Financial Mathematics Instructor: Hyungbin Park

Lecture note 2: Black-Scholes model

1 Black-Scholes formula

Consider a call option with strike K and maturity T. In the Black-Scholes model, the time-t price of this call option is given by $f(t, S_t)$ where

$$f(t,s) := sN(d_1) - Ke^{-r(T-t)}N(d_2)\,.$$

and

$$d_1 = \frac{\ln(se^{r(T-t)}/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\,,\ d_2 = \frac{\ln(se^{r(T-t)}/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\,.$$

The hedging portfolio can be obtained by computing

$$\frac{\partial f}{\partial s}(t,s) = N(d_1) + sN'(d_1)\frac{\partial d_1}{\partial s} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial s} = N(d_1)\,.$$

For the last equality, we used that

$$sN'(d_1)\frac{\partial d_1}{\partial s} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial s} = 0\,. \tag{1.1}$$

The hedging portfolio (ϕ_t, π_t) is

$$\phi_t = f_s(t, S_t) = N(d_1)\,,\ \pi_t = -Ke^{-rT}N(d_2)\,.$$

2 Exercises

Problem 2.1. Consider the following market model with bank account $G = (G_t)_{t\geq 0}$ and risky asset $S = (S_t)_{t\geq 0}$ given by

$$\begin{aligned} dG_t &= rG_t\,dt \\ dS_t &= \mu t^2 S_t\,dt + \sigma(2+\sin t)S_t\,dB_t \end{aligned}$$

for positive constants r, μ, σ. Let $T > 0$.

(i) (5 points) Find a risk-neutral measure.

(ii) (5 points) What is the risk-neutral dynamics of S?

(iii) (10 points) Price and hedge an option whose payoff is $\log S_T$ at maturity $T = 2\pi$.

(iv) (10 points) Fix $K > 0$. Find the time-0 price of a call option whose payoff is $(S_T - K)_+$ at maturity $T = 2\pi$.

Problem 2.2. (10 points) Prove the equality in Eq.(1.1).
Hint: $d_2 = d_1 - \sigma\sqrt{T-t}$, so $\partial d_1/\partial s = \partial d_2/\partial s$, also $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Problem 2.3. Consider a market with bank account $G \equiv 1$ and two stocks $S^{(1)}$, $S^{(2)}$ given as

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = \mu^{(1)}\,dt + v^{(1)}\,dB_t^{(1)}$$
$$\frac{dS_t^{(2)}}{S_t^{(2)}} = \mu^{(2)}\,dt + \sigma^{(1)}\,dB_t^{(1)} + \sigma^{(2)}\,dB_t^{(2)}$$

for $\mu^{(1)}, \mu^{(2)}, \sigma^{(1)} \in \mathbb{R}$ and $v^{(1)}, \sigma^{(2)} \neq 0$.

(i) (5 points) Find the risk-neutral measure.

(ii) (10 points) For fixed strike $K > 0$, find the price of the option with payoff

$$S_T^{(1)}(S_T^{(1)} - S_T^{(2)})_+$$

and maturity T. Hint: Girsanov theorem.

(iii) (10 points) Use the Markov property to find the time-t price for $0 \leq t \leq T$.

(iv) (10 points) Find the hedging portfolio.

References