Homework 4

Due date: 2019. 4. 10.

1. Determine whether the following permutations are as a product of disjoint cycles, and then as a product of 2-cycles.

(1) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

2. What is the order of each of the following permutations?

(1) $(1\,2\,4)(3\,5\,7)$

(2) $(1\,2\,4)(3\,5\,7\,8\,6\,9)$

(3) $(1\,2\,3\,5)(2\,4\,5\,6\,7)$

3. Show by an example that every proper subgroup of a non-Abelian group may be Abelian.

4. Show that A_8 contains an element of order 15.

5. Let α and β belongs to S_n. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation.

6. Let $\beta = (1\,2\,3)(1\,4\,5)$. Write β^{99} in disjoint cycle form.

7. What cycle is $(a_1\, a_2\, \cdots\, a_n)^{-1}$?

8. Prove that S_n is non-Abelian for all $n \geq 3$.

9. Show that every element in A_n for $n \geq 3$ can be expressed as a 3-cycle or a product of three cycles.

10. Show that if α is a cycle of odd length, then α^2 is a cycle.

현대대수1 - HW4

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1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (1\,8)(3\,6\,4)(5\,7) = (1\,8)(3\,4)(3\,6)(5\,7)$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix} = (1\,3\,4)(2\,6)(5\,8\,7) = (1\,4)(1\,3)(2\,6)(5\,7)(5\,8)$

2. (1) $(1\,2\,4)(3\,5\,7)$

$\alpha = (1\,2\,4)$, $\beta = (3\,5\,7)$

α and β are disjoint

length of $\alpha, \beta = 3$.

The order is $\operatorname{lcm}(3,3) = 3$

(2) $(1\,2\,4)(3\,5\,7\,8\,6\,9)$

$\operatorname{lcm}(3,6) = 6$. $\therefore$ order : 6

(3) $(1\,2\,3\,5)(2\,4\,5\,6\,7)$

$= (2\,4\,1)(5\,6\,7\,3)$

$\operatorname{lcm}(3,4) = 12$. $\therefore$ order : 12

3. $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, D, D', H, V\}$

D_4 is non-Abelian group. $(\because R_{90}H = D' \neq D = HR_{90})$

proper subgroup of D_4

order 1 : $\{R_0\}$

order 2 : $\{R_0, R_{180}\}$, $\{R_0, D\}$, $\{R_0, D'\}$, $\{R_0, H\}$, $\{R_0, V\}$

order 4 : $\{R_0, R_{90}, R_{180}, R_{270}\}$, $\{R_0, R_{180}, D, D'\}$, $\{R_0, R_{180}, H, V\}$

Whenever above subgroups are added one more other element

it become non-proper subgroup, D_4.

$\therefore$ Every proper subgroups of D_4 are Abelian

4. $\alpha = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8) \in S_8$

order of α is $\operatorname{lcm}(3, 5) = 15$.

ETS α is even permutation.

$(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)$

$= (1\ 3)(1\ 2)(4\ 8)(4\ 7)(4\ 6)(4\ 5)$

$\therefore \alpha \in A_8$ ▨

5. $\alpha, \beta \in S_n \Rightarrow \alpha^{-1}\beta^{-1}\alpha\beta$ is even permutation

Let $\alpha = (a_1, a_2, \cdots, a_m)$ a_i's 2-cycles

$\beta = (b_1, b_2, \cdots, b_n)$ b_i's 2-cycles.

then $\alpha^{-1} = (a_m^{-1}, a_{m-1}^{-1}, \cdots, a_1^{-1}) = (a_m, a_{m-1}, \cdots, a_1)$

$\beta^{-1} = (b_n^{-1}, \cdots, b_1^{-1}) = (b_n, \cdots, b_1)$

($\because$ A 2-cycle is its own inverse).

$\alpha^{-1}\beta^{-1}\alpha\beta = (a_m \cdots a_1)(b_n \cdots b_1)(a_1 \cdots a_m)(b_1 \cdots b_n)$

$\Rightarrow$ product of $(2m+2n)$ 2-cycles.

$2m+2n = 2(m+n) \Rightarrow$ even number

$\therefore \alpha^{-1}\beta^{-1}\alpha\beta$ is even permutation.

6. $\beta = (1\ 2\ 3)(1\ 4\ 5) = (1\ 4\ 5\ 2\ 3)$

order of β : $5 \Rightarrow \beta^5 = \varepsilon$

$\beta^{99} = (\beta^5)^{19} \cdot \beta^4 = \beta^4$

$\beta^2 = (1\ 4\ 5\ 2\ 3)(1\ 4\ 5\ 2\ 3) = (1\ 5\ 3\ 4\ 2)$

$\beta^4 = (1\ 5\ 3\ 4\ 2)(1\ 5\ 3\ 4\ 2) = (1\ 3\ 2\ 5\ 4)$

$\therefore \beta^4 = (1\ 3\ 2\ 5\ 4)$

7. $\alpha = (a_1\ a_2\ \cdots\ a_n)$
$= (a_1\ \alpha(a_1)\ \alpha(a_2)\ \cdots\ \alpha(a_{n-1}))$

$\alpha(a_1) = a_2 \rightarrow \alpha^{-1}(a_2) = a_1$
$\alpha(a_2) = a_3 \rightarrow \alpha^{-1}(a_3) = a_2$
$\vdots \qquad\qquad \vdots$
$\alpha(a_{n-1}) = a_n \rightarrow \alpha^{-1}(a_n) = a_{n-1}$
$\alpha(a_n) = a_1 \rightarrow \alpha^{-1}(a_1) = a_n$

$\alpha^{-1} = (a_1\ \alpha^{-1}(a_1)\ \alpha^{-1}(a_n)\ \alpha^{-1}(a_{n-1})\ \cdots\ \alpha^{-1}(a_3))$
$= (a_1\ a_n\ a_{n-1}\ a_{n-2}\ \cdots\ a_2)$
$= (a_n\ a_{n-1}\ \cdots\ a_2\ a_1)$

$(a_1\ a_2 \cdots a_{n-1}\ a_n)(a_n\ a_{n-1}\ \cdots\ a_2\ a_1) = \varepsilon.$
$(\because a_n \rightarrow a_{n-1} \rightarrow a_n,\ \cdots,\ a_1 \rightarrow a_n \rightarrow a_1)$
$\therefore (a_1\ a_2\ \cdots\ a_n)^{-1} = (a_n\ \cdots\ a_2\ a_1)$

8. S_n is non-Abelian $(n \geq 3)$
$S_1 = \{\varepsilon\}$
$S_2 = \{\varepsilon, (1\ 2)\}$) Abelian
$S_3 = \{\varepsilon, (1\ 2\ 3)(1\ 3\ 2)(1\ 2)(1\ 3)(2\ 3)\}$

$$\underset{\substack{\| \\ (1\ 3\ 2)}}{(1\ 2)(1\ 3)} \neq \underset{\substack{\| \\ (1\ 2\ 3)}}{(1\ 3)(1\ 2)}$$

$S_n \supset \{(1\ 2), (1\ 3), (1\ 3\ 2), (1\ 2\ 3)\}$
$\therefore S_n$ is non-Abelian $(n \geq 3)$

9. A_n is subgroup of even permutation of S_n

$\therefore \ {}^{\forall}\alpha \in A_n \qquad \alpha = (\ \ )\underbrace{(\ \ )(\ \ )(\ \ )\cdots(\ \ )(}_{\text{even number}}\ \ )$

ETS product of two 2-cycles can be 3-cycle or product of 3-cycles.

$$\left.\begin{array}{ll} \text{i) } (a\ b)(a\ b) = \varepsilon = (a\ b\ c)(a\ c\ b) \\ \text{ii) } (a\ b)(c\ d) = (a\ b\ c)(b\ c\ d) \end{array}\right\} \text{product of 3-cycles.}$$

$$\left.\begin{array}{ll} \text{iii) } (a\ b)(a\ c) = (a\ c\ b) \\ \text{iv) } (a\ b)(b\ c) = (a\ b\ c) \end{array}\right\} \text{a 3-cycle}$$

$\therefore$ Every element of A_n can be expressed as a product of even number of 2-cycles. And we showed that every pair of 2-cycles can be expressed as a 3-cycle or a product of 3-cycles.
Therefore every elements of A_n can be expressed as a 3-cycle or a product of 3-cycles.

10. $\alpha = (a_1\ a_2 \cdots a_n)$

if) $n = 2k+1$

$$\alpha^2 = (a_1\ a_2\ a_3 \cdots a_{2k}\ a_{2k+1})(a_1\ a_2\ a_3 \cdots a_{2k}\ a_{2k+1})$$
$$= (a_1\ a_3\ a_5 \cdots a_{2k+1}\ a_2\ a_4 \cdots a_{2k})$$

$\therefore \ \alpha^2$ is a cycle

if) $n = 2k$

$$\alpha^2 = (a_1\ a_2 \cdots a_{2k-1}\ a_{2k})(a_1\ a_2 \cdots a_{2k-1}\ a_{2k})$$
$$= (a_1\ a_3\ a_5 \cdots a_{2k-1})(a_2\ a_4\ a_6 \cdots a_{2k})$$

$\therefore \ \alpha^2$ is product of two disjoint cycles.