

2020 Fall 351 - HW4

Due: 2020-9/16/2020

1.1. (i).

$$\begin{aligned}
 P(B_1 > 1, B_3 - B_2 > 1) &= P(B_1 > 1) \cdot P(B_3 - B_2 > 1) \quad (\because B_1 \perp B_3 - B_2) \\
 &= P(Z > 1) \cdot P(Z > 1) \quad (\because B_1 \sim Z, B_3 - B_2 \sim Z) \\
 &= (N(-1))^2
 \end{aligned}$$

1.1. (ii)

$$P(2B_1 - B_2 > -1)$$

Since B_t is Gaussian process, $2B_1 - B_2$ is normal

$$E(2B_1 - B_2) = 0.$$

$$\text{Var}(2B_1 - B_2) = \text{Var}(B_1 - (B_2 - B_1)) = \text{Var}(B_1) + \text{Var}(B_2 - B_1) = 2$$

$$2B_1 - B_2 \sim N(0, 2) \approx \sqrt{2}Z$$

$$P(2B_1 - B_2 > -1) = P(Z > -\frac{1}{\sqrt{2}}) = N(\frac{1}{\sqrt{2}})$$

1.1. (iii).

$$P(B_1 < 0, B_2 > (1 - \sqrt{3})B_1) = P(B_1 < 0, (B_2 - B_1) > -\sqrt{3}B_1)$$

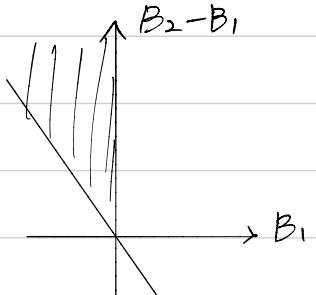
$$B_1 \perp (B_2 - B_1)$$

$$= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \int_{-\sqrt{3}x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \int_{-\infty}^0 \int_{-\sqrt{3}x}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dy dx = \int_0^{\infty} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r d\theta dr$$

$$= \int_0^{\infty} \frac{1}{2\pi} e^{-\frac{r^2}{2}} \cdot r \cdot \frac{\pi}{6} dr$$

$$= \frac{1}{12} \int_0^{\infty} r \cdot e^{-\frac{r^2}{2}} dr = -\frac{1}{12} [e^{-\frac{r^2}{2}}]_0^{\infty} = \frac{1}{12}$$



1.1. (iv).

$$\begin{aligned} P(B_3 < 1 | B_1) &= P(B_3 < 1 | B_1 = x) = P(B_3 - B_1 < 1 - x | B_1 = x) \\ &= P(B_3 - B_1 < 1 - x) = P\left(\frac{1}{\sqrt{2}}(B_3 - B_1) < \frac{1-x}{\sqrt{2}}\right) \\ &= P(Z < \frac{1-x}{\sqrt{2}}) = N\left(\frac{1-x}{\sqrt{2}}\right) = N\left(\frac{1-B_1}{\sqrt{2}}\right) \end{aligned}$$

1.1. (v).

$$\begin{aligned} P(B_1 < 2 | B_2) &= P(2B_1 < 4 | B_2 = x) \\ &= P(2B_1 - B_2 < 4 - x | B_2 = x) \quad \textcircled{1} \\ &= P(2B_1 - B_2 < 4 - x) \\ &= P\left(\frac{1}{\sqrt{2}}(2B_1 - B_2) < \frac{4-x}{\sqrt{2}}\right) \quad \textcircled{2} \\ &= P(Z < \frac{4-x}{\sqrt{2}}) = N\left(\frac{4-B_2}{\sqrt{2}}\right) \end{aligned}$$

① Since B_t is Gaussian process, $\lambda_1 B_1 + \lambda_2 B_2$ is normal for $\forall \lambda_1, \lambda_2 \in \mathbb{R}$
Thus $\mu_1(2B_1 - B_2) + \mu_2 B_2 = 2\mu_1 B_1 + (\mu_2 - \mu_1)B_2$ is normal $\forall \mu_1, \mu_2 \in \mathbb{R}$
 $\Rightarrow (2B_1 - B_2, B_2)$: normal.

$$\text{COV}(2B_1 - B_2, B_2) = E(2B_1 B_2 - B_2^2) = 2 \cdot \min(1, 2) - 2 = 0.$$

$$\therefore 2B_1 - B_2 \perp B_2$$

② ~~$2B_1 - B_2 \sim N(0, 2)$~~

✓ 1.1. (vi) $0 < s < t$

$$\begin{aligned} E(L B_s^2 e^{2B_t}) &= E(B_s^2 e^{2B_s} e^{2(B_t - B_s)}) = E(B_s^2 e^{2B_s}) E(e^{2(B_t - B_s)}) \\ &= E(B_s^2 e^{2B_s}) \cdot e^{2(t-s)} \end{aligned}$$

1.1. (vii).

$$\begin{aligned} \text{Var}(2B_3 - B_2) &= \text{Var}(2(B_3 - B_2) + B_2) = \text{Var}(2\underline{(B_3 - B_2)}) + \text{Var}(\underline{B_2}) \\ &= 4 + 2 = 6. \quad \sim N(0, 1) \quad \sim N(0, 2) \end{aligned}$$

1.1. (Viii).

$$\begin{aligned}\text{Cov}(e^{Bt}, e^{-2Bs}) &= E(e^{Bt-2Bs}) - E(e^{Bt})E(e^{-2Bs}) \\ &= e^{\frac{1}{2}\text{Var}(Bt-2Bs)} - e^{\frac{1}{2}t} \cdot e^{2s} \\ &= e^{\frac{1}{2}t} - e^{\frac{1}{2}t+2s} = e^{\frac{1}{2}t}(1-e^{2s})\end{aligned}$$

1.1. (ix).

$$E(B_1 + B_3 | B_1 - 2B_2).$$

$$\begin{aligned}\text{Cov}(aB_1 + bB_2 + cB_3, B_1 - 2B_2) &= 0. \\ &= aE(B_1^2) + bE(B_1B_2) + cE(B_1B_3) - 2aE(B_1B_2) - 2bE(B_2^2) - 2cE(B_2B_3) \\ &= a + b + c - 2a - 4b - 4c = -a - 3b - 3c = 0. \\ \Rightarrow aB_1 + bB_2 + cB_3 &\perp B_1 - 2B_2 \text{ for } a, b, c \in \mathbb{R} \text{ s.t. } \underline{a+3b+3c=0}. \\ (\because (aB_1 + bB_2 + cB_3, B_1 - 2B_2) &\text{ : normal}).\end{aligned}$$

$$\begin{aligned}E(B_1 + B_3 | B_1 - 2B_2) &= E(B_1 + B_3 + \lambda(B_1 - 2B_2) - \lambda(B_1 - 2B_2) | B_1 - 2B_2) \\ \text{choose } \lambda &= \frac{4}{5}, \\ &= E(B_1 + B_3 + \frac{4}{5}(B_1 - 2B_2) | B_1 - 2B_2) - \frac{4}{5}E(B_1 - 2B_2 | B_1 - 2B_2). \\ &= E(\frac{9}{5}B_1 - \frac{8}{5}B_2 + B_3 | B_1 - 2B_2) - \frac{4}{5}E(B_1 - 2B_2) \\ (\frac{9}{5} - 3 \cdot \frac{8}{5} + 3 &= 0 \Rightarrow \frac{9}{5}B_1 - \frac{8}{5}B_2 + B_3 \perp B_1 - 2B_2). \\ &= E(\frac{9}{5}B_1 - \frac{8}{5}B_2 + B_3) - \frac{4}{5}E(B_1 - 2B_2) = 0.\end{aligned}$$

✓

1.3. (i)

$$(X_t)_{t \geq 0} = (\frac{1}{\sqrt{c}} B_{ct})_{t \geq 0} \text{ B.M. for any } c > 0.$$

① $(\frac{1}{\sqrt{c}} B_{ct})_{t \geq 0}$ Gaussian process.

$$\sum_{i=1}^n \lambda_i \frac{1}{\sqrt{c}} B_{ct_i} = \lambda_1 \frac{1}{\sqrt{c}} B_{ct_1} + \dots + \lambda_n \frac{1}{\sqrt{c}} B_{ct_n} \text{ normal for } \forall \lambda_i, c \in \mathbb{R}.$$

($\because \sum_{i=1}^n \lambda_i B_{ct_i}$ is normal $\forall \lambda_i \in \mathbb{R}$).

$$② E(\frac{1}{\sqrt{c}} B_{ct}) = 0 \text{ for } \forall t, \forall c$$

$$③ \text{cov}(\frac{1}{\sqrt{c}} B_{ct}, \frac{1}{\sqrt{c}} B_{cs}) = E(\frac{1}{c} B_{ct} B_{cs}) = \frac{1}{c} \cdot \min(ct, cs) = \min(t, s)$$

④ $B_t(w)$: continuous for $\forall w \in \Omega$

$B_{ct}(w) = B_t(w)$. (\because sample paths are depend on $w \in \Omega$).

$\frac{1}{\sqrt{c}} B_{ct}(w)$: also conti. \square

1.3. (ii). $X_t = B_{t+s} - B_s$ is B.M.

($\because X_t$ is continuous Gaussian, $E(X_t) = 0$,

$$\begin{aligned} \text{cov}(X_t, X_p) &= \text{cov}(B_{t+s}, B_{p+s}) - \text{cov}(B_{t+s}, B_s) - \text{cov}(B_{p+s}, B_s) + \text{cov}(B_s, B_s) \\ &= \min(t+s, p+s) - s - s + s = \min(t, p). \end{aligned}$$

WLOG, Assume $s = 0$.

$$P\left(\liminf_{t \rightarrow 0^+} \frac{B_t}{\sqrt{2t \ln \ln t}} = -1, \limsup_{t \rightarrow 0^+} \frac{B_t}{\sqrt{2t \ln \ln t}} = 1\right)$$

$$P\left(\liminf_{t \rightarrow 0^+} \frac{\frac{1}{\sqrt{t}} B_t}{\sqrt{\frac{2}{t} \ln \ln t}} = -1, \limsup_{t \rightarrow 0^+} \frac{\frac{1}{\sqrt{t}} B_t}{\sqrt{\frac{2}{t} \ln \ln t}} = 1\right)$$

$$P\left(\liminf_{t \rightarrow \infty} \frac{t B_{1/t}}{\sqrt{2t \ln \ln t}} = -1, \limsup_{t \rightarrow \infty} \frac{t B_{1/t}}{\sqrt{2t \ln \ln t}} = 1\right) = 1$$

(By law of iterated logarithm of Brownian motion,
 $(t B_{1/t})_{t \geq 0}$ is B.M.).

① $\sum_{i=1}^n \lambda_i t B_{1/t}$ is normal ($\because B_{1/t}$: Gaussian) $\Rightarrow t B_{1/t}$ is Gaussian.

$$② E(t B_{1/t}) = t E(B_{1/t}) = 0.$$

$$③ \text{cov}(t B_{1/t}, s B_{1/s}) = ts \text{cov}(B_{1/t}, B_{1/s}) = ts \min(\frac{1}{t}, \frac{1}{s}) = \min(t, s).$$

④ B_t continuous $\Rightarrow t B_{1/t}$ is continuous at $t > 0$.

$\lim_{t \rightarrow 0^+} t B_{1/t} = 0 = X_0 \Rightarrow t B_{1/t}$ is continuous at $t = 0$. \square .

1.4. (i)

$f: [0, T] \rightarrow \mathbb{R}$: RCLL function.

Assume f is unbounded to show contradiction.

Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$. s.t $x_n \rightarrow x \in [0, T]$ $f(x_n) \rightarrow \infty$ as $n \rightarrow \infty$

Split into two subsequences, $\{l_n\} = \{x_n | x_n < x\}$, $\{r_n\} = \{x_n | x_n > x\}$.

WLOG, assume $\{l_n\}, \{r_n\}$ are infinite sequences.

Then $l_n \rightarrow x^-$, $r_n \rightarrow x^+$, $f(l_n) = f(r_n) = \infty$.

This can not happen on RCLL function which have left limit right continuous.

Contradiction was shown. \square .

1.4. (ii).

$(X_t)_{t \geq 0}$: RCLL Gaussian process.

WTS $\sum_{i=1}^n \lambda_i \int_0^{t_i} X_s ds$: normal.

$\Rightarrow \int_0^T \sum_{i=1}^n \lambda_i X_s \mathbb{1}_{[0, t_i]} ds$, $T = \max(t_1, \dots, t_n)$.

RCLL Gaussian

If show that following Lemma, we are done.

Lemma. $(X_t)_{t \geq 0}$: RCLL Gaussian, $T > 0$

$\Rightarrow \int_0^T X_u du$ is normal.

Pf of Lemma).

Consider a partition $\{0 < \frac{T}{n}, \frac{2T}{n}, \dots, \frac{kT}{n}, \dots, T\}$,

an step function $X_t^{(n)}$

$$X_t^{(n)} = \begin{cases} X_{kT/n} & \text{for } \frac{(k-1)T}{n} \leq t < \frac{kT}{n} \\ X_T & \text{for } t = T \end{cases} \quad k = 1, \dots, n.$$

$$= \sum_{k=1}^n X_{kT/n} \mathbb{1}_{[\frac{(k-1)T}{n}, \frac{kT}{n})}(t) + X_T \mathbb{1}_{\{T\}}(t)$$

$X_t^{(n)} \rightarrow X_t$ a.s. as $n \rightarrow \infty$ for $\forall w \in \Omega$

Since X_t is bounded, $\exists C = C(\omega)$ s.t. $|X_u(\omega)| \leq C(\omega) \quad \forall u \in [0, T]$
 $\therefore \int_0^T X_u^{(n)} du \rightarrow \int_0^T X_u du$ (\because By DCT).

fact). $\{Y_n\}$: seq. of normal r.v's.

$Y_n \rightarrow Y$ converges in distribution.

$\Rightarrow Y$ is normal.

$\int_0^T X_u^{(n)} du = \frac{1}{n} \sum_{k=1}^n X_{kT/n}^{(n)}$ is normal ($\because X_t$ is Gaussian).

$\int_0^T X_u^{(n)} du \rightarrow \int_0^T X_u du$ a.s. $\Rightarrow \int_0^T X_u^{(n)} du \rightarrow \int_0^T X_u du$ (in distribution).

$\Rightarrow \int_0^T X_u du$: normal. \square

$$1.4. \text{ (iii). } \int_0^T u B_t du$$

$\sum_{i=1}^n \lambda_i t B_t$: normal. $\Rightarrow t B_t$: continuous Gaussian process $t \in [0, T]$

($t B_t$ defined on compact set $[0, T]$ is bounded.)

By (ii), $(\int_0^t u B_u du)_{t \geq 0}$ is Gaussian. $\Rightarrow \int_0^T u B_u du$: normal.

$$E \int_0^T u B_u du = \int_0^T E(u B_u) du \quad (\text{by Fubini's thm})$$

$$= 0. \quad (\because E(t B_t) = 0)$$

$$\text{Var}(\int_0^T u B_u du) = E((\int_0^T u B_u du)^2) = E(\int_0^T u B_u du \cdot \int_0^T v B_v dv) \quad (u \neq v)$$

$$= E(\int_0^T \int_0^T u v B_u B_v du dv)$$

$$= \int_0^T \int_0^T E(u v B_u B_v) du dv$$

$$= \int_0^T \int_0^T u v \cdot \min(u, v) du dv. \quad (\text{WLOG, } u < v)$$

$$= \int_0^T \int_0^T u v^2 du dv$$

$$= \frac{T^5}{6}$$

$$\therefore \int_0^T u B_u du \sim N(0, \frac{T^5}{6})$$

$$1.4. \text{ (iv). } \int_0^T u B_u du = Y \sim N(0, \frac{T^5}{6}) \quad \text{by (iii)}$$

$$E(e^{\int_0^T u B_u du}) = e^{E \int_0^T u B_u du + \frac{1}{2} \text{Var}(\int_0^T u B_u du)} = e^{\frac{T^5}{12}} \quad (\phi_Y(-i) = e^{E(Y) + \frac{1}{2} \text{Var}(Y)})$$

$$1.5. \text{ (i)}$$

$$B^H = (B_t^H)_{t \geq 0} \quad \text{continuous Gaussian process.}$$

$$E(B_t^H) = 0,$$

$$\text{Cov}(B_s^H, B_t^H) = \frac{1}{2} (s^{2H} + t^{2H} - |t-s|^{2H})$$

$\Rightarrow B^H$ is fractional Brownian motion with parameter H .

WTS : $B^{\frac{1}{2}}$ is standard B.M.

$$\text{ETS : } \text{Cov}(B_s^H, B_t^H) = \min(s, t).$$

(\because already B^H : conti, $E(B^H) = 0$, B^H : Gaussian process).

$$\text{Cov}(B_s^{\frac{1}{2}}, B_t^{\frac{1}{2}}) = \frac{1}{2} (s + t - |t-s|) = \begin{cases} s & \text{if } t > s \\ t & \text{if } t \leq s \end{cases}$$

$$= \min(s, t). \quad \square$$

1.5. (ii).

$$X_t = B_{t+h}^H - B_h^H$$

\mathbb{D} (Gaussian)

$$\sum_{i=1}^n \lambda_i X_{ti} = \lambda_1 B_{t+h}^H + \dots + \lambda_n B_h^H - (\sum_{i=1}^n \lambda_i) B_h^H : \text{normal}$$

($\because B^H$ is Gaussian).

$$\textcircled{2} E(X_t) = E(B_{t+h}^H) - E(B_h^H) = 0.$$

$$\textcircled{3} \text{ COV}(X_s, X_t) = \text{COV}(B_{s+h}^H - B_h^H, B_{t+h}^H - B_h^H)$$

$$= E(B_{s+h}^H B_{t+h}^H) - E(B_{s+h}^H B_h^H) - E(B_{t+h}^H B_h^H) + E(B_h^H B_h^H)$$

$$= \frac{1}{2} ((s+h)^{2H} + (t+h)^{2H} - |(t+h) - (s+h)|^{2H}) - \frac{1}{2} ((s+h)^{2H} + h^{2H} - |h - (s+h)|^{2H}) \\ - \frac{1}{2} ((t+h)^{2H} + h^{2H} - |h - (t+h)|^{2H}) + \frac{1}{2} (2 \cdot h^{2H})$$

$$= \frac{1}{2} (s^{2H} + t^{2H} - |t-s|^{2H}).$$

$\textcircled{4}$ Obviously continuous.

Thus X is a fractional B.M with parameter H .

1.5. (iii).

$$B_t^H - B_s^H : \text{normal. } (\because B^H \text{ is Gaussian}).$$

B_{t-s}^H is normal

$$E(B_t^H - B_s^H) = E(B_{t-s}^H) = 0.$$

$$\begin{aligned} \text{Var}(B_t^H - B_s^H) &= E((B_t^H - B_s^H)^2) = \text{COV}(B_t^H - B_s^H, B_t^H - B_s^H) \\ &= \text{COV}(B_t^H, B_t^H) - 2\text{COV}(B_t^H, B_s^H) + \text{COV}(B_s^H) \\ &= \frac{1}{2} (2t^{2H}) - 2 \cdot \frac{1}{2} (t^{2H} + s^{2H} - |s-t|^{2H}) + \frac{1}{2} (2s^{2H}) \\ &= |s-t|^{2H} = \end{aligned}$$

$$\begin{aligned} \text{Var}(B_{t-s}^H) &= E(B_{t-s}^H \cdot B_{t-s}^H) = \text{COV}(B_{t-s}^H, B_{t-s}^H) \\ &= (t-s)^{2H} \end{aligned}$$

$$\text{Var}(B_t^H - B_s^H) = B(B_{t-s}^H)$$

$$\therefore B_t^H - B_s^H \sim N(0, (t-s)^{2H}) \sim B_{t-s}^H$$

$\therefore B_t^H - B_s^H$ has same distribution with B_{t-s}^H

1.4. (iv). $0 \leq u \leq s \leq t$

$$\begin{aligned} E(B_u^H (B_t^H - B_s^H)) &= E(B_u^H B_{t-s}^H) \quad \text{by (iii).} \\ &= \text{Cov}(B_u^H, B_{t-s}^H) = \frac{1}{2}(u^{2H} + (t-s)^{2H} - |t-s-u|^{2H}). \end{aligned}$$

For $H_0 \in (0, 1)$ s.t. $E(B_u^{H_0} (B_t^{H_0} - B_s^{H_0})) = 0$,
 $B_t^{H_0} - B_s^{H_0} \perp \mathcal{G}(B_u^{H_0})$.

1.5. (v).

$$(\int_0^t B_u^H \mathbf{1}_{[1,2)}(u) - 2B_u^H \mathbf{1}_{[3,\infty)}(u) du)_{t \geq 0}.$$

WTS : $\sum_{i=1}^n \lambda_i (\int_0^{t_i} B_u^H \mathbf{1}_{[1,2)}(u) - 2B_u^H \mathbf{1}_{[3,\infty)}(u) du)$ is normal.

$$\Rightarrow \int_0^T \sum_{i=1}^n \lambda_i (B_u^H \mathbf{1}_{[1,2)}(u) \cdot \mathbf{1}_{[0,t_i)}(u) - 2B_u^H \mathbf{1}_{[3,\infty)}(u) \cdot \mathbf{1}_{[0,t_i)}(u)) du$$

where $T = \max\{t_1, t_2, \dots, t_n\}$

$$B_u^H \mathbf{1}_{[1,2)}(u) \cdot \mathbf{1}_{[0,t_i)}(u) - 2B_u^H \mathbf{1}_{[3,\infty)}(u) \cdot \mathbf{1}_{[0,t_i)}(u)$$

: a RCLL Gaussian process. ($\because B_u^H$ is a conti. Gaussian).

$$\Rightarrow \sum_{i=1}^n \lambda_i (B_u^H \mathbf{1}_{[1,2)}(u) \cdot \mathbf{1}_{[0,t_i)}(u) - 2B_u^H \mathbf{1}_{[3,\infty)}(u) \cdot \mathbf{1}_{[0,t_i)}(u)) : \text{RCLL Gaussian}.$$

By 1.4. (ii),

$$\int_0^T \sum_{i=1}^n \lambda_i (B_u^H \mathbf{1}_{[1,2)}(u) \cdot \mathbf{1}_{[0,t_i)}(u) - 2B_u^H \mathbf{1}_{[3,\infty)}(u) \cdot \mathbf{1}_{[0,t_i)}(u)) du \text{ is Gaussian } \square.$$