금융수학 과제 1 답안

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Problem 5.1. Consider the binomial model

$$R = 0.2\,,\ s = 110\,,\ s_u = 144\,,\ s_d = 96\,,\ p_u = 0.6\,,\ p_d = 0.4\,.$$

(i) (5 points) Price and hedge a call option with strike price $K = 100$ and maturity $t = 1$.

(ii) (5 points) Find the risk-neutral measure, and evaluate the price of this option by using this risk-neutral measure

(i): Let, $X = (S_1 - 100)_+$: payoff of a call-option with strike price $K=100$,
$h = (x, y) \in \mathbb{R}^2$ be a hedging portfolio of X. i.e., $xG_1 + yS_1 = X$.

Note that $G_1 = 1+R = 1.2$, $S_1 = \begin{cases} S_u = 144 \\ S_d = 96 \end{cases}$ $X = \begin{cases} 44 \\ 0 \end{cases}$

$\therefore$ We get 2 equations $\begin{cases} 1.2x + 144y = 44 \\ 1.2x + 96y = 0 \end{cases}$ $\Rightarrow \therefore x = -\frac{220}{3},\ y = \frac{11}{12}$

$\therefore V_0 = xG_0 + yS_0 = -\frac{220}{3} + \frac{11}{12} \times 110 = \frac{330}{12} = \frac{55}{2}$

Ans.: Price of option $= \frac{55}{2}$. Hedging Portfolio $= \left(-\frac{220}{3}, \frac{11}{12}\right)$.

(ii): We will find a Risk neutral measure $\mathbb{Q} : 2^{\{u,d\}} \to \mathbb{R}$.

i.e. find a q_u, q_d s.t. $q_u + q_d = 1$, $0 < q_u, q_d < 1$.

$$\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}[S_1] = \frac{1}{1.2}(144q_u + 96q_d) = 110.$$

$(q_u = \mathbb{Q}(\{u\}),\ q_d = \mathbb{Q}(\{d\}))$

Solving $q_u + q_d = 1$, $12q_u + 8q_d = 11$, we get $q_u = 0.75$, $q_d = 0.25$.

$\therefore$ Price of call-option $= \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}[X] = \frac{1}{1.2}(0.75 \times 44 + 0.25 \times 0) = \frac{33}{1.2} = \frac{55}{2}$

Ans.: Risk neutral msr $\mathbb{Q} : 2^{\{u,d\}} \to \mathbb{R}$ is given by $\mathbb{Q}(\emptyset) = 0$, $\mathbb{Q}(\{u,d\}) = 1$
$\mathbb{Q}(\{u\}) = 0.75$, $\mathbb{Q}(\{d\}) = 0.25$.
Price of a call option $= \frac{55}{2}$

Problem 5.2. Consider the one-period trinomial model: $s = 95$, $s_u = 150$, $s_m = 125$, $s_d = 100$, $R = 0.25$, $p_u = 0.2$, $p_m = 0.2$, $p_d = 0.6$.

(i) (5 points) Define $\Omega = \{u, m, d\}$ and let $\mathbb{P}$ be the probability measure on 2^Ω such that $\mathbb{P}(\{u\}) = 0.2$, $\mathbb{P}(\{m\}) = 0.2$, $\mathbb{P}(\{d\}) = 0.6$. Define bank accounts G_0, G_1 and stock prices S_0, S_1 on this space.

Ans.: $G_0, G_1, S_0, S_1 : \Omega \to \mathbb{R}$ are defined as followings:

① $G_0 \equiv 1$ ② $G_1 \equiv 1+R = 1.25$. ③ $S_0 \equiv s = 95$.

④ $S_1(u) = 150$, $S_1(m) = 125$, $S_1(d) = 100$.

(ii) (5 points) Show a risk-neutral measure $\mathbb{Q}$ exists, but is not unique. Give two examples of $\mathbb{Q}$.

pf) Note that finding risk neutral msr $\mathbb{Q} : 2^\Omega \to \mathbb{R}$

$\Leftrightarrow$ finding $0 < q_u, q_m, q_d < 1$ s.t. $q_u + q_m + q_d = 1$, $\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}[S_1] = S_0$.

$\Leftrightarrow$ 〃 $q_u + q_m + q_d = 1$, $\frac{1}{1.25}(150q_u + 125q_m + 100q_d) = 95$

$\Leftrightarrow$ 〃 $q_u + q_m + q_d = 1$, $6q_u + 5q_m + 4q_d = 4.75$. $- \circledast$

We can easily check that $(q_u, q_m, q_d) = (0.1, 0.55, 0.35)$, $(0.2, 0.35, 0.45)$
both satisfy $\circledast$! $\therefore$ Risk neutral msr is not unique. ☒

(iii) (5 points) Find the super-hedging price of the option with payoff

$$X = \begin{cases} 80 & \text{if } S_1 = 150 \\ 40 & \text{if } S_1 = 125 \\ 0 & \text{if } S_1 = 100 \end{cases}$$

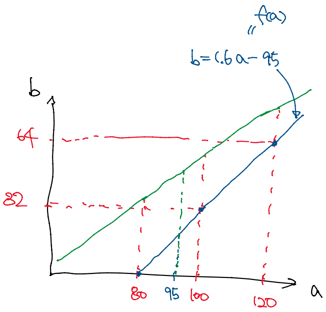
at maturity $t = 1$.

Pf. We may consider X as a random variable. $X : \Omega \to \mathbb{R}$ where $X(u) = 80$, $X(m) = 40$, $X(d) = 0$.

Note that, (super-hedging price of X) $= \inf\{ xB_0 + yS_0 \mid xB_1(\omega) + yS_1(\omega) \geq X(\omega), \forall \omega \in \Omega \}$.

$$= \inf \left\{ x + 95y \;\middle|\; \begin{array}{l} 1.25x + 150y \geq 80 \\ 1.25x + 125y \geq 40 \\ 1.25x + 100y \geq 0 \end{array} \right\}$$

$$= \inf \left\{ x + 95y \;\middle|\; \begin{array}{l} x + 120y \geq 64 \\ x + 100y \geq 32 \\ x + 80y \geq 0 \end{array} \right\} = 24.$$



($\because$ Consider (x, y) as a "line" $b = ya + x$ in xy-plane.

$\Rightarrow$ This line should locate above of points A, B, C...! (as green line)

$\therefore$ (Infimum of "height" of green line at $a = 95$) $= f(95) = 24$)

(iv) (10 points) Calculate

$$\sup\left\{ \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X) \,\middle|\, \mathbb{Q} \text{ is a risk-neutral measure} \right\}$$

and confirm that the superhedging duality holds. Can you find a risk-neutral measure $\mathbb{Q}$ which achieves the supremum?

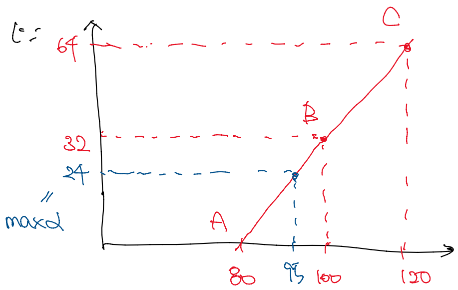
Pf. Set $\mathbb{Q}(\{u\}) = q_u$, $\mathbb{Q}(\{m\}) = q_m$, $\mathbb{Q}(\{d\}) = q_d$.

$$\Rightarrow \sup\left\{ \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X) \,\middle|\, \mathbb{Q} : \text{risk neutral msr} \right\}$$

$$= \sup \left\{ \frac{1}{1.25}(80q_u + 40q_m) \;\middle|\; \begin{array}{l} 0 < q_u, q_m, q_d < 1 \\ q_u + q_m + q_d = 1 \\ 6q_u + 5q_m + 4q_d = 4.75 \end{array} \right\}$$ See ※ in (ii)!

$$= \sup \left\{ \underbrace{64q_u + 32q_m}_{=\alpha} \;\middle|\; \begin{array}{l} 0 < q_u, q_m, q_d < 1 \\ q_u + q_m + q_d = 1 \\ 120q_u + 100q_m + 80q_d = 95 \end{array} \right\} = 24.$$

($\because$



We need to maximize α s.t. $q_u + q_m + q_d = 1$

$$q_u \begin{pmatrix} 120 \\ 64 \end{pmatrix} + q_m \begin{pmatrix} 100 \\ 32 \end{pmatrix} + q_d \begin{pmatrix} 80 \\ 0 \end{pmatrix} = \begin{pmatrix} 95 \\ \alpha \end{pmatrix}$$

$\rightarrow D = (95, \alpha)$ is an interior point of $\triangle ABC =$ segment $\overline{AC}$

$\therefore \max \alpha = \max_{D \in \mathrm{int}\,\triangle ABC} (\text{height of } D) = 24.$)

$\therefore$ Combining the result from (iii), super-hedging price $=$ maximum of risk-neutral price $= 24$

$\therefore$ Combining the result from (iii), super-hedging price = maximum of risk-neutral price = 24

i.e., super-hedging duality holds!

Moreover, we can easily check $(q_u, q_m, q_d) = (0.2, 0.35, 0.45)$ achieves the supremum! ▨.

(v) (10 points) Let $\mathcal{P}$ be set of all probability measures $\mathbb{Q}$ on 2^Ω such that $S_0 = \frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(S_1)$ (not necessarily to satisfy $\mathbb{Q}(\{u\}) > 0$, $\mathbb{Q}(\{m\}) > 0$, $\mathbb{Q}(\{d\}) > 0$). Calculate

$$\sup\left\{\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X)\,\Big|\,\mathbb{Q}\in\mathcal{P}\right\}$$

and confirm that this is equal to the superhedging price. Find the probability measure $\mathbb{Q} \in \mathcal{P}$ which achieves the supremum.

A. Similarly with (iv) (In (iv), we did not use the conditions

$q_u, q_m, q_d > 0$).

we can easily check $\sup\left\{\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}(X) \,\Big|\, \mathbb{Q}\in\mathcal{P}\right\} = 24$.

$(q_u, q_m, q_d) = (0.2, 0.35, 0.45) \in \mathcal{P}$ achieves the supremum! ▨

(vi) (5 points) Let $\mathcal{M}$ be the set of all signed-measures on 2^Ω (easy to check that this space $\mathcal{M}$ is a vector space over $\mathbb{R}$). Show that $\mathcal{P}$ is a convex subset of $\mathcal{M}$.

A. Let, $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{P}$, $0 \le \lambda \le 1$ $\to$ ETS. $\mathbb{Q} = \lambda\mathbb{Q}_1 + (1-\lambda)\mathbb{Q}_2 \in \mathcal{P}$.

$$\frac{1}{1+R}\mathbb{E}^{\mathbb{Q}}[S_1] = \frac{1}{1+R}\left(\lambda\mathbb{E}^{\mathbb{Q}_1}[S_1] + (1-\lambda)\mathbb{E}^{\mathbb{Q}_2}[S_1]\right)$$

$$= \lambda S_0 + (1-\lambda)S_0 = S_0 \quad (\because \mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{P})$$

$\therefore \mathbb{Q} \in \mathcal{P}$. ▨.

Problem 5.3. (15 points) In class, we merely checked that the superhedging duality holds for a specific example. The proof of the superhedging duality is in Eq.(4.1). Explain why these equalities hold except for the "inf sup = sup inf" in the second equality.

$$\begin{aligned}
\inf_{\substack{x_i \le \alpha+\beta s_i\\ i=1,2,3}} \alpha+\beta s &= \inf_{\alpha,\beta}\sup_{p_i>0} \alpha+\beta s+\sum_{i=1}^{3}p_i(x_i-\alpha-\beta s_i)\\
&= \sup_{p_i>0}\inf_{\alpha,\beta} \alpha+\beta s+\sum_{i=1}^{3}p_i(x_i-\alpha-\beta s_i)\\
&= \sup_{p_i>0}\inf_{\alpha,\beta} \alpha(1-\sum_{i=1}^{3}p_i)+\beta(s-\sum_{i=1}^{3}p_is_i)+\sum_{i=1}^{3}p_ix_i\\
&= \sup_{\substack{p_i>0\\ \sum_{i=1}^3 p_i=1\\ \sum_{i=1}^3 p_is_i=s}} \sum_{i=1}^{3}p_ix_i
\end{aligned} \tag{4.1}$$

A. ① First Equality :

For $i = 1,2,3$ $\quad A_i := \{(\alpha,\beta)\in\mathbb{R}^2 \mid \alpha+\beta s_i \ge x_i\}$.

Note that if for some $i=1,2,3$, $(\alpha,\beta)\notin A_i$,

then $\sup_{p_j>0}\left\{(\alpha+\beta S)+\sum_{j=1}^{3}p_j(x_j-\alpha-\beta S_j)\right\}=\infty$. (Fix p_j for $j\neq i$, $p_i\to\infty$)

$$\therefore \inf_{\alpha,\beta}\sup_{p_i>0}\left\{(\alpha+\beta S)+\sum_{i=1}^{3}p_i(x_i-\alpha-\beta S_i)\right\}=\inf_{(\alpha,\beta)\in A_1\cap A_2\cap A_3}\sup_{p_i>0}\left\{(\alpha+\beta S)+\sum_{i=1}^{3}p_i(x_i-\alpha-\beta S_i)\right\}$$

$$=\underbrace{\inf_{(\alpha,\beta)\in A_1\cap A_2\cap A_3}}\ (\alpha+\beta S)=\inf_{\substack{x_i\le\alpha+\beta S_i\\ i=1,2,3}}(\alpha+\beta S)\quad \text{ok!}$$

($\because$ if $(\alpha,\beta)\in A_1\cap A_2\cap A_3$ i.e., $\forall i=1,2,3$, $x_i-\alpha-\beta S_i\le 0$

$\Rightarrow$ $\alpha+\beta S+\sum_{i=1}^{3}p_i(x_i-\alpha-\beta S_i)$ achieves supremum as $p_1,p_2,p_3\to 0+$).

② Third Equality : Obvious! ok!

③ Last Equality : Similarly with ①. let, $B:=\{(p_1,p_2,p_3): p_1+p_2+p_3=1\}$

$C:=\{(p_1,p_2,p_3): p_1S_1+p_2S_2+p_3S_3=S\}$.

if $(p_1,p_2,p_3)\notin B \Rightarrow \inf_{\alpha,\beta}\alpha\left(1-\sum_{i=1}^{3}p_i\right)+\beta\left(S-\sum_{i=1}^{3}p_iS_i\right)+\sum_{i=1}^{3}p_ix_i=-\infty$

($\alpha\to+\infty$ if $\sum_{i=1}^{3}p_i>1$, $\alpha\to-\infty$ if $\sum_{i=1}^{3}p_i<1$)

if $(p_1,p_2,p_3)\notin C \Rightarrow \inf_{\alpha,\beta}\alpha\left(1-\sum_{i=1}^{3}p_i\right)+\beta\left(S-\sum_{i=1}^{3}p_iS_i\right)+\sum_{i=1}^{3}p_ix_i=-\infty$

($\beta\to+\infty$ if $\sum_{i=1}^{3}p_iS_i>S$, $\beta\to-\infty$ if $\sum_{i=1}^{3}p_iS_i<S$)

Thus, $\sup_{p_i>0}\inf_{\alpha,\beta}\alpha\left(1-\sum_{i=1}^{3}p_i\right)+\beta\left(S-\sum_{i=1}^{3}p_iS_i\right)+\sum_{i=1}^{3}p_ix_i$

$$=\sup_{\substack{p_i>0\\(p_1,p_2,p_3)\in B\cap C}}\inf_{\alpha,\beta}\alpha\left(1-\sum_{i=1}^{3}p_i\right)+\beta\left(S-\sum_{i=1}^{3}p_iS_i\right)+\sum_{i=1}^{3}p_ix_i$$

$$=\sup_{\substack{p_i>0\\ \sum p_i=1\\ \sum p_iS_i=S}}\sum_{i=1}^{3}p_ix_i.\quad \text{ok!}\ \blacksquare$$