Modern Algebra I – Midterm Exam

April 24, 2019

1. Show that a group G with identity e and such that $x^2 = e$ for all $x \in G$ is Abelian.
2. The set $GL(n, \mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \,|\, \det(A) \neq 0\}$ under matrix multiplication is a group. Let $M = \{A \in GL(n, \mathbb{R}) \,|\, A^{\mathrm{T}}A = I_n\}$. Show that M is a subgroup of $GL(n, \mathbb{R})$. (Hint: If $A \in M$, then $A^{\mathrm{T}} = A^{-1}$.)
3. In $\mathbb{Z}_{40}$,
   (1) compute the order of 28.
   (2) how many generators does $\mathbb{Z}_{40}$ have?
   (3) find all the subgroup of $\mathbb{Z}_{40}$ and draw their subgroup diagram (or lattice).
4. Compute 8^{41} mod 13.
5. Let
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}.$$
   (1) compute the order of α and β.
   (2) write $\alpha\beta$ as product of 2-cycles.
6. Prove that every group of prime order is cyclic.
7. Show that $\mathbb{R} \approx \mathbb{R}^+$.
8. Prove that $\mathrm{Aut}(\mathbb{Z}_8) \approx \mathrm{Aut}(\mathbb{Z}_{12})$.
9. Let G be a group. Prove that if G has only finitely many subgroups, then G must be a finite group.