

# Probability Theory – Midterm Exam

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**Problem 1.** Let  $(\Omega, \mathcal{F}, \mu)$  be measure space and  $f : \Omega \rightarrow \mathbb{R}$  be measurable functions. Show that  $\{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$  is  $\sigma$ -algebra on  $Y$ . Show also that  $\nu(B) = \mu(f^{-1}(B))$  defines a measure on this  $\sigma$ -algebra.

*Proof.* To show that  $\mathcal{G} = \{B \subset Y : f^{-1}(B) \in \mathcal{F}\}$  is  $\sigma$ -algebra, we need to check that  $\emptyset \in \mathcal{G}$ ,  $\mathcal{G}$  is closed under complements and countable unions. Since  $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}$ ,  $\mathcal{G}$  contains  $\emptyset$ . If  $E \in \mathcal{G}$ ,  $f^{-1}(E) \in \mathcal{F}$ . Since  $\mathcal{F}$  is  $\sigma$ -algebra,  $(f^{-1}(E))^c = f^{-1}(E^c) \in \mathcal{F}$ . Thus,  $E^c \in \mathcal{G}$ . If  $E_i \in \mathcal{G}$  for  $i = 1, 2, \dots$ , then  $f^{-1}(E_i) \in \mathcal{F}$ . Since  $\mathcal{F}$  is closed under countable unions,  $\bigcup_{i=1}^{\infty} f^{-1}(E_i) = f^{-1}(\bigcup_{i=1}^{\infty} E_i) \in \mathcal{F}$ . Therefore,  $\bigcup_{i=1}^{\infty} E_i \in \mathcal{G}$ ,  $\mathcal{G}$  is  $\sigma$ -algebra. To show that  $\nu$  is a measure, we need to show that  $\nu(\emptyset) = 0$  and  $\nu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$  for pairwise disjoint subsets  $E_i \in \mathcal{G}$  ( $i = 1, 2, \dots$ ). Since  $\mu$  is a measure, we get that

$$\begin{aligned}\nu(\emptyset) &= \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0 \\ \nu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(E_i)\right) = \sum_{i=1}^{\infty} \mu(f^{-1}(E_i)) = \sum_{i=1}^{\infty} \nu(E_i).\end{aligned}$$

Hence,  $\nu$  is a measure on  $\mathcal{G}$ . □

**Problem 2.** Let  $f : \Omega \rightarrow \mathbb{R}$  be measurable and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous function. Show that  $g \circ f$  is measurable.

*Proof.* Enough to show that

$$(g \circ f)^{-1}((a, \infty)) = f^{-1}(g^{-1}((a, \infty))) = \{\omega \in \Omega : (g \circ f)(\omega) > a\}$$

is measurable for all  $a \in \mathbb{R}$ . Since  $g$  is continuous, inverse image of open set is open. Since any open subset of  $\mathbb{R}$  is a countable union of disjoint open intervals,  $g^{-1}((a, \infty))$  can be written as  $\bigcup_{n=1}^{\infty} I_n$  where  $I_n$  are disjoint open intervals. Then we get

$$f^{-1}(g^{-1}((a, \infty))) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n).$$

Since  $f$  is measurable, inverse image of interval is measurable, i.e.  $f^{-1}(I_n)$  is measurable. Since countable union of measurable sets is also measurable,  $\bigcup_{n=1}^{\infty} f^{-1}(I_n)$  is measurable. Hence,  $g \circ f$  is measurable. □

**Problem 3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous function. Show that if  $f = 0$  almost everywhere, then  $f = 0$  everywhere.

*Proof.* Suppose that there exists  $c \in [a, b]$  such that  $f(c) > 0$ . Since  $f$  is continuous, for given  $\varepsilon = \frac{f(c)}{2}$ , there exists  $\delta > 0$  such that if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \frac{f(c)}{2}$ . This implies that whenever  $0 < |x - c| < \delta$ , we have  $0 < \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$ . Then,  $m(\{x \in [a, b] : f(x) \neq 0\}) > \delta > 0$ . This is a contradiction. Similarly, for the case of  $f(c) < 0$ , proceed as before. Therefore, there is no point  $c$  such that  $f(c) \neq 0$ , so  $f = 0$  everywhere. □

**Problem 4.** Let  $f$  be non-negative integrable function and  $\alpha$  be positive real number. Show that

$$m(\{x \in E : f(x) > \alpha\}) < \frac{1}{\alpha} \int_E f \, dm.$$

*Proof.* Let  $A = \{x \in E : f(x) > \alpha\}$  and  $\varphi = \alpha \mathbf{1}_A$  be simple function. Note that  $f > \alpha$  on  $A$ . Then we get

$$\int_E \varphi \, dm = \int_E \alpha \mathbf{1}_A \, dm = \int_A \alpha \, dm = \alpha m(A) < \int_A f \, dm \leq \int_E f \, dm.$$

$$\therefore m(A) = m(\{x \in E : f(x) > \alpha\}) < \frac{1}{\alpha} \int_E f \, dm.$$

□

**Problem 5.** Let  $(\Omega, \mathcal{F}, P)$  be a probability space. Prove that if  $H_i$  are pairwise disjoint events such that  $\bigcup_{i=1}^{\infty} H_i = \Omega$ ,  $P(H_i) \neq 0$ , then

$$P(A) = \sum_{i=1}^{\infty} P(A|H_i)P(H_i).$$

*Proof.* Since  $A \subset \Omega$ ,  $P(A) = P(A \cap \Omega) = P(A \cap (\bigcup_{i=1}^{\infty} H_i))$ . Since  $H_i$  are pairwise disjoint,  $A \cap H_i$  are also pairwise disjoint. By countable additivity of probability measure and definition of conditional probability,

$$P(A) = P(A \cap (\bigcup_{i=1}^{\infty} H_i)) = P(\bigcup_{i=1}^{\infty} (A \cap H_i)) = \sum_{i=1}^{\infty} P(A \cap H_i) = \sum_{i=1}^{\infty} P(A|H_i)P(H_i).$$

□

**Problem 6.** Let  $X_1, \dots, X_n$  be random variables and  $a_i \in \mathbb{R}$ . Show that

$$\text{Var}(a_1 X_1 + \dots + a_n X_n) = \sum_{j,k} a_j a_k \text{Cov}(X_j, X_k).$$

*Proof.* Let  $Z := a_1 X_1 + \dots + a_n X_n = \sum_{i=1}^n a_i X_i$ . Then, we get that

$$\begin{aligned} \mathbb{E}(Z) &= \sum_{i=1}^n a_i \mathbb{E}(X_i) \\ Z^2 &= \sum_{j,k} a_j a_k X_j X_k \\ \mathbb{E}(Z^2) &= \sum_{j,k} a_j a_k \mathbb{E}(X_j X_k) \\ \mathbb{E}(Z)^2 &= \sum_{j,k} a_j a_k \mathbb{E}(X_j) \mathbb{E}(X_k) \\ \text{Var}(Z) &= \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = \sum_{j,k} a_j a_k (\mathbb{E}(X_j X_k) - \mathbb{E}(X_j) \mathbb{E}(X_k)) \\ &= \sum_{j,k} a_j a_k \text{Cov}(X_j, X_k). \end{aligned}$$

□

**Problem 7.** Take  $\Omega = [0, 1]$  with Lebesgue measure and let  $X(\omega) = \sin 2\pi\omega$ ,  $Y(\omega) = \cos 2\pi\omega$ . Show that  $X, Y$  are uncorrelated but not independent.

*Proof.* Let Lebesgue measure  $P := m|_{[0,1]}$ . Then we get

$$\begin{aligned}\mathbb{E}(X) &= \int_{\Omega} X \, dP = \int_{\Omega} \sin 2\pi\omega \, dP = \int_0^1 \sin 2\pi\omega \, d\omega = -\frac{1}{2\pi} \cos 2\pi\omega \Big|_0^1 = 0 \\ \mathbb{E}(Y) &= \int_{\Omega} Y \, dP = \int_{\Omega} \cos 2\pi\omega \, dP = \int_0^1 \cos 2\pi\omega \, d\omega = \frac{1}{2\pi} \sin 2\pi\omega \Big|_0^1 = 0 \\ \mathbb{E}(XY) &= \int_{\Omega} XY \, dP = \int_{\Omega} \sin 2\pi\omega \cos 2\pi\omega \, dP \\ &= \int_0^1 \frac{1}{2} (\sin(2\pi\omega + 2\pi\omega) + \sin(2\pi\omega - 2\pi\omega)) \, d\omega \\ &= \frac{1}{2} \int_0^1 \sin 4\pi\omega \, d\omega = -\frac{1}{8\pi} \cos 4\pi\omega \Big|_0^1 = 0 \\ \text{Cov}(X, Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0.\end{aligned}$$

Thus,  $\rho_{X,Y} = 0$ , i.e.  $X$  and  $Y$  are uncorrelated. Take  $a > 0$  so small that the sets  $A = \{\omega : \sin 2\pi\omega < a - 1\}$ ,  $B = \{\omega : \cos 2\pi\omega < a - 1\}$  are disjoint. Then  $P(A \cap B) = 0$  but  $P(A)P(B) \neq 0$ . Thus,  $X$  and  $Y$  are not independent.  $\square$