Probability Theory – Exercise 6

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Problem 1

Find $\limsup_{n\to\infty} A_n$ for a sequence $\{A_n\}$ where $A_n = [\frac{i}{2^k}, \frac{i+1}{2^k}]$ if $n = i + 2^k$, $0 \le i < 2^k$.

Proof. We observe $A_1 = [0,1]$, $A_2 = [0, \frac{1}{2}]$, $A_3 = [\frac{1}{2}, 1]$, $A_4 = [0, \frac{1}{4}]$, $A_5 = [\frac{1}{4}, \frac{1}{2}]$, and so on. Since the union $\bigcup_{m=n}^{\infty} A_m = [0,1]$ for any n, so is their intersection. That is,

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = [0,1]. \qquad \square$$

Problem 2

Let $S_n = X_1 + X_2 + \cdots + X_n$ describe the position after n steps of a symmetric random walk on $\mathbb{Z}^d$. Using the asymtotic formula: $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$ and the Borel-Cantelli lemmas show that the probability of $\{S_n = 0 \text{ i.o.}\}$ is 1 when $d = 1, 2$ and 0 for $d > 2$.

Proof. Let $d = 1$. There are $\binom{2n}{n}$ paths that return to 0, so $P(S_{2n} = 0) = \binom{2n}{n}\frac{1}{2^{2n}}$. Now

$$\frac{(2n)!}{(n!)^2} \sim \frac{(\frac{2n}{e})^{2n}\sqrt{2\pi 2n}}{(\frac{n}{e})^{2n} 2\pi n} = \frac{2n\sqrt{2}}{\sqrt{n\pi}}$$

so $P(S_{2n} = 0) \sim \frac{c}{\sqrt{n}}$ with $c = \sqrt{\frac{2}{\pi}}$. Hence $\sum_{n=1}^{\infty} P(A_n)$ diverges and Borel-Cantelli applies (as (A_n) are independent) so that $P(S_{2n} = 0 \text{ i.o.}) = 1$. Same for $d = 2$ since $P(A_n) \sim \frac{1}{n}$. But for $d > 2$, $P(A_n) \sim \frac{1}{n^{d/2}}$, the series converges and by the first Borel-Cantelli lemma $P(S_{2n} = 0 \text{ i.o.}) = 0$. $\square$

Problem 3

Let $X_1, X_2, \cdots$ be independent random variables with finite expectation. If $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$, show that $\sum_{n=1}^{\infty}(X_n - \mathbb{E}[X_n])$ converges a.s.

Proof. Let $Y_n = X_n - \mathbb{E}(X_n)$ be centred random variables. Clearly $\mathbb{E}(Y_n) = 0$, $\mathrm{Var}(Y_n)$=$\mathrm{Var}(X_n)$. So, $\sum_{n=1}^{\infty} \mathrm{Var}(Y_n) = \sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$. Now consider partial sum $S_N = \sum_{n=1}^{N} Y_n$. To show that $\sum_{n=1}^{\infty} Y_n = \lim_{N\to\infty} S_N$ converges almost surely, it is sufficient to prove that

$$\limsup_{N\to\infty} S_N - \liminf_{N\to\infty} S_N = 0$$

with probability 1. For any $m \in \mathbb{N}$,

$$\limsup_{N\to\infty} S_N - \liminf_{N\to\infty} S_N = \limsup_{N\to\infty}(S_N - S_m) - \liminf_{N\to\infty}(S_N - S_m) \le 2\max_{k\in\mathbb{N}} \left|\sum_{i=1}^{k} Y_{m+i}\right|.$$

Thus, for any $\epsilon > 0$,

$$\begin{aligned} P(\limsup_{N\to\infty} S_N - \liminf_{N\to\infty} S_N \ge \epsilon) &\le P(2\max_{k\in\mathbb{N}} \left|\sum_{i=1}^{k} Y_{m+i}\right| \ge \epsilon) = P(\max_{k\in\mathbb{N}} \left|\sum_{i=1}^{k} Y_{m+i}\right| \ge \frac{\epsilon}{2}) \\ &= P(\lim_{n\to\infty} \max_{1\le k\le n} \left|\sum_{i=1}^{k} Y_{m+i}\right| \ge \frac{\epsilon}{2}) = \lim_{n\to\infty} P(\max_{1\le k\le n} \left|\sum_{i=1}^{k} Y_{m+i}\right| \ge \frac{\epsilon}{2}) \\ &\le \lim_{n\to\infty} \frac{4}{\epsilon^2}\mathrm{Var}(\sum_{i=1}^{n} Y_{m+i}) = \frac{4}{\epsilon^2} \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{Var}(Y_{m+i}). \end{aligned}$$

While the second inequality is due to Kolmogorov's inequality. Since $\sum_{n=1}^{\infty} \mathrm{Var}(Y_n) < \infty$, it follows that the last term tends to 0 as m goes to infinity, for every arbitrary $\epsilon > 0$. $\square$