Mathematical Statistics I – Final Exam

June 18, 2019

1. Let X_1, X_2 be independent random variables, each of which follows an exponential distribution with a mean 1. Let us consider $Y_1 = X_1 - X_2$, $Y_2 = X_1 + X_2$, and $Z = X_1/(X_1 + X_2)$.
   (1) Find the joint pdfs of Y_1 and Y_2.
   (2) Find the marginal pdfs of Y_1 and Y_2.
   (3) Compute $P(Y_2 < 5 | Y_1 = -2)$.
   (4) Find the marginal pdf of Z.
2. Solve each question.
   (1) If two random variables X and Y have the correlation coefficient ρ, show that ρ satisfies the inequality $-1 \leq \rho \leq 1$.
   (2) If X have continuous strictly increasing cdf $F_X(x)$ on the range $a < x < b$, show that $Y = F_X(x)$ has a uniform distribution on $(0, 1)$.
   (3) If (X, Y) has the bivariate normal pdf

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2) \right\}$$

   where $\mathrm{Corr}(X, Y) = \rho$, then find the conditional distribution of X given $Y = y$ and write down its distribution notation (e.g. $X \sim Poi(\theta)$). Here, you don't need to derive the marginal distribution of Y.
3. Let $X \sim N(\mu, \sigma^2)$.
   (1) Derive the moment generating function (mgf) of X.
   (2) Compute the variance of X.
   (3) Find the pdf of $W = (X - \mu)^2/\sigma^2 = V^2$ after showing the pdf of $V = (X - \mu)/\sigma$.
   (4) Derive the mgf of W.
   (5) Compute the mean and variance of W.
4. Let X_1 and X_2 be nonnegative integer random variables such that $X_1 + X_2 \leq n$ with joint pmf

$$f(x_1, x_2) = \frac{n!}{x_1! x_2! (n - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{n - x_1 - x_2},$$

   where $0 < p_1,\ p_2 < 1$ and $p_1 + p_2 \leq 1$.
   (1) Derive the marginal pmf of X_1.
   (2) Find the conditional pmf of X_2, given $X_1 = x_1$.
   (3) Compute the covariance of X_1 and X_2.