Vector Calculus – Midterm Exam

October 26, 2018

1. (1) Find the volume of a tetrahedron which consists of the following vertices:

$$A = (1, 2, 3),\ B = (0, 1, 2),\ C = (1, 3, 5),\ D = (1, 4, 11).$$

   (2) Compute the divergence of the vector field $F(x, y, z) = (xy, yz, zx)$ at $(1, 1, 1)$.

   (3) Compute the curl of the vector field $F(x, y, z) = (y^2 z, e^{xyz}, x^2 y)$ at $(1, 1, 1)$.

2. Consider a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \begin{cases} \dfrac{x^3 y - xy^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

   (1) Find $D_1 f(0, 0)$ and $D_2 f(0, 0)$.

   (2) Prove or disprove that f is differentiable at origin.

3. Show the following matrix is positive definite:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

4. Suppose that there is a circle centered at $(0, r)$ with the radius r. Imagine that this circle is a wheel rolling along x-axis. When the wheel rolls one cycle to right direction, the point on the rim traces a cycloid curve.

   (1) Provide a parametrized curve to this cycloid starting from the origin $(0, 0)$.

   (2) Find the arc length of the curve when the wheel rolls one cycle.

   (3) Compute the curvature of this curve when the curve has a maximum y value.

5. Compute the curvature and the torsion of the following parametrized curve at origin:

$$X(t) = (e^t \cos t, e^t \sin t, e^t).$$

6. Find the second-order Taylor approximation for the following function at origin:

$$f(x, y) = e^{x+xy} \log(1 - xy).$$

7. Consider a function $f(x, y, z) = x + y^2$. Let $P = (1, 1, 1)$ be a point on a level set

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + xy + xyz = 3\}.$$

   Then find the maximum value of $D_v f(P)$ for a unit vector v which is tangent to S at point P.

8. Let $F\colon X \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous force field which is given by a gradient field such that there is a C^1 potential V satisfying $F = -\nabla V$. Let $X\colon I \subset \mathbb{R} \to \mathbb{R}^3$ be a path of a moving particle with the mass m which follows the force field F. Prove that the total energy (kinetic + potential) is conserved.

9. Find all critical points of the following function and determine whether it is a local maximum, a local minimum or a saddle point.

$$f(x, y) = x^3 y + 2xy^2 - xy.$$