

# MAT4004: Topology 2

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Based on lecture by Youngsik Huh in fall 2021

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## Chapter 0

# Review of Topology 1

**Definition 1** (Topology). A topology on a set  $X$  is a collection of subsets of  $X$ ,  $\{\text{open sets}\}$ , which satisfies followings

1.  $\emptyset, X \in \mathcal{T}$ .
2. Arbitrary union of elements in  $\mathcal{T}$  is in  $\mathcal{T}$ .
3. Finite intersection of elements in  $\mathcal{T}$  is in  $\mathcal{T}$ .

Elements in  $\mathcal{T}$  are called open sets.

**Lemma 1.** product topology on  $X \times Y$  is coarsest topology s.t.  $\pi_1, \pi_2$  are continuous.

**Definition 2** (Basis). A basis  $\mathcal{B} \subset \mathcal{P}(X)$  is a collection of subsets of  $X$  s.t.

1.  $\bigcup_{B \in \mathcal{B}} B = X$ .
2. For any  $x \in B_1 \cap B_2$  ( $B_1, B_2 \in \mathcal{B}$ ),  $\exists B \in \mathcal{B}$  such that  $x \in B \subset B_1 \cap B_2$ .

**Definition 3** (Hausdorff). A topological space  $X$  is Hausdorff if  $\forall x_1 \neq x_2$ ,  $\exists$  neighborhood  $U_1 \ni x_1, U_2 \ni x_2$  s.t.  $U_1 \cap U_2 = \emptyset$ .

**Theorem 1** (Tychonoff theorem).  $\prod_{\beta \in B} X_\beta$  is compact.

**Definition 4** (Countable basis).  $X$  has a countable basis of nbds at  $x$  if  $\exists \{O_n\}_{n \in \mathbb{N}}$  of  $x$  s.t. for any nbd  $U$  of  $x$ ,  $\exists O_n \subset U$  for some  $n \in \mathbb{N}$ .

**Definition 5** (First countable).  $X$  is called first countable if  $X$  has countable basis of nbds at every point of  $X$ .

**Example.** Metric space is first countable. For any  $x$ ,  $O_n = B_{\frac{1}{n}}(x)$   $n \in \mathbb{N}$ .

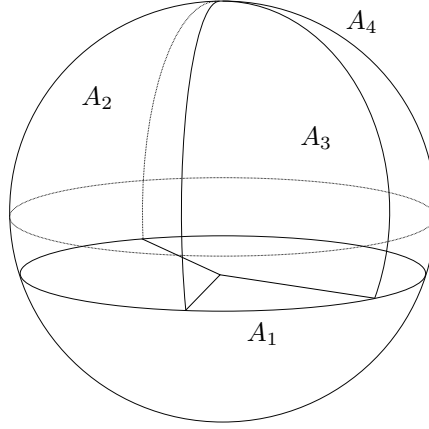


Figure 1: Example with four elements

**Definition 6.** A sequence  $\{x_n\}$  converges to  $y$  if given any open nbd  $U$  of  $y$ ,  $\exists N$  so that if  $n > N$ ,  $x_n \in U$ .

**Theorem 2.**  $A \subset X$  topological space. If  $x_n \in A$  converges to  $y$ , then  $y \in \bar{A}$ . Converse holds if  $X$  is first countable, that is, if  $y \in \bar{A}$ , then  $\exists x_n \in A$  with  $x_n \rightarrow y$ .

**Proof.** First statement is easy. Say  $X$  first countable. Pick  $y \in \bar{A}$ , we will find  $x_n \rightarrow y$ ,  $x_n \in A$ .  $\exists \{O_n\}$  countable basis of nbds of  $y$ . Set

$$\begin{aligned} U_1 &= O_1 \\ U_2 &= O_1 \cap O_2 \\ U_3 &= O_1 \cap O_2 \cap O_3 \\ &\vdots \\ U_1 &\supset U_2 \supset U_3 \cdots \end{aligned}$$

$\{U_n\}_{n \in \mathbb{N}}$  is also countable basis of nbds of  $y$ .

Now,  $y \in \bar{A} \Rightarrow U_n \cap A \neq \emptyset$ . Pick  $x_n \in U_n \cap A$ . Claim:  $x_n \rightarrow y$ . Choose any nbd  $U$  of  $y$ . Then,  $\exists N$  s.t.  $O_N \subset U$ .

**Note.** If  $n > N$ ,  $U_n = O_1 \cap \cdots \cap O_N \cap \cdots \cap O_n \subset O_N \subset U$ .

$\therefore x_n \in U$  for any  $n > N$ .  $\therefore x_n \rightarrow y$ . □

**Definition 7 (Second countable).**  $X$  is called second countable if  $X$  has countable basis (of topology).

**Example.**  $\mathbb{R}$ ,  $\{(a, b) \mid a, b \in \mathbb{Q}\}$ .

**Example.**  $X_1 \times \cdots \times X_n$  ( $X_i$ : second countable) is also second countable.

**Example.** Compact metric space.

**Question** If  $X$  is second countable, does it have a countable dense subset?

**Definition 8 (Separable).**  $X$  is called separable if  $\exists$  countable subset whose closure is  $X$ .

**Proposition 1.** Second countable  $\Rightarrow$  separable.

**Proposition 2.** Separable metric space  $\Rightarrow$  second countable.

**Definition 9 (Normal).**  $X$  is normal if  $X$  is Hausdorff and for any closed subset  $C_1, C_2$  with  $C_1 \cap C_2 = \emptyset$ ,  $\exists$  open sets  $U_1, U_2$  with  $U_1 \supset C_1, U_2 \supset C_2, U_1 \cap U_2 = \emptyset$ .

**Proposition 3.** Every compact Hausdorff space is normal.

**Theorem 3 (Urysohn's lemma).** Let  $X$  be normal and  $C_1, C_2$  disjoint closed subsets. Then  $\exists$  continuous function  $f : X \rightarrow [0, 1]$  such that

1.  $f(x) = 0 \quad \forall x \in A$ .
2.  $f(x) = 1 \quad \forall x \in B$ .

**Definition 10.** Equivalence relation:  $(X, \sim)$  satisfies

1.  $x \sim x$
2.  $x \sim y \Rightarrow y \sim x$
3.  $x \sim y, y \sim z \Rightarrow x \sim z$

$X/\sim$ : the set of equivalence classes

**Definition 11 (Locally compact).**  $X$  is called locally compact if for any  $x \in X$ ,  $\exists$  open nbd  $O$  of  $x$  such that  $\overline{O}$  is compact.

# Chapter 1

## Quotient topology

<https://en.wikipedia.org/wiki/Homotopy>

**Definition 12** (Homotopic). If  $f$  and  $f'$  are continuous maps of the space  $X$  into the space  $Y$ , we say that  $f$  is homotopic to  $f'$  if there is a continuous map  $F: X \times I \rightarrow Y$  such that  $F(x, 0) = f(x)$  and  $F(x, 1) = f'(x)$  for each  $x$ . (Here  $I = [0, 1]$ .) The map  $F$  is called a homotopy between  $f$  and  $f'$ . If  $f$  is homotopic to  $f'$ , we write  $f \simeq f'$ . If  $f \simeq f'$  and  $f'$  is a constant map, we say that  $f$  is nullhomotopic.

**Definition 13** (Evenly covered). Let  $p: E \rightarrow B$ , surjective map (so continuous). Let  $U \subset B$  open. Then  $U$  is evenly covered iff  $p^{-1}(U) = \bigcup_{\alpha \in I} V_{\alpha}$  with

- $V_{\alpha}$  open in  $E$
- $V_{\alpha} \cap V_{\beta} = \emptyset$  if  $\alpha \neq \beta$
- $p|_{V_{\alpha}}: V_{\alpha} \rightarrow U$  is a homeomorphism.

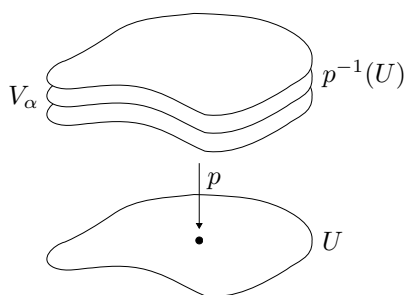


Figure 1.1: Evenly covered

## Chapter 2

# Fundamental group and applications

## Chapter 3

# Jordan curve theorem

[https://en.wikipedia.org/wiki/Jordan\\_curve\\_theorem](https://en.wikipedia.org/wiki/Jordan_curve_theorem)



## Chapter 4

# Seifert–Van Kampen theorem

[https://en.wikipedia.org/wiki/Seifert%E2%80%93Van\\_Kampen\\_theorem](https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem)

## Chapter 5

# Surfaces

## Chapter 6

# Covering spaces