수리통계학1 - HW3

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3.1-5. $Y \sim Unif(0,1)$, $W \sim a+(b-a)Y$

(a) Since $Y \sim U(0,1)$

$$F_Y(y) = \begin{cases} 0 & (-\infty < y \leq 0) \\ \frac{x}{1-0} = x & (0 < y < 1) \\ 1 & (1 \leq y < \infty) \end{cases}$$

$$F_W(w) = P(W \leq w) = P(a+(b-a)Y \leq w)$$
$$= P((b-a)Y \leq w-a)$$
$$= P(Y \leq \frac{w-a}{b-a})$$
$$= F_Y(\frac{w-a}{b-a})$$
$$= \begin{cases} 0 & (-\infty < w \leq a) \\ \frac{w-a}{b-a} & (a < w < b) \\ 1 & (b \leq w < \infty) \end{cases}$$

(b). cdf of W is the cdf of Y such that Uniform distribution on $[a,b]$

$\therefore W \sim Unif(a,b)$.

3.1-8

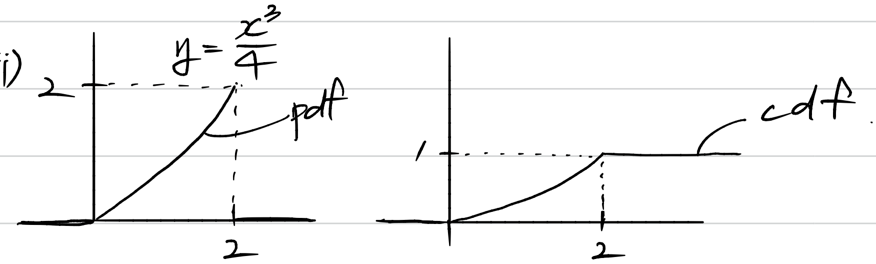
(a) (i) $\int_0^c \frac{1}{4}x^3\,dx = 1$

$\frac{1}{16}c^4 = 1 \quad \therefore c = 2$.

(ii) $\int_0^x \frac{1}{4}t^3\,dt = \frac{1}{16}x^4$

$$F(x) = \begin{cases} 0 & (-\infty < x \leq 0) \\ \frac{1}{16}x^4 & (0 < x < 2) \\ 1 & (2 \leq x < \infty) \end{cases}$$

(iii)



(iv) $\mu = E(X) = \int_0^2 x \cdot \frac{1}{4}x^3\,dx$

$= \frac{1}{4}\int_0^2 x^4\,dx = \frac{1}{20}\left[x^5\right]_0^2 = \frac{1}{20}\cdot 32 = \frac{8}{5}$

$E(X^2) = \int_0^2 x^2 \cdot \frac{1}{4}x^3\,dx = \frac{1}{4}\int_0^2 x^5\,dx$

$= \frac{1}{24}\left[x^6\right]_0^2 = \frac{1}{24}\cdot 64 = \frac{8}{3}$

$\sigma^2 = Var(X) = E(X^2) - \{E(X)\}^2 = \frac{8}{3} - \left(\frac{8}{5}\right)^2 = \frac{8}{75}$.

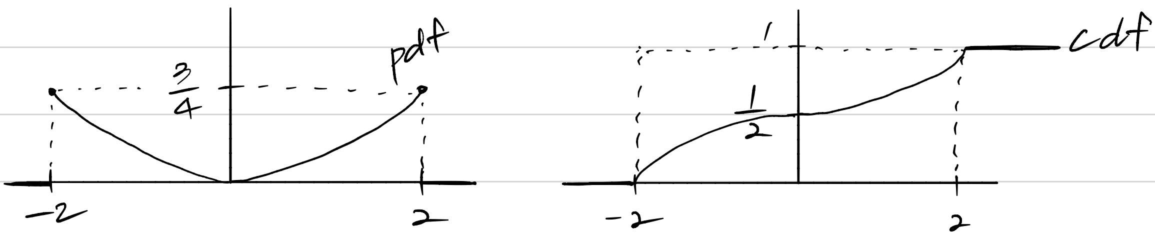
(b). (i) $\int_{-c}^{c} \frac{3}{16}x^2 = 1$

$\frac{1}{16}\left[x^3\right]_{-c}^{c} = \frac{1}{16}\left(c^3 - (-c)^3\right) = \frac{1}{8}c^3 = 1 \qquad \therefore c = 2.$

(ii) $\int_{-2}^{x} \frac{3}{16}t^2\,dt = \frac{1}{16}\left[t^3\right]_{-2}^{x} = \frac{1}{16}(x^3+8) = \frac{x^3}{16} + \frac{1}{2}$

$$F(x) = \begin{cases} 0 & (-\infty < x \leq -2) \\ \frac{1}{16}x^3 + \frac{1}{2} & (-2 < x < 2) \\ 1 & (2 \leq x < \infty) \end{cases}$$

(iii)



(iv) $\mu = E(X) = \int_{-2}^{2} x \cdot \frac{3}{16}x^2\,dx = \frac{3}{16}\int_{-2}^{2} x^3\,dx$

$= \frac{3}{64}\left[x^4\right]_{-2}^{2} = 0.$

$E(X^2) = \int_{-2}^{2} x^2 \cdot \frac{3}{16}x^2 = \frac{3}{16}\int_{-2}^{2} x^4\,dx$

$= \frac{3}{80}\left[x^5\right]_{-2}^{2} = \frac{3}{80}\cdot 64 = \frac{12}{5}$

$\sigma^2 = Var(X) = E(X^2) - \{E(X)\}^2 = \frac{12}{5} - 0 = \frac{12}{5}$

(c) (i) $\int_0^1 c \cdot \frac{1}{\sqrt{x}}\, dx = 1$

$= c \cdot \int_0^1 x^{-\frac{1}{2}}\, dx = c \cdot 2\,[x^{\frac{1}{2}}]_0^1 = 2c = 1 \qquad \therefore c = \frac{1}{2}$

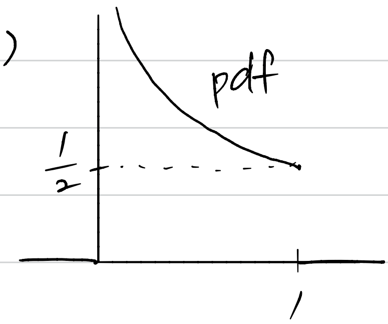
This pdf is unbounded.

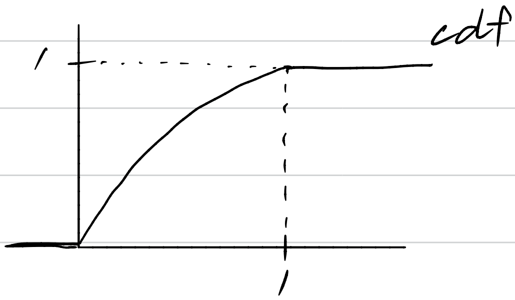
$\left(\because \lim_{x \to 0+} \frac{1}{\sqrt{x}} = \infty\right.$

(ii) $\int_0^x \frac{1}{2} \cdot \frac{1}{\sqrt{t}}\, dt = \frac{1}{2} \cdot 2 \cdot [t^{\frac{1}{2}}]_0^x = \sqrt{x}$

$$F(x) = \begin{cases} 0 & (-\infty < x \leq 0) \\ \sqrt{x} & (0 < x < 1) \\ 1 & (1 \leq x < \infty) \end{cases}$$

(iii)





(iv) $\mu = E(X) = \int_0^1 \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot x\, dx$

$= \frac{1}{2} \int_0^1 \sqrt{x}\, dx = \frac{1}{2} \cdot \frac{2}{3} \cdot [x^{\frac{3}{2}}]_0^1 = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

$E(X^2) = \int_0^1 \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot x^2\, dx = \frac{1}{2} \int_0^1 x\sqrt{x}\, dx = \frac{1}{2} \int_0^1 x^{\frac{3}{2}}\, dx$

$= \frac{1}{2} \cdot \frac{2}{5} \cdot [x^{\frac{5}{2}}]_0^1 = \frac{1}{5}$

$\sigma^2 = Var(X) = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{1}{5} - \frac{1}{9} = \frac{9-5}{45} = \frac{4}{45}$

3.1-10. $f(x) = \frac{c}{x^2}, \quad 1 < x < \infty$

(a) $\int_1^\infty \frac{c}{x^2}\, dx = 1$.

$\lim_{a \to \infty} \int_1^a \frac{c}{x^2}\, dx = \lim_{a \to \infty} c \cdot (-1) \cdot [x^{-1}]_1^a$

$= \lim_{a \to \infty} -c \cdot \left(\frac{1}{a} - 1\right) = c = 1. \qquad \therefore c = 1$

(b) $E(X) = \lim_{a \to \infty} \int_1^a \frac{1}{x^2} x\, dx = \lim_{a \to \infty} \int_1^a \frac{1}{x}\, dx$

$= \lim_{a \to \infty} [\ln x]_1^a = \lim_{a \to \infty} \ln a = \infty$

$\therefore$ $E(X)$ is not finite.

3.2-3. $X \sim Exp(\theta)$

$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 \le x$

$P(X > x+y \mid X > x) = \frac{P(X > x+y \cap X > x)}{P(X > x)} = \frac{P(X > x+y)}{P(X > x)}$

since, $P(X > x) = \int_x^\infty f(t)\,dt = \int_x^\infty \frac{1}{\theta} e^{-\frac{t}{\theta}}\,dt$

$= \frac{1}{\theta}\left[-\theta \cdot e^{-\frac{t}{\theta}}\right]_x^\infty = \frac{1}{\theta} \cdot (\theta \cdot e^{-\frac{x}{\theta}})$

$= e^{-\frac{x}{\theta}}$

$\therefore P(X > x+y \mid X > x) = \frac{P(X > x+y)}{P(X > x)}$

$= \frac{e^{-\frac{x+y}{\theta}}}{e^{-\frac{x}{\theta}}} = e^{-\frac{x+y}{\theta} + \frac{x}{\theta}} = e^{-\frac{y}{\theta}}$

$= P(X > y)$.

3.2-13. $X \sim \chi^2(23) \equiv Gamma\left(\frac{23}{2}, 2\right)$.

(a) $P(14.85 < X < 32.01)$

$P(X < 32.01) = 0.900$

$P(X \le 14.85) = 0.100$

$\therefore P(14.85 < X < 32.01) = 0.900 - 0.100 = 0.800$.

(b) $P(a < X < b) = 0.95$

$P(X < a) = 0.025 \Rightarrow a = 11.69$

$P(a < X < b) = P(X < b) - P(X < a)$

$= P(X < b) - 0.025 = 0.95$

$\therefore P(X < b) = 0.975 \Rightarrow b = 38.08$.

(c) $\mu = E(X) = r = 23$.

$\sigma^2 = 2r = 2 \cdot 23 = 46$

(d) $\chi^2_{0.05}(23) = 35.17$

$\chi^2_{0.95}(23) = 13.09$.

3.2-22. $f(x) = \dfrac{e^{-x}}{(1+e^{-x})^2} \qquad -\infty < x < \infty$

then $Y = \dfrac{1}{1+e^{-x}} \sim U(0.1)$.

pf). cdf of X.

$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \dfrac{e^{-t}}{(1+e^{-t})^2}\,dt$

$(1+e^{-t} = w, \quad -e^{-t}dt = dw)$.

$= \int_{\infty}^{1+e^{-x}} \left(-\dfrac{1}{w^2}\right) dw = \left[\dfrac{1}{w}\right]_{\infty}^{1+e^{-x}} = \dfrac{1}{1+e^{-x}}$

cdf of Y

$F_Y(y) = P(Y \le y) = P\left(\dfrac{1}{1+e^{-x}} \le y\right)$

$= P\left(1+e^{-x} \ge \dfrac{1}{y}\right) = P\left(e^{-x} \ge \dfrac{1}{y} - 1\right)$

$= P\left(-x \ge \ln\left(\dfrac{1}{y}-1\right)\right) = P\left(x \le -\ln\left(\dfrac{1}{y}-1\right)\right)$

$= F_X\left(-\ln\left(\dfrac{1}{y}-1\right)\right)$

$= \dfrac{1}{1+e^{\ln\left(\frac{1}{y}-1\right)}} = \dfrac{1}{1+\frac{1}{y}-1} = y$.

$\therefore$ pdf of $Y = F_Y'(y) = 1 = f_U(u)$.

$\therefore Y \sim U(0.1)$

3.3-6.

$M_X(t) = e^{166t + 200t^2} \Rightarrow$ mgf of normal distribution.

(a) $M(t) = e^{166t + 400t^2/2} \qquad \therefore \mu = 166$

(b) $\dfrac{\sigma^2}{2} = 200 \Rightarrow \sigma^2 = 400$

(c) $P(170 < X < 200) = P\left(\dfrac{170-166}{20} < \dfrac{X-\mu}{\sigma} < \dfrac{200-166}{20}\right)$

$= P(0.2 < Z < 1.7) = 0.3761$

(d) $P(148 \le X \le 172) = P\left(\dfrac{148-166}{20} \le \dfrac{X-\mu}{\sigma} \le \dfrac{172-166}{20}\right)$

$= P(-0.9 \le Z \le 0.3) = 0.4338$.

3.3 − 10.

$X \sim N(\mu, \sigma^2)$

WTS $\quad Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$, $\quad a \neq 0$.

Pf). $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

cdf of Y is

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P(aX \le y - b)$$
$$= P\left(X \le \frac{y-b}{a}\right)$$
$$= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

Let $at + b = w$, $a\,dt = dw \quad dt = \frac{1}{a} dw$.

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\frac{w-b}{a} - \mu)^2}{2\sigma^2}} \cdot \frac{1}{a} dw$$
$$= \int_{-\infty}^{y} \frac{1}{a\sqrt{2\pi\sigma^2}} e^{-\frac{(w-(a\mu+b))^2}{2(a\sigma)^2}} dw$$

It is the distribution $N(b + a\mu, a^2\sigma^2)$.

The case when $a < 0$ can be handled similarly

$\therefore Y = aX + b \sim N(a\mu + b, (a\sigma)^2)$.