# 실변수함수론

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# Lecture 1. motivation of measure theory.

측도론 (measure theory)
힐버트 공간 (Hilbert space theory). | 현대 해석학의 근간.

## (1). Fourier series.

when $f$ is continuously differentiable function on $[-\pi, \pi]$, we have $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$
(the Fourier series of $f$).

where $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

복소지수함수의 선형 합수.
$e^{inx} = \cos(nx) + i\sin(nx)$.

(Parserval's identity)

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty \not\Rightarrow \exists f$ s.t $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ and
$f$ is continuously differentiable

Q1-1. f를 다른 방식으로 표현.

Q1-2. How to integrate $f$ ?

## (2) Limits of continuous function.

$\{f_n\}$ : a sequence of continuous functions on $[0,1]$

assume $\lim_{n\to\infty} f_n(x) = f(x)$. exists for every $x$.

If $f_n \to f$ (uniformly convergence)., $f$ is continuous.

If not, $f$ may not be even Riemann integrable.

Q2. Is there a way to integrate $f$ and guarantee
$\int_0^1 f(x)dx = \lim_{n\to\infty}\int_0^1 f_n(x)dx$ ?

## (3). Length of curves.

$\Gamma$ : continuous curve in $\mathbb{R}^2$

$\Gamma = \{(x(t), y(t)) : a \le t \le b\}$  $x, y$ : continuous fts of $t$.

$\Gamma$ is said to be rectifiable if its lengh $L$ is finite.

If $x(t), y(t)$ are continuously differentiable, $\Gamma \in C^1$

$L = \int_a^b \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2}\, dt.$

Q.3a. What are the conditions on $x(t), y(t)$ that guarantee the rectifiability of $\Gamma$ ?

Q.3b. When does the above formula hold ?

## (4). Differentiation and integration

(Fundamental theorem of calculus).

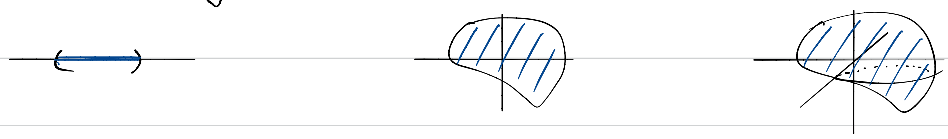
If $f$ is a continuous ft on $[a, b]$, $F$ is an antiderivative of $f$ in $[a, b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

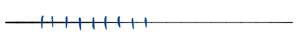
Q4. There are a lot of examples of Riemann integrable fts. that are non-continuous. Can we apply the above formula to them ?

## (5) The problem of measure.

$E \subset \mathbb{R}^n$, $m(E)$ : measure, $n$-dimensional volume

$(n=1)$ : length  $(n=2)$ : area.  $(n=3)$ : volume.





유리수. $\int_E 1(x)\,dx = m(E)$.

non-negative ft.  $m(\mathbb{R}) = +\infty$

Natural condition.

- $E = [a, b]$ $(a \le b)$.  $m(E) = b - a$
- $E = \bigcup_{n=1}^{\infty} E_n$, $E_n$ mutually disjoint, $m(E) = \sum_{n=1}^{\infty} m(E_n)$.
- For $\forall h \in \mathbb{R}$, $m(E) = m(E + h)$.  translation invariance

Q5. The existence and uniqueness of such a measure.

Vitali set. 위에 언급한 세가지 성질을 모두 유지한 채로 실직선의 모든 부분집합의 측도를 정의하는 것이 불가능 하다.

Lebesgue ; possible defining a measure on "measurable" sets.

$\Rightarrow$ If $E_1, E_2, \cdots$ measurable sets, $E = E_1 \cup E_2 \cup \cdots$ measurable.

And if $E_1 \cap E_2 \cap \cdots = \phi$, $m(E) = \sum_{n=1}^{\infty} m(E_n)$

Notation.

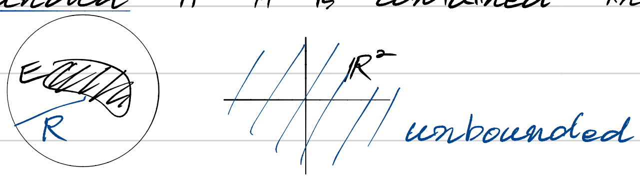
- A point $x \in \mathbb{R}^d$, $x = (x_1, x_2, \cdots, x_d)$, $x_i \in \mathbb{R}$ for $i = 1, \cdots, d$
- The norm of $x$ is $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$
- The distance between two points $x, y$ is $|x - y|$
- The distance between two sets $E$ and $F$ is

  $d(E, F) = \inf \{ |x-y| : x \in E, y \in F \}$.
- The open ball in $\mathbb{R}^n$ centered at $x$ and radius of $r$ is

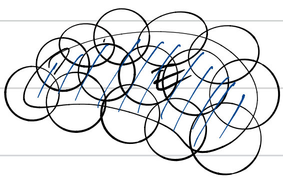
  $B_r(x) = \{ y \in \mathbb{R}^d : |y - x| < r \}$. $\leq$ : closed ball
- A subset $E \subset \mathbb{R}^d$ is open if $^\forall x \in E$, $^\exists r > 0$ s.t $B_r(x) \subset E$
- A set is closed if its complement is open.
- Arbitrary union of open sets is open.

  Finite intersection of open sets is open.

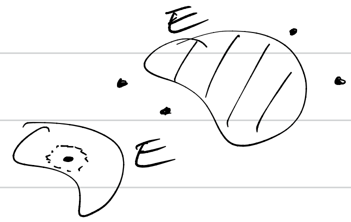
  (countable intersection of open sets may open, may not open).
- A set $E$ $(\in \mathbb{R}^d)$ is bounded if it is contained in some ball of finite radius.




- A set $E$ is compact if any covering of $E$ by a collection of open sets contains a finite subcovering.



- (Heine - Borel theorem).
  A subset $E \subset \mathbb{R}^d$ is compact if and only if it is closed and bounded
- A point $x \in \mathbb{R}^d$ is a limit point of the set $E$ if $\forall r > 0$, $B_r(x)$ contains points of $E$.
- An isolated point of $E$ is a point $x \in E$ s.t there exists an $r > 0$ where $B_r(x) \cap E = \{x\}$
- A point $x \in E$ is an interior point of $E$ if $\exists r > 0$ s.t $B_r(x) \subset E$




- The set of all interior points of $E$ is called the interior of $E$
- The closure $\bar{E}$ of $E$ consist of the union of $E$ and all its limit pts.
- The boundary $\partial E$ of $E$ is the set of pts which are in the closure of $E$ but not in the interior of $E$
- The closure of a set is a closed set.
- Every pt in $E$ is a limit pt of $E$
- A set is closed if and only if it contains all its limit points.

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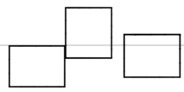
# Lecture 2. Lebesgue measure.

- A closed rectangle $R$ in $\mathbb{R}^d$ is $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ where $a_j \le b_j$ $(j=1, \cdots, d)$ are real numbers.

- Its volume is $|R| = (b_1 - a_1) \cdots (b_d - a_d)$.

- A cube is a rectangle for which $b_1 - a_1 = \cdots = b_d - a_d = \ell$
  $|R| = \ell^d$

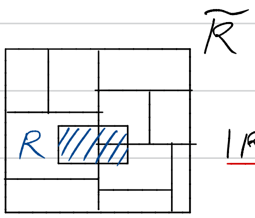
- A union of rectangles is <u>almost disjoint</u> if the interiors of the rectangles are disjoint.



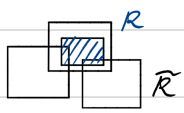
Lemma

(1). If a rectangle is almost disjoint union of finitely many other rectangle, say $R = \bigcup_{k=1}^{N} R_k$.
then $|R| = \sum_{k=1}^{N} |R_k|$

(2) If $R \subset \bigcup_{k=1}^{N} R_k$
(here the union of rectangles need not to be almost disjoint). then,
$|R| \le \sum_{k=1}^{N} |R_k|$.




$|R| \le |\tilde{R}| \le \sum_{k=1}^{N} |R_k|$




Theorem

Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be uniquely written as a countable union of disjoint open intervals.

pf). For each $x \in \mathcal{O}$, $I_x := (a_x, b_x)$, where $a_x = \inf\{a < x \mid (a, x) \subset \mathcal{O}\}$
$b_x = \sup\{b > x \mid (x, b) \subset \mathcal{O}\}$. For $x, y \in \mathcal{O}$, if $I_x \cap I_y \ne \phi$,
$I_x \cup I_y \subset I_x$, $I_x \cup I_y \subset I_y$ $\therefore I_x = I_y$ $\therefore$ distinct interval.
any interval contains rational #. $\Rightarrow$ countable.

- If $O$ is open, there are disjoint open intervals $\{I_j\}$ such that $O = \bigcup_{j=1}^{\infty} I_j$. The measure of $O$ should be $\sum_{j=1}^{\infty} |I_j| = |O|$. It is well-defined since the decomposition is unique.

Theorem.

Every open subset $O$ of $\mathbb{R}^d$, $d \geq 1$ can be uniquely. (X). written as a countable union of almost disjoint closed cubes.

$\forall x \in O$, $\exists$ cube with length $2^{-N} \subset O$. union of all cube covers $O$.

- If $O$ is open, we can write $O = \bigcup_{j=1}^{\infty} R_j$ with almost disjoint rectangles $\{R_j\}$. The measure of $O$ should be $\sum_{j=1}^{\infty} |R_j| = |O| \overset{\checkmark}{=} \sum_{j=1}^{\infty} |\tilde{R}_j|$

- However, the above decomposition is not unique and the sum may be dependent on this decomposition.

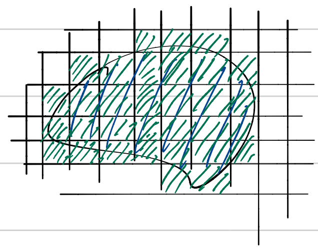
Definition.

If $E$ is any subset of $\mathbb{R}^d$, the exterior measure of $E$ is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

= outer measure.

where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.



- Various examples show that this notion coincides with our earlier intuition.
- However, it lacks the desirable property of additivity when taking the union of disjoint sets.

$E = E_1 \cup E_2$, $E_1 \cap E_2 = \phi$.

$\not\Rightarrow m_*(E) = m_*(E_1) + m_*(E_2)$

countable sets.

- It would not suffice to allow finite sums in the definition of $m_*(E)$. The quantity that would be obtained if one considered only coverings of $E$ by finite union of cubes is in general larger than $m_*(E)$.
- One can replace the coverings by cubes, with coverings by rectangles; or with coverings by balls.

$0 \le m_*(\{point\}) \le |Q| \rightarrow 0.$

Ex) · The exterior measure of a point is zero.

· The exterior measure of a closed cube is equal to its volume.
Indeed, for an arbitrary covering $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by cubes,
it holds that $|Q| \le \sum_{j=1}^{\infty} |Q_j|$ $\quad m_*(Q) \overset{=}{\le} |Q|$

비교: Lemma 2

· If $Q$ is an open cube, the result $m_*(Q) = |Q|$ still holds.

· The exterior measure of a rectangle $R$ is equal to its volume.

· The exterior measure of $\mathbb{R}^d$ is infinite.

Properties.

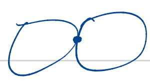
- $\forall \varepsilon > 0$, $\exists$ a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with $\sum_{j=1}^{\infty} m_*(Q_j) \le m_*(E) + \varepsilon$

$m_*(E) = \inf \sum_j |Q_j| = \inf \sum_j m_*(Q_j)$

- If $E_1 \subset E_2$, then $m_*(E_1) \le m_*(E_2)$ Monotonicity.

- If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \le \sum_{j=1}^{\infty} m_*(E_j)$.

- If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf\{m_*(O) : E \subset O \text{ open}\}$



성립X.

- If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$

- If a set $E$ is the countable union of almost disjoint cubes
$E = \bigcup_{j=1}^{\infty} Q_j$, then $\underbrace{m_*(E)}_{=|O|} = \sum_{j=1}^{\infty} |Q_j| = \sum_{j=1}^{\infty} |\tilde{Q}_j|$ $\quad O = \bigcup_{j=1}^{\infty} |Q_j| = \bigcup_{j=1}^{\infty} |\tilde{Q}_j|$

Therefore, the exterior measure of an open set equals the sum of the volumes of the cubes in a decomposition.
In particular, the sum is independent of the decomposition.

Measurable sets.

- Nonetheless, one cannot conclude in general that if $E_1 \cup E_2$ is a disjoint union of subsets of $\mathbb{R}^d$, then
  $m_*(E) = m_*(E_1) + m_*(E_2)$. Vitali : not measurable.

- A subset $E$ of $\mathbb{R}^d$ is (Lebesgue) measurable if for any $\varepsilon > 0$, there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and
  $m_*(\mathcal{O} - E) = m_*(\mathcal{O} \cap E^c) \le \varepsilon$

- If $E$ is measurable, we define its (Lebesgue) measure $m(E)$ by
  $m(E) = m_*(E)$.
  The Lebesgue measure inherits all the features of the exterior measure.
  $m_*(E) \neq m_*(E_1) + m_*(E_2)$, $m(E) = m(E_1) + m(E_2)$.
  $E_1, E_2, E_1 \cup E_2$ measurable.

Lemma
The following sets in $\mathbb{R}^d$ are (Lebesgue) measurable :

- open sets. : It immediately follows from the def.
  $m_*(\mathcal{O} - E) = m_*(E - E) = m_*(\emptyset) = 0 \le \varepsilon$

- null sets $E$, i.e., sets $E$ with $m_*(E) = 0$.
  $E_1 \subset E_2 \Rightarrow \underset{\overset{\Downarrow}{0.}}{0 \le m_*(E_1)} \le m_*(E_2) = 0$.

  $m_*(E) = 0 = \inf\{m_*(\mathcal{O}) : \mathcal{O} \supset E \text{ open}\}$ $\quad\varepsilon > m_*(\mathcal{O}) \ge m_*(\mathcal{O} - E)$
  For any $\varepsilon > 0$, $\exists$ open set $\mathcal{O} \supset E$, $m_*(\mathcal{O}) < \varepsilon$. Hence $m_*(\mathcal{O} - E) \le \varepsilon$

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- subsets of null sets.
- countable unions of measurable sets.
- closed sets.
- complements of measurable sets. $E^c$, $\mathbb{R}^d - E$
- countable intersections of measurable sets.

  It holds that $\bigcap_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j^c\right)^c$

## Lecture 3. Measurable sets and functions

### Additivity.

Theorem.

If $E_1, E_2, \cdots$ are disjoint measurable sets and $E = \bigcup_{j=1}^{\infty} E_j$, then $m(E) = \sum_{j=1}^{\infty} m(E_j)$.

pf). Suppose that each $E_j$ is bounded. $E_j^c$ : measurable.

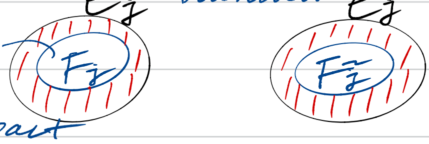
There is an open set $O_j \supset E_j^c$ s.t $m(O_j - E_j^c) \le \frac{\varepsilon}{2^j}$

A closed set $F_j = O_j^c$ satisfies $E_j - F_j = O_j - E_j^c$.

$E_j$ : bounded $E_j^c$

closed

$\Rightarrow$ compact



$E_j$ disjoint $\rightarrow$ $F_j$ disjoint.

$1 \le i, j \le N$

Since $F_1, \cdots, F_N$ are compact and disjoint, $\Rightarrow d(F_i, F_j) > 0$

$m(E) \ge m\left(\bigcup_{j=1}^{N} F_j\right) = \sum_{j=1}^{N} m(F_j) \ge \sum_{j=1}^{N} m(E_j) - \varepsilon$

$\left(E = \bigcup_{j=1}^{\infty} E_j \supseteq \bigcup_{j=1}^{\infty} F_j\right)$. $(d(F_i, F_j) > 0)$.

$m(F_j) = m(O_j^c)$ $\quad E_j - F_j$

$m(E_j) \le m(O_j - E_j^c) + m(F_j)$

$\Rightarrow \sum_{j=1}^{N} m(E_j) \le \sum_{j=1}^{N} m(F_j) + \underbrace{\sum_{j=1}^{N} \frac{\varepsilon}{2^j}}_{\le \varepsilon} \Rightarrow \sum_{j=1}^{N} m(F_j) \ge \sum_{j=1}^{N} m(E_j) - \varepsilon$

Take $N \to \infty$, $\varepsilon \to 0$

$\Rightarrow m(E) \ge \sum_{j=1}^{\infty} m(E_j)$

The reverse inequality holds for the exterior measure.

$m(E) \le \sum_{j=1}^{\infty} m(E_j)$. $\quad \therefore m(E) = \sum_{j=1}^{\infty} m(E_j)$.