Mathematical Statistics 1

Ch.3 Continuous Distributions

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Table of Contents

Ch.3.1 Random Variables of the Continuous Type

1.1 probability density function and cumulative distribution function

Probability density function (pdf)

The *probability density function (pdf)* of a continuous random variable X, with space $\mathcal{X}$ that is an interval or union of intervals, is an integrable function $f_X(x)$ satisfying the following conditions:

- $f_X(x) \geq 0, \quad x \in \mathcal{X}$
- $\displaystyle\int_{\mathcal{X}} f_X(x)dx = 1$
- If $(a, b) \subseteq \mathcal{X}$,

$$P(a < X < b) = \int_a^b f_X(x)dx.$$

cumulative distribution function (cdf) of continuous r.v.

The cumulative distribution function (cdf) of a continuous random variable X is defined as

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt, \quad -\infty < x < \infty$$

Here, if the derivative $F_X'(x)$ exists,

$$f_X(x) = F_X'(x)$$

Example 3.1-1

Let r.v. X be a continuous random variable with pdf

$f_X(x) = 2x,\ 0 < x < 1.$

- Show that $\int_{\mathcal{X}} f_X(x)dx = 1$
- Find the cdf of X.

X discrete

$f_X(x) = P(X=x)$ pmf

$0 < f_X(x) \leq 1 \;,\; x \in \mathcal{X}$

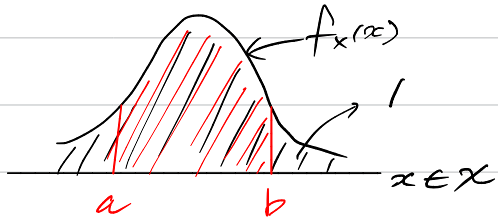
$\sum_{x \in \mathcal{X}} f_X(x) = 1$

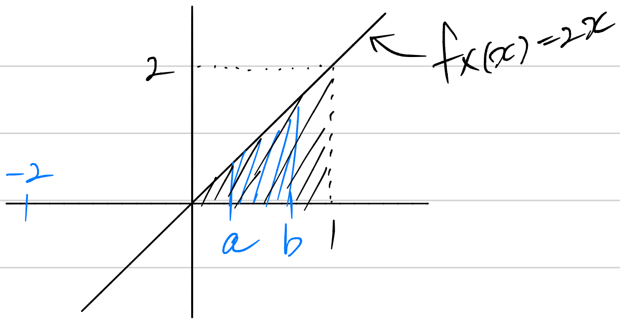
X continuous.

$f_X(x)$: intergrable function

pdf

$$\begin{cases} f_X(x) \geq 0 & x \in \mathcal{X} \\ \int_{\mathcal{X}} f_X(x)\,dx = 1 \\ P(a < x < b) = \int_a^b f_X(x)\,dx \end{cases}$$





$f_X(x) = 2x \quad 0 < x < 1$

$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

cdf. $F_X(x) = P(X \leq x) \quad x \in \mathbb{R}$.

discreat). $F_X(x) = P(X \leq x)$

eg. $X = \begin{cases} 0 & \text{with } 1-p = 0.4 \\ 1 & \text{with. } p = 0.6 \end{cases}$

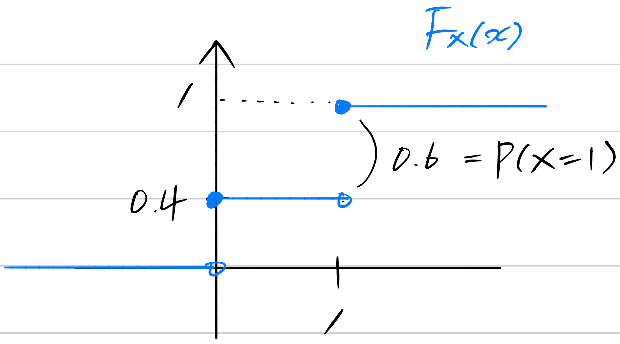
$(0.4)^{1-x}(0.6)^{x}$

$x < 0 \quad F_X(x) = P(X \leq x) = 0$

$x = 0 \quad F_X(x) = P(X \leq 0) = P(X=0) = 0.4$

$0 < x < 1 \quad F_X(x) = 0 \Rightarrow F_X(x) = 0.4. \quad 0 \leq x < 1$

$\left.\begin{array}{ll} x = 1 & F_X(x) = 1 \\ x > 1 & F_X(x) = 1 \end{array}\right\} \Rightarrow \begin{array}{l} x \geq 1 \\ F_X(x) = 1 \end{array}$



continuous)

$F_X(x) = P(X \le x)$: continuous function

$f_X(x) = F_X'(x)$

e.g.) $f_X(x) = 2x \quad 0 < x < 1$

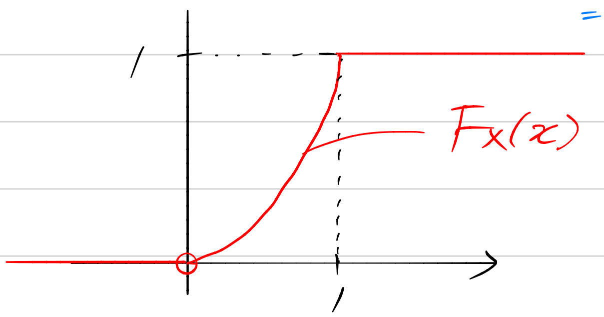
$x \le 0 \quad F_X(x) = P(X < x) = 0$ (Not defined)

$0 < x < 1 \quad F_X(x) = P(X \le x) = \int_0^x f_X(t)\,dt$

$= \int_0^x 2t\,dt = [t^2]_0^x = x^2$

$x \ge 1 \quad F_X(x) = P(X \le x) = P(X \overset{<}{\le} 1) + P(1 \overset{\le}{<} X \le x)$

$= 1 + \underbrace{P(1 \le X \le x)}_{=0} = 1$



continuous case

$P(X \le x) = P(X < x)$

but not discrete case.

$X \sim f_X(x) = c \cdot x \quad 0 < x < 1$

$$\int_0^1 cx\,dx = 1$$

$$P(a \le X \le b) = P(X \le b) - P(X \le a)$$
$$= F_X(b) - F_X(a).$$

Expectation

$E[g(X)]$

discrete).

$$\sum_{x \in \mathcal{X}} g(x) f_X(x)$$

continuous). $\int_{\mathcal{X}} g(x) f_X(x)\,dx$

i) $g(X) = x$

$E(X) = \mu_X = \int x f_X(x)\,dx$

ii) $g(X) = (X - \mu_X)^2$

$E[(X-\mu_X)^2] = \int (x-\mu_X)^2 f_X(x)\,dx = Var(X) = \sigma_X^2$

iii) mgf.

$M_X(t) = E[e^{tX}] = \int e^{tx} f_X(x)\,dx$

Property

Since there are no steps or jumps in $F_X(x)$, a distribution function of the continuous type,

$$P(X = b) = 0, \quad \text{for all real values of } b.$$

Thus, in the continuous case,

$$P(a \le X \le b) = P(a < X < b)$$
$$= P(a \le X < b) = P(a < X \le b) = F(b) - F(a).$$

1.2 Mean, Variance, and mgf

Mean and Variance

The expectation of the function $g(X)$, where X is a continuous random variable, is defined by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Here, the mean and variance of X are

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x) dx,$$

$$\sigma_X^2 = Var(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

Moment generating function

If the moment generating function exists,

$$M_X(t) = E_X[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

Example 3.1-3

For the random variable X in Example 3.1-1, compute the mean and variance.

Example 3.1-4

Let X have the pdf $f(x) = xe^{-x},\ x \geq 0$. Obtain the mgf and then calculate the mean and variance.

$f_X(x) = 2x. \quad 0 < x < 1$

$E[X] = \int_0^1 x \cdot 2x\, dx = 2\int_0^1 x^2 dx = \frac{2}{3}[x^3]_0^1 = \frac{2}{3}.$

$$E[(X-\mu_X)^2] = \int_0^1 (x-\tfrac{2}{3})^2 \cdot 2x\,dx$$
$$= \int_0^1 (x^2 - \tfrac{4}{3}x + \tfrac{4}{9})\, 2x\, dx.$$
$$= \int_0^1 2x^3 - \tfrac{8}{3}x^2 + \tfrac{8}{9}x\, dx$$
$$= \left[\tfrac{1}{2}x^4 - \tfrac{8}{9}x^3 + \tfrac{4}{9}x^2\right]_0^1$$
$$= \tfrac{1}{2} - \tfrac{8}{9} + \tfrac{4}{9} = \tfrac{1}{2} - \tfrac{4}{9} = \tfrac{9}{18} - \tfrac{8}{18} = \tfrac{1}{18}$$

$f_X(x) = x \cdot e^{-x} \quad (x \geq 0)$

mgf : $E[e^{tX}] = \int_x e^{tx} \cdot x e^{-x} dx = \int_0^\infty x \cdot e^{x(t-1)} dx.$

$$\frac{1}{t-1} e^{x(t-1)}$$

$$\left.\frac{x}{t-1} e^{x(t-1)}\right]_0^\infty - \int \frac{1}{t-1} e^{x(t-1)} dx.$$

$$\left. - \frac{1}{(t-1)^2} e^{x(t-1)} \right|_0^\infty$$

$$\frac{+1}{(t-1)^2}$$

$$M^{(1)}(0) = M^{(1)}(t)|_{t=0} = \left.\frac{-2(t-1)}{(t-1)^4}\right|_{t=0} = 2$$

$$M^{(2)}(0) = M^{(2)}(t)|_{t=0} = \left(\frac{-2}{(t-1)^3}\right)' = \frac{2\cdot 3(t-1)^2}{(t-1)^6} = \left.\frac{6}{(t-1)^4}\right|_{t=0}$$
$$= 6.$$

1). $f_X(x) = 2x, \quad 0<x<1$

$$E(X) = \int_0^1 x f_X(x)\, dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

$$E(X^2) = \int_0^1 x^2 f_X(x) dx = \int_0^1 2x^3 dx = \frac{1}{2}$$

$$\Rightarrow var(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

2). $f_X(x) = x \cdot e^{-x} \qquad x \geq 0$

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} x e^{-x} dx = \int_0^\infty \underset{u}{x} \cdot \underset{v'}{\underline{e^{-(1-t)x}}} dx.$$

t 범위 나타내기

$t \neq 1$

$$= \left[-x \frac{1}{1-t} e^{-\overset{0}{(1-t)}x} \right]_0^\infty + \int_0^\infty \frac{1}{(1-t)} e^{-(1-t)x} dx$$

$t<1$

$$= \int_0^\infty \frac{1}{1-t} e^{-(1-t)x} dx = \left[-\frac{1}{(1-t)^2} e^{-(1-t)x} \right]_0^\infty$$

$$= \underline{\frac{1}{(1-t)^2}, \quad t<1}$$

$$M_X^{(1)}(t) = \frac{2(1-t)}{(1-t)^4} \qquad M_X^{(1)}(0) = 2 \qquad E(X) = 2$$

$$var(X) = 2$$

$$M_X^{(2)}(t) = \frac{2 \cdot 3(1-t)^2}{(1-t)^6} \qquad M_X^{(2)}(0) = 6$$

(100p)th percentile

The (100p)th percentile is a number π_p such that the area under $f_X(x)$ to the left of π_p is p.

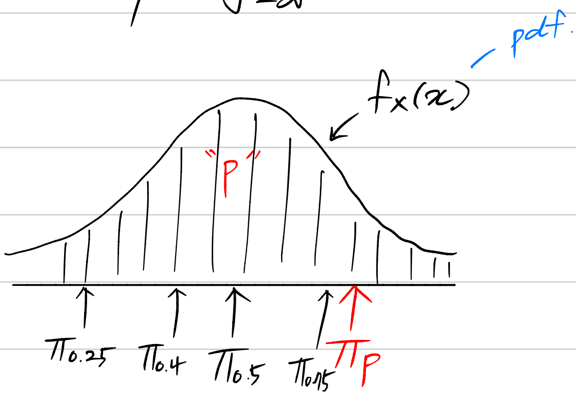
$$p = \int_{-\infty}^{\pi_p} f_X(x)dx = F_X(\pi_p).$$

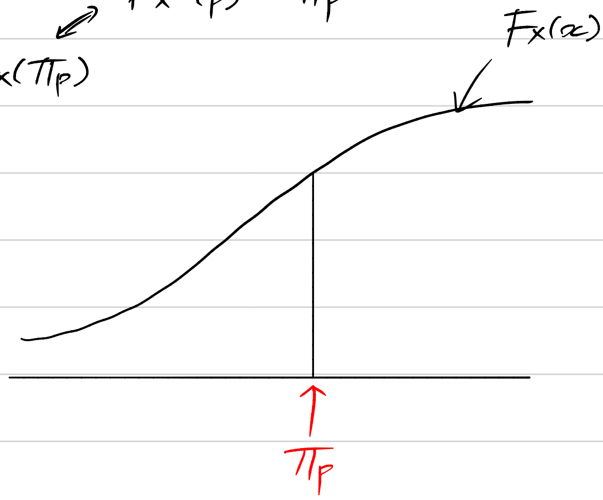
- median=50th percentile=the second quartile, $m = \pi_{0.5}$
- the first quartile=25th percentile, $q_1 = \pi_{0.25}$
- the thrid quartile=75th percentile, $q_3 = \pi_{0.75}$

π_p : $(100p)^{th}$ percentile.

$F_X^{-1}(p) = \pi_p$

$$p = \int_{-\infty}^{\pi_p} f_X(x)\,dx = F_X(\pi_p)$$





1) $p = 0.5$

$\pi_p = \pi_{0.5} = 50^{th}$ percentile. $= Q2$

2) $\pi_{0.25} = 25^{th}$ percentile $= Q1$

3) $\pi_{0.75} = 75^{th}$ percentile. $= Q3$

Example 3.1-6

Let X have the pdf

$$f(x) = e^{-x-1}, \quad -1 < x < \infty$$

- Compute $P(X \geq 1)$.
- Find the mgf of X.
- Find the $E(X)$ and $Var(X)$ of X.
- Find the median.

$P(X \geq 1) = 1 - P(X \leq 1)$

$P(X \leq 1) = \int_{-1}^{1} e^{-x-1} dx$

$= \left[-e^{-x-1}\right]_{-1}^{1} = -e^{-2} + 1 \quad \therefore P(X \geq 1) = e^{-2}$

$M_X(t) = E[e^{tx}] = \int_{-1}^{\infty} e^{tx} \cdot e^{-x-1} dx$

$= \int e^{x(t-1)-1} dx$

$\lim_{a \to \infty} \left[\frac{1}{t-1} e^{x(t-1)-1}\right]_{-1}^{a} \quad \underbrace{(t < 1)}_{//}.$

$= \underbrace{\frac{t}{t-1}}_{//}$ $\quad \cancel{x} - t \cancel{-1}$

$E(X) = M^{(1)}(0) = \left.\frac{(t-1) - t}{(t-1)^2}\right|_{t=0} = -1$

$E(X^2) = M^{(2)}(0) = \left(\frac{-1}{(t-1)^2}\right)' = \left.\frac{2(t-1)}{(t-1)^4}\right|_{t=0} = -2$

$Var(X) = -2 - (-1)^2 = -2 - 1 = -3.$

$f_X(x) = e^{-x-1} \quad -1 < x < \infty$

1) $P(X \geq 1) = \int_1^\infty e^{-x-1}\,dx = e^{-1}[-e^{-x}]_1^\infty = e^{-1}(e^{-1}) = e^{-2}$

2) $M_X(t) = E[e^{tX}] = \int_{-1}^\infty e^{tx}e^{-x-1}\,dx = e^{-1}\int_{-1}^\infty e^{-(1-t)x}\,dx$

$t \neq 1$ $\quad$ $1-t>0$

$= e^{-1}\left[-\frac{1}{(1-t)}e^{-(1-t)x}\right]_{-1}^\infty = e^{-1}\left(\frac{e^{1-t}}{1-t}\right) = e^{-t}(1-t)^{-1}, \quad t<1$

3) $M_X^{(1)}(t) = -e^{-t}(1-t)^{-1} + e^{-t}\frac{1}{(1-t)^2} \Rightarrow M_X^{(1)}(0) = e^0 - e^0 = 0$

$M_X^{(2)}(t) = \underline{\qquad\qquad\qquad} \Rightarrow M_X^{(2)}(0) = 1$

4) $F_X(\pi_{0.5}) = 0.5$

$F_X(x) = \int_{-1}^x f_X(x)\,dt = \int_{-1}^x e^{-t-1}\,dt$

$= e^{-1}\int_{-1}^x e^{-t}\,dt = e^{-1}[-e^{-t}]_{-1}^x$

$= e^{-1}(e - e^{-x}) = 1 - e^{-(1+x)}, \quad x > -1$

$\therefore F_X(\pi_{0.25}) = 0.5 \Rightarrow \pi_{0.25} = \ln 2 - 1$

$1 - e^{-(1+x)} = \frac{1}{2}$

$e^{-(1+x)} = \frac{1}{2}$

$-(1+x) = -\ln 2$

$1+x = \ln 2$

$x = \ln 2 - 1$