Differential Geometry 1 – Final Exam

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June 10, 2020

1. Show that the set $S := \{(x, y, z) \in \mathbb{Z}^3 \mid x^2 + y^2 - z^2 = 0\}$ is not a regular surface.

2. Let S be a compact regular surface. Assume that there is a differentiable function $f: S \to \mathbb{R}$ with at most three critical points. Prove that S is connected.

3. Let $f: S^2 \to (0, +\infty)$ be a positive differentiable function on the unit sphere. Let

$$S_f := \{f(p)p = (f(p)x, f(p)y, f(p)z) \in \mathbb{R}^3 \mid p = (x, y, z) \in S^2\}.$$

   (a) Show that S_f is a regular surface.

   (b) Show that the map $\phi: S^2 \to S_f$ given by $\phi(p) := f(p)p$ is a diffeomorphism.

4. Let S be a regular surface. For a fixed point $p_0 \in \mathbb{R}^3$, let

$$f: S \to \mathbb{R}, \quad f(p) := |p - p_0|^2.$$

   Show that p is a critical point of f if and only if p_0 belongs to the normal line of S at p.

5. Let S be a regular surface given by the graph of a differentiable function $z = f(x, y)$. Let R be a bounded region of S. Show that the area of R is

$$\operatorname{area}(R) = \int_{\pi(Q)} \sqrt{1 + (f_x)^2 + (f_y)^2}\, \mathrm{d}x\, \mathrm{d}y$$

   where $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ given by $\pi(x, y, z) := (x, y)$, and $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.

6. Let S be a regular oriented surface. Show that the mean curvature H at $p \in S$ is equal to

$$H = \frac{1}{\pi} \int_0^{\pi} k_n(\theta)\, \mathrm{d}\theta$$

   where $k_n(\theta)$ denotes the normal curvature at p along a direction making an angle $\theta \in [0, \pi]$ with a fixed direction.

7. Consider the parametrized surface

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

   (a) Compute the coefficients of the first fundamental form.

   (b) Compute the coefficients of the second fundamental form.

   (c) Show that the principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

   (d) Compute the Gaussian curvature K and the mean curvature H at every point.

8. Let S be a regular surface.

   (a) Let $\alpha\colon [-1,1] \to S$ be a geodesic with $|\alpha'(0)| = 1$. Compute that arc length of α.

   (b) Let $\beta\colon [-2,2] \to S$ be another geodesic with $\beta(0) = \alpha(0)$ and $-2\beta'(0) = \alpha'(0)$. Compute the arc length of β and describe how α and β are related.

9. Suppose that S is a regular, compact, connected, orientable surface.

   (a) At any $p \in S$, locally S is the graph of some differentiable function h defined in a neighborhood of 0 in the tangent plane T_pS. ($0 \in T_pS$ is identified with $p \in S$.) Show that the second fundamental form at p equals the Hessian of h at $0 \in T_pS$.

   (b) Show that there is a point $p \in S$ with positive Gaussian curvature $K(p) > 0$.

   (c) Show that if S is not homeomorphic to S^2, then there are points on S where the Gaussian curvature is zero and negative.

10. Let S be a compact regular oriented surface. Prove that the Gauss map $N\colon S \to S^2$ is a local diffeomorphism if and only if S has positive Gaussian curvature everywhere.

11. Let $\alpha\colon [0,1] \to S$ be a differentiable curve.

   (a) Let $P_\alpha\colon T_{\alpha(0)}S \to T_{\alpha(1)}S$ be the parallel transport map along α. Show that P_α is a linear isometry.

   (b) Show that there exist two differentiable vector fields

   $$w_1, w_2\colon [0,1] \to \bigcup_{t\in[0,1]} T_{\alpha(t)}S, \quad w_1(t), w_2(t) \in T_{\alpha(t)}S$$

   along α which form an orthonormal basis of $T_{\alpha(t)}S$ for all $t \in [0,1]$, i.e..

   $$|w_1(t)| = |w_2(t)| = 1, \quad \langle w_1(t), w_2(t)\rangle = 0 \quad \forall t \in [0,1].$$

   (c) Let w be a differentiable vector field along α. Let $P_\alpha^{t_0,t_1} : T_{\alpha(t_0)}S \to T_{\alpha(t_1)}S$ be the parallel transport map along $\alpha|_{[t_0,t_1]}$ for $t_0, t_1 \in (0,1)$. Prove that

   $$\frac{dw}{dt}(t_0) = \left.\frac{d}{dt}\right|_{t=t_0} (P_\alpha^{t_0,t_1})^{-1}(w(t)).$$