2020-2 Probability : Final Exam

1. The joint density function of X and Y is given by
$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & \text{if } 0 < x, y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$
Compute $P(X > 1,\ Y < 1)$ and $P(X < Y)$.

2. If X, Y have joint density $f_{X,Y}$ given by $f_{X,Y}(x,y) = \begin{cases} e^{-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise,} \end{cases}$
compute the density f_{X+Y} of $X+Y$.

3. Let $A_n = \begin{cases} (-\frac{1}{n}, 1], & \text{if } n \text{ is odd,} \\ (-1, \frac{1}{n}], & \text{if } n \text{ is even.} \end{cases}$ Find $\limsup_n A_n$ and $\liminf_n A_n$.

4. Find the characteristic functions of the random variable X with $P(X = k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, $k = 0, 1, 2, 3, \ldots$.

5. Let X_n $(n = 1, 2, \ldots)$ be independent random variables, each uniformly distributed over $[0,1]$. Calculate $\mathbb{E}(S_n)$ and $\mathrm{Var}(S_n)$ where $S_n = X_1 + \cdots + X_n$.

6. Let X and Y be independent Gaussian random variables with expectation 0 and variance 1. Find the value T with $P(3X + 2Y > 5) = 1 - \Phi(T)$ where $\Phi(t) = \displaystyle\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$.

7. Suppose that $X_1, X_2, \ldots$ are independent identically distributed. If $Y_n = X_n \mathbf{1}_{\{\omega : |X_n(\omega)| \le n\}}$ is a truncated random variable, show that $\displaystyle\sum_{n=1}^{\infty} P(Y_n \ne X_n) \le \mathbb{E}(|X_1|)$.

8. Suppose that $X_1, \ldots, X_n$ are independent random variables. Show that for any $a \ge 0$,
$$P\Big(\max_{1 \le k \le n} |S_k| \ge 3a\Big) \le 3 \max_{1 \le k \le n} P(|S_k| \ge a)$$
where $S_k = X_1 + \cdots + X_k$.