

# Topology I – Homework for Chapter 3, 4

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**Problem 1.** Determine, for each of the following topologies on  $\mathbb{R}$ , which of the others it contains.

$\mathcal{T}_1$  = Standard topology

$\mathcal{T}_2$  = Finite complement topology

$\mathcal{T}_3$  = Topology generated by the following basis;

$$B_3 = \{(a, b) \subset \mathbb{R} \mid a, b \in \mathbb{R}\} \cup \{(c, d) - K \in \mathbb{R} \mid c, d \in \mathbb{R}, K = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}\}$$

$\mathcal{T}_4$  = Topology generated by the following basis;

$$B_4 = \{(a, b] \subset \mathbb{R} \mid a, b \in \mathbb{R}\} = \text{basis of upper limit topology}$$

$\mathcal{T}_5$  = Topology generated by the following basis;

$$B_5 = \{(-\infty, a) \subset \mathbb{R} \mid a \in \mathbb{R}\}$$

**Proof.** ( $\mathcal{T}_2 \subset \mathcal{T}_1$ ) Since for  $(a, b) \in \mathcal{T}_1$ ,  $\mathbb{R} \setminus (a, b) = (-\infty, a] \cup [b, +\infty)$  is not finite,  $(a, b) \notin \mathcal{T}_2$ . For  $U \in \mathcal{T}_2$ , it is either  $U = \emptyset \in \mathcal{T}_1$  or  $\mathbb{R} \setminus U$  is finite. So we can put  $\mathbb{R} \setminus U = \{x_1, x_2, \dots, x_n\}$ .

Then  $U = (-\infty, x_1) \cup \bigcup_{i=1}^{n-1} (x_i, x_{i+1}) \cup (x_n, \infty) \in \mathcal{T}_1$ . Since it is an union of open intervals,  $\mathcal{T}_2 \subset \mathcal{T}_1$ .

( $\mathcal{T}_1 \subset \mathcal{T}_3$ ) For basis element  $(a, b) \in \mathcal{T}_1$  and  $x \in (a, b)$ , it is also a basis of  $\mathcal{T}_3$ . However, given the basis element  $B_3 = (c, d) - K \in \mathcal{T}_3$ , there is no open interval that contains 0 and lies in  $B_3$ . Thus,  $\mathcal{T}_3$  is strictly finer than  $\mathcal{T}_1$ .

( $\mathcal{T}_1 \subset \mathcal{T}_4$ ) For  $(a, b) \in \mathcal{T}_1$ ,  $(a, b) = \bigcup (a, x]$  for  $x \in (a, b)$ . It is arbitrary union of basis elements of  $\mathcal{T}_4$ , it is open in  $\mathcal{T}_4$ . But, for any basis element  $(a, b]$  of  $\mathcal{T}_4$ , there is no basis element  $(c, d) \in \mathcal{T}_1$  such that contains b and contained in  $(a, b]$ .

( $\mathcal{T}_5 \subset \mathcal{T}_1$ ) Since for  $(-\infty, a) \in \mathcal{T}_5$ ,  $(-\infty, a) = \bigcup_{n \in \mathbb{N}} (-n, a) \in \mathcal{T}_1$ . However, for  $(a, b) \in \mathcal{T}_1$  and  $x \in (a, b)$ , there is no basis element  $(-\infty, a) \in \mathcal{T}_5$  such that  $x \in (-\infty, a) \subset (a, b)$ . Thus,  $\mathcal{T}_5 \subset \mathcal{T}_1$ .

( $\mathcal{T}_2 \subset \mathcal{T}_3$ ) Since we have proven that  $\mathcal{T}_2 \subset \mathcal{T}_1$  and  $\mathcal{T}_1 \subset \mathcal{T}_3$ , it follows that  $\mathcal{T}_2 \subset \mathcal{T}_3$ .

( $\mathcal{T}_2 \subset \mathcal{T}_4$ ) We have proven that  $\mathcal{T}_2 \subset \mathcal{T}_1$  and  $\mathcal{T}_1 \subset \mathcal{T}_4$ . Thus,  $\mathcal{T}_2 \subset \mathcal{T}_4$ .

( $\mathcal{T}_2 \not\subset \mathcal{T}_5$  and  $\mathcal{T}_5 \not\subset \mathcal{T}_2$ ) For  $U = (-\infty, x) \cup (x, +\infty) \in \mathcal{T}_2$ , there is no basis element  $(-\infty, a) \in \mathcal{T}_5$  containing  $x + 1$ . Thus,  $\mathcal{T}_2 \not\subset \mathcal{T}_5$ . For  $(-\infty, a) \in \mathcal{T}_5$ , and  $a - 1 \in (-\infty, a)$ , there is no basis element  $U \in \mathcal{T}_2$  such that  $a - 1 \in U \subset (-\infty, a)$ . If there exists,  $a - 1 \in U \neq \emptyset$  and  $\mathbb{R} \setminus U$  is finite. However, from  $U \subset (-\infty, a)$ ,  $[a, +\infty) \subset \mathbb{R} \setminus U$ , which is a contradiction to finite set. Thus  $\mathcal{T}_5 \not\subset \mathcal{T}_2$ .

( $\mathcal{T}_3 \subset \mathcal{T}_4$ ) For  $(a, b) \in \mathcal{T}_3$ , we have that  $(a, b) \in \mathcal{T}_1 \subset \mathcal{T}_4$ . For  $(a, b) \setminus K \in \mathcal{T}_3$  such that  $(a, b) \cap K \neq \emptyset$ , we have four cases. If  $a < 0 < 1 < b$ , then we have that  $(a, b) \setminus K = (a, 0] \cup \bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, b) \in \mathcal{T}_4$ . If  $a < 0 < b < 1$ , then  $(a, b) \setminus K = (a, 0] \cup \bigcup_{n=m}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \cup (\frac{1}{m}, b) \in \mathcal{T}_4$  for smallest number  $m \in \mathbb{N}$  such that  $\frac{1}{m} < b$ . If  $0 < a < 1 < b$ , then  $(a, b) \setminus K = (a, \frac{1}{k}) \cup \bigcup_{n=1}^{k-1} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, b) \in \mathcal{T}_4$  for largest number  $k \in \mathbb{N}$  such that  $a < \frac{1}{k}$ . If  $0 < a < b < 1$ , then  $(a, b) \setminus K = (a, \frac{1}{k}) \cup \bigcup_{n=m}^{k-1} (\frac{1}{n+1}, \frac{1}{n}) \cup (\frac{1}{m}, b) \in \mathcal{T}_4$  for smallest number  $m \in \mathbb{N}$  and largest number  $k \in \mathbb{N}$  such that  $a < \frac{1}{k}$ ,  $m \in \mathbb{N}$ .

For  $(a, 0] \in \mathcal{T}_4$ , there is no basis element  $B$  of  $\mathcal{T}_3$  such that  $0 \in B \subset (a, 0]$  because every basis element  $B$  containing 0 contains some positive number. Thus  $\mathcal{T}_3 \subset \mathcal{T}_4$ .

$(\mathcal{T}_5 \subset \mathcal{T}_3)$  We have proven that  $\mathcal{T}_5 \subset \mathcal{T}_1$  and  $\mathcal{T}_1 \subset \mathcal{T}_3$ . Thus,  $\mathcal{T}_5 \subset \mathcal{T}_3$ .

$(\mathcal{T}_5 \subset \mathcal{T}_4)$  We have proven that  $\mathcal{T}_5 \subset \mathcal{T}_1$  and  $\mathcal{T}_1 \subset \mathcal{T}_4$ . Thus,  $\mathcal{T}_5 \subset \mathcal{T}_4$ . ■

**Problem 2.** Let  $D = \{(x, x) \in \mathbb{R}^2 | x \in \mathbb{R}\}$  be the diagonal line in  $\mathbb{R}^2$ . Describe the topology  $D$  inherits as a subspace of  $\mathbb{R}_l \times \mathbb{R}$  and as a subspace of  $\mathbb{R}_l \times \mathbb{R}_l$  where  $\mathbb{R}_l$  is the real line equipped with the lower limit topology.

**Proof.** The basis of  $\mathbb{R}_l \times \mathbb{R}_l$  is composed of the subsets  $[a, b) \times (c, d)$ . For  $(x, x) \in D$ , the intersection of any  $[x, b) \times (c, d)$  with  $D$  is a left-closed, right-open interval with  $x$  as left endpoint, and these intersections form a basis for  $D$  that is homeomorphic to  $\mathbb{R}_l$ . The basis of  $\mathbb{R}_l \times \mathbb{R}$  is composed of the subsets  $[a, b) \times [c, d)$ . The same argument as in the previous case gives us that  $D$  is homeomorphic to  $\mathbb{R}_l$ . ■

**Problem 3.** Show that  $X$  is Hausdorff if and only if  $D = \{(x, x) \in X \times X | x \in X\}$  is closed in  $X \times X$ .

**Proof.** Assume the diagonal  $D$  is closed, and let  $x \neq y$  be two distinct points in  $X$ . Then  $(x, y)$  belongs to the open set  $X \times Y \setminus D$ , hence there is a basis element  $U \times V$  ( $U, V \subset X$  open) such that  $(x, y) \in U \times V \subset X \times X \setminus D$ . Since  $(x, y) \in U \times V$ ,  $U$  is a neighborhood of  $x$  and  $V$  is a neighborhood of  $y$ . Moreover, if  $U$  and  $V$  had non-empty intersection, then  $z \in U \cap V$  would give  $(z, z) \in U \times V \cap D$ , contradiction; so  $U \cap V = \emptyset$ . This proves  $X$  is Hausdorff.

Conversely, if  $X$  is Hausdorff, and  $(x, y) \in X \times X \setminus D$ , then  $x \neq y$  so there exists neighborhoods  $U$  of  $x$  and  $V$  of  $y$  such that  $U \cap V = \emptyset$ ; then  $U \times V$  is a neighborhood of  $(x, y)$  in the product topology, and  $U \times V$  is disjoint from  $D$ . So  $(x, y) \in U \times V \subset X \times X \setminus D$ , which proves that  $X \times X \setminus D$  is open, i.e.  $D$  is closed. ■

**Problem 4.** Fine non-Hausdorff space with non-closed compact subset.

**Proof.** Let  $(\mathbb{R}, \mathcal{T})$  be the real numbers with finite complement topology, namely a set is closed if and only if it is finite; and a set is open if and only if its complement is finite. Consider the natural numbers as a subset of the real line, this is an infinite set, but it is clear that its complement is not finite, so it is neither open nor closed. Suppose that  $\{U_i | i \in I\}$  is an open cover of  $\mathbb{N}$ . There is some  $i_0 \in I$  such that  $0 \in U_{i_0}$ , and since  $U_{i_0}$  is open it means that it contains everything except finitely many points, in particular it must contain all the natural numbers, except maybe finitely many of them. For every  $n \in \mathbb{N} \setminus U_{i_0}$  we can find some  $U_{i_n}$ . We found, therefore, a finite subcover of this open cover, and so  $\mathbb{N}$  is compact. ■

**Problem 5.** Prove that  $X$  is disconnected if and only if there exists a continuous function from  $X$  onto two point set with the discrete topology  $f : X \rightarrow \{a, b\}$ .

**Proof.** Without loss of generality, let two point set be  $\{0, 1\}$ . Let  $X$  be disconnected. Then there exists two non-empty disjoint open subset  $A$  and  $B$  of  $X$  such that  $X = A \cup B$ . Define a mapping  $f$  of  $X$  onto  $\{0, 1\}$  by setting  $f(x) = 0$  if  $x \in A$  and  $f(x) = 1$  if  $x \in B$ . Open sets of  $\{0, 1\}$  on discrete topology are  $\emptyset$ ,  $\{0\}$ ,  $\{1\}$  and  $\{0, 1\}$ . By the definition of  $f$ ,  $f^{-1}(\{0\}) = A$  and  $f^{-1}(\{1\}) = B$ ,  $f^{-1}(\emptyset) = \emptyset$  and  $f^{-1}(\{0, 1\}) = X$ . Thus, we have shown that the inverse image under  $f$  of every open subset of  $\{0, 1\}$  is open in  $X$  and therefore  $f$  is continuous.

Conversely, if there exists such a mapping then  $X$  is disconnected because of continuity. For if  $X$  were connected, the  $\{0, 1\}$  would be connected. But this is impossible since every discrete space is disconnected. ■

**Problem 6.** Show that for a continuous function  $f : S^1 \rightarrow \mathbb{R}$ , there exists a point  $x \in S^1$  with  $f(x) = f(-x)$ .

**Proof.** Assume  $f : S^1 \rightarrow \mathbb{R}$  is continuous. Let  $g : S^1 \rightarrow \mathbb{R}$  be the map defined by  $g(x) = f(x) - f(-x)$ , which is also continuous. If  $g(x) = 0$  for all  $x \in S^1$  then we are done; otherwise, there exists  $x \in S^1$  such that  $g(x) \neq 0$ . Without loss of generality, we can assume that  $g(x) > 0$ ; and then  $g(-x) = f(-x) - f(x) = -g(x) < 0$ . Since  $S^1$  is connected and  $g$  is continuous, the intermediate value theorem implies the existence of  $y \in S^1$  such that  $g(y) = 0$ , i.e.  $f(y) = f(-y)$ . ■

**Proof (Alternative).** If  $f(x) \neq f(-x)$  for all  $x \in S^1$ , then setting  $U = \{x \in S^1 | f(x) < f(-x)\}$  and  $V = \{x \in S^1 | f(x) > f(-x)\}$ , then  $U$  and  $V$  are open (by continuity of  $f$ ) and disjoint, and  $S^1 = U \cup V$ , so by connectedness one of  $U$  and  $V$  must be all of  $S^1$  and the other must be empty. This is impossible since  $x \in U \Leftrightarrow -x \in V$ . ■