Introduction to Differential Geometry I – Homework 9

Junwoo Yang

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Problem 3-3.5 Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\, v - \frac{v^3}{3} + vu^2,\, u^2 - v^2\right)$$

and show that

a. The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

Solution. We calculate

$$\begin{aligned}
\mathbf{x}_u &= (1 - u^2 + v^2,\, 2uv,\, 2u) \\
\mathbf{x}_v &= (2uv,\, 1 - v^2 + u^2,\, -2v)
\end{aligned}$$

Now we have

$$\begin{aligned}
E &= \mathbf{x}_u \cdot \mathbf{x}_u = (1 - u^2 + v^2)^2 + (2uv)^2 + (2u)^2 \\
&= 1 + u^4 + v^4 - 2u^2 + 2v^2 - 2u^2v^2 + 4u^2v^2 + 4u^2 \\
&= 1 + u^4 + v^4 + 2u^2 + 2v^2 + 2u^2v^2 = (1 + u^2 + v^2)^2 \\
F &= \mathbf{x}_u \cdot \mathbf{x}_v = (1 - u^2 + v^2) \cdot 2uv + 2uv(1 - v^2 + u^2) + 2u \cdot (-2v) \\
&= 4uv - 4uv = 0 \\
G &= \mathbf{x}_v \cdot \mathbf{x}_v = (2uv)^2 + (1 - v^2 + u^2)^2 + (-2v)^2 = \cdots = E
\end{aligned}$$

□

b. The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

Solution. We calculate

$$\begin{aligned}
\mathbf{x}_u \wedge \mathbf{x}_v &= \begin{vmatrix} i & j & k \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \end{vmatrix} \\
&= (-4uv^2 - 2u + 2uv^2 - 2u^3)i + (4u^2v + 2v - 2u^2v + 2v^3)j \\
&\quad + (1 - v^2 + u^2 - u^2 + u^2v^2 - u^4 + v^2 - v^4 + v^2u^2 - 4u^2v^2)k \\
&= (-2u(u^2+v^2+1),\, 2v(u^2+v^2+1),\, 1-(u^2+v^2)^2) \\
|\mathbf{x}_u \wedge \mathbf{x}_v|^2 &= 4u^2(u^2+v^2+1)^2 + 4v^2(u^2+v^2+1)^2 + (1-(u^2+v^2)^2)^2 \\
&= (u^2+v^2+1)^4 \\
N &= \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{(-2u,\, 2v,\, 1-u^2-v^2)}{u^2+v^2+1} \\
\mathbf{x}_{uu} &= (-2u,\, 2v,\, 2) \\
\mathbf{x}_{uv} &= (2v,\, 2u,\, 0) \\
\mathbf{x}_{vv} &= (2u,\, -2v,\, -2)
\end{aligned} \tag{1}$$

Thus, we get followings.

$$\begin{aligned}
e = \langle N, \mathbf{x}_{uu} \rangle &= \frac{4u^2 + 4v^2 + 2(1-u^2-v^2)}{u^2+v^2+1} = \frac{2(1+u^2+v^2)}{u^2+v^2+1} = 2 \\
f = \langle N, \mathbf{x}_{uv} \rangle &= \frac{-4uv + 4uv + 0 \cdot (1-u^2-v^2)}{u^2+v^2+1} = 0 \\
g = \langle N, \mathbf{x}_{vv} \rangle &= \frac{-4u^2 - 4v^2 - 2(1-u^2-v^2)}{u^2+v^2+1} = \frac{-2(1+u^2+v^2)}{u^2+v^2+1} = -2
\end{aligned}$$

□

c. The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

Solution. We first calculate

$$\begin{aligned}
K &= \frac{eg - f^2}{EG - F^2} = \frac{2 \cdot (-2) - 0^2}{(1+u^2+v^2)^4 - 0^2} = \frac{-4}{(1+u^2+v^2)^4} \\
H &= \frac{gE - 2fF + eG}{2(EG - F^2)} = \frac{(1+u^2+v^2)^2 \cdot 2 - 2 \cdot 0 \cdot 0 + (1+u^2+v^2)^2(-2)}{2(1+u^2+v^2)^4} = 0
\end{aligned}$$

Specially, the Enneper surface is minimal. Now from the relations

$$\begin{aligned}
K &= k_1 k_2 \\
H &= \frac{k_1 + k_2}{2}
\end{aligned}$$

we get a system of equations for k_1 and k_2. From $H = 0$ we get $k_2 = -k_1$. Now

$$K = -k_1^2 \Rightarrow \frac{-4}{(1+u^2+v^2)^4} = -k_1^2 = k_1 = \frac{\pm 2}{(1+u^2+v^2)^2}$$

and therefore

$$k_2 = -k_1 = \frac{\mp 2}{(1+u^2+v^2)^2}$$

Since the principal curvatures k_1 and k_2 can be switched (the order doesn't matter), we actually have that

$$k_1 = \frac{2}{(1+u^2+v^2)}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

□

d. The lines of curvature are the coordinate curves.

Solution. The curve $c(t) = \mathbf{x}(u(t), v(t))$, which lies on S, is a line of curvature if and only if u and v satisfy the differential equation

$$\begin{vmatrix} v'^2 & -u'v & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

Substituting the values for the Enneper surface we get

$$0 = \begin{vmatrix} v'^2 & -u'v' & u'^2 \\ (1+u^2+v^2)^2 & 0 & (1+u^2+v^2)^2 \\ 2 & 0 & -2 \end{vmatrix} = \text{(Lapl. exp. 2nd col.)}$$
$$= (-1)^4(-u'v') \begin{vmatrix} v'^2 & u'^2 \\ 2 & -2 \end{vmatrix} = -u'v'(-2v'^2 - 2u'^2) = 2u'v'(u'^2 + v'^2)$$

which is equivalent to $u' = 0$ or $v' = 0$ or $u'^2 + v'^2 = 0$. The last case implies $u' = v' = 0$, which is covered by first two cases. Now we have that $j' = 0 \Rightarrow u = \text{const.}$ or $v' = 0 \Rightarrow v = \text{const.}$, which are exactly the coordinate curves. □

e. The asymptotic curves are $u + v =$const., $u - v =$const.

Solution. The curve c is an asymproric curve if and only if u and v satisfy the differential equation

$$eu'^2 + 2fu'v' + gv'^2 = 0$$

Substituting the values for the Enneper surface we get

$$2u'^2 - 2v'^2 = 0 \Rightarrow 2(u' - v')(u' + v') = 0 \Rightarrow u' \pm v' = 0 \Rightarrow u \pm v = \text{const.}$$

□

Claim 1. $q \in S$ is a critical point of h if and only if $n \perp T_qS$ (i.e., $n = \pm N(q)$)

Proof. Suppose q is a critical point of h. Without loss of generality, assume $n = N(q)$. For any $v \in T_qS$, let $\alpha : (-\varepsilon, \varepsilon) \to S$ be s.t. $\alpha(0) = q$, $\alpha'(0) = v$. Then,

$$\begin{aligned} d^2h_q(v) &= \left.\frac{d^2}{dt^2}\right|_{t=0} h(\alpha(t)) = \left.\frac{d^2}{dt^2}\right|_{t=0} \langle \alpha(t) - p, n\rangle = \langle \alpha''(0), N(q)\rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle \alpha'(t), N(\alpha(t))\rangle - \langle v, dN_q(v)\rangle \\ &= \Pi_q(v) \end{aligned}$$

□

Claim 2. There is no compact regular surface with negative Gauss curvature everywhere.

Proof. □