수리통계학1 - HW2

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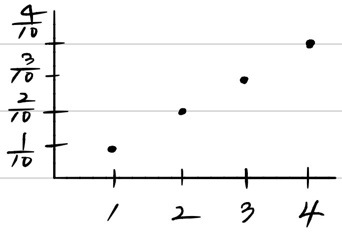
2.1-3

(a) $f(x) = \frac{x}{c}, \quad x = 1, 2, 3, 4.$

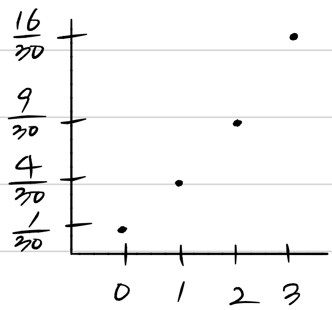
since $\sum_{x=1}^{4} \frac{x}{c} = 1 = \frac{1}{c}(1+2+3+4) = \frac{10}{c} = 1 \qquad \therefore c = 10$



(d) $f(x) = c(x+1)^2, \quad x = 0, 1, 2, 3$

$\sum_{x=0}^{3} c(x+1)^2 = 1 = c(1^2+2^2+3^2+4^2)$

$1 = c(1+4+9+16) = 30c \qquad \therefore c = \frac{1}{30}$



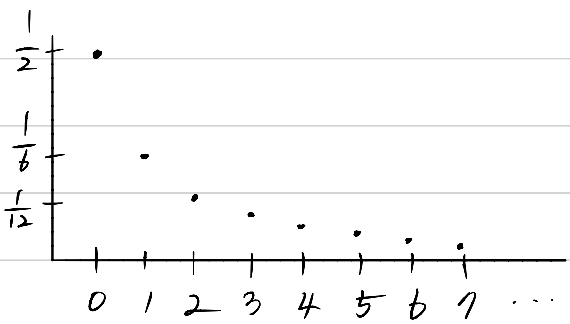
(f) $f(x) = \frac{c}{(x+1)(x+2)}, \quad x = 0, 1, 2, 3, \cdots$

$c \sum_{x=0}^{\infty} \frac{1}{(x+1)(x+2)} = 1$

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots = \frac{1}{c}$

$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots = \frac{1}{c}$

$1 = \frac{1}{c} \qquad \therefore c = 1.$



2.1-12

$$f(x) = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}, \quad x = 0, 1, 2, 3, \cdots$$

$$P(X \geq 4 \mid X \geq 1) = \frac{P(X \geq 4 \cap X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 4)}{P(X \geq 1)}$$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(X \geq 4) = 1 - P(X \leq 3) = 1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{12} - \frac{1}{20} = \frac{1}{5}$$

$$\therefore P(X \geq 4 \mid X \geq 1) = \frac{P(X \geq 4)}{P(X \geq 1)} = \frac{\frac{1}{5}}{\frac{1}{2}} = \frac{2}{5}$$

2.2-2

$$f(x) = \frac{(|x|+1)^2}{27}, \quad x = -2, -1, 0, 1, 2$$

$$E(X) = \sum_{x=-2}^{2} x \cdot \frac{(|x|+1)^2}{27} = \frac{1}{27} \sum_{x=-2}^{2} x(x^2 + 2|x| + 1)$$

$$= \frac{1}{27} \sum_{x=-2}^{2} x^3 + 2x|x| + x$$

$$= \frac{1}{27}((-8-1+0+1+8) + 2(-4-1+0+1+4) + (-2-1+0+1+2))$$

$$= 0.$$

$$E(X^2) = \sum_{x=-2}^{2} x^2 \frac{(|x|+1)^2}{27} = \frac{1}{27} \sum_{x=-2}^{2} x^2(x^2 + 2|x| + 1)$$

$$= \frac{1}{27} \sum_{x=-2}^{2} (x^4 + 2x^2|x| + x^2)$$

$$= \frac{1}{27} \{(16+1+0+1+16) + 2(8+1+0+1+8) + (4+1+0+1+4)\}$$

$$= \frac{1}{27}(34 + 36 + 10) = \frac{80}{27}$$

$$E(X^2 - 3X + 9) = E(X^2) - 3E(X) + 9 = \frac{80}{27} - 0 + 9 = \frac{323}{27}$$

2.2-6

$$f(x) = 6/(\pi^2 x^2), \quad x = 1, 2, 3, \cdots \qquad E(X) = +\infty.$$

$$E(X) = \sum_{x=1}^{\infty} x \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x}$$

$$\left(\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6} \right)$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = \left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots$$

$$> \frac{1}{2}+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{16}\right)+\cdots$$

$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots = \infty$$

Since $\sum_{x=1}^{\infty}\frac{1}{x} = \infty$, $\frac{6}{\pi^2}\sum_{x=1}^{\infty}\frac{1}{x} = \infty$ $\quad \therefore E(X) = \infty$

2.3-3

$E(X+4) = 10, \quad E[(X+4)^2] = 116$

(a) $Var(X+4) = E[(X+4)^2] - \{E(X+4)\}^2 = 116 - 100 = 16.$

(b) $\mu = E(X)$

$E(X+4) = 10, \quad E(X)+4 = 10, \quad \therefore E(X) = 6.$

(c) $\sigma^2 = Var(X)$

$E[X^2+8X+16] = E(X^2) + 8E(X) + 16.$

$= E(X^2) + 8\cdot 6 + 16 = E(X^2) + 64 = 116.$

$\therefore E(X^2) = 52$

$Var(X) = E(X^2) - \{E(X)\}^2 = 52 - 36 = 16.$

2.3-8

$f(x) = \frac{2x-1}{16}, \quad x = 1,2,3,4$

$E(X) = \sum_{x=1}^{4} x\cdot\frac{2x-1}{16} = \frac{1}{16}\sum_{x=1}^{4} 2x^2 - x$

$= \frac{1}{8}(1^2+2^2+3^2+4^2) - \frac{1}{16}(1+2+3+4) = \frac{1}{8}\cdot 30 - \frac{1}{16}\cdot 10 = \frac{25}{8}$

$E(X^2) = \sum_{x=1}^{4} x^2\frac{2x-1}{16} = \frac{1}{16}\sum_{x=1}^{4} 2x^3 - x^2$

$= \frac{1}{8}(1^3+2^3+3^3+4^3) - \frac{1}{16}(1^2+2^2+3^2+4^2) = \frac{1}{8}\cdot 100 - \frac{1}{16}\cdot 30 = \frac{85}{8}$

$Var(X) = E(X^2) - \{E(X)\}^2 = \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64}$

$\sigma_X = \frac{\sqrt{55}}{8}$

2.4-2

$f(5) = \frac{7}{18}$, $f(-5) = \frac{11}{18}$

$E(X) = \frac{7}{18} \cdot 5 - \frac{11}{18} \cdot 5 = 5\left(\frac{7}{18} - \frac{11}{18}\right) = -\frac{20}{18} = -\frac{10}{9}$

$E(X^2) = \frac{7}{18} 25 + \frac{11}{18} 25 = 25\left(\frac{7}{18} + \frac{11}{18}\right) = 25$

$Var(X) = 25 - \frac{100}{81} = \frac{1925}{81} \approx 23.77$

2.4-4

(a) $X \sim Bin(9, 0.21)$

(b) $P(X=0) = \binom{9}{0} \cdot 0.79^9 = 0.79^9 \doteqdot 0.11985$

$P(X=1) = \binom{9}{1} \cdot 0.21 \times 0.79^8 \doteqdot 0.28673$

$P(X=2) = \binom{9}{2} \cdot 0.21^2 \times 0.79^7 \doteqdot 0.30488$

$P(X=3) = \binom{9}{3} \cdot 0.21^3 \times 0.79^6 \doteqdot 0.18910$

$P(X=4) = \binom{9}{4} \cdot 0.21^4 \times 0.79^5 \doteqdot 0.07540$

(i) $P(X \geq 3) = 1 - P(0 \leq X \leq 2)$

$\doteqdot 1 - 0.11985 - 0.28673 - 0.30488$

$\doteqdot 0.28854$

(ii) $P(X=1) \doteqdot 0.28673$

(iii) $P(X \leq 4) = P(0 \leq X \leq 4)$

$\doteqdot 0.11985 + 0.28673 + 0.30488 + 0.18910 + 0.07540$

$\doteqdot 0.97596$

2.4-8

(a) $P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$P(X=2) = \binom{5}{2} 0.97^2 \times 0.03^3 \doteqdot 0.000254043$

$P(X=3) = \binom{5}{3} 0.97^3 \times 0.03^2 \doteqdot 0.008214057$

$P(X=4) = \binom{5}{4} 0.97^4 \times 0.03^1 \doteqdot 0.1327939215$

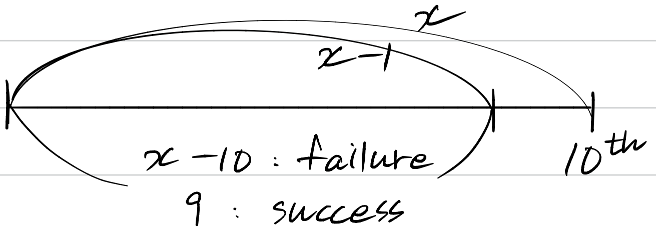
$P(X=5) = \binom{5}{5} 0.97^5 \times 0.03^0 \doteqdot 0.8587340257$

$\therefore P(X \geq 2) \doteqdot 0.9999960472$

(b) $P(X=5) = \binom{5}{5} \cdot 0.97^5 \doteqdot 0.8587340257$

2.5 − 3

(a)



$(x \geq 10)$

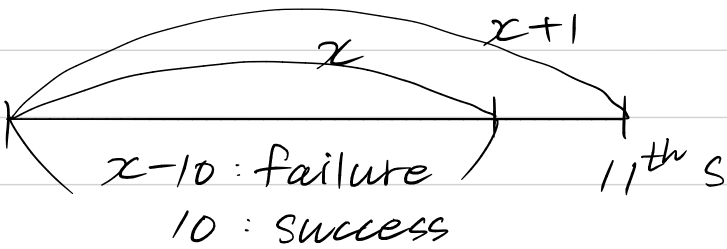
$$f_X(x) = P(X=x) = \binom{x-1}{9} 0.6^{10} \cdot 0.4^{x-10}$$

$$\sum_{x=10}^{\infty} \frac{(x-1)!}{9!\,(x-10)!} 0.6^{10} \times 0.4^{x-10} = 1 \quad (\because \sum_{x \in X} f_X(x) = 1)$$

$$E(X) = \sum_{x=10}^{\infty} x \cdot \frac{(x-1)!}{9!(x-10)!} \cdot 0.6^{10} \times 0.4^{x-10}$$

$$= \sum_{x=10}^{\infty} \frac{x!}{9!\,(x-10)!} 0.6^{10} \times 0.4^{x-10}$$

$$= \frac{10}{0.6} \cdot \sum_{x=10}^{\infty} \frac{x!}{10!\,(x-10)!} 0.6^{11} \times 0.4^{x-10} = \frac{50}{3}$$



$x+1 \geq 11$
$(x \geq 10)$

$$E[X(X+1)] = \sum_{x=10}^{\infty} x(x+1) \frac{(x-1)!}{9!(x-10)!} 0.6^{10} \times 0.4^{x-10}$$

$$= \frac{10 \cdot 11}{0.6^2} \sum_{x=10}^{\infty} \frac{(x+1)!}{11!(x-10)!} 0.6^{12} \times 0.4^{x-10}$$

$$= \frac{10 \cdot 11}{0.6^2} = \frac{110}{0.36} = \frac{11000}{36} = \frac{2750}{9}$$

$$Var(X) = E[X(X+1)] - E(X) - E(X^2)$$

$$= \frac{2750}{9} - \frac{50}{3} - \frac{2500}{9} = \frac{2750 - 150 - 2500}{9} = \frac{100}{9}$$

$$\sigma_X = \sqrt{\frac{100}{9}} = \frac{10}{3}$$

(b) $P(X=16) = \binom{15}{9} \times 0.6^{10} \times 0.4^{6} = 0.1239585632$

2.5-5

$E(X) = M_X^{(1)}(0), \quad Var(X) = M_X^{(2)}(0) - \{M_X^{(1)}(0)\}^2$

$R(t) = \ln M(t)$

$M(t) = E(e^{tX}), \quad M(0) = E(e^0) = 1$

$R(0) = \ln M(0) = \ln 1 = 0$

(a) WTS $M'(0) = R'(0)$

$M(t) = e^{R(t)}$

$M'(t) = R'(t) \cdot e^{R(t)}$

$M'(0) = R'(0) \cdot e^{R(0)} = R'(0) \cdot e^0 = R'(0).$

(b) WTS $M''(0) - \{M'(0)\}^2 = R''(0)$

$M''(t) = R''(t) \cdot e^{R(t)} + R'(t) \cdot R'(t) \cdot e^{R(t)}$

$= R''(t) \cdot e^{R(t)} + \{R'(t)\}^2 \cdot e^{R(t)}$

$M''(0) = R''(0) \cdot e^{R(0)} + \{R'(0)\}^2 \cdot e^{R(0)}$

$= R''(0) + \{R'(0)\}^2$

$M''(0) - \{M'(0)\}^2 = R''(0) + \{R'(0)\}^2 - \{R'(0)\}^2 = R''(0).$ ☒

2.6-4

$X \sim Poi(\lambda), \quad f_X(x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

$P(X=0) = f_X(0) = e^{-\lambda}$

$P(X=1) = f_X(1) = e^{-\lambda} \cdot \lambda \qquad \therefore \lambda = 1.$

$E(X) = \sum_{x=0}^{\infty} x \cdot \dfrac{e^{-\lambda} \cdot \lambda^x}{x!} = \sum_{x=1}^{\infty} \dfrac{e^{-\lambda} \cdot \lambda^x}{(x-1)!} = \lambda \sum_{x=1}^{\infty} \dfrac{e^{-\lambda} \cdot \lambda^{x-1}}{(x-1)!}$

$\overset{x-1=y}{=} \lambda \cdot \sum_{y=0}^{\infty} \dfrac{e^{-\lambda} \cdot \lambda^y}{y!} = \lambda = 1$ ☒

2.6 - 10.

$X \sim Poi(\lambda = 9)$.

$E(X) = \mu = \lambda = 9$. $Var(X) = \lambda = 9$ $\quad \sigma = 3$.

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(3 < X < 15)$

$$= \sum_{x=4}^{14} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} = e^{-9} \cdot \sum_{x=4}^{14} \frac{9^{x}}{x!}$$

$$= e^{-9}\left(\frac{9^4}{4!} + \frac{9^5}{5!} + \frac{9^6}{6!} + \frac{9^7}{7!} + \frac{9^8}{8!} + \cdots + \frac{9^{14}}{14!} \right) \fallingdotseq 0.938$$