Problem 1.1.

(i). $\Omega = \{uu, ud, du, dd\}$. $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \{uu, ud\}, \{du, dd\}, \Omega\}$

$\mathcal{F}_2 = 2^{\Omega}$. $\Rightarrow \{\mathcal{F}_t\}_{t=0,1,2}$ is filtration on Ω.

$\mathbb{P}(\{uu\}) = 0.12$. $\mathbb{P}(\{ud\}) = 0.48$. $\mathbb{P}(\{du\}) = 0.2$. $\mathbb{P}(\{dd\}) = 0.2$.

$S_0 = 100$. $S_1(uu) = S_1(ud) = 160$. $S_1(du) = S_2(dd) = 70$.

$S_2(uu) = 240$. $S_2(ud) = S_2(du) = 96$. $S_2(dd) = 72$

(ii) Let $Y_1 = E^{\mathbb{P}}[S_2/G_2 | \mathcal{F}_1]$ and $\alpha = Y(uu) = Y(ud)$, $\beta = Y(du) = Y(dd)$.

Y is $\mathcal{F}_1$-msble

From def of cond. exp.

$E[Y \mathbb{1}_{\{uu, ud\}}] = E^{\mathbb{P}}(S_2/G_2 \cdot \mathbb{1}_{\{uu, ud\}})$

$\alpha \cdot (0.12 + 0.48) = \frac{240}{1.2} \times 0.12 + \frac{96}{1.2} \times 0.48$.

i.e. $0.6\alpha = 62.4$. $\underline{\alpha = 104}$

In same way, $\underline{\beta = 70}$.

(iii) V_1^h, π_1, ϕ_1 is $\mathcal{F}_1$-msble $\Rightarrow$ Let $V_1^h(uu) = V_1^h(ud) = \alpha$, $V_1^h(du) = V_1^h(dd) = \beta$.

$\pi_1(uu) = \pi_1(ud) = \alpha_\pi$, $\pi_1(du) = \pi_1(dd) = \beta_\pi$.

$\phi_1(uu) = \phi_1(ud) = \alpha_\phi$ $\phi_1(du) = \phi_1(dd) = \beta_\phi$

From $V_1^h = \pi_1 G_1 + \phi_1 S_1$, $\alpha = \alpha_\pi G_1 + \alpha_\phi S_1$, $\beta = \beta_\pi G_1 + \beta_\phi S_1$.

$V_2^h = \pi_2 G_2 + \phi_2 S_2 = \pi_1 G_2 + \phi_1 S_2$.

From $V_2^h = X$,

$$\begin{cases} \alpha_\pi \cdot 1.2 + \alpha_\phi \cdot 240 = 120 & \text{in the case of } uu \\ \alpha_\pi \cdot 1.2 + \alpha_\phi \cdot 96 = 0 & \text{" } ud \end{cases} \Rightarrow \alpha_\pi = -\frac{200}{3}, \quad \alpha_\phi = \frac{5}{6}$$

$$\begin{cases} \beta_\pi \cdot 1.2 + \beta_\phi \cdot 96 = 0 & \text{" } du \\ \beta_\pi \cdot 1.2 + \beta_\phi \cdot 72 = -12 & \text{" } dd \end{cases} \Rightarrow \beta_\pi = -40, \quad \beta_\phi = \frac{1}{2}.$$

$\alpha = -\frac{200}{3} \cdot 1 + \frac{5}{6} \cdot 160 = \frac{200}{3}$, $\beta = -40 \cdot 1 + \frac{1}{2} \cdot 70 = -5$.

From $\pi_0 G_1 + \phi_0 S_1 = V_1^h$,

$$\begin{cases} \pi_0 \cdot 1 + 160 \cdot \phi_0 = \frac{200}{3} & \text{in the case of } uu \text{ or } ud \\ \pi_0 \cdot 1 + 70 \cdot \phi_0 = -5 & \text{" } du \text{ or } dd \end{cases}$$

$\Rightarrow \pi_0 = \frac{-1640}{27}$, $\phi_0 = \frac{43}{54}$. and $X_0 = \pi_0 + 100 \cdot \phi_0 = \frac{170}{9}$.

$\therefore X_0 = \frac{170}{9}$. $X_1(uu) = X_1(ud) = \frac{200}{3}$. $X_1(du) = X_1(dd) = -5$. $X_2(uu) = 120$. $X_2(ud) = X_2(du) = 0$, $X_2(dd) = -12$

$\phi_0 = \frac{43}{54}$. $\pi_0 = \frac{-1640}{27}$. $\phi_1(uu) = \phi_1(ud) = \frac{5}{6}$, $\pi_1(uu) = \pi_1(ud) = -\frac{200}{3}$

$\phi_1(du) = \phi_1(dd) = \frac{1}{2}$. $\pi_1(du) = \pi_1(dd) = -40$.

$\phi_2 = \phi_1$ $\pi_2 = \pi_1$.

(iv) Calculate $V^h_{t+1} = \pi_t G_{t+1} + \phi_t S_{t+1}$ for $t=0,1$

or $V^h_{t+1} - V^h_t = \pi_t(G_{t+1} - G_t) + \phi_t(S_{t+1} - S_t)$ for $t=0,1$.

(v) Is ϕ adapted?

. $\phi_0(uu) = \phi_0(ud) = \phi_0(du) = \phi_0(dd) = \frac{43}{54} \Rightarrow \phi_0^{-1}(\alpha) = \begin{cases} \emptyset & \text{o.w} \\ \Omega & \alpha = 43/54 \end{cases}$.

. $\begin{cases} \phi_1(uu) = \phi_1(ud) = \frac{5}{6} \\ \phi_1(du) = \phi_2(dd) = \frac{1}{2} \end{cases} \Rightarrow \phi_1^{-1}(\alpha) = \begin{cases} \emptyset & \text{o.w} \\ \{uu, ud\} & \alpha = 5/6 \\ \{dd, du\} & \alpha = 1/2 \end{cases}$.

$\Rightarrow$ ϕ_0 is $\mathcal{F}_0$-msble & ϕ_1 is $\mathcal{F}_1$-msble.

$\phi_2 = \phi_1$ & $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Rightarrow \phi_2$ is $\mathcal{F}_2$-msble $\Rightarrow \phi$ is adapted.

(vi). Define a probability measure Q on Ω by

$Q(uu) = \frac{2}{9}$, $Q(ud) = \frac{1}{9}$, $Q(du) = Q(dd) = \frac{1}{3}$.

You can show that

① S_t/G_t is adapted ② $E^Q(S_2/G_2 | \mathcal{F}_1) = S_1/G_1$ ③ $E^Q(S_1/G_1 | \mathcal{F}_0) = S_0/G_0$.

⌈ For example, to show ②, $Y := E^Q(S_2/G_2 | \mathcal{F}_1)$

Y is $\mathcal{F}_1$-msble ; let $a = Y(uu) = Y(ud)$ and $b = Y(du) = Y(dd)$.

Then, $\begin{cases} a \cdot (Q(uu) + Q(ud)) = \frac{240}{1.2} \cdot Q(uu) + \frac{96}{1.2} \cdot Q(ud) \Rightarrow a = 160 \\ b \cdot (Q(du) + Q(dd)) = \frac{96}{1.2} Q(du) + \frac{72}{1.2} Q(dd) \Rightarrow b = 70 \end{cases}$ ⌋

(*) How to find Q.)

From condition ②, $Q(uu), Q(ud)$ satisfy $\cdot\ 160\,(Q(uu) + Q(ud)) = \frac{240}{1.2} Q(uu) + \frac{96}{1.2} Q(ud)$

i.e $Q(uu) : Q(ud) = 2 : 1$

In the same way, $Q(du) : Q(dd) = 1 : 2$.

From condition ③, $\frac{160}{1} \cdot (Q(uu) + Q(ud)) + \frac{70}{1}(Q(du) + Q(dd)) = 100$.

$= 100\,(Q(uu) + Q(ud) + Q(du) + Q(dd))$

$\Rightarrow Q(uu) + Q(ud) : Q(du) + Q(dd) = 1 : 2$.

From above 3 equality, you can find Q.

(vii) Since Q is risk neutral measure, $\frac{S_t}{G_t}$ is martingale under Q.

$\therefore E^Q(S_2/G_2 \mid \mathcal{F}_1) = S_1/G_1$

Since $X_t = V_t^h$ for self-financing portfolio h, $\frac{V_t^h}{G_t}$ is martingale under Q

$\therefore E^Q(X_2/G_2 \mid \mathcal{F}_1) = E^Q(V_2^h/G_2 \mid \mathcal{F}_1) = V_1^h/G_1 = X_1/G_1$.

Problem 1.2 In class, (iii) $\Rightarrow$ (i) is proved.

(i) $\Rightarrow$ (ii).

Suppose (ii) is false. There are 6 cases that (ii) is false and in any case you can construct arbitrage.

For example, if $HR \le S_d/S < S_u/S$, define $h = (\pi_t, \phi_t)$ by

$h_0 = (-S, 1)$, $h_1 = (\frac{V_1^h}{HR}, 0)$, $h_2 = h_1$.

$V_1^h = \pi_1 G_1 + \phi_1 S_1$

$= \frac{V_1^h}{HR} \cdot HR$

Then, $V_0^h = -S\times 1 + S\times 1 = 0$. $V_1^h = \begin{cases} -S(HR) + S_u > 0 & S_1 = S_u \\ -S(HR) + S_d \ge 0 & S_1 = S_d \end{cases}$

$V_2^h = \left(\frac{V_1^h}{HR}\right) \cdot (HR)^2 = (HR) V_1^h \begin{cases} > 0 & S_2 = S_{uu} \text{ or } S_{ud} \\ \ge 0 & S_2 = S_{du} \text{ or } S_{dd} \end{cases}$.

$\therefore$ h is an arbitrage.

In the case of $S_{ud}/S_u < S_{uu}/S_u \le HR$, define $h = (\pi_t, G_t)$ by

$h_0 = (0, 0)$, $h_1 = \begin{cases} (S_u, -(HR)) & S_1 = S_u \\ (0, 0) & S_1 = S_d \end{cases}$ $h_2 = h_1$.

Then h is arbitrage.

(ii) $\Rightarrow$ (iii).

Construct filtered space as problem 1.1 (i), Define measure Q on Ω by

By condition (ii), $Q(uu), \cdots, Q(dd) > 0$.

$Q(uu) = \frac{S(HR) - S_d}{S_u - S_d} \cdot \frac{S_u(HR) - S_{ud}}{S_{uu} - S_{ud}}$, $Q(ud) = \frac{S(HR) - S_d}{S_u - S_d} \cdot \frac{S_{uu} - S_u(HR)}{S_{uu} - S_{ud}}$

$Q(du) = \frac{S_u - S(HR)}{S_u - S_d} \cdot \frac{S_d(HR) - S_{dd}}{S_{du} - S_{dd}}$, $Q(dd) = \frac{S_u - S(HR)}{S_u - S_d} \cdot \frac{S_{du} - S_d(HR)}{S_{du} - S_{dd}}$

You can show that S_t/G_t is martingale under Q.

(*) How to find Q.)

Q must satisfy $E^Q(S_{t+1}/G_{t+1} \mid \mathcal{F}_t) = S_t/G_t$.

Let $q_{uu} = Q(uu)$, $\cdots$, $q_{dd} = Q(dd)$,

$t=1, \{uu, ud\}$ $\Rightarrow$ $\frac{1}{(HR)^2}\left[S_{uu}q_{uu} + S_{ud}q_{ud}\right] = \frac{S_u}{HR}(q_{uu}+q_{ud})$.

$\Leftrightarrow$ $\frac{S_{uu}/S_u \cdot q_{uu} + S_{ud}/S_d \cdot q_{ud}}{q_{uu}+q_{ud}} = HR$.

S_{ud}/S_d — HR — S_{uu}/S_u.

$\Leftrightarrow$ HR은 $(S_{ud}/S_d,\ S_{uu}/S_u)$를 q_{uu} : q_{ud}로 나누는 내분점.

$\Leftrightarrow$ $q_{uu} : q_{ud} = S_{uu}/S_u - (HR) : (HR) - S_{ud}/S_d$.

$t=1, \{du, dd\}$. In the same way, $q_{du} : q_{dd} = \frac{S_{du}}{S_d} - (HR) : (HR) - \frac{S_{dd}}{S_d}$

$t=0$, Ω. $q_{uu}+q_{ud} : q_{du}+q_{dd} = S_u/S - (HR) : (HR) - S_d/S$.

Using $q_{uu}+q_{ud}+q_{du}+q_{dd}=1$, you can find $q_{uu} \cdots q_{dd}$ □

Remark). We can regard 2-period binomial model as 3 1-period binomial model

① $S \xrightarrow{p_u} S_u$, $S \xrightarrow{p_d} S_d$ ② $S_u \xrightarrow{p_{uu}} S_{uu}$, $S_u \xrightarrow{p_{ud}} S_{ud}$ ③ $S_d \xrightarrow{p_{du}} S_{du}$, $S_d \xrightarrow{p_{dd}} S_{dd}$

The condition (ii) is equivalent that ①, ②, ③ have no arbitrage.
Using the def of arbitrage, you can show (ii) $\Rightarrow$ (i) by assuming that 2-period binomial model has arbitrage.

Also, the condition (ii) is equivalent that ①, ② ③ have a risk-neutral measure.
From risk-neutral measure Q on 2-period binomial model,
you can construct a risk-neutral measure of ①, ②, ③.
It proves (iii) $\Rightarrow$ (ii).

Problem 1.3. Since M_t is $\mathcal{F}_t$-msble, M is adapted

tower property $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$

$E[M_{t+1} \mid \mathcal{F}_t] = E[E[X \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] = E[X \mid \mathcal{F}_t] = M_t$.

$\therefore$ M is a martingale.

Problem 1.4. Let $\mathbb{Q}$ be a risk-neutral measure on 2-period binomial model.
Since $(X_t)_{t=0,1,2}$ is the arbitrage-free price process,
there exists self-financing portfolio $h=(\pi_t, \phi_t)_{t=0,1,2}$ s.t. $V_t^h = X_t$.
Therefore, $V_t^h / G_t = X_t / G_t$ is martingale under $\mathbb{Q}$.

Assume $\tilde{h} = (a_t, b_t, c_t)_{t=0,1,2}$ is arbitrage of the market with 3 assests $(G_t)_{t=0,1,2}$, $(S_t)_{t=0,1,2}$, $(X_t)_{t=0,1,2}$.
Then $\mathbb{P}(V_0^{\tilde{h}}=0)=1$, $\mathbb{P}(V_T^{\tilde{h}} \geq 0)=1$, $\mathbb{P}(V_T^{\tilde{h}} > 0) > 0$.
Since $\mathbb{P} \simeq \mathbb{Q}$, $\mathbb{Q}(V_0^{\tilde{h}}=0)=1$, $\mathbb{Q}(V_T^{\tilde{h}} \geq 0)=1$, $\mathbb{Q}(V_T^{\tilde{h}} > 0) > 0$.
$\therefore$ $E^{\mathbb{Q}}[V_T^{\tilde{h}}] > 0$.

Since $\tilde{h}$ is self-financing, $V_{t+1}^{\tilde{h}} = a_t G_{t+1} + b_t S_{t+1} + c_t X_{t+1}$.

$$\Rightarrow E^{\mathbb{Q}}\left[V_{t+1}^{\tilde{h}} / G_{t+1} \mid \mathcal{F}_t\right] = E^{\mathbb{Q}}\left[a_t + b_t \frac{S_{t+1}}{G_{t+1}} + c_t \frac{X_{t+1}}{G_{t+1}} \,\middle|\, \mathcal{F}_t\right].$$

$$= a_t + b_t \cdot \frac{S_t}{G_t} + c_t \cdot \frac{X_t}{G_t} = \frac{V_t^{\tilde{h}}}{G_t} \quad \text{for } t=0,1$$

$\left(\frac{S_t}{G_t}\right)_{t=0,1}$, $\left(\frac{X_t}{G_t}\right)_{t=0,1}$ are martingale under $\mathbb{Q}$.

$$\therefore E^{\mathbb{Q}}[V_T^{\tilde{h}} / G_T] = E^{\mathbb{Q}}[V_0^{\tilde{h}} / G_0] = 0.$$

$\Rightarrow E^{\mathbb{Q}}[V_T^{\tilde{h}}] = 0$. It contradicts.

$\therefore$ A market with three assests (G_t), (S_t), (X_t) is free of arbitrage.