Stochastic Calculus Instructor: Hyungbin Park

Lecture note 3: Stochastic calculus

1 Exercises

Problem 1.1. (10 points) Let $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{G}_t)_{t\geq 0}$ be two filtrations such that $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \geq 0$. Suppose that M is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ and is adapted to $(\mathcal{G}_t)_{t\geq 0}$. Show that M is a martingale with respect to filtration $(\mathcal{G}_t)_{t\geq 0}$.

Problem 1.2. (10 points) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that the following processes M are martingales with respect to the filtration generated by M.

(i) $M_t = B_t^2 - t$

(ii) $M_t = B_t^3 - 3\int_0^t B_u\,du$

Hint: First, show these with respect to the filtration $(\mathcal{F}_t^B)_{t\geq 0}$ generated by B.

Problem 1.3. (20 points) Let $B = (B_t^{(1)}, B_t^{(2)})_{t\geq 0}$ be a two dimensional Brownian motion. Calculate the followings.

(i)
$$\mathbb{P}(B_t^{(1)} > \sqrt{3}B_s^{(2)})$$

(ii)
$$\mathbb{E}(e^{B_t^{(1)} + \int_0^s B_u^{(2)}\,du})$$

(iii) Consider the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by B. Show that the product $(B_t^{(1)}B_t^{(2)})_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

(iv) Show that the process
$$\left(t^2 B_t^{(1)} B_t^{(2)} - 2\int_0^t sB_s^{(1)}B_s^{(2)}\,ds\right)_{t\geq 0}$$
is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Problem 1.4. (15 points) Let B be a Brownian motion. Show that the first order variation of B is $V_T^1(B) = \infty$ almost surely for $T \geq 0$.

Problem 1.5. (10 points) Let $B = (B_t^{(1)}, B_t^{(2)})_{t\geq 0}$ be a two dimensional Brownian motion. Define $X_t = tB_t^{(1)}$ and $Y = t^2 B_t^{(2)}$ for $t \geq 0$. Find the quadratic covariation $\langle X, Y\rangle_T$.

Problem 1.6. (15 points) Show that a right-continuous adapted process is progressively measurable.

Problem 1.7. Solve the following problems.

(i) (10 points) Show that the set

$$\Sigma := \{A \subseteq [0,\infty) \times \Omega : A \text{ is progressive}\}$$

is a σ-algebra.

(ii) (10 points) Let X be a stochastic process. Show that X is progressively measurable if and only if X is Σ-measurable.

References