Homework 4

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Problem 1. Let f be a non-negative integrable function on $\mathbb{R}$ and let m_2 be two dimensional Lebesgue measure on $\mathbb{R}^2$.

(1) Show that $m_2\{(x,y) : 0 \le y \le f(x)\} = m_2\{(x,y) : 0 < y < f(x)\} = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$.

(2) Show that $G(f) = \{(x,y) : y = f(x)\}$ has measure 0.

(3) Let $\phi(t) = m\{x : f(x) \ge t\}$. Show that ϕ is a decreasing function and

$$\int_0^\infty \phi(t)\,\mathrm{d}t = \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

Proof. (1) Let $A = \{(x,y) : 0 \le y \le f(x)\}$. First of all, let's check that A is m_2-measurable. Since f is non-negative measurable, there is a sequence s_n of non-negative simple functions such that $s_n \nearrow f$ which can be written as $s_n = \sum_{k=1}^{k_n} a_{n,k}\mathbf{1}_{A_{n,k}}$. Let $R_{n,k} = A_{n,k} \times [0, c_k]$ and the union of such rectangles is in fact $\int s_n\,\mathrm{d}m$. Then, $\bigcup_{n=1}^\infty \bigcup_{k=1}^{k_n} R_k = A$ so A is m_2-measurable. Now for given x, consider x-section of A, $A_x = \{y : (x,y) \in A\} = \{y : 0 \le y \le f(x)\} = [0, f(x)]$. Its measure is $f(x)$ and by the definition of the product measure

$$m_2(A) = m_2\{(x,y) : 0 \le y \le f(x)\} = \int_{\mathbb{R}} m(A_x)\,\mathrm{d}m(x) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

Since $m([0, f(x)]) = m((0, f(x)) = f(x)$, it doesn't matter whether the endpoint is included.

(2) The x-section of $G(f)$ for given x, $G_x = \{y : (x,y) \in G(f)\} = \{y : y = f(x)\} = \{f(x)\}$ and its measure is zero. Because point set is null. Thus, we get

$$m_2(G(f)) = m_2\{(x,y) : y = f(x)\} = \int_{\mathbb{R}} m(G_x)\,\mathrm{d}m(x) = \int_{\mathbb{R}} 0\,\mathrm{d}m = 0.$$

(3) For any t and $\epsilon > 0$, by the countable additivity of m,

$$\begin{aligned}
\phi(t) &= m\{x : f(x) \ge t\} = m(\{x : f(x) \ge t+\epsilon\} \cup \{x : t \le f(x) < t+\epsilon\}) \\
&= m\{x : f(x) \ge t+\epsilon\} + m\{x : t \le f(x) < t+\epsilon\} \\
&= \phi(t+\epsilon) + m\{x : t \le f(x) < t+\epsilon\} \\
&\ge \phi(t+\epsilon).
\end{aligned}$$

This is exactly the definition of decreasing function. For the second claim consider area under the graph, $A = \{(x,y) : 0 \le y \le f(x)\}$ and t-section of A for given t, $A_t = \{x : (x,t) \in A\} = \{x : t \le f(x)\}$. Then $\phi(t) = m\{x : f(x) \ge t\} = m(A_t)$ for $t \ge 0$. Thus, by the definition of product measure and (1),

$$\int_0^\infty \phi(t)\,\mathrm{d}t = \int_0^\infty m(A_t)\,\mathrm{d}t = \int_0^\infty m(A_t)\,\mathrm{d}m(t) = m_2(A) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

□

Problem 2. Let $f_{(X,Y)}$ be a joint density of random variables X, Y such that

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{50}(x^2+y^2) & \text{if } 0<x<2,\ 1<y<4 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X+Y>4)$ and $P(Y>X)$.

Proof.

$$\begin{aligned} P(X+Y>4) &= \int_A f_{(X,Y)}(x,y)\,\mathrm{d}m_2(x,y) \quad \text{where } A=\{(x,y): y>4-x\}\cap[0,2]\times[1,4] \\ &= \int_0^2\int_{4-x}^4 \frac{1}{50}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{50}\int_0^2 x^2y+\frac{1}{3}y^3\Big|_{4-x}^4\,\mathrm{d}x \\ &= \frac{1}{50}\int_0^2 \frac{4}{3}x^3-4x^2+16x\,\mathrm{d}x = \frac{1}{50}\left(\frac{1}{3}x^4-\frac{4}{3}x^3+8x^2\Big|_0^2\right) = \frac{8}{15} \\ P(Y>X) &= \int_1^2\int_0^y \frac{1}{50}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y + \int_2^4\int_0^2 \frac{1}{50}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{50}\int_1^2 \frac{x^3}{3}+xy^2\Big|_0^y\,\mathrm{d}y + \frac{1}{50}\int_2^4 \frac{x^3}{3}+xy^2\Big|_0^2\,\mathrm{d}y \\ &= \frac{1}{50}\int_1^2 \frac{4}{3}y^3\,\mathrm{d}y + \frac{1}{50}\int_2^4 \frac{8}{3}+2y^2\,\mathrm{d}y = \frac{1}{50}\left(\frac{y^4}{3}\Big|_1^2+\frac{8y}{3}+\frac{2y^3}{3}\Big|_2^4\right) = \frac{143}{150}. \end{aligned}$$

□

Problem 3. Let $f_{(X,Y)}$ be a joint density of random variables X, Y. Find f_{X+Y} if $f_{(X,Y)} = \mathbf{1}_{[0,1]\times[0,1]}$.

Proof. Since X, Y have joint density $f_{(X,Y)}$, the density of their sum is given by

$$f_{X+Y}(z) = \int_{\mathbb{R}} f_{(X,Y)}(x,z-x)\,\mathrm{d}x = \begin{cases} 0 & z<0 \\ z & 0\le z\le 1 \\ 2-z & 1\le z\le 2 \\ 0 & 2<z \end{cases}.$$

□

Problem 4. Let $\Omega=\{1,2,3,4\}$, $\mathcal{F}=2^\Omega$, with the measure P given by $P(\{n\})=1/4$, $(n=1,\cdots,4)$. Let $X:\Omega\to\mathbb{R}$ be a random variable given by $X(1)=X(3)=0$, $X(2)=X(4)=1$, $Y:\Omega\to\mathbb{R}$ be a random variable given by $Y(1)=Y(2)=0$, $Y(3)=Y(4)=1$ and $Z(\omega)=1-X(\omega)$.

(1) Find probability distributions P_X and P_Y.

(2) Find joint distributions $P_{(X,Y)}$ and $P_{(X,Z)}$.

Proof. (1)

$$\begin{aligned} P_X(0) &= P(\{\omega\in\Omega: X(\omega)=0\}) = P(\{1,3\}) = P(\{1\})+P(\{3\}) = \frac14+\frac14=\frac12 \\ P_X(1) &= P(\{\omega\in\Omega: X(\omega)=1\}) = P(\{2,4\}) = P(\{2\})+P(\{4\}) = \frac14+\frac14=\frac12 \\ P_Y(0) &= P(\{\omega\in\Omega: Y(\omega)=0\}) = P(\{1,2\}) = P(\{1\})+P(\{2\}) = \frac14+\frac14=\frac12 \\ P_Y(1) &= P(\{\omega\in\Omega: Y(\omega)=1\}) = P(\{3,4\}) = P(\{3\})+P(\{4\}) = \frac14+\frac14=\frac12. \end{aligned}$$

(2)

$$\begin{aligned}
P_{(X,Y)}(0,0) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (0,0)\}) = P(\{1\}) = \frac{1}{4} \\
P_{(X,Y)}(0,1) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (0,1)\}) = P(\{3\}) = \frac{1}{4} \\
P_{(X,Y)}(1,0) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (1,0)\}) = P(\{2\}) = \frac{1}{4} \\
P_{(X,Y)}(1,1) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (1,1)\}) = P(\{4\}) = \frac{1}{4} \\
P_{(X,Z)}(0,1) &= P(\{\omega \in \Omega : (X(\omega), Z(\omega)) = (0,1)\}) \\
&= P(\{\omega \in \Omega : (X(\omega), 1 - X(\omega)) = (0,1)\}) \\
&= P(\{\omega \in \Omega : X(\omega) = 0\}) = P(\{1,3\}) = P(\{1\}) + P(\{3\}) \\
&= \frac{1}{2} \\
P_{(X,Z)}(1,0) &= P(\{\omega \in \Omega : (X(\omega), Z(\omega)) = (1,0)\}) \\
&= P(\{\omega \in \Omega : (X(\omega), 1 - X(\omega)) = (1,0)\}) \\
&= P(\{\omega \in \Omega : X(\omega) = 1\}) = P(\{2,4\}) = P(\{2\}) + P(\{4\}) \\
&= \frac{1}{2}
\end{aligned}$$

$\square$