Mathematical Statistics 1

Ch.2 Discrete Distributions

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pmf $\qquad E(g(x))$

cdf

mgf

Ch.2.4 Bernoulli Trials and Binomial Distribution

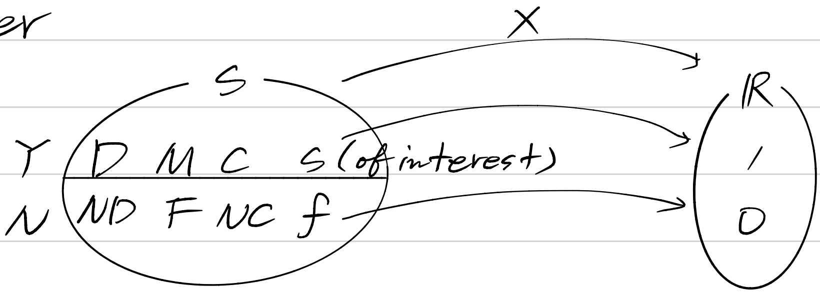
4.1 Bernoulli distribution

A Bernoulli experiment is a random experiment, the outcome of which can be classified in one of two mutually exclusive ways (success or failure). A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several independent times. Let X be a random variable with a Bernoulli trial by defining

$$X(\text{success}) = 1 \quad \text{and} \quad X(\text{failure}) = 0$$
$$P(X = 1) = p \quad \text{and} \quad P(X = 0) = q = 1 - p$$

where p is the probability of success on a Bernoulli trial.

Ber



$X(s) = 1 \quad P(X=1) = P(\{s\}) \overset{let}{=} p$ (prob. of s)

$X(f) = 0 \quad P(X=0) = P(\{f\}) = 1-p$

1) $X \sim Ber(p)$

2) $X \sim f_X(x) = P(X=x)$; pmf.

$= p^x(1-p)^{1-x}, \quad x = 0, 1$

3) $E(X) = \sum_{x=0}^{1} x f_X(x) = f_X(1) = P(X=1) = p$

$Var(X) = E(X^2) - \{E(X)\}^2 = \sum_{x=0}^{1} x^2 f(x) - p^2$

$= f_X(1) - p^2 = p - p^2 = p(1-p)$

Ber(p).

Let r.v. X has a Bernoulli distribution with the success probability p.

- $X \sim \text{Ber}(p)$.
- *pmf*

$$f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1; \quad 0 < p < 1$$

- Mean: $E(X) = p$
- Variance: $\mathrm{Var}(X) = p(1-p)$

Example 2.4-3

Out of millions of instant lottery tickets, suppose that 20% are winners. If five such tickets are purchased, then (0,0,0,1,0) is a possible observed sequence in which the fourth ticket is a winner and the other four are losers. Assuming independence among winning and losing tickets, find the probability of this outcome.

$$0.8^4 \times 0.2$$

4.2 Binomial distribution

- Let r.v X be the total number of successes in n independent Bernoulli trials with the success probability p.
- $X = \sum_{i=1}^{n} X_i,\ X_i \sim \mathrm{Ber}(p).$
- *pmf*

$$\frac{n!}{x!(n-x)!}$$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \cdots, n; \quad 0 < p < 1$$

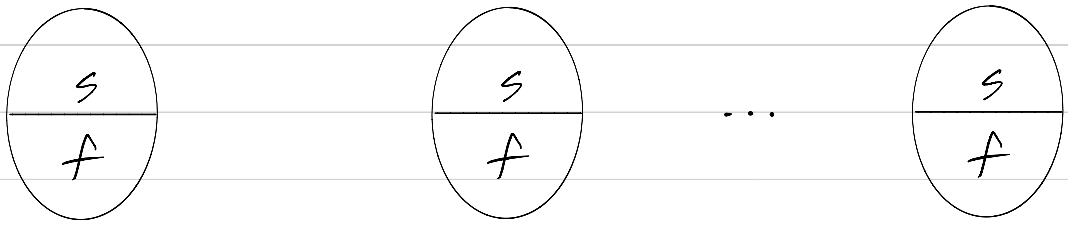
- $X \sim \mathrm{Bin}(n, p)$
- $P(X = x)$: dbinom(x,n,p) in R
- $P(X \leq x)$: pbinom(x,n,p) in R

R

$0 \le x \le n$

X = # of s trials among "n" indep Ber. trials with s.p "p"

1st B trial   2nd Ber. trial n^{th} Ber. trial



$X_1 = \begin{cases} 0 \\ 1 \end{cases}$ $X_2 = \begin{cases} 0 \\ 1 \end{cases}$ $X_n = \begin{cases} 0 \\ 1 \end{cases}$

1) $X = X_1 + X_2 + X_3 + \cdots + X_n$

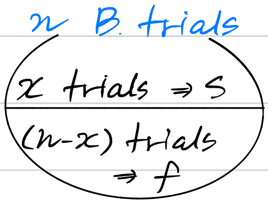
$X_i \sim Ber(p)$ indep

$X \sim Bin(n, p)$ Binomial Distribution Notation

2) $f_X(x) = P(X = x) = {}_nC_x \times p^x \times (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n$

x : s trials among "n" Ber trials

n B. trials



$$E(X) = \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

Properties

- A Bernoulli experiment is performed n times.
- The trials are independent.
- For each trial, the probability of success is a constant p.

Example 2.4-5

In the instant lottery with 20% winning tickets, if X is equal to the number of winning tickets among $n = 8$ that are purchased, compute the probability of purchasing two winning tickets.

$$ {}_8C_2 \cdot 0.2^2 \times 0.8^6 $$

$$\sum_{x=0}^{n} {}_nC_x = 2^n$$

Example 2.4-8

Leghorn chickens are raised for laying eggs. Let $p = 0.5$ be the probability of a female chick hatching. Assuming independence, let X equal the number of female chicks out of 10 newly hatched chicks selected at random.

$$({}_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + {}_{10}C_4 + {}_{10}C_5)\, 2^{-10}$$

- Find the probability of 5 or fewer female chicks. $P(X \le 5)$
- Find the probability of exactly 6 female chicks. $P(X=6) = {}_{10}C_6\, 2^{-10}$
- Find the probability of at least 6 female chicks. $P(X \ge 6)$

$$2^{-10} \cdot ({}_{10}C_6 + {}_{10}C_7 + {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10})$$

Expectation, Variance, and mgf

- Mean: $E(X) = np$
- Variance: $\mathrm{Var}(X) = np(1-p)$
- mgf: $M(t) = \left[(1-p) + pe^t\right]^n, \quad t \in \mathbb{R}$

- Mean

$$E(X) = \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

$x-1 \to y$

$x = y+1$

$$= np \boxed{\sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-y-1)!} p^y (1-p)^{n-y-1}} = np$$

Tip: $\sum_{x \in \mathcal{X}} f_X(x) = 1$

$\Rightarrow$ pmf of $Y \sim B(n-1, p)$

$$\sum_{y=0}^{n-1} f_Y(y) = 1$$

$$= \sum_{y \in Y} f_Y(y) = 1$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$X_i \sim Ber(p)$$

$$E(X_i) = p$$

$$\begin{aligned} E(X) &= E(X_1 + X_2 + \cdots + X_n) \\ &= E(X_1) + E(X_2) + \cdots + E(X_n) \\ &= p + p + \cdots + p \\ &= np \end{aligned}$$

$$\begin{aligned} E(x) &= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \\ &= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \end{aligned}$$

$$x-1 = y \qquad x = y+1$$

$$= np \sum_{y=0}^{n-1} \underbrace{\frac{(n-1)!}{y!(n-1-y)!} p^y (1-p)^{n-1-y}}_{Y \sim Bin(n-1,\, p)}$$

$$np \underbrace{\sum_{y=0}^{n-1} f_Y(y)}_{1} = np$$

- Variance

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1) \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x}$$

$x-2=y$

$x = y+2$

$$= n(n-1)p^2 \sum_{y=0}^{n-2} \frac{(n-2)!}{y!(n-y-2)!} p^y (1-p)^{n-y-2} = n(n-1)p^2$$

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = E[X(X-1)] - \{E(X)\}^2 + E(X)$$

$$= n(n-1)p^2 - (np)^2 + np = np(1-p)$$

$$Var(X) = E(X^2) - \{E(X)\}^2$$

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1)\frac{n!}{x!(n-x)!}p^x\cdot(1-p)^{n-x}.$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!}p^x(1-p)^{n-x}$$

$$= n(n-1)p^2\sum_{x=2}^{n}\frac{(n-2)!}{(x-2)!(n-x)!}p^{x-2}(1-p)^{n-x}$$

$$x-2=y \qquad x=y+2.$$

$$= n(n-1)p^2\sum_{y=0}^{n-2}\frac{(n-2)!}{y!(n-2-y)!}p^y(1-p)^{n-2-y}$$

$$Y \sim B(n-2, p)$$

$$\Rightarrow \sum_{y\in Y} f_Y(y) = 1.$$

$$\Rightarrow E(X(X-1)) = n(n-1)p^2$$

$$= E(X^2-X) = E(X^2) - E(X)$$

$$= n(n-1)p^2$$

$$\therefore E(X^2) - E(X)^2 = E(X^2) - E(X) + E(X) - \{E(X)\}^2$$

$$= n(n-1)p^2 + np - \{np\}^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$\checkmark$

- *Hint:* For positive integer, n,

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}$$

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}.$$

- mgf

$$M(t) = E\left(e^{tX}\right) = \sum_{x=0}^{n} e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= \left[(1-p) + pe^t\right]^n, \qquad t \in \mathbb{R}$$

$$M(t) = E(e^{tx})$$

$$= [(1-p)+pe^t]^n$$

$$M_X(t) = [(1-p)+pe^t]^n$$

$$E(X) = \frac{\partial M}{\partial t}\Big|_{t=0} = npe^t[(1-p)+pe^t]^{n-1}$$

$$E(X) = M_t^{(1)}(0) = \frac{\partial M}{\partial t}\Big|_{t=0}$$

$$= n[(1-p)+pe^t]^{n-1} \cdot pe^t \Big|_{t=0}$$

$$= \underline{npe^t[(1-p)+pe^t]^{n-1}\Big|_{t=0}}$$

$$= np[(1-p)+p]^{n-1}$$

$$= np.$$

$$Var(X) = E(X^2) - \{E(X)\}^2 = M_t^{(2)}(0) - \{M_t^{(1)}(0)\}^2$$

$$M_t^{(2)}(0) = \frac{\partial^2 M}{\partial t^2}\Big|_{t=0}.$$

$$npe^t[(1-p)+pe^t]^{n-1} + npe^t(n-1) \cdot pe^t[(1-p)+pe^t]^{n-2}$$

令 $t=0$. $np + np(n-1)\cdot p = np + \cancel{n^2p^2} - np^2$

$$- \cancel{(np)^2}$$

$$np(1-p)$$

Ch.2.5 Negative Binomial Distribution

5.1 Negative Binomial Distribution $\mathrm{NB}(r, p)$

- Let r.v X be the number of failures before the *r*th success with the success probability *p*.
- *pmf*

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0, 1, \cdots; \quad 0 < p < 1$$

- Mean: $E(X) = \dfrac{r(1-p)}{p}$
- Variance: $\mathrm{Var}(X) = \dfrac{r(1-p)}{p^2}$
- mgf: $M(t) = \left(\dfrac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$

$X \sim NB(r, p)$

X = # of failures until the r^{th} s happen with s.p "p"

$$f_X(x) = P(X=x) = \binom{r-1+x}{x} p^r (1-p)^x \qquad x = 0, 1, \cdots$$

$(r-1+x)$ trials

$\begin{cases} r-1 : s \\ x : \text{failure} \end{cases}$ $\qquad r^{th}$ s

$$\frac{(r+x-1)!}{x!\,(r-1)!} p^r (1-p)^x$$

$$f_X(x) = P(X=x) = \binom{r+x-1}{x} p^r (1-p)^x$$

$$E(X) = \sum_{x \in \mathcal{X}} x \frac{(r+x-1)!}{x!\,(r-1)!} p^r (1-p)^x$$

$$= \sum_{x=1}^{\infty} \frac{(r+x-1)!}{(x-1)!\,(r-1)!} p^r (1-p)^x$$

$$= r \sum_{x=1}^{\infty} \frac{(r+x-1)!}{(x-1)!\,r!} p^r (1-p)^x$$

$$= r \sum_{x=1}^{\infty} \binom{r+x-1}{x-1} p^r (1-p)^x$$

$x-1 = y \qquad x = y+1.$

$$= r \sum_{y=0}^{\infty} \binom{r+y}{y} p^r (1-p)^{y+1}$$

$$= \frac{r(1-p)}{p} \underbrace{\sum_{y=0}^{\infty} \binom{(r+1)+y-1}{y} p^{r+1} (1-p)^y}_{1}$$

$$= \frac{r(1-p)}{p}$$

$f_Y(y).$

$Y \sim NB(r+1, p).$

$\sum_{y \in \mathcal{Y}} f_Y(y) = 1.$

- Mean

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} \frac{(r+x-1)!}{x!(r-1)!} x \cdot p^r (1-p)^x = \sum_{x=1}^{\infty} \frac{(r+x-1)!}{(x-1)!(r-1)!} p^r (1-p)^x \\
&= r \sum_{x=1}^{\infty} \frac{(r+x-1)!}{(x-1)!r!} p^r (1-p)^x = r \sum_{x=1}^{\infty} \binom{r+x-1}{x-1} p^r (1-p)^x \\
&= r \sum_{y=0}^{\infty} \binom{r+y}{y} p^r (1-p)^{y+1} = \frac{r(1-p)}{p} \sum_{y=0}^{\infty} \binom{(r+1)+y-1}{y} p^{r+1} (1-p)^y \\
&= \frac{r(1-p)}{p}
\end{aligned}
$$

$$E[X(X-1)] = \sum_{x=0}^{\infty} \binom{r-1+x}{x} \cdot x(x-1) \cdot p^r (1-p)^x$$

$$= \sum_{x=0}^{\infty} \frac{(r-1+x)!}{x!\,(r-1)!} x(x-1) \cdot p^r (1-p)^x$$

$$= \sum_{x=2}^{\infty} \frac{(r-1+x)!}{(x-2)!\,(r-1)!} p^r (1-p)^x$$

$$= r(1-p)^2 \sum_{x=2}^{\infty} \frac{(r-1+x)!}{(x-2)!\,r!} p^r (1-p)^{x-2}$$

$$= r(1-p)^2 \sum_{y=0}^{\infty} \frac{(r+1+y)!}{y!\,r!} p^r (1-p)^y$$

$$= r(1-p)^2$$

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \frac{(r+x-1)!}{x!\,(r-1)!} p^r (1-p)^x$$

$$= \sum_{x=2}^{\infty} \frac{(r+x-1)!}{(x-2)!\,(r-1)!} p^r (1-p)^x$$

$$= r(r+1) \sum_{x=2}^{\infty} \frac{(r+x-1)!}{(x-2)!\,(r+1)!} p^r (1-p)^x$$

$$= r(r+1) \sum_{x=2}^{\infty} \binom{r+x-1}{x-2} p^r (1-p)^x$$

$$= r(r+1) \sum_{y=0}^{\infty} \binom{r+y+1}{y} p^r (1-p)^{y+2}$$

$$= r(r+1) \sum_{y=0}^{\infty} \binom{(r+2)+y-1}{y} p^r (1-p)^{y+2}$$

$$= \frac{r(r+1)(1-p)^2}{p^2} \underbrace{\sum_{y=0}^{\infty} \binom{(r+2)+y-1}{y} p^{r+2} (1-p)^y}_{1}$$

$$= \frac{r(r+1)(1-p)^2}{p^2}$$

$r^2(1-p)^2$

$$\mathrm{var}(X) = E(X^2) - \{E(X)\}^2 = E(X^2) - E(X) - \{E(X)\}^2 + E(X)$$

$$\frac{r(r+1)(1-p)^2}{p^2} - \frac{r^2(1-p)^2}{p^2} + \frac{r(1-p)}{p} \qquad \frac{r(1-p)}{p^2}$$

$$= \frac{r(1-p)^2}{p^2} + \frac{pr(1-p)}{p^2} = \frac{r(1-p)(1-\cancel{p}+\cancel{p})}{p^2}$$

- $E[X(X-1)]$

$$E[X(X-1)] = \sum_{x=0}^{\infty} \frac{(r+x-1)!}{x!(r-1)!} x(x-1) \cdot p^r (1-p)^x$$

$$= \sum_{x=2}^{\infty} \frac{(r+x-1)!}{(x-2)!(r-1)!} p^r (1-p)^x = r(r+1) \sum_{x=2}^{\infty} \frac{(r+x-1)!}{(x-2)!(r+1)!} p^r (1-p)^x$$

$$= r(r+1) \sum_{x=2}^{\infty} \binom{r+x-1}{x-2} p^r (1-p)^x = r(r+1) \sum_{y=0}^{\infty} \binom{(r+1)+y}{y} p^r (1-p)^{y+2}$$

$$= \frac{r(r+1)(1-p)^2}{p^2} \sum_{y=0}^{\infty} \binom{(r+2)+y-1}{y} p^{r+2} (1-p)^y = \frac{r(r+1)(1-p)^2}{p^2}$$

- Variance

$$\mathrm{Var}(X) = E[X(X-1)] - \{E(X)\}^2 + E(X)$$

$$= \frac{r(r+1)(1-p)^2}{p^2} - \frac{r^2(1-p)^2}{p^2} + \frac{r(1-p)}{p} = \frac{r(1-p)}{p^2}$$

- *Hint* (Maclaurin's series expansion): For $|z| < 1$,

$$(1-z)^{-r} = \sum_{k=0}^{\infty} \binom{r+k-1}{k} z^k.$$

- *mgf*

$$M(t) = E\left(e^{tX}\right) = \sum_{x=0}^{\infty} e^{tx} f(x) = p^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} \left[(1-p)e^t\right]^x$$

$$= \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

$$M^{(1)}(0) = \frac{d}{dt} M(t)\Big|_{t=0} = \frac{d}{dt} \left(\frac{p}{1-(1-p)e^t}\right)^r \Big|_{t=0} = \frac{r(1-p)}{p}$$

$$M^{(2)}(0) = \frac{d^2}{dt^2} M(t)\Big|_{t=0} = \frac{d^2}{dt^2} \left(\frac{p}{1-(1-p)e^t}\right)^r \Big|_{t=0} = \frac{r(1-p)\{1+r(1-p)\}}{p^2}$$

$$\mathrm{Var}(X) = M^{(2)}(0) - \left\{M^{(1)}(0)\right\}^2 = \frac{r(1-p)}{p^2}$$

$$M(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} [(1-p)e^t]^x$$

$$(a+b)^n = \binom{n}{x} b^x a^{n-x}$$

$$(1-p)e^t$$

$$\frac{d}{dt} \left(\frac{p}{1-(1-p)e^t} \right)^r \Bigg|_{t=0}$$

$$p^r \cdot (1-(1-p)e^t)^{-r}$$

$$p^r \cdot (1 - e^t + pe^t)^{-r}$$

$$(2^x + 3^x)^x$$

$$x$$

$$(1-z)^{-r}$$

$$= \sum \binom{}{x}$$

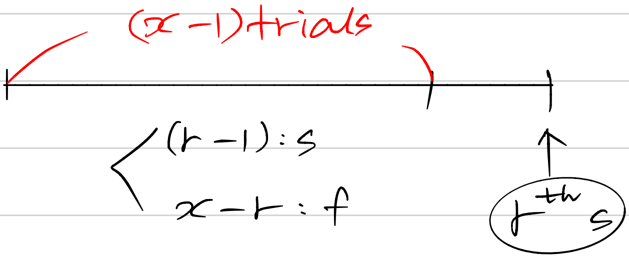
- Suppose that we are interested in the number of trials until the r^{th} success occurs instead of the number of failures (X).
- If we let $Y = X + r$ equal the number of trials required to get the r^{th} success, then $Y \sim NB(r, p)$.

$$f(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}, \quad y = r, r+1, \cdots; \;\; 0 < p < 1.$$

$X \sim NB(r, p)$

X = # of total trials until the r^{th} s happens
with s.p "p"

$X = x$



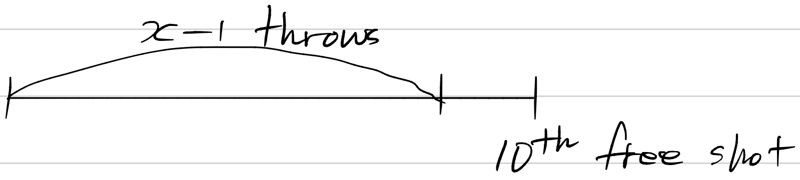
$$f_X(x) = P(X = x) = \binom{x-1}{r-1} p^r \times (1-p)^{x-r}$$

$(1-p)^x p$

Example 2.5-2

Suppose that during practice a basketball player can make a free throw 80% of time. Furthermore, assume that a sequence of free-throw shooting can be thought of as independent Bernoulli trials. Let X equal the minimum number of throws that this player must attempt to make a total of 10 free-throw shoots. Find the pmf, mean, and variance of X.

X = # of throws.



- 9 free shot
- x − 10 : just shot

$$f_X(x) = P(X=x) = \binom{x-1}{9} 0.8^{9} \times 0.2^{x-10} \times 0.8$$

5.2 Geometric Distribution $Geo(p)$

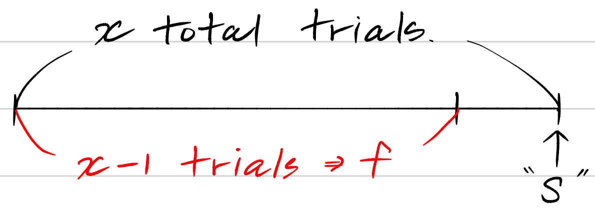
- Let r.v X be the number of trials until the first success with the success probability p.
- *pmf*

$$f(x) = p(1-p)^{x-1}, \quad x = 1, 2, \cdots, ; \quad 0 < p < 1$$

- Mean: $E(X) = \frac{1}{p}$
- Variance: $\mathrm{Var}(X) = \frac{1-p}{p^2}$
- mgf: $M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -log(1-p)$

X = # of totals until the 1^{st} S. happens with s.p. "p"

$X = x$



$$f_X(x) = P(X=x) = (1-p)^{x-1} \cdot p \qquad x = 1, 2, 3, \cdots$$

$$E[X] = \sum_{x \in \mathcal{X}} x f_X = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$= p \cdot [1 + 2(1-p) + 3(1-p)^2 + \cdots]$$

$$= p \cdot \frac{1}{(1-(1-p))^2} \quad (\because (1-p) < 1)$$

$$\left(|x| < 1, \ \sum_{i=1}^{\infty} i \cdot x^{i-1} = \frac{1}{(1-x)^2} \right).$$

$$E[X(X+1)] = \sum_{x \in \mathcal{X}} x(x+1)(1-p)^{x-1} p, \quad x = 1, 2, \cdots$$

$$= p \sum_{x=1}^{\infty} x(x+1)(1-p)^{x-1}$$

$$= p(1 \cdot 2 + 2 \cdot 3 \cdot (1-p) + 3 \cdot 4 (1-p)^2 + \cdots)$$

$$= p \cdot \frac{2}{(1-(1-p))^3} = \frac{2}{p^2}$$

$$Var[X] = E[X(X+1)] - E[X] - \{E[X]\}^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$$

- *Hint:* For $|x| < 1$,

$$\sum_{i=1}^{\infty} x^{i-1} = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$\sum_{i=1}^{\infty} i \cdot x^{i-1} = 1 + 2x + 3x^2 + 4 \cdot x^3 + \cdots = \frac{1}{(1-x)^2}$$

$$\sum_{i=1}^{\infty} (i+1) \cdot i \cdot x^{i-1} = 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots = \frac{2}{(1-x)^3}.$$

$1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ $\quad |x|<1.$

$1+2x+3x^2+4x^3+\cdots = \frac{1}{(1-x)^2}$

$2\cdot1+3\cdot2x+4\cdot3x^3+5\cdot4x^3+\cdots = \frac{2}{(1-x)^3}$

$1-p < 1$

- Mean and Variance

$$\therefore = \frac{1}{(1-(1-p))^2} = \frac{1}{p^2}$$

$$E(X) = p\sum_{x=1}^{\infty} x \cdot (1-p)^{x-1} = p\left[1 + 2(1-p) + 3(1-p)^2 + \cdots\right]$$

$$= p\frac{1}{(1-(1-p))^2} = \frac{1}{p}$$

$1 + 2(1-p) + 3(1-p)^2 + \cdots$

$\therefore E(X) = \frac{1}{p}$

$$E[(X+1)X] = p\sum_{x=1}^{\infty}(x+1)\cdot x\cdot(1-p)^{x-1}$$

$$= p\left[2\cdot 1 + 3\cdot 2(1-p) + 4\cdot 3(1-p)^2 + \cdots\right] = p\frac{2}{(1-(1-p))^3} = \frac{2}{p^2}$$

$$\mathrm{Var}(X) = E[(X+1)X] - \{E(X)\}^2 - E(X) = \frac{2}{p^2} - \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$$

$E((x+1)x) = p\sum_{x=1}^{\infty}(x+1)x\cdot(1-p)^{x-1}$

$(1-x)^2$

$1-p$

$2\cdot 1 + 3\cdot 2(1-p) + 4\cdot 3(1-p)^2$

$= \frac{2}{(1-(1-p))^3} = \frac{2}{p^3}\cdot p = \frac{2}{p^2}$

$M(t) = E(e^{tx}) = \sum e^{tx} p \cdot (1-p)^{x-1}$

$= pe^t \sum (e^t)^{x-1} \cdot (1-p)^{x-1}$

$((1-p)e^t)^{x-1}$

- *mgf*

$$M(t) = E\left(e^{tX}\right) = \sum_{x=1}^{\infty} e^{tx} f(x) = pe^t \sum_{x=1}^{\infty} \{(1-p)e^t\}^{x-1}$$

$$\frac{pe^t}{1-(1-p)e^t}$$

$$= \frac{pe^t}{1-(1-p)e^t}, \quad t < -log(1-p)$$

$$M^{(1)}(0) = \frac{d}{dt} M(t)\bigg|_{t=0} = \frac{d}{dt} \frac{pe^t}{1-(1-p)e^t}\bigg|_{t=0} = \frac{pe^t}{\{1-(1-p)e^t\}^2}\bigg|_{t=0} = \frac{1}{p}$$

$$M^{(2)}(0) = \frac{d^2}{dt^2} M(t)\bigg|_{t=0} = M^{(2)}(0) = \frac{d^2}{dt^2} \frac{pe^t}{1-(1-p)e^t}\bigg|_{t=0}$$

$$= pe^t \frac{1+(1-p)e^t}{\{1-(1-p)e^t\}^3}\bigg|_{t=0} = \frac{2-p}{p^2}$$

$$\text{Var}(X) = M^{(2)}(0) - \left\{M^{(1)}(0)\right\}^2 = \frac{1-p}{p^2}$$

$$M(t) = E[e^{tX}] = \sum_{x \in X} e^{tx} \cdot p(1-p)^{x-1}, \quad x = 1, 2, \dots$$

$$= p \cdot e^t \sum_{x=1}^{\infty} \{(1-p) \cdot e^t\}^{x-1}$$

$$= \frac{p \cdot e^t}{1-(1-p) \cdot e^t} = \frac{p \cdot e^t}{1-e^t+p \cdot e^t}$$

$$(t < -\log(1-p)). \quad \because (1-p) \cdot e^t < 1$$

$$M(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} \cdot p(1-p)^{x-1}$$

$$= p \cdot \sum_{x=1}^{\infty} (e^t)^x (1-p)^{x-1}$$

$$= p \cdot e^t \sum_{x=1}^{\infty} (e^t)^{x-1} \cdot (1-p)^{x-1}$$

$$= p \cdot e^t \cdot \sum_{x=1}^{\infty} \{(1-p)e^t\}^{x-1}$$

$$= \frac{p \cdot e^t}{1-(1-p)e^t} \qquad (t < -\log(1-p))$$

$$M^{(1)}(0) = \frac{d}{dt} M(t)\Big|_{t=0} = \frac{p \cdot e^t (1-(1-p)e^t) - pe^t(-e^t + pe^t)}{(1-(1-p)e^t)^2}$$

$$= \frac{pe^t \cancel{- pe^{2t}} \cancel{+ p^2 e^{2t}} \cancel{+ pe^{2t}} \cancel{- p^2 e^{2t}}}{(1-(1-p)e^t)^2}$$

$$= \frac{pe^t}{(1-(1-p)e^t)^2}\Big|_{t=0}.$$

$$= \frac{p}{p^2} = \frac{1}{p} = M^{(1)}(0) = E(X).$$

$$M^{(2)}(0) = \frac{d^2}{dt^2} M(t)\Big|_{t=0} = \frac{d}{dt}\left(\frac{pe^t}{1-(1-p)e^t)^2}\right)\Big|_{t=0}.$$

$$= \frac{pe^t(1-(1-p)e^t)^2 - pe^t(2(1-p)^2 e^{2t} - 2(1-p)e^t)}{(1-(1-p)e^t)^4}\Bigg|_{t=0}.$$

$$-p(2(1-p)^2 - 2(1-p))$$

$$= \frac{p^3 + 2p^2 - 2p^3}{p^4} \qquad 2(1-p)((1-p)-1)$$

$$2(1-p) \cdot -p.$$

$$= \frac{2p^2 - p^3}{p^4} \qquad p^2(2-2p)$$

$$2p^2 - 2p^3$$

$$= \frac{2-p}{p^2} = E(X^2).$$

$$\therefore Var(X) = M^{(2)}(0) - \{M^{(1)}(0)\}^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$