

Lecture note 3: Stochastic calculus

1 Exercises**Problem 1.1.** (20 points) Define

$$X_t = tB_t^3 + \sin(B_t^2).$$

Show that the Ito integration

$$Y_t = \int_0^t sB_s dX_s$$

is well-defined. Find the dynamics of Y . What are $\langle Y \rangle_T$ and $\langle X, Y \rangle_T$ for $T \geq 0$?**Problem 1.2.** (20 points) Solve the following problems

(i) Show that

$$\arctan B_t + \int_0^t \frac{B_s}{(1+B_s^2)^2} ds, \quad t \geq 0$$

is a martingale.

(ii) Show that $(e^{t/2} \cos B_t)_{t \geq 0}$ is a martingale. Find $\mathbb{E}(\cos B_T)$.**Problem 1.3.** (10 points) Define $X_t = \int_0^t B_s^2 dB_s$ and $Y_t = B_t$. Evaluate $d(t^2 e^{X_t} Y_t^2)$.**Problem 1.4.** (10 points) Let X and Y be two Ito processes given as

$$\begin{aligned} dX_t &= aX_t dt + bX_t dB_t, & X_0 &= 1 \\ dY_t &= cY_t dt + dY_t dB_t, & Y_0 &= 1 \end{aligned}$$

for $a, b, c, d \in \mathbb{R}$. Find the dynamics of $U := XY$ and $V := X/Y$.**Problem 1.5.** Let $B = (B_t^{(1)}, \dots, B_t^{(d)})_{t \geq 0}$ be a d -dimensional Brownian motion. Prove the following Ito formulas. Write down the main idea as we did in class. Do not provide any rigorous proof.(i) (10 points) For $f \in C^{1,2}([0, \infty) \times \mathbb{R}^d)$, show that

$$df(t, B_t) = f_t(t, B_t) dt + \nabla f(t, B_t) dB_t + \frac{1}{2} \Delta f(t, B_t) dt$$

where ∇ is the gradient operator and Δ is the Laplacian operator.(ii) (10 points) For $f \in C^{1,2}([0, \infty) \times \mathbb{R}^d)$ and n -dimensional Ito process $X = (X_t^{(1)}, \dots, X_t^{(n)})_{t \geq 0}$, show that

$$df(t, X_t) = f_t(t, X_t) dt + \sum_{i=1}^n f_{x_i}(t, X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^n f_{x_i x_j}(t, X_t) d\langle X^{(i)}, X^{(j)} \rangle_t$$

Problem 1.6. (20 points) Let $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0}$ be a three-dimensional Brownian motion. Define

- (i) $X_t = \cos(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)})$
- (ii) $Y_t = t^3 \sin B_t^{(2)} + \int_0^t s^2 (B_s^{(1)})^2 dB_s^{(3)}.$

Find the dynamics of these processes. Evaluate the quadratic variations $\langle X \rangle_t$, $\langle Y \rangle_t$, and $\langle X, Y \rangle_t$ for $t \geq 0$.

Problem 1.7. (10 points) Let $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0}$ be a three-dimensional Brownian motion. Show that

$$(B_t^{(1)} B_t^{(2)} B_t^{(3)})_{t \geq 0}$$

is a martingale.

Problem 1.8. (Ornstein-Uhlenbeck process) Consider the stochastic differential equation (SDE)

$$dX_t = a(\theta - X_t) dt + \sigma dB_t, \quad X_0 = x$$

for $a \neq 0$ and $\sigma > 0$.

- (i) (5 points) Solve this SDE. Hint: apply Ito's formula to $d(e^{at} X_t)$.
- (ii) (10 points) Show that the solution $(X_t)_{t \geq 0}$ is a Gaussian process.
- (iii) (5 points) Find $\mathbb{E}(e^{X_T})$

Problem 1.9. (15 points) Solve

$$dX_t = rt dt + \sigma X_t dB_t$$

for $r \in \mathbb{R}$ and $\sigma > 0$. Hint: Calculate $d(XY)$ for $Y_t = e^{-\sigma B_t + \frac{1}{2}\sigma^2 t}$.

References

Prob. 1.1.

$$X_t = tB_t^3 + \sin(B_t^2).$$

To show $Y_t = \int_0^t sB_s dX_s$ is well-defined
need to check

(i) X_t is Ito process.(ii) $tB_t \in \mathcal{H}(X) = \{g: \text{prog. m'ble} | P(\int_0^T |g_{t+u}| + |g_{t+u}|^2 du < \infty) = 1\}$

$$(i) X_t = tB_t^3 + \sin(B_t^2)$$

$$dX_t = df(t, B_t) = f_t dt + f_x dB_t + \frac{1}{2} f_{xx} dt.$$

$$\begin{aligned} &= B_t^3 dt + (2tB_t^2 + 2B_t \cos(B_t^2)) dB_t + (3tB_t - 2B_t^2 \sin(B_t^2)) dt \\ &= (B_t^3 + 3tB_t - 2B_t^2 \sin(B_t^2)) dt + (2tB_t^2 + 2B_t \cos(B_t^2)) dB_t \end{aligned}$$

$b_t = B_t^3 + 3tB_t - 2B_t^2 \sin(B_t^2)$ are conti. & adapted.

$\sigma_t = 2tB_t^2 + 2B_t \cos(B_t^2) \Rightarrow \text{prog. m'ble.} \& b, \sigma \in \mathcal{H}^{loc}$

Since $b, \sigma \in \mathcal{H}^{loc}$, $\int_0^T b_u^2 du < \infty$, $\int_0^T \sigma_u^2 du < \infty$.

$$\int_0^T b_u^2 du < \infty \Rightarrow \int_0^T b_u du < \infty.$$

$$\therefore P(\int_0^T |b_u| + |\sigma_u|^2 du < \infty) = 1.$$

Thus, X_t is Ito process.

(ii) $g_t = tB_t$ is conti. and adapted \Rightarrow prog. m'ble & $g_t \in \mathcal{H}^{loc}$

Since $g_t b_t$, $g_t \sigma_t$ are conti. and adapted $\Rightarrow g_t b_t, g_t \sigma_t \in \mathcal{H}^{loc}$

$$\therefore P(\int_0^T |tB_t b_t| + |tB_t \sigma_t|^2 dt < \infty) = 1.$$

$$\therefore tB_t \in \mathcal{H}(X).$$

Hence, $Y_t = \int_0^t sB_s dX_s$ is well-defined.

$$Y_t = \int_0^t sB_s dX_s = \int_0^t sB_s (b_t dt + \sigma_t dB_t) = \int_0^t sB_s b_s dt + \int_0^t sB_s \sigma_s dB_s$$

i.e., dynamics of Y_t is

$$dY_t = tB_t b_t dt + tB_t \sigma_t dB_t$$

$$= tB_t (B_t^3 + 3tB_t - 2B_t^2 \sin(B_t^2)) dt + tB_t (2tB_t^2 + 2B_t \cos(B_t^2)) dB_t$$

$$= (tB_t^4 + 3t^2 B_t^2 - 2tB_t^3 \sin(B_t^2)) dt + (2t^2 B_t^3 + 2tB_t^2 \cos(B_t^2)) dB_t$$

Since Y_t is also Ito process,

$$\langle Y \rangle_T = \int_0^T (3t^2 B_t^2 + 2t B_t \cos(B_t^2))^2 dt$$

$$\langle X, Y \rangle_T = \int_0^T t B_t G_t^2 dt = \int_0^T t B_t (3t B_t^2 + 2B_t \cos(B_t^2))^2 dt$$

Prob. 1.2.(D).

$$(\arctan B_t + \int_0^t \frac{B_s}{(1+B_s^2)^2} ds)_{t \geq 0}$$

$$f(t) = \arctan t, \quad f \in C^2$$

$$\begin{aligned} df(B_t) &= d(\arctan B_t) = \frac{1}{1+B_t^2} dB_t + \frac{1}{2} \cdot \frac{-2B_t}{(1+B_t^2)^2} d\langle B \rangle_t \\ &= -\frac{B_t}{(1+B_t^2)^2} dt + \frac{1}{1+B_t^2} dB_t \end{aligned}$$

Prob. 1.2.(D).

Prob. 1.3

$$X_t = \int_0^t B_s^2 dB_s, \quad Y_t = B_t = \int_0^t 1 dB_s$$

$$dX_t = B_t^2 dB_t \quad dY_t = dB_t$$

B_s^2 and 1 : continuous and adapted \Rightarrow progr. mble. & $\in \mathcal{H}^{1/2}_{loc}$

$\therefore X_t, Y_t$: Itô processes.

$$\text{Let } f(t, x, y) = t^2 e^x y^2 \in C^{1,2,2}$$

$$\begin{aligned} d(t^2 e^{X_t} Y_t^2) &= 2t e^{X_t} Y_t^2 dt + t^2 e^{X_t} Y_t^2 dX_t + 2t^2 e^{X_t} Y_t dY_t \\ &\quad + \frac{1}{2} \cdot t^2 e^{X_t} Y_t^2 d\langle X \rangle_t + 2t^2 e^{X_t} Y_t d\langle X, Y \rangle_t + \frac{1}{2} \cdot 2t^2 e^{X_t} d\langle Y \rangle_t \\ &= 2t e^{X_t} Y_t^2 dt + \underline{t^2 e^{X_t} Y_t^2 B_t^2 dB_t} + \underline{2t^2 e^{X_t} Y_t dB_t} \\ &\quad + \frac{1}{2} \cdot t^2 e^{X_t} Y_t^2 B_t^4 dt + 2t^2 e^{X_t} Y_t B_t^2 dt + t^2 e^{X_t} dt \\ &= (2t e^{X_t} Y_t^2 + \frac{1}{2} \cdot t^2 e^{X_t} Y_t^2 B_t^4 + 2t^2 e^{X_t} Y_t B_t^2 + t^2 e^{X_t}) dt \\ &\quad + (t^2 e^{X_t} Y_t^2 B_t^2 + 2t^2 e^{X_t} Y_t) dB_t \\ &= (2t e^{X_t} B_t^2 + \frac{1}{2} \cdot t^2 e^{X_t} B_t^6 + 2t^2 e^{X_t} B_t^3 + t^2 e^{X_t}) dt \\ &\quad + (t^2 e^{X_t} B_t^4 + 2t^2 e^{X_t} B_t) dB_t \quad \square \\ &= e^{\int_0^t B_s^2 dB_s} [(2t B_t^2 + \frac{1}{2} \cdot t^2 B_t^6 + 2t^2 B_t^3 + t^2) dt + (t^2 B_t^4 + 2t^2 B_t) dB_t] \end{aligned}$$

Prob. 1.4.

$$dX_t = aX_t dt + bX_t dB_t, \quad X_0 = 1.$$

$$dY_t = cY_t dt + dY_t dB_t, \quad Y_0 = 1.$$

$$dU = d(XY) = Y_t dX_t + X_t dY_t + d\langle X, Y \rangle_t$$

$$= aX_t Y_t dt + bX_t Y_t dB_t + cX_t Y_t dt + dX_t Y_t dB_t + bdX_t Y_t dt$$

$$= (a+c+bd)U_t dt + (b+d)U_t dB_t$$

$$dV = d(X/Y) = \frac{1}{Y_t} dX_t - \frac{X_t}{Y_t^2} dY_t - \frac{1}{Y_t^2} d\langle X, Y \rangle_t + \frac{1}{2} \frac{2X_t}{Y_t^3} d\langle Y \rangle_t$$

$$\begin{aligned} &= \frac{1}{Y_t} (aX_t dt + bX_t dB_t) - \frac{X_t}{Y_t^2} (cY_t dt + dY_t dB_t) - \frac{1}{Y_t^2} (bdX_t Y_t dt) \\ &\quad + \frac{X_t}{Y_t^2} (d^2 Y_t^2 dt) \end{aligned}$$

$$= (a - c - bd + d^2)V_t dt + (b - d)V_t dB_t$$

Prob. 1.5. (i).

$$B = (B_t^{(0)}, \dots, B_t^{(d)})_{t \geq 0}.$$

$$f \in C^{1,2}([0, \infty) \times \mathbb{R}^d)$$

$$df(t, B_t) = f_t(t, B_t) dt + \nabla f(t, B_t) dB_t + \frac{1}{2} \Delta f(t, B_t) dt$$

By Taylor expansion for multivariate function,

$$f(t, X_t) = f(t, X_0^{(0)}, \dots, X_0^{(d)})$$

$$\begin{aligned} &= f(0, X_0) + f_t(t_0, X_0)(t - t_0) + f_{X_0}(t_0, X_0)(X^{(0)} - X_0^{(0)}) \\ &\quad + \dots + f_{X_d}(t_0, X_0)(X^{(d)} - X_0^{(d)}) + \frac{1}{2} \sum_{i=1}^d f_{X_i X_i}(t_0, X_0)(X^{(i)} - X_0^{(i)})^2 \\ &\quad + \sum_{1 \leq i < j \leq d} f_{X_i X_j}(t_0, X_0)(X^{(i)} - X_0^{(i)})(X^{(j)} - X_0^{(j)}) + \dots \end{aligned}$$

$$f(t, B_t) - f(t_0, B_0) = \sum_{i=0}^t (f(t_{i+1}, B_{t_{i+1}}) - f(t_i, B_{t_i}))$$

$$\approx \sum_{i=0}^t \{ f_t(t_i, B_{t_i})(t_{i+1} - t_i) + \sum_{j=1}^d f_{X_j}(t_i, B_{t_i})(B_{t_{i+1}}^{(j)} - B_{t_i}^{(j)}) \}$$

$$+ \frac{1}{2} \sum_{j=1}^d f_{X_j X_j}(t_i, B_{t_i})(B_{t_{i+1}}^{(j)} - B_{t_i}^{(j)})^2 dt$$

$$+ \sum_{1 \leq i < k \leq d} f_{X_i X_k}(t_i, B_{t_i})(B_{t_{i+1}}^{(k)} - B_{t_i}^{(k)})(B_{t_{i+1}}^{(k)} - B_{t_i}^{(k)}) + \dots \}$$

(as $\|\pi_n\| \rightarrow 0$).

$$(dt)^2 = dt dB = 0, \quad (dB)^2 = dt$$

$$\begin{aligned} &\rightarrow \int_0^t f_t(u, B_u) du + \sum_{j=1}^d \int_0^t f_{X_j}(u, B_u) dB_u^{(j)} + \frac{1}{2} \sum_{j=1}^d \int_0^t f_{X_j X_j}(u, B_u) du \\ &= \int_0^t f_t(u, B_u) du + \int_0^t \nabla f(u, B_u) dB_u + \frac{1}{2} \int_0^t \Delta f(u, B_u) du \end{aligned}$$

(i.e., $df(t, B_t) = f_t dt + \nabla f dB_t + \frac{1}{2} \Delta f dt$: dynamics)

Prob. 1.5. (ii).

Similarly to (i),

$$f(t, X_t) - f(0, X_0) = \sum_{i=0}^t (f(t_{i+1}, X_{t_{i+1}}) - f(t_i, X_{t_i}))$$

$$\approx \sum_{i=0}^t \{ f_t(t_i, X_{t_i})(t_{i+1} - t_i) + \sum_{j=1}^d f_{X_j}(t_i, X_{t_i})(X_{t_{i+1}}^{(j)} - X_{t_i}^{(j)}) \}$$

$$+ \frac{1}{2} \sum_{j=1}^d f_{X_j X_j}(t_i, X_{t_i})(X_{t_{i+1}}^{(j)} - X_{t_i}^{(j)})^2$$

$$+ \sum_{1 \leq i < k \leq d} f_{X_i X_k}(t_i, X_{t_i})(X_{t_{i+1}}^{(k)} - X_{t_i}^{(k)})(X_{t_{i+1}}^{(k)} - X_{t_i}^{(k)}) + \dots \}$$

(as $\|\pi_n\| \rightarrow 0$).

$$f(t, X_t) - f(0, X_0) = \int_0^t f_t dt + \sum_{i=1}^d \int_0^t f_{X_i} dX_t^{(i)}$$

$$+ \frac{1}{2} \sum_{i=1}^d f_{X_i X_i} d\langle X^{(i)}, X^{(i)} \rangle_t + \sum_{1 \leq i < k \leq d} \int_0^t f_{X_i X_k} d\langle X^{(i)}, X^{(k)} \rangle_t$$

$$= \int_0^t f_t dt + \sum_{i=1}^d \int_0^t f_{X_i} dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^d \int_0^t f_{X_i X_j} d\langle X^{(i)}, X^{(j)} \rangle_t$$

$$\Rightarrow df(t, X_t) = f_t(t, X_t) dt + \sum_{i=1}^d f_{X_i}(t, X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^d f_{X_i X_j}(t, X_t) d\langle X^{(i)}, X^{(j)} \rangle_t$$

Prob. 1.6.

$$X_t = \cos(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)})$$

$$dX_t = (-\sin(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)}) + \cos(B_t^{(1)}) \cos(B_t^{(1)} B_t^{(2)}) B_t^{(2)}) dB_t^{(1)}$$

$$+ (\cos(B_t^{(1)}) \cos(B_t^{(1)} B_t^{(2)}) B_t^{(1)}) dB_t^{(2)}$$

$$+ \frac{1}{2} (-\cos(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)}) - \sin(B_t^{(1)}) \cos(B_t^{(1)} B_t^{(2)}) B_t^{(2)} - \sin(B_t^{(1)}) \cos(B_t^{(1)} B_t^{(2)}) B_t^{(2)})$$

$$- \cos(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)}) B_t^{(2)^2}) dt$$

$$+ \frac{1}{2} (-\cos(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)}) B_t^{(1)^2}) dt$$

$$= (-\sin(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)}) + \cos(B_t^{(1)}) \cos(B_t^{(1)} B_t^{(2)}) B_t^{(2)}) dB_t^{(1)}$$

$$+ (\cos(B_t^{(1)}) \cos(B_t^{(1)} B_t^{(2)}) B_t^{(1)}) dB_t^{(2)} \quad \text{X}$$

$$+ \frac{1}{2} (-\cos(B_t^{(1)}) \sin(B_t^{(1)} B_t^{(2)}) (B_t^{(1)^2} + B_t^{(2)^2} + 1) - 2 \sin(B_t^{(1)}) \cos(B_t^{(1)} B_t^{(2)}) B_t^{(2)}) dt$$

$$Y_t = t^3 \sin B_t^{(2)} + \int_0^t s^2 (B_s^{(1)})^2 dB_s^{(2)}$$

$$dY_t = d(t^3 \sin B_t^{(2)}) + t^2 B_t^{(1)^2} dB_t^{(2)}$$

$$d(t^3 \sin B_t^{(2)}) = 3t^2 \sin B_t^{(2)} dt + t^3 \cos B_t^{(2)} dB_t^{(2)} - \frac{1}{2} \cdot t^3 \sin B_t^{(2)} dt$$

$$= (-\frac{1}{2}t^3 + 3t^2) \sin B_t^{(2)} dt + t^3 \cos B_t^{(2)} dB_t^{(2)}$$

$$dY_t = (-\frac{1}{2}t^3 + 3t^2) \sin B_t^{(2)} dt + \frac{t^3 \cos B_t^{(2)}}{r} dB_t^{(2)} + \frac{t^2 B_t^{(1)^2}}{f} dB_t^{(2)}$$

$$\langle X \rangle_t = \int_0^t (X_u^2 + \beta^2) du$$

$$\langle Y \rangle_t = \int_0^t (r^2 + f^2) du$$

$$\langle X, Y \rangle_t = \int_0^t \beta r du.$$

Prob. 1.1.

$$d(B_t^{(1)} B_t^{(2)}) = B_t^{(2)} dB_t^{(1)} + B_t^{(1)} dB_t^{(2)}$$

$$d(B_t^{(1)} B_t^{(2)} B_t^{(3)}) = d(B_t^{(1)} B_t^{(2)} \cdot B_t^{(3)})$$

$$= B_t^{(3)} d(B_t^{(1)} B_t^{(2)}) + (B_t^{(1)} B_t^{(2)}) dB_t^{(3)}$$

$$= B_t^{(3)} (B_t^{(1)} dB_t^{(1)} + B_t^{(2)} dB_t^{(2)}) + B_t^{(1)} B_t^{(2)} dB_t^{(3)}$$

$$= B_t^{(1)} B_t^{(2)} dB_t^{(1)} + B_t^{(1)} B_t^{(3)} dB_t^{(2)} + B_t^{(2)} B_t^{(3)} dB_t^{(1)}$$

$$\Rightarrow B_t^{(1)} B_t^{(2)} B_t^{(3)} = 0 + \int_0^t (B_u^{(1)} B_u^{(2)}, B_u^{(2)} B_u^{(3)}, B_u^{(1)} B_u^{(3)}) dB_u$$

$B_t^{(1)} B_t^{(2)}, B_t^{(2)} B_t^{(3)}, B_t^{(1)} B_t^{(3)}$: conti. + adapted \Rightarrow prog. m'ble.

$$E \int_0^T (B_t^{(1)} B_t^{(2)})^2 dt = \int_0^T t^2 dt = \frac{T^3}{3} < \infty \quad \forall i, j \in \{1, 2, 3\}$$

$\therefore (B_t^{(1)} B_t^{(2)}, B_t^{(2)} B_t^{(3)}, B_t^{(1)} B_t^{(3)})_{0 \leq t \leq T} \in \mathcal{H}^2(\mathbb{R} \times [0, T], \mathbb{R}^3, \mathcal{F}, (\mathcal{F}_t), P)$.

$B_t^{(1)} B_t^{(2)} B_t^{(3)} = \int_0^t (B_s^{(1)} B_s^{(2)}, B_s^{(2)} B_s^{(3)}, B_s^{(1)} B_s^{(3)}) dB_s$ is mart. each $T > 0$

$\therefore (B_t^{(1)} B_t^{(2)} B_t^{(3)})_{t \geq 0}$ is a martingale. \square .

Prob. 1.8.(D).

$$dX_t = \theta(\theta - X_t)dt + \sigma dB_t, \quad X_0 = x.$$

$$\text{Let } Y_t = e^{\theta t} X_t$$

$$\begin{aligned} dY_t &= d(e^{\theta t} X_t) = ae^{\theta t} X_t dt + e^{\theta t} dX_t \\ &= ae^{\theta t} X_t dt + e^{\theta t} (\theta(\theta - X_t)dt + \sigma dB_t) \\ &= a\theta e^{\theta t} dt + \sigma e^{\theta t} dB_t \end{aligned}$$

$$\begin{aligned} Y_t &= Y_0 + a\theta \int_0^t e^{au} du + \sigma \int_0^t e^{au} dB_u \\ &= x + a\theta \cdot \frac{1}{a}(e^{\theta t} - 1) + \sigma \int_0^t e^{au} dB_u \end{aligned}$$

$$X_t = e^{-\theta t} Y_t = xe^{-\theta t} + \theta(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-u)} dB_u$$

Prob. 1.8.(D).

$(X_t)_{t \geq 0}$ is Gaussian process

$\Leftrightarrow \sum_{i=1}^n \lambda_i X_{t_i}$ is normal for any $\lambda_i \in \mathbb{R}$, $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T$.

Since $xe^{-\theta t} + \theta(1 - e^{-\theta t})$, $\sigma e^{-\theta t}$ are constants,

$E[\sum_{i=1}^n \lambda_i \int_0^{t_i} e^{au} dB_u]$ is normal. $\Leftrightarrow (\int_0^t e^{au} dB_u)_{0 \leq t \leq T}$ is Gaussian.

$(e^{ax})^{-1}(A) \in B(\mathbb{R})$ for $A \in B(\mathbb{R})$ $\Rightarrow e^{ax}$ is Borel function.

$$\int_0^T e^{2au} du = \frac{1}{2a}(e^{2aT} - 1) < \infty \text{ for each } T > 0.$$

By HWB. Prob. 1.2.(ii), $(\int_0^t e^{au} dB_u)_{0 \leq t \leq T}$ is Gaussian process.

$\therefore (X_t)_{t \geq 0}$ is Gaussian process. \square

Prob. 1.8.(III).

Since $(X_t)_{t \geq 0}$ is Gaussian process, X_T is normal.

$$\text{Then } E[e^{X_T}] = e^{E[X_T] + \frac{1}{2}\text{Var}(X_T)}$$

$$\begin{aligned} E[X_T] &= E[xe^{-\theta T} + \theta(1 - e^{-\theta T}) + \sigma \int_0^T e^{-\theta(T-u)} dB_u] \\ &= xe^{-\theta T} + \theta(1 - e^{-\theta T}) + \sigma e^{-\theta T} E[\int_0^T e^{au} dB_u] \\ &= xe^{-\theta T} + \theta(1 - e^{-\theta T}) + \sigma e^{-\theta T} E[\int_0^0 e^{au} dB_u] \\ &= xe^{-\theta T} + \theta(1 - e^{-\theta T}) = \theta + (x - \theta)e^{-\theta T} \end{aligned}$$

* e^{au} : conti. + adapted \Rightarrow prog. m'ble. & $E[\int_0^T e^{2au} du] = \frac{1}{2a}(e^{2aT} - 1) < \infty$
 $\Rightarrow e^{au} \in \mathcal{H}^2 \Rightarrow (\int_0^t e^{au} dB_u)_{0 \leq t \leq T}$: martingale.

$$\begin{aligned}
\text{Var}(X_T) &= \text{Var}(xe^{-aT} + \theta(1 - e^{-aT}) + 6 \int_0^T e^{-a(T-u)} dB_u) \\
&= 6^2 e^{-2aT} \text{Var}(\int_0^T e^{au} dB_u) \\
&= 6^2 e^{-2aT} (\text{E}(\int_0^T e^{au} dB_u)^2 - (\text{E}(\int_0^T e^{au} dB_u))^2) \\
&= 6^2 e^{-2aT} \text{E}(\int_0^T e^{2au} du) \quad (\because \text{E}(\int_0^T e^{au} dB_u) = 0, e^{au} \in \mathcal{H}^2, \text{ by Ito isometry}) \\
&= 6^2 e^{-2aT} \cdot \frac{1}{2a} (e^{2aT} - 1) \\
&= \frac{6^2}{2a} (1 - e^{-2aT}) \\
\therefore \text{E}(e^{X_T}) &= e^{\text{E}(X_T) + \frac{1}{2}\text{Var}(X_T)} = e^{\theta + (x-\theta)e^{-aT} + \frac{6^2}{4a}(1 - e^{-2aT})}
\end{aligned}$$

Prob. 1.9.

$$dX_t = rt dt + 6X_t dB_t$$

$$\text{Let } Y_t = e^{-6B_t + \frac{1}{2}6^2t}, f(t, x) = e^{-6x + \frac{1}{2}6^2t} \in C^{1,2}$$

$$dY_t = \frac{1}{2}6^2 Y_t dt - 6Y_t dB_t + \frac{1}{2}6^2 Y_t dt$$

$$= 6^2 Y_t dt - 6Y_t dB_t \quad : \text{Ito process.}$$

$$d(X_t Y_t) = Y_t dX_t + X_t dY_t + d\langle X, Y \rangle_t$$

$$= rt Y_t dt + 6X_t Y_t dB_t + 6^2 X_t Y_t dt - 6X_t Y_t dB_t - 6^2 X_t Y_t dt$$

$$= rt Y_t dt$$

$$\Rightarrow X_t Y_t = X_0 Y_0 + \int_0^t r u Y_u du$$

$$X_t = \frac{X_0}{Y_t} + \frac{1}{Y_t} \int_0^t r u Y_u du. \quad (\because Y_0 = 1, Y_t \neq 0)$$

$$= e^{6B_t - \frac{1}{2}6^2t} (X_0 + r \int_0^t e^{-6B_u + \frac{1}{2}6^2u} du)$$