

Topology I – Homework for Chapter 5, 6

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Problem 1. Find a compact topological space which does not satisfy the first countability axiom and justify your answer.

Proof. Let topological space X be \mathbb{R} with finite complement topology. We want to show that X is compact but does not satisfy the first countability axiom.

First, let's check compactness. Let $\{U_i | i \in I\}$ be open cover of X . Take any $U_0 \in \{U_i\}$. Then $X \setminus U_0$ contains only finite number of points $\{x_1, \dots, x_n\}$. For any $1 \leq k \leq n$, choose $U_k \in \{U_i\}$ containing x_k . Such elements exist since $\{U_i\}$ covers X . Then $\{U_0, U_1, \dots, U_n\}$ is a finite subcover of X . Thus, X is compact.

Now, if we show X does not satisfy first countability axiom, we are done. Pick arbitrarily $x \in \mathbb{R}$, and suppose $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ is a countable family of open neighborhoods of x . For each $n \in \mathbb{N}$, there is by definition a finite $F_n \subset \mathbb{R}$ such that $B_n = \mathbb{R} \setminus F_n$. Let $C := \{x\} \cup \bigcup_{n \in \mathbb{N}} F_n$. Since C is the union of countably many finite sets, so C is countable. Now pick $y \in \mathbb{R} \setminus C$. Note that $y \in B_n$ for any n . Let $U := \mathbb{R} \setminus \{y\}$. By definition, U is open, and clearly $x \in U$. However, for any $n \in \mathbb{N}$, $y \in B_n \setminus U$, so $B_n \not\subset U$. This implies that \mathcal{B} is not a local base at x . Since $x \in \mathbb{R}$ and \mathcal{B} was arbitrary, X is not first countable. ■

Problem 2. Show that the real line equipped with the lower limit topology \mathbb{R}_l is not metrizable.

Proof. We first check \mathbb{R}_l is separable. Recall that X is separable if there exists countable subset whose closure is X . Construct subset $\{[z, z+1) : z \in \mathbb{Z}\}$. Then their closure $\overline{\{[z, z+1) : z \in \mathbb{Z}\}} = \mathbb{R}$. Thus, \mathbb{R}_l is separable.

Now, we claim that \mathbb{R}_l is not second countable. Let B be basis of \mathbb{R}_l . By definition of basis, for any open set U and $x \in U$, there exists $B_\alpha \in B$ such that $x \in B_\alpha \subset U$. For all $x \in \mathbb{R}$, pick $U = [x, x+\varepsilon) \in \mathbb{R}_l$ for $\varepsilon > 0$. Then, for each $x \in \mathbb{R}$, there exists $B_x \in B$ such that $x \in B_x \subset [x, x+\varepsilon)$. This B_x has an infimum equal to x , so for different x , the corresponding B_x is different. Thus, the cardinality of B is at least $|\mathbb{R}|$, which is uncountable. So, our claim is proved.

Proposition 5.3. A separable metric space satisfies the second axiom of countability.

If \mathbb{R}_l is metric space (or metrizable), \mathbb{R}_l is second countable from above proposition. This is a contradiction. Hence, \mathbb{R}_l is not metrizable. ■

Problem 3. Let $f : X \rightarrow Y$ be a closed, continuous surjective map between topological spaces. Suppose further that inverse image of any singleton set in Y is compact in X .

- (a) Show that if X is Hausdorff, then so is Y .
- (b) Show that if X is normal, then so is Y .
- (c) Show that if X is locally compact, then so is Y .
- (d) Show that if X satisfy the second countability, then so is Y .

Proof. (a) Suppose Y is not Hausdorff. There exists distinct y_1, y_2 such that there doesn't exist open V_1, V_2 with $V_1 \ni y_1, V_2 \ni y_2, V_1 \cap V_2 = \emptyset$. Namely, every open sets containing y_1, y_2 respectively have non-empty intersection and so are their inverse images, i.e. $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) \neq \emptyset$. Let $C_1 := f^{-1}(y_1), C_2 := f^{-1}(y_2)$. Since C_1 and C_2 are compact and disjoint, by Proposition 5.5, there exists disjoint open sets $U_1 \supset C_1, U_2 \supset C_2$. Now if there exists open sets $W_1 \ni y_1, W_2 \ni y_2$ such that $f^{-1}(W_1) \subset U_1, f^{-1}(W_2) \subset U_2$, we are done. Because if it is, there is contradiction between $W_1 \cap W_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$. Let $W_1 := Y \setminus f(X \setminus U_1)$. Note that $X \setminus U_1$ is closed, $f(X \setminus U_1)$ is also closed, and $Y \setminus f(X \setminus U_1)$ is open in Y . Then, the inverse image of W_1 ,

$$f^{-1}(W_1) = f^{-1}(Y \setminus f(X \setminus U_1)) = f^{-1}(Y) \setminus f^{-1}(f(X \setminus U_1)) = X \setminus f^{-1}(f(X \setminus U_1))$$

where the last equality is due to surjection. Since $f^{-1}(f(X \setminus U_1)) \supset X \setminus U_1$, taking complements gives that $X \setminus f^{-1}(f(X \setminus U_1)) \subset U_1$. We proved $f^{-1}(W_1) \subset U_1$. It is exactly the same with W_2 as well. ■

- (b) Since X is Hausdorff, so is Y by (a). What we need to show is that for any disjoint closed sets $V_1 \ni y_1, V_2 \ni y_2$, there exists open $U_1 \supset V_1, U_2 \supset V_2$ with $U_1 \cap U_2 = \emptyset$. Since f is continuous, $f^{-1}(V_1), f^{-1}(V_2)$ are disjoint closed sets in X . By normality of X , there are open O_1, O_2 with $f^{-1}(V_1) \subset O_1, f^{-1}(V_2) \subset O_2, O_1 \cap O_2 = \emptyset$. Note that $f^{-1}(y_1) \subset f^{-1}(V_1) \subset O_1$. By (a), there are open sets $W_{y_1}^1, W_{y_2}^2 \subset Y$ with $f^{-1}(W_{y_1}^1) \subset O_1, f^{-1}(W_{y_2}^2) \subset O_2$. Let $W^1 := \bigcup_{y_1 \in V_1} W_{y_1}^1, W^2 := \bigcup_{y_2 \in V_2} W_{y_2}^2$. Note that $W^1 \supset V_1, W^2 \supset V_2$, and they are open. Now we claim that $W^1 \cap W^2 = \emptyset$. Suppose that there is $z \in W^1 \cap W^2$. Since $z \in W_{y_1}^1$ for some $y_1 \in V_1, f^{-1}(z) \subset f^{-1}(W_{y_1}^1) \subset O_1$. In same way, since $z \in W_{y_2}^2$ for some $y_2 \in V_2, f^{-1}(z) \subset f^{-1}(W_{y_2}^2) \subset O_2$. This is a contradiction to $O_1 \cap O_2 = \emptyset$ and f is onto map, so $f^{-1}(z) \neq \emptyset$. ■
- (c) What we need to show is that for any $y \in Y$, there is open set $O_y \ni y$ whose closure $\overline{O_y}$ is compact. For arbitrary $y \in Y$, consider $f^{-1}(y)$ which is compact. Since X is locally compact, for $x \in f^{-1}(y)$, there is open U_x such that $\overline{U_x}$ is compact. $\{U_x : x \in f^{-1}(y)\}$ is open covering of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there exists finite subcovering $U = U_{x_1} \cup \dots \cup U_{x_n}$. Note U is open. Let $\overline{U} := \bigcup_{i=1}^n \overline{U_{x_i}}$. \overline{U} is finite union of compact sets, so \overline{U} is compact. By (a) there is $V \subset Y$ such that $y \in V, f^{-1}(V) \subset U$. That is, $y \in V \subset f(U) \subset f(\overline{U})$. Since \overline{U} is compact and f is continuous, $f(\overline{U})$ is compact. Hence, for any $y \in Y$, there is compact subspace $f(\overline{U})$ that contains a neighborhood $f(U)$ of y . ■
- (d) Let $V \subset Y$ open, $y \in V$. Then $f^{-1}(y) \subset f^{-1}(V), f^{-1}(y)$ is compact, and $f^{-1}(V)$ is open. Since X is second countable, there exists countable basis B of X . For every $x \in f^{-1}(y)$, there is $B_x \in B$ with $x \in B_x \subset f^{-1}(V)$. $\{B_x : x \in f^{-1}(y)\}$ is open covering of $f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there is finite subcovering $B_y = \bigcup_{i=1}^n B_i$. By (a), there is open set $W \subset Y$ such that $y \in W, f^{-1}(W) \subset B_i \subset f^{-1}(V)$. Thus, $y \in W \subset V$.

Lemma. Let C be collection of open sets of X . For any open set U of $X, x \in U$, if there is $x \in C_\alpha \subset U$ where $C_\alpha \in C$, then C is basis.

Let W^J be union of all open sets $W \subset Y$ such that $f^{-1}(W) \subset \bigcup B_i$. $\{W_I : I \text{ finite}\}$ is basis of Y . This is finite. Hence, Y is second countable. ■