Linear Algebra I – Midterm Exam

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April 5, 2018

1. Mark each of the following true or false.

   (1) Let $W = \{(a, b) \in \mathbb{R}^2 \mid ab \geq 0\}$, then $W \geq \mathbb{R}^2$.

   (2) Let S_1 and S_2 be subsets of V. If $\langle S_1 \rangle = \langle S_2 \rangle$, then $S_1 = S_2$.

   (3) $\{x^3 - 2x^2 + 1, 4x^2 - x + 3, x + 1, x - 5\}$ is a basis of $P_3(\mathbb{R})$.

   (4) $\dim(U + W) = \dim(U) + \dim(W)$ for $U, W \leq V$.

   (5) Let $T : V \to W$ be a linear transformation. If $\{v_1, \cdots, v_k\} \subset V$ is linearly independent, then $\{T(v_1), \cdots, T(v_k)\}$ is linearly independent.

2. Prove or disprove that if $T : V \to W$ is a linear transformation, then $\ker(T)$ is a subspace of V.

3. Prove or disprove that if U, W are subspaces of V, then $U \cup W$ is a subspace of V.

4. Let U, W be subspaces of V. Prove or disprove that if $U \cap W = \{0\}$, then there exists unique $u \in U$ and unique $w \in W$ such that $v = u + w$ for $v \in U + W$.

5. Let U, V and W be subspaces of $\mathbb{R}^5$ and $\dim W = 4$. Prove or disprove that if $U \oplus W = \mathbb{R}^5 = V \oplus W$, then $U = V$.

6. Let $\{v, w\}$ be a basis of $\mathbb{R}^2$. Prove or disprove that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(v) = 2v + 3w$ and $T(w) = v + 2w$, then T is injective.

7. Let $V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$, $W_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$ and $W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$. Find a basis of $W_1 + W_2$ which contains a basis of $W_1 \cap W_2$. (Explain why the set is a basis.)

8. If linear transformation $T : V \to W$ is injective, then there is a linear transformation $S : W \to V$ such that $S \circ T$ is bijective.

9. Let $T : V \to V$ be a linear transformation and $\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(T \circ T))$. Show that $V = \mathrm{im}(T) \oplus \ker(T)$.

10. Let $T : V \to W$ be a linear transformation. Show that $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$.