Mathematical Statistics 1

Ch.5 Distributions of Functions of Random Variables

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Ch.5.1 Functions of One Random Variable

1.1 Continuous Case

Concept

Let X be a continuous random variable. If we consider a function of X, $Y = u(X)$, Y is also a random variable that has its own distribution:

$$F_Y(y) = P(Y \leq y) = P(u(X) \leq y)$$

$X \sim f_X(x)$: given

$Y = g(X) \sim f_Y(y)$?

$\hookrightarrow$ pmf of New r.v (Y).

discrete r.v $\quad f_Y(y) = P(Y=y) = P(g(X)=y)$

cdf of New r.v (Y).

continuous r.v $\quad F_Y(y) = P(Y \le y) = P(g(X) \le y)$.

1) $g(\cdot)$: increasing function. ex) e^x, $2x$.

$F_Y(y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$.

$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy}$$

$$\Rightarrow f_Y(y) = f_X(x) \frac{dx}{dy}$$

2) $g(\cdot)$: decreasing function.

$F_Y(y) = P((g(X) \le y) = P(X \ge g^{-1}(y)) = 1 - P(X \le g^{-1}(y))$

$= 1 - F_X(g^{-1}(y))$.

$$f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{dg^{-1}(y)}{dy}$$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dg^{-1}(y)}{dy}\right| = f_X(x) \left|\frac{dx}{dy}\right|$$

$\llcorner$ whatever increasing or decreasing.

Example 5.1-1

Let X have a gamma distribution with pdf

$$f_X(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad x > 0$$

Let $Y = e^X$. Find the pdf of Y.

1) $X \sim f_X(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta} \quad x>0$

$Y = e^X$

$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = F_X(\ln y).$

$f_Y(y) = F_Y(y)' = f_X(\ln y)\frac{1}{y} = \frac{1}{\Gamma(\alpha)\theta^\alpha}(\ln y)^{\alpha-1} e^{-\ln y/\theta} \frac{1}{y}$ $\quad \ln y^{-\frac{1}{\theta}}$

$= \frac{1}{\Gamma(\alpha)\theta^\alpha}(\ln y)^{\alpha-1} \frac{1}{y^{1+1/\theta}}, \quad y>1$

2) $Y = (1-X)^3, \quad 1-X = Y^3 \quad X = 1-Y^{\frac{1}{3}}$

$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(1-y^{\frac{1}{3}})\left|\frac{d(1-y^{\frac{1}{3}})}{dy}\right|$

$= 3(1-1+y^{\frac{1}{3}})^2 \cdot \left|-\frac{1}{3}y^{-\frac{2}{3}}\right| = y^{\frac{2}{3}} \cdot y^{-\frac{2}{3}} = 1, \quad 0<y<1$

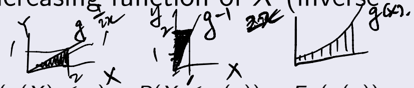
Change of variable technique

Change of variable technique

Let X be a continuous random variable with pdf $f_X(x)$ with support $c_1 < x < c_2$.

$g(x) \le y$ $\quad$ $x \le g^{-1}(y)$.

- $Y = u(X)$: continuous increasing function of X (inverse function: $X = v(Y)$)



$$F_Y(y) = P(Y \le y) = P(u(X) \le y) = P(X \le v(y)) = F_X(v(y)),$$

$$f_Y(y) = f_X(v(y))[v'(y)], \quad u(c_1) < y < u(c_2)$$

- $Y = u(X)$: continuous decreasing function of X (inverse function: $X = v(Y)$)

$$F_Y(y) = P(Y \le y) = P(u(X) \le y) = P(X \ge v(y)) = 1 - F_X(v(y)),$$

$$f_Y(y) = -f_X(v(y))[v'(y)], \quad u(c_2) < y < u(c_1)$$

Change of variable technique

Again, let X be a continuous random variable with pdf $f_X(x)$. The pdf of $Y = u(X)$ is

$$f_Y(y) = f_X(v(y))|v'(y)|, \quad y \in S_y$$

where S_y is the support of Y found by mapping the support of X.

Example 5.1-3

Let X have a pdf $f_X(x) = 3(1-x)^2,\ 0 < x < 1$. Find the pdf of $Y = (1-X)^3$.

$f_X(x) = 3(1-x^2)^2 \quad 0 < x < 1$

$Y = (1-X)^3$ pdf.

$f_Y(y) = P(Y = y) = P((1-X)^3 = y)$

Thm 5.1-2

Let X have continuous cdf $F_X(x)$ that is strictly increasing on the support $a < x < b$. Then, the random variable $Y = F_X(X)$ has a uniform distribution on $(0, 1)$.

$X \sim F_X(x)$ ftn of x

cdf of X

$Y \overset{let}{=} F_X(x) \sim \underset{\sim\sim}{?} f_Y(y)$.

$F_Y(y) = P(Y \le y) = P[F_X(x) \le y] = P(X \le F_X^{-1}(y))$

$= F_X(F_X^{-1}(y)) = y$

$f_Y(y) = \frac{dF_Y(y)}{dy} = 1, \quad 0 \le y \le 1$

$\therefore Y \sim Unif(0,1)$

1.2 Discrete Case

Change of variable technique for discrete case

Let X be a discrete random variable with pmf $f_X(x)$, $x \in S_x = \{c_1, c_2, \ldots\}$. Let $Y = u(X)$ be a one-to-one transformation with inverse $X = v(Y)$ and $y \in S_y = \{u(c_1), u(c_2), \ldots\}$. The pmf of Y is

$$P(Y = y) = P[u(X) = y] = P[X = v(y)], \quad y \in S_y$$

Example 5.1-6

Let X have a Poisson distribution with $\lambda = 4$. Find the pmf of $Y = \sqrt{X}$.

Example 5.1-7

Let X have binomial distribution with parameters n and p. Find the pmf of $Y = X^2$, where $n = 3,\ p = 1/4$.

$X \sim Poi(\lambda = 4)$

$f_X(x) = \frac{4^x e^{-4}}{x!} \quad x = 0, 1, 2, \cdots$

$Y = \sqrt{X}$

$f_Y(y) = P(Y=y) = P(\sqrt{X} = y)$

$= P(X = y^2) = f_X(y^2) = \frac{4^{y^2} e^{-4}}{(y^2)!} \quad y = 0, 1, \sqrt{2}, \cdots$

$X \sim \chi^2(n) \qquad W = \frac{Z}{\theta} = Y$

$Y = \theta X$

$f_Y(y) = P(Y=y) = P(\theta X = y) = P(X = \frac{y}{\theta}) = f_X(\frac{y}{\theta})$

$f_X(x) = \frac{1}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$

$f_X(\frac{y}{\theta}) = \frac{1}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}}} y^{\frac{n}{2}-1} \cdot (\frac{1}{\theta})^{\frac{n}{2}-1} \cdot e^{-\frac{y}{2\theta}}$

$= \frac{1}{\Gamma(\frac{n}{2}) \cdot 2\theta^{\frac{n}{2}}} y^{\frac{n}{2}-1} \cdot \theta \cdot e^{-\frac{y}{2\theta}}$

$\therefore Gamma(\frac{n}{2}, 2\theta) \times \theta$.

$f_X(x) = \frac{1}{\Gamma(\frac{n}{2}) \cdot 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$

$f_Y(y) = f_X(\frac{y}{\theta}) = \frac{1}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}}} (\frac{y}{\theta})^{\frac{n}{2}-1} e^{-\frac{1}{2}\cdot(\frac{y}{\theta})}$

$= \frac{1}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}}} y^{\frac{n}{2}-1} \cdot (\frac{1}{\theta})^{\frac{n}{2}-1} \cdot e^{-\frac{y}{2\theta}}$

$= \frac{1}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}}} y^{\frac{n}{2}-1} (\frac{1}{\theta})^{\frac{n}{2}} (\frac{1}{\theta})^{-1} \cdot e^{-\frac{y}{2\theta}}$

$= \frac{1}{\Gamma(\frac{n}{2})(2\theta)^{\frac{n}{2}}} y^{\frac{n}{2}-1} \theta \cdot e^{-\frac{y}{2\theta}}$

Ch.5.2 Transformations of Two Random Variables

2.1 Bivariate Transformations

Change of variable technique for discrete case

Let (X, Y) be a bivariate random vector with a joint distribution $f_{XY}(x, y)$. Consider a new bivariate random vector (U, V) defined by $U = g_1(X, Y)$ and $V = g_2(X, Y)$.

- (X, Y): discrete bivariate random vector

$$\begin{aligned} f_{UV}(u, v) &= P(U = u, V = v) = P((X, Y) \in A_{uv}) \\ &= \sum_{(x,y)\in A_{uv}} f_{XY}(x, y) \end{aligned}$$

where $A_{uv} = \{(x, y) \in A | g_1(x, y) = u, g_2(x, y) = v\}$

$\Rightarrow (X, Y) \sim f_{XY}(x,y)$: known

$U = g_1(X,Y)$
$V = g_2(X,Y)$ $\sim$ ① $f_{UV}(u,v)$
② $f_U(u)$.

1) discreate r.v
joint pmf
: $f_{UV}(u,v) = P(U=u, V=v)$.

$= P(g_1(X,Y)=u, g_2(X,Y)=v)$

$= P((X,Y) \in A) = \sum_{\substack{g_1(X,Y)=u \\ g_2(X,Y)=v}} f_{XY}(x,y)$.

2) continuous r.v
$f_{UV}(u,v) = f_{XY}(x,y)|J|$ $\quad J = \begin{array}{ccc} u & v & \\ \left|\begin{matrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{matrix}\right| & & \begin{matrix} x \\ y \end{matrix} \end{array}$

$X \sim Poi(\theta)$ independent $(X,Y) \sim f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$
$Y \sim Poi(\lambda)$

$$f_{XY}(x,y) = \frac{\theta^x e^{-\theta}}{x!} \cdot \frac{\lambda^y e^{-\lambda}}{y!} \qquad x = 0,1,2,3,\cdots \quad y = 0,1,2,3,\cdots$$

$U \overset{let}{=} X+Y$ ① $f_{UV}(u,v)$
$V = Y$ ② $f_U(u)$

① $f_{UV}(u,v) = P(U=u, V=v) = P(X+Y=u, Y=v)$
$= P(X=u-v, Y=v) = f_{XY}(u-v, v)$

$$= \frac{\theta^{(u-v)} e^{-\theta}}{(u-v)!} \frac{\lambda^v e^{-\lambda}}{v!} \quad \left(\begin{array}{l} u = v, v+1, v+2, \cdots \\ v = 0,1,2,\cdots \end{array} \right.$$

② $f_U(u) = P(U=u) = \sum_{v=0}^{u} f_{UV}(u,v) = \sum_{v=0}^{u} \frac{e^{-(\theta+\lambda)} \cdot \theta^{(u-v)} \lambda^v}{(u-v)!\,v!} \frac{u!}{u!}$

fixed $\quad$ 0,1,2,⋯
$U = X+V$
$V = U-X$

$$= e^{-(\theta+\lambda)} \times \frac{1}{u!} \sum_{v=0}^{u} \binom{u}{v} \lambda^v \theta^{(u-v)} \qquad u = 0,1,2,\cdots$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} (\lambda+\theta)^u \ : \text{ pmf of } Poi(\lambda+\theta)$$

$\therefore U \sim Poi(\lambda+\theta)$.

* $X \sim Poi(\theta)$, $Y \sim Poi(\lambda)$ indep.
$X+Y \sim Poi(\theta+\lambda)$
indep. $Z \sim Poi(\tau)$
$\Rightarrow X+Y+Z \sim Poi(\theta+\lambda+\tau)$

$X_i \sim Poi(\lambda_i)$

$$\sum_{i=1}^{n} X_i \sim Poi\left(\sum_{i=1}^{n} \lambda_i\right)$$

Change of variable technique for continuous case

The single-valued inverse functions of g_1 and g_2 functions are defined as $x = h_1(u, v)$ and $y = h_2(u, v)$.

- (X, Y): continuous bivariate random vector

$$f_{UV}(u, v) = f_{XY}(h_1(u, v), h_2(u, v))|J|$$

$$J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \frac{dx}{du}\frac{dy}{dv} - \frac{dx}{dv}\frac{dy}{du}$$

where J is the determinant of a matrix of partial derivatives.

Example 5.2-1

Let $X_1,\ X_2$ have the joint pdf $f(x_1, x_2) = 2,\quad 0 < x_1 < x_2 < 1.$

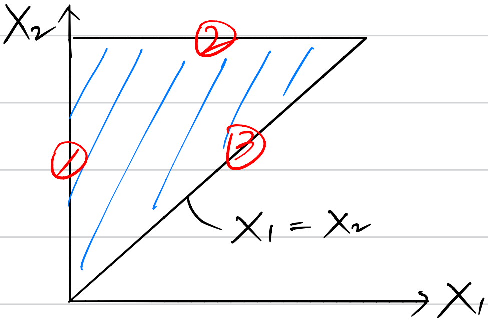
Consider the transformation

$$Y_1 = \frac{X_1}{X_2} \quad Y_2 = X_2.$$

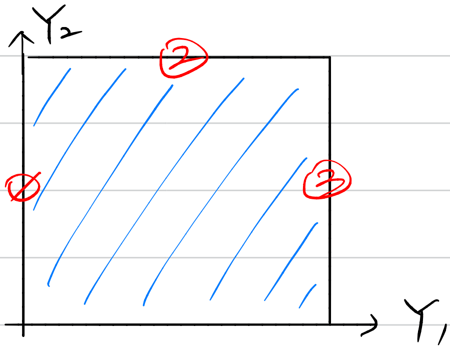
- Find the joint pdf of $Y_1,\ Y_2$.
- Find the marginal pdfs of Y_1 and Y_2.
- Are Y_1 and Y_2 independent?

$(X_1, X_2) \sim f_X(x_1, x_2) = 2 \quad 0 < x_1 < x_2 < 1$

$Y_1 = \frac{X_1}{X_2}, \quad Y_2 = X_2 \quad \Rightarrow \quad X_1 = X_2 Y_1 = Y_1 Y_2, \quad X_2 = Y_2$



$\Rightarrow$



$$\begin{cases} ① \ x_1 = 0, \quad 0 < x_2 < 1 \\ ② \ 0 < x_1 < 1, \quad x_2 = 1 \\ ③ \ 0 < x_1 = x_2 < 1 \end{cases} \Rightarrow \begin{cases} ① \ y_1 = 0, \quad 0 < y_2 < 1 \\ ② \ 0 < y_1 < 1, \quad y_2 = 1 \\ ③ \ y_1 = 1 \quad 0 < y_2 < 1 \end{cases}$$

① $f_Y(y_1, y_2) = f_X(x_1, x_2)\,|J|$

$= f_X(y_1 y_2, y_2)\,|y_2|$

$= 2y_2 \quad 0 < y_1, y_2 < 1$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} \frac{d(y_1 y_2)}{dy_1} & \frac{d(y_1 y_2)}{dy_2} \\ \frac{d(y_2)}{d(y_1)} & \frac{d(y_2)}{d(y_2)} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2.$$

② $f_{Y_1}(y_1) = \int_0^1 f_Y(y_1, y_2)\,dy_2 = \int_0^1 2y_2 = [y_2^2]_0^1 = 1 \quad 0 < y_1 < 1$

$f_{Y_2}(y_2) = \int_0^1 f_Y(y_1, y_2)\,dy_1 = \int_0^1 2y_2\,dy_1 = 2y_2 \quad 0 < y_2 < 1$

③ $f_Y(y_1, y_2) = 2y_2$

$f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = 2y_2$

$\therefore$ Y_1 & Y_2 : independent.

Example 5.2-2

Let $X_1,\ X_2$ be independent random variables, each with pdf $f(x) = e^{-x}$, $x > 0$. Let us consider $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

- Find the joint pdf of $Y_1,\ Y_2$.
- Find the marginal pdfs of Y_1 and Y_2.

y_1, y_2 범위 주의.

Example 5.2-3

Let $X_1,\ X_2$ be independent gamma distributions with $\alpha,\ \theta$ and $\beta,\ \theta$. Let us consider

$$Y_1 = \frac{X_1}{X_1 + X_2} \quad Y_2 = X_1 + X_2.$$

- Find the joint pdf of $Y_1,\ Y_2$.
- Find the marginal pdf of Y_1.

$$X_1 \sim \text{Gamma}(\alpha, \theta) \quad (>0)$$
$$X_2 \sim \text{Gamma}(\beta, \theta)$$
independent.

$$Y_1 = \frac{X_1}{X_1+X_2} \sim ? \quad (0 < Y_1 < 1) \qquad Y_2 = X_1 + X_2 > 0$$

① $f_X(x_1, x_2)$ $\qquad 0 < y_1 < 1,\ y_2 > 0.$

② $f_Y(y_1, y_2) = f_X(x_1, x_2)|J|$

③ $f_{Y_1}(y_1) = \int f_Y(y_1, y_2)\,dy_2 = \int_0^\infty f_Y(y_1, y_2)\,dy_2.$

$$f_{X_1}(x_1) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x_1^{\alpha-1} \cdot e^{-\frac{x_1}{\theta}} \qquad f_{X_2}(x_2) = \frac{1}{\Gamma(\beta)\theta^\beta} x_2^{\beta-1} \cdot e^{-\frac{x_2}{\theta}}$$

② $$f_X(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} x_1^{\alpha-1} \cdot x_2^{\beta-1} \cdot e^{-\frac{x_1+x_2}{\theta}}$$

$$Y_1 = \frac{X_1}{X_1+X_2}, \quad Y_2 = X_1 + X_2 \qquad 0 < y_1 < 1 \quad y_2 > 0.$$

① $\Rightarrow X_1 = Y_1 Y_2 \qquad X_2 = Y_2 - Y_1 Y_2$

$$|J| = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2(1-y_1) + y_1 y_2 = y_2 > 0.$$

③ $f_Y(y_1, y_2) = f_X(y_1 y_2,\ y_2 - y_1 y_2)|J|$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{(\alpha+\beta)}} (y_1 y_2)^{\alpha-1} (y_2 - y_1 y_2)^{\beta-1} \cdot e^{-\frac{y_1 y_2 + (y_2 - y_1 y_2)}{\theta}} \cdot y_2$$

④ $$f_{Y_1} = \int_0^\infty f_Y(y_1, y_2)\,dy_2 = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{(\alpha+\beta)}} \int_0^\infty y_1^{\alpha-1} y_2^{\alpha} y_2^{\beta-1} (1-y_1)^{\beta-1} \cdot e^{-\frac{y_2}{\theta}}\,dy_2$$

$$= \frac{y_1(1-y_1)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)\theta^{(\alpha+\beta)}} \Gamma(\alpha+\beta)\theta^{(\alpha+\beta)} \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha+\beta)\theta^{(\alpha+\beta)}} y_2^{(\alpha+\beta)-1} e^{-\frac{y_2}{\theta}}\,dy_2}_{\text{pdf of Gamma}(\alpha+\beta,\ \theta)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha-1}(1-y_1)^{\beta-1} \quad : \text{pdf of Beta.}$$
$(0 < y_1 < 1)$

$X \sim N(\mu, \sigma^2)$

$$\therefore Y_1 = \frac{X_1}{X_1 + X_2} \sim \text{Beta}(\alpha, \beta)$$

$$Z = \frac{X - \mu}{\sigma_X} \sim N(0,1)$$

$X_1, X_2 \sim$ Gamma $\rightarrow$ Beta(α, β).

Beta distribution

$\Rightarrow$ random variable이 0 or 1 사이 //.

$Z^2 \sim \chi^2(1).$

$$\left(\frac{X_2}{X_1+X_2} \sim \text{Beta}(\beta, \alpha) \right)$$

Example 5.2-4

Let U and V be independent chi-square variables with r_1 and r_2 degrees of freedom, respectively. Consider

$$W = \frac{U/r_1}{V/r_2}$$

- Find the joint pdf of W and $Z = V$.
- Find the marginal pdf of W.

$X_1 \sim \chi^2(r_1) = \text{Gamma}\left(\frac{r_1}{2}, 2\right)$
$X_2 \sim \chi^2(r_2) = \text{Gamma}\left(\frac{r_2}{2}, 2\right)$) independent.

$$F \overset{>0}{=} \frac{X_1/r_1}{X_2/r_2} \sim F\ \text{dist}^n (r_1, r_2).$$

$Z = X_2 > 0.$

① $f_{UV}(u,v) = \frac{1}{\Gamma(\frac{r_1}{2}) 2^{\frac{r_1}{2}}} u^{\frac{r_1}{2}-1} e^{-\frac{u}{2}} \times \frac{1}{\Gamma(\frac{r_2}{2}) 2^{\frac{r_2}{2}}} v^{\frac{r_2}{2}-1} e^{-\frac{v}{2}}$

② $u = \frac{r_1}{r_2} vw = \frac{r_1}{r_2} wz, \quad v = z$

$$J = \begin{array}{c|cc|} u & \frac{r_1}{r_2} z & \frac{r_1}{r_2} w \\ v & 0 & 1 \\ & w & z \end{array} = \frac{r_1}{r_2} z > 0$$

③ $f_{WZ}(w,z) = f_{UV}\left(\frac{r_1}{r_2} wz, z\right) |J|$

$$= \frac{1}{\Gamma(\frac{r_1}{2}) \Gamma(\frac{r_2}{2}) 2^{\frac{r_1+r_2}{2}}} \left(\frac{r_1}{r_2} wz\right)^{\frac{r_1}{2}-1} e^{-\frac{r_1 wz}{2r_2}} z^{\frac{r_2}{2}-1} e^{-\frac{z}{2}} \frac{r_1}{r_2} z$$

$$= \frac{1}{\Gamma(\frac{r_1}{2}) \Gamma(\frac{r_2}{2}) 2^{\frac{r_1+r_2}{2}}} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1} z^{\frac{r_1+r_2}{2}-1} e^{-\frac{z}{2}(1+\frac{r_1}{r_2}w)}$$

④ $f_W(w) = \int_0^\infty f_{WZ}(w,z)\, dz$

$$= \frac{\Gamma(\frac{r_1+r_2}{2}) \left[\frac{1}{2}(1+\frac{r_1}{r_2}w)\right]^{-\frac{r_1+r_2}{2}}}{\Gamma(\frac{r_1}{2}) \Gamma(\frac{r_2}{2}) 2^{\frac{r_1+r_2}{2}}} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1} \int_0^\infty \frac{\left[\frac{1}{2}(1+\frac{r_1}{r_2}w)\right]^{\frac{r_1+r_2}{2}}}{\Gamma(\frac{r_1+r_2}{2})} z^{\frac{r_1+r_2}{2}-1} \times e^{-\frac{z}{2}(1+\frac{r_1}{r_2}w)} dz$$

$$= \frac{\Gamma(\frac{r_1+r_2}{2}) \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}}{\Gamma(\frac{r_1}{2}) \Gamma(\frac{r_2}{2})} \times w^{\frac{r_1}{2}-1} \left(1+\frac{r_1}{r_2}w\right)^{-\frac{r_1+r_2}{2}}, \quad w > 0$$

pdf of Gamma $\left(\frac{r_1+r_2}{2}, \frac{2}{1+\frac{r_1}{r_2}w}\right)$

: pdf of F-dist^n
$W \sim F(r_1, r_2)$

* $\frac{1}{F} = \frac{X_2/r_2}{X_1/r_1} \sim F(r_2, r_1)$

Example 5.2-5

$W \sim F(4,6)$. Let $F_\alpha(r_1, r_2)$ be the $100(1-\alpha)$ percentile of $F(r_1, r_2)$. Find the values of $F_{0.05}(4,6)$, $F_{0.01}(4,6)$, $F_{0.95}(4,6)$, and $P(W \leq 9.15)$.

Example 5.2-6 (Box-Muller transformation)

If X_1 and X_2 are independent and $X_i \sim \mathrm{Unif}(0,1)$, then find the joint pdf of $Z_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2)$ and $Z_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2)$.

$$\begin{cases} X_1 \sim Unif(0,1) \\ X_2 \sim Unif(0,1) \end{cases} \text{ independent.}$$

$$\begin{cases} Z_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2) \\ Z_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2) \end{cases}$$

$\Rightarrow f_Z(z_1, z_2) = f_{Z_1}(z_1) \times f_{Z_2}(z_2)$ (i.e. Z_1, Z_2 independent).

$\Rightarrow Z_1 \sim N(0,1)$, $Z_2 \sim N(0,1)$.

반드시 풀어보기! $|J|$에 arctan 사용.