Discrete-time models

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Lecture note 2: Multi-period models

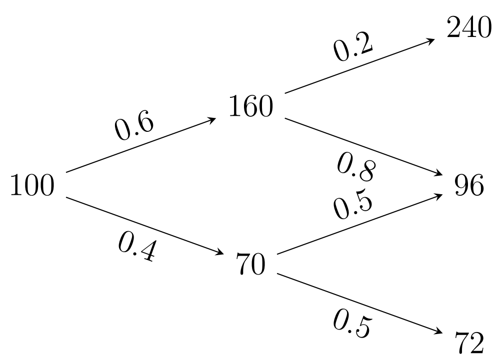
References:
CH 2, 3 in Björk (2004)

1 Exercises

Problem 1.1. (30 points) Consider a two-period binomial model with the time index $t = 0, 1, 2$. This model has two underlying assets; a bank account process $(G_t)_{t=0,1,2}$ given by

$$G_0 = G_1 = 1\,,\ G_2 = 1.2$$

and stock with price process given by the following tree.



(i) (5 points) Construct a sample space Ω, a probability $\mathbb{P}$, a filtration $(\mathcal{F}_t)_{t=0,1,2}$, and a stochastic process $(S_t)_{t=0,1,2}$ representing this stock price tree.

(ii) (5 points) Evaluate the conditional expectation $\mathbb{E}^{\mathbb{P}}(S_2/G_2|\mathcal{F}_1)$.

(iii) (5 points) Consider an option with payoff

$$X_2 = \begin{cases} 120 & \text{if } S_2 = 240 \\ 0 & \text{if } S_2 = 96 \\ -12 & \text{if } S_2 = 72 \end{cases}$$

at maturity $t = 2$. Find the price process $(X_t)_{t=0,1,2}$ and the hedging portfolio $h = (\pi_t, \phi_t)_{t=0,1,2}$ of this option.

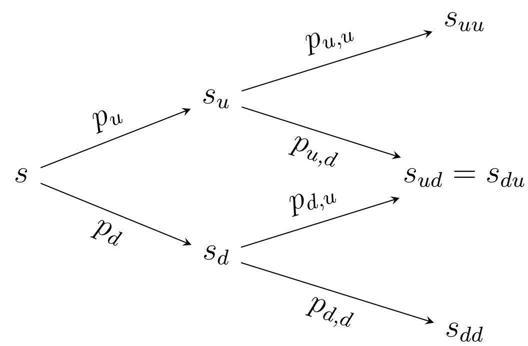
(iv) (5 points) Show that h is self-financing (Recall that we defined $(\pi_2, \phi_2) := (\pi_1, \phi_1)$ in class).

(v) (5 points) Show that the process $\phi = (\phi_t)_{t=0,1,2}$ is adapted .

(vi) (5 points) Find the risk-neutral measure $\mathbb{Q}$.

(vii) (5 points) Evaluate the conditional expectations $\mathbb{E}^{\mathbb{Q}}(S_2/G_2|\mathcal{F}_1)$ and $\mathbb{E}^{\mathbb{Q}}(X_2/G_2|\mathcal{F}_1)$.

Problem 1.2. (30 points) Consider the binomial model: $(G_t)_{t=0,1,2} = ((1+R)^t)_{t=0,1,2}$ and $(S_t)_{t=0,1,2}$ is given as



for $R \geq 0$, $p_u, p_d, p_{u,u}, p_{u,d}, p_{d,u}, p_{d,d} > 0$, $p_u + p_d = p_{u,u} + p_{u,d} = p_{d,u} + p_{d,d} = 1$, $s_u > s_d$, $s_{uu} > s_{ud} = s_{du} > s_{dd}$. Show that the followings are equivalent.

(i) This market satisfies the no-arbitrage condition.

(ii) $\frac{s_d}{s} < 1 + R < \frac{s_u}{s}$, $\frac{s_{ud}}{s_u} < 1 + R < \frac{s_{uu}}{s_u}$, $\frac{s_{dd}}{s_d} < 1 + R < \frac{s_{du}}{s_d}$.

(iii) A risk-neutral measure exists.

Problem 1.3. (5 points) For a finite set Ω and a time-index set $\mathbb{T} = \{0, 1, \cdots, T\}$, consider a filtered probability space $(\Omega, 2^\Omega, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$. Let $X : \Omega \to \mathbb{R}$ be a random variable. Show that a process M defined by

$$M_t = \mathbb{E}(X|\mathcal{F}_t)\,,\ t = 0, 1, \cdots, T$$

is a martingale.

Problem 1.4. (10 points) Consider the two-period ($t = 0, 1, 2$) binomial model satisfying the no-arbitrage condition (thus, a risk-neutral measure exists). Let X_2 be an option payoff with maturity $T = 2$, and let $(X_t)_{t=0,1,2}$ be the arbitrage-free price process of this option. Show that a market with three assets $(G_t)_{t=0,1,2}$, $(S_t)_{t=0,1,2}$, $(X_t)_{t=0,1,2}$ is free of arbitrage. (Here, a portfolio is a three-dimensional adapted process.) Hint: Get an idea from the proof of "The existence of a RN measure implies the no-arbitrage condition" we studied in class.

References

Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2004.