Differential Equations – Final Exam

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1. Solve the given initial value problems.

   (a) $y'' + y = \sec t$, $\quad y(0) = 1$, $\quad y'(0) = 0$.

   (b) $y'' + 2y' + 3y = 3\delta(t - 2\pi) + u(t - \pi)\cos t$, $\quad y(0) = 0$, $\quad y'(0) = 1$.

2. Let y be the solution of $y' = y^2 - ty$, $y(0) = 1$. Estimate $y(1)$ using the Improved Euler method with the step size $h = 1$.

3. Find a second solution y_2 of the given differential equation

$$t^2 y'' - 2t(t + 1)y' + 2(t + 1)y = 0, \quad t > 0, \quad y_1(t) = t$$

   by using the method of the reduction of order.

4. For $x > 0$, find two (linearly independent) solutions of $x^2 y'' + (x^2 + x)y' - y = 0$.

5. Find the general solution of

$$X' = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} X + \begin{pmatrix} -4\cos t \\ 3\cos t \end{pmatrix}.$$

6. For the following two problems

   (a) $\begin{cases} x' = y - x^2 \\ y' = (x - y^2)(1 - 4y) \end{cases}$ $\quad$ and $\quad$ (b) $\begin{cases} x' = -4x + y + x^2 \\ y' = y - x - 2 \end{cases}$

   (a) Find all critical points and classify the type of each critical point.

   (b) Draw the phase plane, sketch a few of the trajectories in the neighborhood of each critical point.

7. Find the solution of

$$\begin{cases} u_{xx} = u_t, & 0 < x < 20, \quad t > 0 \\ u(0, t) = 30, \quad u(20, t) = 10, & t > 0 \\ u(x, 0) = x(20 - x)/10, & 0 \le x \le 20 \end{cases}.$$

8. If u_1 and u_2 are solutions of

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, \quad t > 0 \\ u(x, 0) = f(x), & 0 < x < L \\ u(0, t) = g_1(t), \quad u(L, t) = g_2(t), & t \ge 0 \end{cases},$$

   then show that $u_1 = u_2$.