Mathematical Statistics II

Ch.8 Tests of Statistical Hypotheses

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Type II error. $\beta = P(\text{accept } H_0 \mid H_1 \text{ is true})$.

vs

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true})$.

eg). $H_0 : \mu = 50$ $\quad H_1 : \mu > 50$.

In general, β depends on the values selected from the composite alternative hypothesis

$\Rightarrow$ $1-\beta$ (power) is a function of the parameters in hypothesis

Ch8.5 Power of a Statistical Test

Power of a Statistical Test

In general, alternative hypothesis is composite. Thus, Type II error has different values, with different values selected from the composite alternative hypothesis. That means, power is function of the parameter in the hypothesis, which is called the power function of the test.

Example 8.5-2

Let $X_i \sim N(\mu, 100)$, $i = 1, \ldots, n$, be a random sample. We wish to decide between $H_0 : \mu = 60$ and $H_1 : \mu > 60$. We use a rule to reject H_0 if $\overline{X} \geq 62$. Find the power function of the test, $K(\mu)$, when $n = 25$.

$X_i \sim N(\mu, 100) \quad i = 1, \cdots, 25$

$\Rightarrow \boxed{\bar{X} \sim N(\mu, \frac{100}{25} = 4) \quad \mu > 60.}$ under H_1 : true

$H_0 : \mu = 60 \qquad H_1 : \mu > 60$

Rejection region $\bar{x} \geq 62$

$K(\mu) = 1 - \beta = P(\text{reject } H_0 \mid H_1 \text{ is true}).$

$= P(\bar{X} \geq 62 \mid \mu > 60).$

$= P\left(Z = \frac{\bar{X} - \mu}{2} \geq \frac{62 - \mu}{2}\right), \quad Z \sim N(0, 1)$

$= 1 - \Phi\left(\frac{62 - \mu}{2}\right), \quad \mu > 60$

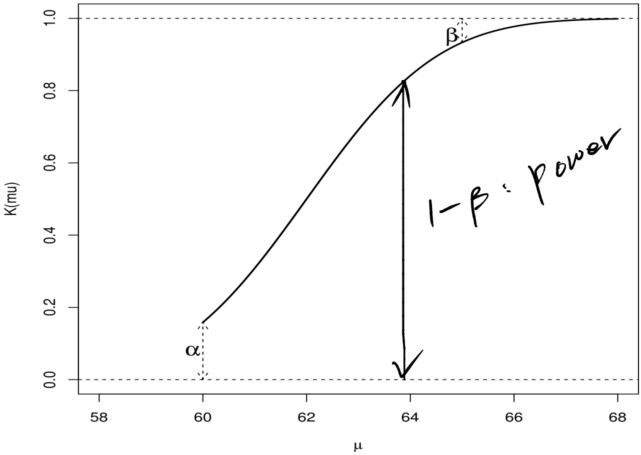
cf). $\alpha = P(\bar{X} \geq 62 \mid \mu = 60).$

$= P\left(\frac{\bar{X} - 60}{10/5} \geq \frac{62 - 60}{2}\right)$

$= P\left(\frac{\bar{X} - 60}{2} \geq 1\right).$

$= P(Z \geq 1).$

Power function



Ch8.6 Best Critical Regions

: Selecting a good critical region of size α

Simple.

Definition 8.6-1

Consider the test, $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$.

A set C is a best critical region of size α if

1) There are many subsets A_i with $\alpha = P(A_i|\theta_0)$.
2) Among A_i's, find the best critical region (C) with the greatest power. Smallest Type II error

$$P(C|\theta_1) \geq P(A_i|\theta_1).$$

Example

Let $X \sim Bin(5, \theta)$. We wish to decide between $H_0 : \theta = 1/2$ and $H_1 : \theta = 3/4$. Consider $\alpha = 1/32$. Find the best critical region.

* Best critical region (BCR)

$(x_1, \cdots x_n) \in C$

① $P[(X_1, \cdots X_n) \in A_i] = \alpha$

many subsets A_i $\qquad C(BCR)$

② Among A_i's, find the A_i $=$

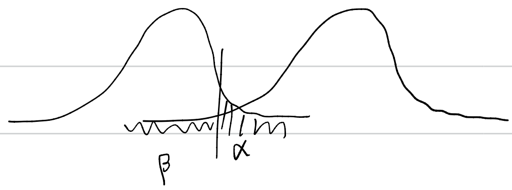
with the greatest power $\qquad P(C|\theta_1) \geq P(A_i|\theta_1)$

cf). UMVUE

$MSE(\hat{\theta}) = Var(\hat{\theta}) + \underbrace{\{Bias(\hat{\theta})\}^2}_{=0}$

i) U.Es

ii) Among U.Es, U.E with smallest Var.



1) Under $H_0: \theta = \frac{1}{2}$, $X \sim Bin(5, \frac{1}{2})$

$P(X = 0) = P(X = 5) = \frac{1}{32}$

$A_1 = \{0\}$. $A_2 = \{5\}$

ii) Calculate the power

$P(X=0 \mid \theta = \frac{3}{4}) = (\frac{1}{4})^5$

$P(X=5 \mid \theta = \frac{3}{4}) = (\frac{3}{4})^5$

Thm. 8.6-1 (Neyman-Pearson Lemma)

Let $X_1, \dots, X_n \sim f(x|\theta)$. Test

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1.$$

The likelihood function is denoted by $L(\theta)$. If there exist a positive constant k and a subset C of the sample space such that

(a) $P((X_1, \dots, X_n) \in C|\theta_0) = \alpha$

(b) $\xcancel{\dfrac{L(\theta_0)}{L(\theta_1)}} = \dfrac{L(\theta_0|H_1)}{L(\theta_1|H_1)} \leq k$ for $(x_1, \dots, x_n) \in C$

(c) $\xcancel{\dfrac{L(\theta_0)}{L(\theta_1)}} = \dfrac{L(\theta_0|H_0)}{L(\theta_1|H_0)} \geq k$ for $(x_1, \dots, x_n) \in C'$,

then C is the best critical region of size α.

best critical region

$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1$

rejection region

To show : $P(C \mid \theta_1) \geq P(D \mid \theta_1)$ for any D s.t

$P(D \mid \theta_0) = \alpha$

Let $\underset{\sim}{x} = (x_1, \cdots, x_n) \quad d\underset{\sim}{x} = dx_1 \cdots dx_n$

$I_C(x) = \begin{cases} 0 & \text{if } x \notin C \\ 1 & \text{if } x \in C \end{cases}$

$[\underbrace{f(\underset{\sim}{x} \mid \theta_0) - k f(\underset{\sim}{x} \mid \theta_1)}_{L(\theta_0) - kL(\theta_1)}][\underbrace{I_C(\underset{\sim}{x})}_{1 \text{ or } 0} - \underbrace{I_D(\underset{\sim}{x})}_{1 \text{ or } 0}] \leq 0$ for $\forall \underset{\sim}{x}$

$\underset{\sim}{x} \in C$: $-$ or 0 $\qquad + \text{ or } 0 \qquad \leq 0$ by (b)

$\underset{\sim}{x} \in C'$: $+$ or 0 $\qquad - \text{ or } 0 \qquad \leq 0$ by (c)

$$\Rightarrow \int [f(\underset{\sim}{x} \mid \theta_0) - k f(\underset{\sim}{x} \mid \theta_1)][I_C(\underset{\sim}{x}) - I_D(\underset{\sim}{x})] \, d\underset{\sim}{x} \leq 0$$

$$\int f(\underset{\sim}{x} \mid \theta_0)[I_C(\underset{\sim}{x}) - I_D(\underset{\sim}{x})] \, d\underset{\sim}{x} \leq k \int f(\underset{\sim}{x} \mid \theta_1)[I_C(\underset{\sim}{x}) - I_D(\underset{\sim}{x})] \, d\underset{\sim}{x}$$

$$P(C \mid \theta_0) - P(D \mid \theta_0) = \alpha - \alpha = 0 \leq k[P(C \mid \theta_1) - P(D \mid \theta_1)]$$

Since k is positive value

$P(C \mid \theta_1) \geq P(D \mid \theta_1)$

$\therefore$ C is BCR

↳ Best Critical Region.

[illegible] 12/11 12-14:30.

(Estimation (P.E & C.I)
Testing.

$$\frac{L(\theta_0 | H_1)}{L(\theta_1 | H_1)} \leq K \qquad \text{if} \quad (x_1, \cdots, x_n) \in C$$

$\Rightarrow$ accept H_1

$$\underset{\substack{\uparrow \\ \text{from } H_0}}{L(\theta_0} | H_1) \leq K \underset{\substack{\uparrow \\ \text{from } H_1}}{L(\theta_1} | H_1)$$

$$\frac{L(\theta_0 | H_0)}{L(\theta_1 | H_0)} \geq K.$$

if $(x_1, \cdots, x_n) \in C'$

$$L(\theta_0 | H_0) \geq K L(\theta_1 | H_0)$$

$\Rightarrow$ accept H_0

Example 8.6-2

Let $X_1, \ldots, X_n \sim N(\mu, 36)$. For testing $H_0 : \mu = 50$ vs $H_1 : \mu = 55$,

1) find the best critical region of $\alpha = 0.05$ and $n = 16$.
2) calculate Type I error when $n = 16$ and the critical region $\overline{X} \geq 53$.

$X_i \sim N(\mu, 36)$. iid

(Test) $H_0 : \mu = 50$ vs $H_1 : \mu = 55$

$$L(\mu) = f(x_1, \cdots, x_n \mid \mu)$$

$$= \left(\frac{1}{\sqrt{2\pi 6^2}}\right)^n \exp\left\{ - \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{72} \right\}$$

For $(x_1, \cdots, x_n) \in C$

$$\frac{L(50 \mid H_1)}{L(55 \mid H_1)} = \frac{(72\pi)^{-\frac{n}{2}}}{(72\pi)^{-\frac{n}{2}}} \frac{\exp\left\{ - \frac{\sum(x_i - 50)^2}{72} \right\}}{\exp\left\{ - \frac{\sum(x_i - 55)^2}{72} \right\}}$$

$$= \exp\left[-\frac{1}{72} \left\{ \sum x_i^2 - 100 \sum x_i + 50^2 n - \sum x_i^2 + 110 \sum x_i - 55^2 n \right] \right.$$

$$= \exp\left\{ -\frac{1}{72} (10 \sum x_i - 525 n) \right\} \leq k$$

By taking the log

$$-10 \sum x_i + 525 n \leq 72 \ln k$$

$$10 \sum x_i \geq 525 n - 72 \ln k$$

$$\bar{x} = \frac{\sum x_i}{n} \geq \underbrace{52.5 - \frac{7.2}{n} \ln k}_{=: c}$$

By NP Lemma, the best critical region is

$$C = \{ (x_1, \cdots, x_n) \mid \bar{x} \geq c \}$$

where c is selected s.t $P(C \mid \mu = 50) = \alpha$.

① $\alpha = 0.05$ & $n = 16$

$$0.05 = P(\bar{X} \geq c \mid \mu = 50)$$
$$= P\left(\frac{\bar{X} - 50}{3/2} = Z \geq \frac{c - 50}{3/2}\right), \quad Z \sim N(0,1)$$
$$= 1 - \Phi\left(\underline{\frac{c-50}{1.5}}\right)$$
$$= 1.645.$$
$$c = 50 + 1.5 \times 1.645 \fallingdotseq 52.47$$

BCR of $\alpha = 0.05$

$$C = \{(x_1, \cdots, x_n) \mid \bar{x} \geq 52.47\}$$

②.

Definition 8.6-2

When $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$ are both simple hypotheses, a critical region of size α is a best critical region if the probability of rejection H_0 when H_1 is true is a maximum compared with all other critical regions of size α.

- most powerful test: test using the best critical region (simple vs simple) statistical Test
- uniformly most powerful (UMP) test: test defined by a critical region C of size α (simple vs composite)
- Power of test depends on each simple alternative in H_1.

Example 8.6-4

Let $X_1, \ldots, X_n \sim N(\mu, 36)$. For testing $H_0 : \mu = 50$ vs $H_1 : \mu = \mu_1 > 50$, find the best critical region of $\alpha = 0.05$ and $n = 16$.

H_0	vs	H_1		
Simple		Simple	$\Rightarrow$	MP test
Simple		Composite	$\Rightarrow$	UMP test

$X_i \sim N(\mu, 36)$. iid $\Rightarrow$ $\bar{X} \sim (\mu, \frac{36}{16})$

(Test). $H_0 : \mu = 50$ vs $H_1 : \mu = 55$

$H_0 : \mu = 50$ vs $H_1 : \mu = \underset{\substack{\uparrow \\ \text{unknown.}}}{\mu_1} > 50$

$(\mu = \mu_1 \text{ s.t } \mu_1 > 50)$

If $(x_1, \cdots, x_n) \in C$,

$$\frac{L(50|H_1)}{L(\mu_1|H_1)} = \frac{(72\pi)^{-\frac{n}{2}} \exp\left\{-\frac{\Sigma(x_i-50)^2}{72}\right\}}{(72\pi)^{-\frac{n}{2}} \exp\left\{-\frac{\Sigma(x_i-\mu_1)^2}{72}\right\}}$$

$$= \exp\left(-\frac{1}{72}\left[2(\mu_1-50)\Sigma x_i + n(50^2-\mu_1^2)\right]\right) \le K$$

K : positive value

$-2(\mu_1-50)\Sigma x_i - n(50^2-\mu_1^2) \le 72 \ln k$

$2(\mu_1-50)\Sigma x_i \ge -n(50^2-\mu_1^2) - 72\ln k$

$$\bar{x} = \frac{\Sigma x_i}{n} \ge \frac{(\mu_1+50)(\mu_1-50)}{2(\mu_1-50)} - \frac{72\ln k}{2n(\mu_1-50)}$$

$$= \frac{\mu_1+50}{2} - \frac{72\ln k}{2n(\mu_1-50)} =: C$$

결과적으로 당연해 보이나 이게 원래 맞는 순서.

(Rejection (critical) region).

BCR $C = \{(x_1, \cdots, x_n) \mid \bar{x} \ge c\}$ where

c is selected s.t $P(\bar{X} \ge c \mid \mu = 50) = \alpha$

$0.05 = P(\bar{X} \ge c \mid \mu = 50)$ $\therefore$ $c = 52.47$.

UMP Test : reject H_0 if $\underbrace{\bar{X} \ge 52.47}_{BCR}$

$X_1 \cdots X_n \sim f(x) \quad x_i = 0, 1, 2, \cdots$

$H_0 : f_0(x) = \frac{e^{-1}}{x!}$, $Poi(1)$

vs

$H_1 : f_1(x) = \left(\frac{1}{2}\right)^{x+1}$, $Geo\left(\frac{1}{2}\right)$

cf). $H_0 : \mu = 50 \quad H_1 : \mu = 55 \quad \bar{X} \geq C.$ Rejection region.

If $(x_1, \cdots, x_n) \in C$

$$\frac{f_0(x_1, \cdots, x_n)}{f_1(x_1, \cdots, x_n)} = \frac{\prod_{i=1}^{n} \frac{e^{-1}}{x_i!}}{\prod_{i=1}^{n} \left(\frac{1}{2}\right)^{x_i+1}} \leq k \qquad k : \text{positive value.}$$

$$\ln\left[\frac{f_0(x_1, \cdots, x_n)}{f_1(x_1, \cdots, x_n)}\right] = -n - \sum_{i=1}^{n} \ln x_i! - \sum (x_i+1) \ln\left(\frac{1}{2}\right) \leq \ln k$$

$$\sum_{i=1}^{n} \ln x_i! - \sum (x_i+1) \ln 2 \geq \underbrace{-n - \ln k}_{c}.$$

$$\Rightarrow \sum_{i=1}^{n} \ln x_i! - \sum x_i \ln 2 \geq c$$

Rejection region is

$C = \{(x_1, \cdots, x_n) \mid \sum_{i=1}^{n} \ln x_i! - \sum_{i=1}^{n} x_i \ln 2 \geq c\}$

where c is selected s.t $P(C \mid H_0) = \alpha$.

8.7. (option). 시간 날 때 읽어볼 것.

Likelihood Ratio Test