Introduction to Differential Geometry I – Homework 8

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Problem 3-2.6 Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

Solution. Let $v \in T_p(S)$, $|v| = 1$. By using the orthonormal basis $\{e_1, e_2\}$, we can write v as $v_1 = \cos(\theta)e_1 + \sin(\theta)e_2$ and orthogonal vector to v as $v_2 = \sin(\theta)e_1 - \cos(\theta)e_2$. Note that $|v_2| = 1$ and $v_1 \cdot v_2 = 0$. The value of the second fundamental form Π_p for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v. So, by Euler formula on v_1 and v_2, we get following.

$$\begin{aligned} k_n(v_1) &= \Pi_p(v_1) = k_1 \cos^2(\theta) + k_2 \sin^2(\theta) \\ k_n(v_2) &= \Pi_p(v_2) = k_1 \sin^2(\theta) + k_2 \cos^2(\theta) \\ k_n(v_1) + k_n(v_2) &= k_1(\cos^2(\theta) + \sin^2(\theta)) + k_2(\cos^2(\theta) + \sin^2(\theta)) = k_1 + k_2 \end{aligned}$$

Thus, the sum of the normal curvatures for any pair of orthogonal directions at p is constant. $\square$