Vector Calculus – Final Exam

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1. Evaluate the following integrals:

   (1) $\displaystyle\int_0^1 \int_y^1 x^2 \sin xy \,\mathrm{d}x\,\mathrm{d}y$

   (2) $\displaystyle\int_0^2 \int_{\frac{x}{2}}^{\frac{x}{2}+1} x^5(2y-x)e^{(2y-x)^2}\,\mathrm{d}y\,\mathrm{d}x$

2. Find the volume of the solid bounded by $z = 2 - y^2$ and $4x^2 + y^2 = z$.

3. Consider a region bounded by $(z-8)^2 = 4(x^2+y^2)$ and $z = 0$. Find the centroid of this region.

4. Explain the numerical integration with the Simpson's rule (or 2-point Gaussian quadrature rule, if you prefer) and use it to compute the integration:

   $$\int_0^2 e^x \sin x \,\mathrm{d}x.$$

   You don't need to make a finer partition for the interval $[0, 2]$.

5. Let F be a vector field in $\mathbb{R}^3$ given by $F(x, y, z) = (e^x \sin y - yz, e^x \cos y - xz, z - xy)$. Suppose $\mathbf{x}\colon [0,1] \to \mathbb{R}^3$ be a curve such that $\mathbf{x}(t) = (t, t^2, t^3)$. Evaluate the work done by F along the path $\mathbf{x}$.

6. Let D be a closed, bounded region in $\mathbb{R}^2$ where ∂D is a simple, closed, C^1 curve. Let F be a C^1 vector field defined on $\mathbb{R}^2$.

   (1) State the Green's theorem and the divergence theorem.

   (2) Prove the divergence theorem using the Green's theorem.

7. Let S be a surface defined by $S = \{(x, y, z) \in \mathbb{R}^3 : z = 4e^{-(x^2+y^2)} - 1, z \geq 0\}$. The surface S is oriented by the normal vector $\mathbf{n}$ such that $\mathbf{n} \cdot \mathbf{k} \geq 0$. Let F be a vector field defined in $\mathbb{R}^3$ such that $F(x, y, z) = (e^{y+z} - 2y, xe^{y+z} + y, e^{x+y})$. Then, evaluate the integration:

   $$\iint_S \nabla \times F \cdot \mathrm{d}S.$$

8. Let S be a surface defined by $S = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1\}$ and $\mathbf{n}$ be the outward unit normal vector on S. Let f be a function defined on $\mathbb{R}^3 \setminus \{0\}$ such that $f(x, y, z) = \log(x^2 + y^2 + z^2)$. Then, evaluate the integration:

   $$\iint_S \nabla f \cdot \mathbf{n} \,\mathrm{d}S.$$