Introduction to Real Analysis – Midterm Exam

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Problem 1.1

In our course, we defined that a subset E of $\mathbb{R}^d$ is **(Lebesgue) measurable** if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(E - F) \leq \epsilon$. Show that E is measurable if and only if for any $\epsilon > 0$ there is a closed set F with $F \subset E$ and $m_*(E - F) \leq \epsilon$.

Proof. To show that, we use following property.

Property. *The complement of a measurable set is measurable.*

Suppose that E is measurable. Above property says that E^c is measurable. Then for any $\epsilon > 0$, there is a open set $\mathcal{O}_\epsilon$ with $E^c \subset \mathcal{O}_\epsilon$ and $m_*(\mathcal{O}_\epsilon - E^c) \leq \epsilon$. The complement F_ϵ of $\mathcal{O}_\epsilon$ is closed. Now we can observe that $F_\epsilon \subset E$ and $\mathcal{O}_\epsilon - E^c = E - F_\epsilon$. Thus $m_*(\mathcal{O}_\epsilon - E^c) = m_*(E - F_\epsilon) \leq \epsilon$ for any $\epsilon > 0$. Conversely, let F_ϵ be closed set with $F \subset E$ and $m_*(E - F_\epsilon) \leq \epsilon$ for any $\epsilon > 0$. The complement $\mathcal{O}_\epsilon$ of F_ϵ is open. We can observe that $E^c \subset \mathcal{O}_\epsilon$ and $E - F_\epsilon = \mathcal{O}_\epsilon - E^c$. Then $m_*(E - F_\epsilon) = m_*(\mathcal{O}_\epsilon - E^c) \leq \epsilon$. This implies that E^c is measurable. Thus E is measurable. □

Problem 1.2

Describe the relationship between the original definition of the measurability and the exterior measure. By using this, discuss what would be the natural definition of the interior measure. Explain also the relationship between the measurability condition in 1.1 and the interior measure.

Proof. The definition of exterior measure is as following.

Definition. If E is *any* subset of $\mathbb{R}^d$, the **exterior measure** of E is $m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$, where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.

The exterior measure can't guarantee that $m_*(E) = m_*(E_1) + m_*(E_2)$ for $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$. It is also said that an exterior measure cannot have countable additivity property. A family of Lebesgue measurealbe sets, however, always holds above identity. In other words, exterior measures can be defined for all subset of $\mathbb{R}^d$, but not lebesgue measures. We say that E is Lebesgue measurable when there is an open set that makes the difference in exterior measure smaller than $\epsilon > 0$. Similarly from this perspective, the interior measure can be defined naturally as following.

Definition. If E is *any* subset of $\mathbb{R}^d$, the **interior measure** of E is $m_*(E) = \sup \sum_{j=1}^{\infty} |Q_j|$, where the supremum is taken over all countable union $\bigcup_{j=1}^{\infty} Q_j \subset E$ by closed cubes.

A Lebesgue measure can be derived from this definition. E is measurable if there is a closed $F \subset E$ makes the difference in measure less than ϵ. In other words, it is possible to derive the condition 1.1 which equivalent condition with original definition of measurablility from the interior measure. $\square$

Problem 1.3

A set E in $\mathbb{R}^d$ is **Caratheodory measurable** if

$$m_*(A) = m_*(E \cap A) + m_*(E^c \cap A) \text{ for every } A \subset \mathbb{R}^d.$$

Prove that this notion is equivalent to the original definition of the measurability. What is its advantage and disadvantage?

Proof. ($\Rightarrow$) By definition of m_*, for any $\epsilon > 0$, there exists an open set $\mathcal{O}_\epsilon \supset E$ such that $m_*(\mathcal{O}_\epsilon) \leq m_*(E) + \epsilon$. Since E is Caratheodory measurable, for any ϵ, $m_*(\mathcal{O}_\epsilon) = m_*(E \cap \mathcal{O}_\epsilon) + m_*(E^c \cap \mathcal{O}_\epsilon) = m_*(E) + m_*(\mathcal{O}_\epsilon - E) \leq m_*(E) + \epsilon$. Thus we get $m_*(\mathcal{O}_\epsilon - E) \leq \epsilon$, and this implies E is measurable.

($\Leftarrow$) We want to show that for measurable set E and all $A \in \mathbb{R}^d$, $m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$. By countable sub-additivity of m_*, it is clear that $m_*(A) \leq m_*(A \cap E) + m_*(A \cap E^c)$. So, if we show the reverse inequality, we are done. Consider countable covering $\bigcup_{j=1}^\infty Q_j \supset A$ such that $\sum_{j=1}^\infty m_*(Q_j) < m_*(A) + \epsilon$ where Q_j is measurable. Since countable union of measurable sets is measurable, $Q_j \cup E$ and $Q_j \cup E^c$ are measurable.

$$\begin{aligned}
m_*(A) + \epsilon > \sum_{j=1}^\infty m_*(Q_j) &= \sum_{j=1}^\infty m_*((Q_j \cap E) \cup (Q_j \cap E^c)) & (1)\\
&= \sum_{j=1}^\infty m_*(Q_j \cap E) + m_*(Q_j \cap E^c) & (2)\\
&= \sum_{j=1}^\infty m_*(Q_j \cap E) + \sum_{j=1}^\infty m_*(Q_j \cap E^c) & (3)\\
&\geq m_*((\bigcup_{j=1}^\infty Q_j) \cap E) + m_*((\bigcup_{j=1}^\infty Q_j) \cap E^c) & (4)\\
&\geq m_*(A \cap E) + m_*(A \cap E^c) & (5)
\end{aligned}$$

Take $\epsilon \to 0$ to get $m_*(A) \geq m_*(A \cap E) + m_*(A \cap E^c)$. Details are as follows.
(1)$\to$(2): By below theorem.
Theorem. *If $E_1, E_2, \cdots,$ are disjoint measurable sets, and $E = \bigcup_{j=1}^\infty E_j$, then $m(E) = \sum_{j=1}^\infty m(E_j)$.*
(3)$\to$(4): By below observation.
Observation. (Countable sub-additivity) *If $E = \bigcup_{j=1}^\infty E_j$, then $m_*(E) \leq \sum_{j=1}^\infty m_*(E_j)$.*
(4)$\to$(5): By the fact that $A \subset \bigcup_{j=1}^\infty Q_j$ and below observation.
Observation. (Monotonicity) *If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.*

If E satisfies the Caratheodory condition, E can be used as a powerful tool to divide arbitrary set A into two sets. The obvious reason is that "divide, measure each parts, and add up" is so essential to make a measure useful. If E satisfies the Caratheodory condition, all sets are neatly divided into two, whereas if it does not hold, some sets will be divided into two odd-shaped subsets. □

Problem 2.1

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of complex-valued functions on $\mathbb{R}^d$ such that $f_n \to f$ a.e. as $n \to \infty$. Also, assume that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |f_n| \le C$$

for a constant $C > 0$. Derive

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} (|f_n| - |f_n - f| - |f|) = 0,$$

and interpret it in terms of Fatou's lemma.

Proof. First to show that $\int_{\mathbb{R}^d} |f| \le C$, i.e., $f \in L^1$, we use Fatou's lemma.

Fatou's lemma. *Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then $\int f \le \liminf_{n\to\infty} \int f_n$.*

Since $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |f_n| \le C$, by Fatou's lemma,

$$\int_{\mathbb{R}^d} |f| = \int_{\mathbb{R}^d} \lim_{n\to\infty} |f_n| \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} |f_n| \le C.$$

By triangle inequality, $|f_n| = |f + f_n - f| \le |f| + |f_n - f|$. Let $g_n = |f_n| - |f_n - f| - |f| \le 0$.
Then $|g_n| = |f| + |f_n - f| - |f_n| \le |f| + |f_n| + |f| - |f_n| = 2|f|$. Since $|f| \in L^1$, $|g_n| \in L^1$.
We are now ready to use the dominated convergence theorem.

Lebesgue's dominated convergence theorem. *Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x, as n tends to infinity. If $|f_n(x)| \le g(x)$, where g is integrable, then $\int |f_n - f| \to 0$ as $n \to \infty$, and consequently $\int f_n \to \int f$ as $n \to \infty$.*

By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g_n = \int_{\mathbb{R}^d} \lim_{n\to\infty} g_n = \int_{\mathbb{R}^d} 0 = 0.$$ □

Problem 2.2

By inspecting the proof of the dominated convergence theorem, prove that if

$$\begin{cases} g_n,\ h_n,\ g,\ h \in L^1(\mathbb{R}^d), \\ g_n \to g \text{ a.e. as } n \to \infty, \\ h_n \to h \text{ a.e. as } n \to \infty, \\ |g_n| \le h_n \text{ for all } n \in \mathbb{N}, \\ \displaystyle\int_{\mathbb{R}^d} h_n \to \int_{\mathbb{R}^d} h \text{ as } n \to \infty, \end{cases}$$

then

$$\int_{\mathbb{R}^d} g_n \to \int_{\mathbb{R}^d} g \text{ as } n \to \infty.$$

Proof. Since $|g_n| \le h_n$, $h_n + g_n$ is non-negative measurable function. Then, by Fatou's lemma,

$$\begin{aligned} \int h + \int g = \int (h+g) = \int \lim_{n\to\infty} (h_n + g_n) \le \liminf_{n\to\infty} \int (h_n + g_n) = \liminf_{n\to\infty} (\int h_n + \int g_n) \\ = \liminf_{n\to\infty} \int h_n + \liminf_{n\to\infty} \int g_n = \int h + \liminf_{n\to\infty} \int g_n. \end{aligned}$$

Thus, $\int g \le \liminf_{n\to\infty} \int g_n$. Since $h_n - g_n$ is also non-negative, similarly,

$$\begin{aligned} \int h - \int g = \int (h-g) = \int \lim_{n\to\infty} (h_n - g_n) \le \liminf_{n\to\infty} \int (h_n - g_n) = \liminf_{n\to\infty} (\int h_n - \int g_n) \\ = \liminf_{n\to\infty} \int h_n + \liminf_{n\to\infty} \int -g_n = \int h + \liminf_{n\to\infty} \int -g_n = \int h - \limsup_{n\to\infty} \int g_n. \end{aligned}$$

Thus, $\limsup_{n\to\infty} \int g_n \le \int g$. Hence,

$$\liminf_{n\to\infty} \int g_n \le \limsup_{n\to\infty} \int g_n \le \int g \le \liminf_{n\to\infty} \int g_n \le \limsup_{n\to\infty} \int g_n.$$

This means all of them have same value. Thus $\int_{\mathbb{R}^d} g_n \to \int_{R^d} g$ as $n \to \infty$. □

Problem 2.3

Applying 2.1 and 2.2, deduce that if $F_n, F \in L^1(\mathbb{R}^d)$ and $F_n \to F$ a.e. as $n \to \infty$, then

$$\int_{\mathbb{R}^d} |F_n - F| \to 0 \text{ if and only if } \int_{\mathbb{R}^d} |F_n| \to \int_{\mathbb{R}^d} |F|$$

as $n \to \infty$.

Proof. Let $G_N = |F_n - F| + |F|$. Then, $|F_n| \le G_n$.
($\Rightarrow$) Since $G_n = |F_n - F| + |F|$, $\int G_n \to \int |F|$ is clear.
By 2.2, letting $g_n = |F_n|$, $h_n = G_n$, $g = |F|$, $h = |F|$, then $\int_{\mathbb{R}^d} |F_n| \to \int_{\mathbb{R}^d} |F|$.
($\Leftarrow$) There exists N such that for $n \ge N$, $\int_{\mathbb{R}^d} |F_n| < \int_{\mathbb{R}^d} |F| + 1$.

Let $C = \max\{\int_{\mathbb{R}^d} |F| + 1, \int_{\mathbb{R}^d} |F_1|, \cdots, \int_{\mathbb{R}^d} |F_N|\}$. Then $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |F_n| \le C$.
Then by 2.1, $\lim_{n\to\infty} \int_{\mathbb{R}^d}(|F_n| - |F_n - F| - |F|) = 0$.
Since $F_n \to F \Rightarrow |F_n| \to |F|$, $\lim_{n\to\infty} \int_{\mathbb{R}^d} |F_n - F| = 0$. □

Problem 3.1

Let f be any non-negative measurable function in $\mathbb{R}^d$. Employing Fubini's theorem, derive

$$\int_{\mathbb{R}^d} f(x)^p \, \mathrm{d}x = p \int_0^\infty t^{p-1} m(\{x \in \mathbb{R}^d : f(x) > t\}) \, \mathrm{d}t$$

for any $p > 1$. Describe its relationship with Cavalieri's principle.

Proof.

$$\begin{aligned}\int_0^\infty pt^{p-1} m(\{x \in \mathbb{R}^d : f(x) > t\}) \, \mathrm{d}t &= \int_0^\infty \int_{\mathbb{R}^d} pt^{p-1} \mathbb{I}_{\{f(x)>t\}} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\mathbb{R}^d} \int_0^{f(x)} pt^{p-1} \, \mathrm{d}t \, \mathrm{d}x = \int_{\mathbb{R}^d} f(x)^p \, \mathrm{d}x\end{aligned}$$

where the second equality is due to Fubini's theorem.

Fubini's theorem. *Suppose $f(x,y)$ is integrable on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:*

(i) *The slice f^y is integrable on $\mathbb{R}^{d_1}$.*

(ii) *The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x) \, \mathrm{d}x$ is integrable on $\mathbb{R}^{d_2}$.*

(iii) $\int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f(x,y) \, \mathrm{d}x) \, \mathrm{d}y = \int_{\mathbb{R}^d} f$.

Cavalieri's principle. *(2-dimensional case) Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.* □

Problem 3.2

Let T be an element of the general linear group $GL(d, \mathbb{R})$ of degree d. Check that if f is a measurable function in $\mathbb{R}^d$, so is $f \circ T$. Also, prove that if $f \in L^1(\mathbb{R}^d)$, then $f \circ T \in L^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f = |\det T| \int_{\mathbb{R}^d} f \circ T. \tag{1}$$

Proof. Since linear map on finite dimensional space is continuous, T^{-1} is continuous mapping. By below property, T^{-1} is measurable.

Property. *If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and Φ is continuous, then $\Phi \circ f$ is measurable.*

For $E \in \mathcal{B}_{\mathbb{R}^d}$, $(f \circ T)^{-1}(E) = T^{-1}(f^{-1}(E)) \in L(\mathbb{R}^n)$. Thus $f \circ T$ is Lebesgue measurable.
If $|f|$ is measurable, $|f \circ T|$ is measurable.

If (1) is true for $T, S \in GL(d, \mathbb{R})$, it is true for $T \circ S$, since

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= |\mathrm{det}T| \int f \circ T(x)\,\mathrm{d}x = |\mathrm{det}T||\mathrm{det}S| \int f \circ T \circ S(x)\,\mathrm{d}x \\ &= |\mathrm{det}(T \circ S)| \int f \circ T \circ S(x)\,\mathrm{d}x.\end{aligned}$$

Every $T \in GL(d, \mathbb{R})$ can be written as the product of finitely many transformations of the types T_1, T_2, T_3 given below. Hence it suffices to prove (1) when T is of the types T_1, T_2, T_3

$$\begin{aligned}T_1(x_1, x_2, \cdots, x_n) &= (x_1, \cdots, cx_j, \cdots, x_n) \\ T_2(x_1, x_2, \cdots, x_n) &= (x_1, \cdots, x_j + cx_k, \cdots, x_n) \\ T_3(x_1, x_2, \cdots, x_j, \cdots, x_k, \cdots, x_n) &= (x_1, x_2, \cdots, x_k, \cdots, x_j, \cdots, x_n)\end{aligned}$$

However, this is simple consequence of the Fubini-Tonelli theorem. □

Problem 3.3

A map $G : \mathbb{R}^d \to \mathbb{R}^d$ is called a $C^1(\mathbb{R}^d)$-*diffeomorphism* if G is injective and its derivative $DG(x)$ is invertible for all $x \in \mathbb{R}^d$. Discuss how the identity in 3.2 becomes if T is replaced with a $C^1(\mathbb{R}^d)$-*diffeomorphism* G and justify why your answer should holds.

Proof. It is clear by following theorem.

Theorem. *Let $G : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ be C^1-diffeomorphism, $G = (g_1, g_2, \cdots, g_n)$. Then,*

(i) *If f is Lebesgue measurable on $G(\Omega)$, $f \circ G$ is Lebesgue measurable on Ω.*

(ii) *If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then $\int_{G(\Omega)} f(x)\,\mathrm{d}x = \int_\Omega f \circ G(x)|\mathrm{det}DxG|\,\mathrm{d}x$.* □