

Introduction to Real Analysis – Final Exam

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1. (1) Let

$$\|f\|_{L^{1,w}(\mathbb{R}^d)} = \sup_{\alpha > 0} \alpha \cdot m(\{x \in \mathbb{R}^d : |f(x)| > \alpha\})$$

where m stands for the Lebesgue measure on \mathbb{R}^d . Check that

$$\|f\|_{L^{1,w}(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

- (2) Give an example of a function g in $(0, \infty)$ such that

$$\|g\|_{L^{1,w}((0,\infty))} = 1 \quad \text{and} \quad \|g\|_{L^1((0,\infty))} = +\infty.$$

2. (1) Suppose that F is a \mathbb{R} -valued absolutely continuous function on $[a, b]$. Prove that

$$T_F(a, b) = \int_a^b |F'(t)| dt.$$

- (2) Suppose that F is a \mathbb{R} -valued continuous function on $[a, b]$. Show that

$$T_F(a, b) = \lim_{\varepsilon \rightarrow 0+} T_F(a + \varepsilon, b).$$

- (3) Determine whether

$$F(x) = (x - 1)^{2022} \sin((x - 1)^{-2020}) \quad \text{for } x \in [0, 2]$$

is of bounded variation on $[0, 2]$ or not.

3. (1) For a fixed number $\xi \in (0, 1)$, we construct a subset \mathcal{C}_ξ of \mathbb{R} in the following manner:
- In the first stage of the construction, we remove the middle ξ from $[0, 1]$ so that the remaining set is $[0, \frac{1-\xi}{2}] \cup [\frac{1+\xi}{2}, 1]$.
 - In the second stage, we remove the middle ξ^2 from each of $[0, \frac{1-\xi}{2}]$ and $[\frac{1+\xi}{2}, 1]$.
 - By repeating this process countably many times, we obtain the set \mathcal{C}_ξ . Note that $\mathcal{C}_{\frac{1}{3}}$ is the Cantor set.

Compute the (strict) Hausdorff dimension of the set \mathcal{C}_ξ .

- (2) Prove that there exists a subset of \mathbb{R} having Hausdorff dimension γ for any $\gamma \in (0, 1)$.
- (3) Compute the Hausdorff dimension and the Minkowski dimension of the compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of \mathbb{R} .