

Topology II – Homework 1

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Exercise 77.1. Let X be a space obtained by pasting the edges of a polygonal region together in pairs. Show that X is homeomorphic to exactly one of the spaces in the following lists:

- (a) $S^2, P^2, K, T_n, T_n \# P^2, T_n \# K$, where K is the Klein bottle and $n \geq 1$.
- (b) $S^2, T_n, P^2, K_m, P^2 \# K_m$, where K_m is the m -fold connected sum of K with itself and $m \geq 1$.

Proof. By the classification theorem, X is homeomorphic to exactly one of S^2 , T_n , and $(P^2)_n$ with $n \geq 1$. Both lists contain S^2 , P^2 , and T_n with $n \geq 1$. Thus, we just need to check that the remaining surfaces on the lists are homeomorphic to $(P^2)_n$ with $n \geq 2$. Let us first show that the Klein bottle is homeomorphic to $P^2 \# P^2$. For this, consider the following cut and paste procedure:

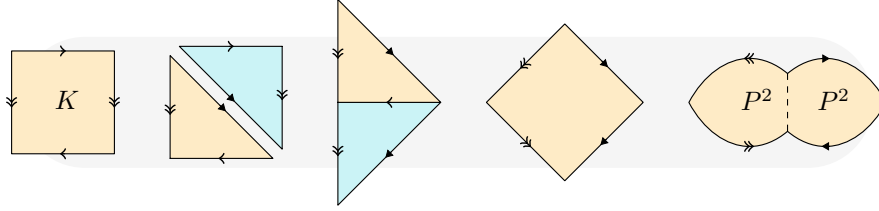


Figure 1: $K = P^2 \# P^2$

Thus, $K_m = (P^2)_{2m}$ and $P^2 \# K_m = P^2 \# (P^2)_{2m} = (P^2)_{2m+1}$ with $m \geq 1$. The proof of (b) is complete.

Let us now prove that $T \# P^2 = K \# P^2$, hence $T \# P^2 = P^2 \# P^2 \# P^2$. It is already quite a bit harder to devise a cut and paste method for showing this, but it can be done as illustrated in Figure 2. Note that this shows that the connected sum does not have a cancellation law: $T \# P^2 = K \# P^2$ does not imply $T = K$. Thus, we have the following relation

$$T_n \# P^2 = \underbrace{T \# \cdots \# T}_{n-1} \# T \# P^2 = \underbrace{T \# \cdots \# T}_{n-1} \# P^2 \# P^2 \# P^2 = \cdots = (P^2)_{2n+1}$$

with $n \geq 1$, and hence $T_n \# K = T_n \# P^2 \# P^2 = (P^2)_{2n+1} \# P^2 = (P^2)_{2n}$ with $n \geq 2$. This completes the proof of (a). \square

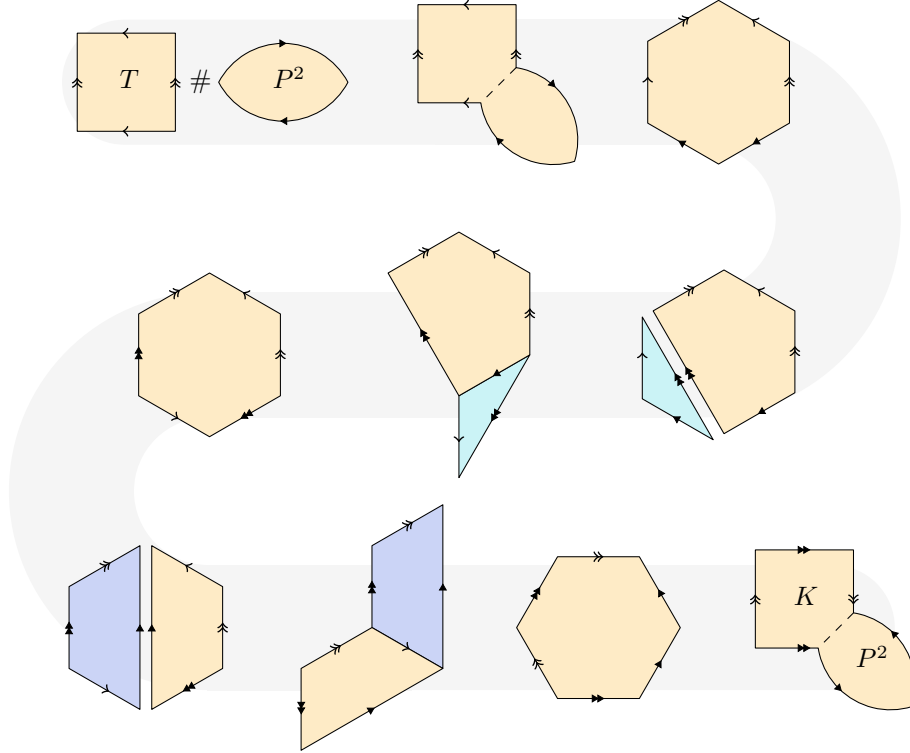


Figure 2: $T \# P^2 = K \# P^2$

Exercise 77.4. Let w be a proper labelling scheme for a 10-sided polygonal region. If w is of projective type, which of the list of spaces in Theorem 77.5 can it represent? What if w is of torus type?

Proof. Assuming w is of projective type, by Corollary 77.2, w can be decomposed as $a_1 a_1 a_2 a_2 \dots a_\ell a_\ell w_1$ where $\ell \geq 1$ and w_1 is empty or of torus type. Note that w has to start with at least one pair of an alphabet with same exponent, so w_1 can have the length of up to 8. Thus, w is never the same as S^2 and T_n . First we cancel all pairs like aa^{-1} in w_1 . If w_1 were all composed of these pairs, w would be $(P^2)_n$ with $1 \leq n \leq 5$. Now by Lemma 77.3, w_1 is equivalent to a scheme starting with a torus like $aba^{-1}b^{-1}$ whose length may be less by cancelling of tail terms. Then w has $ccaba^{-1}b^{-1}$ term which is equivalent to $aabbcc$ by Lemma 77.4 (also can be understood as $P^2 \# T = (P^2)_3$). This implies it doesn't matter how many tori w_1 generates. It only depends on the number of cancelling. Hence the resulting surface of w is homeomorphic to $(P^2)_n$ with $1 \leq n \leq 5$.

Now, let w be of torus type. Since w doesn't have any pair of an alphabet with same exponent like aa , w is never the same as $(P^2)_n$. First cancel all pairs like aa^{-1} . If w were all composed of these pairs, it would be S^2 . Now then, applying Lemma 77.3 again, w equivalent to a scheme starting with a torus like $aba^{-1}b^{-1}$. If we repeat this process one more time, w will correspond to one of S^2 , T , and $T \# T$. \square