Topology II – Homework 3

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Exercise 52.1 A subset A of $\mathbb{R}^n$ is said to be *star convex* if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

(a) Find a start convex set that is not convex.

(b) Show that if A is star convex, A is simply connected.

Proof. (a) There are many non-convex and star convex subsets of $\mathbb{R}^2$: for example a star ★, two circles ●● whose intersection is not empty, etc.

(b) Assuming $A \subset \mathbb{R}^n$ is star convex, it is easy to show that A is path-connected since there are paths from a_0 to all points of A, for example straight line segment. It remains to show that $\pi_1(A, a_0)$ is a trivial group. Given a loop $f\colon I \to A$ based at a_0, the map $F(s,t) = (1-t)f(s) + ta_0$ takes values in A (since for any given s, $F(s,t)$ lies on the line segment from $f(s) \in A$ to a_0, which is contained in A since A is star-convex), and F gives a path homotopy from f to the constant path at a_0.

□

Exercise 52.2 Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. For an arbitrary loop f based at x_0,

$$\begin{aligned}\hat{\gamma}([f]) &= [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] = [\overline{\beta} * \overline{\alpha}] * [f] * [\alpha * \beta] = [\overline{\beta}] * [\overline{\alpha}] * [f] * [\alpha] * [\beta] \\ &= [\overline{\beta}] * \hat{\alpha}([f]) * [\beta] = \hat{\beta}(\hat{\alpha}([f])) = \hat{\beta} \circ \hat{\alpha}([f]).\end{aligned}$$

□

Exercise 52.3 Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. For every α, β, and a loop f based at x_0,

$$\begin{aligned}\hat{\alpha} = \hat{\beta} &\iff \hat{\alpha}([f]) = \hat{\beta}([f]) \\ &\iff [\overline{\alpha}] * [f] * [\alpha] = [\overline{\beta}] * [f] * [\beta] \\ &\iff [f] * [\alpha] * [\overline{\beta}] = [\alpha] * [\overline{\beta}] * [f] \\ &\iff [f] * [\alpha * \overline{\beta}] = [\alpha * \overline{\beta}] * [f].\end{aligned}$$

Note that $\alpha * \overline{\beta}$ is an arbitrary loop based at x_0 that passes through x_1. So, if the group of the path homotopy classes of loops based at x_0 is abelian, then the right hand side expression holds for arbitrary α, β and f, therefore, $\hat{\alpha} = \hat{\beta}$ for every α, β. Conversely, if $\hat{\alpha} = \hat{\beta}$ for every α, β, then we have shown that the group is commutative when at least one of the terms is a path homotopy class of a loop passing through x_1. So, take arbitrary $[f], [g] \in \pi_1(X, x_0)$ and take any path α from x_0 to x_1. Then, $g * \alpha * \overline{\alpha}$ is a loop based at x_0 passing through x_1. Then, $[f] * [g * \alpha * \overline{\alpha}] = [g * \alpha * \overline{\alpha}] * [f]$, but $[g * \alpha * \overline{\alpha}] = [g]$. □

Exercise 52.4 Let $A \subset X$; suppose $r \colon X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a *retraction* of X onto A.) If $a_0 \in A$, show that $r_* \colon \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.

Proof. Let $r \colon X \to A$ be a retraction, and $i \colon A \to X$ the inclusion map. Since $r(a) = a$ for each $a \in A$, we have that $r \circ i = \mathrm{id}_A$. The induced maps $r_* \colon \pi_1(X, a_0) \to \pi_1(A, a_0)$ and $i_* \colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ compose to $r_* \circ i_* = (r \circ i)_* = \mathrm{id}_{\pi_1(A, a_0)}$. This implies that r_* is surjective: for any $[f] \in \pi_1(A, a_0)$, we can write $[f] = r_*(i_*([f]))$ so there exists an element of $\pi_1(X, a_0)$, namely $i_*([f])$, which maps to $[f]$ under r_*. □

Exercise 53.4 Let $q \colon X \to Y$ and $r \colon Y \to Z$ be covering maps; let $p = r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then p is a covering map.

Proof. Assume $q \colon X \to Y$ and $r \colon Y \to Z$ are covering maps, and $r^{-1}(z)$ is finite for $z \in Z$. Then $p = r \circ q \colon X \to Z$ is continuous and surjective, and any point $z \in Z$ has a neighborhood $U \subset Z$ which is evenly covered by a finite number of slices $V_1, \ldots, V_k \subset Y$ (disjoint neighborhoods of the points of $r^{-1}(z) = \{y_1, \ldots, y_k\}$) which are homeomorphic to U respectively. Now, because q is a covering map, each y_i has a neighborhood W_i which is evenly covered by q, so $q^{-1}(W_i)$ is a union of slices each mapping homeomorphically to W_i. Replacing W_i by $W_i \cap V_i$ if necessary, we can assume that $W_i \subset V_i$.

Since $W_i \subset V_i$ is a neighborhood of y_i and r maps V_i homeomorphically to U, $r(W_i) \subset U$ is a neighborhood of z, and so the finite intersection $U' = r(W_1) \cap \cdots \cap r(W_k)$ is also a neighborhood of z, with $z \in U' \subset U$. Now, $r^{-1}(U')$ is the union of k disjoint open subsets $V_1', \ldots, V_k'$ of Y, each satisfying $y_i \in V_i' \subset W_i$. Thus V_i' is also evenly covered by q, i.e. $q^{-1}(V_i')$ is a union of disjoint open subsets of X which map homeomorphically to V_i'. Considering all of these for $1 \leq i \leq k$, we find that $p^{-1}(U') = q^{-1}(r^{-1}(U')) = \bigcup_i q^{-1}(V_i')$ is a disjoint union of slices which are mapped homeomorphically onto V_i' by q and hence onto U' by $p = r \circ q$, i.e. $\{q^{-1}(V_i')\}$ is a partition of $p^{-1}(U')$ into slices. So p is a covering map. □

Exercise 53.5 Show that the map of Example 3 is a covering map. Generalize to the map $p(z) = z^n$.

Proof. Consider a general map $p \colon S^1 \to S^1$ defined by $p(z) = z^n$. Let $U_1 = \{z = e^{i\phi} \mid \phi \in (a, b) \subset (0, 2\pi)\}$, then $p^{-1}(U_1) = \{z = e^{i\phi} \mid \phi \in (\frac{a}{n} + \frac{2\pi k}{n}, \frac{b}{n} + \frac{2\pi k}{n}), k = 0, \ldots, n-1\}$ is the union of n disjoint open arcs in S^1, and each arc is homeomorphic to U_1 which is an open arc too. Similarly, for $U_2 = \{z = e^{i\phi} \mid \phi \in (a, b), -\pi \leq a < 0 < b \leq \pi\}$, $p^{-1}(U_2) = \{z = e^{i\phi} \mid \phi \in$

$(\frac{a}{n}+\frac{2\pi k}{n}, \frac{b}{n}+\frac{2\pi k}{n},), k=1,\dots,n-1\}$ is also the union of n disjoint open arcs in S^1 which are homeomorphic to U_2. Overall, every point of S^1 has such an evenly covered open neighborhood. $\square$

Exercise 53.6 Let $p\colon E \to B$ be a covering map.

(a) If B is Hausdorff, regular, completely regular, or locally compact Hausdorff, then so is E. (Hint: If $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices, and C is a closed set of B such that $C \subset U$, then $p^{-1}(C) \cap V_\alpha$ is a closed set of E.)

(b) If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

Proof. Regarding the hint: $p^{-1}(C)$ is closed in E, so that it has no limit points outside of $p^{-1}(U)$, but $p^{-1}(C) \cap V_\alpha$ is closed in $p^{-1}(U)$. The hint immediately implies that if B is a T_1-space then so is E.

(a) If B is Hausdorff, take two points $x, y \in E$, and if $p(x) = p(y) = z$, then z has a neighborhood U evenly covered by p so that x and y are in different open sets of $p^{-1}(z)$ homeomorphic to U, but if $z = p(x) \neq z' = p(y)$ then z and z' have disjoint open neighborhoods and their preimages are disjoint open neighborhoods of x and y.

If B is regular (locally compact Hausdorff), then using the fact that a T_1-space (Hausdorff space) is regular (locally compact) iff for every point x and its open neighborhood U there is an open neighborhood V of x such that $\overline{V} \subset U$ (is compact) by Lemma 31.1(a) (Theorem 29.2), and knowing already that E is a T_1-space (Hausdorff space), if $y = p(x) \in p(U) \subset B$ where $p(U)$ is open (p is open), we take a neighborhood $V \subset B$ of y evenly covered by p so that its preimage $p^{-1}(V) = \bigcup_\alpha V_\alpha \subset E$, some α such that $x \in V_\alpha$, and finally, a neighborhood $W \subset V \cap p(U)$ such that $\overline{W} \subset V \cap p(U)$ (is compact), to have $p' = p|_{V\alpha \cap U}$, a homeomorphism between $V_\alpha \cap U$ and $V \cap p(U)$, and $U' = p'^{-1}(W)$, an open neighborhood of x such that $\overline{U'} = p'^{-1}(\overline{W}) \subset V_\alpha \cap U \subset U$ (p' is a homeomorphism) is a closed (compact) subset of $V_\alpha \cup U$, and of E (using hint).

If B is completely regular, for every point $x \in E$ and its open neighborhood U, if $y = p(x)$ and its open neighborhood V is evenly covered by p ($p^{-1}(V) = \bigcup_\alpha V_\alpha$, $x \in V_\beta$), there is f such that $f(y) = 1$ and $f(B - W) = f(B - V \cap p(U)) = \{0\}$ (where $p(U)$, and hence, W, is open as p is open), and $f'\colon E \to [0,1]$ such that $f'(e) = f \circ p(e)$ if $e \in U \cap V_\beta$ and $f'(e) = 0$ otherwise is continuous on both closed sets $E - \bigcup_{\alpha\neq\beta} V_\alpha$ (where it equals $f \circ p$) and $E - V_\beta$ (where it equals 0), hence, on E, and is as required.

(b) If $\{U_\alpha\}$ covers E, for a point $y \in B$ choose, first, V^y evenly covered by p, $p^{-1}(V^y) = \bigcup_{i=1}^{n_y} V_i^y$, and then $V_y = \bigcap_{i=1}^{n_y} p(W_i^y)$ where $W_i^y = V_i^y \cap U_{\alpha_i^y}$ for some α_i^y such that $W_i^y \cap p^{-1}(y) \neq \emptyset$, noting that V_y is then an open neighborhood of y evenly covered by p, then take a finite subcover of B, $\{V_{y_j}\}_{j=1}^n$, and consider $\{U_{\alpha_k^{y_j}}\}_{j\in\overline{1,n},\, k\in\overline{1,n_{y_j}}}$ such that if $x \in E$ and $p(x) \in V_{y_j}$, then $x \in p^{-1}(V_{y_j}) = \bigcap_{k=1}^{n_{y_j}} p^{-1}(p(W_k^{y_j})) \subset \bigcup_{i=1}^{n_{y_j}} U_{\alpha_i^{y_j}}$.

$\square$