Stochastic Calculus Instructor: Hyungbin Park

Lecture note 3: Stochastic calculus

1 Exercises

Problem 1.1. (10 points) Evaluate the variance of $\int_0^T tB_t\, dB_t$.

Problem 1.2. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $f : [0,T] \to \mathbb{R}$ be a Borel function with $\int_0^T f^2(u)\, du < \infty$.

(i) (10 points) Show that as a stochastic process the map

$$f : \Omega \times [0,T] \to \mathbb{R}$$

is progressively measurable. Hint: See the definition of progressively measurable and use the fact that the product σ-algebra $\mathcal{F}_t \otimes \mathcal{B}[0,t]$ is generated by measurable rectangles. For each $t \in [0,T]$, since $f : [0,t] \to \mathbb{R}$ is a Borel function, we know $f^{-1}(A)$ is in $\mathcal{B}[0,t]$ for Borel set $A \subseteq \mathbb{R}$.

(ii) (10 points) Show that the process

$$\Big(\int_0^t f(u)\, dB_u\Big)_{0\leq t\leq T}$$

is a Gaussian process. You may use, without proof, the fact that the limit of normal distributions is normal.

(iii) (5 points) Evaluate $\mathbb{E}(e^{\int_0^T t\, dB_t})$.

Problem 1.3. Solve the following problems.

(i) (10 points) Let $f : [0,T] \to \mathbb{R}$ be a Borel function with $\int_0^T f^2(t)\, dt < \infty$. Show that a process

$$M_t := e^{\int_0^t f(s)\, dB_s - \frac{1}{2}\int_0^t f^2(s)\, ds},\ 0 \leq t \leq T$$

is a martingale.

(ii) (10 points) Let $\theta \in \mathcal{H}^2_{\text{loc}}$. Show that a process

$$M_t := e^{\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds},\ 0 \leq t \leq T$$

is a local martingale.

Problem 1.4. Let $B = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t\geq 0}$ be a three dimensional Brownian motion, and consider the filtration $(\mathcal{F}_t^B)_{t\geq 0}$ generated by B.

(i) (10 points) For each $T > 0$, show that

$$(B^{(1)}B^{(2)}, tB^{(3)}, 0) \in \mathcal{H}^2(\Omega \times [0,T], \mathbb{R}^3, \mathcal{F}, (\mathcal{F}_t^B), \mathbb{P}),$$

and deduce that

$$\Big(\int_0^t B_s^{(1)} B_s^{(2)}\, dB_s^{(1)} + \int_0^t sB_s^{(3)}\, dB_s^{(2)}\Big)_{t\geq 0}$$

is a martingale.

(ii) (10 points) Find the mean and the variance of

$$\int_0^T B_s^{(1)} B_s^{(2)}\, dB_s^{(1)} + \int_0^T sB_s^{(3)}\, dB_s^{(2)}$$

(iii) (10 points) Show that

$$\Big(B_t^{(1)} - 2B_t^{(2)} + \int_0^t B_s^{(1)}\, dB_s^{(3)}\Big)^2 - \int_0^t (B_s^{(1)})^2\, ds - 5t\,,\ t \geq 0$$

is a martingale.

Problem 1.5. Let $B = (B_t^{(1)}, \cdots, B_t^{(d)})_{t\geq 0}^{\top}$ be a d-dimensional Brownian motion.

(i) (10 points) Let $g, h \in \mathcal{H}^2([0,T] \times \Omega, \mathbb{R}^d, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Show that

$$\mathbb{E}\Big(\int_0^T g_t\, dB_t \int_0^T h_t\, dB_t\Big) = \mathbb{E}\Big(\int_0^T g_t \cdot h_t\, dt\Big).$$

Hint: Apply the Ito isometry to $\int_0^T (g_t + h_t)\, dB_t$.

(ii) (10 points) Evaluate

$$\mathbb{E}\Big(B_{T/2}^{(1)} \int_0^T (B_t^{(2)})^2\, dB_t^{(1)}\Big)\,,\ \mathbb{E}\Big(B_T^{(1)} B_T^{(2)} \int_0^T e^{B_t^{(2)}}\, dB_t^{(1)}\Big).$$

Problem 1.6. (15 points) Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and define

$$X_t = -t^2 + \sin(B_t^2)$$
$$Y_t = \int_0^t B_s\, ds + \int_0^t sB_s^2\, dB_s$$

Find the quadratic variations and covariations $\langle X\rangle$, $\langle Y\rangle$, and $\langle X, Y\rangle$.

Problem 1.7. (20 points) Let $B = (B_t^{(1)}, \cdots, B_t^{(d)})_{t\geq 0}^{\top}$ be a d-dimensional Brownian motion. Consider two Ito processes X and Y given as

$$X_t = x + \int_0^t b_u^X\, du + \int_0^t \sigma_u^X\, dB_u$$
$$Y_t = y + \int_0^t b_u^Y\, du + \int_0^t \sigma_u^Y\, dB_u$$

where b^X, b^Y, σ^X, σ^Y are progressively measurable and

$$\mathbb{P}\Big(\int_0^T |b_u^X| + |b_u^Y| + \|\sigma_u^X\|^2 + \|\sigma_u^Y\|^2 \, du < \infty\Big) = 1$$

for each $T > 0$. Show that the quadratic covariation is

$$\langle X, Y\rangle_t = \int_0^t \sigma_u^X \cdot \sigma_u^Y \, du \, .$$

Here, $\cdot$ is the usual dot product. Write down the main idea of the proof as we did in class. Do not provide any rigorous proof.

Problem 1.8. Let B be a d-dimensional Brownian motion.

(i) (10 points) Suppose that

$$C + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dB_s = 0 \, , \ 0 \leq t \leq T$$

where C is a constant, and b and σ are 1-dimensional and d-dimensional progressively measurable processes, respectively, satisfying $\int_0^T |b_s| + \|\sigma_s\|^2 \, ds < \infty$ almost surely. Show that $C = 0$, and $b = 0$, $\sigma = 0$ almost surely on $\Omega \times [0, T]$. Hint: quadratic variation

(ii) (5 points) Let X be an Ito process. Deduce that the decomposition

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dB_s \, , \ 0 \leq t \leq T$$

is unique.

References