

# Problem 1.1.

(i)  $\Omega = \{uu, ud, du, dd\}$ .  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ ,  $\mathcal{F}_1 = \{\emptyset, \{uu, ud\}, \{du, dd\}, \Omega\}$   
 $\mathcal{F}_2 = 2^\Omega$ .  $\Rightarrow \{\mathcal{F}_t\}_{t=0,1,2}$  is filtration on  $\Omega$ .

$P(\{uu\}) = 0.12$ ,  $P(\{ud\}) = 0.48$ ,  $P(\{du\}) = 0.2$ ,  $P(\{dd\}) = 0.2$ .

$S_0 = 100$ ,  $S_1(uu) = S_1(ud) = 160$ ,  $S_1(du) = S_1(dd) = 70$ .

$S_2(uu) = 240$ ,  $S_2(ud) = S_2(du) = 96$ ,  $S_2(dd) = 72$ .

(ii) Let  $\gamma_i = E^P[S_2/G_2 | \mathcal{F}_1]$  and  $\alpha = \gamma(uu) = \gamma(ud)$ ,  $\beta = \gamma(du) = \gamma(dd)$ .

From def of cond. exp.

$E[\gamma \mathbb{I}_{\{uu, ud\}}] = E^P[S_2/G_2 \cdot \mathbb{I}_{\{uu, ud\}}]$

$\alpha \cdot (0.12 + 0.48)$

$\frac{240}{1.2} \times 0.12 + \frac{96}{1.2} \times 0.48$

i.e.  $0.6\alpha = 62.4$ ,  $\alpha = 104$

In same way,  $\beta = 70$ .

(iii)  $V_1^h, \pi_1, \phi_1$  is  $\mathcal{F}_1$ -msble  $\Rightarrow$  Let  $V_1^h(uu) = V_1^h(ud) = \alpha$ ,  $V_1^h(du) = V_1^h(dd) = \beta$ .

$\pi_1(uu) = \pi_1(ud) = \alpha\pi$ ,  $\pi_1(du) = \pi_1(dd) = \beta\pi$ .

$\phi_1(uu) = \phi_1(ud) = \alpha\phi$ ,  $\phi_1(du) = \phi_1(dd) = \beta\phi$ .

From  $V_1^h = \pi_1 G_1 + \phi_1 S_1$ ,  $\alpha = \alpha\pi G_1 + \alpha\phi S_1$ ,  $\beta = \beta\pi G_1 + \beta\phi S_1$ .

$V_2^h = \pi_2 G_2 + \phi_2 S_2$

From  $V_2^h = X_1$ ,

$= \pi_1 G_2 + \phi_1 S_2$

$\begin{cases} \alpha\pi \cdot 1.2 + \alpha\phi \cdot 240 = 120 \\ \alpha\pi \cdot 1.2 + \alpha\phi \cdot 96 = 0 \end{cases}$

In the case of  $uu$ ,  $\Rightarrow \alpha\pi = -\frac{200}{3}$   
 $\phi = \frac{5}{6}$

$\begin{cases} \beta\pi \cdot 1.2 + \beta\phi \cdot 96 = 0 \\ \beta\pi \cdot 1.2 + \beta\phi \cdot 72 = -12 \end{cases}$

$\phi = \frac{5}{6}$ ,  $\beta\pi = -40$ ,  $\beta\phi = \frac{1}{2}$ .

$\alpha = -\frac{200}{3} \cdot 1 + \frac{5}{6} \cdot 160 = \frac{200}{3}$ ,  $\beta = -40 \cdot 1 + \frac{1}{2} \cdot 70 = -5$ .

From  $\pi_0 G_1 + \phi_0 S_1 = V_1^h$ ,

$\begin{cases} \pi_0 \cdot 1 + 160 \cdot \phi_0 = \frac{200}{3} \\ \pi_0 \cdot 1 + 70 \cdot \phi_0 = -5 \end{cases}$

In the case of  $uu$  or  $ud$ ,  $\phi_0 = \frac{1}{2}$

$\Rightarrow \pi_0 = \frac{-1640}{21}$ ,  $\phi_0 = \frac{43}{54}$ .

and  $X_0 = \pi_0 + 100 \cdot \phi_0 = \frac{170}{9}$ .

$X_2(dd) = -12$

$\therefore X_0 = \frac{170}{9}$ ,  $X_1(uu) = X_1(ud) = \frac{200}{3}$ ,  $X_1(du) = X_1(dd) = -5$ ,  $X_2(uu) = 120$ ,  $X_2(ud) = X_2(du) = 96$ .

$$\phi_0 = \frac{43}{54}, \pi_0 = \frac{-1640}{27}, \phi_1(uu) = \phi_1(ud) = \frac{5}{6}, \pi_1(uu) = \pi_1(ud) = -\frac{200}{3}$$

$$\phi_1(du) = \phi_1(dd) = \frac{1}{2}, \pi_1(du) = \pi_1(dd) = -40.$$

$$\phi_2 = \phi_1, \pi_2 = \pi_1.$$

(iv) calculate  $V_{t+1}^h = \pi_t G_{t+1} + \phi_t S_{t+1}$  for  $t=0,1$   
 or  $V_{t+1}^h - V_t^h = \pi_t(G_{t+1} - G_t) + \phi_t(S_{t+1} - S_t)$  for  $t=0,1$ .

(v) Is  $\phi$  adapted?

$$\phi_0(uu) = \phi_0(ud) = \phi_0(du) = \phi_0(dd) = \frac{43}{54} \Rightarrow \phi_0^+(\omega) = \begin{cases} \phi & \text{o.w} \\ -2 & \omega = 43/54 \end{cases}$$

$$\begin{cases} \phi_1(uu) = \phi_1(ud) = \frac{5}{6} \\ \phi_1(du) = \phi_1(dd) = \frac{1}{2} \end{cases} \Rightarrow \phi_1^+(\omega) = \begin{cases} \phi & \text{o.w} \\ \{uu, ud\} & \omega = 5/6 \\ \{dd, du\} & \omega = 1/2 \end{cases}$$

$\Rightarrow \phi$  is  $\mathcal{F}_0$ -msble &  $\phi_1$  is  $\mathcal{F}_1$ -msble.  $\Rightarrow \phi$  is adapted.  
 $\phi_2 = \phi_1$  &  $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Rightarrow \phi_2$  is  $\mathcal{F}_2$ -msble

(vi) Define a probability measure  $\mathbb{Q}$  on  $\Omega$  by  
 $\mathbb{Q}(uu) = \frac{2}{9}, \mathbb{Q}(ud) = \frac{1}{9}, \mathbb{Q}(du) = \mathbb{Q}(dd) = \frac{1}{3}.$

You can show that

①  $S_t/G_t$  is adapted ②  $E^{\mathbb{Q}}(S_2/G_2 | \mathcal{F}_1) = S_1/G_1$  ③  $E^{\mathbb{Q}}(S_1/G_1 | \mathcal{F}_0) = S_0/G_0.$

For example, to show ②,  $Y_i = E^{\mathbb{Q}}(S_2/G_2 | \mathcal{F}_1).$

$Y$  is  $\mathcal{F}_1$ -msble; let  $a = Y(uu) = Y(ud)$  and  $b = Y(du) = Y(dd).$

$$\begin{aligned} \text{Then, } a \cdot (\mathbb{Q}(uu) + \mathbb{Q}(ud)) &= \frac{240}{1.2} \cdot \mathbb{Q}(uu) + \frac{96}{1.2} \cdot \mathbb{Q}(ud) \Rightarrow a = 160 \\ b \cdot (\mathbb{Q}(du) + \mathbb{Q}(dd)) &= \frac{96}{1.2} \mathbb{Q}(du) + \frac{72}{1.2} \mathbb{Q}(dd) \Rightarrow b = 70 \end{aligned}$$

⊛ How to find  $\mathbb{Q}.$

From condition ②,  $\mathbb{Q}(uu), \mathbb{Q}(ud)$  satisfy  $\cdot 160 (\mathbb{Q}(uu) + \mathbb{Q}(ud)) = \frac{240}{1.2} \mathbb{Q}(uu) + \frac{96}{1.2} \mathbb{Q}(ud)$   
 i.e  $\mathbb{Q}(uu) : \mathbb{Q}(ud) = 2 : 1$

In the same way,  $\mathbb{Q}(du) : \mathbb{Q}(dd) = 1 : 2.$

From condition ③,  $\frac{160}{1} \cdot (\mathbb{Q}(uu) + \mathbb{Q}(ud)) + \frac{70}{1} (\mathbb{Q}(du) + \mathbb{Q}(dd)) = 100.$

$$\Rightarrow \mathbb{Q}(uu) + \mathbb{Q}(ud) : \mathbb{Q}(du) + \mathbb{Q}(dd) = 1 : 2.$$

From above 3 equality, you can find  $\mathbb{Q}.$



(V15) Since  $\theta$  is risk neutral measure,  $\frac{S_t}{G_t}$  is martingale under  $\mathbb{Q}$ .

$$\therefore E^{\mathbb{Q}}(S_2/G_2 | \mathcal{F}_1) = S_1/G_1.$$

Since  $X_t = V_t^h$  for self-financing portfolio  $h$ ,  $\frac{V_t^h}{G_t}$  is martingale under  $\mathbb{Q}$ .

$$\therefore E^{\mathbb{Q}}(X_2/G_2 | \mathcal{F}_1) = E^{\mathbb{Q}}(V_2^h/G_2 | \mathcal{F}_1) = V_1^h/G_1 = X_1/G_1.$$

Problem 1.2. In class,  $(V15) \Rightarrow (C1)$  is proved.

$(C1) \Rightarrow (V15)$ .

Suppose  $(V15)$  is false. There are 6 cases that  $(C1)$  is false and in any case you can construct arbitrage.

For example, if  $HR \leq \frac{S_d}{S} < \frac{S_u}{S}$ , define  $h = (\pi_t, \phi_t)$  by  $h_0 = (-S, 1)$ ,  $h_1 = (\frac{V_1^h}{HR}, 0)$ ,  $h_2 = h_1$ .

$$\text{Then, } V_0^h = -S \times 1 + S \times 1 = 0. \quad V_1^h = \begin{cases} -S(HR) + S_u > 0 & S_1 = S_u \\ -S(HR) + S_d \geq 0 & S_1 = S_d \end{cases}$$

$$\begin{aligned} \phi V_1^h &= \pi_1 G_1 + \phi_1 S \\ &= \frac{V_1^h}{HR} \cdot HR \end{aligned}$$

$$V_2^h = \left( \frac{V_1^h}{HR} \right) \cdot (HR)^2 \geq (HR) V_1^h \quad \begin{cases} > 0 & S_2 = S_{uu} \text{ or } S_{ud} \\ \geq 0 & S_2 = S_{du} \text{ or } S_{dd} \end{cases}$$

$\therefore h$  is an arbitrage.

In the case of  $\frac{S_{ud}}{S_u} < \frac{S_{uu}}{S_u} \leq HR$ , define  $h = (\pi_t, \phi_t)$  by

$$h_0 = (0, 0), \quad h_1 = \begin{cases} (S_u, -(HR)) & S_1 = S_u \\ (0, 0) & S_1 = S_d \end{cases} \quad h_2 = h_1.$$

Then  $h$  is arbitrage.

$(V15) \Rightarrow (C15)$ .

Construct filtered space as problem 1.1 (C1). Define measure  $\mathbb{Q}$  on  $\mathcal{L}$  by

By condition (C15),  $\mathbb{Q}(uu), \dots, \mathbb{Q}(dd) > 0$ .

$$\mathbb{Q}(uu) = \frac{S(HR) - S_d}{S_u - S_d} \cdot \frac{S_u(HR) - S_{dd}}{S_{uu} - S_{ud}}, \quad \mathbb{Q}(ud) = \frac{S(HR) - S_d}{S_u - S_d} \cdot \frac{S_{uu} - S_u(HR)}{S_{uu} - S_{ud}}$$

$$\mathbb{Q}(du) = \frac{S_u - S(HR)}{S_u - S_d} \cdot \frac{S_d(HR) - S_{dd}}{S_{du} - S_{dd}}, \quad \mathbb{Q}(dd) = \frac{S_u - S(HR)}{S_u - S_d} \cdot \frac{S_{dd} - S_d(HR)}{S_{du} - S_{dd}}$$

You can show that  $S_t/G_t$  is martingale under  $\mathbb{Q}$ .

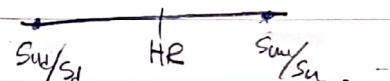
(\*) How to find  $\mathbb{Q}$

$$\mathbb{Q} \text{ must satisfy } E^{\mathbb{Q}}(S_{t+1}/G_{t+1} | \mathcal{F}_t) = S_t/G_t.$$

$$\text{Let } q_{uu} = \mathbb{Q}(uu), \dots, q_{dd} = \mathbb{Q}(dd).$$

$$t=1, \{u, d\} \Rightarrow \frac{1}{(HR)^2} [S_{uu}q_{uu} + S_{ud}q_{ud}] = \frac{S_u}{HR} (q_{uu} + q_{ud}).$$

$$\Leftrightarrow \frac{S_{uu}/S_u \cdot q_{uu} + S_{ud}/S_u \cdot q_{ud}}{q_{uu} + q_{ud}} = HR.$$



$$\Leftrightarrow HR \in (S_{ud}/S_d, S_{uu}/S_u) \Leftrightarrow q_{uu} > 0, q_{ud} > 0 \text{ (이는 내보림)}$$

$$\Leftrightarrow q_{uu} : q_{ud} = S_{uu}/S_u - (HR) : (HR) - S_{ud}/S_d.$$

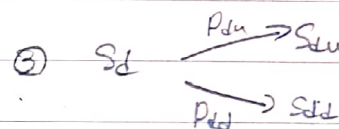
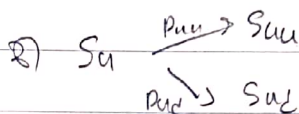
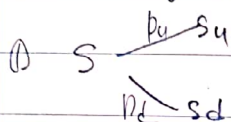
$$t=1, \{du, dd\}. \quad \text{In the same way, } q_{du} : q_{dd} = \frac{S_{du}}{S_d} - (HR) : (HR) - \frac{S_{dd}}{S_d}$$

$t=0, \Omega$ .

$$q_{uu} + q_{ud} : q_{du} + q_{dd} = S_u/S - (HR) : (HR) - S_d/S.$$

Using  $q_{uu} + q_{ud} + q_{du} + q_{dd} = 1$ , you can find  $q_{uu}, \dots, q_{dd}$ .  $\square$

Remark). We can regard 2-period binomial model as 3 1-period binomial model



The condition (iii) is equivalent that ①, ②, ③ have no arbitrage.

Using the def of arbitrage, you can show (ii)  $\Rightarrow$  (i) by assuming that 2-period binomial model has arbitrage.

Also, the condition (ii) is equivalent that ①, ②, ③ have a risk-neutral measure.

From risk-neutral measure  $Q$  on 2-period binomial model, you can construct a risk-neutral measure of ①, ②, ③.

It proves (iii)  $\Rightarrow$  (ii).

Problem 1.8. Since  $M_t$  is  $\mathcal{F}_t$ -insble,  $M$  is adapted. Lower property  $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$

$$E[M_{t+1} | \mathcal{F}_t] = E[E[X | \mathcal{F}_{t+1}] | \mathcal{F}_t] = E[X | \mathcal{F}_t] = M_t.$$

$\therefore M$  is a martingale.

Problem 1.4. Let  $\mathbb{Q}$  be a risk-neutral measure on 2-period binomial model.

Since  $(X_t)_{t=0,1,2}$  is the arbitrage-free price process, there exists self-financing portfolio  $\tilde{\pi} = (\pi_t, \phi_t)_{t=0,1,2}$  s.t.  $V_t^{\tilde{\pi}} = X_t$ .  
Therefore,  $V_t^{\tilde{\pi}}/G_t = X_t/G_t$  is martingale under  $\mathbb{Q}$ .

Assume  $\tilde{\pi} = (a_t, b_t, c_t)_{t=0,1,2}$  is arbitrage of the market with 3 assets  $(G_t)_{t=0,1,2}$ ,  $(S_t)_{t=0,1,2}$ ,  $(X_t)_{t=0,1,2}$ .

Then  $\mathbb{P}(V_0^{\tilde{\pi}} = 0) = 1$ ,  $\mathbb{P}(V_T^{\tilde{\pi}} \geq 0) = 1$ ,  $\mathbb{P}(V_T^{\tilde{\pi}} > 0) > 0$ .

Since  $\mathbb{P} \sim \mathbb{Q}$ ,  $\mathbb{Q}(V_0^{\tilde{\pi}} = 0) = 1$ ,  $\mathbb{Q}(V_T^{\tilde{\pi}} \geq 0) = 1$ ,  $\mathbb{Q}(V_T^{\tilde{\pi}} > 0) > 0$ .

$$\therefore E^{\mathbb{Q}}[V_T^{\tilde{\pi}}] > 0.$$

Since  $\tilde{\pi}$  is self-financing,  $V_{t+1}^{\tilde{\pi}} = a_t G_{t+1} + b_t S_{t+1} + c_t X_{t+1}$ .

$$\Rightarrow E^{\mathbb{Q}}[V_{t+1}^{\tilde{\pi}}/G_{t+1} | \mathcal{F}_t] = E^{\mathbb{Q}}[a_t + b_t \frac{S_{t+1}}{G_{t+1}} + c_t \frac{X_{t+1}}{G_{t+1}} | \mathcal{F}_t].$$

$$= a_t + b_t \cdot \frac{S_t}{G_t} + c_t \cdot \frac{X_t}{G_t} = \frac{V_t^{\tilde{\pi}}}{G_t} \quad \text{for } t=0,1$$

$$\left(\frac{S_t}{G_t}\right)_{t=0,1}$$

$$\left(\frac{X_t}{G_t}\right)_{t=0,1}$$

are martingale under  $\mathbb{Q}$ .

$$\therefore E^{\mathbb{Q}}[V_T^{\tilde{\pi}}/G_T] = E^{\mathbb{Q}}[V_0^{\tilde{\pi}}/G_0] = 0.$$

$$\Rightarrow E^{\mathbb{Q}}[V_T^{\tilde{\pi}}] = 0. \quad \text{It contradicts.}$$

$\therefore$  A market with three assets  $(G_t), (S_t), (X_t)$  is free of arbitrage.