

Topology 2

Notes taken by Junwoo Yang

Based on a lecture by Youngsik Huh in fall 2021

Contents

0	Introduction	2
0.1	Quotient topology	2
0.2	What is algebraic topology?	2
0.3	Fundamental group	3
9	Fundamental group	5
9.51	Homotopy of paths	5
9.52	Fundamental group	6
9.53	Covering spaces	8
9.54	Fundamental group of the circle	9
10	Separation theorems in the plane	10
10.63	Jordan curve theorem	10
11	Seifert–Van Kampen theorem	11
11.70	The Seifert–Van Kampen theorem	12
12	Classification of surfaces	15
13	Classification of covering spaces	16

Chapter 0

Introduction

0.1 Quotient topology

Definition 1 (Equivalence relation). An equivalence relation is a relation $x \sim y$ so that $x \sim x$; if $x \sim y$ then $y \sim x$; and if $x \sim y$ and $y \sim z$, then $x \sim z$. Given an equivalence relation defined on X , X/\sim is the set of equivalence classes.

Definition 2 (Quotient topology). Let $f: X \rightarrow Y$ be a surjective map from the topological space X to the set Y . Then, we define a topology on Y , called the quotient topology, by requiring that $O \subset Y$ be open if and only if $f^{-1}(O)$ is actually an open set of X . One checks trivially that this defines a topology on Y .

Example. Let X be the closed unit ball, $\{(x, y) : x^2 + y^2 \leq 1\}$, in \mathbb{R}^2 and X^* be the partition of X consisting of all the one-point sets $\{(x, y)\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Then X^* is homeomorphic with the subspace of \mathbb{R}^3 called the unit 2-sphere.

0.2 What is algebraic topology?

Functor from category of topological spaces to the category of groups.

- Category: set of spaces and morphisms.
- Functor: $X \rightsquigarrow G_X$ and $f: X \rightarrow Y \rightsquigarrow f_*: G_X \rightarrow G_Y$ such that
 - $(f \circ g)_* = f_* \circ g_*$
 - $(1_X)_* = 1_{G_X}$

Two systems we'll discuss:

- fundamental groups
- homology groups

Example. Suppose we have a functor. If $G_X \not\cong G_Y$, then X and Y are not homeomorphic. If ‘shadows’ are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f: X \rightarrow Y$ and $g: Y \rightarrow X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_*: G_X \rightarrow G_Y$ and $g_*: G_Y \rightarrow G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_*: G_X \rightarrow G_Y$ is an isomorphism. \diamond

0.3 Fundamental group

Pick a base point x_0 and consider it fixed. (The fundamental group will not depend on it. We assume all spaces are path connected) $X \rightsquigarrow \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f: I = [0, 1] \rightarrow X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

Example. Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the ‘constant’ loop.

Example. Let $X = S^1$ and pick x_0 on the circle. Two options:

- The loop is trivial equivalent to the constant loop
- The loop goes around the circle.
- The loop goes around the circle, twice.
- The loop goes around the circle, clockwise, once
- ...

$\pi(S^1) \cong \mathbb{Z}$ (proof will follow)

The composition of loops is simply pasting them. In the case of the circle, the loop $-1 \circ$ the loop 2 is the loop 1 .

Suppose $\alpha: I \rightarrow X$ and $f: X \rightarrow Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

Theorem 1 (Fixed point theorem of Brouwer). Any continuous map from a rectangle to itself has at least one fixed point.

Proof. Suppose there is no fixed point, so $f(x) \neq x$ for all $x \in B^2$. Then we can construct map $r: B^2 \rightarrow S^1$ as follows: take the intersection of the boundary and half ray between $f(x)$ and x . If x lies on the boundary, we have the identity map. This map is continuous. Then we have $S^1 \rightarrow B^2 \rightarrow S^1$, via the inclusion and r . Looking at the fundamental groups:

$$\pi(S^1) = \mathbb{Z} \rightarrow \pi(B^2) = 1 \rightarrow \pi(S^1) = \mathbb{Z}.$$

The map from $\pi(S^1) \rightarrow \pi(S^1)$ is the identity map, but the first map maps everything on 1. \square

Chapter 9

Fundamental group

See wikipedia¹ for a brief introduction.

9.51 Homotopy of paths

Definition 3 (Homotopic). If f and f' are continuous maps of the space X into the space Y , we say that f is homotopic to f' if there is a continuous map $F: X \times I \rightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each x . (Here $I = [0, 1]$.) The map F is called a homotopy between f and f' . If f is homotopic to f' , we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is nullhomotopic.

Definition 4 (Path homotopy). Let $f, g: I \rightarrow X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H: I \times I \rightarrow X$ is a path homotopy between f and g , if and only if

- $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ (homotopy between maps)
- $H(0, t) = x_0$ and $H(1, t) = x_1$ (start and end points fixed)

Notation: $f \simeq_p g$.

Lemma 1. \simeq and \simeq_p are equivalence relations.

Proof. • Reflective: $F(x, t) = f(x)$

- Symmetric: $G(x, t) = H(x, 1 - t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x, t) = \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

□

¹<https://en.wikipedia.org/wiki/Homotopy>

Example (Trivial, but important). Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g: X \rightarrow C$ are homotopic.
- Any two paths $f, g: I \rightarrow C$ with $f(0) = g(0)$ and $f(1) = g(1)$ are path homotopic.

Choose $H: X \times I \rightarrow C$ defined by $(x, t) \mapsto H(x, t) = (1 - t)f(x) + tg(x)$.

Product of paths

Let $f: I \rightarrow X$, $g: I \rightarrow X$ be paths, $f(1) = g(0)$. Define

$$f * g: I \rightarrow X \text{ given by } s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g: I \rightarrow X \mid g \simeq_p f\}$.

- Theorem 2.**
1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
 2. Let e_x denote the constant path $e_x: I \rightarrow X$ given by $s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
 3. Let $\bar{f}: I \rightarrow X$ given by $s \mapsto f(1 - s)$. Then $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

9.52 Fundamental group

Definition 5. Let X be a space and $x_0 \in X$, then the fundamental group of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f: I \rightarrow X, f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\bar{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

Theorem 3 (52.1). Let X be a space, $x_0, x_1 \in X$ and $\alpha: I \rightarrow X$ a path from x_0 to x_1 . Then

$$\begin{aligned}\hat{\alpha}: \pi(X, x_0) &\longrightarrow \pi(X, x_1) \\ [f] &\longmapsto [\bar{\alpha}] * [f] * [\alpha].\end{aligned}$$

is an isomorphism of groups. Note however that this isomorphism depends on α .

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * [f] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}[f] * \hat{\alpha}[g].\end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic. \square

Remark. If $f: (X, x_0) \rightarrow (Y, y_0)$ is a map of pointed topological spaces ($f: X \rightarrow Y$ continuous and $f(x_0) = y_0$). Then

$$f_*: \pi(X, x_0) \rightarrow \pi(Y, y_0) \text{ given by } [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two ‘rules’ discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \quad (1_X)_* = 1_{\pi(X, x_0)}.$$

Definition 6. Let X be a topological space, then X is simply connected iff X is path connected and $\pi_1(X, x_0) = 1$ for some $x_0 \in X$.

Remark. If trivial for one base point, it’s trivial for all base points.

Example. Any convex subset $C \subset \mathbb{R}^n$ is simply connected.

Lemma 2. Suppose X is simply connected and $\alpha, \beta: I \rightarrow X$ two paths with same start and end points. Then $\alpha \simeq_p \beta$.

Proof. Simply connected implies loops are homotopic? Consider $\alpha * \bar{\beta} \simeq_p e_{x_0}$, since the space is simply connected.

$$\begin{aligned}([\alpha] * [\bar{\beta}]) * [\beta] &= [e_{x_0}] * [\beta] = [\beta] \\ [\alpha] * ([\bar{\beta}] * [\beta]) &= [\alpha] * [e_{x_0}] = [\alpha].\end{aligned}$$

And therefore $\alpha \simeq_p \beta$. (Note: make sure end and start point matches when using $*$) \square

9.53 Covering spaces

Definition 7 (Evenly covered). Let $p: E \rightarrow B$, surjective map (so continuous). Let $U \subset B$ open. Then U is evenly covered iff $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$ with

- V_α open in E
- $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$
- $p|_{V_\alpha}: V_\alpha \rightarrow U$ is a homeomorphism.

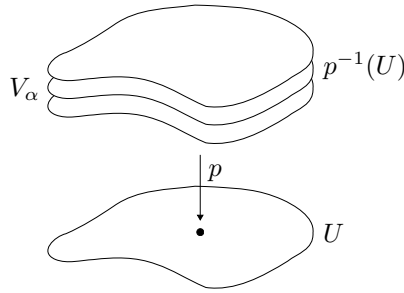


Figure 9.1: Evenly covered

Remark. If $U' \subset U$, also open and U is evenly covered, then also U' .

Definition 8. Let $p: E \rightarrow B$ be a surjective map. Then p is a covering projection iff $\forall b \in B, \exists U \subset B$ open, containing b such that U is evenly covered by p . Then (E, p) is called a covering space.

Example. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Take $p: \mathbb{R} \rightarrow S^1$ given by $t \mapsto e^{2\pi it}$. Note that \mathbb{R} is an easier space than S^1 , and so will be π_1 (1 vs \mathbb{Z}).

Proposition 1. A covering map is always a open map.

Proof. Exercise. □

Proposition 2. For any $b \in B$, $p^{-1}(b)$ is a discrete subset of E . (No accumulation point)

Proof. Indeed for any $\alpha \in I$, $V_\alpha \cap p^{-1}(b)$ is exactly one point. □

Remark. A covering is always local homeomorphism. But there are surjective local homeomorphism which are not covering maps. A covering map is more than a surjective local homeomorphism.

For example, $p: \mathbb{R}_0^+ \rightarrow S^1$ given by $t \mapsto e^{2\pi it}$. Consider the inverse image of a neighborhood around 1. When we restrict p to the part around 0, it is no longer a homeomorphism (we don't get the whole neighborhood around one).

Creating new covering spaces out of old ones

- Suppose $p: E \rightarrow B$ is a covering and $B_0 \subset B$ is a subspace with the subspace topology. Let $E_0 = p^{-1}(B_0)$ and $p_0 = p|_{E_0}$. Then (E_0, p_0) is a covering of B_0 .
- Suppose that (E, p) is a covering of B and (E', p') is a covering of B' , then $(E \times E', p \times p')$ is a covering of $B \times B'$.

Example. Let $T^2 = S^1 \times S^1$.

- $p: \mathbb{R}^2 \rightarrow S^1 \times S^1$ given by $(t, s) \mapsto (e^{ait}, e^{bis})$.
- $p': \mathbb{R} \times S^1 \rightarrow T^2$ given by $(t, z) \mapsto (e^{ait}, z^n)$.
- $p: S^1 \times S^1 \rightarrow T^2$ given by $(z_1, z_2) \mapsto (z_1^n, z_2^m)$.

These are the only types of coverings of the torus. We'll prove this later on.

9.54 Fundamental group of the circle

Given f , when can f be 'lifted' to E ? I.e. when does there exist an $\tilde{f}: X \rightarrow E$ such that $p \circ \tilde{f} = f$? In this section, we'll only consider $X = [0, 1]$, $X = [0, 1]^2$.

Definition 9. Let $p: E \rightarrow B$ be a map. If f is a continuous mapping of some space X into B , a lifting of f is a map $\tilde{f}: X \rightarrow E$ such that $p \circ \tilde{f} = f$.

$$\begin{array}{ccc} & & E \\ & \nearrow \tilde{f} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

Chapter 10

Separation theorems in the plane

10.63 Jordan curve theorem

https://en.wikipedia.org/wiki/Jordan_curve_theorem

Chapter 11

Seifert–Van Kampen theorem

https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem

Note. This doesn't follow the book very well.

Definition 10. A free group on a set X consists of a group F_X and a map: $i: X \rightarrow F_X$ such that the following holds: For any group G and any map $f: X \rightarrow G$, there exists a unique morphism of groups $\phi: F_X \rightarrow G$ such that

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow f & \downarrow \exists! \phi \\ & & G \end{array} .$$

Note. The free group of a set is unique. Suppose $i: X \rightarrow F_X$ and $j: X \rightarrow F'_X$ are free groups.

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow j & \downarrow \exists \phi \\ & & F'_X \end{array} \quad \begin{array}{ccc} X & \xrightarrow{j} & F'_X \\ & \searrow i & \downarrow \exists \psi \\ & & F_X \end{array} .$$

Then

$$\begin{array}{ccc} X & \xrightarrow{i} & F_X \\ & \searrow i & \downarrow \psi \circ \phi \\ & & F_X \end{array} .$$

Then by uniqueness, $\psi \circ \phi$ is 1_{F_X} , and likewise for $\phi \circ \psi$.

Note. The free group on a set always exists. You can construct it using “irreducible words”.

Example. Consider $X = \{a, b\}$. An example of a word is $aaba^{-1}baa^{-1}bbb^{-1}a$. This is not a irreducible word. The reduced form is $aaba^{-1}bba = a^2ba^{-1}b^2a$. Then F_X is the set of irreducible words.

Example. If $X = \{a\}$, then $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. Exercise: check that \mathbb{Z} satisfies the universal property.

Example. If $X = \emptyset$, then $F_X = 1$.

Definition 11 (Free product of a collection of groups). Let G_i with $i \in I$, be a set of groups. Then the free product of these groups denoted by $*_{i \in I} G_i$ is a group G together with morphisms $j_i: G_i \rightarrow G$ such that the following universal property holds: Given any group H and a collection of morphisms $f_i: G_i \rightarrow H$, then there exists a unique morphism $f: G \rightarrow H$, such that for all $i \in I$, the following diagram commutes:

$$\begin{array}{ccc} G_i & \xrightarrow{j_i} & G \\ & \searrow f_i & \downarrow \exists! f \\ & & H \end{array} .$$

Note. Again, $*_{i \in I} G_i$ is unique.

11.70 The Seifert–Van Kampen theorem

Theorem 4 (70.1, Seifert–Van Kampen theorem). Let $X = U \cup V$ where $U, V, U \cap V$ are open and path connected.^a Let $x_0 \in U \cap V$. For any group H and 2 morphisms $\Phi_1: \pi(U, x_0) \rightarrow H$ and $\Phi_2: \pi(V, x_0) \rightarrow H$ such that $\Phi_1 \circ i_1$ and $\Phi_2 \circ i_2$, there exists exactly one $\Phi: \pi(X, x_0) \rightarrow H$ making the diagram commute

$$\begin{array}{ccccc} & & \pi(U, x_0) & & \\ & \nearrow i_1 & \downarrow j_1 & \searrow \Phi_1 & \\ \pi(U \cap V, x_0) & \xrightarrow{i} & \pi(x, x_0) & \xrightarrow{\Phi} & H \\ & \searrow i_2 & \uparrow j_2 & \nearrow \Phi_2 & \\ & & \pi(V, x_0) & & \end{array} .$$

i_1, i_2, i, j_1, j_2 are induced by inclusions.

^aNote that U, V should also be path connected!

Theorem 5 (70.2, Seifert–Van Kampen theorem (classical version)). Assume the hypotheses of the Theorem 4. Let $j: \pi(U, x_0) * \pi(V, x_0) \rightarrow \pi(X, x_0)$ (induced by j_1 and j_2). On elements of $\pi(U, x_0)$ it acts like j_1 , on elements of $\pi(V, x_0)$ it acts like j_2 .

$$\begin{array}{ccc}
 G_1 & & \\
 \downarrow & \searrow f_1 & \\
 G_1 * G_2 & \xrightarrow{f} & H \\
 \uparrow & \nearrow f_2 & \\
 G_2 & &
 \end{array}$$

Then j is surjective^a and the kernel of j is the normal subgroup of $\pi(U, x_0) * \pi(V, x_0)$ generated by all elements of the form $i_1(g)^{-1}i_2(g)$, where $g \in \pi(U \cap V, x_0)$.

^aThis is the only place where algebraic topology is used. We've proved this last week. The groups U and V generate the whole group. The rest of this theorem follows from the previous theorem.

Proof. • j is surjective. (later)

- Let N be the normal subgroup generated by $i_1(g)^{-1}i_2(g)$. Then we claim that $N \subset \ker(j)$. This means we have to show that $i_1(g)^{-1}i_2(g) \in \ker j$. $j(i_1(g)) = j_1(i_1(g))$ by definition of j . Looking at the diagram, we find that $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$. Therefore $j(i_1(g)^{-1}i_2(g)) = 1$, which proves that elements of the form $i_1(g)^{-1}i_2(g)$ are in the kernel.
- Since $N \subset \ker j$, there is an induced morphism

$$\begin{aligned}
 k: (\pi_1(U, x_0) * \pi_1(V, x_0))/N &\longrightarrow \pi_1(X, x_0) \\
 gN &\longmapsto j(g).
 \end{aligned}$$

To prove that $N = \ker j$, we have to show that k is injective. Because this would mean that we've divided out the whole kernel of j .

Now we're ready to use the previous theorem. Let $H = (\pi(U) * \pi(V))/N$. Repeating the diagram:

$$\begin{array}{ccccc}
 & & \pi(U, x_0) & & \\
 & \nearrow i_1 & \downarrow j_1 & \searrow \Phi_1 & \\
 \pi(U \cap V, x_0) & \xrightarrow{i} & \pi(x, x_0) & \xleftarrow[\Phi]{k} & H \\
 & \searrow i_2 & \uparrow j_2 & \nearrow \Phi_2 & \\
 & & \pi(V, x_0) & &
 \end{array}$$

Now, we define $\Phi_1: \pi(U, x_0) \rightarrow H$ given by $g \mapsto gN$, and $\Phi_2: \pi(V, x_0) \rightarrow H$ given by $g \mapsto gN$. For the theorem to work, we needed that $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$. This is indeed the case: let

$g \in \pi(U \cap V)$. Then $\Phi_1(i_1(g)) = i_1(g)N$ and $\Phi_2(i_2(g)) = i_2(g)N$ and $i_1(g)N = i_2(g)N$ because $i_1(g)^{-1}i_2(g) \in N$.

The conditions of the previous theorem are satisfied, so there exists a Φ such that the diagram commutes.

Note that we also have $k: H \rightarrow \pi(X)$. We claim that $\Phi \circ k = 1_H$, which would mean that k is injective, concluding the proof. It's enough to prove that

□

Corollary 5.1. Suppose $U \cap V$ is simply connected, so $\pi_1(U \cap V, x_0)$ is the trivial group. In this case $N = \ker j = 1$, hence $\pi(U, x_0) * \pi(V, x_0) \rightarrow \pi(X, x_0)$ is an isomorphism.

Corollary 5.2. Suppose U is simply connected. Then $\pi(X, x_0) \cong \pi(V, x_0)/N$ where N is the normal subgroup generated by the image of $i_2: \pi(U \cap V) \rightarrow \pi(V, x_0)$.

Example. Let X be the figure 8 space.

Chapter 12

Classification of surfaces

Chapter 13

Classification of covering spaces

Lemma 3 (79.1, General lifting lemma). Let $p: E \rightarrow B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.^a Let $f: Y \rightarrow B$, $y_0 \in Y$, $b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f}: Y \rightarrow E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{array}{ccc} & (E, e_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (Y, y_0) & \xrightarrow{f} & (B, b_0) \end{array}$$

iff $f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$. If \tilde{f} exists then it is unique.

^aFrom now on, all spaces are locally path connected: Every neighborhood contains an open that is path connected.

Example. Take $Y = [0, 1]$. Then f is a path, then we showed that every map can be lifted. And indeed, the condition holds: $f_*(\pi(Y, y_0)) = 1$, the trivial group, which is a subgroup of all groups.

Proof. Suppose \tilde{f} exists. Then $p \circ \tilde{f} = f$, so $(p \circ \tilde{f})_*\pi(Y, y_0) = \pi(Y, y_0)$. The left hand side is of course $p_*(\tilde{f}_*(\pi(Y, y_0))) \subset p_*(\pi(E, e_0))$, so $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$.

Conversely, we'll show the uniqueness first. Suppose \tilde{f} exists. $p \circ (\tilde{f} \circ \alpha) = f \circ \alpha$, so $\tilde{f} \circ \alpha$ is the unique lift of $f \circ \alpha$ starting at e_0 . Hence $\tilde{f}(y)$ the endpoint of the unique lift of $f \circ \alpha$ to E starting at e_0 . This also shows how you can define \tilde{f} : choose a path α from y_0 to y . Lift $f \circ \alpha$ to a path starting at e_0 . Define $\tilde{f}(y) =$ the end point of this lift. Problem: is this well defined? A second problem: is \tilde{f} continuous?

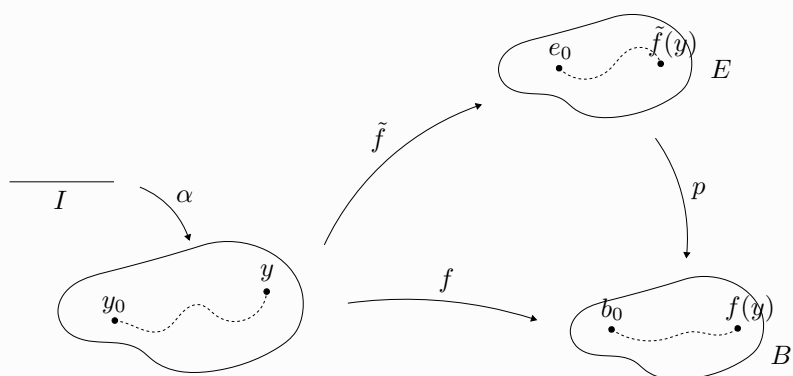


Figure 13.1: General lifting lemma

◇