

Prop. 4.6.) If  $X$  and  $Y$  are connected topological spaces,  
so is  $X \times Y$ .

pf.) Suppose  $X \times Y = U \cup V$ ,  $U, V$  : disjoint open sets.  
and  $U \neq \emptyset$ .

Want to show  $V = \emptyset$ .

(because, If  $V \neq \emptyset$ ,  $X \times Y$  is disconnected).

Pick  $(a, b) \in U$ .

with subspace topology,

$(X \times \{b\} \cap U)$ ,  $(X \times \{b\} \cap V)$  : two open disjoint sets in  $X \times \{b\}$ .

Since  $X$  is connected and  $X \times \{b\}$  is homeomorphic to  $X$   
 $X \times \{b\}$  is connected.

$(a, b) \in X \times \{b\} \cap U \neq \emptyset \Rightarrow X \times \{b\} \cap V = \emptyset$ .

( $\because$  otherwise,  $X \times \{b\}$  is disconnected.)

$\therefore X \times \{b\} \subset U$

For any  $x \in X$ , take  $\{x\} \times Y$ . (Note:  $(x, b) \in U$ ).

proceed as before  $\Rightarrow \{x\} \times Y \subset U$  ( $\because Y$  is connected).

$\Rightarrow X \times Y = \bigcup_{x \in X} \{x\} \times Y \subset U$

$\therefore V = \emptyset$ .

$\therefore X \times Y$  connected.  $\square$