MAT4004: Topology 2

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Based on lecture by Youngsik Huh in fall 2021

Contents

Chapter 0

Review of Topology 1

Definition 1 (Topology)**.** A topology on a set X is a collection of subsets of X, {open sets}, which satisfies followings

1. $\emptyset, X \in \mathcal{T}$.
2. Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$.
3. Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$.

Elements in $\mathcal{T}$ are called open sets.

Lemma 1. product topology on $X \times Y$ is coarest topology s.t. π_1, π_2 are continuous.

Definition 2 (Basis)**.** A basis $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X s.t.

1. $\bigcup_{B \in \mathcal{B}} B = X$.
2. For any $x \in B_1 \cap B_2$ $(B_1, B_2 \in \mathcal{B})$, $\exists B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Definition 3 (Hausdorff)**.** A topological space X is Hausdorff if $\forall x_1 \neq x_2$, $\exists$neighborhood $U_1 \ni x_1$, $U_2 \ni x_2$ s.t. $U_1 \cap U_2 = \emptyset$.

Theorem 1 (Tychonoff theorem)**.** $\Pi_{\beta \in B} X_\beta$ is compact.

Definition 4 (Countable basis)**.** X has a countable basis of nbds at x if $\exists \{O_n\}_{n \in \mathbb{N}}$ of x s.t. for any nbd U of x, $\exists O_n \subset U$ for some $n \in \mathbb{N}$.

Definition 5 (First countable)**.** X is called first countable if X has countable basis of nbds at every point of X.

Example. Metric space is first countable. For any x, $O_n = B_{\frac{1}{n}}(x)$ $n \in \mathbb{N}$.

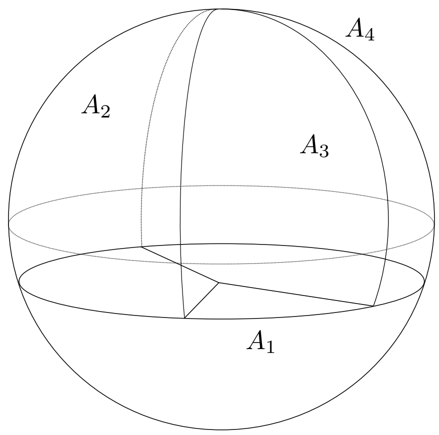

Figure 1: Example with four elements

Definition 6. A sequence $\{x_n\}$ converges to y if given any open nbd U of y, $\exists N$ so that if $n > N$, $x_n \in U$.

Theorem 2. $A \subset X$ topological space. If $x_n \in A$ converges to y, then $y \in \overline{A}$. Converse holds if X is first countable, that is, if $y \in \overline{A}$, then $\exists x_n \in A$ with $x_n \to y$.

Proof. First statement is easy. Say X first countable. Pick $y \in \overline{A}$, we will find $x_n \to y$, $x_n \in A$. $\exists \{O_n\}$ countable basis of nbds of y. Set

$$\begin{aligned} U_1 &= O_1 \\ U_2 &= O_1 \cap O_2 \\ U_3 &= O_1 \cap O_2 \cap O_3 \\ &\vdots \end{aligned}$$

Note that $U_1 \supset U_2 \supset U_3 \cdots$. $\{U_n\}_{n\in\mathbb{N}}$ is also countable basis of nbds of y.

Now, $y \in \overline{A}$, $\Rightarrow U_n \cap A \neq \emptyset$. Pick $x_n \in U_n \cap A$.Claim is that $x_n \to y$. Choose any nbd U of y. Then, $\exists N$ s.t. $O_n \subset U$. Note that If $n > N$, $U_n = O_1 \cap \cdots \cap O_N \cap \cdots \cap O_n \subset O_N \subset U$. $\therefore x_n \in U$ for any $n > N$. $\therefore x_n \to y$. □

Definition 7 (Second countable)**.** X is called second countable if X has countable basis (of topology).

Example. $\mathbb{R}$, $\{(a,b) \mid a, b \in \mathbb{Q}\}$.

Example. $X_1 \times \cdots \times X_n$ (X_i: second countable) is also second countable.

Example. Compact metric space.

Question If X is second countable, does it have a countable dense subset?

Definition 8 (Separable). X is called separable if $\exists$countable subset whose closure is X.

Proposition 1. Second countable $\Rightarrow$ separable.

Proposition 2. Separable metric space $\Rightarrow$ second countable.

Definition 9 (Normal). X is normal if X is Hausdorff and for any closed subset C_1, C_2 with $C_1 \cap C_2 = \emptyset$, $\exists$open sets U_1, U_2 with $U_1 \supset C_1$, $U_2 \supset C_2$, $U_1 \cap U_2 = \emptyset$.

Proposition 3. Every compact Hausdorff space is normal.

Theorem 3 (Urysohn's lemma). Let X be normal and C_1, C_2 disjoint closed subsets. Then $\exists$continuous function $f : X \to [0, 1]$ such that

1. $f(x) = 0 \quad \forall x \in A$.
2. $f(x) = 1 \quad \forall x \in B$.

Definition 10. Equivalence relation: $(X, \sim)$ satisfies

1. $x \sim x$
2. $x \sim y \Rightarrow y \sim x$
3. $x \sim y$, $y \sim z \Rightarrow x \sim z$

$X/_\sim$: the set of equivalence classes

Definition 11 (Locally compact). X is called locally compact if for any $x \in X$, $\exists$open nbd O of x such that $\overline{O}$ is compact.

Chapter 1

Quotient topology

https://en.wikipedia.org/wiki/Homotopy

Definition 12 (Homotopic)**.** If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there is a continuous map $F\colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each x. (Here $I = [0, 1]$.) The map F is called a homotopy between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is nulhomotopic.

Definition 13 (Evenly covered)**.** Let $p\colon E \to B$, surjective map (so continuous). Let $U \subset B$ open. Then U is evenly covered iff $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$ with

- V_α open in E
- $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$
- $p|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism.

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g\colon I \to X \,|\, g \simeq_p f\}$.

Chapter 2

Fundamental group and applications

Pick a base point x_0 and consider it fixed. (The fundamental gruop will not depend on it. We assume all spaces are path connected) $X \to \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f\colon I = [0, 1] \to X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

> **Example.** Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the 'constant' loop.

The composition of loops is simply pasting them. In the case of the circle, the loop $-1\circ$ the loop 2 is the loop 1.

Suppose $\alpha\colon I \to X$ and $f\colon X \to Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

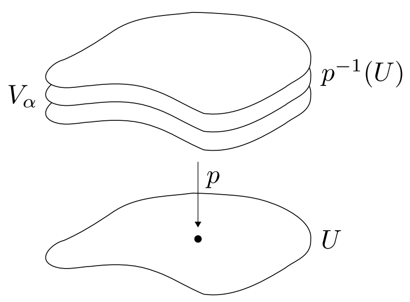

Figure 2.1: Evenly covered

Chapter 3

Jordan curve theorem

https://en.wikipedia.org/wiki/Jordan_curve_theorem

Chapter 4

Seifert–Van Kampen theorem

https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem

Chapter 5

Surfaces

Chapter 6

Covering spaces