Probability Theory – Exercise 4

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Problem 1

Let f be a non-negative integrable function on $\mathbb{R}$ and let m_2 be two dimensional Lebesgue measure on $\mathbb{R}^2$.

(1) Show that $m_2\{(x,y) : 0 \le y \le f(x)\} = m_2\{(x,y) : 0 < y < f(x)\} = \int_{\mathbb{R}} f(x)\,\mathrm{d}x$.

Proof. Let $A = \{(x,y) : 0 \le y \le f(x)\}$. First of all, let's check that A is m_2-measurable. Since f is non-negative measurable, there is a sequence s_n of non-negative simple functions such that $s_n \nearrow f$ which can be written as $s_n = \sum_{k=1}^{k_n} a_{n,k}\mathbf{1}_{A_{n,k}}$. Let $R_{n,k} = A_{n,k} \times [0, c_k]$ and the union of such rectangles is in fact $\int s_n\,\mathrm{d}m$. Then, $\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{k_n} R_k = A$ so A is m_2-measurable. Now for given x, consider x-section of A, $A_x = \{y : (x,y) \in A\} = \{y : 0 \le y \le f(x)\} = [0, f(x)]$. Its measure is $f(x)$ and by the definition of the product measure

$$m_2(A) = m_2\{(x,y) : 0 \le y \le f(x)\} = \int_{\mathbb{R}} m(A_x)\,\mathrm{d}m(x) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

Since $m([0, f(x)]) = m((0, f(x)) = f(x)$, it doesn't matter whether the endpoint is included. □

(2) Show that $G(f) = \{(x,y) : y = f(x)\}$ has measure 0.

Proof. The x-section of $G(f)$ for given x, $G_x = \{y : (x,y) \in G(f)\} = \{y : y = f(x)\} = \{f(x)\}$ and its measure is zero. Because point set is null. Thus, we get

$$m_2(G(f)) = m_2\{(x,y) : y = f(x)\} = \int_{\mathbb{R}} m(G_x)\,\mathrm{d}m(x) = \int_{\mathbb{R}} 0\,\mathrm{d}m = 0. \qquad \square$$

(3) Let $\phi(t) = m\{x : f(x) \ge t\}$. Show that ϕ is a decreasing function and

$$\int_0^{\infty} \phi(t)\,\mathrm{d}t = \int_{\mathbb{R}} f(x)\,\mathrm{d}x.$$

Proof. For any t and $\epsilon > 0$, by the countable additivity of m,

$$\begin{aligned}\phi(t) = m\{x : f(x) \ge t\} &= m(\{x : f(x) \ge t+\epsilon\} \cup \{x : t \le f(x) < t+\epsilon\}) \\ &= m\{x : f(x) \ge t+\epsilon\} + m\{x : t \le f(x) < t+\epsilon\}\end{aligned}$$

$$
\begin{aligned}
&= \phi(t+\epsilon) + m\{x : t \le f(x) < t+\epsilon\} \\
&\ge \phi(t+\epsilon).
\end{aligned}
$$

This is exactly the definition of decreasing function. For the second claim consider area under the graph, $A = \{(x,y) : 0 \le y \le f(x)\}$ and t-section of A for given t, $A_t = \{x : (x,t) \in A\} = \{x : t \le f(x)\}$. Then $\phi(t) = m\{x : f(x) \ge t\} = m(A_t)$ for $t \ge 0$. Thus, by the definition of product measure and (1),

$$
\int_0^\infty \phi(t)\,\mathrm{d}t = \int_0^\infty m(A_t)\,\mathrm{d}t = \int_0^\infty m(A_t)\,\mathrm{d}m(t) = m_2(A) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x. \qquad \square
$$

Problem 2

Let $f_{(X,Y)}$ be a joint density of random variables X, Y such that

$$
f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{50}(x^2+y^2), & \text{if } 0 < x < 2,\ 1 < y < 4 \\ 0, & \text{otherwise.} \end{cases}
$$

Find $P(X+Y>4)$ and $P(Y>X)$.

Proof.

$$
\begin{aligned}
P(X+Y>4) &= \int_A f_{(X,Y)}(x,y)\,\mathrm{d}m_2(x,y) \quad \text{where } A = \{(x,y) : y > 4-x\} \cap [0,2]\times[1,4] \\
&= \int_0^2 \int_{4-x}^4 \frac{1}{50}(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{50}\int_0^2 x^2y + \frac{1}{3}y^3\Big|_{4-x}^4 \,\mathrm{d}x \\
&= \frac{1}{50}\int_0^2 \frac{4}{3}x^3 - 4x^2 + 16x\,\mathrm{d}x = \frac{1}{50}\left(\frac{1}{3}x^4 - \frac{4}{3}x^3 + 8x^2\Big|_0^2\right) = \frac{8}{15} \\
P(Y>X) &= \int_1^2\int_0^y \frac{1}{50}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y + \int_2^4\int_0^2 \frac{1}{50}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{50}\int_1^2 \frac{x^3}{3} + xy^2\Big|_0^y \,\mathrm{d}y + \frac{1}{50}\int_2^4 \frac{x^3}{3} + xy^2\Big|_0^2 \,\mathrm{d}y \\
&= \frac{1}{50}\int_1^2 \frac{4}{3}y^3\,\mathrm{d}y + \frac{1}{50}\int_2^4 \frac{8}{3} + 2y^2\,\mathrm{d}y = \frac{1}{50}\left(\frac{y^4}{3}\Big|_1^2 + \frac{8y}{3} + \frac{2y^3}{3}\Big|_2^4\right) = \frac{143}{150}. \qquad \square
\end{aligned}
$$

Problem 3

Let $f_{(X,Y)}$ be a joint density of random variables X, Y. Find f_{X+Y} if $f_{(X,Y)} = \mathbf{1}_{[0,1]\times[0,1]}$.

Proof. Since X, Y have joint density $f_{(X,Y)}$, the density of their sum is given by

$$f_{X+Y}(z) = \int_{\mathbb{R}} f_{(X,Y)}(x, z-x)\,\mathrm{d}x = \begin{cases} 0 & z < 0 \\ z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \\ 0 & 2 < z \end{cases}. \qquad \square$$

Problem 4

Let $\Omega = \{1,2,3,4\}$, $\mathcal{F} = 2^{\Omega}$, with the measure P given by $P(\{n\}) = 1/4$, $(n = 1, \cdots, 4)$. Let $X : \Omega \to \mathbb{R}$ be a random variable given by $X(1) = X(3) = 0$, $X(2) = X(4) = 1$, $Y : \Omega \to \mathbb{R}$ be a random variable given by $Y(1) = Y(2) = 0$, $Y(3) = Y(4) = 1$ and $Z(\omega) = 1 - X(\omega)$.

(1) Find probability distributions P_X and P_Y.

Proof.

$$\begin{aligned} P_X(0) &= P(\{\omega \in \Omega : X(\omega) = 0\}) = P(\{1,3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ P_X(1) &= P(\{\omega \in \Omega : X(\omega) = 1\}) = P(\{2,4\}) = P(\{2\}) + P(\{4\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ P_Y(0) &= P(\{\omega \in \Omega : Y(\omega) = 0\}) = P(\{1,2\}) = P(\{1\}) + P(\{2\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ P_Y(1) &= P(\{\omega \in \Omega : Y(\omega) = 1\}) = P(\{3,4\}) = P(\{3\}) + P(\{4\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \qquad \square \end{aligned}$$

(2) Find joint distributions $P_{(X,Y)}$ and $P_{(X,Z)}$.

Proof.

$$\begin{aligned} P_{(X,Y)}(0,0) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (0,0)\}) = P(\{1\}) = \frac{1}{4} \\ P_{(X,Y)}(0,1) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (0,1)\}) = P(\{3\}) = \frac{1}{4} \\ P_{(X,Y)}(1,0) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (1,0)\}) = P(\{2\}) = \frac{1}{4} \\ P_{(X,Y)}(1,1) &= P(\{\omega \in \Omega : (X(\omega), Y(\omega)) = (1,1)\}) = P(\{4\}) = \frac{1}{4}. \end{aligned}$$

$$\begin{aligned} P_{(X,Z)}(0,1) &= P(\{\omega \in \Omega : (X(\omega), Z(\omega)) = (0,1)\}) = P(\{\omega \in \Omega : (X(\omega), 1 - X(\omega)) = (0,1)\}) \\ &= P(\{\omega \in \Omega : X(\omega) = 0\}) = P(\{1,3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{2} \\ P_{(X,Z)}(1,0) &= P(\{\omega \in \Omega : (X(\omega), Z(\omega)) = (1,0)\}) = P(\{\omega \in \Omega : (X(\omega), 1 - X(\omega)) = (1,0)\}) \\ &= P(\{\omega \in \Omega : X(\omega) = 1\}) = P(\{2,4\}) = P(\{2\}) + P(\{4\}) = \frac{1}{2}. \qquad \square \end{aligned}$$