

# Differential Equations – Final Exam

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1. Solve the given initial value problems.

(a)  $y'' + y = \sec t$ ,  $y(0) = 1$ ,  $y'(0) = 0$ .

(b)  $y'' + 2y' + 3y = 3\delta(t - 2\pi) + u(t - \pi)\cos t$ ,  $y(0) = 0$ ,  $y'(0) = 1$ .

2. Let  $y$  be the solution of  $y' = y^2 - ty$ ,  $y(0) = 1$ . Estimate  $y(1)$  using the Improved Euler method with the step size  $h = 1$ .

3. Find a second solution  $y_2$  of the given differential equation

$$t^2 y'' - 2t(t+1)y' + 2(t+1)y = 0, \quad t > 0, \quad y_1(t) = t$$

by using the method of the reduction of order.

4. For  $x > 0$ , find two (linearly independent) solutions of  $x^2 y'' + (x^2 + x)y' - y = 0$ .

5. Find the general solution of

$$X' = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} X + \begin{pmatrix} -4 \cos t \\ 3 \cos t \end{pmatrix}.$$

6. For the following two problems

$$(a) \begin{cases} x' = y - x^2 \\ y' = (x - y^2)(1 - 4y) \end{cases} \quad \text{and} \quad (b) \begin{cases} x' = -4x + y + x^2 \\ y' = y - x - 2 \end{cases}$$

- (a) Find all critical points and classify the type of each critical point.  
(b) Draw the phase plane, sketch a few of the trajectories in the neighborhood of each critical point.

7. Find the solution of

$$\begin{cases} u_{xx} = u_t, & 0 < x < 20, \quad t > 0 \\ u(0, t) = 30, \quad u(20, t) = 10, & t > 0 \\ u(x, 0) = x(20 - x)/10, & 0 \leq x \leq 20 \end{cases}.$$

8. If  $u_1$  and  $u_2$  are solutions of

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, \quad t > 0 \\ u(x, 0) = f(x), & 0 < x < L \\ u(0, t) = g_1(t), \quad u(L, t) = g_2(t), & t \geq 0 \end{cases},$$

then show that  $u_1 = u_2$ .