Vector Calculus – Midterm Exam

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1. (1) Find the volume of a tetrahedron which consists of the following vertices:
$$A = (1,2,3),\ B = (0,1,2),\ C = (1,3,5),\ D = (1,4,11).$$
   (2) Compute the divergence of the vector field $F(x,y,z) = (xy, yz, zx)$ at $(1,1,1)$.
   (3) Compute the curl of the vector field $F(x,y,z) = (y^2 z, e^{xyz}, x^2 y)$ at $(1,1,1)$.

2. Consider a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$
$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$
   (1) Find $D_1 f(0,0)$ and $D_2 f(0,0)$.
   (2) Prove or disprove that f is differentiable at origin.

3. Show the following matrix is positive definite:
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

4. Suppose that there is a circle centered at $(0,r)$ with the radius r. Imagine that this circle is a wheel rolling along x-axis. When the wheel rolls one cycle to right direction, the point on the rim traces a cycloid curve.
   (1) Provide a parametrized curve to this cycloid starting from the origin $(0,0)$.
   (2) Find the arc length of the curve when the wheel rolls one cycle.
   (3) Compute the curvature of this curve when the curve has a maximum y value.

5. Compute the curvature and the torsion of the following parametrized curve at origin:
$$X(t) = (e^t \cos t, e^t \sin t, e^t).$$

6. Find the second-order Taylor approximation for the following function at origin:
$$f(x,y) = e^{x+xy} \log(1-xy).$$

7. Consider a function $f(x,y,z) = x + y^2$. Let $P = (1,1,1)$ be a point on a level set
$$S = \{(x,y,z) \in \mathbb{R}^3 : x + xy + xyz = 3\}.$$
   Then find the maximum value of $D_v f(P)$ for a unit vector v which is tangent to S at point P.

8. Let $F\colon X \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous force field which is given by a gradient field such that there is a C^1 potential V satisfying $F = -\nabla V$. Let $X\colon I \subset \mathbb{R} \to \mathbb{R}^3$ be a path of a moving particle with the mass m which follows the force field F. Prove that the total energy (kinetic + potential) is conserved.

9. Find all critical points of the following function and determine whether it is a local maximum, a local minimum or a saddle point.
$$f(x,y) = x^3y + 2xy^2 - xy.$$