4-bar linkage problem

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● $R_1\cos(\alpha) - R_2\cos\phi + R_3 - \cos(\alpha - \phi) = 0$

$R_1 = \frac{5}{3}, R_2 = \frac{5}{2}, R_3 = \frac{11}{6}$.

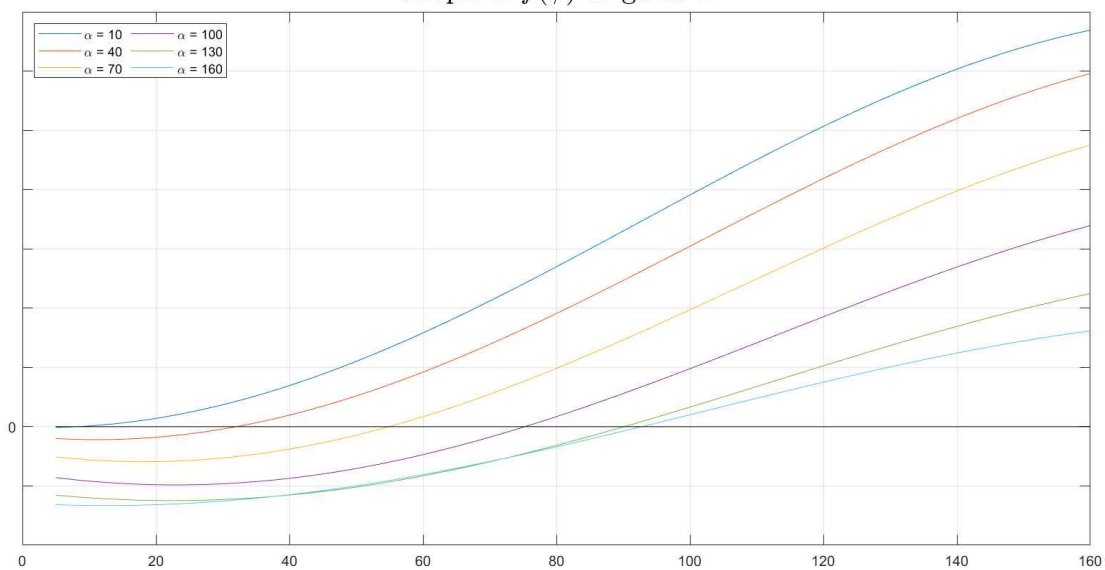
● $f(\phi) = R_1\cos(\alpha) - R_2\cos\phi + R_3 - \cos(\alpha - \phi)$.

We use 4 kinds of numerical analysis to solve this problem using matlab, bisection method, newton mentod, secant method and fixed-point iteration.

● **Solution of the 4-bar linkage problem**

α	ϕ
10°	8.06934531362
40°	32.01518035933
70°	54.88776318878
100°	75.27087338607
130°	90.12408036526
160°	92.73496285135

Graph of $f(\phi)$ at given α



Bisection method

● Input $\alpha = 70^\circ$

Convergence criterion : 10^{-6}

Initial angle : $\phi_a = 50^\circ, \phi_b = 60^\circ$

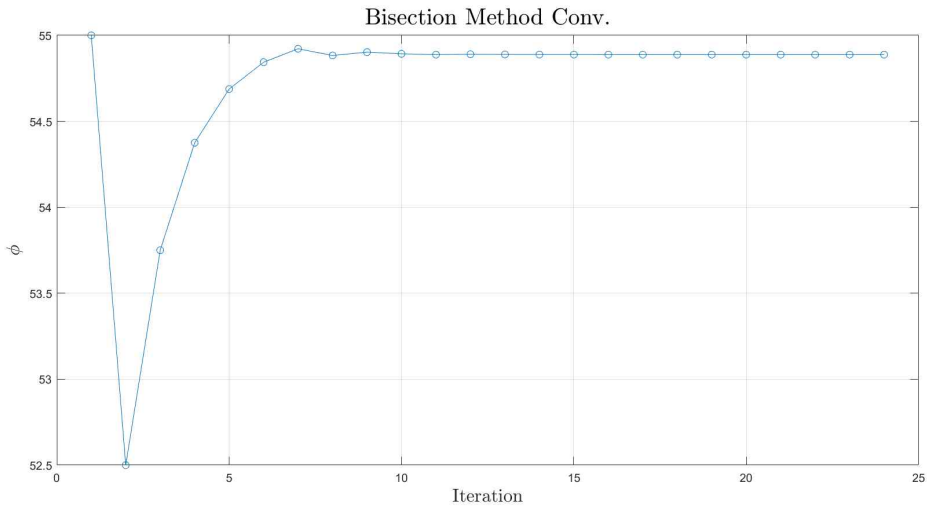
● Code

```
clc;
clear all;
close all;
format long
alpha = 70;
tol = 1e-6;
f = @(x) (5*cos((pi*alpha)/180))/3 - (5*cos((pi*x)/180))/2 - cos((pi*(alpha - x))/180)
+ 11/6;
a = 50;
b = 60;
c = (a+b)/2;
e = (b-a)/2;
n = 1;
while e > tol
    c = (a+b)/2;
    e = (b-a)/2;
    i(n) = n;
    ai(n) = a;
    fa(n) = f(a);
    bi(n) = b;
    fb(n) = f(b);
    ci(n) = c;
    ei(n) = e;
    if f(c) > 0;
        b = c;
    else f(c) < 0;
        a = c;
    end
    n = n + 1;
end
fprintf('Root of given equation is %f',c)
T = table(i', ai', fa', bi', fb', ci', ei');
T.Properties.VariableNames = [{'i'},{'a'},{'fa'},{'b'},{'fb'}, {'c'}, {'e'}
```

● Solution by Bisection method

i	ϕ_a	$f(\phi_a)$	ϕ_b	$f(\phi_b)$	ϕ_c	$error$
0	50.00000000000	-0.14329473946	60.00000000000	0.16855915253	55.00000000000	5.00000000000
1	50.00000000000	-0.14329473946	55.00000000000	0.00349998838	52.50000000000	2.50000000000
2	52.50000000000	-0.07225361773	55.00000000000	0.00349998838	53.75000000000	1.25000000000
3	53.75000000000	-0.03495706975	55.00000000000	0.00349998838	54.37500000000	0.62500000000
4	54.37500000000	-0.01587247333	55.00000000000	0.00349998838	54.68750000000	0.31250000000
5	54.68750000000	-0.00622208237	55.00000000000	0.00349998838	54.84375000000	0.15625000000
6	54.84375000000	-0.00136998894	55.00000000000	0.00349998838	54.92187500000	0.07812500000
7	54.84375000000	-0.00136998894	54.92187500000	0.00106276649	54.88281250000	0.03906250000
8	54.88281250000	-0.00015416982	54.92187500000	0.00106276649	54.90234375000	0.01953125000
9	54.88281250000	-0.00015416982	54.90234375000	0.00045415872	54.89257812500	0.00976562500
10	54.88281250000	-0.00015416982	54.89257812500	0.00014995955	54.88769531250	0.00488281250
11	54.88769531250	-0.00000211386	54.89257812500	0.00014995955	54.89013671875	0.00244140625
12	54.88769531250	-0.00000211386	54.89013671875	0.00007392066	54.88891601563	0.00122070313
13	54.88769531250	-0.00000211386	54.88891601563	0.00003590285	54.88830566406	0.00061035156
14	54.88769531250	-0.00000211386	54.88830566406	0.00001689436	54.88800048828	0.00030517578
15	54.88769531250	-0.00000211386	54.88800048828	0.00000739021	54.88784790039	0.00015258789
16	54.88769531250	-0.00000211386	54.88784790039	0.00000263817	54.88777160645	0.00007629395
17	54.88769531250	-0.00000211386	54.88777160645	0.00000026215	54.88773345947	0.00003814697
18	54.88773345947	-0.00000092586	54.88777160645	0.00000026215	54.88775253296	0.00001907349
19	54.88775253296	-0.00000033185	54.88777160645	0.00000026215	54.88776206970	0.00000953674
20	54.88776206970	-0.00000003485	54.88777160645	0.00000026215	54.88776683807	0.00000476837
21	54.88776206970	-0.00000003485	54.88776683807	0.00000011365	54.88776445389	0.00000238419
22	54.88776206970	-0.00000003485	54.88776445389	0.00000003940	54.88776326180	0.00000119209
23	54.88776206970	-0.00000003485	54.88776326180	0.00000000227	54.88776266575	0.00000059605

● Convergence analysis



We took two intervals $[50,60], [60,70]$.

It is easily verified that sign of $f(x)$ does not change in interval $[60, 70]$.

Hence we choose the interval $[50,60]$.

$$\frac{\phi_a - \phi_b}{2^{n+1}} < 10^{-6} \quad \Rightarrow \quad 10^7 < 2^{n+1} \quad \Rightarrow \quad n \geq 23$$

n has to be greater than or equal 23.

$$|r - \phi_c| \leq \left| \frac{\phi_a - \phi_b}{2^{n+1}} \right| = \frac{10}{2^{24}} = 5.98 \times 10^{-7} < 10^{-6}$$

$$(r - \phi_c = 54.8877631 - 54.8877626 = 5.22 \times 10^{-7})$$

$$e_{i+1} \leq \frac{1}{2} e_i$$

$\Rightarrow$ Hence, ϕ_C linearly converges to root.

Newton's method

● Input $\alpha = 70^\circ$

Initial value $\phi_1 = 50^\circ$

$f(\phi_1) = -0.143294$

$f'(\phi_1) = 0.027455$

$\phi_2 = \phi_1 - \dfrac{f(\phi_1)}{f'(\phi_1)} = 55.219141$

⋮

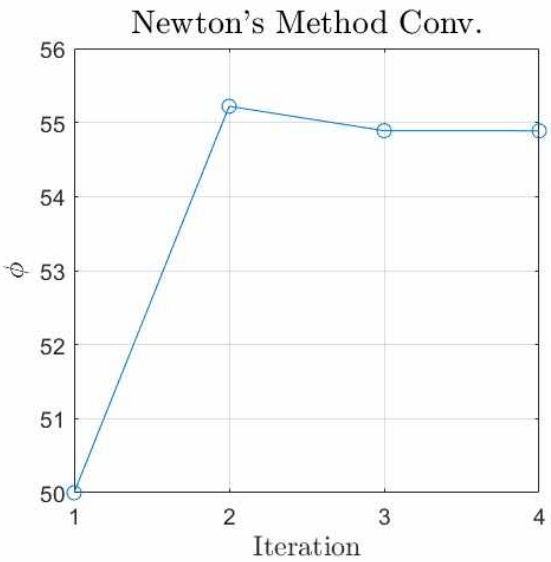
● Code

```
clc;
clear all;
close all;
format long
alpha = 160;
tol = 1e-6;
f = @(x) (5*cos((pi*alpha)/180))/3 - (5*cos((pi*x)/180))/2 - cos((pi*(alpha - x))/180)
+ 11/6;
fp = @(x) (pi*sin((pi*x)/180))/72 - (pi*sin((pi*(alpha - x))/180))/180;
x = 100;
e = 1;
n = 1;
while  e > tol
    i(n) = n;
    xi(n) = x;
    fx(n) = f(x);
    fpx(n) = fp(x);
    x = x - f(x)/fp(x);
    xj(n) = x;
    e = abs(xi(n) - xj(n));
    ei(n) = e;
    n = n + 1;
end
fprintf('Root of given equation is %f',x)
T = table(i', xi', fx', fpx', xj', ei');
T.Properties.VariableNames = [{'i'},{'xi'},{'fx'},{'fpx'},{'xj'}, {'ei'}]
```

● Solution by Newton's method

i	ϕ_i	$f(\phi_i)$	$f'(\phi_i)$	ϕ_{i+1}	$error$
1	50.00000000000	-0.14329473946	0.02745561676	55.21914115768	5.21914115768
2	55.21914115768	0.01036020900	0.03138497856	54.88904027900	0.33010087868
3	54.88904027900	0.00003977289	0.03114383148	54.88776320795	0.00127707105
4	54.88776320795	0.00000000060	0.03114289653	54.88776318878	0.00000001917

● Convergence analysis



$$e_{i+1} = \frac{1}{2}\frac{f''(r)}{f'(r)}e_i^2$$

$$e_4 = \frac{1}{2}(\frac{f''(r)}{f'(r)})e_3^2,$$

$$\frac{f'(r)}{2f''(r)} = -\frac{0.0007321}{2\times 0.0311428} = -0.0117539$$

$$\frac{e_4}{e_3^2} = \frac{1.91695\times 10^{-8}}{0.001277^2} = 0.011755 \approx \frac{f''(r)}{2f'(r)}$$

It converges quadratically.

Secant method

● Input $\alpha = 70^\circ$

Initial value $\phi_0 = 50^\circ, \phi_1 = 60^\circ$

$f(\phi_0) = -0.143294$

$f(\phi_1) = 0.168559$

$$g'(\phi_1) = \frac{f(\phi_1) - f(\phi_0)}{\phi_1 - \phi_0} = 0.031185$$

$$\phi_2 = \phi_1 - \frac{f(\phi_1)}{g'(\phi_1)} = 54.594931$$

⋮

● Code

```
clc;
clear all;
close all;
format long
alpha = 60;
tol = 1e-6;
f = @(x) (5*cos((pi*alpha)/180))/3 - (5*cos((pi*x)/180))/2 - cos((pi*(alpha - x))/180)
+ 11/6;
a = 40;
b = 50;
gp = (f(b)-f(a))/(b-a);
c = b - f(b)/gp;
e = abs(c-b);
i(1) = 0;
i(2) = 1;
xi(1)= a;
xi(2)= b;
fx(1) = f(a);
fx(2) = f(b);
gpx(1) = NaN;
gpx(2) = gp;
ci(1) = NaN;
ci(2) = c;
ei(1) = NaN;
ei(2) = e;
n = 3;
while e > 10^(-6);
```

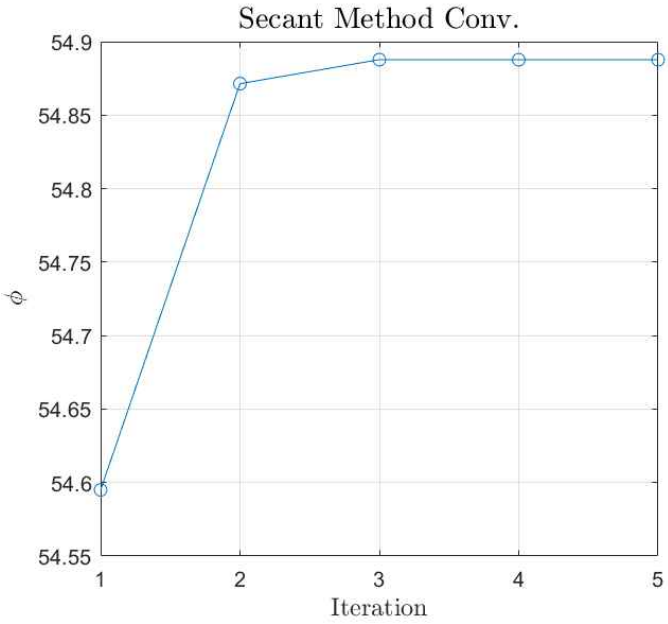
```
    a = b;
    b = c;
    gp = (f(b)-f(a))/(b-a);
    c = b - f(b)/gp;
    e = abs(c-b);

    i(n) = n-1;
    xi(n) = b;
    fx(n) = f(b);
    gpx(n) = gp;
    ci(n) = c;
    ei(n) = e;
    n = n+1;
end
fprintf('Root of given equation is %f',c)
T = table(i',xi', fx', gpx', ci', ei');
T.Properties.VariableNames = [{'i'}, {'xi'}, {'fx'}, {'gpx'}, {'ci'}, {'ei'}]
```

● Solution by Secant Method

i	ϕ_i	$f(\phi_i)$	$g'(\phi_i)$	ϕ_{i+1}	$error$
0	50.00000000000	-0.14329473946	NaN	NaN	NaN
1	60.00000000000	0.16855915253	0.03118538920	54.59493189407	5.40506810593
2	54.59493189407	-0.00908818591	0.03286680851	54.87144751326	0.27651561919
3	54.87144751326	-0.00050801994	0.03102958881	54.88781962655	0.01637211329
4	54.88781962655	0.00000175764	0.03113694434	54.88776317795	0.00005644860
5	54.88776317795	-0.00000000034	0.03114291718	54.88776318878	0.00000001083

● Convergence Analysis



$$e_{i+1} = \left[\frac{1}{2}\frac{f''(r)}{f'(r)}\right]^{0.62} e_i^{1.62}$$

$$|e_{n+1}| = |e_5| = 1.083 \times 10^{-8}$$

$$\left|\frac{f''(r)}{2f'(r)}\right|^{0.62} \cdot |e_4|^{1.62} = \left|\frac{f''(r)}{2f'(r)}\right|^{0.62} \cdot |e_4|^{1.62}$$
$$= 0.0117539^{0.62} \times (5.644860 \times 10^{-5})^{1.62}$$
$$= 0.834 \times 10^{-8}$$

$$\Rightarrow |e_{n+1}| \approx \left|\frac{f''(r)}{2f'(r)}\right|^{0.62} \cdot |e_n|^{1.62}$$

The convergence is superlinear.

Seant method (which degree is 1.62) is slow than Newton's method(degree 2). But It is faster than Bisection(degree 1).

Fixed-Point Iteration

- $R_2\cos\phi = R_1\cos\alpha + R_3 - \cos(\alpha - \phi)$

  Set $\phi = \cos^{-1}(u(\phi)) = g(\phi)$,

  where $u(\phi) = \frac{1}{R_2}(R_1\cos\alpha + R_3 - \cos(\alpha - \phi))$

- Initial value $\phi_1 = 50^\circ$, Input $\alpha = 70^\circ$

  $\phi_2 = g(\phi_1)$

  $\Rightarrow g(50^\circ) = \cos^{-1}(u(50^\circ))$

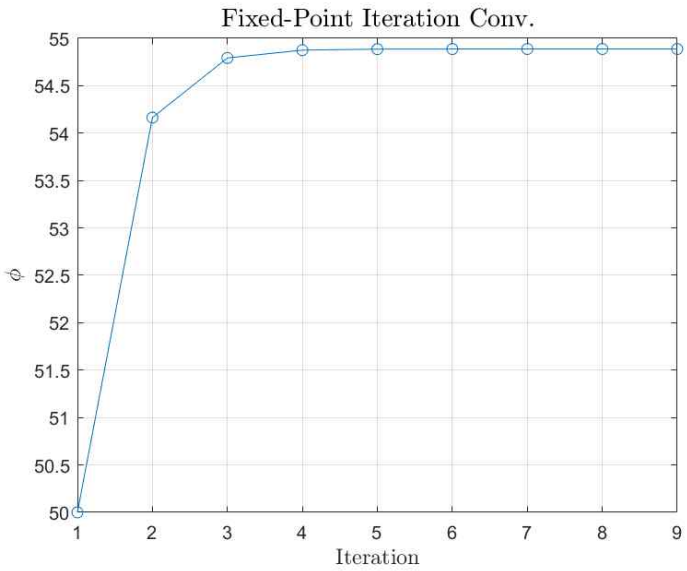
- Code

```
clc;
clear all;
close all;
format long
alpha = 60;
tol = 1e-6;
f = @(x) (5*cos((pi*alpha)/180))/3 - (5*cos((pi*x)/180))/2 - cos((pi*(alpha - x))/180)
+ 11/6;
u = @(x) (2*cos((pi*alpha)/180))/3 - (2*cos((pi*(alpha - x))/180))/5 + 11/15;
g = @(x) acosd(u(x));
gp = @(x) (2*sin((pi*(alpha - x))/180))/(5*(1 - ((2*cos((alpha*pi)/180))/3 -
(2*cos((pi*(alpha - x))/180))/5 + 11/15)^2)^(1/2));
x = 50;
e = x - g(x);
n = 1;
while abs(e) > tol
    i(n) = n;
    xi(n) = x;
    ux(n) = u(x);
    xj(n) = g(x);
    gpx(n) = gp(x);
    x = g(x);
    fxj(n) = f(x);
    e = xi(n) - xj(n);
    ei(n) = e;
    n = n + 1;
end
fprintf('Root of given equation is %f',x)
T = table(i', xi', ux', xj', fxj', gpx', ei');
T.Properties.VariableNames = [{'i'}, {'xi'}, {'ux'}, {'xj'}, {'fxj'}, {'gpx'}, {'ei'}]
```

● Solution by fixed-point iteration

i	ϕ_i	$u(\phi_i)$	ϕ_{i+1}	$f(\phi_{i+1})$	$g'(\phi_i)$	$error$
1	50.00000000000	0.58546971390	54.16381992127	-0.02235324642	0.16875419115	-4.16381992127
2	54.16381992127	0.57652841533	54.79326153179	-0.00293978492	0.13359218930	-0.62944161052
3	54.79326153179	0.57535250136	54.87567818091	-0.00037630869	0.12828017557	-0.08241664912
4	54.87567818091	0.57520197789	54.88622194307	-0.00004799799	0.12758468530	-0.01054376216
5	54.88622194307	0.57518277870	54.88756669659	-0.00000611932	0.12749571024	-0.00134475351
6	54.88756669659	0.57518033097	54.88773813926	-0.00000078011	0.12748436235	-0.00017144267
7	54.88773813926	0.57518001892	54.88775999540	-0.00000009945	0.12748291561	-0.00002185614
8	54.88775999540	0.57517997914	54.88776278168	-0.00000001268	0.12748273117	-0.00000278628
9	54.88776278168	0.57517997407	54.88776313688	-0.00000000162	0.12748270766	-0.00000035520

● Convergence analysis

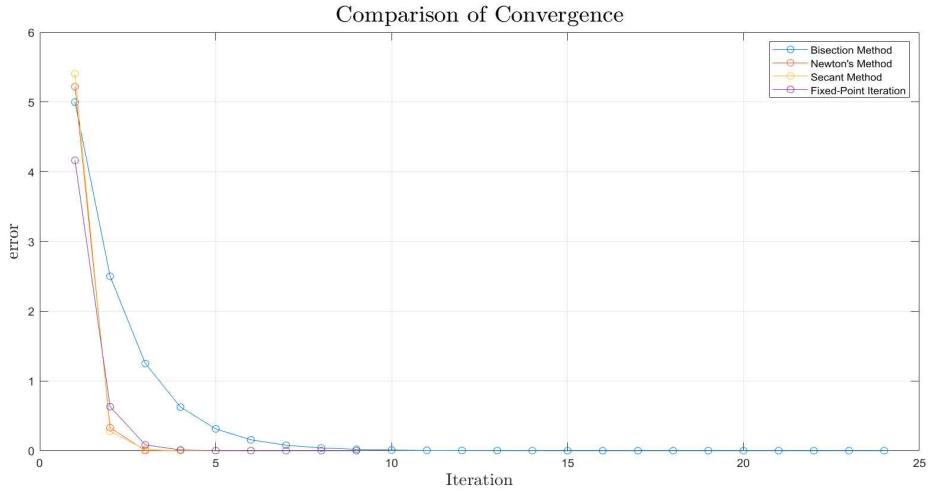


Set $\phi = g(\phi) = \cos^{-1}(u(\phi))$ where $u(\phi) = \frac{1}{R_2}(R_1\cos(70^\circ) + R_3 - \cos(70^\circ - \phi))$

$$g'(\phi) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{d\phi} = \frac{\sin(70^\circ - \phi)}{R_2\sqrt{1-u^2}}$$

$$g'(50) = \frac{2}{5}\frac{\sin(70^\circ - 50^\circ)}{\sqrt{1-0.58546^2}} = 0.16875 \ (<1) \ : \text{ locally convergent}$$

Conclusion



Speed of convergernce

Newton's method > secant method > Fixed-point iteration >Bisection method

Bisection method

This method is simple and robust. But the convergence rate is relatively slow. Because of this, it is often used to obtain a rough approximation to a solution which is then used as a starting point for more rapidly converging methods.
In case that finding root is the first purpose, bisection method is the most suitable method. Because premise of bisection is that there exists at least one root. Then existence of root is assured.

Newton's method

Convergence rate is faster than other methods. (quadratically conv.)
However there are three cases that fail, which means it has the lowest probability of success compared to other methods. Therefore, using newton's method, setting the initial value is very important.

Secant method

The convergence rate is faster than bisection method, but it is slower than newton's method. Like newton's method, there are cases that we can fail. The reason for using this method is that there are times when it is too difficult to obtain a derivative from the newton's method. Compared with newton's method, it has low convergence degree which leads more calculation, the secant method can be better in the calculation cost.

Fixed-point iteration

Fixed-point method is easy to find root.

This method need several fixed-point theorems to guarantee the existence of the fixed point.

And we have to test if an iteration converges or not.