Topology II – Homework 1

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Exercise 77.1. Let X be a space obtained by pasting the edges of a polygonal region together in pairs. Show that X is homeomorphic to exactly one of the spaces in the following lists:

(a) $S^2, P^2, K, T_n, T_n\#P^2, T_n\#K$, where K is the Klein bottle and $n \geq 1$.

(b) $S^2, T_n, P^2, K_m, P^2\#K_m$, where K_m is the m-fold connected sum of K with itself and $m \geq 1$.

Proof. By the classification theorem, X is homeomorphic to exactly one of S^2, T_n, and $(P^2)_n$ with $n \geq 1$. Both lists contain S^2, P^2, and T_n with $n \geq 1$. Thus, we just need to check that the remaining surfaces on the lists are homeomorphic to $(P^2)_n$ with $n \geq 2$. Let us first show that the Klein bottle is homeomorphic to $P^2\#P^2$. For this, consider the following cut and paste procedure:

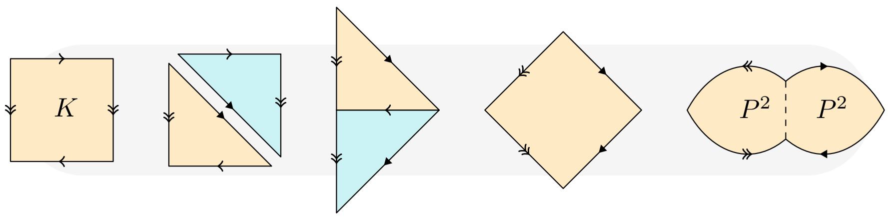

Figure 1: $K = P^2\#P^2$

Thus, $K_m = (P^2)_{2m}$ and $P^2\#K_m = P^2\#(P^2)_{2m} = (P^2)_{2m+1}$ with $m \geq 1$. The proof of (b) is complete.

Let us now prove that $T\#P^2 = K\#P^2$, hence $T\#P^2 = P^2\#P^2\#P^2$. It is already quite a bit harder to devise a cut and paste method for showing this, but it can be done as illustrated in Figure 2. Note that this shows that the connected sum does not have a cancellation law: $T\#P^2 = K\#P^2$ does not imply $T = K$. Thus, we have the following relation

$$T_n\#P^2 = \underbrace{T\#\cdots\#T}_{n-1}\#T\#P^2 = \underbrace{T\#\cdots\#T}_{n-1}\#P^2\#P^2\#P^2 = \cdots = (P^2)_{2n+1}$$

with $n \geq 1$, and hence $T_n\#K = T_n\#P^2\#P^2 = (P^2)_{2n+1}\#P^2 = (P^2)_{2n}$ with $n \geq 2$. This completes the proof of (a). $\square$

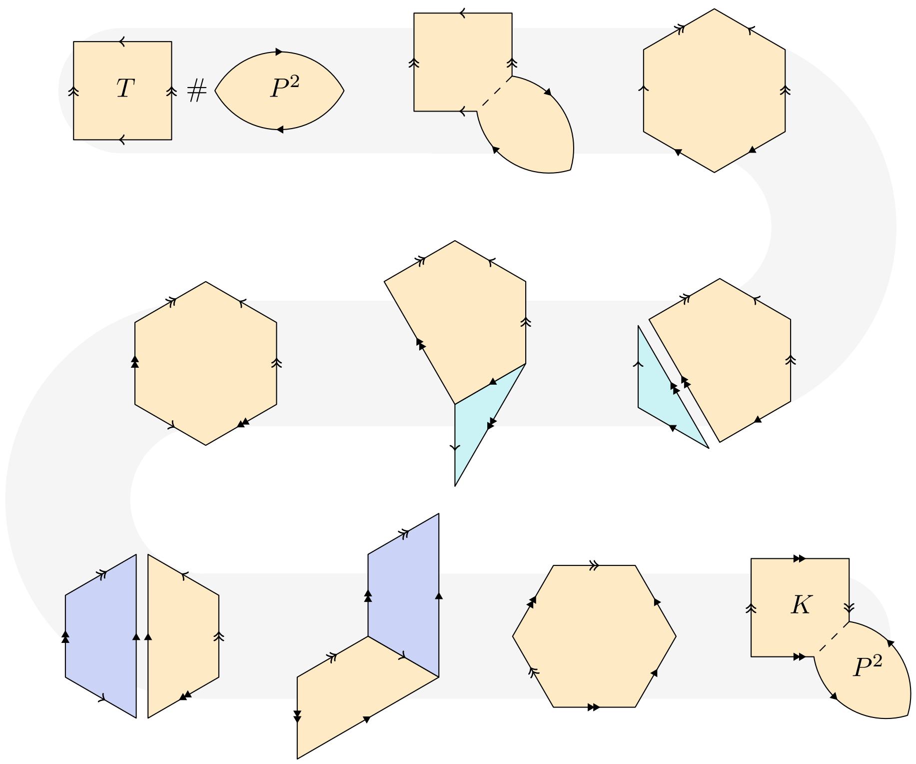

Figure 2: $T\#P^2 = K\#P^2$

Exercise 77.4. Let w be a proper labelling scheme for a 10-sided polygonal region. If w is of projective type, which of the list of spaces in Theorem 77.5 can it represent? What if w is of torus type?

Proof. Assuming w is of projective type, by Corollary 77.2, w can be decomposed as $a_1a_1a_2a_2\dots a_\ell a_\ell w_1$ where $\ell \geq 1$ and w_1 is empty or of torus type. Note that w has to start with at least one pair of an alphabet with same exponent, so w_1 can have the length of up to 8. Thus, w is never the same as S^2 and T_n. First we cancel all pairs like aa^{-1} in w_1. If w_1 were all composed of these pairs, w would be $(P^2)_n$ with $1 \leq n \leq 5$. Now by Lemma 77.3, w_1 is equivalent to a scheme starting with a torus like $aba^{-1}b^{-1}$ whose length may be less by cancelling of tail terms. Then w has $ccaba^{-1}b^{-1}$ term which is equivalent to $aabbcc$ by Lemma 77.4 (also can be understood as $P^2\#T = (P^2)_3$). This implies it doesn't matter how many tori w_1 generates. It only depends on the number of cancelling. Hence the resulting surface of w is homeomorphic to $(P^2)_n$ with $1 \leq n \leq 5$.

Now, let w be of torus type. Since w doesn't have any pair of an alphabet with same exponent like aa, w is never the same as $(P^2)_n$. First cancel all pairs like aa^{-1}. If w were all composed of these pairs, it would be S^2. Now then, applying Lemma 77.3 again, w equivalent to a scheme starting with a torus like $aba^{-1}b^{-1}$. If we repeat this process one more time, w will correspond to one of S^2, T, and $T\#T$. $\square$