ADVANCED CALCULUS 1
ASSIGNMENT # 2 : 2019 SPRING

§2.1. # 2. Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy > 1\}$. Show that S is open.

§2.3. # 5. Let $S = \{x \in \mathbb{R} \mid x \text{ is irrational}\}$. Is S closed?

§2.4. # 3. Find the accumulation points of the following sets in $\mathbb{R}^2$:
a. $\{(m, n) \mid m, n \text{ integers}\}$
b. $\{(p, q) \mid p, q \text{ rational}\}$
c. $\{(\frac{m}{n}, \frac{1}{n}) \mid m, n \text{ integers},\ n \neq 0\}$
d. $\{(\frac{1}{n} + \frac{1}{m}, 0) \mid n, m \text{ integers},\ n \neq 0,\ m \neq 0\}$

§2.6. # 5. Let $A \subset \mathbb{R}$ be bounded and nonempty and let $x = \sup(A)$. Is $x \in \mathrm{bd}(A)$?

§2.8. # 2. Let (M, d) be a metric space with the property that every bounded sequence has a convergent subsequence. Prove that M is complete

(Exercises for Chapter 2)

18. If $x, y \in M$ and $x \neq y$, then prove that there exist open sets U and V such that $x \in U, y \in V$, and $U \cap V = \varnothing$

29. Let $A, B \subset \mathbb{R}^n$ and x be an accumulation point of $A \cup B$. Must x be an accumulation point of either A or B?

Advanced Calculus I - HW2

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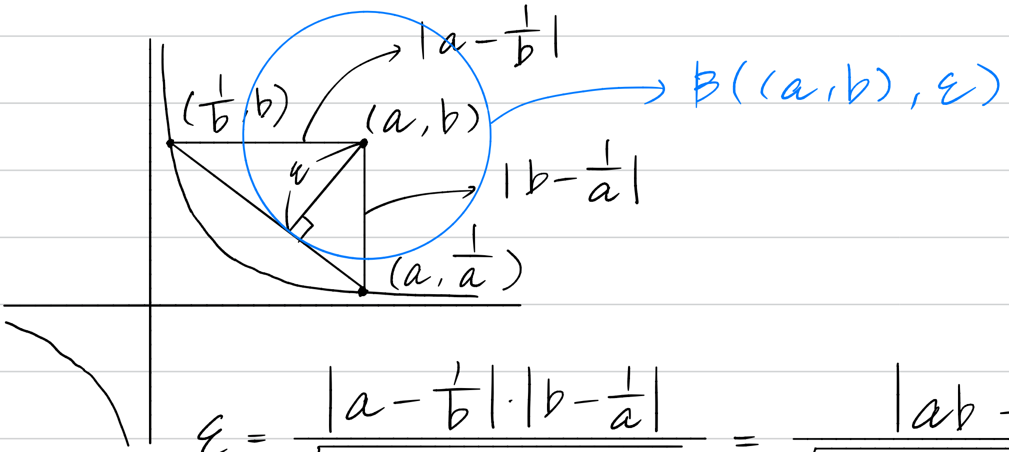
§2.1 #2. $S = \{(x,y) \in \mathbb{R}^2 \mid xy > 1\}$ open.



$$\varepsilon = \frac{|a-\frac{1}{b}|\cdot|b-\frac{1}{a}|}{\sqrt{(a-\frac{1}{b})^2+(b-\frac{1}{a})^2}} = \frac{|ab-2+\frac{1}{ab}|}{\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}-\frac{2a}{b}-\frac{2b}{a}}}$$

$$= \frac{|ab-2+\frac{1}{ab}|}{\sqrt{a^2+b^2+\frac{a^2+b^2}{a^2b^2}-\frac{2}{ab}(a^2+b^2)}} = \frac{ab-1}{\sqrt{a^2+b^2}}$$

$$\left(\because\ |ab-2+\tfrac{1}{ab}| = (1-\tfrac{1}{ab})(ab-1),\right.$$
$$\left.(a^2+b^2)(1+\tfrac{1}{a^2+b^2}-\tfrac{2}{ab}) = (a^2+b^2)(1-\tfrac{1}{ab})^2\right)$$

For any $(a,b)\in S$, $\varepsilon := \frac{ab-1}{\sqrt{a^2+b^2}}$

$B((a,b),\varepsilon) \subset S \Rightarrow S$ is open. ∎

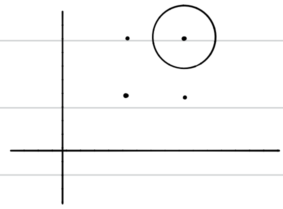
§2.3 #5. $S = \{x \in \mathbb{R} \mid x \text{ is irrational}\}$

$\mathbb{R}\setminus S = \{y \in \mathbb{R} \mid y \text{ is ratinal}\}$

$\forall y \in \mathbb{R}\setminus S$, $\forall \varepsilon > 0$, $B(y,\varepsilon)$ contains irrationl number

$\therefore$ $\mathbb{R}\setminus A$ is not open. and S is not closed.

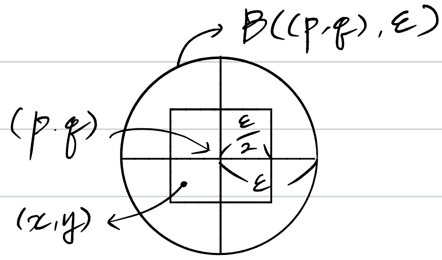
§2.4 #3. find the accumulation point.



(a) $A = \{(m,n) \mid m,n \text{ integers}\}$

$\forall\, 0 < \varepsilon \leq 1,\ B((m,n),\varepsilon) \cap A = \phi$

$\therefore$ A does not have accumulation point.



(b) $B = \{(p,q) \mid p,q \text{ rational}\}$

$\forall \varepsilon > 0$, choose $(x,y) \in B$

s.t $0 < |p-x| \leq \frac{\varepsilon}{2}$, $0 < |q-y| \leq \frac{\varepsilon}{2}$

It must exist because of Archimedian property.

Then $(x,y) \in B((p,q),\varepsilon)$

$\therefore$ all accumulation points of B is $\mathbb{R}^2$

(c) $C = \{(m/n, 1/n) \mid m,n \text{ integers}, n \neq 0\} : \mathbb{R}^2$

consider $\{m/n \in \mathbb{R} \mid m.n \in \mathbb{Z}, n \neq 0\}$

Since m/n is rational,

all accumulation points is $\mathbb{R}$

Similarly consider $\{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z} \setminus \{0\}\}$

It has the only accumulation point $\{0\}$.

$\therefore$ all accumulation points of C is X-axis.

(d) $D = \{(1/n + 1/m, 0) \mid n,m \text{ integers}, n \neq 0, m \neq 0\}$

$\forall \varepsilon > 0$, choose $x,y \in \mathbb{Z} \setminus \{0\}$ s.t $x = n$, $\left|\frac{1}{y}\right| < \varepsilon$

then $(1/x + 1/y, 0) \in B((1/n + 1/m, 0), \varepsilon)$

$\therefore \{(1/t, 0) \mid t \in \mathbb{Z} \setminus \{0\}\}$ is accumulation points of D.

$\forall \varepsilon > 0$, choose $x,y \in \mathbb{Z} \setminus \{0\}$ s.t $x = y$, $x > \frac{2}{\varepsilon}$

then $(1/x + 1/y, 0) \in B((0,0), \varepsilon)$

$\therefore (0,0)$ is also accumulation point of D.

$\Rightarrow$ all accumulation points $= \{(\frac{1}{t}, 0) \mid t \in \mathbb{Z} \setminus \{0\}\} \cup \{(0,0)\}$

§2.6 #5 Yes. $x = \sup(A)$.

i) $x \in A$

$x \in cl(A)$. $cl((x,\infty)) = [x,\infty)$ $\therefore x \in cl(\mathbb{R} \setminus A)$

$\therefore x \in cl(A) \cap cl(\mathbb{R}\setminus A) = bd(A)$

ii) $x \notin A$

$\forall \varepsilon > 0$, $\exists y \in A$ s.t $y \in (x-\varepsilon, x)$

($\because$ if $\nexists y \in (x-\varepsilon, x)$ for some $\varepsilon > 0$, then $\sup A = x - \varepsilon$)

$\therefore x$ is accumulation point $\Rightarrow x \in cl(A)$.

$cl([x,\infty)) = [x,\infty)$ $x \in cl(\mathbb{R}\setminus A)$

$\therefore x \in cl(A) \cap cl(\mathbb{R}\setminus A) = bd(A)$

§2.8 #2.

M is complete iff Every Cauchy sequence in M Converges to x in M.

Thus we want to show that Every Cauchy sequence $x_n \in M$ converges to $x \in M$.

Suppose x_n is Cauchy sequence in M.

A Cauchy sequence in a metric space must be bounded.

x_n is bounded.

By given hypothesis, x_n has a subsequence converging to $x \in M$

We know that if a subsequence of Cauchy sequence converges to x, then the sequence converges to x.

Thus Every Cauchy sequence in M converges to $x \in M$.

$\therefore$ M is complete.

#18. $\forall \varepsilon > 0$, $B(x, \varepsilon)$ is open.

ETS $\exists \varepsilon_1, \varepsilon_2 > 0$ s.t $B(x, \varepsilon_1) \cap B(y, \varepsilon_2) = \phi$

Let $k = \frac{1}{2} \cdot d(x, y)$

since $x, y \in M$, $x \neq y$, $d(x, y) > 0$

choose $\varepsilon_1, \varepsilon_2 \leq k$

then, $B(x, \varepsilon_1) \cap B(y, \varepsilon_2) = \phi$ for $0 < \varepsilon_1, \varepsilon_2 \leq k$

#29. Answer is Yes.

ETS contraposition is True.

x is not accumulation point of both A and B

$\Leftrightarrow \exists \varepsilon > 0$ s.t $B(x, \varepsilon) \cap A = \phi$, $B(x, \varepsilon) \cap B = \phi$

$\Rightarrow B(x, \varepsilon) \subset M \setminus A$, $B(x, \varepsilon) \subset M \setminus B$

$\Rightarrow B(x, \varepsilon) \subset (M \setminus A) \cap (M \setminus B)$

$\Rightarrow B(x, \varepsilon) \subset M \setminus (A \cup B)$

$\Rightarrow B(x, \varepsilon) \cap (A \cup B) = \phi$

$\therefore$ x is not accumulation point of $A \cup B$.