Homework 10

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Exercise 3-3.13 Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the map (a similarity) defined by $F(p) = cp$, $p \in \mathbb{R}^3$, c a positive constant. Let $S \subset \mathbb{R}^3$ be a regular surface and set $F(S) = \overline{S}$. Show that $\overline{S}$ is a regular surface, and find formulas relating the Gaussian and mean curvatures, K and H, of S with the Gaussian and mean curvatures, $\overline{K}$ and $\overline{H}$, of $\overline{S}$.

Proof. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a local parametrization. Then the map

$$\overline{\mathbf{x}}(u, v) = (F \circ \mathbf{x})(u, v) = c\mathbf{x}(u, v)$$

locally parametrizes the surface $\overline{S}$. Specially, $\overline{\mathbf{x}}_u = c\mathbf{x}_u$ and $\overline{\mathbf{x}}_v = c\mathbf{x}_v$. Now since S is regular, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent, which implies that $\overline{\mathbf{x}}_u$ and $\overline{\mathbf{x}}_v$ are linearly independent. Hence, $\overline{S}$ is regular. We also have that

$$\overline{E} = \overline{\mathbf{x}}_u \cdot \overline{\mathbf{x}}_v = (c\mathbf{x}_u) \cdot (c\mathbf{x}_u) = c^2(\mathbf{x}_u \cdot \mathbf{x}_u) = c^2 E$$

and the same way $\overline{F} = c^2 F$ and $G = c^2 \overline{G}$. We also have

$$\overline{n} = \frac{\overline{\mathbf{x}}_u \times \overline{\mathbf{x}}_v}{\|\overline{\mathbf{x}}_u \times \overline{\mathbf{x}}_v\|} = \frac{c^2(\mathbf{x}_u \times \mathbf{x}_v)}{c^2\|\mathbf{x}_u \times \mathbf{x}_v\|} = n$$

Hence,

$$\overline{L} = \overline{n} \cdot \overline{\mathbf{x}}_{uu} = n \cdot (c\mathbf{x}_{uu}) = c(n \cdot \mathbf{x}_{uu}) = cL$$

and the same way $\overline{M} = cM$ and $\overline{N} = cN$. Therefore,

$$\overline{K} = \frac{\overline{LN} - \overline{M}^2}{\overline{EG} - \overline{F}^2} = \frac{cL \cdot cN - (cM)^2}{c^2E \cdot c^2G - (c^2F)^2} = \frac{1}{c^2} \cdot \frac{LN - M^2}{EG - F^2} = \frac{K}{c^2}$$
$$\overline{H} = \frac{\overline{EN} - 2\overline{FM} + \overline{GL}}{2(\overline{EG} - \overline{F}^2)} = \frac{c^3}{c^4} \cdot \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{H}{c}$$

$\square$