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Lecture note 1: Black-Scholes model

1 Black-Scholes model

In mathematical finance, a financial market is defined as a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ having a Brownian motion $(B_t)_{t\geq 0}$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual condition. The measure $\mathbb{P}$ is referred to as the *objective measure* of the market. Assume that in the market there are a money-market account $G = (G_t)_{t\geq 0}$ and a risky asset $S = (S_t)_{t\geq 0}$ defined as below.

A money-market account represents a locally riskless investments, where profit is accrued continuously at the risk-free rate prevailing in the market at every instant.

Definition 1.1. *A* money-market account *G is a deterministic process defined by $G_t = e^{rt}, t \geq 0$ where $r \geq 0$ is a constant.*

The money-market account is expressed as

$$dG_t = rG_t\,dt\,,\ G_0 = 1\,.$$

The nonnegative number r is called the *short interest rate.*

Definition 1.2. *A risky asset is a stochastic process $S = (S_t)_{t\geq 0}$ given by*

$$S_t = S_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B_t}\,.$$

The risky asset S is an Ito process satisfying the SDE

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t\,.$$

The real numbers μ and σ are referred to as the *drift* and the *volatility* of the asset, respectively.

2 Portfolio dynamics

Definition 2.1. *A portfolio is a pair $h = (\pi_t, \phi_t)_{0\leq t\leq T}$ of progressively measurable processes satisfying*

$$\int_0^T \phi_u^2 + |\pi_u|\,du < \infty \quad \text{a.s.}$$

The portfolio value process is defined by

$$V_t^h = \phi_t S_t + \pi_t G_t\,,\ t \geq 0\,.$$

A portfolio $h = (\pi_t, \phi_t)$ is called self-financing *if the value process V_t^h satisfies*

$$dV_t^h = \phi_t\,dS_t + \pi_t\,dG_t\,.$$

From the above integrability condition, we note that the stochastic integrals

$$\int_0^t \phi_u \, dS_u = \mu \int_0^t \phi_u S_u \, du + \sigma \int_0^t \phi_u S_u \, dB_u$$
$$\int_0^t \pi_u \, dG_u = r \int_0^t \pi_u G_u \, du$$

are well-defined since S and G are continuous processes. A portfolio is self-financing if there is no exogenous infusion or withdrawal of money; the purchase of a new asset must be financed by the sale of an old one.

Remark 2.1. Since the value of a self-financing portfolio

$$V_t = V_0 + \int_0^t \phi_u \, dS_u + \int_0^t \pi_u \, dG_u$$

is the sum of an Ito integral and a Riemann integral, the process $(V_t)_{t\geq 0}$ is an Ito process.

Theorem 2.1. *Let $V_t = V_t^h$ be the value of a self-financing portfolio $h = (\phi_t, \pi_t)$. Then*

$$d\Big(\frac{V_t}{G_t}\Big) = \phi_t \, d\Big(\frac{S_t}{G_t}\Big) \, .$$

Theorem 2.2. *For any $x \in \mathbb{R}$ and any progressively measurable process ϕ with $\int_0^t \phi_u^2 \, du < \infty$ a.s. for each $t \geq 0$, there exists a unique progressively measurable process π such that $\int_0^t \pi_u^2 \, du < \infty$ a.s. for each $t \geq 0$, and $(\phi_t, \pi_t)_{t\geq 0}$ is self-financing. In this case, the value process V_t and π_t are given by*

$$V_t = xG_t + G_t \int_0^t \phi_u \, d\left(\frac{S_u}{G_u}\right) \, , \quad \pi_t = \frac{V_t - \phi_t S_t}{G_t} \, .$$

Based on this theorem, usually a self-financing portfolio is specified by only $\phi = (\phi_t)_{t\geq 0}$.

Proof. Fix $t > 0$. Motivated by the equation in Theorem 2.1, we define

$$V_t := xG_t + G_t \int_0^t \phi_u \, d\left(\frac{S_u}{G_u}\right) \, .$$

This stochastic integral is well-defined since ϕ is progressively measurable and $\int_0^t \phi_u^2 \, du < \infty$ a.s. Define

$$\pi_t := \frac{V_t - \phi_t S_t}{G_t} \, .$$

It is easy to check that π is progressively measurable and $\int_0^t \pi_u^2 \, du < \infty$ a.s. Clearly, V_t is the value process of the portfolio (ϕ_t, π_t). It remains to show that this portfolio is self-financing.

$$\begin{aligned} dV_t = d\left(\frac{V_t}{G_t} G_t\right) &= G_t \, d\left(\frac{V_t}{G_t}\right) + \frac{V_t}{G_t} \, dG_t \\ &= \phi_t G_t \, d\left(\frac{S_t}{G_t}\right) + \phi_t \frac{S_t}{G_t} \, dG_t + \pi_t \frac{G_t}{G_t} \, dG_t \\ &= \phi_t \, dS_t + \pi_t \, dG_t \, . \end{aligned}$$

This gives the desired result. □

Corollary 2.3. *In particular, if the short interest rate is zero (i.e., $G_t \equiv 1$), then the processes V and π in the above theorem are*

$$V_t = x + \int_0^t \phi_u \, dS_u \,, \quad \pi_t = V_t - \phi_t S_t \,.$$

Definition 2.2. *Fix a terminal time $T > 0$. An* arbitrage *is a self-financing portfolio h such that*

$$\begin{aligned} &V_0^h = 0 \,, \\ &\mathbb{P}(V_T^h \geq 0) = 1 \,, \\ &\mathbb{P}(V_T^h > 0) > 0 \,. \end{aligned}$$

We say a market is arbitrage free *if there is no arbitrage.*

Definition 2.3. *Fix a terminal time $T > 0$. A self-financing portfolio is said to be* admissible *if there exists a constant $\alpha > 0$ such that the value process*

$$V_t \geq -\alpha$$

for all $0 \leq t \leq T$ a.s.

This definition is to avoid the doubling strategy. Refer to page 9 in Karatzas et al. (1998). We assume that every self-financing portfolio is admissible.

3 Classical approach

This section presents a heuristic argument to derive the Black-Scholes formula. Recall that the payoff of a call option is $(S_T - K)_+$.

Proposition 3.1. Consider the Black-Scholes model. Let V be a self-financing portfolio with a zero diffusion term, that is,

$$dV_t = k_t \, dt$$

for some $(k_t)_{t \geq 0}$. Then $k_t = rV_t$ for $t \geq 0$.

Proof. Let $(\phi_t, \pi_t)_{t \geq 0}$ be the self-financing portfolio of V. Since dS_t has a dW_t-term, we know that

$$dV_t = \phi_t \, dS_t + \pi_t \, dG_t$$

has a zero diffusion term if and only if $\phi_t = 0$ for $t \geq 0$. This implies that

$$V_t = V_0 G_t$$

by Theorem 2.2. Thus, $dV_t = rV_0 G_t \, dt = rV_t \, dt$. □

Assumption 1. *Let V_t be the value of the call option $(S_T - K)_+$ at time t. We assume the followings.*

(i) $V_t = f(t, S_t)$ for some function f.

(ii) $f \in C^{1,2}$

(iii) $\sup_{0\le t\le T} |f_s(t,s)|$ *has polynomial growth in* s.

We now want to find this function f. Consider a market with three assets (V_t), (S_t) and (G_t). Construct the following portfolio.

- Buy one call option V_t
- Sell $f_s(t, S_t)$ number of stocks
- Finance this transaction from the bank.

Then the portfolio is $(1, -f_s(t, S_t), \pi_t)$ where π_t is the amount financed from the bank. The value of this portfolio is

$$\hat{V}_t = V_t - f_s(t, S_t)S_t + \pi_t G_t \,.$$

Since this is self-financing, we get

$$\begin{aligned}\hat{V}_t &= dV_t - f_s(t, S_t)dS_t + \pi_t dG_t \\ &= df(t, S_t) - f_s(t, S_t)dS_t + \pi_t dG_t \\ &= \left(f_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{ss}(t, S_t) + r\pi_t G_t\right) dt\end{aligned}$$

By Proposition 3.1, it follows that

$$\begin{aligned}\hat{V}_t &= r\hat{V}_t\, dt \\ &= r(V_t - f_s(t, S_t)S_t + \pi_t G_t)\, dt \\ &= r(f(t, S_t) - f_s(t, S_t)S_t + \pi_t G_t)\, dt\end{aligned}$$

By comparing the above two equations,

$$f_t(t, S_t) + rS_t f_s(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 f_{ss}(t, S_t) - rf(t, S_t) = 0 \,.$$

Since $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$ ranges $(0, \infty)$, we obtain

$$f_t(t, s) + rsf_s(t, s) + \frac{1}{2}\sigma^2 s^2 f_{ss}(t, s) - rf(t, s) = 0 \text{ for } s > 0 \,.$$

At the terminal time, $f(T, s) = (s - K)_+$ is clear.

4 Feynman-Kac formula

The above PDE can be solved by the Feynman-Kac formula below. This section is indebted to Section 6.3 in Baudoin (2014).

Assumption 2. *Assume that there is a positive number* C *such that*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \le C|x - y|$$

for all $0 \le t \le T$.

Theorem 4.1. *Let W be a Brownian motion and fix $t \geq 0$. Under Assumption 2, the SDE*

$$\begin{aligned} dX_s &= b(s, X_s)\, ds + \sigma(s, X_s)\, dW_s\,, s > t \\ X_t &= x \end{aligned}$$

has a unique solution $(X_s)_{s \geq t} = (X_s^{t,x})_{s \geq t}$. Moreover, for every $T > 0$ and $p \geq 1$,

$$\mathbb{E}(\sup_{t \leq s \leq T} |X_s|^p) < \infty\,.$$

Theorem 4.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function with polynomial growth and $r : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a bounded function. Under Assumption 2, define*

$$f(t, x) = \mathbb{E}(e^{-\int_t^T r(s, X_s^{t,x})\, ds} g(X_T^{t,x}))\,.$$

and assume $f \in C^{1,2}$. Then

$$f_t + \frac{1}{2}\sigma^2(t, x) f_{xx} + b(t, x) f_x - r(t, x) f = 0$$

with the terminal condition $f(T, x) = g(x)$.

Theorem 4.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function with polynomial growth and $r : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a bounded function. Under Assumption 2, suppose that $f \in C^{1,2}$ solves*

$$f_t + \frac{1}{2}\sigma^2(t, x) f_{xx} + b(t, x) f_x - r(t, x) f = 0$$

with the terminal condition $f(T, x) = g(x)$. If $\max_{0 \leq t \leq T} |f_x(t, x)|$ has polynomial growth, then

$$f(t, x) = \mathbb{E}(e^{-\int_t^T r(s, X_s^{t,x})\, ds} g(X_T^{t,x}))$$

where $(X^{t,x})$ is the solution in Theorem 4.1.

5 Exercises

Problem 5.1. (15 points) Consider a market with one bank account $G = (e^{rt})_{0 \leq t \leq T}$ and N stocks $S^{(1)}, \cdots, S^{(N)}$ which are positive Ito processes. A portfolio is defined as a $N + 1$ dimensional progressively measurable process $h = (\pi_t, \phi_t)_{0 \leq t \leq T} = (\pi_t, \phi_t^{(1)}, \cdots, \phi_t^{(N)})_{0 \leq t \leq T}$ with

$$\int_0^T |\pi_t| + \|\phi_t\|^2\, dt < \infty \text{ a.s.}$$

Show that for any $x \in \mathbb{R}$ and any N-dimensional progressively measurable process ϕ there is a unique progressively measurable π such that $h = (\pi_t, \phi_t)_{0 \leq t \leq T}$ is self-financing, $V_0^h = x$, and $\int_0^T |\pi_t|\, dt < \infty$ a.s.

Problem 5.2. (10 points) Consider the Black-Scholes stock model

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}\,, t \geq 0$$

for $\mu \in \mathbb{R}$, $\sigma > 0$ and a Brownian motion $(B_t)_{t \geq 0}$.

(i) Show that $\mathcal{F}_t^B = \mathcal{F}_t^S$ for all $t \geq 0$, that is, the natural filtrations of $(B_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ coincide.

(ii) Evaluate $\mathbb{E}(\int_0^T S_u\, du | \mathcal{F}_t^S)$.

Problem 5.3. (15 points) Consider the Black-Scholes model. Find the self-financing portfolio value V_t and the amount π_t financed from the bank in the following cases. Let $r \geq 0$ be the short rate.

(i) The initial portfolio value is $V_0 = 1$ and $\phi_t = 1/S_t^2$.

(ii) Assume $r = 0$. The initial portfolio value is $V_0 = 0$ and $\phi_t = S_t$ for $0 \leq t \leq 1$ and $\phi_t = 1$ when $t > 1$.

(iii) Assume $r = 0$. The initial portfolio value is $V_0 = 1$ and $\phi_t = tS_t$.

Problem 5.4. (10 points) Use the put-call parity to find the put price in the Black-Scholes model.

Problem 5.5. (10 points) Let $T > 0$. Find a solution $f \in C^{1,2}$ to the PDE

$$\begin{aligned} &f_t + bxf_x + \frac{1}{2}\sigma^2 f_{xx} = 0 \\ &f(T, x) = (1 - e^x)_+ \end{aligned}$$

such that $\max_{0\leq t\leq T} |f_x(t, x)|$ has polynomial growth in x.

Problem 5.6. Consider the multi-dimensional BS model. Let $(B_t^{(1)}, B_t^{(2)})_{t\geq 0}$ be a two-dimensional Brownian motion. The bank account is $G_t = e^{rt}$ and two stocks are given as

$$\begin{aligned} \frac{dS_t^{(1)}}{S_t^{(1)}} &= \mu^{(1)}\, dt + \sigma_1^{(1)}\, dB_t^{(1)} + \sigma_2^{(1)}\, dB_t^{(2)}\,,\ S_0^{(1)} > 0 \\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= \mu^{(2)}\, dt + \sigma_2^{(2)}\, dB_t^{(1)} + \sigma_2^{(2)}\, dB_t^{(2)}\,,\ S_0^{(2)} > 0\,. \end{aligned}$$

(i) (5 points) Solve the above SDEs and find $S^{(1)}$ and $S^{(2)}$.

(ii) (15 points) Consider an option with payoff $(S_T^{(1)} - S_T^{(2)})_+$ and maturity T. Use the heuristic argument to derive the Black-Scholes PDE: Let $f(t, x_1, x_2)$ be the function such that the time-t price of option is $f(t, S_t^{(1)}, S_t^{(2)})$. Then

$$f_t + rx_1 f_{x_1} + rx_2 f_{x_2} + \frac{1}{2}|\sigma^{(1)}|^2 x_1^2 f_{x_1x_1} + + \frac{1}{2}|\sigma^{(2)}|^2 x_2^2 f_{x_2x_2} + \sigma^{(1)} \cdot \sigma^{(2)} x_1 x_2 f_{x_1x_2} - rf = 0$$

with the terminal condition $f(T, x_1, x_2) = (x_1 - x_2)_+$. Here $\sigma^{(1)} := (\sigma_1^{(1)}, \sigma_2^{(1)})$ and $\sigma^{(2)} := (\sigma_1^{(2)}, \sigma_2^{(2)})$.

Problem 5.7. Solve the following problems.

(i) (10 points) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Evaluate $\mathbb{E}(e^{\int_0^{T/2} t^2\, dB_t} \int_0^T t\, dB_t)$ and $\mathbb{E}(e^{\int_0^T t\, dB_t} \int_0^T B_t^2\, dt)$.

(ii) (5 points) Let $(B_t^{(1)}, B_t^{(2)})_{t\geq 0}$ be a two-dimensional Brownian motion. Find

$$\mathbb{E}(e^{\int_0^T t\, dB_t^{(1)} + \int_0^T t^2\, dB_t^{(2)}} \int_0^T B_t^{(1)}\, dB_t^{(2)})\,.$$

Problem 5.8. (15 points) In the Black-Scholes model, consider an option whose payoff is $X = (\ln S_{T/2})^2$ at maturity T. Find the time-t price and the hedging portfolio of this option.

Problem 5.9. (15 points) Assume the Black-Scholes market model. Let $K > 0$. Consider an option whose payoff is

$$(S_T^2(S_T^2 - K))_+$$

at maturity $T > 0$. Evaluate the time-0 price of this option. Hint: Use the Girsanov theorem.

References

Fabrice Baudoin. *Diffusion processes and stochastic calculus.* 2014.

Ioannis Karatzas, Steven E Shreve, I Karatzas, and Steven E Shreve. *Methods of mathematical finance*, volume 39. Springer, 1998.