Advanced Calculus I – Assignment 2

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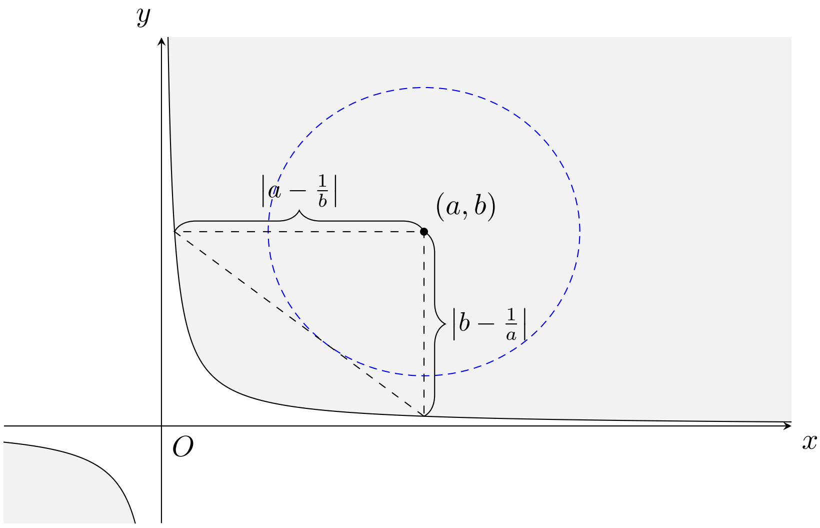
§2.1 #2. Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy > 1\}$. Show that S is open.

Proof. The set in question is the set of points "outside" the hyperbola $xy = 1$. We need to show that each point in S is surrounded by some small disk completely contained in S. This seems reasonably clear from Figure 2-2. Any radius shorter than the distance from our point to the closest point on the curve will do.



One could actually try to find the closest point on the hyperbola to the origin and its distance using the methods of beginning calculus. This would give the largest radius which will do. Any smaller radius will also work. Here is a geometric argument which shows that such a radius exists. It assumes that the branch of the hyperbola in the first quadrant is the graph of a decreasing function and is concave up. This is easily checked for $f(x) = 1/x$. The argument is based on the figures sketched in Figure 2-3.

$$\begin{aligned}\varepsilon &= \frac{|a - \frac{1}{b}| \cdot |b - \frac{1}{a}|}{\sqrt{(a - \frac{1}{b})^2 + (b - \frac{1}{a})^2}} = \frac{|ab - 2 + \frac{1}{ab}|}{\sqrt{a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} - \frac{2a}{b} - \frac{2b}{a}}} \\ &= \frac{|ab - 2 + \frac{1}{ab}|}{\sqrt{a^2 + b^2 + \frac{a^2+b^2}{a^2+b^2} - \frac{2}{ab}(a^2 + b^2)}} = \frac{ab - 1}{\sqrt{a^2 + b^2}}.\end{aligned}$$

□

§2.3 #5. Let $S = \{x \in \mathbb{R} \mid x \text{ is irrational}\}$. Is S closed?

Proof. $\square$

§2.4 #3. Find the accumulation points of the following sets in $\mathbb{R}^2$:

Proof. $\square$