Topology 2

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Contents

Chapter 0

Introduction

0.1 Quotient topology

Definition 1. An ***equivalence relation*** is a relation $x \sim y$ so that $x \sim x$; if $x \sim y$ then $y \sim x$; and if $x \sim y$ and $y \sim z$, then $x \sim z$. Given an equivalence relation defined on X, $X/_\sim$ is the set of ***equivalence classes***.

Definition 2. Let $f\colon X \to Y$ be a surjective map from the topological space X to the set Y. Then, we define a topology on Y, called the ***quotient topology***, by requiring that $O \subset Y$ be open if and only if $f^{-1}(O)$ is actually an open set of X. One checks trivially that this defines a topology on Y.

Example. Let X be the closed unit ball, $\{(x, y) : x^2 + y^2 \leq 1\}$, in $\mathbb{R}$ and X^* be the partition of X consisting of all the one-point sets $\{(x, y)\}$ for which $x^2 + y^2 < 1$, along with the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Then X^* is homeomorphic with the subspace of $\mathbb{R}^3$ called the unit 2-sphere.

0.2 What is algebraic topology?

Functor from category of topological spaces to the category of groups.

- Category: set of spaces and morphisms.
- Functor: $X \rightsquigarrow G_X$ and $f\colon X \to Y \rightsquigarrow f_*\colon G_X \to G_Y$ such that
  - $(f \circ g)_* = f_* \circ g_*$
  - $(1_X)_* = 1_{G_X}$

Two systems we'll discuss:

- fundamental groups
- homology groups

Example. Suppose we have a functor. If $G_X \ncong G_Y$, then X and Y are not homeomorphic. If 'shadows' are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f \colon X \to Y$ and $g \colon Y \to X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* \colon G_X \to G_Y$ and $g_* \colon G_Y \to G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* \colon G_X \to G_Y$ is an isomorphism. ◇

0.3 Fundamental group

Pick a base point x_0 and consider it fixed. (The fundamental gruop will not depend on it. We assume all spaces are path connected) $X \rightsquigarrow \pi(X)$.

- A loop based at $x_0 \in X$ is a map $f \colon I = [0,1] \to X$, $f(0) = f(1) = x_0$.
- Loops are equivalent if one can be deformed in the other in a continuous way, with the base point fixed.
- The fundamental group consists of equivalent classes of loops.

Example. Let $X = B^2$ (2 dimensional disk). Then $\pi(B^2) = 1$, because every loop is equivalent to the 'constant' loop.

Example. Let $X = S^1$ and pick x_0 on the circle. Two options:

- The loop is trivial equivalent to the constant loop
- The loop goes around the circle.
- The loop goes around the circle, twice.
- The loop goes around the circle, clockwise, once
- ...

$\pi(S^1) \cong \mathbb{Z}$ (proof will follow)

The composition of loops is simply pasting them. In the case of the circle, the loop $-1\circ$ the loop 2 is the loop 1.
Suppose $\alpha \colon I \to X$ and $f \colon X \to Y$. Then we define

$$f_*[\alpha] = [f \circ \alpha].$$

Theorem 1 (Fixed point theorem of Brouwer)**.** Any continuous map from a rectangle to itself has at least one fixed point.

Proof. Suppose there is no fixed point, so $f(x) \neq x$ for all $x \in B^2$. Then we can construct map $r\colon B^2 \to S_1$ as follows: take the intersection of the boundary and half ray between $f(x)$ and x. If x lies on the boundary, we have the identity map. This map is continuous. Then we have $S^1 \to B^2 \to S^1$, via the inclusion and r. Looking at the fundamental groups:

$$\pi(S^1) = \mathbb{Z} \to \pi(B^2) = 1 \to \pi(S^1) = \mathbb{Z}.$$

The map from $\pi(S^1) \to \pi(S^1)$ is the identity map, but the first map maps everything on 1. $\square$

Chapter 9

Fundamental group

See wikipedia[1] for a brief introduction.

9.51 Homotopy of paths

Definition 3. If f and f' are continuous maps of the space X into the space Y, we say that f is ***homotopic*** to f' if there is a continuous map $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$ for each x. (Here $I = [0,1]$.) The map F is called a ***homotopy*** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is ***null homotopic***.

Definition 4. Let $f, g\colon I \to X$ be two paths such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Then $H\colon I \times I \to X$ is a ***path homotopy*** between f and g, if and only if

- $H(s,0) = f(s)$ and $H(s,1) = g(s)$ (homotopy between maps)
- $H(0,t) = x_0$ and $H(1,t) = x_1$ (start and end points fixed)

Notation: $f \simeq_p g$.

Lemma 1. $\simeq$ and $\simeq_p$ are equivalence relations.

Proof.
- Reflective: $F(x,t) = f(x)$
- Symmetric: $G(x,t) = H(x, 1-t)$
- Transitive: Suppose $f \simeq g$ and $g \simeq h$, with H_1, H_2 resp.

$$H(x,t) = \begin{cases} H_1(x,2t) & 0 \le t \le \frac{1}{2} \\ H_2(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

□

[1] https://en.wikipedia.org/wiki/Homotopy

Example (Trivial, but important)**.** Let $C \subset \mathbb{R}^n$ be a convex subset.

- Any two maps $f, g \colon X \to C$ are homotopic.
- Any two paths $f, g \colon I \to C$ with $f(0) = g(0)$ and $g(1) = f(1)$ are path homopotic.

Choose $H \colon X \times I \to C$ defined by $(x, t) \mapsto H(x, t) = (1 - t)f(x) + tg(x)$.

Product of paths

Let $f \colon I \to X$, $g \colon I \to X$ be paths, $f(1) = g(0)$. Define

$$f * g \colon I \to X \text{ given by } s \mapsto \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark. If f is path homotopic to f' and g path homotopic to g' (which means that $f(1) = f'(1) = g(0) = g'(0)$), then $f * g \simeq_p f' * g'$.

So we can define $[f] * [g] := [f * g]$ with $[f] := \{g \colon I \to X \,|\, g \simeq_p f\}$.

Theorem 2.

1. $[f] * ([g] * [h])$ is defined iff $([f] * [g]) * [h]$ is defined and in that case, they are equal.
2. Let e_x denote the constant path $e_x \colon I \to X$ given by $s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.
3. Let $\overline{f} \colon I \to X$ given by $s \mapsto f(1 - s)$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Proof. First two observations

- Suppose $f \simeq_p g$ via homotopy H, $f, g \colon I \to X$. Let $k \colon X \to Y$. Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.
- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

2. Take $e_0 \colon I \to I$ given by $s \mapsto 0$. Take $i \colon I \to I$ given by $s \mapsto s$. Then $e_0 * i$ is a path from 0 to $1 \in I$. The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 * \simeq i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

3. Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

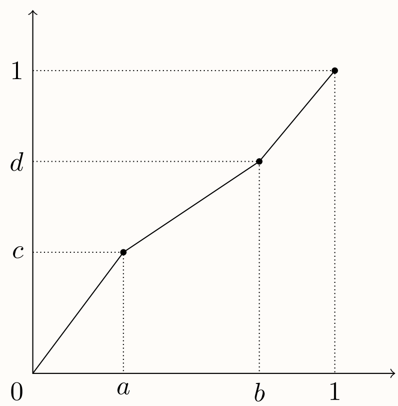
$$
\begin{aligned}
f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\
f * \overline{f} &\simeq_p e_{x_0} \\
[f] * [\overline{f}] &= [e_{x_0}].
\end{aligned}
$$

1. Remark: Only defined if $f(1) = g(0)$, $g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

   Assume the product is defined. Suppose $[a, b]$, $[c, d]$ are intervals in $\mathbb{R}$. Then there is a unique positive (positive slope), linear map from $[a, b] \to [c, d]$. For any $a, b \in [0, 1)$ with $0 < a < b < 1$, we define a path

$$
\begin{aligned}
k_{a,b} \colon [0, 1] &\longrightarrow X \\
[0, a] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{f} X \\
[a, b] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{g} X \\
[b, 0] &\xrightarrow{\text{lin.}} [0, 1] \xrightarrow{h} X
\end{aligned}
$$

   Then $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$.
   Let γ be that path $\gamma \colon I \to I$ with the following graphs:



   Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$
\begin{aligned}
k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\
k_{a,b} &\simeq_p k_{c,d},
\end{aligned}
$$

   which is what we wanted to show.

$\square$

9.52 Fundamental group

Definition 5. Let X be a space and $x_0 \in X$, then the ***fundamental group*** of X based at x_0 is

$$\pi(X, x_0) = \{[f] \mid f\colon I \to X,\ f(0) = f(1) = x_0\}.$$

(Also $\pi_1(X, x_0)$ is used, first homotopy group of X based at x_0)

For $[f], [g] \in \pi(X, x_0)$, $[f] * [g]$ is always defined, $[e_{x_0}]$ is an identity element, $*$ is associative and $[f]^{-1} = [\overline{f}]$. This makes $(\pi(X, x_0), *)$ a group.

Example. If $C \subset \mathbb{R}^n$, convex then $\pi(X, x_0) = 1$. E.g. $\pi(B^2, x_0) = 1$.

Remark. All groups are a fundamental group of some space.

Question: how does the group depend on the base point?

Theorem 3 (52.1)**.** Let X be a space, $x_0, x_1 \in X$ and $\alpha\colon I \to X$ a path from x_0 to x_1. Then

$$\begin{aligned} \hat{\alpha}\colon \pi(X, x_0) &\longrightarrow \pi(x, x_1) \\ [f] &\longmapsto [\overline{\alpha}] * [f] * [\alpha]. \end{aligned}$$

is an isomorphisms of groups. Note however that this isomorphism depends on α.

Proof. Let $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned} \hat{\alpha}([f] * [g]) &= [\overline{\alpha}] * [f] * [g] * [\alpha] \\ &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}[f] * \hat{\alpha}[g]. \end{aligned}$$

We can also construct the inverse, proving that these groups are isomorphic. $\square$

Remark. If : $(x, x_0) \to (Y, y_0)$ is a map of pointed topology spaces ($f\colon X \to Y$ continuous and $f(x_0) = y_0$). Then

$$f_*\colon \pi(X, x_0) \to \pi(Y, y_0) \text{ given by } [\gamma] \mapsto [f \circ \gamma]$$

is a morphism of groups, because of the two 'rules' discussed previously, with

$$(f \circ g)_* = f_* \circ g_* \quad (1_X)_* = 1_{\pi(X, x_0)}.$$

Definition 6. Let X be a topological space, then X is ***simply connected*** iff X is path connected and $\pi_1(X, x_0) = 1$ for some $x_0 \in X$.

Remark. If trivial for one base point, it's trivial for all base points.

Example. Any convex subset $C \subset \mathbb{R}^n$ is simply connected.

Example (Wrong proof of $\pi(S^2)$ being trivial)**.** Let f be a path from $[0,1] \to S^2$. Then pick $y_0 \in \operatorname{Im}(f)$. Then $S^2 \setminus \{y_0\} \approx \mathbb{R}^2$. Then use $\mathbb{R}^2$.

This is wrong because we cannot always find $y_0 \in \operatorname{Im}(f)$. Space filling loops! We'll see the correct proof later on.

Lemma 2 (52.3)**.** Suppose X is simply connected and $\alpha, \beta \colon I \to X$ two paths with same start and end points. Then $\alpha \simeq_p \beta$.

Proof. Simply connected implies loops are homotopic? Consider $\alpha * \overline{\beta} \simeq_p e_{x_0}$, since the space is imply connected.

$$
\begin{aligned}
([\alpha] * [\overline{\beta}]) * [\beta] = [e_{x_0}] * [\beta] = [\beta] \\
[\alpha] * ([\overline{\beta}] * [\beta]) = [\alpha] * [e_{x_0}] = [\alpha].
\end{aligned}
$$

And therefore $\alpha \simeq_p \beta$. (Note: make sure end and start point matchs when using $*$) ☐

9.53 Covering spaces

Definition 7. Let $p \colon E \to B$, surjective map (so continuous). Let $U \subset B$ open. Then U is ***evenly covered*** iff $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$ with

- V_α open in E
- $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$
- $p|_{V_\alpha} \colon V_\alpha \to U$ is a homeomorphism.

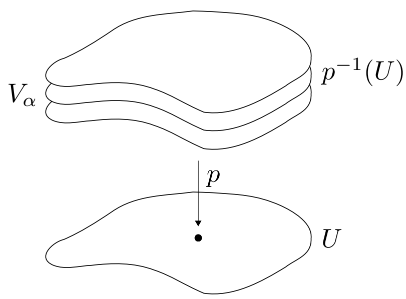


Figure 9.1: Evenly covered

Remark. If $U' \subset U$, also open and U is evenly covered, then also U'.

Definition 8. Let $p\colon E \to B$ be a surjective map. Then p is a ***covering projection*** iff $\forall b \in B$, $\exists U \subset B$ open, containing b such that U is evenly covered by p. Then (E, p) is called a ***covering space***.

Example. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Take $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. Note that $\mathbb{R}$ is an easier space than S^1, and so will be π_1 (1 vs $\mathbb{Z}$).

Proposition 1. A covering map is always a open map.

Proof. Exercise. $\square$

Proposition 2. For any $b \in B$, $p^{-1}(b)$ is a discrete subset of E. (No accumulation point)

Proof. Indeed for any $\alpha \in I$, $V_\alpha \cap p^{-1}(b)$ is exactly one point. $\square$

Remark. A covering is always local homeomorphism. But there are surjective local homeomorphism which are not covering maps. A covering map is more than a surjective local homeomorphism.

For example, $p\colon \mathbb{R}_0^+ \to S^1$ given by $t \mapsto e^{2\pi i t}$. Consider the inverse image of a neighborhood around 1. When we restrict p to the part around 0, it is no longer a homeomorphism (we don't get the whole neighborhood around one).

Creating new covering spaces out of old ones

- Suppose $p\colon E \to B$ is a covering and $B_0 \subset B$ is a subspace with the subspace topology. Let $E_0 = p^{-1}(B_0)$ and $p_0 = p|_{E_0}$. Then (E_0, p_0) is a covering of B_0.
- Suppose that (E, p) is a covering of B and (E', p') is a covering of B', then $(E \times E', p \times p')$ is a covering of $B \times B'$.

Example. Let $T^2 = S^1 \times S^1$.

- $p\colon \mathbb{R}^2 \to S^1 \times S^1$ given by $(t, s) \mapsto (e^{ait}, e^{bis})$.
- $p'\colon \mathbb{R} \times S^1 \to T^2$ given by $(t, z) \mapsto (e^{ait}, z^n)$.
- $p\colon S^1 \times S^1 \to T^2$ given by $(z_1, z_2) \mapsto (z_1^n, z_2^m)$.

These are the only types of coverings of the torus. We'll prove this later on.

9.54 Fundamental group of the circle

Given f, when can f be 'lifted' to E? I.e. when does there exist an $\tilde{f}\colon X \to E$ such that $p \circ \tilde{f} = f$? In this section, we'll only consider $X = [0, 1]$, $X = [0, 1]^2$.

Definition 9. Let $p\colon E \to B$ be a map. If f is a continuous mapping of some space X into B, a ***lifting*** of f is a map $\tilde{f}\colon X \to E$ such that $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Lemma 3 (54.1, Important result). Suppose (E,p) is a covering of B, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$. Suppose that $f\colon I \to B$ is a path starting at b_0. Then there exists a unique lift $\tilde{f}\colon I \to E$ of f with $\tilde{f}(0) = e_0$.

Proof. For any b of B, we choose an open U_b such that U_b is evenly covered by p. Then $\{f^{-1}(U_b) \mid b \in B\}$ is an open cover of I, which is compact. There is a number $\delta > 0$ such that any subset of I of diameter $\leq \delta$ is contained entirely in one of these opens $f^{-1}(U_b)$. (Lebesgue number lemma). Now, we divide the interval into pieces $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $|s_{i+1} - s_i| \leq \delta$. For nay i, we have that $f([s_i, s_{i+1}]) \subset U_b$ for some b. □

Lemma 4 (54.2). (E,p) is a covering of B, $b_0 \in B$, $e_0 \in E$, with $p(e_0) = b_0$. Suppose $F\colon I \times I \to B$ is a continuous map with $f(0,0) = b_0$, then there is a unique $\tilde{F}\colon I \times I \to E$. Moreover, if F is a path homotopy, then also $\tilde{F}$ is a path homotopy.

Proof. Same as in the one dimensional case. □

Theorem 4 (54.3). Let (E,p) be a covering of B, $b_0 \in B$, $e_0 \in E$ with $p(e_0) = b_0$. Let f, g be two paths in B starting in b_0 s.t. $f \simeq_p g$ (so f and g end at the same point). Let $\tilde{f}, \tilde{g}$ be the unique lifts of f, g starting at e_0. Then $\tilde{f} \simeq_p \tilde{g}$, and so $\tilde{f}(1) = \tilde{g}(1)$.

Proof. $F\colon I \times I \to B$ is a path homotopy between f and g. Then $\tilde{F}\colon I \times I \to E$ with $\tilde{F}(0,0) = e_0$. Then $\tilde{F}$ is a path homotopy, by the previous result, between $\tilde{F}(\cdot, 0)$ and $\tilde{F}(\cdot, 1)$. Note that $p \circ \tilde{F}(t,0) = F(t,0) = f(t)$ and $p \circ \tilde{F}(t,1) = F(t,1) = g(t)$. By uniqueness $\tilde{F}(\cdot,0) = \tilde{f}$, $\tilde{F}(\cdot,1) = \tilde{g}$. □

We've Shown that homotopy from below lifts to above. The converse is easy. Now we're ready to discuss the relation between groups and covering spaces.

Definition 10. Let (E,p) be a covering of B. $b_0 \in B$, $e_0 \in E$ and $p(e_0) = b_0$. Then the ***lifting correspondence*** is the map

$$\begin{aligned} \phi\colon \pi(B, b_0) &\longrightarrow p^{-1}(b_0) \\ [f] &\longmapsto \tilde{f}(1) \end{aligned}$$

where $\tilde{f}$ is the unique lift of f, starting at e_0. This is well-defined because $[f] = [g] \Rightarrow \tilde{f} \simeq_p \tilde{g} \Rightarrow \tilde{f}(1) = \tilde{g}(1)$. This ϕ depends on the choice of e_0.

Theorem 5 (54.4)**.** It E is path connected, then ϕ is a surjective map. If E is simply connected, then ϕ is a bijective map.

Proof. Suppose E is path connected, and let $e_0, e_1 \in p^{-1}(b_0)$. Consider a path $\tilde{f}: I \to E$ with $\tilde{f}(0) = e_0$ and $\tilde{f}(1) = e_1$. This is possible because E is path connected. Let $f = p \circ \tilde{f}: I \to B$ with $f(0) = p(e_0) = b_0$ and $f(1) = p(e_1) = b_0$, so f is a loop based at b_0. So f is a loop at b_0 and its unique lift to E starting at e_0 is $\tilde{f}$. Hence $\phi[f] = \tilde{f}(1) = e_1$, which shows that ϕ is surjective.

Now assume that E is simply connected (group is trivial). Consider $[f], [g] \in \pi(B_0)$ with $\phi[f] = \phi[g]$. This implies $\tilde{f}(1) = \tilde{g}(1)$. These start at e_0. It follows from Lemma 2 that $\tilde{f} \simeq_p \tilde{g}$. □

9.55 Retractions and fixed points

Definition 11. Let $A \subset X$, then A is a ***retract*** of X iff there exists a map $r: X \to A$ such that $r|_A = 1|_A$, i.e. $r(a) = a$ for all $a \in A$. The map r is called a ***retraction***.

Chapter 10

Separation theorems in the plane

10.63 Jordan curve theorem

https://en.wikipedia.org/wiki/Jordan_curve_theorem

Chapter 11

Seifert–Van Kampen theorem

https://en.wikipedia.org/wiki/Seifert%E2%80%93Van_Kampen_theorem

Note. This doesn't follow the book very well.

Definition 12. A ***free group*** on a set X consists of a group F_X and a map $i\colon X \to F_X$ such that the following holds: For any group G and any map $f\colon X \to G$, there exists a unique morphism of groups $\phi\colon F_X \to G$ such that

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F_X \arrow[d, "\exists!\phi"] \\ & G \end{tikzcd}.$$

Note. The free group of a set is unique. Suppose $i\colon X \to F_X$ and $j\colon X \to F'_X$ are free groups.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"'] & F_X \arrow[d, "\exists\phi"] \\ & F'_X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "i"'] & F'_X \arrow[d, "\exists\psi"] \\ & F_X \end{tikzcd}.$$

Then

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i"'] & F_X \arrow[d, "\psi \circ \phi"] \\ & F_X \end{tikzcd}.$$

Then by uniqueness, $\psi \circ \phi$ is 1_{F_X}, and likewise for $\phi \circ \psi$.

Note. The free group on a set always exists. You can construct it using "irreducible words".

Example. Consider $X = \{a, b\}$. An example of a word is $aaba^{-1}baa^{-1}bbb^{-1}a$. This is not a irreducible word. The reduced form is $aaba^{-1}bba = a^2ba^{-1}b^2a$. Then F_X is the set of irreducible words.

Example. If $X = \{a\}$, then $F_x = \{a^z \mid z \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. Exercise: check that $\mathbb{Z}$ satisfies the universal property.

Example. If $X = \emptyset$, then $F_X = 1$.

Definition 13. Let G_i with $i \in I$, be a set of groups. Then the ***free product*** of these groups denoted by $*_{i\in I} G_i$ is a group G together with morphisms $j_i \colon G_i \to G$ such that the following universal property holds: Given any group H and a collection of morphisms $f_i \colon G_i \to H$, then there exists a unique morphism $f \colon G \to H$, such that for all $i \in I$, the following diagram commutes:

$$\begin{tikzcd} G_i \arrow[r, "j_i"] \arrow[dr, "f_i"'] & G \arrow[d, "\exists! f"] \\ & H \end{tikzcd} .$$

Note. Again, $*_{i\in I} G_i$ is unique.

11.70 The Seifert–Van Kampen theorem

Theorem 6 (70.1, Seifert–Van Kampen theorem)**.** Let $X = U \cup V$ where $U, V, U \cap V$ are open and path connected.[a] Let $x_0 \in U \cap V$. For any group H and 2 morphisms $\Phi_1 \colon \pi(U, x_0) \to H$ and $\Phi_2 \colon \pi(V, x_0) \to H$ such that $\Phi_1 \circ i_1$ and $\Phi_2 \circ i_2$, there exists exactly one $\Phi \colon \pi(X, x_0) \to H$ making the diagram commute

$$\begin{tikzcd} & \pi(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\ \pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x, x_0) \arrow[r, dashed, "\Phi"] & H \\ & \pi(V, x_0) \arrow[u, "j_2"] \arrow[ur, "\Phi_2"'] & \end{tikzcd} .$$

i_1, i_2, i, j_1, j_2 are induced by inclusions.

[a] Note that U, V should also be path connected!

Theorem 7 (70.2, Seifert–Van Kampen theorem (classical version))**.** Assume the hypotheses of the Theorem 6. Let $j\colon \pi(U,x_0) * \pi(V,x_0) \to \pi(X,x_0)$ (induced by j_1 and j_2). On elements of $\pi(U,x_0)$ it acts like j_1, on elements of $\pi(V,x_0)$ it acts like j_2.

$$\begin{tikzcd}
G_1 \arrow[d] \arrow[dr, "f_1"] & \\
G_1 * G_2 \arrow[r, dashed, "f"] & H \\
G_2 \arrow[u] \arrow[ur, "f_2"'] &
\end{tikzcd}.$$

Then j is surjective[a] and the kernel of j is the normal subgroup of $\pi(U,x_0) * \pi(U,x_0)$ generated by all elements of the form $i_1(g)^{-1}i_2(g)$, were $g \in \pi(U \cap V, x_0)$.

[a]This is the only place where algebraic topology is used. We've proved this last week. The groups U and V generate the whole group. The rest of this theorem follows from the previous theorem.

Proof.
- j is surjective. (later)
- Let N be the normal subgroup generated by $i_1(g)^{-1}i_2(g)$. Then we claim that $N \subset \ker(j)$. This means we have to show that $i_1(g)^{-1}i_2(g) \in \ker j$. $j(i_1(g)) = j_1(i_1(g))$ by definition of j. Looking at the diagram, we find that $j_1(i_1(g)) = j_2(i_2(g)) = i(g) = j(i_2(g))$. Therefore $j(i_1(g)^{-1}i_2(g)) = 1$, which proves that elements of the form $i_1(g)^{-1}i_2(g)$ are in the kernel.
- Since $N \subset \ker j$, there is an induced morphism

  $$\begin{aligned} k\colon (\pi_1(U,x_0) * \pi_1((V,x_0))/N &\longrightarrow \pi_1(X,x_0) \\ gN &\longmapsto j(g). \end{aligned}$$

  To prove that $N = \ker j$, we have to show that k is injective. Because this would mean that we've divided out the whole kernel of j.

  Now we're ready to use the previous theorem. Let $H = (\pi(U) * \pi(V))/N$. Repeating the diagram:

  $$\begin{tikzcd}
  & \pi(U,x_0) \arrow[d, "j_1"] \arrow[dr, "\Phi_1"] & \\
  \pi(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r, "i"] \arrow[dr, "i_2"'] & \pi(x,x_0) \arrow[r, dashed, shift left, "\Phi"] & H \arrow[l, shift left, "k"] \\
  & \pi(V,x_0) \arrow[u, "j_2"'] \arrow[ur, "\Phi_2"'] &
  \end{tikzcd}.$$

  Now, we define $\Phi_1\colon \pi(U,x_0) \to H$ given by $g \mapsto gN$, and $\Phi_2\colon \pi(V,x_0) \to H$ given by $g \mapsto gN$. For the theorem to work, we needed that $\Phi_1 \circ i_1 = \Phi_2 \circ i_2$. This is indeed the case: let

$g \in \pi(U \cap V)$. Then $\Phi_1(i_1(g)) = i_1(g)N$ and $\Phi_2(i_2(g)) = i_2(g)N$ and $i_1(g)N = i_2(g)N$ because $i_1(g)^{-1}i_2(g) \in N$.

The conditions of the previous theorem are satisfied, so there exists a Φ such that the diagram commutes.

Note that we also have $k\colon H \to \pi(X)$. We claim that $\Phi \circ k = 1_H$, which would mean that k is injective, concluding the proof. It's enough to prove that

□

Corollary 7.1. Suppose $U \cap V$ is simply connected, so $\pi_1(U \cap V, x_0)$ is the trivial group. In this case $N = \ker j = 1$, hence $\pi(U, x_0) * \pi(V, x_0) \to \pi(X, x_0)$ is an isomorphism.

Corollary 7.2. Suppose U is simply connected. Then $\pi(X, x_0) \cong \pi(V, x_0)/N$ where N is the normal subgroup generated by the image of $i_2\colon \pi(U \cap V) \to \pi(V, x_0)$.

Example. Let X be the figure 8 space.

Chapter 12

Classification of surfaces

Chapter 13

Classification of covering spaces

Lemma 5 (79.1, General lifting lemma)**.** Let $p\colon E \to B$ be a covering, Y a space. Assume B, E, Y are path connected, and locally path connected.[a] Let $f\colon Y \to B$, $y_0 \in Y$, $b_0 = f(y_0)$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then $\exists \tilde{f}\colon Y \to E$ with $\tilde{f}(y_0) = e_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

iff $f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$. If $\tilde{f}$ exists then it is unique.

[a] From now on, all spaces are locally path connected: Every neighborhood contains an open that is path connected.

Proof. Suppose $\tilde{f}$ exists. Then $p \circ \tilde{f} = f$, so $(p \circ \tilde{f})_* \pi(Y, y_0) = \pi(Y, y_0)$. The left hand side is of course $p_*(\tilde{f}_*(\pi(Y, y_0)) \subset p_*(\pi(E, e_0))$, so $p_*(\pi(E, e_0)) \subset f_*(\pi(Y, y_0))$.

Conversely, we'll show the uniqueness first. Suppose $\tilde{f}$ exists. $p \circ (\tilde{f} \circ \alpha) = f \circ \alpha$, so $\tilde{f} \circ \alpha$ is the unique lift of $f \circ \alpha$ starting at e_0. Hence $f(y)$ the endpoint of the unique lift of $f \circ \alpha$ to E starting at e_0. This also shows how you can define $\tilde{f}$: choose a path α from y_0 to y. Lift $f \circ \alpha$ to a path starting at e_0. Define $\tilde{f}(y) =$ the end point of this lift. Problem: is this well defined? A second problem: is $\tilde{f}$ continuous?

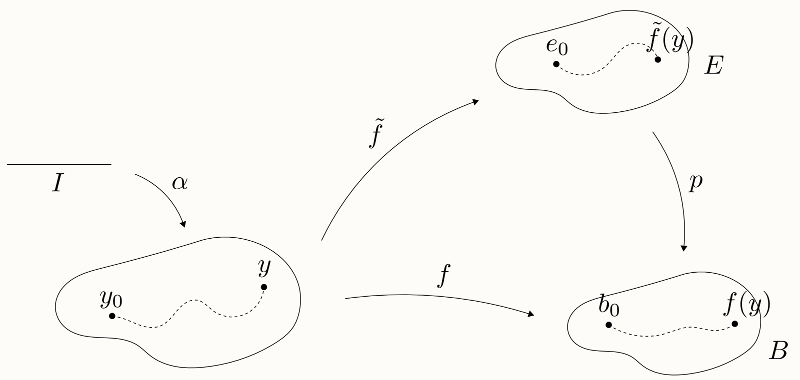


Figure 13.1: General lifting lemma

We prove that $\tilde{f}$ is continuous.

- Choose a neighborhood of $\tilde{f}(y_1)$, say N.
- Take U, a path connected open neighborhood of $f(y_1)$ which is evenly covered, such that the slice $p^{-1}(U)$ containing $\tilde{f}(y_1)$ is completely contained in N.

□

Example. Take $Y = [0, 1]$. Then f is a path, then we showed that every map can be lifted. And indeed, the condition holds: $f_*(\pi(Y, y_0)) = 1$, the trivial group, which is a subgroup of all groups.

Lemma 6 (General lifting lemma, short statement)**.** Short statement:

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

$\exists! \tilde{f} \iff f_*(\pi(Y, y_0)) \subset p_*\pi(E, e_0)$.

Definition 14. Let (E, p) and (E', p') be two coverings of a space B. An ***equivalence*** between (E, p) and (E', p') is a homeomorphism $h \colon E \to E'$ such that

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & B \end{tikzcd}$$

is commutative. $p' \circ h = p$.

Theorem 8 (79.2)**.** Let $p\colon (E,e_0)\to(B,b_0)$ and $p'\colon(E',e_0')\to(B,b_0)$ be coverings, and $H_0=p_*\pi(E,e_0)$ and $H_0'=p_*'\pi(E',e_0')\leq\pi(B,b_0)$. Then there exists and equivalence $h\colon(E,p)\to(E',p')$ with $h(e_0)=e_0'$ iff H_0-H_0'. Not isomorphic, but really the same as a subgroup of $\pi(B,b_0)$. In that case, h is unique.

Proof. $\boxed{\Rightarrow}$ Suppose h exists. Then

$$\begin{tikzcd}(E,e_0) \arrow[r, "h"] \arrow[dr, "p"] & (E',e_0') \arrow[d, "p'"] \\ & (B,b_0)\end{tikzcd}.$$

Therefore $p_*\pi(E,e_0)=p_*'(h_*\pi(E,e_0))$. Since h is a homeomorphism, it induces an isomorphism, so $p_*'(h_*\pi(E,e_0))=p_*'(\pi(E',e_0')$.

$\boxed{\Leftarrow}$

$$\begin{tikzcd} & (E',e_0') \arrow[d, "p'"] \\ (E,e_0) \arrow[r, "p"] \arrow[ur, dashed, "k"] & (B,b_0)\end{tikzcd}$$

By the previous lemma, there exists a unique k iff $p_*\pi(E,e_0)\subset p_*'\pi(E',e_0')$ or equivalently $H_0\subset H_0'$, which is the case. Reversing the roles, we get

$$\begin{tikzcd} & (E,e_0) \arrow[d, "p"] \\ (E',e_0') \arrow[r, "p'"] \arrow[ur, "l"] & (B,b_0)\end{tikzcd}$$

for the same reasoning, l exists. Now, composing the diagrams

$$\begin{tikzcd} & (E,e_0) \arrow[d, "p"] \\ (E,e_0) \arrow[r, "p"] \arrow[ur, dashed, "l\circ k"] & (B,b_0)\end{tikzcd} \qquad \begin{tikzcd} & (E',e_0') \arrow[d, "p'"] \\ (E',e_0') \arrow[r, "p'"] \arrow[ur, dashed, "k\circ l"] & (B,b_0)\end{tikzcd}$$

But placing the identity in place of $l\circ k$ or $k\circ l$, this diagram also commutes! By unicity, we have that $l\circ k=1_E$ and $k\circ l=1_{E'}$. Therefore, k and l are homeomorphism $k(e_0)=e_0'$.

Uniqueness is trivial, because of the general lifting theorem.

$\square$

Note that this doesn't answer the question 'is there a equivalence between two coverings', it only answers the question 'is there an equivalence between two coverings mapping $e_0\to e_0'$'. So now, we seek to understand the dependence of H_0 on the base point.

Lemma 7 (79.3)**.** Let (E, p) be a covering of B. Let $e_0, e_1 \in p^{-1}(b_0)$. Let $H_0 = p_*\pi(E, e_0)$, $H_1 = p_*\pi(E, e_1)$.

- Let γ be a path from e_0 to e_1 and let $p \circ \gamma = \alpha$ be the induced *loop* at b_0. Then $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$, so H_0 and H_1 are conjugate inside $\pi(B, b_0)$.
- Let H be a subgroup of $\pi(B, b_0)$ which is conjugate to H_0, then there is a point $e \in p^{-1}(b_0)$ such that $H = p_*\pi(E, e)$.

So a covering space induces a conjugacy class of a subgroup of $\pi(B, b_0)$.