실변수함수론 HW3

파이낸스경영학과 *2017004093* 양준우

Cantor set

0. Construction

Let $C_0 := [0,1]$: closed unit interval

Let I_1 be the middle third of C_0. $I_1 = [\frac{1}{3}, \frac{2}{3}]$

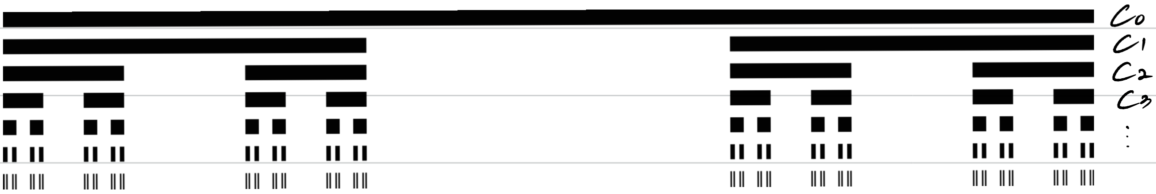
Define $C_1 := C_0 \setminus I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

Let I_2 be the union of the middle thirds of each intervals of C_1

Define $C_2 := C_1 \setminus I_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

We repeat this process for defining $C_k = C_{k-1} \setminus I_k$

We define the cantor set $C := \bigcap_{k=0}^{\infty} C_k$



1. Topological properties

① C is compact.

C_k : closed set.

$C = \bigcap_{k=0}^{\infty} C_k$: an arbitrary intersection of closed sets.

Thus, C is closed.

$C \subset [0,1]$ bounded.

By Heine - Borel theorem, C is compact.

② C is totally disconnected

${}^{\forall} x, y \in C$, ${}^{\exists}$ open set, U, V s.t $x \in U$, $y \in V$, $U \cap V = \phi$

③ C is perfect.

Def. $A \subset T$ is perfect

$\Leftrightarrow$ A is closed and has no isolated point.

Def. $x \in A$ is an isolated point

$\Leftrightarrow \exists \varepsilon > 0$ s.t $B_\varepsilon(x) \cap A = \{x\}$

We know that the boundary of C_k belong to C.

Let $x \in C$, $\varepsilon > 0$. Then $x \in C_k$ $\forall k$

We know that the length of C_k goes to zero as $k \to \infty$

Consider $B_\varepsilon(x)$.

$\exists N \in \mathbb{N}$ s.t the interval of C_k containing x is included in $B_\varepsilon(x)$ for $k \geq N$

In other word, there is an interval of C_k containing x and contained in $B_\varepsilon(x)$ for $k \geq N$

The boundary of this interval belongs to C.

So, x is accumulation point of C. ($\because B_\varepsilon(x) \cap C_k = \{x, \partial C_k\}$)

Every point in C is an accumulation point of C.

There is no isolated point.

C is closed and has no isolated point.

Thus C is perfect. $\cdot$ $int(C) = \phi$.

2. Cardinality and Measure

① C is uncountable.

Moreover, there is 1−1 correspondence between C and $[0,1]$

Write the number in $[0,1]$ with ternary number system (3-adic number system).

For $a \in [0,1]$

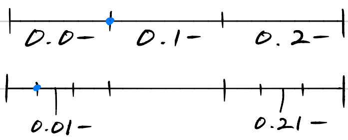
$a \in I_k \Rightarrow a = 0.\cdots$ the first 1 appears at k-th digit.

Thus, $c \in C \Rightarrow c$ can be expressed by 0 and 2.

Consider the binary number system for $[0,1]$

$b \in [0,1] \Rightarrow b$ can be expressed by 0 and 1

Construct a 1-1 correspondence so that
binary number expression of $[0,1]$ $\quad 0 \updownarrow 0 \quad 1 \updownarrow 2$
ternary number expression for $c \in C$



$0.1 \Rightarrow 0.02222\cdots$

$0.01 \Rightarrow 0.00222\cdots$

$\forall c \in C$ can be expressed by 0, 2.

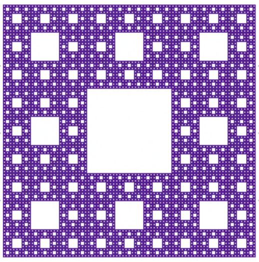
$\therefore$ C has the cardinality of the continuum
$= 2^{\aleph_0} = \aleph_1$: uncountable.

② C has zero measure.
(C has length zero).
The length of $\bigcup_{k=1}^{\infty} I_k$ = the sum of the length of $I_k = 1$
length of $C_k \to 0$ as $k \to \infty$.
$l_k :=$ the length of I_k.
$l_1 = \frac{1}{3}, \ l_2 = \frac{2}{9}, \ l_3 = \frac{4}{27}, \ \cdots$
$l_k = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$
$\sum_{k=1}^{\infty} l_k = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$
The length of $C_k = \left(\frac{2}{3}\right)^k \to 0$ as $k \to \infty$

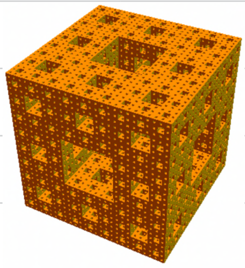
· measure zero
$\Leftrightarrow \forall \varepsilon > 0, \ \exists$ countable S_i s.t $A \subset \bigcup_{i=1}^{\infty} S_i, \ \sum_{i=1}^{\infty} V(S_i) < \varepsilon$
We can take N so large that $\sum_{k=1}^{N} \underbrace{\frac{1}{3}\left(\frac{2}{3}\right)^{k-1}}_{l_k} > 1 - \varepsilon$
then, $C \subset C_N < \varepsilon$.
countable
C_N is a disjoint union of 2^N intervals of length 3^{-N}
$\therefore$ for $\forall \varepsilon > 0$, statement satisfied
$\therefore$ C is measure zero.

3. Self-similarilty

Fractal is a self-similar subset of Euclidean space whose fractal dimension strictly exceeds its topological dimension.
Cantor set is a fractal with self-similarity. If we reduce the C to set of size of $\frac{1}{3}$, it is same as the left part of C.
Sierpinski carpet is 2-dimensional generalization of C while Menger sponge is 3-dimensional generalization of C. They are also fractal with self-similarity.



Sierpinski carpet



Menger sponge

4. Fat Cantor set

① construction.

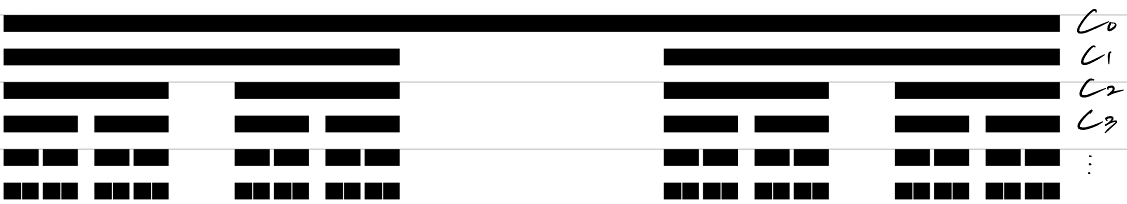
$C_0 = [0,1]$.

$C_1 = C_0 \setminus \overset{= I_1}{\{\text{middle } \tfrac{1}{4}\}} = [0, \tfrac{3}{8}] \cup [\tfrac{5}{8}, 1]$

$I_2 =$ union of middle $\left(\frac{1}{4}\right)^2$ of each subintervals.

$C_2 = C_1 \setminus I_2 = [0, \tfrac{5}{32}] \cup [\tfrac{7}{32}, \tfrac{3}{8}] \cup [\tfrac{5}{8}, \tfrac{25}{32}] \cup [\tfrac{27}{32}, 1]$

and so on. $C_k = C_{k-1} \setminus I_k$. $C = \bigcap_{k=0}^{\infty} C_k$



② lengh.

l_k = length of I_k $= 2^{k-1} \cdot \left(\frac{1}{4}\right)^k = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1}$

$\sum_{k=1}^{\infty} l_k = \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{2}$

lengh of $C = 1 - \sum_{k=1}^{\infty} l_k = \frac{1}{2}$.

$\therefore$ measure of $C = \frac{1}{2}$.

③ $\text{int}(C) = \phi$.

WANT. $\forall x \in C$, $\nexists \varepsilon > 0$ s.t $B_\varepsilon(x) \subset C$

We know that length of subinterval of C_k goes zero

for $\forall x \in C$, $\forall \varepsilon > 0$, $B_\varepsilon(x)$, $\exists N \in \mathbb{N}$ s.t $C_k \subset B_\varepsilon(x)$ for $k \geq N$.

$\therefore$ There is no interior point.

If $x, y \in C$, $\exists$ U, V s.t $x \in U$, $y \in V$, $U \cap V = \phi$.

($\because$ x, y must be boundary of C_k for large k.

If x, y are in same subinterval of C_k, x, y are disconnected in C_{k+1}).

Thus C is totally disconnected.

Hence fat Cantor set is a nowhere dense set of measure $\frac{1}{2}$