

# Differential Geometry 1 – Final Exam

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June 10, 2020

1. Show that the set  $S := \{(x, y, z) \in \mathbb{Z}^3 \mid x^2 + y^2 - z^2 = 0\}$  is not a regular surface.
2. Let  $S$  be a compact regular surface. Assume that there is a differentiable function  $f: S \rightarrow \mathbb{R}$  with at most three critical points. Prove that  $S$  is connected.
3. Let  $f: S^2 \rightarrow (0, +\infty)$  be a positive differentiable function on the unit sphere. Let

$$S_f := \{f(p)p = (f(p)x, f(p)y, f(p)z) \in \mathbb{R}^3 \mid p = (x, y, z) \in S^2\}.$$

- (a) Show that  $S_f$  is a regular surface.
  - (b) Show that the map  $\phi: S^2 \rightarrow S_f$  given by  $\phi(p) := f(p)p$  is a diffeomorphism.
4. Let  $S$  be a regular surface. For a fixed point  $p_0 \in \mathbb{R}^3$ , let

$$f: S \rightarrow \mathbb{R}, \quad f(p) := |p - p_0|^2.$$

Show that  $p$  is a critical point of  $f$  if and only if  $p_0$  belongs to the normal line of  $S$  at  $p$ .

5. Let  $S$  be a regular surface given by the graph of a differentiable function  $z = f(x, y)$ . Let  $R$  be a bounded region of  $S$ . Show that the area of  $R$  is

$$\text{area}(R) = \int_{\pi(Q)} \sqrt{1 + (f_x)^2 + (f_y)^2} \, dx \, dy$$

where  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by  $\pi(x, y, z) := (x, y)$ , and  $f_x = \frac{\partial f}{\partial x}$  and  $f_y = \frac{\partial f}{\partial y}$ .

6. Let  $S$  be a regular oriented surface. Show that the mean curvature  $H$  at  $p \in S$  is equal to

$$H = \frac{1}{\pi} \int_0^\pi k_n(\theta) \, d\theta$$

where  $k_n(\theta)$  denotes the normal curvature at  $p$  along a direction making an angle  $\theta \in [0, \pi]$  with a fixed direction.

7. Consider the parametrized surface

$$\mathbf{x}(u, v) = \left( u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right).$$

- (a) Compute the coefficients of the first fundamental form.
- (b) Compute the coefficients of the second fundamental form.
- (c) Show that the principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

- (d) Compute the Gaussian curvature  $K$  and the mean curvature  $H$  at every point.

8. Let  $S$  be a regular surface.

- (a) Let  $\alpha: [-1, 1] \rightarrow S$  be a geodesic with  $|\alpha'(0)| = 1$ . Compute the arc length of  $\alpha$ .
- (b) Let  $\beta: [-2, 2] \rightarrow S$  be another geodesic with  $\beta(0) = \alpha(0)$  and  $-2\beta'(0) = \alpha'(0)$ . Compute the arc length of  $\beta$  and describe how  $\alpha$  and  $\beta$  are related.

9. Suppose that  $S$  is a regular, compact, connected, orientable surface.

- (a) At any  $p \in S$ , locally  $S$  is the graph of some differentiable function  $h$  defined in a neighborhood of  $0$  in the tangent plane  $T_p S$ . ( $0 \in T_p S$  is identified with  $p \in S$ .) Show that the second fundamental form at  $p$  equals the Hessian of  $h$  at  $0 \in T_p S$ .
- (b) Show that there is a point  $p \in S$  with positive Gaussian curvature  $K(p) > 0$ .
- (c) Show that if  $S$  is not homeomorphic to  $S^2$ , then there are points on  $S$  where the Gaussian curvature is zero and negative.

10. Let  $S$  be a compact regular oriented surface. Prove that the Gauss map  $N: S \rightarrow S^2$  is a local diffeomorphism if and only if  $S$  has positive Gaussian curvature everywhere.

11. Let  $\alpha: [0, 1] \rightarrow S$  be a differentiable curve.

- (a) Let  $P_\alpha: T_{\alpha(0)}S \rightarrow T_{\alpha(1)}S$  be the parallel transport map along  $\alpha$ . Show that  $P_\alpha$  is a linear isometry.
- (b) Show that there exist two differentiable vector fields

$$w_1, w_2: [0, 1] \rightarrow \bigcup_{t \in [0, 1]} T_{\alpha(t)}S, \quad w_1(t), w_2(t) \in T_{\alpha(t)}S$$

along  $\alpha$  which form an orthonormal basis of  $T_{\alpha(t)}S$  for all  $t \in [0, 1]$ , i.e.,

$$|w_1(t)| = |w_2(t)| = 1, \quad \langle w_1(t), w_2(t) \rangle = 0 \quad \forall t \in [0, 1].$$

- (c) Let  $w$  be a differentiable vector field along  $\alpha$ . Let  $P_\alpha^{t_0, t_1}: T_{\alpha(t_0)}S \rightarrow T_{\alpha(t_1)}S$  be the parallel transport map along  $\alpha|_{[t_0, t_1]}$  for  $t_0, t_1 \in (0, 1)$ . Prove that

$$\frac{dw}{dt}(t_0) = \frac{d}{dt} \Big|_{t=t_0} (P_\alpha^{t_0, t_1})^{-1}(w(t)).$$