미분기하학개론 1 - HW5 양준우 (2020-91659)

2-4. 15

Let $S \subseteq \mathbb{R}^3$ be connected surface and $\mathbb{R}\cdot N(p)$ contain $p_0 \in \mathbb{R}^3$ for $\forall p \in S$. ($p_0 \notin S$ trivial).

WTS : $S \subseteq$ sphere centered p_0.

Let $f : \mathbb{R}^3 \to \mathbb{R}$, $p \mapsto |p - p_0|$

ETS : $\forall p \in S$ is critical point of f.

Because if it is, $df_p = 0$ for $\forall p \in S$.

Then $|p - p_0| = c$ for $\forall p \in S$ where c is positive constant.

This implies that S is contained in a sphere of radius c with centered at p_0. ($\because S$ is connected).

Thus if we show that $\forall p \in S$ is critical point of f, we are done.

Let's pick any $p \in S$ and $w \in T_pS$.

Let $\alpha : (-\varepsilon, \varepsilon) \to S$ s.t $\alpha(0) = p$, $\alpha'(0) = w$

Note that $w \perp N(p) \Rightarrow w \perp (p - p_0)$. for $\forall w \in T_pS$

($\because w$ is tangent vector at p and $\mathbb{R}\cdot N(p)$ contains p_0).

$$
\begin{aligned}
df_p(w) &= (f \circ \alpha)'(0) = \frac{d}{dt}\left(\|\alpha(t) - p_0\|\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\sqrt{(\alpha(t) - p_0)\cdot(\alpha(t) - p_0)}\right)\Big|_{t=0} \\
&= \frac{1}{2\sqrt{(\alpha(t) - p_0)\cdot(\alpha(t) - p_0)}} \cdot 2\alpha'(t)(\alpha(t) - p_0)\Big|_{t=0} \\
&= \frac{1}{\|\alpha(0) - p_0\|}\,\alpha'(0)(\alpha(0) - p_0) \\
&= \frac{1}{\|p - p_0\|} \cdot w \cdot (p - p_0) \\
&= 0 \quad (\because w \perp (p - p_0))
\end{aligned}
$$

Since $df_p(T_pS) = 0$, p is arbitrary,

$\forall p \in S$ is critical point of f. $\square$

2-4.24

$\varphi : S_1 \to S_2$ diff.

$\psi : S_2 \to S_3$ diff.

$\Rightarrow d(\psi \circ \varphi)_p = d\psi_{\varphi(p)} \circ d\varphi_p$

pf). Let $w \in T_pS_1$ for $p \in S_1$.

There exists curve $\alpha : (-\varepsilon, \varepsilon) \to S_1$ s.t $\alpha(0) = p$, $\alpha'(0) = w$.

Let $\beta(t) := \varphi(\alpha(t)) = \varphi \circ \alpha(t)$

$q := \beta(0) = \varphi(p)$.

$v := \beta'(0) = \frac{d}{dt}\Big|_{t=0} \varphi \circ \alpha(t) = (\varphi \circ \alpha)'(0)$ $(v \in T_qS_2)$

$$d(\psi \circ \varphi)_p(w) = \frac{d}{dt}\Big|_{t=0} (\psi \circ \varphi) \circ \alpha(t) = ((\psi \circ \varphi) \circ \alpha)'(0) = (\psi \circ (\varphi \circ \alpha))'(0)$$

$$\begin{aligned}(d\psi_{\varphi(p)} \circ d\varphi_p)(w) &= d\psi_{\varphi(p)}(d\varphi_p(w)) = d\psi_{\varphi(p)}((\varphi \circ \alpha)'(0)) \\ &= d\psi_q(v) = \frac{d}{dt}\Big|_{t=0} (\psi \circ \beta)(t) = (\psi \circ \beta)'(0) \\ &= (\psi \circ (\varphi \circ \alpha))'(0)\end{aligned}$$

$\therefore d(\psi \circ \varphi)_p(w) = (d\psi_{\varphi(t)} \circ d\varphi_p)(w)$ for $\forall w \in T_pS_1$

$\therefore d(\psi \circ \varphi)_p = d\psi_{\varphi(t)} \circ d\varphi_p$ $\square$.