Mathematical Statistics 1

Ch.1 Probability

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3.1 Conditional probability

Example 1.3-1

Suppose that we are given 20 tulip bulbs that are similar in appearance and told that 8 will bloom early, 12 will bloom late, 13 will be red, and 7 will be yellow. Also, we know that 5 red tulip will bloom early. Find the probability that it will produce a red tulip given that it has bloomed early if one bulb is selected at random.

	Early	Late	Total
Red	5	8	13
Yellow	3	4	7
Total	8	12	20

$\#(S) = 20 \Rightarrow P(E\&R) = \frac{5}{20}$

$\#(E) = 8 \Rightarrow \frac{5}{8} = \frac{\#(E\&R)}{\#(E)} \underset{\text{define}}{=} P(R|E) = \frac{P(R\cap E)}{P(E)} \frac{5/\cancel{20}}{8/\cancel{20}} = \frac{5}{8}$

new sample space.

original sample space

$B \subset S$

Definition

If A and B are events in $\mathcal{S}$, and $P(B) > 0$, then the *conditional probability of A given B*, written $P(A|B)$, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \neq \phi$$

new sample space

1
1, 2
2 3
3 4
4

Example 1.3-4

A pair of four-sided dice is rolled and the sum is determined. Let A be the event that a sum of 3 is rolled, and let B be the event that a sum of 3 or a sum of 5 is rolled. Compute the probability that a sum of 3 is rolled given that a sum of 3 or 5 has occurred in a sequence of rolls.

$n(S) = 16$

$n(A) = 2$

$n(B) = 6$

$\frac{1}{3}$

$P(A|B) = \frac{2}{6} = \frac{1}{3}$

Properties

- $0 \leq P(A|B) \leq 1$
- $P(A|A) = 1$
- If A and B are disjoint, then $P(A|B) = P(B|A) = 0$
- If $A_1, A_2, \ldots$ are mutually exclusive events, then
  $P(A_1 \cup A_2 \cup \cdots | B) = P(A_1|B) + P(A_2|B) + \cdots$
- $P(A^c|B) = 1 - P(A|B)$

$$P(A|A) = \frac{P(A \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$$P(S|S) = 1$$

$$P(A|S) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)}{1} = P(A)$$

disjoint : $A \cap B = \emptyset$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = 0$$

If $A_1, A_2, \cdots$ are mutually exclusive events,

$A_i \cap A_j = \emptyset$ for $i \neq j$ $\cdots$ ①

$$P(A_1 \cup A_2 \cup \cdots | B) = \frac{P[(A_1 \cup A_2 \cup \cdots) \cap B]}{P(B)} = \frac{P[(A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots]}{P(B)}$$

For $i \neq j$ $(A_i \cap B) \cap (A_j \cap B) = (A_i \cap A_j) \cap B \underset{\text{by ①}}{=} \emptyset \cap B = \emptyset$

$\therefore (A_1 \cap B), (A_2 \cap B), (A_3 \cap B) \cdots$ are also mutually exclusive

$$\therefore P(A_1 \cup A_2 \cup \cdots | B) = \frac{P(A_1 \cap B) + P(A_2 \cap B) + \cdots}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + \cdots$$

$$= P(A_1|B) + P(A_2|B) + \cdots$$

$$* \quad P(A^c|B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = 1 - P(A|B)$$

Definition

If A and B are events in $\mathcal{S}$, and $P(B) > 0$ and $P(A) > 0$

$$P(A \cap B) = P(A)P(B|A)$$
$$P(A \cap B) = P(B)P(A|B)$$

Example 1.3-6

A bowl contains seven blue chips and three red chips. Two chips are to be drawn successively at random and without replacement. Compute the probability that the first draw results in a red chip and the second draw results in a blue chip.

$$\frac{3}{10} \times \underbrace{\frac{7}{9}}_{\text{new sample space}} = \frac{7}{30}$$

Properties

If A, B, and C are events in $\mathcal{S}$,

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

$$P(\underbrace{A \cap B}_{=D} \cap C) = P(D) \cdot P(C|D)$$

$$= P(A \cap B) \cdot P(C | A \cap B)$$

$$= P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

Ch.1.4 Independent Events

4.1 Independent event

cf) A & B

i) disjoint : $A \cap B = \phi$

ii) independent : $P(A \cap B) = P(A) \cdot P(B)$

Definition

Two events, A and B, are (statistically) independent if

$$P(A \cap B) = P(A)P(B).$$

⇒ This is our independent.

take this

Otherwise, A and B are dependent.

If A & B : indep does not related (affected)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} = P(A)$$

if) $P(B) > 0$

↳ 이 조건이 붙기 때문에 간주.

Example 1.4-2

Let's toss a red die and a white die. Let event $A = \{4 \text{ on the red die}\}$ and event $B = \{\text{sum of dice is odd}\}$. Verify that A and B are independent.

$P(A) = \frac{1}{6}$ $P(B) = \frac{1}{2}$ $P(A)P(B) = \frac{1}{12}$

$P(A \cap B) = \frac{1}{12}$

Example 1.4-3

Let's toss a red die and a white die. Let event $C = \{5 \text{ on the red die}\}$ and event $D = \{\text{sum of dice is } 11\}$. Check if C and D are independent.

5 6
6 5

$P(C) = \frac{1}{6}$ $P(D) = \frac{2}{36} = \frac{1}{18}$

$P(C \cap D) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$P(C)P(D) \neq P(C \cap D)$.

$S\,(R, W)$

$(1,1)$
(1.2)
$\vdots$
$(4,1)$
$(4,6)$
$\vdots$
(6.6)

$P(A) = \frac{6}{36} = \frac{1}{6}$

$P(B) = \frac{18}{36} = \frac{1}{2}$

$A \cap B = \{4$ on the red die & sum of dice is odd$\}$

$= \{(4,1), (4,3), (4,5)\}$

$P(A \cap B) = \frac{3}{36} = \frac{1}{12} = P(A) \cdot P(B)$

$P(C) = \frac{6}{36} = \frac{1}{6}$ $\quad D = \{(5,6), (6,5)\}$ $\quad P(D) = \frac{2}{36} = \frac{1}{18}$

$C \cap D = \{(5,6)\}$ $\quad P(C \cap D) = \frac{1}{36} \neq P(C)P(D) = \frac{1}{6} \times \frac{1}{18}$

$\therefore$ C & D are dependent

Theorem 1.4-1

If A and B are independent events, then the following pairs are also independent.

complement

- A and B^c
- A^c and B
- A^c and B^c

Since A & B are indep.

$$P(A \cap B) = P(A) \cdot P(B) \quad \& \quad P(B|A) \overset{P(A)>0}{=} \frac{P(A \cap B)}{P(A)} = P(B)$$

i) $P(A \cap B^c) = P(A) \cdot P(B^c | A)$ under $P(A) > 0$

$= P(A)[1 - P(B|A)]$

$= P(A)[1 - P(B)]$

$= P(A) P(B^c)$

ii) If $P(A) = 0$ $\quad (A \cap B^c) \subset A$

$0 \le P(A \cap B^c) \le P(A) = 0$

$\therefore P(A \cap B^c) = 0 = P(A) \cdot P(B^c) = 0$

A, B are independent. A^c, B^c tho.

$P(A \cap B) = P(A) \cdot P(B)$.

i) $P(A^c) \neq 0$

$P(A \cap B) = P(A) \cdot P(B|A)$.

$P(A^c \cap B^c) = P(A^c) P(B^c | A^c)$.

$= P(A^c)[1 - P(B|A^c)]$ $\quad A^c, B$ indep.

$P(A^c)[1 - P(B)]$

$P(A^c) \cdot P(B^c)$.

i) $P(A^c) = 0. \rightarrow P(A^c) = 0.$

$(A^c \cap B^c) \subset A^c$

$0 \le P(A^c \cap B^c) \le P(A^c) = 0.$

$\therefore P(A^c \cap B^c) = 0.$

$P(A^c \cap B^c) = 0 = \underset{0}{P(A^c)} \cdot P(B^c) = 0.$

Example 1.4-4 - pairwise independence

An urn contains four balls numbered 1, 2, 3, and 4. One ball is to be drawn at random from the urn. Let the events A, B, and C be defined by $A = \{1,2\}$, $B = \{1,3\}$, and $C = \{1,4\}$. Verify that A, B, and C are independent in pairs (called pairwise independence).

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

$$A \cap B = \{1\} = B \cap C = A \cap C$$

$$\begin{array}{ccccccc} P(A \cap B) & = & P(B \cap C) & = & P(A \cap C) & = & \frac{1}{4} \\ \parallel & & \parallel & & \parallel & & \\ P(A) \cdot P(B) & & P(B) \cdot P(C) & & P(A) \cdot P(C) & & \end{array}$$

mutually exclusive : disjoint.

Definition

Events A, B, and C are mutually independent if and only if the following two conditions hold:

- A, B, and C are pairwise independent: that is,

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C),$$
$$P(A \cap C) = P(A)P(C)$$

- $P(A \cap B \cap C) = P(A)P(B)P(C)$

A, B, C : mutually indep.

① pairwise indep $\Leftrightarrow \begin{cases} P(A\cap B) = P(A)\cdot P(B) \\ P(A\cap C) = P(A)\cdot P(C) \\ P(B\cap C) = P(B)\cdot P(C) \end{cases}$

② $P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)$

Ex 1.44. $P(A\cap B\cap C) = P(\{1\}) = \frac{1}{4} \neq P(A)\cdot P(B)\cdot P(C) = \frac{1}{8}$

$\therefore$ A, B and C are not mutually independent

Example 1.4-5

A rocket has a built-in redundant system. In this system, if component K_1 fails, it is bypassed and component K_2 is used. If component K_2 fails, it is bypassed and component K_3 is used. Suppose that the probability of failure of any one component is 0.15, and assume that the failures of these components are mutually independent events. The system fails if K_1 fails and K_2 fails and K_3 fails. What is the probability that the system does not fail?

$P(k_1 \text{ fails}) = P(k_2 \text{ fails}) = P(k_3 \text{ fails}) = 0.15$

$\quad A_1 \qquad\qquad A_2 \qquad\qquad A_3$

$P[(A_1 \cap A_2 \cap A_3)^c] = 1 - P(A_1 \cap A_2 \cap A_3) = 1 - P(A) \cdot P(B) \cdot P(C)$

$= 1 - (0.15)^3 = 0.9966$

P(None accident from Jan to Mar & accident in Apr)

$= (1-0.31)(1-0.28)(1-0.31) \times 0.3 = 0.103$

Example 1.4-7

The probability that a company's workforce has at least one accident during a certain month is $(0.01)k$, where k is the number of days in that month (say, February has 28 days). Assume that the numbers of accidents is independent from month to month. Compute the probability that the first accident is in April if the company's year starts with January.

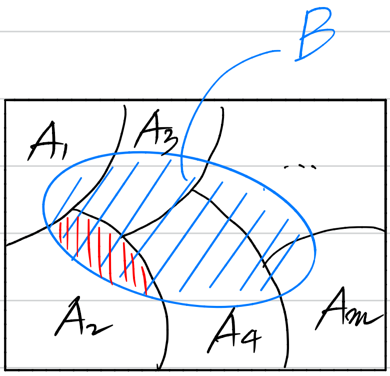
Ch.1.5 Bayes' Theorem

5.1 Bayes' Theorem

Bayes' Theorem

Let $A_1, A_2, \ldots, A_m$ be a partition of the sample space $\mathcal{S}$, and let B be any set. Then, for each $i = 1, 2, \ldots, m$

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{m} P(B|A_j)P(A_j)}$$



$$\bigcup_{i=1}^{m} A_i = S \quad \text{and} \quad A_i \cap A_j = \emptyset \quad \text{for } i \neq j$$

$$\underline{P(A_i|B)} = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)\cdot P(A_i)}{P(B \cap S)} = \frac{P(B|A_i)P(A_i)}{P[B \cap (\bigcup_{i=1}^{m} A_i)]}$$

$$= \frac{P(B|A_i)P(A_i)}{P[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots (A_m \cap B)]} = \frac{P(B|A_i)\cdot P(A_i)}{\sum_{j=1}^{m} P(A_j \cap B)}$$

$$= \frac{P(B|A_i)\cdot P(A_i)}{\sum_{j=1}^{m} P(B|A_j)\cdot P(A_j)}$$

For $i \neq j$, $(A_i \cap B) \cap (A_j \cap B) = (A_i \cap A_j) \cap B \neq \emptyset$

$(A_1 \cap B)$, $(A_2 \cap B)$ $\ldots$ $(A_m \cap B)$ are mutually exclusive.

For $\forall i$, $P(B|A_i)$ & $P(A_i)$

Example 1.5-2

In a certain factory, machines I, II, and III are all producing springs of the same length. Of their production, machines I, II, and III, respectively produce 2%, 1%, and 3% defective springs. Of the total production of springs in the factory, machine I produces 35%, machine II produces 25%, and machine III produces 40%. If one spring is selected at random from the total springs produced in a day, what is the probability that it is defective? Also, if the selected spring is defective, find the probability that it was produced by machine III.

$P(M1 \text{ produces springs}) = P(A_1) = 0.35$
$P(M2 \quad \text{"} \quad) = P(A_2) = 0.25$
$P(M3 \quad \text{"} \quad) = P(A_3) = 0.40$
$P(A_1 \cup A_2 \cup A_3) = 1$

B = defective spring appears

P(defective spring among the springs from M1) = 0.02

$= P(B \mid A_1) = 0.02$

$P(B \mid A_2) = 0.01, \quad P(B \mid A_3) = 0.03$

$P(B) = P(B \cap S) = P(B \cap (A_1 \cup A_2 \cup A_3)) = P[(B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)]$

$= P(B \cap A_1) + P(B \cap A_2) + \underline{P(B \cap A_3)}$
$= P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A_2) + \underline{P(B \mid A_3) \cdot P(A_3)}$

$P(M3 \text{ prod} \mid \text{defective spring}) = P(A_3 \mid B)$

$$= \frac{P(A_3 \cap B)}{P(B)} = \frac{P(B \mid A_3) \cdot P(A_3)}{P(B)}$$

Example 1.5-3

A Pap smear is a screening procedure used to detect cervical cancer. For women with this cancer, there are about 16% negatives. For women without cancer, there are about 19% positives. In the US, there are about 8 women in 100,000 who have this cancer. What is the probability that a woman with positive Pap smears has cervical cancer?

$$P(\text{Test-Neg.} \mid \text{cancer}) = 0.16$$

$$P(\text{Test-Pos.} \mid \text{None-Cancer}) = 0.19$$

$$P(\text{cancer}) = 0.00008$$

$$P(\text{cancer} \mid \text{Test-Pos}) = \frac{P(\text{cancer} \cap \text{Test-pos})}{P(\text{Test-Pos})}$$

$$= \frac{P(\text{cancer}) \cdot P(\text{Test-Pos} \mid \text{cancer})}{P(\text{Test-Pos} \mid \text{cancer}) \cdot P(\text{cancer}) + P(\text{Test-pos} \mid \text{None-cancer}) \cdot P(\text{none-cancer})}$$

$$= \frac{0.00008 \times 0.84}{0.00008 \times 0.84 + 0.99992 \times 0.19} \fallingdotseq 0.000354$$