

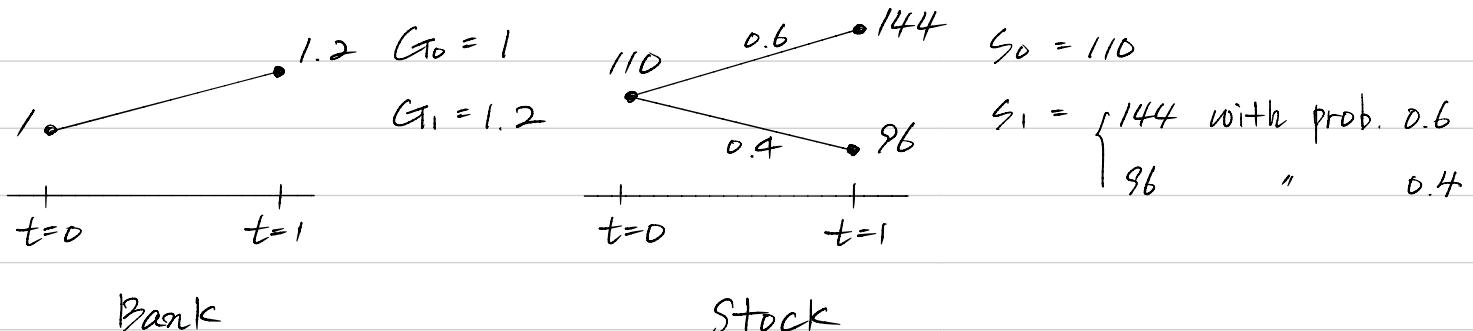
2028년 3월 1 - HWI

주 21 차 학부

2020-91659

영 35 9

5.1. (i)



$$\text{Call option payoff } X = (S_1 - 100)_+ = \begin{cases} 44 & \text{if } S_1 = 144 \\ 0 & \text{if } S_1 = 96 \end{cases}$$

hedging portfolio $h = (x, y) \in \mathbb{R}^2$ s.t. $V_1^h = X$

$$V_1^h = \begin{cases} xG_1 + yS_u = 1.2x + 144y = 44 \Leftrightarrow (x, y) = \left(-\frac{220}{3}, \frac{11}{12}\right) \\ xG_1 + yS_d = 1.2x + 96y = 0. \end{cases}$$

$$V_0^h = -\frac{220}{3}G_0 + \frac{11}{12}S_0 = -\frac{220}{3} \cdot 1 + \frac{11}{12} \cdot 110 = 27.5$$

5.1. (ii). Risk-neutral measure \mathbb{Q} s.t. $S_0 = \frac{1}{1+r} E^{\mathbb{Q}}(S_1)$

$$\begin{cases} q_u + q_d = 1 \\ 144q_u + 96q_d = 1.2 \times 110 = 132 \Leftrightarrow (q_u, q_d) = \left(\frac{3}{4}, \frac{1}{4}\right) \end{cases}$$

$$\frac{1}{1+r} E^{\mathbb{Q}}(X) = \frac{1}{1.2} \left(\frac{3}{4} \cdot 44 + \frac{1}{4} \cdot 0 \right) = 27.5$$

\therefore The risk-neutral measure is given by $\mathbb{Q}(u) = \frac{3}{4}$, $\mathbb{Q}(d) = \frac{1}{4}$, and the arbitrage-free price of option is 27.5

5.2. (i) $\Omega = \{u, m, d\}$

2^Ω = the power set of Ω

$P : 2^\Omega \rightarrow \mathbb{R}$, $P(\{u\}) = 0.2$, $P(\{m\}) = 0.2$, $P(\{d\}) = 0.6$

$\Rightarrow (\Omega, 2^\Omega, P)$: probability space.

$G_0, G_1, S_0, S_1 : \Omega \rightarrow \mathbb{R}$ random variables.

$$\left\{ \begin{array}{l} G_0 = 1 \text{ (i.e., } G_0(u) = G_0(m) = G_0(d) = 1) \\ G_1 = 1.25 \end{array} \right.$$

$$S_0 = 95$$

$$S_1(u) = 150, S_1(m) = 125, S_1(d) = 100$$

5.2. (ii). Risk-neutral measure \bar{Q} should satisfy that

$$\emptyset \bar{Q}(\{u\}) > 0, \bar{Q}(\{m\}) > 0, \bar{Q}(\{d\}) > 0.$$

$$\textcircled{2} S = \frac{1}{1+r} E^{\bar{Q}}(S_1)$$

$$\text{Let } \bar{Q}(\{u\}) = q_u, \bar{Q}(\{m\}) = q_m, \bar{Q}(\{d\}) = q_d.$$

$$\left\{ \begin{array}{l} 0 < q_u, q_m, q_d < 1 \\ q_u + q_m + q_d = 1 \end{array} \right.$$

$$150q_u + 125q_m + 100q_d = 95 \times 1.25 = 118.75$$

Intersection between two planes with normal vector $(1, 1, 1)$ and $(150, 125, 100)$ respectively generate line.

It is perpendicular to both of normal vectors.

Thus it is parallel to $(1, 1, 1) \wedge (150, 125, 100)$.

The line L is $t(1, -2, 1) + \left(\frac{3}{8}, 0, \frac{5}{8}\right)$ and the

symmetric equation of L is $8x - 3 = -4y = 8z - 5$

In this case, (q_u, q_m, q_d) is L on first octant.

Hence, $\bar{Q} = (q_u, q_m, q_d)$ is not unique obviously,

and two example of \bar{Q} is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{1}{8}, \frac{1}{2}, \frac{3}{8})$.

Actually, risk-neutral measure $\bar{Q} = (q_u, q_m, q_d)$
 $= t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8})$ for $t \in (0, 1)$.

5.2. (iii). super-hedging price = $\inf \{V_0^h \mid X \leq V_1^h\}$, $h = (\alpha, \beta)$.

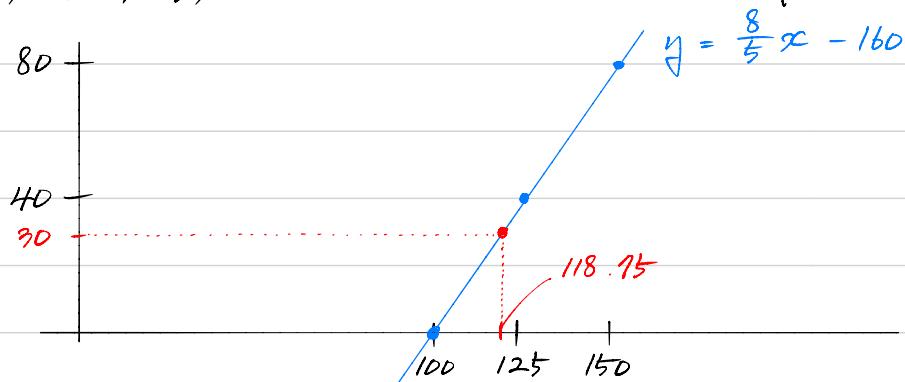
$$\text{where } X = \begin{cases} 80 & \text{if } S_1 = 150 \\ 40 & \text{if } S_1 = 125 \\ 0 & \text{if } S_1 = 100 \end{cases}$$

$$\inf \{V_0^h \mid X \leq V_1^h\} = \inf \left\{ \alpha + 95\beta \mid \begin{array}{l} 80 \leq 1.25\alpha + 150\beta \\ 40 \leq 1.25\alpha + 125\beta \\ 0 \leq 1.25\alpha + 100\beta \end{array} \right\}$$

$$\Downarrow$$

$$\frac{1.25\alpha + 118.75\beta}{1.25}$$

We want to find line $y = 1.25\alpha + \beta x$ above three points, $(150, 80), (125, 40), (100, 0)$ while $1.25\alpha + 118.75\beta$ be minimum.



$$\inf \{1.25\alpha + 118.75\beta \mid X \leq V_1^h\} = 30.$$

\therefore Super-hedging price is $\inf \{\alpha + 95\beta \mid X \leq V_1^h\} = \frac{30}{1.25} = 24$

Actually, in this case, we can construct hedge portfolio $h = (-128, \frac{8}{5})$, and price is $V_0^h = 24$. \therefore superhedging price

5.2. (iv). $\sup \left\{ \frac{1}{1+r} E^Q(X) \mid Q: \text{risk-neutral measure} \right\}$ is also arbitrage-free price.

$$\begin{cases} 0 < q_u, q_m, q_d < 1 \\ q_u + q_m + q_d = 1 \end{cases}$$

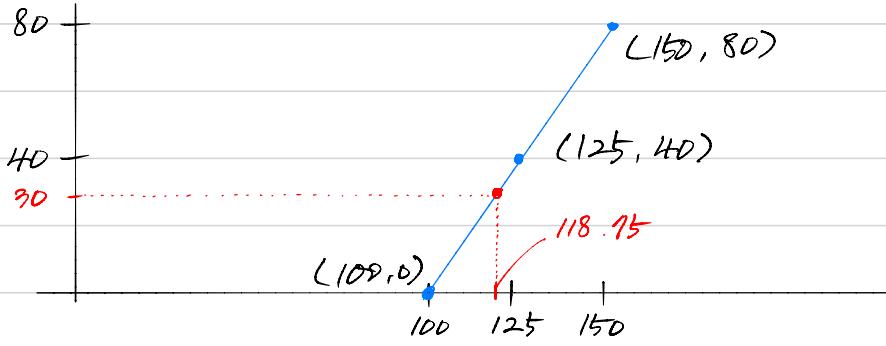
$$150q_u + 125q_m + 100q_d = 118.75$$

$$80q_u + 40q_m + 0q_d = E^Q(X)$$

$$\rightarrow \left(\frac{150}{80} \right) q_u + \left(\frac{125}{40} \right) q_m + \left(\frac{100}{0} \right) q_d = \left(\frac{118.75}{E^Q(X)} \right)$$

$(150, 80), (125, 40), (100, 0)$ 은 헤지포트를 통해 가격을 맞춘다

$(118.75, E^Q(X))$ 는 $E^Q(X)$ 의 가격이다.



$$(118.75, E^Q(X)) = (118.75, 30).$$

$$\therefore \sup \left\{ \frac{1}{1.25} E^Q(X) \mid Q: \text{R.N.} \right\} = \frac{30}{1.25} = 24$$

$$\inf \left\{ V_0^h \mid X \leq V_0^h \right\} = \sup \left\{ \frac{1}{1+R} E^Q(X) \mid Q: \text{R.N.} \right\}$$

∴ super-hedging duality holds.

Risk-neutral measure Q s.t. above supremum is achieved should satisfies that

$$\begin{cases} q_u + q_m + q_d = 1 \\ 150q_u + 125q_m + 100q_d = 118.75 \\ 80q_u + 40q_m + 0q_d = 30 \end{cases}$$

\Leftrightarrow

$$\begin{pmatrix} 1 & 1 & 1 \\ 150 & 125 & 100 \\ 80 & 40 & 0 \end{pmatrix} \begin{pmatrix} q_u \\ q_m \\ q_d \end{pmatrix} = \begin{pmatrix} 1 \\ 118.75 \\ 30 \end{pmatrix} \Rightarrow Aq = b. \\ \det(A) = 0.$$

Two examples in 5.2 (ii), $Q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{1}{8}, \frac{1}{2}, \frac{3}{8})$ satisfy above three equations. So, we can find risk-neutral measure Q which achieves supremum. And it is not unique because of $\det(A) = 0$.

In this case, all risk-neutral measure Q get supremum.

$$\begin{aligned} t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8}) &= (0, \frac{3}{4}t, \frac{1}{4}t) + (\frac{3}{8} - \frac{3}{8}t, 0, \frac{5}{8} - \frac{5}{8}t) \\ &= (\frac{3}{8} - \frac{3}{8}t, \frac{3}{4}t, \frac{5}{8} - \frac{5}{8}t) =: lt \text{ for } t \in (0, 1). \\ lt \cdot (80, 40, 0) &= 80(\frac{3}{8} - \frac{3}{8}t) + 40 \cdot \frac{3}{4}t = 30. \quad \forall t \in (0, 1). \end{aligned}$$

5.2.(V) P : set of all $\mathbb{Q} : 2^{\Omega} \rightarrow \mathbb{R}$ s.t. $S = \frac{1}{1+R} E^{\mathbb{Q}}(S)$

It may be $\mathbb{Q}(u) = 0$ or $\mathbb{Q}(m) = 0$ or $\mathbb{Q}(d) = 0$.

$$\sup \left\{ \frac{1}{1+R} E^{\mathbb{Q}}(X) \mid \mathbb{Q} \in P \right\} = \sup \left\{ \frac{1}{1.25} (80q_u + 40q_m + 100q_d) \mid \begin{array}{l} 0 \leq q_u, q_m, q_d \leq 1 \\ q_u + q_m + q_d = 1 \\ 150q_u + 125q_m + 100q_d = 118.75 \end{array} \right\}$$

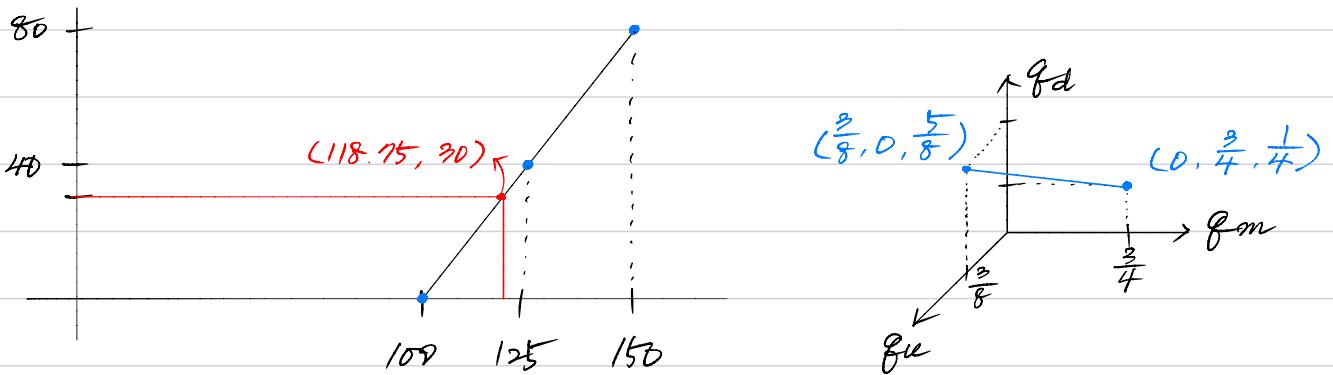
$$= \sup \left\{ \frac{80q_u + 40q_m}{1.25} \mid \begin{array}{l} 0 \leq q_u, q_m, q_d \leq 1 \\ q_u + q_m + q_d = 1 \\ 150q_u + 125q_m + 100q_d = 118.75 \end{array} \right\}$$

$$\Rightarrow \begin{cases} 150q_u + 125q_m + 100q_d = 118.75 \\ 80q_u + 40q_m + 0q_d = E^{\mathbb{Q}}(X) \end{cases}$$

$$\Rightarrow \left(\frac{150}{80} \right) q_u + \left(\frac{125}{40} \right) q_m + \left(\frac{100}{0} \right) q_d = \left(\frac{118.75}{E^{\mathbb{Q}}(X)} \right)$$

$\Rightarrow (150, 80), (125, 40), (100, 0)$ 를 축거제로 하는 선형방정식

여기 $(118.75, E^{\mathbb{Q}}(X))$ 를 \approx $E^{\mathbb{Q}}(X)$ 의 해로 볼 수 있다.



$$\sup \left\{ \frac{E^{\mathbb{Q}}(X)}{1.25} \mid \mathbb{Q} \in P \right\} = \frac{30}{1.25} = 24$$

superhedging price don't change from 5.2.(iii).

Thus it hold $\inf \{ V_0^h \mid X \leq V_1^h \} = \sup \left\{ \frac{E^{\mathbb{Q}}(X)}{1+R} \mid \mathbb{Q} \in P \right\} = 24$.

$\mathbb{Q} \in P$ that achieves the supremum is not unique because of $\det(A) = 0$. All of \mathbb{Q} is line from $(0, \frac{3}{4}, \frac{1}{4})$ to $(\frac{3}{8}, 0, \frac{5}{8})$ in \mathbb{R}^3 . That is, $\mathbb{Q} = (q_u, q_m, q_d)$ is $t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8})$ for $t \in [0, 1]$

5.2. (vi)

$$M = \{f \mid f: 2^{\omega} \rightarrow \mathbb{R}\} = \{f = (f_u, f_m, f_d) \mid f_u, f_m, f_d \in \mathbb{R}\}$$

$$P = \{g \mid g: 2^{\omega} \rightarrow [0, 1], g(\Omega) = 1, \frac{Eg(g)}{1+R} = S\}$$

$$= \{g = (g_u, g_m, g_d) \mid g = t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8})\}$$

Note. $f(u) = f_u$. $g(u) = g_u$

$$f(u, m) = f_u + f_m. \quad g(u, m) = g_u + g_m$$

for $f_1, f_2 \in M, c \in \mathbb{R}$,

$$cf_1 + f_2 = (cf_{1u} + f_{2u}, cf_{1m} + f_{2m}, cf_{1d} + f_{2d}) \in M$$

$\therefore M$ is vector space.

P is a convex subset of M if $P \subseteq M, \forall g_1, g_2 \in P$,
line segment containing g_1, g_2 contained in P

We can consider $M, P \subseteq \mathbb{R}^3$

$$M = \{(f_u, f_m, f_d) \mid f_u, f_m, f_d \in \mathbb{R}\} \cong \mathbb{R}^3 \text{ (isomorphic).}$$

$$P = \{(g_u, g_m, g_d) \mid t(0, \frac{3}{4}, \frac{1}{4}) + (1-t)(\frac{3}{8}, 0, \frac{5}{8}), t \in [0, 1]\}$$

$P \subseteq M$ (trivial).

$$\forall g_1, g_2 \in P, \exists t_1, t_2 \in [0, 1] \text{ s.t. } g_1 = t_1(0, \frac{3}{4}, \frac{1}{4}) + (1-t_1)(\frac{3}{8}, 0, \frac{5}{8})$$

$$g_2 = t_2(0, \frac{3}{4}, \frac{1}{4}) + (1-t_2)(\frac{3}{8}, 0, \frac{5}{8})$$

the line segment containing g_1, g_2

$$tg_1 + (1-t)g_2 \text{ for } t \in [0, 1]$$

$$= t(t_1(0, \frac{3}{4}, \frac{1}{4}) + (1-t_1)(\frac{3}{8}, 0, \frac{5}{8})) + (1-t)(t_2(0, \frac{3}{4}, \frac{1}{4}) + (1-t_2)(\frac{3}{8}, 0, \frac{5}{8}))$$

$$= \underline{(tt_1 + t_2 - tt_2)(0, \frac{3}{4}, \frac{1}{4})} + \underline{(1-t_2 - tt_1 + tt_2)(\frac{3}{8}, 0, \frac{5}{8})}$$

$$(tt_1 + t_2 - tt_2) + (1-t_2 - tt_1 + tt_2) = 1.$$

Thus line segment containing g_1, g_2 is contained in
line segment containing $(0, \frac{3}{4}, \frac{1}{4}), (\frac{3}{8}, 0, \frac{5}{8})$, that is, P
 $\therefore P$ is convex subset of M .

5.3. $R = 0$.

$$\inf \{ V_0^h \mid X \leq V_1^h \}$$

probability measure.

$$P_i > 0, i = 1, 2, 3, \Rightarrow E^P(X) \leq E^P(V_1^h)$$

$$\Rightarrow E^P(X) - E^P(V_1^h) \leq 0.$$

$$= \inf \{ \alpha + \beta s \mid X_i \leq \alpha + \beta s_i, i = 1, 2, 3 \}$$

$$= \inf_{i=1,2,3} \{ \alpha + \beta s_i \}$$

$x_i \leq \alpha + \beta s_i$ 제약조건을 만족하면서 $\alpha + \beta s$ 의 최소.

$$= \inf_{\alpha, \beta} \left(\sup_{P_i > 0} \{ \alpha + \beta s + (E^P(X) - E^P(V_1^h)) \} \right) \leq 0$$

상수인 α 와 s 를 \sup 으로
제한 $E^P(X) - E^P(V_1^h) \rightarrow 0$.

$$= \inf_{\alpha, \beta} \left(\sup_{P_i > 0} \left| \alpha + \beta s + \sum_{i=1}^3 P_i(x_i - \alpha - \beta s_i) \right| \right) \leq 0$$

$$= \sup_{P_i > 0} \left| \inf_{\alpha, \beta} \left| \alpha + \beta s + \sum_{i=1}^3 P_i(x_i - \alpha - \beta s_i) \right| \right|$$

$$= \sup_{P_i > 0} \left| \inf_{\alpha, \beta} \left| \alpha \left(1 - \sum_{i=1}^3 P_i \right) + \beta \left(s - \sum_{i=1}^3 P_i s_i \right) + \sum_{i=1}^3 P_i x_i \right| \right|$$

$\alpha, \beta \in \mathbb{R}$

만약 α 가 예상해 $\sum_{i=1}^3 P_i = 1, \sum_{i=1}^3 P_i s_i = s$ 가 되도록 α . $P_i > 0, x_i \geq 0$.

\Rightarrow Risk-neutral measure. α, β .

$$= \sup_{\substack{P_i > 0 \\ \sum_{i=1}^3 P_i = 1 \\ \sum_{i=1}^3 P_i s_i = s}} \left\{ \sum_{i=1}^3 P_i x_i \right\}$$

$$= \sup \{ E^\alpha(X) \mid \alpha: \text{risk-neutral measure} \}.$$