

# Advanced Calculus 2 – Final Exam

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1. State the definition of a complex inner product space.
2. State the definition of  $L^2$  norm of a function  $f: A \subset \mathbb{R} \rightarrow \mathbb{R}$ .
3. Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous and differentiable on  $(a, b)$ . Assume  $f(a) = 0$ ,  $f(b) = -1$ , and  $\int_a^b f(x) dx = 0$ . Prove that there is a  $c \in (a, b)$  such that  $f'(c) = 0$ .
4. Let  $\mathcal{V}$  be an inner product space and  $\phi_0, \phi_1, \dots, \phi_n$  a set of orthonormal vectors in  $\mathcal{V}$ . Prove that for each set of numbers  $t_0, t_1, \dots, t_n$ ,

$$\left\| f - \sum_{k=0}^n t_k \phi_k \right\| \geq \left\| f - \sum_{k=0}^n \langle f, \phi_k \rangle \phi_k \right\|.$$

5. Suppose that the sets  $A_1, A_2, \dots$  have measure zero in  $\mathbb{R}^n$ . Prove that  $A_1 \cup A_2 \cup \dots$  has measure zero in  $\mathbb{R}^n$ .
6. Let  $f: [a, b] \times [c, d] \rightarrow \mathbb{R}$  be a  $C^1$  function and  $u, v: [c, d] \rightarrow [a, b]$  be  $C^1$  functions. Suppose

$$F(t) = \int_{u(t)}^{v(t)} f(x, t) dx.$$

Find  $F'(t)$ .

7. Let  $f(x) = |\sin x|$ . Find the Fourier series of  $f$ .  
(Hint:  $f$  is even and  $\pi$  periodic.  $\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}$ )
8. Let  $f, g: A \subset \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions. Prove that

$$\int_A |f(x)g(x)| dx \leq \left( \int_A |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_A |g(x)|^q dx \right)^{\frac{1}{q}}$$

for  $p, q \in (1, \infty)$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ .