Introduction to Real Analysis – Final Exam

June 17, 2020

1. (1) Let
$$\|f\|_{L^{1,w}(\mathbb{R}^d)} = \sup_{\alpha>0} \alpha \cdot m\left(\left\{x \in \mathbb{R}^d : |f(x)| > \alpha\right\}\right)$$
where m stands for the Lebesgue measure on $\mathbb{R}^d$. Check that
$$\|f\|_{L^{1,w}(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

   (2) Give an example of a function g in $(0, \infty)$ such that
$$\|g\|_{L^{1,w}((0,\infty))} = 1 \quad \text{and} \quad \|g\|_{L^1((0,\infty))} = +\infty.$$

2. (1) Suppose that F is a $\mathbb{R}$-valued absolutely continuous function on $[a, b]$. Prove that
$$T_F(a, b) = \int_a^b |F'(t)| \, \mathrm{d}t.$$

   (2) Suppose that F is a $\mathbb{R}$-valued continuous function on $[a, b]$. Show that
$$T_F(a, b) = \lim_{\varepsilon \to 0+} T_F(a + \varepsilon, b).$$

   (3) Determine whether
$$F(x) = (x-1)^{2022} \sin((x-1)^{-2020}) \quad \text{for } x \in [0, 2]$$
is of bounded variation on $[0, 2]$ or not.

3. (1) For a fixed number $\xi \in (0, 1)$, we construct a subset $\mathcal{C}_\xi$ of $\mathbb{R}$ in the following manner:
   - In the first stage of the construction, we remove the middle ξ from $[0, 1]$ so that the remaining set is $[0, \frac{1-\xi}{2}] \cup [\frac{1+\xi}{2}, 1]$.
   - In the second stage, we remove the middle ξ^2 from each of $[0, \frac{1-\xi}{2}]$ and $[\frac{1+\xi}{2}, 1]$.
   - By repeating this process countably many times, we obtain the set $\mathcal{C}_\xi$. Note that $\mathcal{C}_{\frac{1}{3}}$ is the Cantor set.

   Compute the (strict) Hausdorff dimension of the set $\mathcal{C}_\xi$.

   (2) Prove that there exists a subset of $\mathbb{R}$ having Hausdorff dimension γ for any $\gamma \in (0, 1)$.

   (3) Compute the Hausdorff dimension and the Minkowski dimension of the compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $\mathbb{R}$.