## Advanced Calculus 1 – Final Exam

## Junwoo Yang

## June 20, 2019

- 1. Let  $f \colon A \subset \mathbb{R} \to \mathbb{R}$  be a function. Write the definitions of the following statements.
  - (1) f is continuous.
  - (2) f is uniformly continuous.
  - (3) f is Lipschitz continuous.
- 2. Prove or disprove the following statements.
  - (1) If f is Lipschitz continuous, then f is continuous.
  - (2) If f is uniformly continuous, then f is Lipschitz continuous.
- 3. Consider a sequence of functions  $\{g_n\}$  where  $g_n(x) := \left(\frac{x^n}{n!}\right)^2$  for  $x \in [-a, a], a \in \mathbb{R}$ . Prove that  $\sum_{n=0}^{\infty} g_n(x)$  is continuous on [-a, a].
- 4. Prove the following statements.
  - (1) Let  $f: \mathbb{R} \to \mathbb{R}$  and  $g: \mathbb{R} \to \mathbb{R}$  be continuous at  $a \in \mathbb{R}$ . Then, the sum  $f+g: \mathbb{R} \to \mathbb{R}$  is continuous at a.
  - (2) Let  $f: \mathbb{R} \to \mathbb{R}$  and  $g: \mathbb{R} \to \mathbb{R}$  be continuous at  $a \in \mathbb{R}$ . Then, the product  $f \cdot g: \mathbb{R} \to \mathbb{R}$  is continuous at a.
  - (3) Let  $f: A \subset \mathbb{R} \to \mathbb{R}$  and  $g: B \subset \mathbb{R} \to \mathbb{R}$  be continuous functions such that  $f(A) \subset B$ . Then the composition  $g \circ f: A \subset \mathbb{R} \to \mathbb{R}$  is continuous.
- 5. Let M be a complete normed space and  $\Phi \colon M \to M$  be a mapping. Suppose that there is a constant  $\alpha \in [0,1)$  such that  $\|\Phi(x) \Phi(y)\| \le \alpha \|x y\|$  for all  $x,y \in M$ . Prove that there exists a fixed point in M.
- 6. Let M be a normed space and  $K \subset M$  be a compact set. Suppose that  $f: M \to M$  is a continuous function. Prove that f(K) is compact.
- 7. Prove or disprove the following statements.
  - (1) Let  $\{f_n\}$  be a sequence of Riemann integrable functions on  $[a, b] \in \mathbb{R}$ . Suppose  $f_n$  converges to f uniformly on [a, b]. Then, f is Riemann integrable on [a, b].
  - (2) Let  $\{f_n\}$  be a sequence of Riemann integrable functions on  $\mathbb{R}$  such that  $f_n(x)$  converges to 0 for all  $x \in \mathbb{R}$  as  $n \to \infty$ . Then,  $\int_{-\infty}^{\infty} f_n(x) dx$  converges as  $n \to \infty$ .