Probability Theory – Exercise 2

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Problem 1

Let E be measurable with $0 < m(E) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on E, and let f be a measurable function on E. Show that $f_n \to f$ in measure on E if and only if

$$\lim_{n\to\infty} \int_E \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) = 0.$$

Proof. ($\Rightarrow$) For $\epsilon > 0$, there exist N such that for $n \geq N$, we have the set

$$X = \left\{ x : |f_n(x) - f(x)| > \frac{\epsilon}{1 + m(E)} \right\}$$

whose measure is $\mu(X) \leq \dfrac{\epsilon}{1 + m(E)}$. Then we have

$$\begin{aligned}
\int_E \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) &= \int_{E\cap X} \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) + \int_{E\cap X^c} \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) \\
&\leq \int_{E\cap X} 1 \,\mathrm{d}m(x) + \int_{E\cap X^c} \frac{\epsilon}{1 + m(E)} \,\mathrm{d}m(x) \\
&\leq \frac{\epsilon}{1 + m(E)} + \frac{\epsilon}{1 + m(E)} m(E) = \epsilon
\end{aligned}$$

Since ϵ is an arbitrary chosen positive constant, we have

$$\lim_{n\to\infty} \int_E \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) = 0.$$

($\Leftarrow$) We can find N such that for $n \geq N$,

$$\int_E \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) \leq \frac{\epsilon^2}{1 + \epsilon}$$

Let $X = \{x : |f_n(x) - f(x)| > \epsilon\}$, then we have

$$\mu(X) \frac{\epsilon}{1 + \epsilon} \leq \int_X \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) \leq \int_E \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|} \,\mathrm{d}m(x) \leq \frac{\epsilon^2}{1 + \epsilon}$$

Thus, $\mu(X) = \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \epsilon$ which implies $f_n \to f$ in measure. $\square$

Problem 2

Let $f : \mathbb{R} \to [0, \infty)$ be an integrable function. For any $k \in \mathbb{N}$, define $f_k : \mathbb{R} \to \mathbb{R}$ by $f_k(x) = f(x + k)$ and set $h(x) = \liminf_{k\to\infty} f_k(x)$. Show that $h = 0$ almost everywhere.

Proof. By definition of h, h is 1-periodic function. ETS $h = 0$ a.e. on [0,1). i.e. $\int_{[0,1)} h(x)\,\mathrm{d}x = 0$. Since f_k are non-negative measurable functions, by Fatou's lemma,

$$\int_{[0,1)} h(x)\,\mathrm{d}x = \int_{[0,1)} \liminf_{k\to\infty} f_k(x)\,\mathrm{d}x \le \liminf_{k\to\infty} \int_{[0,1)} f_k(x)\,\mathrm{d}x = \liminf_{k\to\infty} \int_{[k,k+1)} f(x)\,\mathrm{d}x = 0.$$

$$(\because a_k = \int_{[k,k+1)} f(x)\,\mathrm{d}x \ \to \sum_{k\in\mathbb{Z}} a_k = \int_{\mathbb{R}} f(x)\,\mathrm{d}x < \infty. \quad \therefore \lim_{k\to\infty} a_k = 0).$$ □

Problem 3

Let E be measurable with $0 < m(E) < \infty$ and f be measurable with ess sup $|f| < \infty$. Show that

$$\lim_{n\to\infty} \left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} = \operatorname{ess\,sup} |f|.$$

Proof. Let $\alpha = \operatorname{ess\,sup} |f| < \infty$. We want to show that $\lim_{n\to\infty} \left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} = \alpha$. By definition of essential supremum, $|f| \le k$ a.e. for all $k > \alpha$. Thus, we get

$$\left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} \le \left(\int_E k^n \,\mathrm{d}m \right)^{\frac{1}{n}} = k m(E)^{\frac{1}{n}}$$
$$\limsup_{n\to\infty} \left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} \le \limsup_{n\to\infty} k m(E)^{\frac{1}{n}} = k \text{ for all } k > \alpha$$

Thus, $\limsup_{n\to\infty} \left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} \le \alpha$. Now, it is enough to show that $\liminf_{n\to\infty} \left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} \ge \alpha$. For all $\epsilon > 0$, we know that $m(A_\epsilon) > 0$ where $A_\epsilon = \{x \in E : |f| \ge \alpha - \epsilon\}$. Thus,

$$\left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} \ge \left(\int_{A_\epsilon} (\alpha - \epsilon)^n \,\mathrm{d}m \right)^{\frac{1}{n}} = (\alpha - \epsilon) m(A_\epsilon)^{\frac{1}{n}}$$
$$\liminf_{n\to\infty} \left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} \ge \liminf_{n\to\infty} (\alpha - \epsilon) m(A_\epsilon)^{\frac{1}{n}} = \alpha - \epsilon \quad \forall \epsilon > 0.$$

Thus,

$$\liminf_{n\to\infty} \left(\int_E |f|^n \,\mathrm{d}m \right)^{\frac{1}{n}} \ge \alpha.$$ □

Problem 4

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable and X be a random variable. Show that

$$\mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x)\,\mathrm{d}P_X(x)$$

in that if either side exists then so does the other and they are equal.

Proof. ∎