ADVANCED CALCULUS 1
ASSIGNMENT # 1 : 2019 SPRING

§1.2. # 2. Show that $\dfrac{3^n}{n!}$ converges to 0.

§1.2. # 3. Let $x_n = \sqrt{n^2+1} - n$. Compute $\lim_{n\to\infty} x_n$.

§1.3. # 4. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded below and define $A+B = \{\, x+y \mid x \in A \text{ and } y \in B\}$. Is it true that $\inf(A+B) = \inf A + \inf B$?

(Exercises for Chapter 1)

4. Show that $d = \inf(S)$ iff d is a lower bound for S and for any $\varepsilon > 0$ there is an $x \in S$ such that $d \geq x - \varepsilon$.

10. Verify that the bounded metric in Example 1.7.2d is indeed a metric.

12. In an inner product space show that
(a) $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ (parallelogram law)
(b) $\|x+y\|\,\|x-y\| \leq \|x\|^2 + \|y\|^2$
(c) $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$ (polarization identity).
Interpret these results geometrically in terms of the parallelogram formed by x and y.

15. Let x_n be a sequence in $\mathbb{R}$ such that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)/2$. Show that x_n is a Cauchy sequence.

17. Let $S \in \mathbb{R}$ be bounded below and nonempty. Show that $\inf(S) = \sup\{\, x \in \mathbb{R} \mid x \text{ is a lower bound for } S\}$.

22.
(a) If x_n and y_n are bounded sequences in $\mathbb{R}$, prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

(b) Is the product rule true for lim sups?

32.
(a) Give a reasonable definition for what $\lim_{n\to\infty} x_n = \infty$ should mean.
(b) Let $x_1 = 1$ and define inductively $x_{n+1} = (x_1 + \cdots + x_n)/2$. Prove that $x_n \to \infty$.