Probability Theory – Final Exam

December 15, 2020

1. The joint density function of X and Y is given by

$$f(x,y)=\begin{cases}2e^{-x}e^{-2y} & \text{if } 0<x,y<\infty\\ 0 & \text{otherwise.}\end{cases}$$

Compute $P(X>1, Y<1)$ and $P(X<Y)$.

2. If X, Y have joint density $f_{X,Y}$ given by

$$f_{X,Y}(x,y)=\begin{cases}e^{-y} & \text{if } 0<x<y<\infty\\ 0 & \text{otherwise,}\end{cases}$$

compute the density f_{X+Y} of $X+Y$.

3. Let

$$A_n=\begin{cases}(-\frac{1}{n},1] & \text{if } n \text{ is odd}\\ (-1,\frac{1}{n}] & \text{if } n \text{ is even.}\end{cases}$$

Find $\limsup_n A_n$ and $\liminf_n A_n$.

4. Find the characteristic functions of random variable X with

$$P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!},\quad k=0,1,2,3,\dots.$$

5. Let X_n $(n=1,2,\dots)$ be independent random variables, each uniformly distributed over $[0,1]$. Calculate $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$ where $S_n=X_1+\cdots+X_n$.

6. Let X and Y be independent Gaussian random variables with expectation 0 and variance 1. Find the value T with $P(3X+2Y>5)=1-\Phi(T)$ where

$$\Phi(t)=\int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx.$$

7. Suppose that $X_1, X_2, \dots$ are i.i.d. If $Y_n=X_n\mathbb{I}_{\{\omega:|X_n(\omega)|\le n\}}$ is a truncated random variable, show that

$$\sum_{n=1}^{\infty}P(Y_n\neq X_n)\le\mathbb{E}(|X_1|).$$

8. Suppose that $X_1,\dots,X_n$ are independent random variables. Show that for any $a\ge 0$,

$$P\left(\max_{1\le k\le n}|S_k|\ge 3a\right)\le 3\max_{1\le k\le n}P(|S_k|\ge a)$$

where $S_k=X_1+\cdots+X_k$.