

20284351 - HW2

9월 21주 목부

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1.1. (i)

$$\Omega = \{uu, ud, du, dd\}.$$

2^Ω : the power set of Ω

$P : 2^\Omega \rightarrow \mathbb{R}$ probability measure with $P(uu) = 0.12$,

$$P(ud) = 0.48, P(du) = 0.2, P(dd) = 0.2$$

$(F_t)_{t=0,1,2}$: filtration. $F_0 = \{\emptyset, \Omega\}$.

$$F_1 = \{\emptyset, \{uu, ud\}, \{du, dd\}, \Omega\}$$

$(\Omega, 2^\Omega, (F_t), P)$: filtered space

$$F_2 = 2^\Omega$$

$(S_t)_{t=0,1,2} : \Omega \times \mathbb{T} \rightarrow \mathbb{R}$. stochastic process.

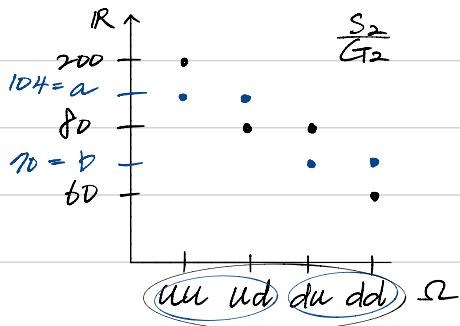
$$S_0 = 100.$$

$$S_1(uu) = S_1(ud) = 160, S_1(du) = S_1(dd) = 70$$

$$S_2(uu) = 240, S_2(ud) = S_2(du) = 96, S_2(dd) = 72$$

1.1. (ii)

$$E^P\left(\frac{S_2}{G_2} | F_1\right)$$



$$\frac{S_2}{G_2}(uu) = \frac{240}{1.2} = 200$$

$$\frac{S_2}{G_2}(ud) = \frac{96}{1.2} = 80$$

$$\frac{S_2}{G_2}(dd) = \frac{72}{1.2} = 60$$

$$200 \times P(uu) + 80 \times P(ud) = a \times P(uu) + a \times P(ud).$$

$$200 \times 0.12 + 80 \times 0.48 = 0.6a \Rightarrow a = 104.$$

$$80 \times 0.2 + 60 \times 0.2 = 0.4b \Rightarrow b = 70.$$

$$\therefore \begin{cases} E^P\left(\frac{S_2}{G_2} | F_1\right)(uu) = E^P\left(\frac{S_2}{G_2} | F_1\right)(ud) = 104 \\ E^P\left(\frac{S_2}{G_2} | F_1\right)(du) = E^P\left(\frac{S_2}{G_2} | F_1\right)(dd) = 70. \end{cases}$$

1.1. (iii)

$$\begin{cases} x_{1u} = -\frac{200}{3} \\ y_{1u} = \frac{5}{6} \end{cases} \quad 240 \rightarrow 120$$

$$100 \xrightarrow{\text{D}} \begin{cases} x_0 = -\frac{1640}{27} \\ y_0 = \frac{43}{54} \end{cases} \quad 160 \xrightarrow{\text{U}} \begin{cases} x_{1d} = -40 \\ y_{1d} = \frac{1}{2} \end{cases} \quad 96 \rightarrow 0$$

$$G_0 = 1 \quad G_1 = 1 \quad G_2 = 1.2$$

$$\begin{cases} 1.2x_{1u} + 240y_{1u} = 120 \\ 1.2x_{1u} + 96y_{1u} = 0 \end{cases} \Leftrightarrow \begin{cases} x_{1u} = -\frac{200}{3} \\ y_{1u} = \frac{5}{6} \end{cases}$$

$$\Rightarrow \text{price} = x_{1u} + 160y_{1u} = \frac{200}{3}$$

$$\begin{cases} 1.2x_{1d} + 96y_{1d} = 0 \\ 1.2x_{1d} + 12y_{1d} = -12 \end{cases} \Leftrightarrow \begin{cases} x_{1d} = -40 \\ y_{1d} = \frac{1}{2} \end{cases}$$

$$\Rightarrow \text{price} = x_{1d} + 100y_{1d} = -5$$

$$\begin{cases} x_0 + 160y_0 = \frac{200}{3} \\ x_0 + 100y_0 = -5 \end{cases} \Leftrightarrow \begin{cases} x_0 = -\frac{1640}{27} \\ y_0 = \frac{43}{54} \end{cases}$$

$$\text{price} = x_0 + 100y_0 = \frac{170}{9}$$

$(X_t)_{t=0,1,2} : \Omega \times \mathbb{T} \rightarrow \mathbb{R}$ stochastic process.

$$X_0 = \frac{170}{9}$$

$$X_1(uu) = X_1(ud) = \frac{200}{3}, \quad X_1(du) = X_1(dd) = -5$$

$$X_2(uu) = 120, \quad X_2(ud) = X_2(du) = 0, \quad X_2(dd) = -12$$

$\pi = (\pi_t, \phi_t)_{t=0,1,2} : \Omega \times \mathbb{T} \rightarrow \mathbb{R}$ stochastic process.

$$(\pi_0, \phi_0) = \left(-\frac{1640}{27}, \frac{43}{54}\right)$$

$$(\pi_1, \phi_1)(uu) = (\pi_1, \phi_1)(ud) = \left(-\frac{200}{3}, \frac{5}{6}\right)$$

$$(\pi_1, \phi_1)(du) = (\pi_1, \phi_1)(dd) = (-40, \frac{1}{2})$$

$$(\pi_2, \phi_2) = (\pi_1, \phi_1)$$

1.1. (IV)

h is self-financing

$$\text{if } V_{t+1}^h - V_t^h = x_t(G_{t+1} - G_t) + y_t(S_{t+1} - S_t).$$

$$1). V_0^h - V_0^h = x_0(G_1 - G_0) + y_0(S_1 - S_0)$$

$$\textcircled{1} \quad \frac{200}{3} - \frac{110}{9} = \frac{430}{9} = -\frac{1640}{27}(1-1) + \frac{43}{54}(160-100) = \frac{1290}{27} = \frac{430}{9}$$

$$\textcircled{2} \quad -5 - \frac{110}{9} = -\frac{215}{9} = -\frac{1640}{27}(1-1) + \frac{43}{54}(10-100) = -\frac{645}{27} = -\frac{215}{9}$$

$$2) V_2^h - V_1^h = x_1(G_2 - G_1) + y_1(S_2 - S_1)$$

$$\textcircled{3} \quad 120 - \frac{200}{3} = \frac{160}{3} = -\frac{200}{3}(1.2-1) + \frac{5}{6}(240-160) = -\frac{40}{3} + \frac{200}{3} = \frac{160}{3}$$

$$\textcircled{4} \quad 0 - \frac{200}{3} = -\frac{200}{3} = -\frac{200}{3}(1.2-1) + \frac{5}{6}(96-160) = -\frac{40}{3} - \frac{160}{3} = -\frac{200}{3}$$

$$\textcircled{5} \quad 0 - (-5) = 5 = -40(1.2-1) + \frac{1}{2}(96-10) = -8 + 13 = 5$$

$$\textcircled{6} \quad -12 - (-5) = -7 = -40(1.2-1) + \frac{1}{2}(72-10) = -8 + 1 = -7.$$

$\therefore h$ is self-financing

1.1. (V)

$$\phi = (\phi_t)_{t=0,1,2} \quad \left| \begin{array}{l} \phi_0 = \frac{43}{54} \\ \phi_1(uu) = \phi_1(uu) = \frac{5}{6}, \quad \phi_1(du) = \phi_1(dd) = \frac{1}{2} \\ \phi_2 = \phi_1 \end{array} \right.$$

$$\phi \text{ is adapted} \Leftrightarrow \phi_t^{-1}(a) \in F_t \text{ for } a \in \mathbb{R}, t \in \{0,1,2\}$$

case 1) ϕ_0 is F_0 -measurable. $F_0 = \{\emptyset, \Omega\}$

$$\phi_0^{-1}\left(\frac{43}{54}\right) = \Omega \in F_0, \text{ for } a \in \mathbb{R} \text{ s.t. } a \neq \frac{43}{54} \quad \phi_0^{-1}(a) = \emptyset \in F_0.$$

case 2) ϕ_1 is F_1 -measurable. $F_1 = \{\emptyset, \{uu, ud\}, \{du, dd\}, \Omega\}$.

$$\phi_1^{-1}\left(\frac{5}{6}\right) = \{uu, ud\} \in F_1, \quad \phi_1^{-1}\left(\frac{1}{2}\right) = \{du, dd\} \in F_1,$$

$$\text{for } a \in \mathbb{R} \text{ s.t. } a \neq \frac{5}{6}, a \neq \frac{1}{2}, \quad \phi_1^{-1}(a) = \emptyset \in F_1.$$

Case 3) ϕ_2 is F_2 -measurable. $F_2 = 2^\omega$

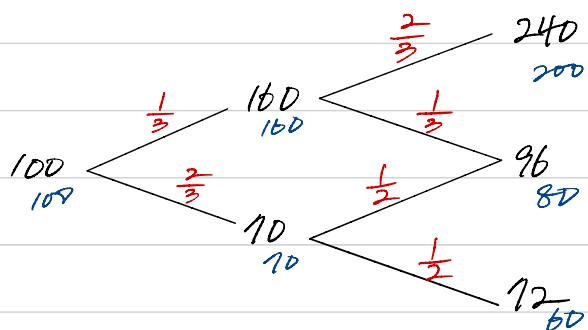
$$\phi_2^{-1}\left(\frac{5}{6}\right) = \{uu, ud\} \in F_2, \quad \phi_2^{-1}\left(\frac{1}{2}\right) = \{du, dd\} \in F_2$$

$$\text{for } a \in \mathbb{R} \text{ s.t. } a \neq \frac{5}{6}, a \neq \frac{1}{2} \quad \phi_2^{-1}(a) = \emptyset \in F_2.$$

$\therefore \phi$ is adapted.

1.1. (vi). Q.

$(\frac{S_t}{G_t})$.



$$\left\{ \begin{array}{l} Q(\text{uuu}) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9} \\ Q(\text{uud}) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \\ Q(\text{udu}) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \\ Q(\text{udd}) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \end{array} \right.$$

Q is risk-neutral measure.

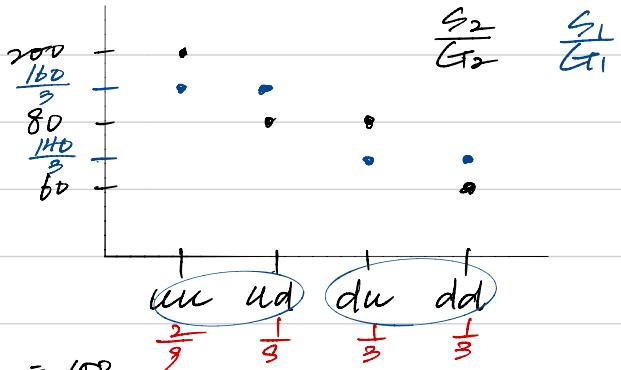
$\Leftrightarrow P \simeq Q$: trivial

$(\frac{S_t}{G_t})_{t=0,1,2}$: martingale on Q. $\Rightarrow E^Q(\frac{S_{t+1}}{G_{t+1}} | F_t) = \frac{S_t}{G_t}$

i.e. ETS

$$\left\{ \begin{array}{l} E^Q(\frac{S_2}{G_2} | F_1) = \frac{S_1}{G_1} \\ E^Q(\frac{S_1}{G_1} | F_0) = \frac{S_0}{G_0} \end{array} \right.$$

$$\Rightarrow E^Q(\frac{S_2}{G_2}) = E^Q(\frac{S_1}{G_1}) = E^Q(\frac{S_0}{G_0})$$



$$E^Q(\frac{S_2}{G_2}) = \frac{2}{3} \cdot 200 + \frac{1}{3} \cdot 80 + \frac{1}{3} \cdot 80 + \frac{1}{3} \cdot 60 = 100$$

$$E^Q(\frac{S_1}{G_1}) = \frac{2}{3} \cdot 160 + \frac{1}{3} \cdot 160 + \frac{1}{3} \cdot 100 + \frac{1}{3} \cdot 100 = 100$$

$$E^Q(\frac{S_0}{G_0}) = 100.$$

$\therefore Q$ is RN.

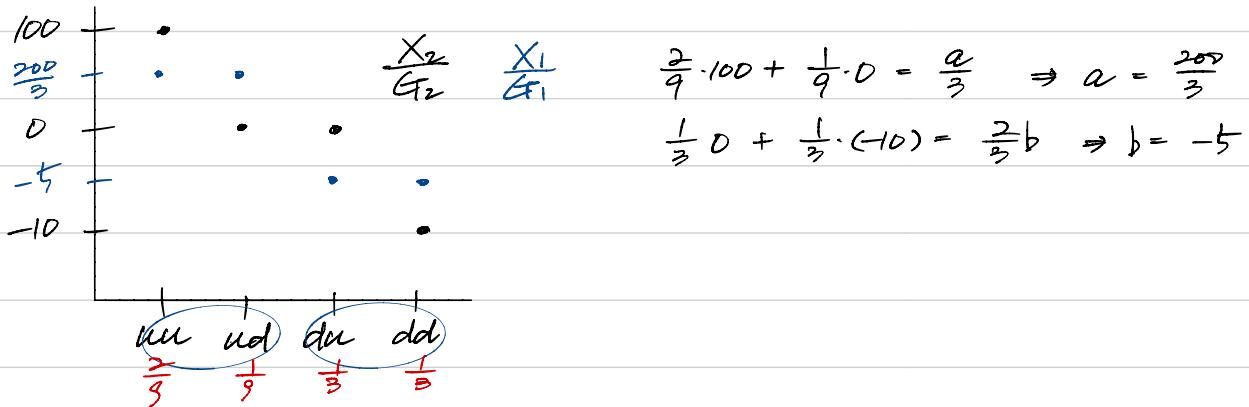
1.1. (vii)

$$E^Q\left(\frac{S_2}{G_2} \mid F_1\right) = \frac{S_1}{G_1} \quad (\because Q \text{ is RN}, \left(\frac{S_t}{G_t}\right)_{t=0,1,2} \text{ is martingale.})$$

$$\Rightarrow E^Q\left(\frac{S_2}{G_2} \mid F_1\right)(uu) = E^Q\left(\frac{S_2}{G_2} \mid F_1\right)(ud) = 160$$

$$E^Q\left(\frac{S_2}{G_2} \mid F_1\right)(du) = E^Q\left(\frac{S_2}{G_2} \mid F_1\right)(dd) = 10.$$

$$E^Q\left(\frac{X_2}{G_2} \mid F_1\right) = \begin{cases} E^Q\left(\frac{X_2}{G_2} \mid F_1\right)(uu) = E^Q\left(\frac{X_2}{G_2} \mid F_1\right)(ud) = \frac{200}{3} \\ E^Q\left(\frac{X_2}{G_2} \mid F_1\right)(du) = E^Q\left(\frac{X_2}{G_2} \mid F_1\right)(dd) = -5 \end{cases}$$



1.2. (i) \Leftrightarrow (ii)

(\Rightarrow). a). Suppose $\frac{s_d}{s} \geq 1+R \Rightarrow s_d \geq (1+R)s$

Let $h = (-s, 1) \Rightarrow V_0^h = -s \cdot 1 + 1 \cdot s = 0$.

$$V_1^h = \begin{cases} -s(1+R) + s_u \geq 0 & (\because s_u \geq s_d) \\ -s(1+R) + s_d \geq 0 & (\because s_d \geq (1+R)s) \end{cases}$$

$\Rightarrow h$ is arbitrage.

b). Suppose $1+R \geq \frac{s_u}{s} \Rightarrow (1+R)s \geq s_u$

Let $h = (s, -1)$.

$$\Rightarrow V_0^h = s \cdot 1 - 1 \cdot s = 0.$$

$$V_1^h = \begin{cases} s(1+R) - s_u \geq 0 & (\because (1+R)s \geq s_u) \\ s(1+R) - s_d \geq 0 & (\because s_u \geq s_d) \end{cases}$$

$\Rightarrow h$ is arbitrage.

We can show above for $\frac{s_d}{s_u} < 1+R < \frac{s_u}{s_d}$ and $\frac{s_d}{s_d} < 1+R < \frac{s_u}{s_d}$

Similarly. \square .

(\Leftarrow). Suppose h is arbitrage s.t $P(V_0^h = 0) = 1 \quad h = (x, y).$

$$P(V_2^h \geq 0) = 1.$$

$$V_0^h = x + sy = 0.$$

$$P(V_2^h \geq 0) \geq 0$$

$$\begin{aligned} V_2^h &= (1+r)^2 x + S_u y > 0 \quad (\because S_u > S_d > S_{dd}) \\ &\quad (1+r)^2 x + S_u y > 0 \\ &\quad (1+r)^2 x + S_d y \geq 0. \end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{l} x + \frac{S_u}{(1+r)^2} y > 0 \\ x + \frac{S_d}{(1+r)^2} y \geq 0 \\ x + \frac{S_{dd}}{(1+r)^2} y \geq 0 \end{array} \right. \begin{array}{l} \leftarrow x + \frac{S_u}{1+r} y \quad (\because \frac{S_d}{S_u} < 1+r < \frac{S_u}{S_d}) \\ \leftarrow x + \frac{S_d}{1+r} y \quad (\because \frac{S_{dd}}{S_d} < 1+r < \frac{S_u}{S_d}) \\ \end{array} \quad x + sy = 0 \quad (\because \frac{S_d}{S} < 1+r < \frac{S_u}{S}).$$

$\not\Rightarrow \square$.

1. 2. (ii) \Leftrightarrow (iii).

1. 2.

1.4