are contractible.
- (b) Show that a contractible space is path connected.
- (c) Show that if  $Y$  is contractible, then for any  $X$ , the set  $[X, Y]$  has a single element.
- (d) Show that if  $X$  is contractible and  $Y$  is path connected, then  $[X, Y]$  has a single element.

*Proof.* (a) Let  $F: I \times I \rightarrow I$  be defined by  $F(s, t) = st$  and  $G: \mathbb{R} \times I \rightarrow \mathbb{R}$  by  $G(s, t) = st$ . These are homotopies between the constant map at 0 and identity map, so both spaces are contractible.

- (b) Recall that if there is a path between  $a, b$  and a path between  $b, c$ , then there is a path between  $a, c$ . It therefore suffices to show that all points can be connected to a given point by a path. Assuming  $X$  is contractible, there is a homotopy  $F: X \times I \rightarrow X$  between identity map  $\text{id}_X$  and the constant map  $f_0$  mapping every point  $x \in X$  to the same point  $x_0 \in X$  s.t.  $F(x, 0) = f_0(x) = x_0$  and  $F(x, 1) = \text{id}_X(x) = x$  for all  $x \in X$ . Then, the map  $g: I \rightarrow X$  defined by  $g(t) = F(x, t)$  is continuous and determines a path from  $g(0) = x_0$  to  $g(1) = x$ .
- (c) Assume  $Y$  is contractible, and let  $F: Y \times I \rightarrow Y$  be a homotopy s.t.  $F(y, 1) = y$  is the identity map and  $F(y, 0) = y_0 \in Y$  is a constant map sending every point to some point  $y_0 \in Y$ . Then given any map  $g: X \rightarrow Y$ , we consider  $G: X \times I \rightarrow Y$  defined by  $G(x, t) = F(g(x), t)$ . This is continuous, and defines a homotopy between  $g$  and the constant map  $g_0$  which maps every point of  $X$  to  $y_0$ . Indeed,  $G(x, 1) = F(g(x), 1) = g(x)$ , and  $G(x, 0) = F(g(x), 0) = y_0$ . It follows that every map from  $X$  to  $Y$  is homotopic to the constant map  $g_0$ , and hence that any two maps from  $X$  to  $Y$  are homotopic to each other.
- (d) Since  $X$  is contractible,  $\text{id}_X$  is homotopic to a constant map  $g(x) = x_0$  by a homotopy  $G: X \times I \rightarrow X$  s.t.  $G(x, 0) = x_0$ ,  $G(x, 1) = x$  for all  $x \in X$ . First we show that every continuous map  $f: X \rightarrow Y$  is homotopic to the constant map  $X \rightarrow Y$  which maps every element of  $X$  to  $f(x_0)$ . Indeed, define a continuous map  $F: X \times I \rightarrow Y$  by  $F(x, t) = f(G(x, t))$ . This is a homotopy between the constant map  $F(x, 0) = f(G(x, 0)) = f(x_0)$  and  $F(x, 1) = f(G(x, 1)) = f(x)$ .

Next, we show that if  $Y$  is path connected then constant maps (sending every point of  $X$  to the same point of  $Y$ ) are homotopic to each other. Indeed, given two points  $y_0, y_1 \in Y$ , let  $f_0, f_1: X \rightarrow Y$  be the constant maps taking the values  $f_0(x) = y_0$  and  $f_1(x) = y_1$  for all  $x \in X$ . Since  $Y$  is path-connected, there exists a path  $g: I \rightarrow Y$  s.t.  $g(0) = y_0$  and  $g(1) = y_1$ . We then consider the map  $F: X \times I \rightarrow Y$  defined by  $F(x, t) = g(t)$ , which gives a homotopy between  $F(x, 0) = g(0) = y_0 = f_0(x)$  and  $F(x, 1) = g(1) = y_1 = f_1(x)$ .

Thus, assuming  $X$  contractible and  $Y$  path-connected, any continuous map of  $X$  into  $Y$  is homotopic to a constant map, and any two constant maps are homotopic to each other. It follows that any two continuous maps from  $X$  to  $Y$  are homotopic to each other.

□