Financial Mathematics I (SNU) – Exam 2

June 4, 2020

1. For a two-dimensional Brownian motion $(B_t^{(1)}, B_t^{(2)})_{t\geq 0}$, calculate
$$\mathbb{E}(e^{\int_0^T \sqrt{t}\,dB_t^{(1)} + B_T^{(2)}}).$$

2. Consider the progressive σ-algebra
$$\Sigma \coloneqq \{A \subseteq [0,\infty) \times \Omega : A \text{ is progressive}\}.$$
Show that a stochastic process X is progressively measurable if and only if X is Σ-measurable.

3. Let $\theta \in \mathcal{H}^2_{\text{loc}}$. Show that a process
$$M_t \coloneqq e^{\int_0^t \theta_s\,dB_s - \frac{1}{2}\int_0^t \theta_s^2\,ds}, \quad 0 \leq t \leq T$$
is a local martingale.

4. Show that a left-continuous adapted process is progressively measurable.

5. Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that the process
$$\left(\cos B_t + \frac{1}{2}\int_0^t \cos B_s\,ds\right)_{t\geq 0}$$
is a martingale.

6. Let $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t\geq 0}$ be a three-dimensional Brownian motion. Define
$$X_t = te^{B_t^{(2)}} + \int_0^t sB_s^{(1)}\,dB_s^{(3)}.$$
Find the quadratic variation $\langle X \rangle_t$ for $t \geq 0$.