수리통계학2 - HW2

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6.6 - 1

$X_i \sim N(\theta, 20) \quad i=1,\cdots,10 \quad iid$

(a). $Y = \frac{1}{2}(X_1 + X_2)$

$E[Y] = E[\frac{1}{2}(X_1+X_2)] = \frac{1}{2}E[X_1] + \frac{1}{2}E[X_2] = \frac{1}{2}\theta + \frac{1}{2}\theta = \theta$

$\therefore$ Y is UE of θ.

(b) $I(\theta) = E[\{l'(\theta)\}^2] = -E[l''(\theta)]$

$L_{10}(\theta) = \left(\frac{1}{\sqrt{40\pi}}\right)^{10} \cdot \exp\left(-\frac{\Sigma(x_i-\theta)^2}{40}\right)$

$l_{10}(\theta) = \log L_{10}(\theta) = -5\log 40\pi - \frac{\Sigma(x_i-\theta)^2}{40}$

$l'(\theta) = \frac{\Sigma(x_i-\theta)}{20} = \frac{1}{20}\left(\sum_{i=1}^{10} x_i - 10\theta\right) = \frac{1}{2}\cdot\frac{1}{10}\sum_{i=1}^{10} x_i - \frac{\theta}{2}$

$= \frac{\bar{X}}{2} - \frac{\theta}{2}$

$l''(\theta) = -\frac{1}{2}$

$I(\theta) = -E[l''(\theta)] = -E[-\frac{1}{2}] = \frac{1}{2}$

$Var[\hat{\theta}_{UE}] \geq \frac{1}{I(\theta)} = 2$

$\therefore$ Rao-Cramér lower bound : 2.

(c) $Var[Y] = E[Y^2] - \{E[Y]\}^2 = E[\frac{1}{4}(X_1+X_2)^2] - \theta^2$

$= \frac{1}{4}E[X_1^2 + 2X_1X_2 + X_2^2] - \theta^2$

$= \frac{1}{4}(E[X_1^2] + 2E[X_1]E[X_2] + E[X_2^2]) - \theta^2 \quad (\because iid)$

$= \frac{1}{4}(Var[X_1] + \{E[X_1]\}^2 + 2E[X_1]E[X_2]$

$+ Var[X_2] + \{E[X_2]\}^2) - \theta^2$

$= \frac{1}{4}(20 + \theta^2 + 2\theta^2 + 20 + \theta^2) - \theta^2$

$= \frac{1}{4}(40 + 4\theta^2) - \theta^2 = 10 + \theta^2 - \theta^2 = 10.$

efficiency of Y : $2 / Var[Y] = \frac{1}{5}$

6.6 - 2

(a) $X_i \sim B(2, p) \quad i = 1, \cdots, 20 \quad iid$

$L(p) = \frac{2!}{x!(2-x)!} p^x (1-p)^{2-x}$

$l(p) = \log \frac{2}{x!(2-x)!} + x \log p + (2-x) \log(1-p)$

$l'(p) = \frac{x}{p} - \frac{2-x}{(1-p)}$

$l''(p) = -\frac{x}{p^2} - \frac{2-x}{(1-p)^2}$

$I(p) = -E[l''(p)] = \frac{2p}{p^2} + \frac{2-2p}{(1-p)^2} = \frac{2}{p} + \frac{2}{1-p} = \frac{2}{p(1-p)}$

$I_{20}(p) = \frac{40}{p(1-p)}$

$\therefore$ Rao - Cramer lower bound : $\frac{1}{40} \cdot p(1-p) = \frac{1}{80} \cdot Var(X_i)$

(b) $Var(\bar{X}) = \frac{1}{n} \cdot Var(X) = \frac{1}{20} \cdot 2p(1-p) = \frac{1}{10} p(1-p)$.

efficiency of $\bar{X} = (\frac{1}{40} p(1-p)) / Var(\bar{X}) = \frac{1/40}{1/10} = \frac{1}{4}$.

6.7 - 4

$X_i \sim f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1 \quad 0 < \theta \quad i = 1, \cdots, n \quad iid$

(a) $\theta x^{\theta-1} = \exp(\log\theta + (\theta-1)\log x)$

$\therefore$ exponential family

Thus $\sum_{i=1}^{n} \log x_i = \log(\prod_{i=1}^{n} x_i)$ is S.S for θ

Since log function is one-to-one function.

$\prod_{i=1}^{n} x_i$ is also S.S for θ.

(b) $L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n (\prod_{i=1}^{n} x_i)^{\theta-1}$

$l(\theta) = n \log\theta + (\theta-1) \log \prod_{i=1}^{n} x_i$

$= n \log\theta + (\theta-1) \cdot \sum_{i=1}^{n} \log x_i$

$l'(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0$

$\therefore \hat{\theta} = -\frac{n}{\sum \log x_i} = -\frac{n}{\log(\prod x_i)}$

(c) $\hat{\theta}$ contains $\prod x_i$ or $\log(\prod x_i)$.

$\therefore \hat{\theta}$ also S.S for θ.

6.7 - 5

$X_i \sim \text{Gamma}(1, \frac{1}{\theta})$. $i = 1, \cdots, n$. iid.

$f(x|\theta) = \theta e^{-\theta x} = \exp(-\theta x(\log\theta + 1))$

: exponential family.

Thus $\sum_{i=1}^{n} x_i$ is S.S for θ $\quad Y = \sum_{i=1}^{n} X_i$

Mgf of Y, $M_Y(t) = E[e^{Yt}] = E[e^{\Sigma X_i t}] = \prod_{i=1}^{n} M_{X_i}(t)$

$= [M_X(t)]^n = (1 - \frac{1}{\theta}t)^{-n}$

: mgf of Gamma$(n, \frac{1}{\theta})$.

$\therefore$ $Y \sim \text{Gamma}(n, \frac{1}{\theta})$.

$$f_Y(y) = \frac{y^{n-1} \cdot e^{-\theta y}}{\Gamma(n)(1/\theta)^n} = \frac{\theta^n \cdot y^{n-1} \cdot e^{-\theta y}}{(n-1)!}$$

$$E[\tfrac{1}{Y}] = \int_0^\infty \frac{1}{y} \frac{\theta^n \cdot y^{n-1} e^{-\theta y}}{(n-1)!} dy.$$

$$= \frac{\theta}{n-1} \int_0^\infty \underbrace{\frac{\theta^{n-1} y^{n-2} e^{-\theta y}}{(n-2)!}}_{\text{pdf of Gamma}(n-1, \frac{1}{\theta})} dy = \frac{\theta}{n-1}$$

$\therefore$ $E[\frac{n-1}{Y}] = n-1 \cdot \frac{\theta}{n-1} = \theta$.

$\therefore$ $\frac{n-1}{Y}$ is UE of θ

6.7 - 7

$X_i \sim \text{Geo}(p)$. $i = 1, \cdots, n$ iid. $\quad E[X] = \frac{1}{p}$

$f(x|\theta) = p(1-p)^{1-x} \quad x = 1, 2, \cdots$

(a) $f(x|\theta) = \exp(\log p + (1-x)\log(1-p))$

$= \exp(\log p(1-p) - x\log(1-p))$.

: exponential family $\Rightarrow$ $\sum_{i=1}^{n} X_i$ is S.S for p.

(b) $E[Y] = E[\sum_{i=1}^{n} X_i] = n \cdot E[X] = n \cdot \frac{1}{p} = \frac{n}{p}$

$E[\frac{Y}{n}] = \frac{1}{p}$

$\therefore$ $\frac{Y}{n}$ is UE of $\theta = \frac{1}{p}$