Probability Theory – Exercise 7

Junwoo Yang

December 6, 2020

Problem 1

Let (A_n) be events in $\mathcal{F}$ and let $\mathcal{A}$ be the smallest σ-field containing each of these events. If B is an event in $\mathcal{A}$ with the property that, for any integers $i_1, \cdots, i_k$ the events B and $A_{i_1} \cap \cdots \cap A_{i_k}$ are independent, prove that $P(B)$ is either 0 or 1.

Proof. Consider the smallest σ-field $\mathcal{B}$ containing all $B \in \mathcal{A}$. That is,

$$\mathcal{B} = \bigcap \{\mathcal{G} : \mathcal{G} = \sigma\text{-field containing all } B \in \mathcal{A}\}.$$

Similarly, Let $\mathcal{H}$ be the smallest σ-field containing all $A_{i_1} \cap \cdots \cap A_{i_k}$.

Claim 1. $\mathcal{H} = \mathcal{A}$.
Clearly, $\mathcal{H} \subset \mathcal{A}$ since σ-field is closed under finite intersections. Conversely, if $k = 1$, then $\mathcal{H}$ is exactly same with $\mathcal{A}$. Thus, $B \in \mathcal{H}$.

Claim 2. $\mathcal{B}$ and $\mathcal{H}$ are independent.

$$\begin{aligned} P(B \cap (A_{i_1} \cap \cdots \cap A_{i_k})^c) &= P(B) - P(B \cap (A_{i_1} \cap \cdots \cap A_{i_k})) \\ &= P(B) - P(B)P(A_{i_1} \cap \cdots \cap A_{i_k}) \\ &= P(B)(1 - P(A_{i_1} \cap \cdots \cap A_{i_k})) \\ &= P(B)P((A_{i_1} \cap \cdots \cap A_{i_k})^c). \end{aligned}$$

$$\begin{aligned} P(B \cap ((A_{i_1} \cap \cdots \cap A_{i_k}) \cap (A_{j_1} \cap \cdots \cap A_{j_l}))) &= P(B \cap (A_{i_1} \cap \cdots \cap A_{i_k} \cap A_{j_1} \cap \cdots \cap A_{j_l})) \\ &= P(B)P(A_{i_1} \cap \cdots \cap A_{i_k} \cap A_{j_1} \cap \cdots \cap A_{j_l}) \\ &= P(B)P((A_{i_1} \cap \cdots \cap A_{i_k}) \cap (A_{j_1} \cap \cdots \cap A_{j_l})). \end{aligned}$$

$$\begin{aligned} P(B \cap ((A_{i_1} \cap \cdots \cap A_{i_k}) \cup (A_{j_1} \cap \cdots \cap A_{j_l}))) &= P((B \cap (A_{i_1} \cap \cdots \cap A_{i_k})) \cup (B \cap (A_{j_1} \cap \cdots \cap A_{j_l}))) \\ &= P(B \cap (A_{i_1} \cap \cdots \cap A_{i_k})) + P(B \cap (A_{j_1} \cap \cdots \cap A_{j_l})) \\ &\quad - P(B \cap ((A_{i_1} \cap \cdots \cap A_{i_k}) \cap (A_{j_1} \cap \cdots \cap A_{j_l}))) \\ &= P(B)P(A_{i_1} \cap \cdots \cap A_{i_k}) + P(B)P(A_{j_1} \cap \cdots \cap A_{j_l}) \end{aligned}$$

$$
\begin{aligned}
&- P(B)P((A_{i_1} \cap \cdots \cap A_{i_k}) \cap (A_{j_1} \cap \cdots \cap A_{j_l})) \\
&= P(B)(P(A_{i_1} \cap \cdots \cap A_{i_k}) + P(A_{j_1} \cap \cdots \cap A_{j_l}) \\
&\quad - P((A_{i_1} \cap \cdots \cap A_{i_k}) \cap (A_{j_1} \cap \cdots \cap A_{j_l}))) \\
&= P(B)P((A_{i_1} \cap \cdots \cap A_{i_k}) \cup (A_{j_1} \cap \cdots \cap A_{j_l})).
\end{aligned}
$$

Since $B \in \mathcal{B}$ and $B \in \mathcal{H}$, we get $P(B) = P(B \cap B) = P(B)P(B)$. Thus, $P(B)$ is either 0 or 1. $\square$

Problem 2

Let X be a random variable with uniform distribution on [0,1] and let A_n be the event $\{X < \frac{1}{n}\}$. Show that

$$\sum_{n=1}^{\infty} P(A_n) = \infty, \quad \text{but that} \quad P(\limsup_{n\to\infty} A_n) = 0.$$

Proof.

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} 1 \, \mathrm{d}P = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Since $\bigcup_{m=n}^{\infty} A_m = \{X < \frac{1}{n}\}$ for any $n \in \mathbb{N}$,

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{X < \lim_{n\to\infty} \frac{1}{n}\} = \{X < 0\}.$$

Therefore,

$$P(\limsup_{n\to\infty} A_n) = P(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m) = \int_0^0 1 \, \mathrm{d}P = 0.$$

$\square$

Problem 3

If X is a nonnegative random variable, show that

$$\sum_{n=1}^{\infty} P(X \geq n) \leq \mathbb{E}(X) \leq 1 + \sum_{n=1}^{\infty} P(X \geq n).$$

Proof. Let $I_n = \begin{cases} 1 & \text{if } X \geq n \\ 0 & \text{otherwise} \end{cases}$ for $n \in \mathbb{N}$.

Then,

$$\sum_{n=1}^{\infty} I_n \leq X \leq 1 + \sum_{n=1}^{\infty} I_n.$$

Taking expectations gives

$$\sum_{n=1}^{\infty} \mathbb{E}(I_n) \leq \mathbb{E}(X) \leq 1 + \sum_{n=1}^{\infty} \mathbb{E}(I_n).$$

Note that above inequality holds due to linearity of expectation.

Also, $\mathbb{E}(I_n) = 1 \cdot P(X \geq n) + 0 \cdot P(X < n) = P(X \geq n)$. Therefore, we are done. $\square$

Problem 4

Let X and Y be independent random variables whose values are nonnegative integers, and write

$$a_i = P(X = i), \quad b_i = P(Y = i).$$

If $Z = X + Y$, prove that $P(Z = n) = \sum_{i=0}^{n} a_i b_{n-i}$.

Proof.

$$\begin{aligned}
P(Z = n) &= P(X + Y = n) \\
&= \sum_{i=0}^{n} P(X + Y = n | X = i) P(X = i) \\
&= \sum_{i=0}^{n} P(i + Y = n) P(X = i) \quad (\because X \text{ and } Y \text{ are independent.}) \\
&= \sum_{i=0}^{n} P(Y = n - i) P(X = i) \\
&= \sum_{i=0}^{n} b_{n-i} a_i = \sum_{i=0}^{n} a_i b_{n-i}.
\end{aligned}$$

□