Modern Algebra I – Homework 8

Junwoo Yang

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Problem 1. Let a belong to a ring R. Let $S = \{x \in R : ax = 0\}$. Show that S is a subring of R.

Proof. Since $a \cdot 0 = 0$, $0 \in S$. S is nonempty set. Let $x, y \in S$. then $ax = 0$, $ay = 0$. $a(x-y) = ax-ay = 0-0$. So, $x-y \in X$. $a(xy) = (ax)y = 0 \cdot y = 0$. So, $xy \in S$. Therefore S is a subring of R.

Problem 2. Let m and n be positive integers and let k be the least common multiple of m and n. Show that $m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$.

Proof. Since every multiple of k is obviously multiple of both m and n, $k\mathbb{Z} \subseteq m\mathbb{Z} \cap n\mathbb{Z}$ is trivial. Let $x = am = bn$, i.e. $x \in m\mathbb{Z} \cap n\mathbb{Z}$. Let $x = qk+r$, $r < k$. Since x, k are both multiples of m, n, then so is $r = x - qk$. k is the least natural number, therefore this is a contradiction. Thus, x is multiple of k. $m\mathbb{Z} \cap n\mathbb{Z} \subset k\mathbb{Z}$. $\therefore m\mathbb{Z} \cap n\mathbb{Z} = k\mathbb{Z}$.

Problem 3. Give an example of a finite non-commutative ring. Give an example of an infinite non-commutative ring that does not have a unity.

Proof. Consider $M_2(\mathbb{Z}_p) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Z}_p\}$ in which p is prime. $M_2(\mathbb{Z}_p)$ is commutative group under addition. But matirx multiplication is not commutative. Also, it satisfies that for all $x, y \in M_2(\mathbb{Z}_p)$, $(xy)z = x(yz)$, $(x+y)z = xz+yz$. Thus, $M_2(\mathbb{Z}_p)$ is non-commutative ring. $M_2(2\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in 2\mathbb{Z}\}$, meanwhile, is infinite non-commutative ring without unity.

Problem 4. Describe all the subrings of the ring of integers.

Proof.

Problem 5. Let $R = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $S = \{(a, b, c) \in R : a + b = c\}$. Prove or disprove that S is a subring of R.

Proof.

Problem 6. Find a zero-divisor in $\mathbb{Z}_5[i] = \{a + bi : a, b \in \mathbb{Z}_5\}$.

Proof.

Problem 7. Find all solutions of the equation $x^3 - 2x^2 - 3x = 0$ in $\mathbb{Z}_{12}$.

Proof.

Problem 8. Find all solutions of $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_7$.

Proof.

Problem 9. Let x and y belong to a commutative ring R with prime characteristic p. Show that $(x + y)^p = x^p + y^p$.

Proof.

Problem 10. Show that $\mathbb{Z}_7[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Z}_7\}$ is a field.

Proof.

Problem 11. Let F be a field of order 2^n. Prove that $\operatorname{char} F = 2$.

Proof.