Topology II – Homework 4

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Lemma 54.2 Prove that $I \times I$ can be divided into subrectangles so that $F(I_i \times J_j)$ is contained in an evenly-covered open subset of B.

Proof. Let $\{U_\alpha\}$ be an open covering of B consisting of evenly-covered open subsets. Then, $\{F^{-1}(U_\alpha)\}$ is an open covering of the compact space $I \times I$. By Lebesgue number lemma, there exists $\varepsilon > 0$ s.t. for any point of $I \times I$, open ball with the radius ε is contained in some $F^{-1}(U_\alpha)$. Then we have a subrectangles $I_i \times J_j$ so that $F(I_i \times J_j) \subset U_\alpha$ for some α by setting $I_i - I_{i-1}$ and $J_j - J_{j-1}$ smaller than $\frac{\varepsilon}{2}$. $\square$

Exercise 54.3 Let $p\colon E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

Proof. If $t \le \frac{1}{2}$, then $p \circ (\tilde{\alpha} * \tilde{\beta})(t) = (p \circ \tilde{\alpha})(2t) = \alpha(2t)$, similarly for $t \ge \frac{1}{2}$, $p \circ (\tilde{\alpha} * \tilde{\beta})(t) = (p \circ \tilde{\beta})(2t-1) = \beta(2t-1)$. Thus, $p \circ (\tilde{\alpha} * \tilde{\beta}) = \alpha * \beta$. $\square$

Exercise 54.5 Consider the covering map $p \times p\colon \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ of Example 4 of §53. Consider the path

$$f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$$

in $S^1 \times S^1$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\tilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$, and sketch it.

Proof. Since $(p \times p)^{-1}(f(0)) = \mathbb{Z} \times \mathbb{Z}$, for each $z, z' \in \mathbb{Z}$, $\tilde{f}(t) = (z + t, z' + 2t)$. Note that $\tilde{f}$ sketched in Figure 1 is an unique path for each base point (z, z'). $\square$

Exercise 54.6 Consider the maps $g, h\colon S^1 \to S^1$ given $g(z) = z^n$ and $h(z) = 1/z^n$. (here we represent S^1 as the set of complex numbers z of absolute value 1.) Compute the induced homomorphisms g_*, h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself. [Hint: Recall the equation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.]

Proof. The group is cyclic, a generator is $[f]$, where $f(t) = (\cos 2\pi t, \sin 2\pi t)$, and $g_*([f]) = [g \circ f]$ where $g \circ f(t) = (\cos 2\pi nt, \sin 2\pi nt)$, i.e.

$$g \circ f = \underbrace{f * \cdots * f}_{n} \text{ and } g_*([f]) = \underbrace{[f] * \cdots * [f]}_{n}.$$

If we consider isomorphism i between $\pi_1(S^1, b_0)$ and $(\mathbb{Z}, +)$, then corresponding $i \circ g_*(z) = nz$. Similarly, $h_*([f]) = [\overline{f}] * \cdots * [\overline{f}]$, and $i \circ h_*(z) = -nz$. $\square$

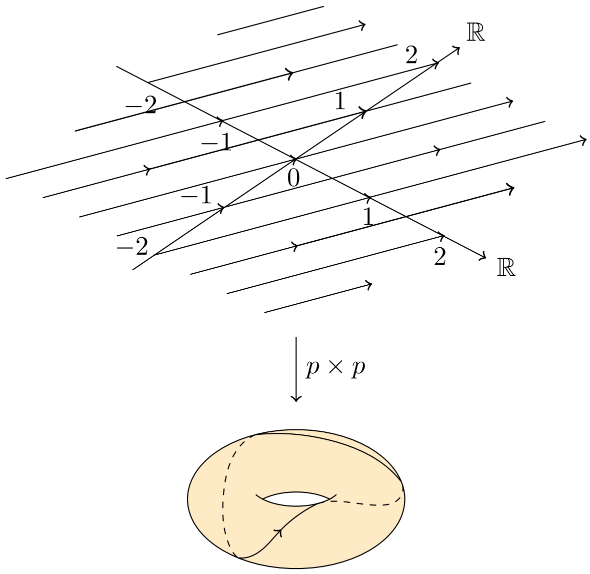

Figure 1: A path in the torus and its liftings

Exercise 54.8 Let $p\colon E \to B$ be a covering map, with E path connected. Show that if B is simply connected, then p is a homeomorphism.

Proof. Assume $p\colon E \to B$ is a covering map, with E path connected and B simply connected. Consider $e_0 \in E$, and $b_0 = p(e_0) \in B$. We claim that $p^{-1}(b_0) = \{e_0\}$. Indeed, let $e \in p^{-1}(b_0)$. Since E is path connected, there exists a path $\tilde{f}\colon I \to E$ from e_0 to e. The projection $f = p \circ \tilde{f}\colon I \to B$ is a path in B from b_0 to itself (since $p(e) = p(e_0) = b_0$), whose lift to E beginning at e_0 is $\tilde{f}$.

Since B is simply connected, the loop f at b_0 is path-homotopic to the constant loop at b_0, by a path homotopy $F\colon I \times I \to B$. By Lemma 54.2, F admits a unique continuous lift $\tilde{F}\colon I \times I \to E$ with $\tilde{F}(0,0) = e_0$; this left is a path homotopy in E from $\tilde{f}$ (the lift of f beginning at e_0) to the constant path at e_0 (the lift of the constant path). It follows in particular that $\tilde{f}$ and the constant path at e_0 have the same end points, so $e = e_0$.

Having shown that $p^{-1}(b_0)$ consists of a single point, we can conclude that E is a 1-fold covering of B by Exercise 53.3 (we proved in class), i.e. $p^{-1}(b)$ has a single element for each $b \in B$. Thus $p\colon E \to B$ is a bijection, and a local homeomorphism. It is therefore a homoemorphism. $\square$

Exercise 55.1 Show that if A is a retract of B^2, then every continuous map $f\colon A \to A$ has a fixed point.

Proof. Let $A \subset B^2$ be a retract, with $r\colon B^2 \to A$ the retraction, and $i\colon A \to B^2$ the inclusion. Given a continuous map $f\colon A \to A$, the composition $F = i \circ f \circ r$ is a continuous map from B^2 to itself, with $F(B^2) \subset A$, and $F|_A = f$. By the Brouwer fixed point theorem, F has a fixed point, i.e. there exists $x \in B^2$ such that $F(x) = x$. By the way, since F takes values in A, this implies that in fact $x \in A$, and then $f(x) = F(x) = x$. So x is a fixed point of f. $\square$

Exercise 55.2 Show that if $h\colon S^1 \to S^1$ is null-homotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. If $h\colon S^1 \to S^1$ is null-homotopic, then by Lemma 55.3 it extends to a continuous map $k\colon B^2 \to S^1$. The composition of k with the inclusion i of S^1 into B^2 is a continuous map from B^2 to itself, so by the Brouwer fixed point theorem, there exists $x \in B^2$ such that $i(k(x)) = x$. However, this equality implies that $x \in S^1$, so in fact there exists $x \in S^1$ such that $k(x) = h(x) = x$, and we conclude that h has a fixed point.

Denote by $\alpha\colon S^1 \to S^1$ the antipodal map $\alpha(x) = -x$. If $h\colon S^1 \to S^1$ is null-homotopic, then so is $\alpha \circ h$ (namely, if H is a homotopy from h to a constant map then $\alpha \circ H$ is a homotopy from $\alpha \circ h$ to a constant map). By the previous result, $\alpha \circ h$ has a fixed point, so there exits $x \in S^1$ such that $\alpha(h(x)) = x$, i.e. $h(x) = -x$. $\square$