

## Advanced Calculus I – Midterm Exam

April 25, 2019

1. Let  $A \subset \mathbb{R}$  be a bounded set. Show the following equality:

$$\sup A - \inf A = \sup\{x - y : x, y \in A\}.$$

2. Determine whether the following sequences converge or diverge. If it converges, attain the limit.

(a)  $a_n = \sqrt{n+1} - \sqrt{n}$

(b)  $a_n = \sum_{k=1}^n \frac{k}{k^2+1}$

(c)  $a_n = \sum_{k=1}^n \frac{k!}{3^k}$

3. Show that every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

4. Let  $\{a_n\}$  be a sequence such that  $a_n > 0$  for all  $n \in \mathbb{N}$ . Show the following inequality.

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} (a_n)^{\frac{1}{n}}.$$

5. Prove or disprove the following statements:

(a) Every inner product space is a normed space.

(b) Every normed space is an inner product space.

6. Let  $\{a_n\}$  be a bounded sequence in  $\mathbb{R}$  satisfying  $2a_n \leq a_{n-1} + a_{n+1}$ . Show that

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0.$$

7. Let  $M$  be a metric space. Suppose  $A$  is a closed set in  $M$  and  $B$  is a compact set in  $M$ . Show that  $A \cap B$  is compact.

8. Let  $A \subset \mathbb{R}^n$  be a compact set. Show  $A$  is closed and bounded without the Heine–Borel theorem.

9. Show that a closed interval  $[a, b] \subset \mathbb{R}$  is connected by using the definition of connectedness.