실변수함수론

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2020. 03. 16. 월

Lecture 1. motivation of measure theory.

측도론 (measure theory)
힐버트 공간 (Hilbert space theory). | 현대 해석학의 근간.

(1). Fourier series.

when f is continuously differentiable function on $[-\pi, \pi]$, we have $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$

(the Fourier series of f).

복소지수함수의 선형 결합.

where $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

$e^{inx} = \cos(nx) + i\sin(nx)$.

(Parserval's identity)

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty \not\Rightarrow \exists f$ s.t $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ and f is continuously differentiable

Q1-1. f를 다른 방식으로 표현.

Q1-2. How to integrate f ?

(2) Limits of continuous function.

$\{f_n\}$: a sequence of continuous functions on $[0,1]$

assume $\lim_{n\to\infty} f_n(x) = f(x)$. exists for every x.

If $f_n \to f$ (uniformly convergence)., f is continuous.

If not, f may not be even Riemann integrable.

Q2. Is there a way to integrate f and guarantee

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \ ?$$

(3). Length of curves.

Γ : continuous curve in $\mathbb{R}^2$

$\Gamma = \{(x(t), y(t)) : a \le t \le b\}$ x, y : continuous fts of t.

Γ is said to be rectifiable if its lengh L is finite.

If $x(t), y(t)$ are continuously differentiable, $\Gamma \in C^1$

$L = \int_a^b \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2}\, dt.$

Q3a. What are the conditions on $x(t), y(t)$ that guarantee the rectifiability of Γ ?

Q.3b. When does the above formula hold ?

(4). Differentiation and integration

(Fundamental theorem of calculus).

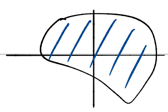
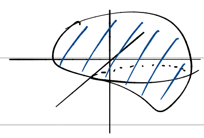
If f is a continuous ft on $[a,b]$, F is an antiderivative of f in $[a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

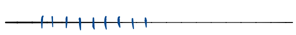
Q4. There are a lot of examples of Riemann integrable fts. that are non-continuous. Can we apply the above formula to them ?

(5) The problem of measure.

$E \subset \mathbb{R}^n$, $m(E)$: measure, n-dimensional volume

$(n=1)$: length $(n=2)$: area. $(n=3)$: volume.



[illegible]. $\int_E 1(x)\,dx = m(E)$.

non-negative ft. $m(\mathbb{R}) = +\infty$

Natural condition.

- $E = [a,b]$ $(a \le b)$. $m(E) = b - a$
- $E = \bigcup_{n=1}^{\infty} E_n$, E_n mutually disjoint, $m(E) = \sum_{n=1}^{\infty} m(E_n)$.
- For $\forall h \in \mathbb{R}$, $m(E) = m(E+h)$. translation invariance

Q5. The existence and uniqueness of such a measure.

Vitali set. 위에 언급한 세가지 성질을 모두 유지한 채로 실직선의 모든 부분집합의 측도를 정의하는 것이 불가능 하다.

Lebesgue ; possible defining a measure on "measurable" sets.

$\Rightarrow$ If $E_1, E_2, \cdots$ measurable sets, $E = E_1 \cup E_2 \cup \cdots$ measurable.

And if $E_1 \cap E_2 \cap \cdots = \phi$, $m(E) = \sum_{n=1}^{\infty} m(E_n)$

Notation.

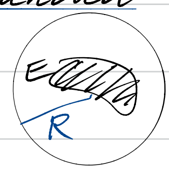
- A point $x \in \mathbb{R}^d$, $x = (x_1, x_2, \cdots, x_d)$, $x_i \in \mathbb{R}$ for $i = 1, \cdots, d$
- The norm of x is $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$
- The distance between two points x, y is $|x-y|$
- The distance between two sets E and F is

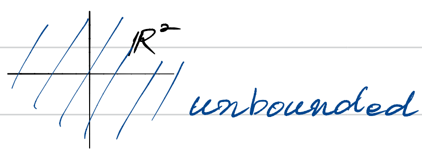
  $d(E, F) = \inf\{|x-y| : x \in E, y \in F\}$.
- The open ball in $\mathbb{R}^n$ centered at x and radius of r is

  $B_r(x) = \{y \in \mathbb{R}^d : |y-x| < r\}$. $\leq$: closed ball
- A subset $E \subset \mathbb{R}^d$ is open if $\forall x \in E$, $\exists r > 0$ s.t $B_r(x) \subset E$
- A set is closed if its complement is open.
- Arbitrary union of open sets is open.

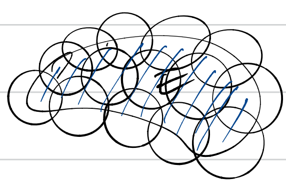
  Finite intersection of open sets is open.

  (countable intersection of open sets may open, may not open).
- A set E $(\in \mathbb{R}^d)$ is bounded if it is contained in some ball of finite radius.





- A set E is compact if any covering of E by a collection of open sets contains a finite subcovering.

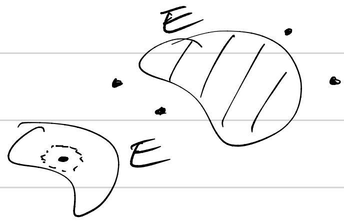


- (Heine - Borel theorem).

  A subset $E \subset \mathbb{R}^d$ is compact if and only if it is closed and bounded

- A point $x \in \mathbb{R}^d$ is a limit point of the set E if $\forall r > 0$, $B_r(x)$ contains points of E.

- An isolated point of E is a point $x \in E$ s.t there exists an $r > 0$ where $B_r(x) \cap E = \{x\}$



- A point $x \in E$ is an interior point of E if $\exists r > 0$ s.t $B_r(x) \subset E$

- The set of all interior points of E is called the interior of E

- The closure $\bar{E}$ of E consist of the union of E and all its limit pts.

- The boundary ∂E of E is the set of pts which are in the closure of E but not in the interior of E

- The closure of a set is a closed set.

- Every pt in E is a limit pt of E

- A set is closed if and only if it contains all its limit points.

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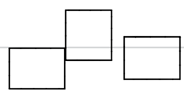
Lecture 2. Lebesgue measure.

- A closed rectangle R in $\mathbb{R}^d$ is $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ where $a_j \le b_j$ $(j = 1, \cdots, d)$ are real numbers.

- Its volume is $|R| = (b_1 - a_1) \cdots (b_d - a_d)$.

- A cube is a rectangle for which $b_1 - a_1 = \cdots = b_d - a_d = \ell$
  $|R| = \ell^d$

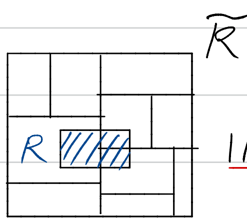
- A union of rectangles is <u>almost disjoint</u> if the interiors of the rectangles are disjoint.



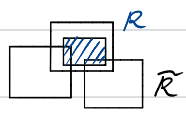
Lemma

(1). If a rectangle is almost disjoint union of finitely many other rectangle, say $R = \bigcup_{k=1}^{N} R_k$.
then $|R| = \sum_{k=1}^{N} |R_k|$

(2) If $R \subset \bigcup_{k=1}^{N} R_k$
(here the union of rectangles need not to be almost disjoint). then,
$|R| \le \sum_{k=1}^{N} |R_k|$.



$|R| \le |\tilde{R}| \le \sum_{k=1}^{N} |R_k|$



Theorem

Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be uniquely written as a countable union of disjoint open intervals.

pf). For each $x \in \mathcal{O}$, $I_x := (a_x, b_x)$, where $a_x = \inf\{a < x \mid (a, x) \subset \mathcal{O}\}$
$b_x = \sup\{b > x \mid (x, b) \subset \mathcal{O}\}$. For $x, y \in \mathcal{O}$, if $I_x \cap I_y \ne \phi$,
$I_x \cup I_y \subset I_x$, $I_x \cup I_y \subset I_y$ $\therefore I_x = I_y$ $\therefore$ distinct interval.
any interval contains rational #. $\Rightarrow$ countable.

- If $\mathcal{O}$ is open, there are disjoint open intervals $\{I_j\}$ such that $\mathcal{O} = \bigcup_{j=1}^{\infty} I_j$. The measure of $\mathcal{O}$ should be $\sum_{j=1}^{\infty} |I_j| = |\mathcal{O}|$
It is well-defined since the decomposition is unique.

Theorem.

uniquely. (X).

Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$ can be written as a countable union of almost disjoint closed cubes.

$\forall x \in \mathcal{O}$, $\exists$ cube with length $2^{-N} \subset \mathcal{O}$. union of all cube covers $\mathcal{O}$.

- If $\mathcal{O}$ is open, we can write $\mathcal{O} = \bigcup_{j=1}^{\infty} R_j$ with almost disjoint rectangles $\{R_j\}$. The measure of $\mathcal{O}$ should be $\sum_{j=1}^{\infty} |R_j| = |\mathcal{O}| \overset{?}{=} \sum_{j=1}^{\infty} |\tilde{R}_j|$

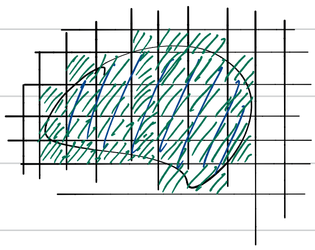
- However, the above decomposition is not unique and the sum may be dependent on this decomposition.

Definition.

If E is any subset of $\mathbb{R}^d$, the exterior measure of E is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j| \qquad = \text{outer measure.}$$

where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.



- Various examples show that this notion coincides with our earlier intuition.
- However, it lacks the desirable property of additivity when taking the union of disjoint sets.

$E = E_1 \cup E_2$, $E_1 \cap E_2 = \phi$.

$\not\Rightarrow m_*(E) = m_*(E_1) + m_*(E_2)$

countable sets.

- It would not suffice to allow finite sums in the definition of $m_*(E)$. The quantity that would be obtained if one considered only coverings of E by finite union of cubes is in general larger than $m_*(E)$.
- One can replace the coverings by cubes, with coverings by rectangles; or with coverings by balls.

$0 \le m_*(\{point\}) \le |Q| \to 0.$

Ex). · The exterior measure of a point is zero

· The exterior measure of a closed cube is equal to its volume.
Indeed, for an arbitrary covering $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by cubes,
it holds that $|Q| \le \sum_{j=1}^{\infty} |Q_j|$ $m_*(Q) \overset{=}{\le} |Q|$

비교: Lemma 2

· If Q is an open cube, the result $m_*(Q) = |Q|$ still holds.

· The exterior measure of a rectangle R is equal to its volume.

· The exterior measure of $\mathbb{R}^d$ is infinite.

Properties.

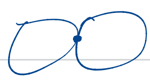
- $\forall \varepsilon > 0$, $\exists$ a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with $\sum_{j=1}^{\infty} m_*(Q_j) \le m_*(E) + \varepsilon$

$m_*(E) = \inf \sum_j |Q_j| = \inf \sum_j m_*(Q_j)$

- If $E_1 \subset E_2$, then $m_*(E_1) \le m_*(E_2)$ Monotonicity.

- If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \le \sum_{j=1}^{\infty} m_*(E_j)$.

- If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf\{ m_*(\mathcal{O}) : E \subset \mathcal{O} \text{ open} \}$



성립X.

- If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$

- If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then $\underset{|\mathcal{O}|}{\overset{}{m_*(E)}} = \sum_{j=1}^{\infty} |Q_j| = \sum_{j=1}^{\infty} |\tilde{Q}_j|$ $\mathcal{O} = \bigcup_{j=1}^{\infty} |Q_j| = \bigcup_{j=1}^{\infty} |\tilde{Q}_j|$

Therefore, the exterior measure of an open set equals the sum of the volumes of the cubes in a decomposition.
In particular, the sum is independent of the decomposition.

Measurable sets.

- Nonetheless, one cannot conclude in general that if $E_1 \cup E_2$ is a disjoint union of subsets of $\mathbb{R}^d$, then
  $m_*(E) = m_*(E_1) + m_*(E_2)$. Vitali : not measurable.

- A subset E of $\mathbb{R}^d$ is (Lebesgue) measurable if for any $\varepsilon > 0$, there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and
  $m_*(\mathcal{O} - E) = m_*(\mathcal{O} \cap E^c) \leq \varepsilon$

- If E is measurable, we define its (Lebesgue) measure $m(E)$ by
  $m(E) = m_*(E)$.
  The Lebesgue measure inherits all the features of the exterior measure.
  $m_*(E) \neq m_*(E_1) + m_*(E_2)$, $m(E) = m(E_1) + m(E_2)$.
  $E_1, E_2, E_1 \cup E_2$ measurable.

Lemma

The following sets in $\mathbb{R}^d$ are (Lebesgue) measurable :

- open sets. : It immediately follows from the def.
  $m_*(\mathcal{O} - E) = m_*(E - E) = m_*(\emptyset) = 0 \leq \varepsilon$

- null sets E, i.e., sets E with $m_*(E) = 0$.
  $E_1 \subset E_2 \Rightarrow \underbrace{0 \leq m_*(E_1)}_{0.} \leq m_*(E_2) = 0$.

  $m_*(E) = 0 = \inf\{m_*(\mathcal{O}) : \mathcal{O} \supset E \text{ open}\}$ $\quad \varepsilon > m_*(\mathcal{O}) \geq m_*(\mathcal{O} - E)$
  For any $\varepsilon > 0$, $\exists$ open set $\mathcal{O} \supset E$, $m_*(\mathcal{O}) < \varepsilon$. Hence $m_*(\mathcal{O} - E) \leq \varepsilon$

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- subsets of null sets.
- countable unions of measurable sets.
- closed sets.
- complements of measurable sets. E^c, $\mathbb{R}^d - E$
- countable intersections of measurable sets.

  It holds that $\bigcap_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j^c\right)^c$

Lecture 3. Measurable sets and functions

Additivity.

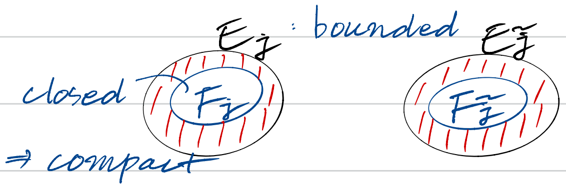
Theorem.

If $E_1, E_2, \cdots$ are disjoint measurable sets and $E = \bigcup_{j=1}^{\infty} E_j$,
then $m(E) = \sum_{j=1}^{\infty} m(E_j)$.

pf). Suppose that each E_j is bounded. E_j^c : measurable.

There is an open set $O_j \supset E_j^c$ s.t $m(O_j - E_j^c) \le \frac{\varepsilon}{2^j}$

A closed set $F_j = O_j^c$ satisfies $E_j - F_j = O_j - E_j^c$.



E_j disjoint → F_j disjoint.

$1 \le i, j \le N$

Since $F_1, \cdots, F_N$ are compact and disjoint, $\Rightarrow d(F_i, F_j) > 0$

$m(E) \ge m\left(\bigcup_{j=1}^{N} F_j\right) = \sum_{j=1}^{N} m(F_j) \ge \sum_{j=1}^{N} m(E_j) - \varepsilon$

$\left(E = \bigcup_{j=1}^{\infty} E_j \supseteq \bigcup_{j=1}^{\infty} F_j\right)$. $(d(F_i, F_j) > 0)$.

$m(F_j) = m(O_j^c)$ $\quad E_j - F_j$

$m(E_j) \le m(O_j - E_j^c) + m(F_j)$

$\Rightarrow \sum_{j=1}^{N} m(E_j) \le \sum_{j=1}^{N} m(F_j) + \underbrace{\sum_{j=1}^{N} \frac{\varepsilon}{2^j}}_{\le \varepsilon} \Rightarrow \sum_{j=1}^{N} m(F_j) \ge \sum_{j=1}^{N} m(E_j) - \varepsilon$

Take $N \to \infty$, $\varepsilon \to 0$

$\Rightarrow m(E) \ge \sum_{j=1}^{\infty} m(E_j)$

The reverse inequality holds for the exterior measure.

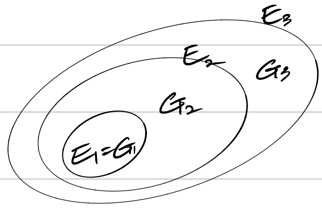
$m(E) \le \sum_{j=1}^{\infty} m(E_j)$. $\quad \therefore m(E) = \sum_{j=1}^{\infty} m(E_j)$.

Continuity.

Theorem

If $\{E_j\}$ satisfies $E_k \subset E_{k+1}$ for all k, and $E = \bigcup_{k=1}^{\infty} E_k$, then we write $E_k \nearrow E$. Suppose that $E_1, E_2, \cdots$ are measurable subsets of $\mathbb{R}^d$ and $E_k \nearrow E$. Then $m(E) = \lim_{N\to\infty} m(E_N)$.

pf). Let $G_1 = E_1$ and $G_k = E_k - E_{k-1}$ for $k \geq 2$.



G_k are measurable, disjoint and $E = \bigcup_{k=1}^{\infty} G_k$

Therefore, $m(E) = \sum_{k=1}^{\infty} m(G_k) = \lim_{N\to\infty} \sum_{k=1}^{N} m(G_k)$.

$$= \lim_{N\to\infty} m\left(\bigcup_{k=1}^{N} G_k\right) = \lim_{N\to\infty} m(E_N).$$

Theorem.

If $\{E_j\}$ satisfies $E_k \supset E_{k+1}$ for all k, and $E = \bigcap_{k=1}^{\infty} E_k$, then we write $E_k \searrow E$. Suppose that $E_1, E_2, \cdots$ are measurable subsets of $\mathbb{R}^d$, $m(E_k) < \infty$ for some k and $E_k \searrow E$. Then $m(E) = \lim_{N\to\infty} m(E_N)$.

Approximation.

Theorem.

Suppose E is a measurable subset of $\mathbb{R}^d$. Then, for every $\varepsilon > 0$,

(i) There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \varepsilon$

(ii) There exists a closed set F with $F \subset E$ and $m(E - F) \leq \varepsilon$

(iii) If $m(E) < \infty$, there exists a compact set K with $K \subset E$ and $m(E - K) \leq \varepsilon$

check iii). Pick a closed set $F \subset E$ and $m(E - F) \leq \frac{\varepsilon}{2}$ by ii).

Set compact sets $K_n = F \cap \overline{B_n(0)}$ radius : n, center : $0 \in \mathbb{R}^d$

Since $E - K_n \searrow E - F$ and $m(E) < \infty$, $m(E - K_n) \leq \varepsilon$ for some n.

$m(E) \geq m(E - K_1) < \infty \Rightarrow m(E - K_n) \approx m(E - F) \leq \frac{\varepsilon}{2}$

σ-algebra and Borel sets.

- A σ-algebra of sets is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements.

- The collection of all subsets in $\mathbb{R}^d$ and that of all measurable sets in $\mathbb{R}^d$ form σ-algebra

  Lebesgue σ-algebra. ← 너무 많은 elements 를 포함.

- The Borel σ-algebra in $\mathbb{R}^d$, denoted by $\mathcal{B}_{\mathbb{R}^d}$, is the smallest σ-algebra that contains all open sets.

  Borel σ-algebra $\subsetneq$ Lebesgue σ-algebra.

- Elements of this σ-algebra are called Borel sets.

- Countable intersections of open sets are called G_δ sets.

  open or not open

- Countable union of closed sets are called F_σ sets.

  closed or not closed

Lemma.

A subset E of $\mathbb{R}^d$ is measurable. $\qquad E = E_0 \cup N = F_0 \cup N$

(i) if and only if E differs from a $\underbrace{G_\delta \text{ set}}_{E_0}$ by a $\underbrace{\text{set of}}_{N}$ measure zero,

(ii) if and only if E differs from an F_σ set by a set of measure zero.

Theorem.

The Lebesgue sets arise as the completion of the Borel σ-algebra, that is, by adjoining all subsets of Borel set of measure zero.

(Lebesgue σ-algebra) = (Borel σ-algebra) ∪ (null).

check (i). If E measurable, $\exists O_n$ s.t $m(O_n - E) \le \frac{1}{n}$, $E \subset O_n$.

$S = \bigcap_{n=1}^{\infty} O_n$, $(S-E) \subset (O_n - E)$ for all n.

$m(S-E) \le m(O_n - E) \le \frac{1}{n}$ for all n.

$\therefore m(S-E) \le \frac{1}{n}$ $\forall n$. measure zero.

E; measurable $\Leftrightarrow$ $\exists O \in G_\delta$ s.t $m(O-E) = 0$.

$\Leftrightarrow$ $\exists O \in F_\sigma$ s.t $m(O-E) = 0$.

Measurable functions.

- A characteristic function of a set E is a function defined by $\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$ $\sim \mathbb{1}_E(x)$.

  Chi

- A simple function is a function of the form $f = \sum_{k=1}^{N} a_k \chi_{E_k}$

  where E_k is a measurable set with $m(E_k) < \infty$, $a_k \in \mathbb{R}$

Definition.

We allow for f to take on the infinite values $\pm\infty$.

We say that f is finite-valued if $-\infty < f(x) < \infty$ $\forall x$.

A function f defined on a measurable subset E of $\mathbb{R}^d$ is measurable if the set $f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$ $= \{f < a\}$ is measurable for all $a \in \mathbb{R}$.

If f is measurable, then the sets $\{f \le a\} = \bigcap_{k=1}^{\infty} \{f < a + \frac{1}{k}\}$ are measurable for all $a \in \mathbb{R}$.

Properties.

Reversely, if $\{f \le a\}$ are all measurable, then f is measurable. This is because $\{f < a\} = \bigcup_{k=1}^{\infty} \{f \le a - \frac{1}{k}\}$

$\Rightarrow$ f : measurable $\Leftrightarrow$ $\{f \le a\}$: measurable $\forall a \in \mathbb{R}$.

Similarly, f is measurable if and only if $\{f \ge a\} = \{f < a\}^c$ are measurable for all $a \in \mathbb{R}$.

If f is finite-valued, it is measurable if and only if $\{a < f < b\}$ are measurable for all $a, b \in \mathbb{R}$.

In general, the finite-valued function f is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$, and if and only if $f^{-1}(F)$ is measurable for every closed set F.

- If f is continuous on $\mathbb{R}^d$, then f is measurable.

  measurable $\subset$ continuous $\subset$ uniformly continuous $\subset$ differentiable.

- For a sequence $\{f_n\}$ of measurable functions, $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ are all measurable. In particular, if $f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$ exists, it is measurable.

- If f and g are measurable and finite-valued, then $f+g$ and fg are measurable. f/g.

- If $f(x) = g(x)$ almost everywhere (a.e.), i.e., $m(\{x : f(x) \neq g(x)\}) = 0$, and f is measurable, then g is measurable.

2020. 03. 25. 수

Lecture 4. Lebesgue integral (1).

Approximation of measurable functions.

Theorem.

(i) Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\varphi_k\}$ that converges pointwise to f, namely, $\varphi_k(x) \le \varphi_{k+1}(x)$ and $\lim_{k\to\infty} \varphi_k(x) = f(x)$ for all x.

(ii). Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\varphi_k\}$ that converges pointwise to $f(x)$ for almost every x.

simple fts

Littlewood's three principle

- Littlewood summarized the connections between measurability and the order concepts such as continuity.

(i) Every set is nearly a finite union of intervals.

(ii) Every function is nearly continuous.

(iii) Every convergent sequence is nearly uniformly convergent.

Approximation of measurable sets.

Theorem

Suppose E is measurable subset of $\mathbb{R}^d$. If $m(E) < \infty$, there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes s.t

$m(E \triangle F) = m(E-F) + m(F-E) \le \varepsilon$

exterior measure 사용

$\underset{(E-F)\cup(F-E)}{\overset{\Vert}{E \triangle F}}$: symmetric difference.

pf). Choose $\{Q_j\}$ so that $E \subset \bigcup_{j=1}^{\infty} Q_j$ and $\sum_{j=1}^{\infty} |Q_j| \le m(E) + \frac{\varepsilon}{2}$

Since $m(E) < \infty$, $\sum_{j=N+1}^{\infty} |Q_j| < \frac{\varepsilon}{2}$ for some N.

We have $m(E-F) + m(F-E) \leq \sum_{j=N+1}^{\infty} |Q_j| + \left(\sum_{j=1}^{\infty} |Q_j| - m(E)\right) \leq \varepsilon$

$$\overset{\text{''}}{m(F) - m(E)} \leq \sum_{j=1}^{N} |Q_j| - m(E) \leq \sum_{j=1}^{\infty} |Q_j| - m(E)$$

$$\leq \underbrace{\sum_{j=N+1}^{\infty} |Q_j|}_{\leq \frac{\varepsilon}{2}} + \underbrace{\left(\sum_{j=1}^{\infty} |Q_j| - m(E)\right)}_{\leq \frac{\varepsilon}{2}} \leq \varepsilon$$

Lusin's theorem (Littlewood' principle ii).

Theorem

$-\infty < f(x) < \infty \quad \forall x \in E \qquad m(E) < \infty$

Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\varepsilon > 0$ there exists a closed set F_ε, with $F_\varepsilon \subset E$ and $m(E - F_\varepsilon) \leq \varepsilon$ and such that $f|_{F_\varepsilon}$ is continuous. $f|_{F_\varepsilon}(x) = f(x)$ on $x \in F_\varepsilon$; restriction

Egorov's theorem

Theorem

Suppose $\{f_k\}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e on E. Given $\varepsilon > 0$, we can find a closed set $A_\varepsilon \subset E$ such that $m(E - A_\varepsilon) \leq \varepsilon$ and $f_k \to f$ uniformly on A_ε

$m(\{x \in E : f_k(x) \nrightarrow f(x)\}) = 0$.

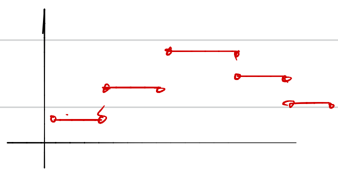
Lebesgue integral for simple functions : Definition.

- A simple function φ is a function of the form $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ where E_k is a measurable set with $m(E_k) < \infty$ and $a_k \in \mathbb{R}$

- The canonical form of φ is the unique decomposition as above, where the numbers a_k are distinct and non-zero, and the sets E_k are disjoint.

ex). $\varphi = 1 \cdot \chi_{E_1} + 1 \cdot \chi_{E_2} = \chi_{E_1 \cup E_2}$ ← canonical form.

not distinct.



- If φ is a simple function with canonical form $\varphi(x) = \sum_{k=1}^{M} c_k \chi_{F_k}(x)$, then the Lebesgue integral of φ is $\int_{\mathbb{R}^d} \varphi(x)\,dx = \sum_{k=1}^{M} c_k m(F_k)$.

- If E is a measurable subset of $\mathbb{R}^d$ with $m(E) < \infty$, then $\varphi \chi_E$ is also a simple function, and we define $\int_E \varphi(x)\,dx = \int_{\mathbb{R}^d} \varphi(x) \chi_E(x)\,dx = \int \varphi \chi_E$.

  $\varphi = \chi_{E_1} \Rightarrow \varphi \chi_E = \chi_{E_1} \chi_E = \chi_{E_1 \cap E}$

- The integral is independent of the representation of the simple function.

  $\varphi = \Sigma a_k \chi_{E_k} = \Sigma b_k \chi_{F_k} \Rightarrow \int \varphi = \Sigma a_k m(E_k) = \Sigma b_k m(F_k)$

  canonical form 이 아니더라도 앞뒤의 simple function 이 같기만 하면 르벡적분 값은 같다.

Properties.

- Suppose that φ, ψ are simple, E and F are disjoint, and $a, b \in \mathbb{R}$: $\int (a\varphi + b\psi) = a\int\varphi + b\int\psi$, $\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi$.

- If $\varphi \leq \psi$, then $\int \varphi \leq \int \psi$.

- If φ is a simple function, then so is $|\varphi|$, and $|\int \varphi| \leq \int |\varphi|$

  $E_1 \cap E_2 = \phi$, $\varphi = \chi_{E_1} - 2\chi_{E_2} \Rightarrow |\varphi| = \chi_{E_1} + 2\chi_{E_2}$

Lebesgue integral for bounded functions with finite support.

A convergence Lemma.

- The support of a measurable function f is $supp(f) = \{x : f(x) \neq 0\}$

supported on a set of finite measure

- We also say that f is supported on E if $f(x) = 0$ whenever $x \notin E$. $supp(f) \subset E$

Lemma

Let f be a bounded function supported on a set E of finite measure. If $\{\varphi_n\}$ is any sequence of simple functions bounded by M, supported on E, and with $\varphi_n(x) \to f(x)$ for a.e. x, then

(i) The limit $\lim_{n\to\infty} \int \varphi_n$ exists.

(ii) If $f = 0$ a.e., then the limit $\lim_{n\to\infty} \int \varphi_n$ equals 0.

Definition. on E with $m(E) < \infty$

- For any bounded function f supported on a set of finite measure, we define its Lebesgue integral by $\int f = \lim_{n\to\infty} \int \varphi_n$ $\varphi_n \to f$

- In order for the integral to be well-defined, we must show that f is independent of the limiting sequence $\{\varphi_n\}$ used.

- The basic properties of the integral of simple functions such as linearity, additivity, monotonicity, and triangle inequality continue to hold.
  $\int (af + bg) = a\int f + b\int g$, $\int_{E\cup F} f = \int_E f + \int_F f$ $(E \cap F = \emptyset)$.
  $f \le g \Rightarrow \int f \le \int g$, $|f|$ also bounded, supported on a set of finite measure, and $|\int f| \le \int |f|$

Bounded convergence theorem

Theorem.

$|f_n(x)| \le M$ $\quad$ $supp\{f_n\} \subset E$

Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M, are supported on a set E of finite measure, $m(E) < \infty$, and $f_n(x) \to f(x)$ a.e. x as $n \to \infty$

Then f is measurable, bounded, supported on E for a.e. x, and $\int |f_n - f| \to 0$ as $n \to \infty$.

$|f| \le M$, $supp\{f\} \subset E$

Consequently, $\int f_n \to \int f$ as $n \to \infty$.

$$|\int f_n - \int f| = |\int (f_n - f)| \le \int |f_n - f| \to 0.$$

pf). Let us check $\int |f_n - f| \to 0$ as $n \to \infty$

Note that $f_n \rightrightarrows f \Rightarrow \int f_n \to \int f$

Given $\varepsilon > 0$, by Egorov's theorem, there is a measurable subset A_ε of E such that $m(E - A_\varepsilon) \le \varepsilon$ and $f_n \to f$ uniformly on A_ε.

For large n, $|f_n(x) - f(x)| \le \varepsilon$ for all $x \in A_\varepsilon$

(def of uniformly convergence).

$= \int_E |\ | = \int \chi_E(x) |\ |$ $\quad$ $\le m(A_\varepsilon) \cdot \varepsilon$ $\quad$ $\le m(E) \cdot \varepsilon$ $\quad$ $(\int_A + \int_B) f dx = \int_A f + \int_B f$

$$\text{Therefore, } \int |f_n(x) - f(x)| = \left(\int_{A_\varepsilon} + \int_{E - A_\varepsilon}\right) |f_n(x) - f(x)| \, dx$$
$$\le \varepsilon \cdot m(E) + 2M \underbrace{m(E - A_\varepsilon)}_{\le \varepsilon}$$
$$\le \varepsilon (m(E) + 2M)$$

$|f_n(x)| \le M$, $|f(x)| \le M$

Take $\varepsilon \to 0$.

$\therefore \int |f_n(x) - f(x)| \to 0$.

Riemann integral

Theorem.

Suppose f is Reimann integrable on the closed interval $[a,b]$. Then f is measurable, and

$$\int^{\mathcal{R}}_{[a,b]} f(x) dx = \int^{\mathcal{L}}_{[a,b]} f(x) dx$$

where the integral on the left-hand side is the standard Reimann integral, and that on right hand side is the Lebesgue integral.

$\mathcal{R} \Rightarrow \mathcal{L} \not\Rightarrow \mathcal{R}$

ex). $f = \chi_{\mathbb{Q}\cap[0,1]} = \begin{cases} 1 & x \in \mathbb{Q}\cap[0,1] \\ 0 & x \notin \mathbb{Q}\cap[0,1] \end{cases}$

$\overline{\int} f = 1$, $\underline{\int} f = 0$. $\therefore$ non-Reimann integrable.

$\int_{[0,1]} f\,dx = \int_{[0,1]} \chi_{\mathbb{Q}}\,dx = m(\mathbb{Q}) = \sum_{q\in\mathbb{Q}} \underbrace{m(\{q\})}_{=0} = 0$

$\therefore$ Lebesgue integrable. $\because$ point set.

$\mathbb{Q}\cap[0,1] = \bigcup_{q\in[0,1]} \{q\}$