미분방정식 HW 2-2

수리과학부
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양 준 우

3.6-4

$y'' + y = 2\tan t \quad (0 < t < \frac{\pi}{2})$

i) Complementary solution y_c.

$y'' + y = 0$. $\quad r^2 + 1 = 0$. $\quad r = \pm i \quad y_1 = \cos t \quad y_2 = \sin t$

$\Rightarrow y_c = c_1 \cos t + c_2 \sin t$

$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = 1$

ii). particular solution y_p.

Let $y_p := u_1 + u_2 e^{-t}$ $\quad$ Assume $u_1' y_1 + u_2' y_2 = 0$.

$$u_1' = \frac{-y_2 g}{W} = -2\sin t \cdot \tan t = -2 \cdot \frac{1 - \cos^2 t}{\cos t} = -2\sec t + 2\cos t$$

$$u_1 = \int (-2\sec t + 2\cos t)\,dt = -2\ln(\sec t + \tan t) + 2\sin t \quad (\because 0 < t < \tfrac{\pi}{2})$$

$$u_2' = \frac{y_1 g}{W} = 2\cos t \cdot \tan t = 2\sin t \Rightarrow u_2 = -2\cos t$$

$$u_1 y_1 + u_2 y_2 = (-2\ln(\sec t + \tan t) + 2\sin t)\cos t - 2\cos t \sin t$$

$$= -2\cos t \cdot \ln(\sec t + \tan t). = y_p$$

$\therefore y = c_1 \cos t + c_2 \sin t - 2\cos t \cdot \ln(\sec t + \tan t)$.

3.6-6.

$y'' + 4y' + 4y = 2t^{-2} e^{-2t} \quad (t > 0)$

i) $r^2 + 4r + 4 = 0$. $\quad r = -2$. $\quad \Rightarrow y_c = c_1 e^{-2t} + c_2 t e^{-2t}$

$W[y_1, y_2] = e^{-2t}(e^{-2t} - 2te^{-2t}) - te^{-2t} \cdot -2e^{-2t} = e^{-4t}$

ii). $y_p := u_1 y_1 + u_2 y_2$ and $u_1' y_1 + u_2' y_2 = 0$.

$\Rightarrow u_1' = -e^{4t} \cdot te^{-2t} \cdot 2t^{-2} e^{-2t} = -\frac{2}{t} \Rightarrow u_1 = -2\ln t \quad (\because t > 0)$.

$u_2' = e^{4t} \cdot e^{-2t} \cdot 2t^{-2} e^{-2t} = 2t^{-2} \Rightarrow u_2 = -\frac{2}{t}$

$\Rightarrow u_1 y_1 + u_2 y_2 = -2e^{-2t} \ln t - 2e^{-2t} \qquad \therefore y = -2e^{-2t} \ln t + c_1 e^{-2t} + c_2 t e^{-2t}$

3.6 - 12

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t} \quad (0<t<1) \quad y_1 = e^t, \quad y_2 = t$$

$$\Rightarrow y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = \underbrace{-2(t-1)e^{-t}}_{g}, \quad y_1 = y_1' = y_1'' = e^t, \quad y_2' = 1, \quad y_2'' = 0.$$

$$\left.\begin{aligned} y_1'' + \frac{t}{1-t}y_1' - \frac{1}{1-t}y_1 &= e^t\left(1 + \frac{t}{1-t} - \frac{1}{1-t}\right) = 0. \\ y_2'' + \frac{t}{1-t}y_2' - \frac{1}{1-t}y_2 &= \frac{t}{1-t} - \frac{t}{1-t} = 0. \end{aligned}\right) \Rightarrow y_1, y_2 : \text{sol. of homo. eq.}$$

Let $y_p = u_1 y_1 + u_2 y_2$ s.t $u_1' y_1 + u_2' y_2 = 0.$

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = e^t - te^t \neq 0 \quad (\because 0<t<1)$$

$$u_1' = -\frac{y_2 g}{W} = \frac{2t(t-1)e^{-t}}{e^t - te^t} = \frac{2t(t-1)e^{-t}}{e^t(1-t)} = -2te^{-2t}$$

$$\Rightarrow u_1 = \int -2e^{-2t} \cdot t = e^{-2t}t - \int e^{-2t} \cdot 1 \cdot dt = te^{-2t} + \frac{1}{2}e^{-2t}$$

$$u_2' = \frac{y_1 g}{W} = \frac{2(1-t)}{e^t - te^t} = 2e^{-t} \Rightarrow u_2 = -2e^{-t}$$

$$\therefore y_p = u_1 y_1 + u_2 y_2 = te^{-t} + \frac{1}{2}e^{-t} - 2te^{-t} = \frac{1}{2}e^{-t} - te^{-t}$$

3.6 - 13.

$$x^2 y'' + xy' + (x^2 - 0.25)y = 3x^{\frac{3}{2}}\sin x \quad (x>0) \quad y_1 = x^{-\frac{1}{2}} \cdot \sin x$$

$$\Rightarrow y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = \underbrace{3x^{-\frac{1}{2}}\sin x}_{g} \quad y_2 = x^{-\frac{1}{2}}\cos x$$

$$y_1' = -\frac{1}{2}x^{-\frac{3}{2}}\sin x + x^{-\frac{1}{2}}\cos x$$

$$y_1'' = \left(\frac{3}{4}x^{-\frac{5}{2}} - x^{-\frac{1}{2}}\right)\sin x - x^{-\frac{3}{2}}\cos x$$

$$\begin{aligned} y_1'' + \frac{1}{x}y_1' + \left(1 - \frac{1}{4x^2}\right)y_1 = &\left(\cancel{\frac{3}{4}x^{-\frac{5}{2}}} - \cancel{x^{-\frac{1}{2}}}\right)\sin x - \cancel{x^{-\frac{3}{2}}\cos x} \\ &- \cancel{\frac{1}{2}x^{-\frac{5}{2}}\sin x} + \cancel{x^{-\frac{3}{2}}\cos x} \\ &+ \cancel{x^{-\frac{1}{2}}}\sin x - \cancel{\frac{1}{4}x^{-\frac{5}{2}}\sin x} = 0. \end{aligned}$$

$$y_2' = -\frac{1}{2}x^{-\frac{3}{2}}\cos x - x^{-\frac{1}{2}}\sin x$$

$$y_2'' = \left(\frac{3}{4}x^{-\frac{5}{2}} - x^{-\frac{1}{2}}\right)\cos x + x^{-\frac{3}{2}}\sin x$$

$$\begin{aligned} y_2'' + \frac{1}{x}y_2' + \left(1 - \frac{1}{4x^2}\right)y_2 = &\left(\cancel{\frac{3}{4}x^{-\frac{5}{2}}} - \cancel{x^{-\frac{1}{2}}}\right)\cos x + \cancel{x^{-\frac{3}{2}}\sin x} \\ &- \cancel{\frac{1}{2}x^{-\frac{5}{2}}\cos x} - \cancel{x^{-\frac{3}{2}}\sin x} \\ &\cancel{x^{-\frac{1}{2}}}\cos x - \cancel{\frac{1}{4}x^{-\frac{5}{2}}\cos x} = 0. \end{aligned}$$

$\therefore y_1$ and y_2 are solutions of homogeneous equation.

Let $y_p = u_1 y_1 + u_2 y_2$ and $u_1' y_1 + u_2' y_2 = 0$.

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1' = x^{-\frac{1}{2}} \cdot \sin x \left(\cancel{-\frac{1}{2}x^{-\frac{3}{2}}\cos x} - x^{-\frac{1}{2}}\sin x\right)$$
$$- x^{-\frac{1}{2}}\cos x \cdot \left(\cancel{-\frac{1}{2}x^{-\frac{3}{2}}\sin x} + x^{-\frac{1}{2}}\cos x\right) = -\frac{1}{x}$$

$$u_1' = -\frac{y_2 g}{W} = x \cdot x^{-\frac{1}{2}}\cos x \cdot 3x^{-\frac{1}{2}}\sin x = 3\sin x \cdot \cos x = \frac{3}{2}\sin 2x$$

$$u_1 = -\frac{3}{4}\cos 2x$$

$$u_2' = \frac{y_1 g}{W} = -x \cdot x^{-\frac{1}{2}}\sin x \cdot 3x^{-\frac{1}{2}}\sin x = -3\sin^2 x = \frac{3}{2}(\cos 2x - 1)$$

$$u_2 = \frac{3}{4}\sin 2x - \frac{3}{2}x$$

$$u_1 y_1 + u_2 y_2 = -\frac{3}{4}\cos 2x \cdot x^{-\frac{1}{2}}\sin x + \left(\frac{3}{4}\sin 2x - \frac{3}{2}x\right)x^{-\frac{1}{2}}\cos x$$
$$= \frac{3}{4}x^{-\frac{1}{2}}\left(\sin 2x \cdot \cos(-x) + \cos 2x \sin(-x)\right) - \frac{3}{2}x^{\frac{1}{2}}\cos x$$
$$= \underline{\frac{3}{4}x^{-\frac{1}{2}}\sin x} - \frac{3}{2}x^{\frac{1}{2}}\cos x$$

$c' y_1$

$$\therefore y_p = -\frac{3}{2}x^{\frac{1}{2}}\cos x$$

3.8 - 9.

$$u'' + u = F(t) = \begin{cases} F_0 t & (0 \le t \le \pi) \\ F_0(2\pi - t) & (\pi < t \le 2\pi) \\ 0 & (2\pi < t) \end{cases} \qquad u(0) = u'(0) = 0.$$

i) $0 \le t \le \pi$

$u'' + u = F_0 t \Rightarrow u_c = c_1 \cos t + c_2 \sin t$

$u_p = At + B = F_0 t$

$u = F_0 t + c_1 \cos t + c_2 \sin t \qquad u(0) = c_1 = 0.$

$u' = F_0 + c_2 \cos t - c_1 \sin t \qquad u'(0) = F_0 + c_2 = 0. \qquad c_2 = -F_0$

$\therefore u = F_0 t - F_0 \sin t \qquad (u(\pi) = F_0 \pi \quad u'(\pi) = 2F_0)$

ii) $\pi < t \le 2\pi$

$u'' + u = F_0(2\pi - t). \quad \underline{(u(\pi) = F_0\pi,\ u'(\pi) = 2F_0)}$

$\hookrightarrow$ solution $u \in C^2$

$u_p = At + B. = -F_0 t + 2F_0 \pi$

$u = -F_0 t + 2F_0\pi + c_1 \cos t + c_2 \sin t \Rightarrow u(\pi) = F_0\pi - c_1 = F_0\pi. \quad \therefore c_1 = 0.$

$u' = -F_0 + c_2 \cos t - c_1 \sin t \Rightarrow u'(\pi) = -F_0 - c_2 = 2F_0 \quad \therefore c_2 = -3F_0$

$\therefore\ u = -F_0 t + 2F_0\pi - 3F_0 \sin t$

$u(2\pi) = 0, \quad u'(2\pi) = -4F_0$

iii) $t > 2\pi$

$u'' + u = 0 \quad (u(2\pi) = 0, \quad u'(2\pi) = -4F_0).$

$u = c_1 \cos t + c_2 \sin t \quad \Rightarrow \quad u(2\pi) = c_1 = 0.$

$u' = c_2 \cos t - c_1 \sin t \quad \Rightarrow \quad u'(2\pi) = c_2 = -4F_0$

$\therefore\ u = -4F_0 \sin t$

$$\therefore\ u = \begin{cases} F_0 t - F_0 \sin t & (0 \leq t \leq \pi) \\ -F_0 t + 2F_0\pi - 3F_0 \sin t & (\pi \leq t \leq 2\pi) \\ -4F_0 \sin t & (2\pi \leq t). \end{cases}$$