Financial Mathematics I (SNU) – Exam 3

June 18, 2020

1. Let $T > 0$. Consider the market model with bank account $(G_t)_{0\le t\le T} = (e^{rt})_{0\le t\le T}$ and risky asset $(S_t)_{0\le t\le T}$ given by SDE

$$dS_t = \mu(t, S_t)S_t\,dt + \sigma(t, S_t)S_t\,dB_t.$$

Assume that this SDE has a unique solution.

(a) Consider an European option with payoff $g(S_T)$ for a Borel function g having polynomial growth. Use the heuristic argument to derive the Black-Scholes PDE: Let $f(t, x)$ be the function such that the time-t price of option is $f(t, S_t)$. Then

$$f_t + rxf_x(t, x) + \frac{1}{2}\sigma^2(t, x)x^2 f_{xx}(t, x) - rf(t, x) = 0$$

with the terminal condition $f(T, x) = g(x)$.

(b) Apply the Feynman-Kac formula and express $f(t, x)$ as an expectation form.

2. Consider the following market model with bank account $G = (G_t)_{t\ge 0}$ and risky asset $S = (S_t)_{t\ge 0}$ given by

$$\begin{aligned} dG_t &= rtG_t\,dt \\ dS_t &= \mu S_t\,dt + \sigma S_t\,dB_t \end{aligned}$$

for positive constants r, μ, σ. Let $T > 0$.

(a) Find a risk-neutral measure.

(b) Calculate the time-0 price of an option whose payoff is $\log S_T$ at maturity T.

3. Fix $T > 0$. Consider the market $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ with bank account $G \equiv 1$ and two stocks $S^{(1)}$, $S^{(2)}$ given as

$$\begin{aligned} \frac{dS_t^{(1)}}{S_t^{(1)}} &= \mu^{(1)}\,dt + v^{(1)}\,dB_t^{(1)} \\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= \mu^{(2)}\,dt + \sigma^{(1)}\,dB_t^{(1)} + \sigma^{(2)}\,dB_t^{(2)} \end{aligned}$$

for $\mu^{(1)}, \mu^{(2)}, \sigma^{(1)} \in \mathbb{R}$ and $v^{(1)}, \sigma^{(2)} \neq 0$.

(a) Find the time-t price of the option with payoff

$$S_T^{(1)} S_T^{(2)} \mathbb{I}_{S_T^{(1)} > K}$$

for $K > 0$ and maturity T. Use the cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\,dz.$$

(b) Find the hedging portfolio.

4. Let X be the solution of the SDE

$$dX_t = -\theta X_t\,dt + \sigma\,dB_t$$

for $\theta \neq 0$, $\sigma > 0$ and $X_0 \in \mathbb{R}$. Calculate

$$\mathbb{R}(e^{X_T}(e^{X_T} - K)_+ | \mathcal{F}_t^X)$$

for $0 \le t \le T$.