

Vector Calculus – Final Exam

Junwoo Yang

December 20, 2018

1. Evaluate the following integrals:

(1) $\int_0^1 \int_y^1 x^2 \sin xy \, dx \, dy$

(2) $\int_0^2 \int_{\frac{x}{2}}^{\frac{x}{2}+1} x^5 (2y-x) e^{(2y-x)^2} \, dy \, dx$

2. Find the volume of the solid bounded by $z = 2 - y^2$ and $4x^2 + y^2 = z$.
3. Consider a region bounded by $(z - 8)^2 = 4(x^2 + y^2)$ and $z = 0$. Find the centroid of this region.
4. Explain the numerical integration with the Simpson's rule (or 2-point Gaussian quadrature rule, if you prefer) and use it to compute the following integration

$$\int_0^2 e^x \sin x \, dx.$$

You don't need to make a finer partition for the interval $[0, 2]$.

5. Let F be a vector field in \mathbb{R}^3 given by $F(x, y, z) = (e^x \sin y - yz, e^x \cos y - xz, z - xy)$. Suppose $\mathbf{x} : [0, 1] \rightarrow \mathbb{R}^3$ be a curve such that $\mathbf{x}(t) = (t, t^2, t^3)$. Evaluate the work done by F along the path \mathbf{x} .
6. Let D be a closed, bounded region in \mathbb{R}^2 where ∂D is a simple, closed, C^1 curve. Let F be a C^1 vector field defined on \mathbb{R}^2 .

(1) State the Green's theorem and the divergence theorem.

(2) Prove the divergence theorem using the Green's theorem.

7. Let S be a surface defined by $S = \{(x, y, z) \in \mathbb{R}^3 : z = 4e^{-(x^2+y^2)} - 1, z \geq 0\}$. The surface S is oriented by the normal vector \mathbf{n} such that $\mathbf{n} \cdot \mathbf{k} \geq 0$. Let F be a vector field defined in \mathbb{R}^3 such that $F(x, y, z) = (e^{y+z} - 2y, xe^{y+z} + y, e^{x+y})$. Then, evaluate the following integration.

$$\iint_S \nabla \times F \cdot d\mathbf{S}$$

8. Let S be a surface defined by $S = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1\}$ and \mathbf{n} be the outward unit normal vector on S . Let f be a function defined on $\mathbb{R}^3 \setminus \{0\}$ such that $f(x, y, z) = \log(x^2 + y^2 + z^2)$. Then, evaluate the following integration.

$$\iiint_S \nabla f \cdot \mathbf{n} \, dS$$