Discrete-time models Instructor: Hyungbin Park

Lecture note 1: One-period models

References:
CH 2, 3 in Björk (2004)

1 Introduction

In this section, we consider a simple binomial model. Let

$$\Omega = \{u, d\}$$

and define a probability measure $\mathbb{P}$ on 2^{Ω} by

$$\mathbb{P}(\varnothing) = 0\,,\ \mathbb{P}(\{u\}) = p_u\,,\ \mathbb{P}(\{d\}) = p_d\,,\ \mathbb{P}(\Omega) = 1$$

where $0 < p_u, p_d < 1$ and $p_u + p_d = 1$. Running time is denoted by t, and we have two points in time, $t = 0$ (today) and $t = 1$ (tomorrow). There are two assets in the market. One is a bank account and the other is a stock.

Definition 1.1. *A bank account is a sequence of deterministic random variables $G_0, G_1 : \Omega \to \mathbb{R}$ given by*

$$\begin{aligned} G_0 &= 1 \\ G_1 &= 1 + R \end{aligned}$$

Here, R is the short interest rate.

Definition 1.2. *The stock price is a sequence of random variables $S_0, S_1 : \Omega \to \mathbb{R}$, and its dynamic behavior is described by*

$$S_0 = s$$
$$\begin{cases} S_1(u) = s_u \\ S_1(d) = s_d \end{cases}$$

where $s, s_d, s_u > 0$ and $s_d < s_u$.

Definition 1.3. *A portfolio is a vector $h = (x, y)$ in $\mathbb{R}^2$. The value process of the portfolio is defined by*

$$V_t^h = xG_t + yS_t\,,\ t = 0, 1\,.$$

Definition 1.4. *An arbitrage is a portfolion h such that*

$$\begin{aligned} &V_0^h = 0 \\ &V_1^h \geq 0 \quad \text{with probability 1} \\ &V_1^h > 0 \quad \text{with positive probability} \end{aligned}$$

Theorem 1.1. *The binomial model above is free of arbitrage if and only if*

$$\frac{s_d}{s} < 1 + R < \frac{s_u}{s}\,.$$

2 Option pricing

An option is a contract which gives the buyer a specified amount, depending on the value of the underlier, at a specified date. Options are characterized by the payoff and the maturity.

Definition 2.1. *An option payoff is a random variable*

$$X : \Omega \to \mathbb{R} .$$

One of the main purposes of this note is to price options.

Definition 2.2. *We say a portfolio h is the* hedging portfolio *or the* replicating portfolio *of an option X if*

$$V_1^h = X .$$

Theorem 2.1. *An arbitrage-free price of an option is V_0^h where h is the hedging portfolio of the option.*

3 Risk-neutral measures

Definition 3.1. *A risk-neutral measure is a probability measure $\mathbb{Q}$ on Ω such that*

$$S_0 = \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(S_1)$$

and $\mathbb{Q}(\{u\}) > 0$, $\mathbb{Q}(\{d\}) > 0$.

Theorem 3.1. *The binomial model above is arbitrage-free if and only if a risk-neutral measure exists. In this case,*

$$\mathbb{Q}(\{u\}) = \frac{(1+R)s - s_d}{s_u - s_d}\,, \; \mathbb{Q}(\{d\}) = \frac{s_u - (1+R)s_d}{s_u - s_d}$$

Theorem 3.2. *Consider an option with payoff X with maturity $t = 1$. The arbitrage-free price is*

$$\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X) .$$

4 Super-hedging duality

Consider a one-period ($t = 0$ or T) trinomial model. The initial stock price is $S_0 = s$, and there are three possible prices at T: $S_T = s_3$, $S_T = s_2$ and $S_T = s_1$, with probabilities p_u, p_m and p_d, respectively. Assume that $s_1 < s_2 < s_3$ and $p_u, p_m, p_d > 0$. The back account earns zero short interest rate.

In class, we studied that the super-hedging price of an option whose payoff is

$$X = \begin{cases} x_3 & \text{if } S_T = s_3 \\ x_2 & \text{if } S_T = s_2 \\ x_1 & \text{if } S_T = s_1 \end{cases}$$

at maturity T satisfies the super-hedging duality;

$$\inf\{\alpha+\beta s \,|\, X \le \alpha+\beta S_T\} = \sup\{\mathbb{E}^{\mathbb{Q}}(X) \,|\, \mathbb{Q} \text{ is a risk-neutral measure}\}\,.$$

The proof is as follows

$$\begin{aligned}
\inf_{\substack{x_i\le\alpha+\beta s_i\\ i=1,2,3}} \alpha+\beta s &= \inf_{\alpha,\beta}\sup_{p_i>0} \alpha+\beta s+\sum_{i=1}^{3} p_i(x_i-\alpha-\beta s_i)\\
&= \sup_{p_i>0}\inf_{\alpha,\beta} \alpha+\beta s+\sum_{i=1}^{3} p_i(x_i-\alpha-\beta s_i)\\
&= \sup_{p_i>0}\inf_{\alpha,\beta} \alpha(1-\sum_{i=1}^{3} p_i)+\beta(s-\sum_{i=1}^{3} p_i s_i)+\sum_{i=1}^{3} p_i x_i\\
&= \sup_{\substack{p_i>0\\ \sum_{i=1}^{3} p_i=1\\ \sum_{i=1}^{3} p_i s_i=s}} \sum_{i=1}^{3} p_i x_i
\end{aligned} \tag{4.1}$$

We have four equalities in these equations. The "inf sup = sup inf" in the second equality is not trivial and can be proven by using the "linear programing".

5 Exercises

Problem 5.1. Consider the binomial model

$$R=0.2\,,\ s=110\,,\ s_u=144\,,\ s_d=96\,,\ p_u=0.6\,,\ p_d=0.4\,.$$

(i) (5 points) Price and hedge a call option with strike price $K=100$ and maturity $t=1$.

(ii) (5 points) Find the risk-neutral measure, and evaluate the price of this option by using this risk-neutral measure

Problem 5.2. Consider the one-period trinomial model: $s=95$, $s_u=150$, $s_m=125$, $s_d=100$, $R=0.25$, $p_u=0.2$, $p_m=0.2$, $p_d=0.6$.

(i) (5 points) Define $\Omega=\{u,m,d\}$ and let $\mathbb{P}$ be the probability measure on 2^{Ω} such that $\mathbb{P}(\{u\})=0.2$, $\mathbb{P}(\{m\})=0.2$, $\mathbb{P}(\{d\})=0.6$. Define bank accounts G_0, G_1 and stock prices S_0, S_1 on this space.

(ii) (5 points) Show a risk-neutral measure $\mathbb{Q}$ exists, but is not unique. Give two examples of $\mathbb{Q}$.

(iii) (5 points) Find the super-hedging price of the option with payoff

$$X=\begin{cases}80 & \text{if } S_1=150\\ 40 & \text{if } S_1=125\\ 0 & \text{if } S_1=100\end{cases}$$

at maturity $t=1$.

(iv) (10 points) Calculate

$$\sup\left\{\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X)\,\middle|\,\mathbb{Q} \text{ is a risk-neutral measure}\right\}$$

and confirm that the superhedging duality holds. Can you find a risk-neutral measure $\mathbb{Q}$ which achieves the supremum?

(v) (10 points) Let $\mathcal{P}$ be set of all probability measures $\mathbb{Q}$ on 2^{Ω} such that $S_0 = \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(S_1)$ (not necessarily to satisfy $\mathbb{Q}(\{u\}) > 0$, $\mathbb{Q}(\{m\}) > 0$, $\mathbb{Q}(\{d\}) > 0$). Calculate

$$\sup\left\{\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X)\,\middle|\,\mathbb{Q} \in \mathcal{P}\right\}$$

and confirm that this is equal to the superhedging price. Find the probability measure $\mathbb{Q} \in \mathcal{P}$ which achieves the supremum.

(vi) (5 points) Let $\mathcal{M}$ be the set of all signed-measures on 2^{Ω} (easy to check that this space $\mathcal{M}$ is a vector space over $\mathbb{R}$). Show that $\mathcal{P}$ is a convex subset of $\mathcal{M}$.

Problem 5.3. (15 points) In class, we merely checked that the superhedging duality holds for a specific example. The proof of the superhedging duality is in Eq.(4.1). Explain why these equalities hold except for the "inf sup = sup inf" in the second equality.

References

Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2004.