

# Vector Calculus – Final Exam

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1. Evaluate the following integrals:

(1)  $\int_0^1 \int_y^1 x^2 \sin xy \, dx \, dy$

(2)  $\int_0^2 \int_{\frac{x}{2}}^{\frac{x}{2}+1} x^5(2y-x)e^{(2y-x)^2} \, dy \, dx$

2. Find the volume of the solid bounded by  $z = 2 - y^2$  and  $4x^2 + y^2 = z$ .
3. Consider a region bounded by  $(z - 8)^2 = 4(x^2 + y^2)$  and  $z = 0$ . Find the centroid of this region.
4. Explain the numerical integration with the Simpson's rule (or 2-point Gaussian quadrature rule, if you prefer) and use it to compute the integration:

$$\int_0^2 e^x \sin x \, dx.$$

You don't need to make a finer partition for the interval  $[0, 2]$ .

5. Let  $F$  be a vector field in  $\mathbb{R}^3$  given by  $F(x, y, z) = (e^x \sin y - yz, e^x \cos y - xz, z - xy)$ . Suppose  $\mathbf{x}: [0, 1] \rightarrow \mathbb{R}^3$  be a curve such that  $\mathbf{x}(t) = (t, t^2, t^3)$ . Evaluate the work done by  $F$  along the path  $\mathbf{x}$ .
6. Let  $D$  be a closed, bounded region in  $\mathbb{R}^2$  where  $\partial D$  is a simple, closed,  $C^1$  curve. Let  $F$  be a  $C^1$  vector field defined on  $\mathbb{R}^2$ .
- (1) State the Green's theorem and the divergence theorem.
- (2) Prove the divergence theorem using the Green's theorem.
7. Let  $S$  be a surface defined by  $S = \{(x, y, z) \in \mathbb{R}^3 : z = 4e^{-(x^2+y^2)} - 1, z \geq 0\}$ . The surface  $S$  is oriented by the normal vector  $\mathbf{n}$  such that  $\mathbf{n} \cdot \mathbf{k} \geq 0$ . Let  $F$  be a vector field defined in  $\mathbb{R}^3$  such that  $F(x, y, z) = (e^{y+z} - 2y, xe^{y+z} + y, e^{x+y})$ . Then, evaluate the integration:

$$\iint_S \nabla \times F \cdot d\mathbf{S}.$$

8. Let  $S$  be a surface defined by  $S = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1\}$  and  $\mathbf{n}$  be the outward unit normal vector on  $S$ . Let  $f$  be a function defined on  $\mathbb{R}^3 \setminus \{0\}$  such that  $f(x, y, z) = \log(x^2 + y^2 + z^2)$ . Then, evaluate the integration:

$$\iint_S \nabla f \cdot \mathbf{n} \, dS.$$