Stochastic calculus Instructor: Hyungbin Park

Lecture note 3: Stochastic calculus

1 Exercise

Problem 1.1. (45 points) Consider a Brownian motion $B = (B_t)_{t\geq 0}$. For $0 < s < t$, evaluate the followings. Use the cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$$

of the standard normal density if needed.

(i) $\mathbb{P}(B_1 > 1, B_3 - B_2 > 1)$

(ii) $\mathbb{P}(2B_1 - B_2 > -1)$

(iii) $\mathbb{P}(B_1 < 0, B_2 > (1 - \sqrt{3})B_1)$

(iv) $\mathbb{P}(B_3 < 1 \,|\, B_1)$

(v) $\mathbb{P}(B_1 < 2 \,|\, B_2)$

(vi) $\mathbb{E}(B_s^2 e^{2B_t})$

(vii) $\mathrm{Var}(2B_3 - B_2)$

(viii) $\mathrm{Cov}(e^{B_t}, e^{-2B_s})$

(ix) $\mathbb{E}(B_1 + B_3 | B_1 - 2B_2)$.

For random variable X, Y and a Borel set A, the notation $\mathbb{P}(X \in A|Y)$ means $\mathbb{E}(\mathbb{I}_{\{X\in A\}}|\sigma(Y))$.

Problem 1.2. (10 points) Let $0 \leq s < t$. Show that $B_t - B_s$ is independent of $\sigma(B_u | 0 \leq u \leq s)$. Read Problem 1.4 on page 49 in (Karatzas and Shreve, Brownian Motion and Stochastic Calculus, 1991). You can find the solution from the book.

Problem 1.3. Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(i) (10 points) Show that $(X_t)_{t\geq 0} = (\frac{1}{\sqrt{c}} B_{ct})_{t\geq 0}$ is a Brownian motion for any $c > 0$,

(ii) (5 points) Use the time inversion formula and the law of iterated logarithm of Brownian motion to show that for $s \geq 0$

$$\mathbb{P}\left(\liminf_{t\to 0+} \frac{B_{t+s} - B_s}{\sqrt{2t \ln\ln \frac{1}{t}}} = -1\,,\ \limsup_{t\to 0+} \frac{B_{t+s} - B_s}{\sqrt{2t \ln\ln \frac{1}{t}}} = 1 \right) = 1\,.$$

Problem 1.4. Solve the following problems.

(i) (5 points) Let $f : [0, T] \to \mathbb{R}$ be a RCLL function. Show that f is bounded.

(ii) (10 points) Let $(X_t)_{t\geq 0}$ be a RCLL Gaussian process. Show that

$$\Big(\int_0^t X_s\,ds\Big)_{t\geq 0}$$

is a continuous Gaussian process. You may use, without proof, the fact that the limit (in the sense of convergence in distribution) of a sequence of normal random variables is normal.

(iii) (5 points) For $T > 0$, find the distribution of

$$\int_0^T uB_u\,du\,.$$

(iv) (5 points) Calculate

$$\mathbb{E}(e^{\int_0^T uB_u\,du})$$

Problem 1.5. (Fractional Brownian motion) (25 points) Let $0 < H < 1$. A continuous Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with mean zero and covariance

$$\mathrm{cov}(B_s^H, B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$$

is called a fractional Brownian motion with parameter H.

(i) Show that if $H = 1/2$, then B^H is the standard Brownian motion.

(ii) Let B^H be a fractional Brownian motion with parameter H. Show that for any $h > 0$, the process X given by

$$X_t = B_{t+h}^H - B_h^H$$

is a fractional Brownian motion with parameter H.

(iii) Deduce that a fractional Brownian motion has stationary increments, that is, $B_t^H - B_s^H$ has the same distribution with B_{t-s}^H for $0 < s < t$.

(iv) Let $0 \leq u \leq s \leq t$. Evaluate $\mathbb{E}(B_u^H(B_t^H - B_s^H))$. For which H the increment $B_t^H - B_s^H$ is independent of the past $\sigma(B_u^H : 0 \leq B_u^H \leq s)$?

(v) Show that

$$\Big(\int_0^t B_u^H \mathbb{I}_{[1,2)}(u) - 2B_u^H \mathbb{I}_{[3,\infty)}(u)\,du\Big)_{t\geq 0}$$

is a Gaussian process.

Hint: The proofs of the above problems are similar with the Brownian motion case we did in class.

References

금융수학1 - HW4 양준우 (2020-91659)

1.1.(i).

$$P(B_1>1,\ B_3-B_2>1) = P(B_1>1)\cdot P(B_3-B_2>1) \quad (\because B_1 \perp B_3-B_2).$$

$$= P(Z>1)\cdot P(Z>1) \quad (\because B_1 \sim Z,\ B_3-B_2 \sim Z).$$

$$= (N(-1))^2$$

1.1.(ii)

$P(2B_1-B_2>-1)$

Since B_t is Gaussian process, $2B_1-B_2$ is normal

$E(2B_1-B_2)=0.$

$Var(2B_1-B_2) = Var(B_1-(B_2-B_1)) = Var(B_1)+Var(B_2-B_1) = 2$

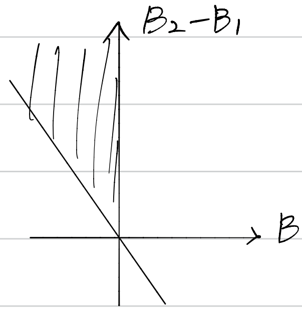
$2B_1-B_2 \sim N(0,2) \simeq \sqrt{2}Z$

$P(2B_1-B_2>-1) = P(Z>-\frac{1}{\sqrt{2}}) = N(\frac{1}{\sqrt{2}}).$

1.1.(iii).

$P(B_1<0,\ B_2>(1-\sqrt{3})B_1) = P(B_1<0,\ (B_2-B_1)>-\sqrt{3}B_1)$

$B_1 \perp (B_2-B_1)$



$$= \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\int_{-\sqrt{3}x}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$

$$= \int_{-\infty}^{0}\int_{-\sqrt{3}x}^{\infty}\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dydx = \int_{0}^{\infty}\int_{\frac{\pi}{2}}^{\frac{2}{3}\pi}\frac{1}{2\pi}e^{-\frac{r^2}{2}}rd\theta dr$$

$$= \int_{0}^{\infty}\frac{1}{2\pi}e^{-\frac{r^2}{2}}\cdot r\cdot\frac{\pi}{6}dr$$

$$= \frac{1}{12}\int_{0}^{\infty}r\cdot e^{-\frac{r^2}{2}} = -\frac{1}{12}\left[e^{-\frac{r^2}{2}}\right]_{0}^{\infty} = \frac{1}{12}$$

1.1. (iv).

$$P(B_3<1|B_1) = P(B_3<1|B_1=x) = P(B_3-B_1<1-x|B_1=x)$$
$$= P(B_3-B_1<1-x) = P\left(\frac{1}{\sqrt{2}}(B_3-B_1)<\frac{1-x}{\sqrt{2}}\right)$$
$$= P\left(Z<\frac{1-x}{\sqrt{2}}\right) = N\left(\frac{1-x}{\sqrt{2}}\right) = N\left(\frac{1-B_1}{\sqrt{2}}\right)$$

1.1. (v).

$$P(B_1<2|B_2) = P(2B_1<4|B_2=x)$$
$$= P(2B_1-B_2<4-x|B_2=x). \quad ①$$
$$= P(2B_1-B_2<4-x)$$
$$= P\left(\frac{1}{\sqrt{2}}(2B_1-B_2)<\frac{4-x}{\sqrt{2}}\right) \quad ②$$
$$= P\left(Z<\frac{4-x}{\sqrt{2}}\right) = N\left(\frac{4-B_2}{\sqrt{2}}\right)$$

① Since B_t is Gaussian process, $\lambda_1 B_1+\lambda_2 B_2$ is normal for $\forall \lambda_1,\lambda_2\in\mathbb{R}$

Thus $\mu_1(2B_1-B_2)+\mu_2 B_2 = 2\mu_1 B_1+(\mu_2-\mu_1)B_2$ is normal $\forall \mu_1,\mu_2\in\mathbb{R}$

$\Rightarrow (2B_1-B_2, B_2)$: normal.

$Cov(2B_1-B_2, B_2) = E(2B_1B_2-B_2^2) = 2\cdot \min(1,2)-2 = 0.$

$\therefore 2B_1-B_2 \perp B_2$

② $2B_1-B_2 \sim N(0,2)$

1.1. (vi) $\quad 0<s<t$

$$E(B_s^2 e^{2B_t}) = E(B_s^2 e^{2B_s} e^{2(B_t-B_s)}) = E(B_s^2 e^{2B_s})E(e^{2(B_t-B_s)})$$
$$= E(B_s^2 e^{2B_s})\cdot e^{2(t-s)}$$

1.1. (vii).

$$Var(2B_3-B_2) = Var(2(B_3-B_2)+B_2) = Var(2\underbrace{(B_3-B_2)}_{\sim N(0,1)}) + Var(\underbrace{B_2}_{\sim N(0,2)}).$$
$$= 4+2 = 6.$$

1.1.(viii).

$$Cov(e^{B_t}, e^{-2B_s}) = E(e^{B_t-2B_s}) - E(e^{B_t})E(e^{-2B_s})$$
$$= e^{\frac{1}{2}var(B_t-2B_s)} - e^{\frac{1}{2}t} \cdot e^{2s}$$
$$= e^{\frac{1}{2}t} - e^{\frac{1}{2}t+2s} = e^{\frac{1}{2}t}(1-e^{2s})$$

1.1.(ix).

$E(B_1+B_3 \mid B_1-2B_2)$.

$Cov(aB_1+bB_2+cB_3,\ B_1-2B_2) = 0$.

$= aE(B_1^2) + bE(B_1B_2) + cE(B_1B_3) - 2aE(B_1B_2) - 2bE(B_2^2) - 2cE(B_2B_3)$

$= a+b+c-2a-4b-4c = -a-3b-3c = 0$.

$\Rightarrow aB_1+bB_2+cB_3 \perp B_1-2B_2$ for $a,b,c \in \mathbb{R}$ s.t $\underline{a+3b+3c=0}$.

$(\because (aB_1+bB_2+cB_3,\ B_1-2B_2)$: normal$)$.

$E(B_1+B_3 \mid B_1-2B_2) = E(B_1+B_3+\lambda(B_1-2B_2)-\lambda(B_1-2B_2) \mid B_1-2B_2)$

choose $\lambda = \frac{4}{5}$,

$= E(B_1+B_3+\frac{4}{5}(B_1-2B_2) \mid B_1-2B_2) - \frac{4}{5}E(B_1-2B_2 \mid B_1-2B_2)$.

$= E(\frac{9}{5}B_1 - \frac{8}{5}B_2 + B_3 \mid B_1-2B_2) - \frac{4}{5}E(B_1-2B_2)$

$(\frac{9}{5} - 3\cdot\frac{8}{5} + 3 = 0 \Rightarrow \frac{9}{5}B_1 - \frac{8}{5}B_2 + B_3 \perp B_1-2B_2)$.

$= E(\frac{9}{5}B_1 - \frac{8}{5}B_2 + B_3) - \frac{4}{5}E(B_1-2B_2) = 0$.

1.2

1.3. (i)

$(X_t)_{t\geq 0} = (\frac{1}{\sqrt{c}}B_{ct})_{t\geq 0}$ B.M for any $c>0$.

① $(\frac{1}{\sqrt{c}}B_{ct})_{t\geq 0}$ Gaussian process.

$\sum_{i=1}^{n} \lambda_i \frac{1}{\sqrt{c}} B_{ct_i} = \lambda_1 \frac{1}{\sqrt{c}} B_{ct_1} + \cdots + \lambda_n \frac{1}{\sqrt{c}} B_{ct_n}$: normal for $^\forall \lambda_i, c \in \mathbb{R}$.

($\because \sum_{i=1}^{n} u_i B_{ct_i}$ is normal $^\forall u_i \in \mathbb{R}$).

② $E(\frac{1}{\sqrt{c}}B_{ct}) = 0$. for $^\forall t$, $^\forall c$

③ $Cov(\frac{1}{\sqrt{c}}B_{ct}, \frac{1}{\sqrt{c}}B_{cs}) = E(\frac{1}{c}B_{ct}B_{cs}) = \frac{1}{c}\cdot \min(ct, cs) = \min(t,s)$

④ $B_t(\omega)$: continuous for $^\forall \omega \in \Omega$

$B_{ct}(\omega) = B_t(\omega)$. ($\because$ sample paths are depend on $\omega \in \Omega$).

$\frac{1}{\sqrt{c}}B_{ct}(\omega)$: also conti. $\square$

1.3. (ii). $X_t = B_{t+s} - B_s$ is B.M.

($\because X_t$ is continuous Gaussian, $E(X_t) = 0$,

$Cov(X_t, X_p) = Cov(B_{t+s}, B_{p+s}) - Cov(B_{t+s}, B_s) - Cov(B_{p+s}, B_s) + Cov(B_s, B_s)$.

$= \min(t+s, p+s) - s - s + s = \min(t,p)$.).

WLOG, Assume $s=0$.

$$P\left(\liminf_{t\to 0+} \frac{B_t}{\sqrt{2t \ln\ln \frac{1}{t}}} = -1, \limsup_{t\to 0+} \frac{B_t}{\sqrt{2t\ln\ln\frac{1}{t}}} = 1 \right)$$

$$P\left(\liminf_{t\to 0+} \frac{\frac{1}{t}B_t}{\sqrt{\frac{2}{t}\ln\ln\frac{1}{t}}} = -1, \limsup_{t\to 0+} \frac{\frac{1}{t}B_t}{\sqrt{\frac{2}{t}\ln\ln\frac{1}{t}}} = 1 \right)$$

$$P\left(\liminf_{t\to \infty} \frac{tB_{1/t}}{\sqrt{2t \ln\ln t}} = -1, \limsup_{t\to \infty} \frac{tB_{1/t}}{\sqrt{2t\ln\ln t}} = 1 \right) = 1$$

(By law of iterated logarithm of Brownian motion,

$(tB_{\frac{1}{t}})_{t\geq 0}$ is B.M.).

① $\sum_{i=1}^{n} \lambda_i t B_{\frac{1}{t}}$ is normal ($\because B_t$: Gaussian) $\Rightarrow tB_{\frac{1}{t}}$ is Gaussian.

② $E(tB_{\frac{1}{t}}) = tE(B_{\frac{1}{t}}) = 0$.

③ $Cov(tB_{\frac{1}{t}}, sB_{\frac{1}{s}}) = ts\, Cov(B_{\frac{1}{t}}, B_{\frac{1}{s}}) = ts \min(\frac{1}{t}, \frac{1}{s}) = \min(t,s)$.

④ B_t continuous $\Rightarrow tB_{\frac{1}{t}}$ is continuous at $t>0$.

$\lim_{t\to 0} tB_{\frac{1}{t}} = 0 = X_0 \Rightarrow tB_{\frac{1}{t}}$ is continuous at $t=0$. $\square$

1.4. (i).

$f : [0,T] \to \mathbb{R}$: RCLL function.

Assume f is unbounded to show contradiction.

Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$. s.t $x_n \to x \in [0,T]$ $f(x_n) \to \infty$ as $n \to \infty$

Split into two subsequences, $\{l_n\} = \{x_n \mid x_n < x\}$, $\{r_n\} = \{x_n \mid x_n > x\}$.

WLOG, assume $\{l_n\}$, $\{r_n\}$ are infinite sequence.

Then $l_n \to x^-$, $r_n \to x^+$, $f(l_n) = f(r_n) = \infty$.

This can not be happen on RCLL function which have left limit right continuous.

Contradiction was shown. $\square$.

1.4.(ii).

$(X_t)_{t\geq 0}$: RCLL Gaussian process.

WTS $\sum_{i=1}^{n} \lambda_i \int_0^{t_i} X_s ds$: normal.

$\Rightarrow \int_0^T \underbrace{\sum_{i=1}^{n} \lambda_i X_s \mathbb{1}_{[0,t_i]}}_{\text{RCLL Gaussian}} ds$, $T = \max\{t_1, \cdots, t_n\}$.

If show that following Lemma, we are done.

Lemma. $(X_t)_{t\geq 0}$: RCLL Gaussian, $T>0$

$\Rightarrow \int_0^T X_u du$ is normal.

pf of Lemma).

Consider a partition $\{0 < \frac{T}{n}, \frac{2T}{n}, \cdots, \frac{kT}{n}, \cdots, T\}$,

an step function $X_t^{(n)}$

$$X_t^{(n)} = \begin{cases} X_{kT/n} & \text{for } \frac{(k-1)T}{n} \leq t < \frac{kT}{n} \quad k = 1, \cdots, n. \\ X_T & \text{for } t = T \end{cases}$$

$$= \sum_{k=1}^{n} X_{\frac{kT}{n}} \mathbb{1}_{[\frac{(k-1)}{n}T, \frac{kT}{n})}(t) + X_T \mathbb{1}_{\{T\}}(t)$$

$X_t^{(n)} \to X_t$ a.s. as $n \to \infty$ for $\forall \omega \in \Omega$

Since X_t is bounded, $\exists C = C(\omega)$ s.t $|X_u(\omega)| \leq C(\omega)$ $\forall u \in [0,T]$

$\therefore \int_0^T X_u^{(n)} du \to \int_0^T X_u du$ ($\because$ By DCT).

fact). $\{Y_n\}$: seq. of normal r.v's.

$Y_n \to Y$ converges in distribution.

$\Rightarrow Y$ is normal.

$\int_0^T X_u^{(n)} du = \frac{T}{n} \sum_{k=1}^{n} X_{\frac{kT}{n}}$ is normal ($\because X_t$ is Gaussian).

$\int_0^T X_u^{(n)} du \to \int_0^T X_u du$ a.s $\Rightarrow \int_0^T X_u^{(n)} du \to \int_0^T X_u du$ (in distribution).

$\Rightarrow \int_0^T X_u du$: normal. $\square$

1.4. (iii). $\int_0^T uB_u du$

$\sum_{i=1}^{n} \lambda_i t B_t$: normal. $\Rightarrow$ tB_t : continuous Gaussian process $t \in [0,T]$

(tB_t defined on compact set $[0,T]$ is bounded.)

By (ii), $(\int_0^t uB_u du)_{t \geq 0}$ is Gaussian. $\Rightarrow$ $\int_0^T uB_u du$: normal.

$E\int_0^T uB_u du = \int_0^T E(uB_u) du$ (by Fubini's thm).

$= 0$. ($\because$ $E(tB_t) = 0$).

$Var(\int_0^T uB_u du) = E((\int_0^T uB_u du)^2) = E(\int_0^T uB_u du \cdot \int_0^T vB_v dv)$ $(u \neq v)$.

$= E(\int_0^T\int_0^T uv B_u B_v du dv)$

$= \int_0^T\int_0^T E(uvB_uB_v) du dv$.

$= \int_0^T\int_0^T uv \cdot min(u,v) du dv$. (WLOG, $u < v$).

$= \int_0^T\int_0^T uv^2 du dv$

$= \frac{T^5}{6}$

$\therefore$ $\int_0^T uB_u du \sim N(0, \frac{T^5}{6})$

1.4. (iv). $\int_0^T uB_u du =: Y \sim N(0, \frac{T^5}{6})$ by (iii)

$E(e^{\int_0^T uB_u du}) = e^{E\int_0^T uB_u du + \frac{1}{2}Var(\int_0^T uB_u du)} = e^{\frac{T^5}{12}}$ $(\phi_Y(-i) = e^{E(Y) + \frac{1}{2}Var(Y)})$

1.5. (i).

$B^H = (B_t^H)_{t \geq 0}$: continuous Gaussian process.

$E(B_t^H) = 0$,

$cov(B_s^H, B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$

$\Rightarrow$ B^H is fractional Brownian motion with parameter H.

WTS : $B^{\frac{1}{2}}$ is standard B.M.

ETS : $cov(B_s^H, B_t^H) = min(s,t)$.

($\because$ already B^H: conti, $E(B^H) = 0$, B^H : Gaussian proess).

$$cov(B_s^{\frac{1}{2}}, B_t^{\frac{1}{2}}) = \frac{1}{2} \cdot (s + t - |t-s|) = \begin{cases} s & \text{if } t \geq s \\ t & \text{if } t \leq s. \end{cases}$$

$= min(s,t)$. $\square$

1.5. (ii).

$X_t = B^H_{t+h} - B^H_h$

① (Gaussian).

$\sum_{i=1}^{n} \lambda_i X_{t_i} = \lambda_1 B^H_{t_1+h} + \cdots + \lambda_n B^H_{t_n} - \left(\sum_{i=1}^{n} \lambda_i\right) B^H_h$: normal.

($\because$ B^H is Gaussian).

② $E(X_t) = E(B^H_{t+h}) - E(B^H_h) = 0$.

③ $cov(X_s, X_t) = cov(B^H_{s+h} - B^H_h,\ B^H_{t+h} - B^H_h)$

$= E((B^H_{s+h} - B^H_h)(B^H_{t+h} - B^H_h))$

$= E(B^H_{s+h} B^H_{t+h}) - E(B^H_{s+h} B^H_h) - E(B^H_{t+h} B^H_h) + E((B^H_h B^H_h)$

$= \frac{1}{2}(\cancel{(s+h)^{2H}} + \cancel{(t+h)^{2H}} - |(t+h)-(s+h)|^{2H}) - \frac{1}{2}(\cancel{(s+h)^{2H}} + \cancel{h^{2H}} - |h-(s+h)|^{2H})$

$- \frac{1}{2}(\cancel{(t+h)^{2H}} + \cancel{h^{2H}} - |h-(t+h)|^{2H}) + \cancel{\frac{1}{2}(2 \cdot h^{2H})}$.

$= \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$.

④ Obviously continuous.

Thus X is a fractional B.M with parameter H.

1.5. (iii).

$B^H_t - B^H_s$: normal. ($\because$ B^H is Gaussian).

B^H_{t-s} is normal

$E(B^H_t - B^H_s) = E(B^H_{t-s}) = 0$.

$Var(B^H_t - B^H_s) = E((B^H_t - B^H_s)^2) = cov(B^H_t - B^H_s,\ B^H_t - B^H_s)$.

$= cov(B^H_t, B^H_t) - 2cov(B^H_t, B^H_s) + cov(B^H_s)$.

$= \frac{1}{2}(2t^{2H}) - 2 \cdot \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H}) + \frac{1}{2}(2s^{2H})$

$= |s-t|^{2H} =$

$Var(B^H_{t-s}) = E(B^H_{t-s} \cdot B^H_{t-s}) = cov(B^H_{t-s}, B^H_{t-s})$

$= (t-s)^{2H}$

$Var(B^H_t - B^H_s) = B(B^H_{t-s})$

$\therefore$ $B^H_t - B^H_s \sim N(0, (t-s)^{2H}) \sim B^H_{t-s}$

$\therefore$ $B^H_t - B^H_s$ has same distribution with B^H_{t-s}.

1.5. (iv). $0 \le u \le s \le t$

$$E(B_u^H(B_t^H - B_s^H)) = E(B_u^H B_{t-s}^H) \quad \text{by (iii)}.$$

$$= Cov(B_u^H, B_{t-s}^H) = \frac{1}{2}(u^{2H} + (t-s)^{2H} - |t-s-u|^{2H}).$$

For $H_0 \in (0,1)$ s.t $E(B_u^{H_0}(B_t^{H_0} - B_s^{H_0})) = 0$.,

$B_t^{H_0} - B_s^{H_0} \perp \sigma(B_u^{H_0})$.

1.5. (v).

$(\int_0^t B_u^H \mathbb{1}_{[1,2)}(u) - 2B_u^H \mathbb{1}_{[3,\infty)}(u)\,du)_{t \ge 0}$.

WTS : $\sum_{i=1}^{n} \lambda_i (\int_0^{t_i} B_u^H \mathbb{1}_{[1,2)}(u) - 2B_u^H \mathbb{1}_{[3,\infty)}(u)\,du)$ is normal.

$\Rightarrow \int_0^T \sum_{i=1}^{n} \lambda_i (B_u^H \mathbb{1}_{[1,2)}(u) \cdot \mathbb{1}_{[0,t_i)}(u) - 2B_u^H \mathbb{1}_{[3,\infty)}(u) \cdot \mathbb{1}_{[0,t_i)}(u))du$

where $T = \max\{t_1, t_2, \cdots, t_n\}$

$B_u^H \mathbb{1}_{[1,2)}(u) \cdot \mathbb{1}_{[0,t_i)}(u) - 2B_u^H \mathbb{1}_{[3,\infty)}(u) \cdot \mathbb{1}_{[0,t_i)}(u)$

: a RCLL Gaussian process. ($\because B_u^H$ is a conti. Gaussian).

$\Rightarrow \sum_{i=1}^{n} \lambda_i (B_u^H \mathbb{1}_{[1,2)}(u) \cdot \mathbb{1}_{[0,t_i)}(u) - 2B_u^H \mathbb{1}_{[3,\infty)}(u) \cdot \mathbb{1}_{[0,t_i)}(u))$: RCLL Gaussian.

By 1.4. (ii),

$\int_0^T \sum_{i=1}^{n} \lambda_i (B_u^H {}_{[1,2)}(u) \cdot \mathbb{1}_{[0,t_i)}(u) - 2B_u^H \mathbb{1}_{[3,\infty)}(u) \mathbb{1}_{[0,t_i)}(u))du$ is Gaussian $\square$.