Mathematical Statistics 1

Ch.1 Probability

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Basic concepts

0.1 Sample space

Definition

The set, $\mathcal{S}$, of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

Examples

- Toss a coin. $S = \{x \mid x \geq 0\}$
- Toss a dice. $S = \{A, B, AB, O\}$
- Toss two dice. 36
- Reaction time to a certain stimulus. $S = \{t \mid t \geq 0\}$

0.2 Event

↳ subset of sample space

Definition

An *event* is a part of the collection of possible outcomes of an experiment, that is, any subset of $\mathcal{S}$ (including $\mathcal{S}$ itself).

Examples

Let's toss a dice.

$A = \{2, 4, 6\} \subset S$

- A is the event that the number from the dice is even.
- B is the event that the number from the dice is a multiple of 3.

0.3 Random Variable

Definition

A *random variable* is a function from a sample space $\mathcal{S}$ into the real numbers.

Examples

Let X be the random variable.

(a) Toss two dice. X=sum of the numbers

(b) Toss a coin 5 times. X=number of heads in 5 tosses.

Ch.1.1 Properties of Probability

1.1 Set operations

Let A and B be two events (or sets).

- $\varnothing$: null or empty set
- Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x | x \in A \text{ and } x \in B\}$
- Complementation: $A^c = \{x | x \notin A, x \in \mathcal{S}\}$

Theorem

For any three events, A, B,and C, defined on a sample space $\mathcal{S}$,

- Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$,
  $$A \cap (B \cap C) = (A \cap B) \cap C$$
- Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
  $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
- DeMorgan's Laws: $(A \cap B)^c = A^c \cup B^c$
  $$(A \cup B)^c = A^c \cap B^c$$

$A_1, A_2, A_3, \ldots, A_k$ is a collection of sets, all defined on a sample space $\mathcal{S}$, then

- $\cup_{i=1}^{k} A_i = \{x \in \mathcal{S} : x \in A_i, \text{ for some } i, i = 1, \ldots, k\}$
- $\cap_{i=1}^{k} A_i = \{x \in \mathcal{S} : x \in A_i, \text{ for all } i, i = 1, \ldots, k\}$

1.2 disjoint/partition

Definition of disjoint

Two events A and B are *disjoint* (or mutually exclusive) if $A \cap B = \varnothing$. The events $A_1, A_2, \ldots, A_k$ are *pairwise disjoint* (or mutually exclusive) if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Example

Let $\mathcal{S} = [0, 5)$ be the sample space.

Events $A_i = [i, i+1),\ i = 0, 1, \ldots, 4$ are given.

Definition of partition

If $A_1, A_2, \ldots, A_k$ are pairwise disjoint and $\cup_{i=1}^{k} A_i = \mathcal{S}$, then the collection $A_1, A_2, \ldots, A_k$ forms a *partition* of $\mathcal{S}$.

1.3 Probability

Definition 1

Let $\mathcal{S}$ be a finite sample space. Assume that all the outcomes in $\mathcal{S}$ are equally likely. Suppose that $\mathcal{S} = \{s_1, \cdots, s_N\}$ and $P(\{s_i\}) = 1/N$.

any event ; subspace of S

$$P(A) = \sum_{s_i \in A} P(\{s_i\}) = \sum_{s_i \in A} \frac{1}{N} = \frac{\#\text{ of elements in } A}{\#\text{ of elements in } S}$$

Example

Let's toss a dice. $\mathcal{S} = \{1, 2, \cdots, 6\}$. $A = \{2, 4, 6\}$.

$$P(A) = 3/6.$$

$P : \overset{\text{set}}{K} \to \mathbb{R}$) function.
$X : S \to \mathbb{R}$

Definition 2 (Axioms of Probability)

Let A be an event in the sample space $\mathcal{S}$. Probability is a real-valued set function P that satisfies

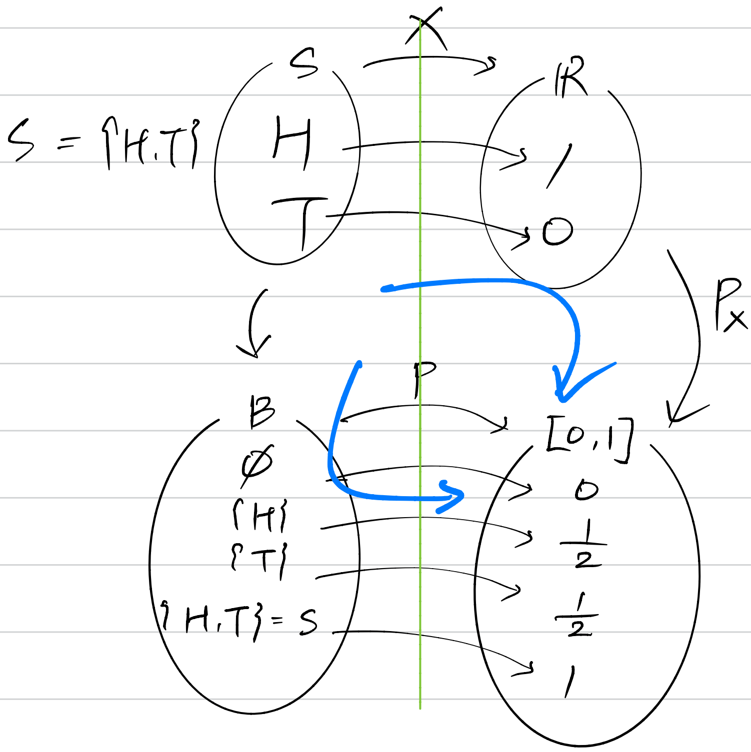
- $P(A) \geq 0$
- $P(\mathcal{S}) = 1$
- If $A_1, A_2, \ldots$ are pairwise disjoint events, then

① $A_i \cap A_j = \phi$ for any $i \neq j$
② $\cup_{i=1}^{k} A_i = S$

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

$\mathcal{B}$ (σ-algebra or Borel field)

: collection of subsets of the sample space S



data → number.

real world → mathematics world

$P(\{H\}) = P(X=1) = \frac{1}{2}$

$P : \mathcal{B} \to [0, \ ]$

비워둔 이유 : upper limit을 아직 정의하지 않았기 때문.

$\mathcal{B} = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$

$P(\emptyset) = 0 \quad \dfrac{\#\text{ of ele in }\emptyset}{\#\text{ of ele in } S}$

$P(\{H,T\} = S) = 1$

$P(\{H\}) = \frac{1}{2} = P_X(X=1)$

$P(\{T\}) = \frac{1}{2} = P_X(X=0)$

$A_1 = \{H\}$

$A_2 = \{T\}$

$P(A_1 \cup A_2) = P(\{H,T\}) = 1$

$P_X : \mathbb{R} \to [0,1]$

* $P : \mathcal{B} \to [0, \ ]$

$P_X : \mathbb{R} \to [0, \ ]$)

(domain이 달라도)

data set이 달라도 random valuable을 따르면 (P_X)

같은 형태로 표기 가능

Theorem 1.1-1,2,3,4, and 5

If P is a probability function and A and B are any events, then

- $P(A^c) = 1 - P(A)$
- $P(\varnothing) = 0$
- If $A \subset B$, then $P(A) \leq P(B)$
- $P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1) Since $A \cup A^c = S$ and A and A^c are disjoint

$$\underset{\text{by 2) condition of prob}}{1 = P(S)} = \underset{\text{by 3) condition of prob}}{P(A \cup A^c) = P(A) + P(A^c)}$$

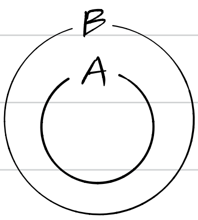
$$P(A^c) = 1 - P(A)$$

2) Let $A^c = \phi$ Then $A = S$

In 1), $P(A^c) = P(\phi) = 1 - P(A) = 1 - P(S) = 1 - 1 = 0$

3) $A \subset B \Rightarrow P(A) \le P(B)$

pf)



$$B = \underbrace{A \cup (A^c \cap B)}_{\text{disjoint}}$$

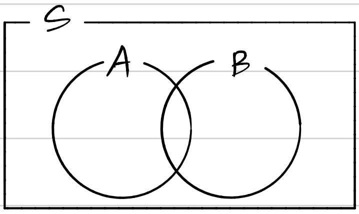
Then $P(B) = P[A \cup (A^c \cap B)] = P(A) + \overset{\ge 0}{P(A^c \cap B)} \ge P(A)$

$\therefore P(B) \ge P(A)$

4) Let $B = S$ in 3) $P(A) \le P(B) = P(S) = 1$

$\therefore P(A) \le 1 \Rightarrow 0 \le P(A) \le 1$

5.



i) $A \cup B = \underbrace{A \cup (A^c \cap B)}_{\text{disjoint}}$

$P(A \cup B) = P(A) + P(A^c \cap B)$

ii) $B = \underbrace{(A \cap B) \cup (A^c \cap B)}_{\text{disjoint}}$

대입

$\therefore P(B) = P(A \cap B) + P(A^c \cap B)$ $\Rightarrow$ $P(B) - P(A \cap B) = P(A^c \cap B)$

By ① & ② $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Key seperate two disjoint ~~part~~ event

Theorem

If P is a probability function and A and B are any events, then

- $P(B \cap A^c) = P(B) - P(A \cap B) = P(A \cup B) - P(A)$
- (Bonferroni's Inequality)

$$P(A \cap B) \geq P(A) + P(B) - 1$$

Since $(A \cup B) \subset S$,

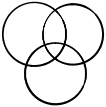
$$P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq P(S) = 1$$

$$P(A \cap B) \geq P(A) + P(B) - 1$$

Example 1.1-4

A faculty leader was meeting two students in Paris, on arriving by train from Amsterdam and the other arriving by train from Brussels at approximately the same time. Let A and B be the events that the respective trains are on time. Suppose we know from past experience that $P(A) = 0.93, P(B) = 0.89$, and $P(A \cap B) = 0.87$. Find the probability that at least one train is on time.

$A \cup B$

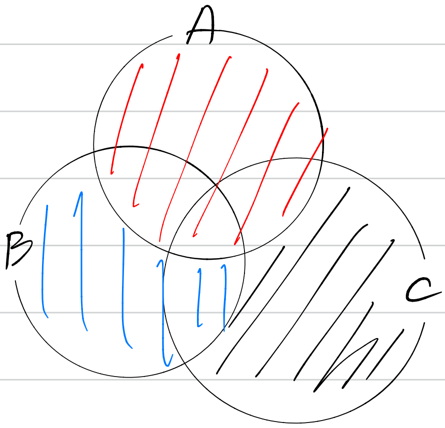


Theorem 1.1-6

If A, B, and C are any three events, then

$$\begin{aligned} P(A \cup B \cup C) = \quad & P(A) + P(B) + P(C) - P(A \cap B) \\ & - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

OH : 화, 목 11-12



$$A \cup B \cup C = \underline{A} \cup \underline{(B \cap A^c)} \cup \underline{(C \cap (A \cup B)^c)}$$

pairwise disjoint

$$P(A \cup B \cup C) = P(A) + P(B \cap A^c) + P(C \cap (A \cup B)^c)$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P(C \cap (A \cup B)) \quad \cdots ①$$

Since $C \cap (A \cup B) = \overbrace{(C \cap A)}^{D} \cup \overbrace{(C \cap B)}^{E}$

$$P(C \cap (A \cup B)) = \underbrace{P(C \cap A)}_{D} + \underbrace{P(C \cap B)}_{E} - \underbrace{P(A \cap B \cap C)}_{D \cap E} \quad \cdots ②$$

by ① & ②,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$\begin{cases} P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ P(A \cap B^c) = P(A) - P(A \cap B) \end{cases}$$

Ch.1.2 Methods of Enumeration

2.1 permutation/combination

$n(n-1)\cdots(n-r+1)$ $\quad {}_{10}P_3 = 10\cdot 9\cdot 8$

Definition

- ${}_n\mathrm{P}_r$: permutation of n objects taken r at a time
- ${}_n\mathrm{C}_r = \binom{n}{k} = \frac{n!}{r!(n-r)!}$: combination of n objects taken r at a time

$${}_nC_r = \frac{n!}{r!.(n-r)!}$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

2.2 Sampling

Definition

- *Sampling with replacement* occurs when an object is selected and then replaced before the next object is selected.
- *Sampling without replacement* occurs when an object is not replaced after it has been selected.