Advanced Calculus II – Final Exam

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December 16, 2019

1. State the definition of a complex inner product space.

2. State the definition of L^2 norm of a function $f : A \subset \mathbb{R} \to \mathbb{R}$.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Assume $f(a) = 0$, $f(b) = -1$, and $\int_a^b f(x)\,\mathrm{d}x = 0$. Prove that there is a $c \in (a, b)$ such that $f'(c) = 0$.

4. Let $\mathcal{V}$ be an inner product space and $\phi_0, \phi_1, \cdots, \phi_n$ a set of orthonormal vectors in $\mathcal{V}$. Prove that for each set of numbers $t_0, t_1, \cdots, t_n$,
$$\left\| f - \sum_{k=0}^{n} t_k \phi_k \right\| \geq \left\| f - \sum_{k=0}^{n} \langle f, \phi_k \rangle \phi_k \right\|.$$

5. Suppose that the sets $A_1, A_2, \cdots$ have measure zero in $\mathbb{R}^n$. Prove that $A_1 \cup A_2 \cup \cdots$ has measure zero in $\mathbb{R}^n$.

6. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a C^1 function and $u, v : [c, d] \to [a, b]$ be C^1 functions. Suppose
$$F(t) = \int_{u(t)}^{v(t)} f(x, t)\,\mathrm{d}x.$$
Find $F'(t)$.

7. Let $f(x) = |\sin x|$. Find the Fourier series of f.
(Hint: f is even and π periodic. $\sin a \cos b = \frac{\sin(a+b)+\sin(a-b)}{2}$)

8. Let $f, g : A \subset \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that
$$\int_A |f(x)g(x)|\,\mathrm{d}x \leq \left(\int_A |f(x)|^p\,\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_A |g(x)|^q\,\mathrm{d}x \right)^{\frac{1}{q}}$$
for $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.