Topology 2

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Contents

Chapter 0

Review of Topology 1

Definition 1 (Topology). A topology on a set X is a collection of subsets of X, {open sets}, which satisfies followings

1. $\emptyset, X \in \mathcal{T}$.
2. Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$.
3. Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$.

Elements in $\mathcal{T}$ are called open sets.

Lemma 1. product topology on $X \times Y$ is coarest topology s.t. π_1, π_2 are continuous.

Definition 2 (Basis). A basis $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X s.t.

1. $\bigcup_{B \in \mathcal{B}} B = X$.
2. For any $x \in B_1 \cap B_2$ $(B_1, B_2 \in \mathcal{B})$, $\exists B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Definition 3 (Hausdorff). A topological space X is Hausdorff if $\forall x_1 \neq x_2$, $\exists$neighborhood $U_1 \ni x_1$, $U_2 \ni x_2$ s.t. $U_1 \cap U_2 = \emptyset$.

Theorem 1 (Tychonoff theorem). $\Pi_{\beta \in B} X_\beta$ is compact.

Definition 4 (Countable basis). X has a countable basis of nbds at x if $\exists \{O_n\}_{n \in \mathbb{N}}$ of x s.t. for any nbd U of x, $\exists O_n \subset U$ for some $n \in \mathbb{N}$.

Definition 5 (First countable). X is called first countable if X has countable basis of nbds at every point of X.

Example. Metric space is first countable. For any x, $O_n = B_{\frac{1}{n}}(x)$ $n \in \mathbb{N}$.

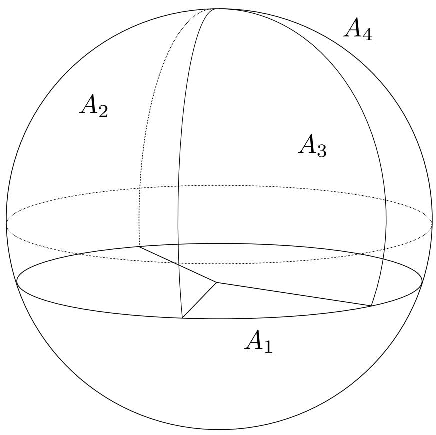

Figure 1: Example with four elements

Definition 6. A sequence $\{x_n\}$ converges to y if given any open nbd U of y, $\exists N$ so that if $n > N$, $x_n \in U$.

Theorem 2. $A \subset X$ topological space. If $x_n \in A$ converges to y, then $y \in \overline{A}$. Converse holds if X is first countable, that is, if $y \in \overline{A}$, then $\exists x_n \in A$ with $x_n \to y$.

Proof. First statement is easy. Say X first countable. Pick $y \in \overline{A}$, we will find $x_n \to y$, $x_n \in A$. $\exists \{O_n\}$ countable basis of nbds of y. Set

$$\begin{aligned} U_1 &= O_1 \\ U_2 &= O_1 \cap O_2 \\ U_3 &= O_1 \cap O_2 \cap O_3 \\ &\vdots \end{aligned}$$

Note that $U_1 \supset U_2 \supset U_3 \cdots$. $\{U_n\}_{n \in \mathbb{N}}$ is also countable basis of nbds of y.

Now, $y \in \overline{A}$, $\Rightarrow U_n \cap A \neq \emptyset$. Pick $x_n \in U_n \cap A$.Claim is that $x_n \to y$. Choose any nbd U of y. Then, $\exists N$ s.t. $O_n \subset U$. Note that If $n > N$, $U_n = O_1 \cap \cdots \cap O_N \cap \cdots \cap O_n \subset O_N \subset U$. $\therefore x_n \in U$ for any $n > N$. $\therefore x_n \to y$.

Definition 7 (Second countable)**.** X is called second countable if X has countable basis (of topology).

Example. $\mathbb{R}$, $\{(a, b) \,|\, a, b \in \mathbb{Q}\}$.

Example. $X_1 \times \cdots \times X_n$ (X_i: second countable) is also second countable.

Example. Compact metric space.

Question If X is second countable, does it have a countable dense subset?

Definition 8 (Separable)**.** X is called separable if $\exists$countable subset whose closure is X.

Proposition 1. Second countable $\Rightarrow$ separable.

Proposition 2. Separable metric space $\Rightarrow$ second countable.

Definition 9 (Normal)**.** X is normal if X is Hausdorff and for any closed subset C_1, C_2 with $C_1 \cap C_2 = \emptyset$, $\exists$open sets U_1, U_2 with $U_1 \supset C_1$, $U_2 \supset C_2$, $U_1 \cap U_2 = \emptyset$.

Proposition 3. Every compact Hausdorff space is normal.

Theorem 3 (Urysohn's lemma)**.** Let X be normal and C_1, C_2 disjoint closed subsets. Then $\exists$continuous function $f : X \to [0, 1]$ such that

1. $f(x) = 0 \quad \forall x \in A$.
2. $f(x) = 1 \quad \forall x \in B$.

Definition 10. Equivalence relation: $(X, \sim)$ satisfies

1. $x \sim x$
2. $x \sim y \Rightarrow y \sim x$
3. $x \sim y$, $y \sim z \Rightarrow x \sim z$

$X/_\sim$: the set of equivalence classes

Definition 11 (Locally compact)**.** X is called locally compact if for any $x \in X$, $\exists$open nbd O of x such that $\overline{O}$ is compact.

Chapter 1

Quotient topology

Pick a base point x_0 and consider it fixed. (The fundamental gruop will not depend on it. We assume all spaces are path connected) $X \rightsquigarrow \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f\colon I = [0,1] \to X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

Example. Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the 'constant' loop.

The composition of loops is simply pasting them. In the case of the circle, the loop $-1\circ$ the loop 2 is the loop 1.

Suppose $\alpha\colon I \to X$ and $f\colon X \to Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

Chapter 9

Fundamental group

See wikipedia[1] for a brief introduction.

9.51 Homotopy of paths

Definition 12 (Homotopic). If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there is a continuous map $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$ for each x. (Here $I = [0,1]$.) The map F is called a homotopy between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is nulhomotopic.

Definition 13 (Path homotopy). Let $f, g\colon I \to X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H\colon I \times I \to X$ is a path homotopy between f and g, if and only if

- $H(s,0) = f(s)$ and $H(s,1) = g(s)$ (homotopy between maps)
- $H(0,t) = x_0$ and $H(1,t) = x_1$ (start and end points fixed)

Notation: $f \simeq_p g$.

Lemma 2. $\simeq$ and $\simeq_p$ are equivalence relations.

Proof.

- Reflective: $F(x,t) = f(x)$
- Symmetric: $G(x,t) = H(x, 1-t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x,t) = \begin{cases} H_1(x, 2t) & 0 \le t \le \frac{1}{2} \\ H_2(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

□

[1] https://en.wikipedia.org/wiki/Homotopy

Example (Trivial, but important). Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g\colon X \to C$ are homotopic.
- Any two paths $f, g\colon I \to C$ with $f(0) = g(0)$ and $g(1) = f(1)$ are path homopotic.

Choose $H\colon X \times I \to C$ defined by $(x,t) \mapsto H(x,t) = (1-t)f(x) + tg(x)$.

Product of paths

Let $f\colon I \to X$, $g\colon I \to X$ be paths, $f(1) = g(0)$. Define

$$f * g\colon I \to X \text{ given by } s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g\colon I \to X \,|\, g \simeq_p f\}$.

Theorem 4.
1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
2. Let e_x denote the constant path $e_x\colon I \to X$ given by $s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Let $\overline{f}\colon I \to X$ given by $s \mapsto f(1-s)$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Fundamental group

Definition 14. Let X be a space and $x_0 \in X$, then the fundamental group of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f\colon I \to X,\ f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\overline{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Covering spaces

Definition 15 (Evenly covered)**.** Let $p\colon E \to B$, surjective map (so continuous). Let $U \subset B$ open. Then U is evenly covered iff $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$ with

- V_α open in E
- $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$
- $p|_{V_\alpha}: V_\alpha \to U$ is a homeomorphism.

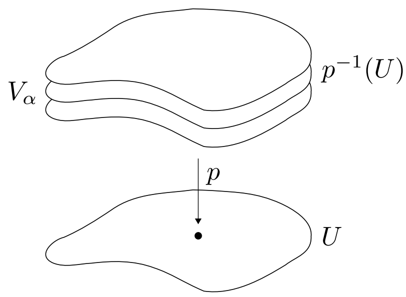

Figure 9.1: Evenly covered

Chapter 10

Separation theorems in the plane

10.63 Jordan curve theorem

https://en.wikipedia.org/wiki/Jordan_curve_theorem

Chapter 11

Seifert–Van Kampen theorem

`https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem`

Note. This doesn't follow the book very well.

Definition 16. A free group on a set X consists of a group F_x and a map: $i\colon X \to F_X$ such that the following holds: For any group G and any map $f\colon X \to G$, there exists a unique morphism of groups $\phi\colon F_X \to G$ such that

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F_x \arrow[d, "\exists!\phi"] \\ & G \end{tikzcd}.$$

Note. The free group of a set is unique. Suppose $i\colon X \to F_X$ and $j\colon X \to F'_X$ are free groups.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"'] & F_X \arrow[d, "\exists\phi"] \\ & F'_X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "i"'] & F'_X \arrow[d, "\exists\psi"] \\ & F_X \end{tikzcd}.$$

Then

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i"'] & F_X \arrow[d, "\psi\circ\phi"] \\ & F_X \end{tikzcd}.$$

Then by uniqueness, $\psi \circ \phi$ is 1_{F_X}, and likewise for $\phi \circ \psi$.

Note. The free group on a set always exists. You can construct it using "irreducible words".

Example. Consider $X = \{a, b\}$. An example of a word is $aaba^{-1}baa^{-1}bbb^{-1}a$. This is not a irreducible word. The reduced form is $aaba^{-1}bba = a^2ba^{-1}b^2a$. Then F_X is the set of irreducible words.

Example. If $X = \{a\}$, then $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. Exercise: check that $\mathbb{Z}$ satisfies the universal property.

Example. If $X = \emptyset$, then $F_X = 1$.

Definition 17 (Free product of a collection of groups)**.** Let G_i with $i \in I$, be a set of groups. Then the free product of these groups denoted by $*_{i \in I} G_i$ is a group G together with morphisms $j_i \colon G_i \to G$ such that the following universal property holds: Given any group H and a collection of morphisms $f_i \colon G_i \to H$, then there exists a unique morphism $f \colon G \to H$, such that for all $i \in I$, the following diagram commutes:

$$\begin{tikzcd}
G_i \arrow[r, "j_i"] \arrow[dr, "f_i"'] & G \arrow[d, "\exists! f"] \\
& H
\end{tikzcd}.$$

Note. Again, $*_{i \in I} G_i$ is unique.

11.70 The Seifert–Van Kampen theorem

Theorem 5 (70.1, Seifert–Van Kampen Theorem)**.** Let $X = U \cup V$ where $U, V, U \cap V$ are open and path connected.[a] Let $x_0 \in U \cap V$. For any group H and 2 morphisms $\Phi_1 \colon \pi(U, x_0) \to H$ and $\Phi_2 \colon \pi(V, x_0) \to H$ such that $\Phi_1 \circ i_1$ and $\Phi_2 \circ i_2$, there exists exactly one $\Phi \colon \pi(X, x_0) \to H$ making the diagram commute

$$\begin{tikzcd}
& \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, "\Phi"] & H \\
& \pi(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}.$$

i_1, i_2, i, j_1, j_2 are induced by inclusions.

[a]Note that U, V should also be path connected!

Theorem 6 (70.2, Seifert–Van Kampen (classic version)). Let $X = U \cup V$ as before ($U, V, U \cap V$, path connected) and $x_0 \in U \cap V$. Let $j\colon \pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ (induced by j_1 and j_2). On elements of $\pi(U, x_0)$ it acts like j_1, on elements of $\pi(V, x_0)$ it acts like j_2.

$$\begin{tikzcd}
G_1 \arrow[d] \arrow[dr, "f_1"] & \\
G_1 * G_2 \arrow[r, dashed, "f"] & H \\
G_2 \arrow[u] \arrow[ur, "f_2"] &
\end{tikzcd}.$$

Then j is surjective[a] and the kernel of j is the normal subgroup of $\pi(U, x_0) * \pi(U, x_0)$ generated by all elements of the form $i_1(g)^{-1} i_2(g)$, were $g \in \pi(U \cap V, x_0)$.

[a] This is the only place where algebraic topology is used. We've proved this last week. The groups U and V generate the whole group. The rest of this theorem follows from the previous theorem.

Proof.
- j is surjective (later)
- Let N be the normal subgroup generated by $i_1(g)^{-1} i_2(g)$. Then we claim that $N \subset \ker(j)$. This means we have to show that $i_1(g)^{-1} i_2(g) \in \ker j$. $j(i_1(g)) = j_1(i_1(g))$ by definition of j. Looking at the diagram, we find that $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$. Therefore $j(i_1(g)^{-1} i_2(g)) = 1$, which proves that elements of the form $i_1(g)^{-1} i_2(g)$ are in the kernel.
- Since $N \subset \ker j$, there is an induced morphism

$$\begin{aligned} k\colon (\pi_1(U, x_0) * \pi_1((V, x_0))/N &\to \pi_1(X, x_0) \\ gN &\mapsto j(g). \end{aligned}$$

To prove that $N = \ker j$, we have to show that k is injective. Because this would mean that we've divided out the whole kernel of j.

Now we're ready to use the previous theorem. Let $H = (\pi(U) * \pi(V))/N$. Repeating the diagram:

$$\begin{tikzcd}
& \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
\pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, shift left, "\Phi"] & H \arrow[l, shift left, "k"] \\
& \pi(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
\end{tikzcd}.$$

Now, we define $\Phi_1\colon \pi(U, x_0) \to H$ given by $g \mapsto gN$, and $\Phi_2\colon \pi(V, x_0) \to H$ given by $g \mapsto gN$. For the theorem to work, we needed that $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$. This is indeed the case: let

$g \in \pi(U \cap V)$. Then $\Phi_1(i_1(g)) = i_1(g)N$ and $\Phi_2(i_2(g)) = i_2(g)N$ and $i_1(g)N = i_2(g)N$ because $i_1(g)^{-1}i_2(g) \in N$.

The conditions of the previous theorem are satisfied, so there exists a Φ such that the diagram commutes.

Note that we also have $k\colon H \to \pi(X)$. We claim that $\Phi \circ k = 1_H$, which would mean that k is injective, concluding the proof. It's enough to prove that

$\square$

Corollary 6.1. Suppose $U \cap V$ is simply connected, so $\pi_1(U \cap V, x_0)$ is the trivial group. In this case $N = \ker j = 1$, hence $\pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ is an isomorphism.

Corollary 6.2. Suppose U is simply connected. Then $\pi(X, x_0) \cong \pi(V, x_0)/N$ where N is the normal subgroup generated by the image of $i_2\colon \pi(U \cap V) \to \pi(V, x_0)$.

Example. Let X be the figure 8 space.

Chapter 12

Classification of surfaces

Chapter 13

Classification of covering spaces

Lemma 3 (79.1, General lifting lemma)**.** Let $p\colon E \to B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.[a] Let $f\colon Y \to B$, $y_0 \in Y$, $b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f}\colon Y \to E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

iff $f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$. If $\tilde{f}$ exists then it is unique.

[a]From now on, all spaces are locally path connected: Every neighborhood contains an open that is path connected.

Example. Take $Y = [0, 1]$. Then f is a path, then we showed that every map can be lifted. And indeed, the condition holds: $f_*(\pi(Y, y_0)) = 1$, the trivial group, which is a subgroup of all groups.

Proof. Suppose $\tilde{f}$ exists. Then $p \circ \tilde{f} = f$, so $(p \circ \tilde{f})_*\pi(Y, y_0) = \pi(Y, y_0)$. The left hand side is of course $p_*(\tilde{f}_*(\pi(Y, y_0)) \subset p_*(\pi(E, e_0))$, so $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$. $\diamond$