Mathematical Statistics II

Ch.6 Estimation

Jungsoon Choi

jungsoonchoi@hanyang.ac.kr

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Ch6.4 Maximum Likelihood Estimation

Problem

Suppose that $X_1, \ldots, X_n$ are random samples from the distribution, $f(x|\theta)$, where $\theta \in \Omega$ is the unknown parameter of interest, where Ω is called the parameter space.

Problem: Given the data $(X_1, \ldots, X_n) = (x_1, \ldots, x_n)$, how to estimate θ? Estimator of θ is a function of the sample (statistic), $\hat{\theta} = u(X_1, \ldots, X_n)$.

pop.

$N(\mu, \sigma^2)$ → $X_1, \cdots, X_n$: random sample.

guess → estimate "~~find~~"

unknown
fixed

[Descriptive stat : plot, table.
[Statistical inference

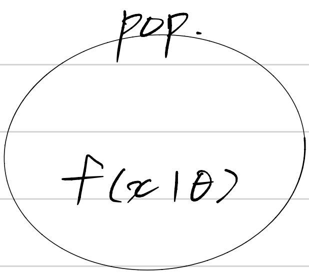
1) Estimation
   ch6 point estimation 3.5
   ch7 interval estimation (,)
   (cofidence I.E.).

ch8 2) Testing

point estimation methods
1) MLE
2) U.E
3) MoM (MME) method of moment
4) LSE Least square estimation

$\theta = (\mu, \sigma^2). \in \Omega$ (parameter space).

$Poi(\lambda) \longrightarrow \theta = \lambda \in \Omega$



$X_i \sim f(x)$. given parameter

$f(x|\theta)$. conditional distribution

$X_i \sim \text{Poi}(\lambda)$. given λ impormation

$f(x|\lambda)$

θ : parameter $\in \Omega$ (parameter space).

↑ unknown ↑ guess (estimate)

$X_i|\theta \sim f(x|\theta)$ $i=1,\cdots,n$ iid

$X_1,\cdots,X_n$

$\hat{\theta}$: estimator for θ

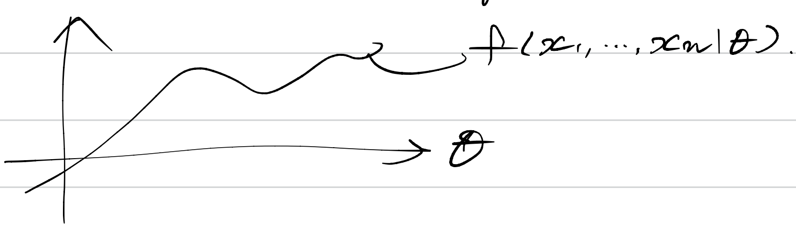
$= g(X_1,\cdots,X_n)$. function of random sample (statistic.).

$X_i \sim f(x|\theta)$.

$X_1,\cdots,X_n \sim \underline{f(x_1,\cdots x_n|\theta)} = \prod_{i=1}^{n} f(x_i|\theta)$.

f^{tn} of θ after collecting the data

$f(x_1,\cdots,x_n|\theta)$ f^{tn} of θ.



* Likelihood f^{th} $L(\theta) = f(x_1,\cdots,x_n|\theta)$.

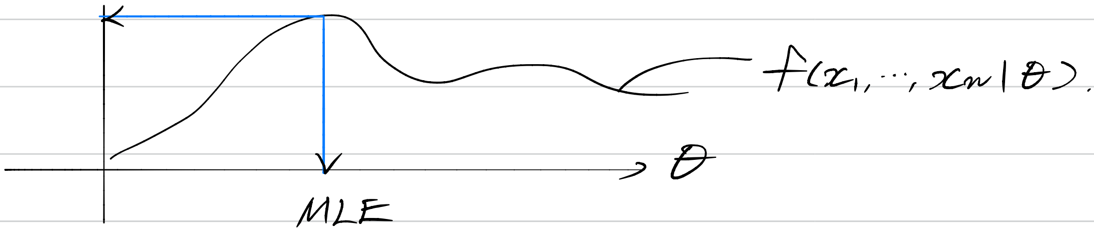
focus parameter.

just value

$L(\theta) = L(\theta|x_1,\cdots,x_n)$

Likelihood function 값이 최대가 되게 하는 θ 값.

Maximum Likelihood Estimator (MLE).



$x_i < \begin{smallmatrix}0\\1\end{smallmatrix}$

(e.g.). $X_1, \cdots, X_n \sim Ber(p)$ iid.

$f(x_i) = p^{x_i}(1-p)^{1-x_i}$ $\quad i = 1, \cdots, n.$

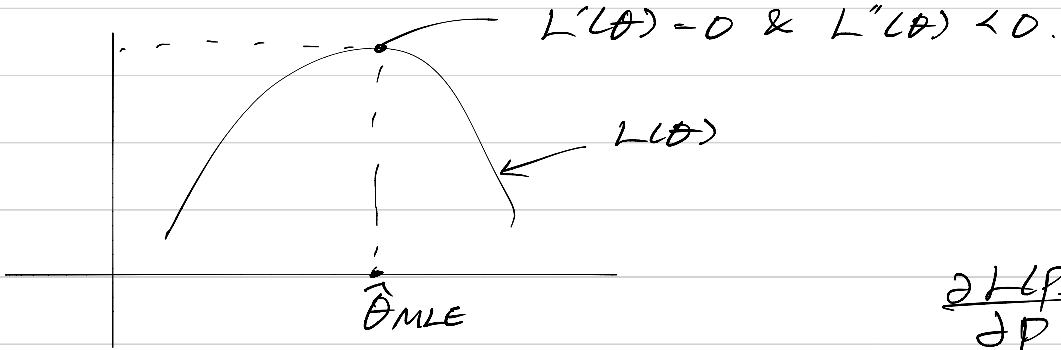
$$\underline{f(x_1, \cdots, x_n | p)} = \prod_{i=1}^{n} f(x_i|p) = \underline{p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}}$$

$= P(X_1 = x_1, \cdots, X_n = x_n)$. $\quad f^{tn}$ of p : $L(p)$.

good estimator of p should be maximizing the joint function of $X_1, \cdots, X_n$

↳ joint probability.

$$\hat{\theta}_{MLE} = \underset{\theta \in \Omega}{\operatorname{argmax}} L(\theta)$$



$$\frac{\partial L(p)}{\partial p} = 0.$$

1) $L(\theta) = L(\theta | x_1, \cdots, x_n). = f(x_1, \cdots, x_n | \theta).$

2) $\dfrac{\partial L(\theta)}{\partial \theta} = 0$

3) $L''(\theta) < 0 \quad \Rightarrow \hat{\theta}_{MLE}$: MLE of θ

6.4.1 Maximum Likelihood Estimator (MLE)

Definition of likelihood function

The likelihood function is the joint pdf (or pmf) of the samples $(x_1, \ldots, x_n)$ regarded as a function of θ.

$$L(\theta) = L(\theta | x_1, \ldots, x_n) = f(x_1, \ldots, x_n | \theta)$$

If X_i is iid from $f(x_i|\theta)$, then

$$L(\theta) = f(x_1, \ldots, x_n | \theta) = \Pi_{i=1}^{n} f(x_i | \theta)$$

Definition of MLE

A Maximum Likelihood Estimator (MLE) of the parameter θ based on the data $(x_1, \ldots, x_n)$ denoted by $\hat{\theta} = u(X_1, \ldots, X_n)$ is a value at which $L(\theta)$ attains its maximum as a function of θ.

Note

To find the MLE of θ, it is easy way to take a log function of $L(\theta)$ and then to find θ satisfying $\dfrac{d \ln L(\theta)}{d\theta} = 0$ (likelihood equation), which is a function of the data and is the MLE of θ.

maximum value 는 중요 X.

$\therefore$ $\ln L(\theta)$ 에서 maximize value 만 찾아주면 됨.

* Process to find the MLE of θ

1) Compute likelihood function
$L(\theta) = f(x_1, \cdots, x_n | \theta)$

2) Take a log function of $L(\theta)$
$l(\theta) = \ln L(\theta)$: log likelihood f^{tn}

3) Find θ statisfying $\frac{\partial l(\theta)}{\partial \theta} = 0$
$\hat{\theta}$: f^{tn} of $X_1, \cdots, X_n$ (f^{tn} of data).

4) $L''(\hat{\theta}) < 0$ ($\Leftrightarrow l''(\hat{\theta}) < 0$) $\therefore \hat{\theta}$: MLE of θ

Example Ber(p)

Let $X_1, \ldots, X_n \sim \mathrm{Ber}(p)$ (iid) and find the MLE of p.

Example 6.4-1

Let $X_1, \ldots, X_n \sim \mathrm{Exp}(\theta)$ (iid) and find the MLE of θ.

Example 6.4-2

Let $X_1, \ldots, X_n \sim \mathrm{Geo}(p)$ (iid), where $f(x|p) = (1-p)^{x-1}p$ for $x = 1, 2, \ldots,$ and find the MLE of p.

Example 6.4-3 (two parameters)

Let $X_1, \ldots, X_n \sim \mathrm{N}(\theta_1 = \mu, \theta_2 = \sigma^2)$ (iid) and find the MLEs of θ_1 and θ_2.

Ber(p).

1) $f(x_1, \cdots, x_n | p) = \prod_{i=1}^{n} f(x_i | p) = p^{\Sigma x_i} (1-p)^{n - \Sigma x_i}$

2) $l(p) = \ln L(p) = \sum_{i=1}^{n} x_i \ln p + (n - \sum_{i=1}^{n} x_i) \ln (1-p)$.

3) $\frac{\partial l(p)}{\partial p} = \frac{\Sigma x_i}{p} - \frac{n - \Sigma x_i}{1-p} = 0 \Rightarrow (1-p)\Sigma x_i = p(n - \Sigma x_i)$

$\Rightarrow \Sigma x_i = pn$

4) $l''(p) = \frac{\partial^2 l(p)}{\partial p^2} = -\frac{\Sigma x_i}{p^2} - \frac{(n - \Sigma x_i)}{(1-p)^2} < 0 \quad \therefore p = \frac{\Sigma x_i}{n}$

아직 data collecting 전 이므로 X_i (capital).

$\hat{P}_{MLE} = \frac{\Sigma X_i}{n}$: MLE of P.

$X_1 \cdots X_{100} \sim$ Ber(p)

0, 1, ⋯, 0, 1

$\Rightarrow$

No.	# of counts
0	40
1	60
	100

prob. of obesity people.

Using my data, $\hat{P}_{MLE} = \frac{60}{100} = 0.6$

$X_i \sim Exp(\theta) \quad i=1, \cdots, n \quad iid.$

$f(x|\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0.$

1) $L(\theta) = f(x_1, \cdots, x_n|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x}{\theta}} = \frac{1}{\theta^n} e^{-\frac{\Sigma x_i}{\theta}}$

2) $l(\theta) = \ln L(\theta) = -n \ln\theta - \frac{\Sigma x_i}{\theta}$

3) $\frac{\partial \ln l(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0 \qquad \theta \cdot n = \Sigma x_i \qquad \therefore \hat{\theta} = \frac{\Sigma x_i}{n}$

4) $l''(\theta) = \frac{n}{\theta^2} + \frac{\Sigma x_i(-2\theta)}{\theta^4} = \frac{n}{\theta^2} - \frac{2\Sigma x_i}{\theta^3}$

$l''(\hat{\theta}) = \frac{1}{\hat{\theta}^2}(n - \frac{2\Sigma x_i}{\hat{\theta}}) = \frac{1}{\hat{\theta}^2}(n-2n) = -\frac{n}{\hat{\theta}^2} < 0$

$\therefore \hat{\theta}_{MLE} = \frac{\Sigma X_i}{n} = \overline{X}$: MLE of θ.

$X_1, \cdots, X_n \sim \exp(\theta).$

$$f(x_i|\theta) = \frac{1}{\theta} \cdot e^{-\frac{x_i}{\theta}}$$

$$f(x_1, \cdots, x_n|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \frac{1}{\theta^n} e^{-\frac{\Sigma x_i}{\theta}}$$

$$\ln L(\theta) = \ln \frac{1}{\theta^n} - \frac{\Sigma x_i}{\theta}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0.$$

$$\frac{\Sigma x_i}{\theta^2} = \frac{n}{\theta} \qquad \Sigma x_i = n\theta. \qquad \theta = \frac{\Sigma x_i}{n}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2\Sigma x_i}{\theta^3} = \frac{n}{\theta^2} - \frac{2n\theta}{\theta^3} = \frac{n-2n}{\theta^2} = \frac{-n}{\theta^2} < 0$$

$$\therefore \hat{\theta}_{MLE} = \frac{\Sigma x_i}{n}$$

Gamma $\quad \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$

$$\frac{1}{\Gamma(\alpha)^n \beta^{\alpha n}} \prod_{i=1}^{n} (x_i)^{\alpha-1} \cdot e^{-\frac{\Sigma x_i}{\beta}}$$

Geo. $\quad p(1-p)^{x-1}$

$$L(p) = p^n (1-p)^{\Sigma x_i - n} \qquad \ln p^n + \ln (1-p)^{\Sigma x - n}$$

$$(\Sigma x_i - n) \ln p(1-p)$$

$$(\Sigma x_i - n) \frac{1-2p}{p(1-p)} = 0. \qquad p - p^2 \qquad 1-2p$$

$p(\Sigma x_i - n + n)$

$n = p \Sigma x_i \quad \frac{\Sigma x_i}{n}$

p

$$n - np = p\Sigma x_i - np \qquad \frac{n}{p} = \frac{\Sigma x_i - n}{1-p}$$

$$p = \frac{n}{\Sigma x_i}$$

$n(1-p) = p(\Sigma x_i - n).$

$n - np = p$

$$X_i \sim N(\theta_1, \theta_2) = N(\mu, \sigma^2) \ , \quad \underset{\sim}{\theta} = (\theta_1, \theta_2)$$

$$f(x_i \mid \underset{\sim}{\theta}) = \frac{1}{\sqrt{2\pi\theta_2}} \exp\left\{-\frac{(x_i-\theta_1)^2}{2\theta_2}\right\}$$

1) $$L(\underset{\sim}{\theta}) = L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} \exp\left\{-\frac{(x_i-\theta_1)^2}{2\theta_2}\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta_2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^{n}(x_i-\theta_1)^2}{2\theta_2}\right\}$$

2) $$l(\underset{\sim}{\theta}) = \ln L(\underset{\sim}{\theta}) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta_2) - \frac{\sum(x_i-\theta_1)^2}{2\theta_2}$$

3) $$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = \frac{2\sum(x_i-\theta_1)}{2\theta_2} = 0 \Rightarrow \sum x_i - n\theta_1 = 0 \quad \therefore \hat{\theta}_1 = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_2} = -\frac{n}{2\theta_2} + \frac{\sum(x_i-\theta_1)^2}{2\theta_2^2} = 0$$

$$\therefore \hat{\theta}_2 = \frac{\sum(x_i-\hat{\theta}_1)^2}{n} = \frac{\sum(x_i-\bar{x})^2}{n} = \frac{(n-1)s^2}{n}$$

4) $$\frac{\partial^2 l(\theta_1,\theta_2)}{\partial\theta_1^2} \qquad \frac{\partial^2 l(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} \qquad \frac{\partial^2 l(\theta_1,\theta_2)}{\partial\theta_2^2}$$

$$\Rightarrow \underset{=}{H}$$

시험에서 two-variable 일때 4) skip 해도 됨.

$$\hat{\theta}_{1MLE} = \bar{X} \ , \quad \hat{\theta}_{2MLE} = \frac{(n-1)s^2}{n}$$

are MLEs of (θ_1, θ_2)

$\nRightarrow$ $\theta_1 = \mu$ $\quad \theta_2 = \sigma$ $\quad$ my job.

(σ^2 말고 σ).

6.4.2 Unbiased Estimator

Definition

If $\hat{\theta} = u(X_1, \ldots, X_n)$ and $E[\hat{\theta}] = \theta$ for all $\theta \in \Omega$, then $\hat{\theta}$ is an unbiased estimator of θ. Otherwise, it is biased.

Example 6.4-4

Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample $X_1, \ldots, X_4$ from a uniform distribution with pdf $f(x) = 1/\theta$, $0 < x \leq \theta$. Find the MLE and UE of θ.

pop.

$f(x|\theta)$

$X_1, \cdots, X_n$

$\hat{\theta} = g(X_1, \cdots, X_n)$: statistic

$E[\hat{\theta}] = \theta \Rightarrow \hat{\theta}$: U.E of θ

unbiased estimator

biased 〃 : $E[\hat{\theta}] \neq \theta$

bias $= E[\hat{\theta}] - \theta$

e.x. 6.4.4.

$X_1, \cdots, X_4 \sim Unif(0, \theta]$

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 < x \leq \theta \\ 0 & \text{o.w} \end{cases}$$

1) $L(\theta) = \prod_{i=1}^{4} \left(\frac{1}{\theta}\right) \boxed{I(0 < x_i \leq \theta)} = \left(\frac{1}{\theta}\right)^4 \boxed{I(0 < \forall x_i \leq \theta)}$

2) To maximize $L(\theta)$: a value of θ ~~should be small~~ is the smallest.

and $0 < \forall x_i \leq \theta$

$x_1 < \theta$
$x_2 < \theta$
$x_3 < \theta$
$x_4 < \theta$

$\therefore \hat{\theta}_{MLE} = \max\{X_i\}$

$= Y_4 = X_{(4)}$: MLE of θ

cf). $L(\theta) = \left(\frac{1}{\theta}\right)^4$

indicator function.

$I(x_i > 0)$

$x_i = 0, 1.$

$l(\theta) = \ln L(\theta) = -4 \ln \theta$

$\frac{\partial l(\theta)}{\partial \theta} = \frac{-4}{\theta} = 0$?

x 값이 θ 에 따라 변하기 때문 !

이전에는 x 값이 θ에 관계 없이 정해 졌음.

$E[\hat{\theta}_{MLE}] = \theta + 2$

$E[\underbrace{\hat{\theta}_{MLE} - 2}_{\hat{\theta}_{U.E}}] = \theta$

$E[Y_4] = \int y_4 f_{Y_4}(y)\, dy$

$f_{Y_4}(y) = \binom{4}{3}\left(\frac{y}{\theta}\right)^3\left(\frac{1}{\theta}\right) = 4\frac{y^3}{\theta^4}, \quad 0 < y \leq \theta$

$E[Y_4] = \int_0^\theta y f_{Y_4}(y)\, dy = \frac{4}{5}\theta$: f^{tn} of θ

$\Rightarrow E[\underbrace{Y_4 \frac{5}{4}}] = \theta.$

$\hat{\theta}_{U.E} = \frac{5}{4}Y_4 = \frac{5}{4}\max\{X_i\}$

: U.E of θ

- $E[X_i] = \frac{\theta}{2} \Rightarrow 2X_1, 2X_2$

$\Rightarrow E[\sum_{i=1}^{4} X_i] = 2\theta$

$E\left[\frac{\sum_{i=1}^{4} X_i}{2}\right] = E[2\bar{X}] = \theta$

$2\bar{X} = \frac{\sum_{i=1}^{4} X_i}{2} = \hat{\theta}_{U.E}$: U.E of θ

Several U.E.

Example 6.4-5

Let $X_1, \ldots, X_n \sim \mathrm{N}(\theta_1 = \mu, \theta_2 = \sigma^2)$ (iid). Check if the MLEs of θ_1 and θ_2 are unbiased estimators.

$X_1, \dots, X_4 \sim N(\theta_1 = \mu,\ \theta_2 = \sigma^2).$

$$\hat{\theta}_{1,MLE} = \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$
$$\hat{\theta}_{2,MLE} = \frac{(n-1)S^2}{n} = \frac{\Sigma(X_i-\bar{X})^2}{n} \sim \chi^2(n-1)$$

$$E[\hat{\theta}_{1,MLE}] = \theta_1 \ ? \Rightarrow \mu = \theta_1 \Rightarrow \hat{\theta}_{1,MLE} = \hat{\theta}_{1,UE}$$

$\bar{X}$: MLE & UE of μ.

$$E[\hat{\theta}_{2,MLE}] = \theta_2 \ ?$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1),$$

$$E\left[\frac{(n-1)S^2}{\sigma^2}\right] = n-1$$
$$\frac{1}{\sigma^2}E(S^2) = 1$$
$$\Rightarrow E[S^2] = \sigma^2$$

$$E[\hat{\theta}_{2,MLE}] = E\left[\frac{(n-1)S^2}{n}\right] = \frac{(n-1)}{n}E(S^2) = \frac{(n-1)}{n}\sigma^2 \neq \sigma^2$$

$\hat{\theta}_{2,MLE} = \frac{\Sigma(X_i-\bar{X})^2}{n}$ is not UE of θ_2

parameter	MLE	UE
μ	$\bar{X}$	$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}).$
σ^2	$\frac{\Sigma(X_i-\bar{X})^2}{n}$	$S^2 = \frac{\Sigma(X_i-\bar{X})^2}{n-1}$

S^2의 정의를 $\frac{\Sigma(X_i-\bar{X})^2}{n-1}$ 로 사용하는 이유 !

$E[\hat{\theta}] = \theta$

U.E of σ^2

6.4.3 Method of Moments Estimator

Definition

Let $X_1, \ldots, X_n$ are random samples from the distribution, $f(x|\theta_1, \ldots, \theta_r)$, where $(\theta_1, \ldots, \theta_r) \in \Omega$ are the parameters of interest. For $k = 1, 2, 3, \ldots, r$, using

$$E[X^k] = \frac{\sum_{i=1}^n X_i^k}{n},$$

find the estimator of θ_i, which is denoted by $\tilde{\theta}$. This is called method of moments estimator.

pop

$f(x|\theta)$

f^{tn} of θ

$X_1, \cdots, X_n$ — $X_1, \cdots, X_n$ (sample / data).

$E[X^1] = \frac{\sum X_i}{n} \Rightarrow \hat{\theta}_{MME}$: MME of θ

$E[X^2] = \frac{\sum X_i^2}{n}$

from $f(x|\theta)$.

해결기.

$$E[X] = \int_0^1 x f_X(x)\,dx = \frac{\theta}{\theta+1}$$
$$= \int_0^1 \theta x^{\theta}\,dx$$

Example 6.4-6

Let $X_1, \ldots, X_n \sim f(x_i|\theta) = \theta x_i^{\theta-1}$ (iid), where $0 < x < 1,\ \theta > 0$.

Find the estimator of θ using the method of moments.

Example 6.4-7

Let $X_1, \ldots, X_n \sim \mathrm{N}(\mu, \sigma^2)$ (iid) and find the estimators of μ and σ^2 using the method of moments.

pop.

$f(x|\theta)$ $\rightarrow$ $X_1, \cdots, X_n$

$\Downarrow$ $//$ $g(X_1, \cdots, X_n)$.

$\hat{\theta}$
$//$
estimator of θ

- MLE
- U.E $\quad E[\hat{\theta}] = \theta$
- MME $\quad E[\underbrace{g(X_1, \cdots, X_n)}_{U.E}] = \theta$

$\tilde{\theta} = \hat{\theta}_{MLE}$ f^{tn} of θ $\quad \hat{\theta}_{MLE}, \hat{\theta}_{UE}, \hat{\theta}_{MME}$

$$\left.\begin{aligned} E(X) &= \frac{\sum X_i}{n} \\ E(X^2) &= \frac{\sum X_i^2}{n} \\ &\vdots \end{aligned}\right) \Rightarrow \text{For } K \text{ equations} = \# \text{ of parameter}$$

$X_1, \cdots, X_n \sim f(x_i|\theta) = \theta x_i^{\theta-1} \quad 0 < x < 1, \quad \theta > 0$

$E(X) = \frac{\Sigma X_i}{n} = \overline{X}$

$E[X] = \int_0^1 x \theta x^{\theta-1} dx = \int_0^1 \theta x^{\theta} dx = \frac{\theta}{\theta+1}$

$\therefore \frac{\theta}{\theta+1} = \overline{X} \Rightarrow \therefore \hat{\theta}_{MME} = \frac{\overline{X}}{1-\overline{X}}$

$X_i \sim N(\mu, \sigma^2)$.

$E[X] = \overline{X} \Rightarrow \mu = \overline{X}$

$E[X^2] = \frac{\Sigma X_i^2}{n} \Rightarrow \sigma^2 + \mu^2 = \frac{\Sigma X_i^2}{n} \Rightarrow \sigma^2 = \frac{\Sigma X_i^2}{n} - \left(\frac{\Sigma X_i}{n}\right)^2 = \frac{\Sigma(X_i - \overline{X})^2}{n}$

check!

$\therefore \hat{\mu}_{MME} = \overline{X}, \quad \hat{\sigma}^2_{MME} = \frac{\Sigma(X_i-\overline{X})^2}{n}$

$N(\mu, \sigma^2)$	μ	σ^2
MLE	$\overline{X}$	$\frac{\Sigma(X_i-\overline{X})^2}{n} = \frac{(n-1)S^2}{n}$
UE	$\overline{X}$	$\frac{\Sigma(X_i-\overline{X})^2}{n-1} = S^2$
MME	$\overline{X}$	$\frac{\Sigma(X_i-\overline{X})^2}{n}$

6.4 - 11. $X \sim (\mu, \sigma^2)$. $\bar{X} \sim (\mu, \frac{\sigma^2}{n})$.

$S^2 = \frac{1}{n-1}\Sigma(X_i - \bar{X})^2$ is U.E of σ^2 ?

$$S^2 = \frac{\Sigma(X_i-\bar{X})^2}{n-1} = \frac{1}{n-1}(\Sigma X^2 - n\bar{X}^2)$$

$\hat{\sigma}^2_m = \frac{\Sigma(X-\mu)^2}{n}$

$$E[S^2] = \frac{1}{n-1}E(\Sigma X^2 - n\bar{X}^2)$$
$$= \frac{1}{n-1}\left[E[\Sigma X^2] - nE[\bar{X}^2]\right]$$

$\Sigma(X_i - \bar{X})^2$

$= \Sigma(X_i^2 - 2X_i\bar{X} + \bar{X}^2)$

$= \Sigma X_i^2 - 2\bar{X}\Sigma X_i + n\bar{X}^2$

$= \Sigma X_i^2 - 2n\bar{X}^2 + n\bar{X}^2$

$= \Sigma X_i^2 - n\bar{X}^2$

$\therefore \frac{\Sigma(X_i-\bar{X})^2}{n} = \frac{\Sigma X_i^2}{n} - \left(\frac{\Sigma X_i}{n}\right)^2$

$\frac{\Sigma X_i^2}{n} - \left(\frac{\Sigma X_i}{n}\right)^2$

$= \frac{\Sigma X_i^2}{n} - \bar{X}^2$

$= \frac{1}{n}(\Sigma X_i^2 - n\bar{X}^2)$

$= \frac{1}{n}(\Sigma X_i^2 - 2n\bar{X}^2 + n\bar{X}^2)$

$= \frac{1}{n}(\Sigma X_i^2 - 2\bar{X}\Sigma X_i + n\bar{X}^2)$

$= \frac{1}{n}(\Sigma X_i^2 - 2\bar{X}\Sigma X_i + \Sigma\bar{X}^2)$

$= \frac{1}{n}\Sigma(X_i^2 - 2\bar{X}X_i + \bar{X}^2)$

$= \frac{1}{n}\Sigma(X_i - \bar{X})^2$

$Var[X] = \sigma^2 < \infty$. show that $E[S^2] = \sigma^2$

$$
\begin{aligned}
E[S^2] &= E[\tfrac{1}{n-1}\Sigma(X_i - \overline{X})^2] = \tfrac{1}{n-1} E[\Sigma(X_i^2 - 2\overline{X}X_i + \overline{X}^2)] \\
&= \tfrac{1}{n-1} E[\sum_{i=1}^{n} X_i^2 - 2\overline{X}\sum_{i=1}^{n} X_i + \sum_{i=1}^{n}\overline{X}^2] \\
&= \tfrac{1}{n-1} E[\sum_{i=1}^{n} X_i^2 - 2\overline{X}\cdot n\overline{X} + n\overline{X}^2] \\
&= \tfrac{1}{n-1} E[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2] \\
&= \tfrac{1}{n-1}(\sum_{i=1}^{n} E[X_i^2] - nE[\overline{X}^2]) \\
&= \tfrac{1}{n-1}(n\cdot E[X^2] - nE[\overline{X}^2]) \\
&= \tfrac{n}{n-1}(E[X^2] - E[\overline{X}^2]) \\
&= \tfrac{n}{n-1}(Var[X] + \{E[X]\}^2 - Var[\overline{X}] - \{E[\overline{X}]\}^2) \\
&= \tfrac{n}{n-1}(\sigma^2 + \mu^2 - \tfrac{\sigma^2}{n} - \mu^2\} \\
&= \tfrac{n}{n-1}(\sigma^2 - \tfrac{\sigma^2}{n}) \\
&= \tfrac{n}{n-1}(\tfrac{n-1}{n}\sigma^2) = \sigma^2
\end{aligned}
$$

$$
\begin{aligned}
E[cS] &= E[\tfrac{c\sigma}{\sqrt{n-1}}\left(\tfrac{(n-1)S^2}{\sigma^2}\right)^{\frac{1}{2}}] \\
&= \tfrac{c\sigma}{\sqrt{n-1}}\int_0^\infty v^{\frac{1}{2}}\cdot\frac{1}{\Gamma(\frac{n-1}{2})\cdot 2^{\frac{n-1}{2}}}\cdot v^{\frac{n-1}{2}-1}\cdot e^{-\frac{v}{2}}dv \\
&= \tfrac{c\sigma}{\sqrt{n-1}}\int_0^\infty \frac{1}{\Gamma(\frac{n}{2})\cdot 2^{\frac{n}{2}}}\cdot v^{\frac{n}{2}-1}\cdot e^{-\frac{v}{2}}dv\cdot\sqrt{2}\cdot\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \\
&= \tfrac{c\sigma}{\sqrt{n-1}}\cdot\frac{\sqrt{2}\,\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}
\end{aligned}
$$

$$\therefore c = \frac{\sqrt{n-1}\cdot\Gamma(\frac{n-1}{2})}{\sqrt{2}\,\Gamma(\frac{n}{2})}.$$