Project B : Polynomial Interpolation

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In this project we use function $f(x)=\frac{1}{1+x^2}$ on the interval $[-5,5]$. We construct the polynomial interpolants $P_n(x)$ for $n=5,7,8,10,15,20$, where $P_n(x)$ interpolates f at the equally spaced points, $x_i=-5,-5+\frac{10}{n},-5+2\frac{10}{n},\ldots,5$.

● MATLAB code

```
clc;
clear all;
close all;

n = input('Input a degree of polynomial: ');
f = @(x) 1./(1 + x.^2);

% equally spaced nodes vector
i = 0;
while i <= n;
    nodes(i+1) = -5 + (10/n)*i;
    i = i + 1;
end

% value of equally spaced nodes vector
value_nodes = f(nodes);

% coefficient vector of equally spaced nodes
j = 0;
while j <= n;
    E(:,j+1) = nodes.^j;
    j = j + 1;
end
coef = inv(E)*value_nodes';

%polynomial interpolating equally spaced nodes by Horner's rule
k = n;
pe = @(x) coef(n+1);
while k >= 1;
    pe = @(x) pe(x).*x + coef(k);
    k = k - 1;
end


% chebyshev nodes
i = 0;
while i <= n;
    cheby_nodes(i+1) = 5*cos((2*i+1)*pi/(2*n+2));
    i = i + 1;
end

% value of chebyshev nodes vector
value_cheby = f(cheby_nodes);
```

```
% coefficient vector of chebyshev nodes
j = 0;
while j <= n;
    C(:,j+1) = cheby_nodes.^j;
    j = j + 1;
end
coef_cheby = inv(C)*value_cheby';

% polynomial interpolating chebyshev nodes by Horner's rule
k = n;
pc = @(x) coef_cheby(n+1);
while k >= 1;
    pc = @(x) pc(x).*x + coef_cheby(k);
    k = k - 1;
end

%graph
t = -5:0.01:5;
figure('Color','white')
plot(t,f(t),t,pe(t),nodes,value_nodes,'o',t,pc(t),cheby_nodes,value_cheby,'o')
title(['Polynomial Interpolation $n=$ ',num2str(n)],'FontSize',20,'Interpreter','latex')
legend('$f(x)$',['$p_{',num2str(n),'}(x)$'],'equally spaced nodes',['Chebyshev
$p_{',num2str(n),'}(x)$'],'Chebyshev nodes','Interpreter','latex')
grid on;

% error
ee = max(abs(pe(t)-f(t)))
ec = max(abs(pc(t)-f(t)))
```

● Runge phenomenon

Runge phenomenon is a problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree over a set of equally spaced interpolation points.

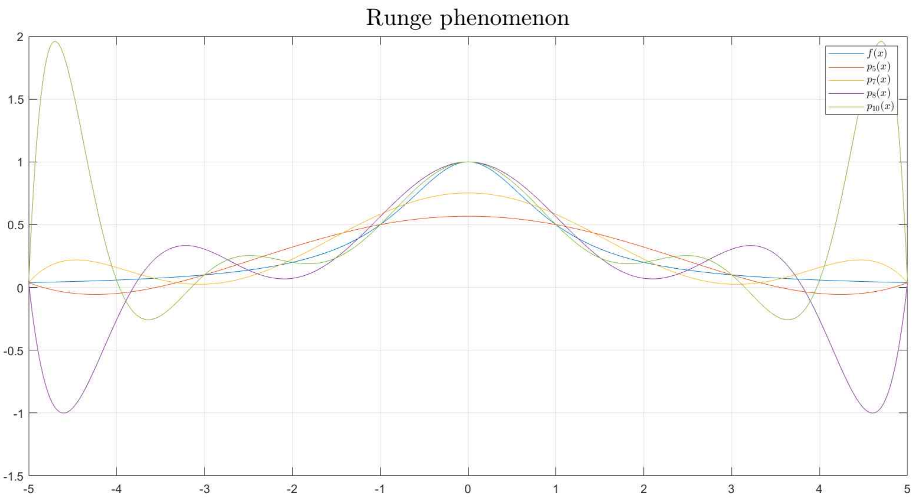


Figure : Interpolation of Runge's function at equally spaced nodes.

● Interpolation error analysis

Thm Set $f(x)$ in $[a,b]$ include $n+1$ mutual different nodes, $x_0, x_1, x_2, \cdots, x_n$. The interval x_n at $[a,b]$ has N order continuous derivative, and in (a,b) exists within $n+1$ order derivative. Then for any $x \in [a,b]$, we have

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} W_{n+1}(x)$$

where, $\xi \in (a,b)$, $W_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

● The analysis of Runge phenomenon

The error between the generating function and the interpolating polynomial of order n is given by

$f(x) - P_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=1}^{n+1} (x - x_i)$ for some ξ in (-5,5). Thus,

$$\max_{-5 \le x \le 5} |f(x) - p(x)| \le \max_{-5 \le x \le 5} \frac{|f^{(n+1)}(x)|}{(n+1)!} \max_{-5 \le x \le 5} \prod_{i=0}^{n} |x - x_i|.$$

Denote by $w_n(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

and let W_n be the maximum of the w_n function:

$W_n = \max_{-5 \le x \le 5} w_n(x)$

Then it can be proved that, if equidstant nodes are used, then $W_n \le h^n \dfrac{(n-1)!}{4}$

where $h = \dfrac{10}{n-1}$ is the step size.

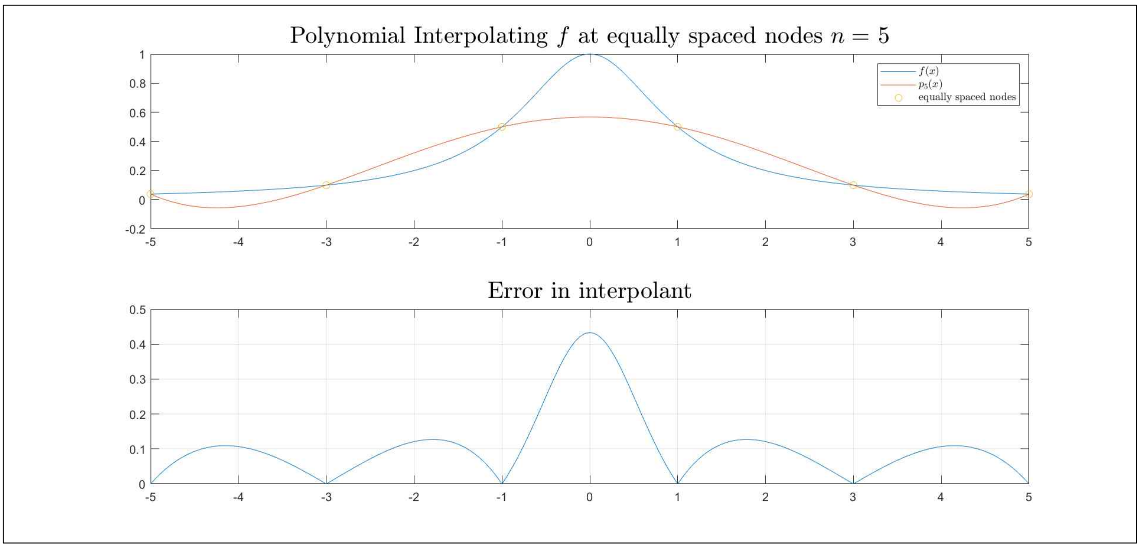
Moreover, assume that the n-th derivative of f is bounded, $\max_{-5 \le x \le 5} f^{(n)}(x) \le M_n$.

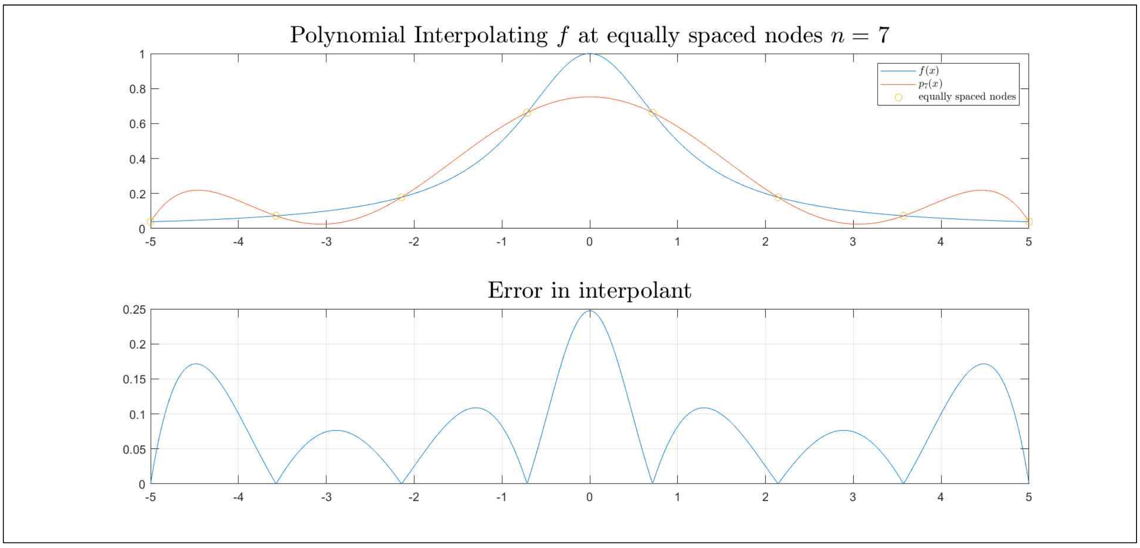
Therefore, $\max_{-5 \le x \le 5} |f(x) - P_{n-1}(x)| \le M_n \dfrac{h^n}{4n}$.

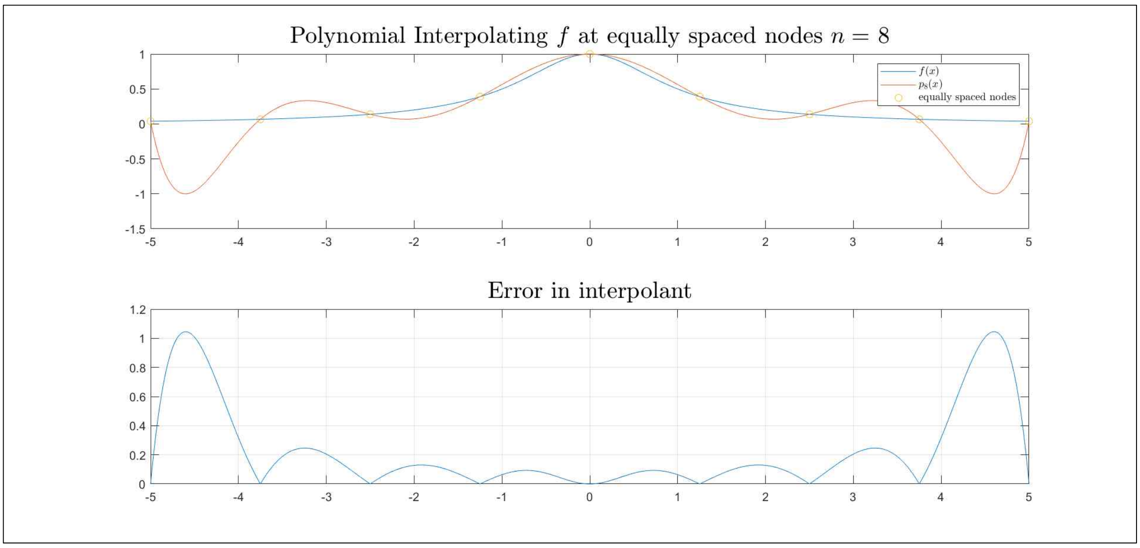
But the magnitude of the n-th derivative of Runge's function increases when n increases, and very fast. The result is that the product in the previous equation tends to infinity when n tends to infinity.

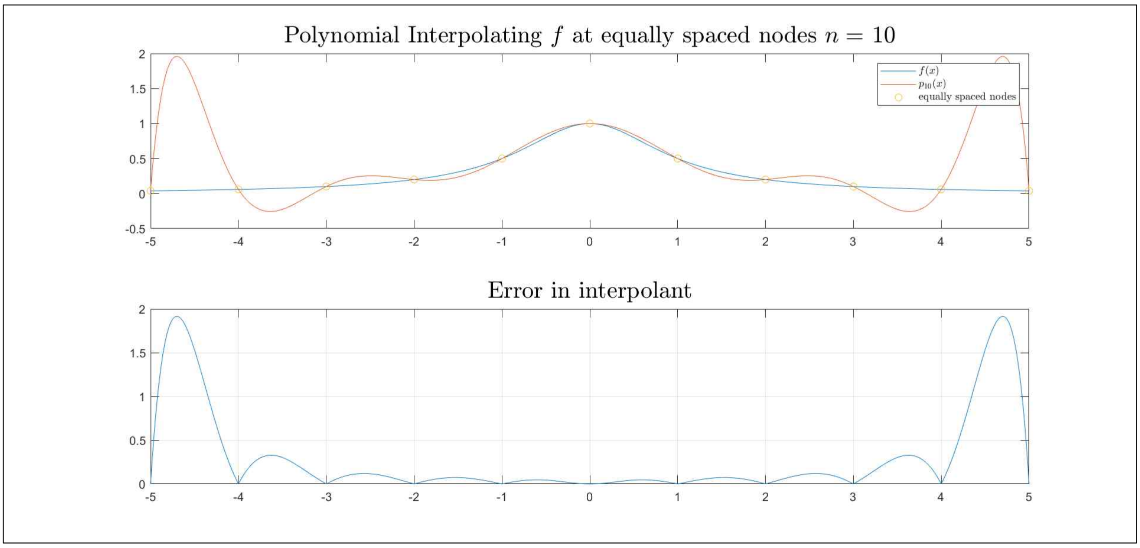
The fact that the upper bound of the error goes to infinity does not necessarily imply, of course, that the error itself also diverges with n.

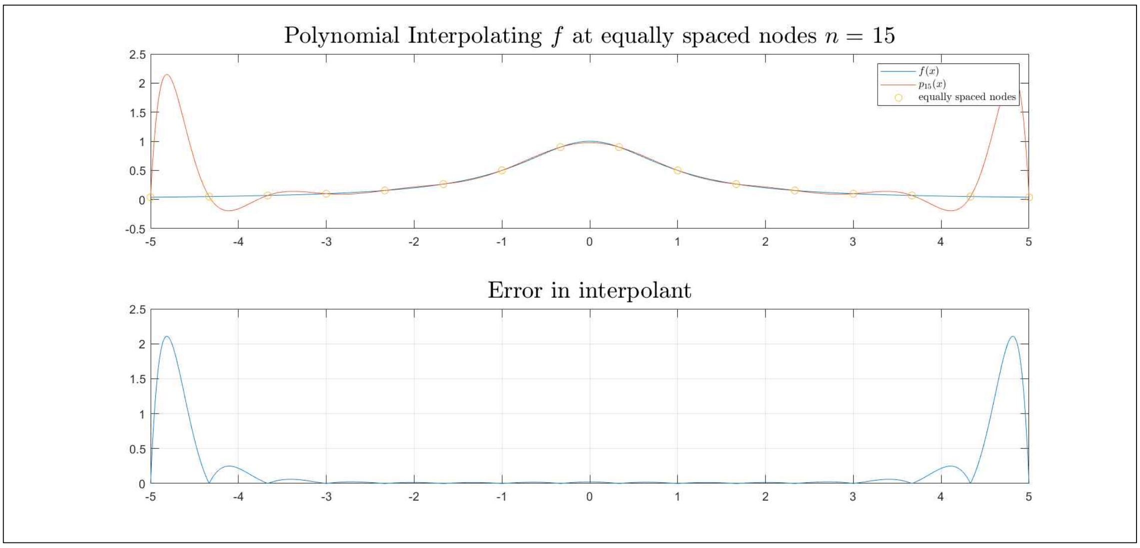
● Polynomial interpolation in $n = 5, 7, 8, 10, 15, 20$

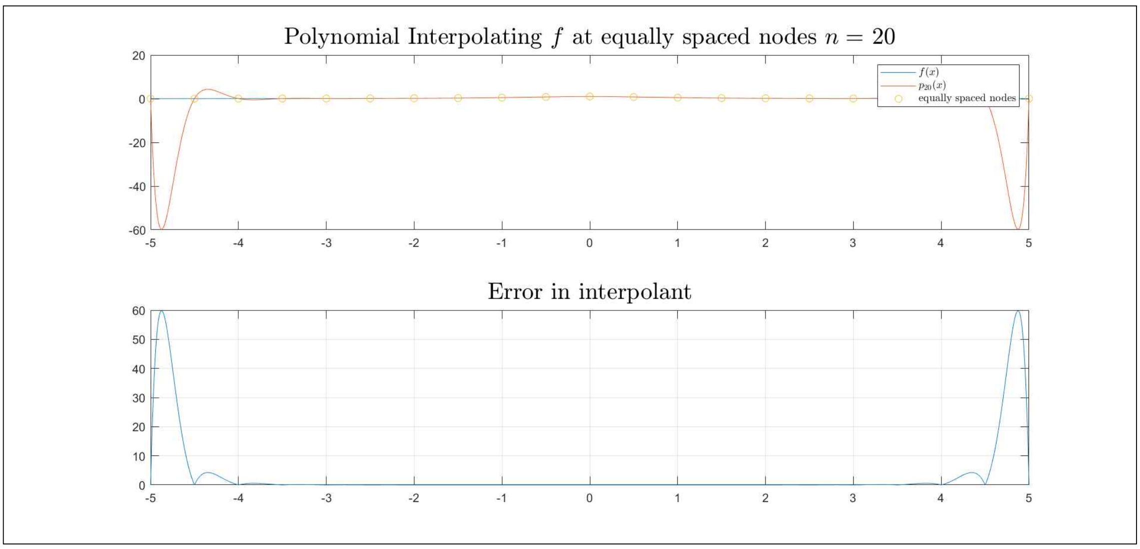







Polynomial Interpolating f at equally spaced nodes $n = 10$
$f(x)$
$p_{10}(x)$
equally spaced nodes
Error in interpolant


Polynomial Interpolating f at equally spaced nodes $n = 15$
$f(x)$
$p_{15}(x)$
equally spaced nodes
Error in interpolant


Polynomial Interpolating f at equally spaced nodes $n = 20$
$f(x)$
$p_{20}(x)$
equally spaced nodes
Error in interpolant

● How to avoid

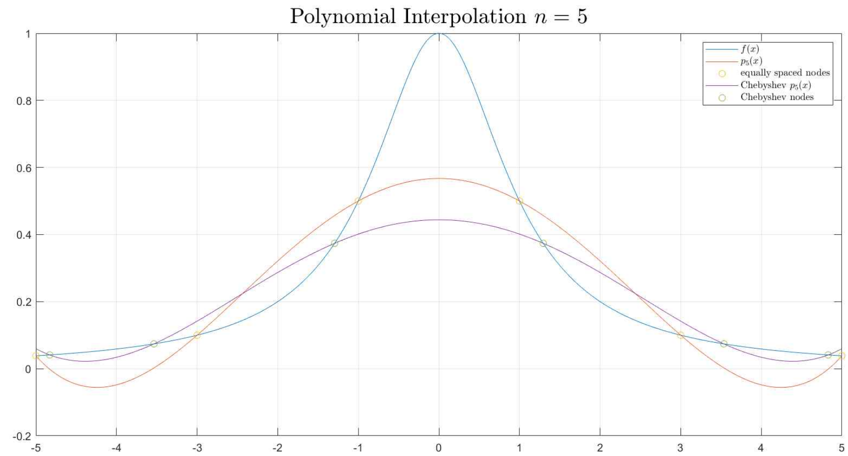
We use two kinds of method to avoid Runge phenomenon. The first way is to take the node as Chebyshev nodes, which are

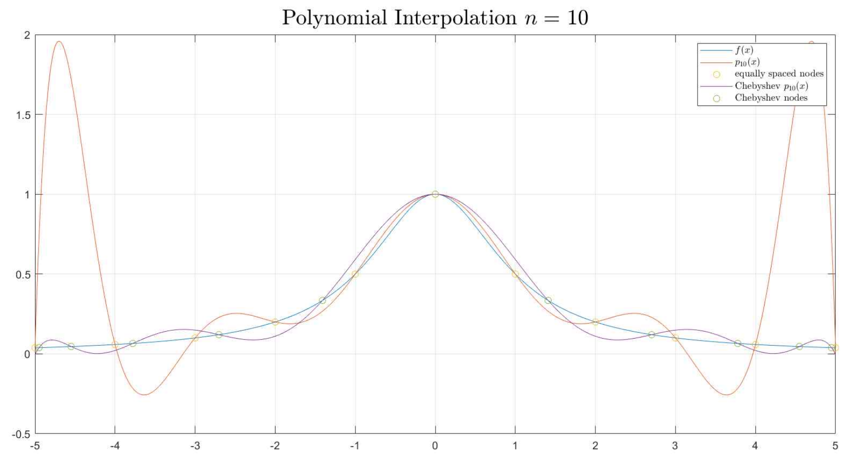
$x_k = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos(\frac{2k-1}{2n}\pi), k = 1,\dots,n$ for interval $[a,b]$.

Our interval is $[-5,5]$ so we choose nodes as,

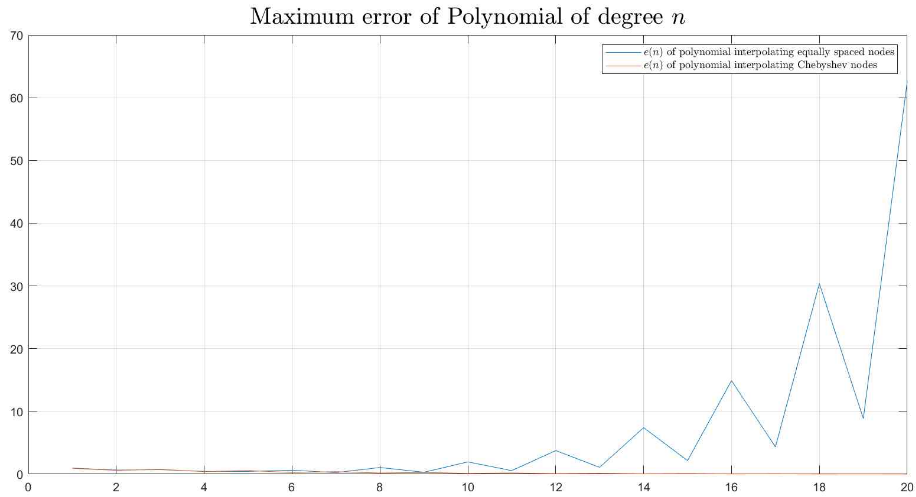
$x_k = 5\cos(\frac{2k-1}{2n}\pi),\ \ k = 1,\dots,n.$

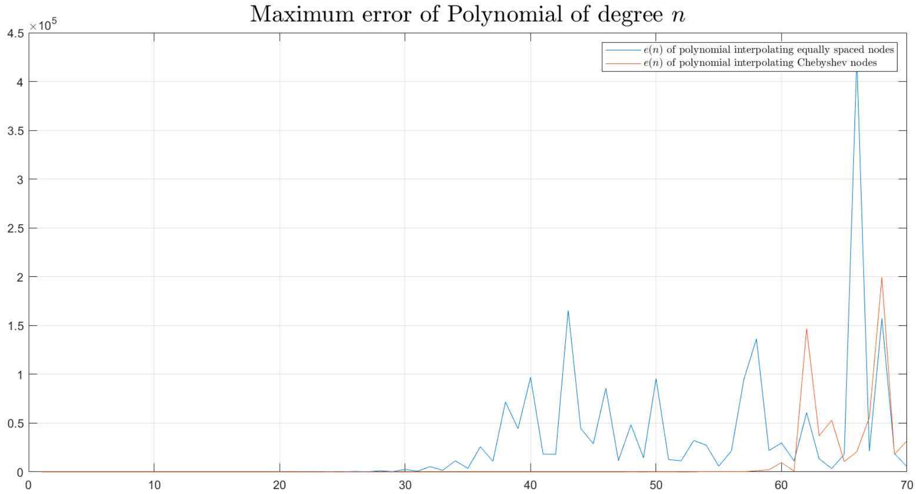
As you can see in the previous case, in equally nodes case, it has a large oscillation in part of the edge. However, choosing nodes as chebyshev, you can see that the above phenomenon does not occur.





● Divergence of maximum error





● Conclusion & Suggestion

우리의 목표는 주어진 함수인, runge function과 가장 일치하는 polynomial을 찾는 것이다. 오차 그래프를 살펴보면, 우리의 기대와는 다르게 Chebyshev nodes를 사용한 polynomial 역시 equally spaced node를 사용한 polynomial과 마찬가지로 node의 개수를 늘릴수록 오차가 무한대로 발산하는 것은 마찬가지이다. 따라서 최대오차가 가장 작게 나오는 degree값으로 polynomial을 구할 수 있지만, error가 0으로 수렴하는 완벽한 polynomial을 구할 수는 없다.

Equally spaced nodes와 Chebyshev nodes를 사용하여 polynomial을 construct한 결과, 두 polynomial 모두 node의 개수가 증가할수록 구간의 양 끝 부분에서 오차가 점점 더 커짐을 알 수 있는 동시에, 가운데 부분은 원함수와 점점 같아지는 것을 확인할 수 있다. 즉, node가 늘어날수록 [-3,3] 구간의 점들은 오차가 줄어드는 것처럼 보이지 않는가.

이 점에 착안하여, 구간을 [-5,5]에서 [-10,10]으로 늘리고 Chebyshev polynomial을 구한 후, [-5,5]에서의 최대오차를 계산해보았다. [-10,10] 정도의 구간이면 최대오차는 [-10,-5], [5,10]에서 발생하고 [-5,5]에서의 오차는 0으로 수렴하길 기대했기 때문이다.

실행결과 이 경우도 마찬가지로 최대오차는 발산한다. 하지만 구간이 [-5,5]로 수렴할수록 최대오차가 작아졌으며, [-5.05,5.05]의 구간에서 최대오차가 더 작게 나왔다. 즉, [-5,5] 구간 상의 polynomial은 최대오차가 9.496980102086106e-04이지만, [-5.05,5.05] 구간의 polynomial은 최대오차가 그보다 작은 6.879283965312011e-04 이다. 하지만 [-5.05,5.05] 보다 좁은 구간에서 다시 [-5,5]구간 상의 최대오차 값을 초과했다.

따라서 구간을 [-5.05,5.05]로 수정한 후에, Chebyshev nodes로 node choosing을 하면 [-5,5] 구간에서 상당히 작은 최대오차를 가진 polynomial을 얻을 수 있다. 즉 [-5.05,5.05] 구간에서 Chebyshev polynomial을 그린 후 [-5.05,-5), (5,5.05] 구간을 잘라버리는 것이다.