Topology I – Homework 2

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Problem 2.1 Let Y be a subset of a metric space X. Show that Y is closed if $X \setminus Y$ is open.

Proof. Suppose Y is not closed. Then $Y \neq \overline{Y}$. This means that there exists $y \in \overline{Y}$ such that $y \notin Y$ because $Y \subset \overline{Y}$. Thus y belongs to $X \setminus Y$. Since every point of $\overline{Y}$ is adherent to Y, $B(y, r) \cap Y \neq \emptyset$ for all $r > 0$. Thus $X \setminus Y$ has a point y where $B(y, r) \not\subset X \setminus Y$ for all $r > 0$. Therefore $X \setminus Y$ is not open. This is a contradiction. □

Problem 2.2 Let E be subset of a metric space (X, d) and $\partial E = \overline{E} \cap (\overline{X \setminus E})$. Show that

$$E \text{ is open} \iff E \cap \partial E = \emptyset, \tag{1}$$

$$E \text{ is closed} \iff \partial E \subseteq E. \tag{2}$$

Proof. (1) ($\Rightarrow$) Since E is open, $X \setminus E$ is closed by 2.1. Thus $X \setminus E = \overline{X \setminus E}$.
$E \cap \partial E = E \cap \overline{E} \cap (\overline{X \setminus E}) = E \cap (\overline{X \setminus E}) = E \cap (X \setminus E) = \emptyset$.
($\Leftarrow$) $E \cap (\overline{X \setminus E}) = \emptyset$. Suppose E is not open. Then, there exists $x \in E$ such that $B(x, r) \not\subset E$ for all $r > 0$. In other word, $B(x, r) \cap (X \setminus E) \neq \emptyset$ for all $r > 0$. Thus, x is adherent to $(X \setminus E)$, so $x \in \overline{X \setminus E}$. Then $E \cap (\overline{X \setminus E}) \neq \emptyset$. This is a contradiction. Hence E is open.
(2) ($\Rightarrow$) Since E is closed, $E = \overline{E}$. $\partial E = \overline{E} \cap (\overline{X \setminus E}) \subseteq \overline{E} = E$.
($\Leftarrow$) $\overline{E} \cap (\overline{X \setminus E}) \subseteq E$. Suppose E is not closed. Since $E \subsetneq \overline{E}$, there exists $x \in \overline{E}$ such that $x \notin E$. So, x belongs to $X \setminus E$. Then x also belongs to $\overline{X \setminus E}$. Thus, $x \in \overline{E} \cap (\overline{X \setminus E})$ but $x \notin E$. Since this is a contradiction, E is closed. □

Problem 2.3 Let $Y = \{(t, \sin \frac{1}{t}) | t > 0\} \subset \mathbb{R}^2$. Show that Y is not closed in the Euclidean space $\mathbb{R}^2$.

Solution. Suppose Y is closed in $\mathbb{R}^2$. Then $Y = \overline{Y}$. Consider $B(0, r)$ for $0 \in \mathbb{R}^2$, $r > 0$. For $\delta < r$, there exists $(\delta, \sin \frac{1}{\delta} = 0) \in B(0, r)$. Thus 0 is adherent point of Y. Since $Y = \overline{Y}$, 0 belongs to Y. However, $Y = \{(t, \sin \frac{1}{t}) | t > 0\}$ does not have origin point. This is a contradiction. Hence, Y is not closed in $\mathbb{R}^2$. □