ADVANCED CALCULUS 1
ASSIGNMENT # 1 : 2019 SPRING

§1.2. # 2. Show that $\dfrac{3^n}{n!}$ converges to 0.

§1.2. # 3. Let $x_n = \sqrt{n^2+1} - n$. Compute $\lim_{n\to\infty} x_n$.

§1.3. # 4. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded below and define $A+B = \{\, x+y \mid x \in A \text{ and } y \in B\}$. Is it true that $\inf(A+B) = \inf A + \inf B$?

(Exercises for Chapter 1)

4. Show that $d = \inf(S)$ iff d is a lower bound for S and for any $\varepsilon > 0$ there is an $x \in S$ such that $d \geq x - \varepsilon$.

10. Verify that the bounded metric in Example 1.7.2d is indeed a metric.

12. In an inner product space show that
(a) $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ (parallelogram law)
(b) $\|x+y\|\,\|x-y\| \leq \|x\|^2 + \|y\|^2$
(c) $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$ (polarization identity).
Interpret these results geometrically in terms of the parallelogram formed by x and y.

15. Let x_n be a sequence in $\mathbb{R}$ such that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)/2$. Show that x_n is a Cauchy sequence.

17. Let $S \in \mathbb{R}$ be bounded below and nonempty. Show that $\inf(S) = \sup\{\, x \in \mathbb{R} \mid x \text{ is a lower bound for } S\}$.

22.
(a) If x_n and y_n are bounded sequences in $\mathbb{R}$, prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

(b) Is the product rule true for lim sups?

32.
(a) Give a reasonable definition for what $\lim_{n\to\infty} x_n = \infty$ should mean.
(b) Let $x_1 = 1$ and define inductively $x_{n+1} = (x_1 + \cdots + x_n)/2$. Prove that $x_n \to \infty$.

Advanced Calculus I - HW1

파이낸스경영학과
2017004093
양 준 우

§1.2 #2 $\frac{3^n}{n!} \to 0$

For any $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t $N > -\log_2 \varepsilon$

then for $n \geq N$, $\left|\frac{1}{2^n} - 0\right| < \varepsilon$

$\therefore \frac{1}{2^n} \to 0$ as $n \to \infty$

$$\frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdot \frac{3}{4} \cdot \frac{3}{5} \cdot \frac{3}{6} \cdot \frac{3}{7} \cdots \frac{3}{n}$$

$$= \frac{81}{40} \cdot \frac{3}{6} \cdot \frac{3}{7} \cdots \frac{3}{n} < \frac{81}{40} \cdot \left(\frac{1}{2}\right)^{n-5} = \frac{81}{40} \cdot 2^5 \cdot \left(\frac{1}{2}\right)^n = \frac{324}{5} \cdot \frac{1}{2^n}$$

$\lim_{n\to\infty} \frac{324}{5} \cdot \frac{1}{2^n} = 0 \quad \left(\because \frac{1}{2^n} \to 0 \text{ as } n \to \infty\right)$

By Sandwich Lemma

$0 < \frac{3^n}{n!} < \frac{324}{5} \cdot \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{3^n}{n!} \leq \lim_{n\to\infty} \frac{324}{5} \cdot \frac{1}{2^n}$

$0 \leq \lim_{n\to\infty} \frac{3^n}{n!} \leq 0$

$\therefore \frac{3^n}{n!} \to 0$ as $n \to \infty$

§1.2 #3 $x_n = \sqrt{n^2+1} - n$, $\lim_{n\to\infty} x_n = 0$.

pf) $\left|\sqrt{n^2+1} - n\right| = \left|\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}\right| = \left|\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}\right| = \left|\frac{1}{\sqrt{n^2+1}+n}\right|$

$< \left|\frac{1}{\sqrt{n^2}+n}\right| = \left|\frac{1}{2n}\right| < \left|\frac{1}{n}\right| = \frac{1}{n}$

$0 < x_n < \frac{1}{n} \Rightarrow \lim_{n\to\infty} 0 \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} \frac{1}{n}$ (by sandwich lemma)

$\Rightarrow 0 \leq \lim_{n\to\infty} x_n \leq 0$

$\therefore \lim_{n\to\infty} x_n = 0$

§1.3 #4 $\inf(A+B) = \inf A + \inf B$

i) Let $x \in A$, $y \in B$.

then $\forall z \in A+B$, $\exists x, y$ s.t $x+y=z$

since $\inf A \le x$, $\inf B \le y$

$\underline{\inf A + \inf B} \le x+y = z$

$\Rightarrow$ lower bound of $A+B$.

$\therefore \inf A + \inf B \le \inf(A+B)$

ii) $\forall \varepsilon > 0$, $\exists x$ s.t $\inf A \le x < \inf A + \frac{\varepsilon}{2}$

$\exists y$ s.t $\inf B \le y < \inf B + \frac{\varepsilon}{2}$

for $x+y \in A+B$

$\inf(A+B) \le x+y < \inf A + \inf B + \varepsilon$

for any $\varepsilon > 0$, $\inf(A+B) < \inf A + \inf B + \varepsilon$

$\therefore \inf(A+B) \le \inf A + \inf B$

by i), ii), $\inf(A+B) = \inf A + \inf B$

Ex. #4 $d = \inf(S) \iff \exists x \in S$ s.t $d > x - \varepsilon$ for $\forall \varepsilon > 0$

$(\Rightarrow)$ For $\forall x \in S$, $\forall \varepsilon > 0$, $d \le x$

if $x = d + \frac{\varepsilon}{2}$, $d > d + \frac{\varepsilon}{2} - \varepsilon \Rightarrow \varepsilon > 0$

$(\Leftarrow)$ d is lower bound of S $(d \le x)$

Let $d + \varepsilon = c$, $\varepsilon = c - d$ $(\varepsilon > 0)$

From $d \ge x - \frac{\varepsilon}{2}$,

$d \ge x - (c-d) + \frac{\varepsilon}{2} \Rightarrow d \ge x - c + d + \frac{\varepsilon}{2}$

$c \ge x + \frac{\varepsilon}{2} \Rightarrow c > c - \frac{\varepsilon}{2} \ge x \Rightarrow x < c$

$\therefore c$ can't be lower bound.

$\Rightarrow d$ is $\inf(S)$

Ex. #10. $\rho(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$

we want to show that $\rho(x,y)$ satisfies properties of metric space (M,d)

i) positivity

$d(x,y) \geq 0 \Rightarrow 0 \leq d(x,y) < 1 + d(x,y)$

$\Rightarrow 0 \leq \dfrac{d(x,y)}{1+d(x,y)} < 1$

$\therefore \rho(x,y) \geq 0$ and $\rho(x,y)$ is bounded by 1.

ii) nongeneracy

WTS $\rho(x,y) = 0 \Leftrightarrow x = y$

$(\Leftarrow)$ $x = y \Rightarrow d(x,y) = 0 \Rightarrow \dfrac{d(x,y)}{1+d(x,y)} = 0$

$(\Rightarrow)$ $\rho(x,y) \Rightarrow \dfrac{d(x,y)}{1+d(x,y)} = 0 \Rightarrow d(x,y) = 0 \Rightarrow x = y$

$(\because d(x,y) \geq 0,\ 1 + d(x,y) \neq 0)$

iii) Symmetry

We know that $d(x,y) = d(y,x)$

$\rho(y,x) = \dfrac{d(y,x)}{1+d(y,x)} = \dfrac{d(x,y)}{1+d(x,y)} = \rho(x,y)$

iv) Triangle Inequality

$d(x,y) \leq d(x,z) + d(z,y)$

WTS $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$

$\dfrac{d(x,y)}{1+d(x,y)} \leq \dfrac{d(x,z)}{1+d(x,z)} + \dfrac{d(z,y)}{1+d(z,y)}$

Let $d(x,y) = c$, $d(x,z) = a$, $d(z,y) = b$

then $\dfrac{c}{1+c} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$

$\Rightarrow \dfrac{c(1+a)(1+b)}{(1+a)(1+b)(1+c)} \leq \dfrac{a(1+b)(1+c) + b(1+a)(1+c)}{(1+a)(1+b)(1+c)}$

$\Rightarrow c(1+a)(1+b) \leq a(1+b)(1+c) + b(1+a)(1+c)$

$\Rightarrow c(1+a+b+ab) \leq a(1+b+c+bc) + b(1+a+c+ac)$

$$\Rightarrow \quad c + \cancel{ac} + \cancel{bc} + \cancel{abc} \leq a + ab + \cancel{ac} + \cancel{abc} + b + ab + \cancel{bc} + abc$$

$$\Rightarrow \quad c \leq a + b + 2ab + abc$$

$$\Rightarrow \quad d(x,y) \leq d(x,z) + d(z,y) + 2d(x,z)\cdot d(z,y) + d(x,y)\cdot d(x,z)\cdot d(z,y)$$

It is true because $d(x,y) \leq d(x,z) + d(z,y)$, $d(x,y), d(x,z), d(z,y) \geq 0$

$\therefore$ $\rho(x,y)$ is metric on the space M

Ex. #12.

(1) $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$= \langle x+y, x\rangle + \langle x+y, y\rangle + \langle x-y, x\rangle - \langle x-y, y\rangle$

$= \langle x,x\rangle + \cancel{\langle y,x\rangle} + \cancel{\langle x,y\rangle} + \langle y,y\rangle + \langle x,x\rangle - \cancel{\langle y,x\rangle} - \cancel{\langle x,y\rangle} + \langle y,y\rangle$

$= 2\langle x,x\rangle + 2\langle y,y\rangle = 2\|x\|^2 + 2\|y\|^2$

(2) $0 \leq (\|x+y\| - \|x-y\|)^2$

$= \|x+y\|^2 - 2\|x+y\|\|x-y\| + \|x-y\|^2$ by (1)

$= 2\|x\|^2 - 2\|x+y\|\|x-y\| + 2\|y\|^2$

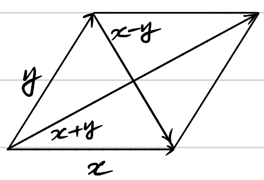
$\therefore$ $\|x+y\|\|x-y\| \leq \|x\|^2 + \|y\|^2$

(3) $\|x+y\|^2 - \|x-y\|^2 = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle$

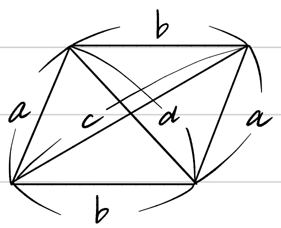
$= \langle x+y, x\rangle + \langle x+y, y\rangle - \langle x-y, x\rangle + \langle x-y, y\rangle$

$= \cancel{\langle x,x\rangle} + \langle y,x\rangle + \langle x,y\rangle + \cancel{\langle y,y\rangle} - \cancel{\langle x,x\rangle} + \langle y,x\rangle + \langle x,y\rangle - \cancel{\langle y,y\rangle}$

$= 4\langle x,y\rangle$



x, y : vector



(1) $2a^2 + 2b^2 = c^2 + d^2$

(2) $cd \leq a^2 + b^2$

(3) $4\langle x,y\rangle = c^2 - d^2$

Ex. # 15. $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)/2 \rightarrow x_n$ is cauchy sequence.

step 1). WTS $d(x_n, x_{n+1}) \leq c^{n-1} \cdot d(x_1, x_2)$ for $0 \leq c < 1$

$d(x_n, x_{n+1}) \leq c \cdot d(x_{n-1}, x_n) \leq c^2 \cdot d(x_{n-2}, x_{n-1}) \cdots$

$\leq c^{n-1} \cdot d(x_1, x_2)$

step 2). cauchy sequence

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $|x_n - x_m| < \varepsilon$ for $n, m \geq N$

Let $m = n + p$. $(p \in \mathbb{N}, p > 0)$

ETS $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $d(x_n, x_{n+p}) < \varepsilon$ for $n \geq N$

$(\because |x_n - x_m| = d(x_n, x_{n+p}))$

step 3). By Cauchy Inequality

$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p})$

$\leq c^{n-1} d(x_1, x_2) + \cdots + c^{n+p-2} d(x_1, x_2)$

$= d(x_1, x_2) \cdot (c^{n-1} + c^n + c^{n+1} + \cdots c^{n+p-2})$

$\leq d(x_1, x_2) \cdot \frac{c^{n-1}}{1-c}$ $\quad (\because \sum_{n=1}^{\infty} c^{n-1} = \frac{c^{n-1}}{1-c})$

It goes 0 as $n \to \infty$ because $c^{n-1} \to 0$ $(0 \leq c < 1)$

$\therefore$ $\forall \varepsilon > 0$, choose N s.t $d(x_1, x_2) \frac{c^{N-1}}{1-c} < \varepsilon$

then $d(x_n, x_{n+p}) < \varepsilon$ for $n \geq N$

$\Rightarrow x_n$ is cauchy sequence.

Ex. # 17. $x \in S$, for $\forall x$, $\exists b \in \mathbb{R}$ s.t $b \leq x$.

then b is called lower bound for S.

Let $B := \{b \in \mathbb{R} \mid b$ is lower bound for $S\}$, $y \in B$

If $u \in B$, $v \in S$, $u \leq v < v+1$

$v+1$ is upper bound for B.

$\therefore$ B is bounded above and exists $\sup B$.

$\inf S$ is greatest lower bound for S.
$\therefore y \leq \inf S$
then $\inf S$ is also upper bound for B.
$\Rightarrow \sup B \leq \inf S$
($\because \sup B$ is least upper bound of B)

$\inf S \leq x \Rightarrow \inf S$ is lower bound of S. $\Rightarrow \inf S \in B$
$\Rightarrow \inf S \leq \sup B$ ($\because y \leq \sup B$)

$\therefore \sup B \leq \inf S$ and $\inf S \leq \sup B \Rightarrow \inf S = \sup B$

Ex. #22.

(1) $\limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n$
Let $A = \limsup x_n$, $B = \limsup y_n$
Since x_n, y_n are bounded sequences,
A, B are both finite real numbers.
$\forall \varepsilon > 0$, $\exists N_1$ s.t $x_n < A + \frac{\varepsilon}{2}$ for $n \geq N_1$
$\exists N_2$ s.t $y_n < B + \frac{\varepsilon}{2}$ for $n \geq N_2$
Let $N := \max(N_1, N_2)$
$x_n + y_n < A + B + \varepsilon$ for $n \geq N$
$\therefore x_n + y_n$ can have no cluster points larger than $A+B$
$\Rightarrow \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n$
inequality)
if $x_n = 1, -1, 1, -1, \cdots$ $y_n = -1, 1, -1, 1, \cdots$, then $x_n + y_n = 0$
$\limsup x_n + y_n = 0 < \limsup x_n + \limsup y_n = 1 + 1 = 2$

(2) Product Rule.
$\limsup (x_n \cdot y_n) = \limsup x_n + \limsup y_n$

i) equality

if $x_n = 1$, $y_n = 1$, then $x_n y_n = 1$

$\limsup(x_n \cdot y_n) = 1 = \limsup x_n \cdot \limsup y_n = 1 \cdot 1 = 1$

ii) "<"

if $x_n = 1, 0, 1, 0, \cdots$ $y_n = 0, 1, 0, 1, \cdots$, then $x_n \cdot y_n = 0$

$\limsup(x_n \cdot y_n) = 0 < \limsup x_n \cdot \limsup y_n = 1 \cdot 1 = 1$

iii) ">"

if $x_n = 0, -1, 0, -1, \cdots$ $y_n = 0, -1\ 0, -1, \cdots$ $x_n \cdot y_n = 0, 1, 0, 1, \cdots$

$\limsup x_n \cdot y_n = 1 > \limsup x_n \cdot \limsup y_n = 0 \cdot 0 = 0$

Ex. #32.

(a) $x_n \to \infty$ as $n \to \infty$

$\Leftrightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $x_n > \varepsilon$ for $n \geq N$

(b) $x_1 = 1$ $x_2 = \frac{1}{2}$

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(x_1 + x_2 + \cdots + x_n) \\ &= \frac{1}{2}(x_1 + x_2 + \cdots + x_{n-1}) + \frac{1}{2}x_n \\ &= x_n + \frac{1}{2}x_n = \frac{3}{2}x_n \quad (n \geq 2) \end{aligned}$$

$x_n = 1, \frac{1}{2}, \frac{3}{4}, \frac{9}{8}, \frac{27}{16}, \cdots$

$x_n = \frac{1}{2}\left(\frac{3}{2}\right)^{n-2} \quad (n \geq 2)$

By (a) definition,

$\forall \varepsilon > 0$, choose $N > \dfrac{\log(2\varepsilon)}{\log(\frac{3}{2})} + 2$

$$\left\{ \begin{aligned} &\frac{1}{2}\left(\frac{3}{2}\right)^{n-2} > \varepsilon \\ &\left(\frac{3}{2}\right)^{n-2} > 2\varepsilon \\ &(n-2)\log\left(\frac{3}{2}\right) > \log 2\varepsilon \end{aligned} \right.$$

$x_n > \varepsilon$ for $n \geq N$

$\therefore x_n \to \infty$ as $n \to \infty$