Topology II – Homework 2

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1 Proof of Theorem 51.2

Theorem 51.2. The product $*$ has the following properties:

(i) Associative: $([f] * [g]) * [h] = [f] * ([g] * [h])$

(ii) Let e_x denote the constant path $e_x \colon I \to X$ given by $s \mapsto x$, $x \in X$. If $f(0) = x_0$ and $f(1) = x_1$ then $[e_{x_0}] * [f] = [f]$ and $[f] * [e_{x_1}] = [f]$.

(iii) Let $\overline{f} \colon I \to X$ given by $s \mapsto f(1-s)$. Then $[f] * [\overline{f}] = [e_{x_0}]$ and $[\overline{f}] * [f] = [e_{x_1}]$.

Proof. First two observations

- Suppose $f \simeq_p g$ via homotopy H, $f, g \colon I \to X$. Let $k \colon X \to Y$. Then $k \circ f \simeq_p k \circ g$ using $k \circ H$.
- If $f * g$ (not necessarily path homotopic). Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Now, the proof

(ii) Take $e_0 \colon I \to I$ given by $s \mapsto 0$. Take $i \colon I \to I$ given by $s \mapsto s$. Then $e_0 * i$ is a path from 0 to $1 \in I$. The path i is also such a path. Because I is a convex subset, $e_0 * i$ and i are path homotopic, $e_0 * \simeq i$. Using one of our observations, we find that

$$\begin{aligned} f \circ (e_0 * i) &\simeq_p f \circ i \\ (f \circ e_0) * (f \circ i) &\simeq_p f \\ e_{x_0} * f &\simeq_p f \\ [e_{x_0}] * [f] &= [f]. \end{aligned}$$

(iii) Note that $i * \bar{i} \simeq_p e_0$. Now, applying the same rules, we get

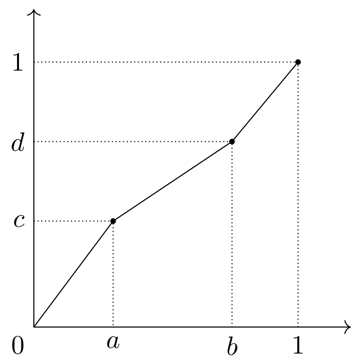
$$\begin{aligned} f \circ (i * \bar{i}) &\simeq_p f \circ e_0 \\ f * \overline{f} &\simeq_p e_{x_0} \\ [f] * [\overline{f}] &= [e_{x_0}]. \end{aligned}$$

(i) Remark: Only defined if $f(1) = g(0)$, $g(1) = h(0)$. Note that $f * (g * h) \neq (f * g) * h$. The trajectory is the same, but the speed is not.

Assume the product is defined. Suppose $[a, b]$, $[c, d]$ are intervals in $\mathbb{R}$. Then there is a unique positive (positive slope), linear map from $[a, b] \to [c, d]$. For any $a, b \in [0, 1)$ with $0 < a < b < 1$, we define a path

$$\begin{aligned} k_{a,b} \colon [0,1] &\longrightarrow X \\ [0, a] &\xrightarrow{\text{lin.}} [0,1] \xrightarrow{f} X \\ [a, b] &\xrightarrow{\text{lin.}} [0,1] \xrightarrow{g} X \\ [b, 0] &\xrightarrow{\text{lin.}} [0,1] \xrightarrow{h} X \end{aligned}$$

Then $f * (g * h) = k_{\frac{1}{2}, \frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4}, \frac{1}{2}}$.
Let γ be that path $\gamma \colon I \to I$ with the following graphs:



Note that $\gamma \simeq_p i$. Now, using the fact that composition of positive linear maps is positive linear.

$$\begin{aligned} k_{c,d} \circ \gamma &\simeq_p k_{c,d} \circ i \\ k_{a,b} &\simeq_p k_{c,d}, \end{aligned}$$

which is what we wanted to show.

2 Exercises

Exercise 51.1. Show that if $h, h' \colon X \to Y$ are homotopic and $k, k' \colon Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof. Let $H \colon X \times I \to Y$ and $K \colon Y \times I \to Z$ be homotopies between h, h' and k, k' respectively, i.e. $H(x, 0) = h(x)$, $H(x, 1) = h'(x)$, $K(y, 0) = k(y)$, and $K(y, 1) = k'(y)$. Then, define the map $F \colon X \times I \to Z$ by $F(x, t) = K(H(x, t), t)$. This is continuous and defines a homotopy between $F(x, 0) = K(H(x, 0), 0) = K(h(x), 0) = k(h(x)) = k \circ h$ and $F(x, 1) = K(H(x, 1), 1) = K(h'(x), 1) = k'(h'(x)) = k' \circ h'$. $\square$

Exercise 51.2. Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.

(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Proof. To explain more about homotopy class, given two topological spaces X and Y, place an equivalence relation on the continuous maps $f\colon X \to Y$ using homotopies, and write $f_1 \sim f_2$ if f_1 is homotopic to f_2.

(a) We need to show that all continuous maps of X into I are homotopic to each other; we do this by showing that every continuous map $f\colon X \to I$ is homotopic to the constant map $f_0\colon X \to I$ defined by $f_0(x) = 0$ for all $x \in X$. This is indeed the case, and an explicit homotopy is given by $F\colon X \times I \to I$ defined by $F(x, t) = tf(x)$, which is clearly continuous, and satisfies $F(x, 0) = 0 = f_0(x)$ and $F(x, 1) = f(x)$.

(b) Assuming Y is path-connected, we need to show that any two continuous maps from I to Y are homotopic. First we show that every continuous map $f\colon I \to Y$ is homotpic to the constant map $I \to Y$ which maps every element of I to $f(0)$. Indeed, consider $F\colon I \times I \to Y$ given by $F(s, t) = f(st)$, which is continuous. This is a homotopy between the constant map $F(s, 0) = f(0)$ and $F(s, 1) = f(s)$. (In other terms: we have shown that every path in Y can be homotoped (not fixing the end points) to the constant path at its starting point).

Next, given two points $y, y' \in Y$, let $f, f'\colon I \to Y$ be the constant maps taking the values $f(s) = y$ and $f'(s) = y'$ $\forall s \in I$. Since Y is path-connected, there exists a path $g\colon I \to Y$ such that $g(0) = y$ and $g(1) = y'$. We then consider the map $F\colon I \times I \to Y$ defined by $F(s, t) = g(t)$, which gives a homotopy between $F(s, 0) = g(0) = y = f(s)$ and $F(s, 1) = g(1) = y' = f'(s)$. Thus, any path is homotopic to a constant path, and any two constant paths are homotopic to each other (again, not fixing the end points); it follows that any two maps $I \to Y$ are homotopic.

$\square$

Exercise 51.3. A space X is said to be contractible if the identity map $i_X\colon X \to X$ is null-homotopic.

(a) Show that I and $\mathbb{R}$ are contractible.

(b) Show that a contractible space is path connected.

(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Proof. (a) Let $F\colon I \times I \to I$ be defined by $F(s, t) = st$ and $G\colon \mathbb{R} \times I \to \mathbb{R}$ by $G(s, t) = st$. These are homotopies between the constant map at 0 and identity map, so both spaces are contractible.

(b) Recall that if there is a path between a, b and a path between b, c, then there is a path between a, c. It therefore suffices to show that all points can be connected to a given point by a path. Assuming X is contractible, there is a homotopy $F\colon X \times I \to X$ between identity map id_X and the constant map f_0 mapping every point $x \in X$ to the same point $x_0 \in X$ s.t. $F(x, 0) = f_0(x) = x_0$ and $F(x, 1) = \mathrm{id}_X(x) = x$ for all $x \in X$. Then, the map $g\colon I \to X$ defined by $g(t) = F(x, t)$ is continuous and determines a path from $g(0) = x_0$ to $g(1) = x$.

(c) Assume Y is contractible, and let $F\colon Y \times I \to Y$ be a homotopy s.t. $F(y, 1) = y$ is the identity map and $F(y, 0) = y_0 \in Y$ is a constant map sending every point to some point $y_0 \in Y$. Then given any map $g\colon X \to Y$, we consider $G\colon X \times I \to Y$ defined by $G(x, t) = F(g(x), t)$. This is continuous, and defines a homotopy between g and the constant map g_0 which maps every point of X to y_0. Indeed, $G(x, 1) = F(g(x), 1) = g(x)$, and $G(x, 0) = F(g(x), 0) = y_0$. It follows that every map from X to Y is homotopic to the constant map g_0, and hence that any two maps from X to Y are homotopic to each other.

(d) Since X is contractible, id_X is homotopic to a constant map $g(x) = x_0$ by a homotopy $G\colon X \times I \to X$ s.t. $G(x, 0) = x_0$, $G(x, 1) = x$ for all $x \in X$. First we show that every continuous map $f\colon X \to Y$ is homotopic to the constant map $X \to Y$ which maps every element of X to $f(x_0)$. Indeed, define a continuous map $F\colon X \times I \to Y$ by $F(x, t) = f(G(x, t))$. This is a homotopy between the constant map $F(x, 0) = f(G(x, 0)) = f(x_0)$ and $F(x, 1) = f(G(x, 1)) = f(x)$.

Next, we show that if Y is path connected then constant maps (sending every point of X to the same point of Y) are homotopic to each other. Indeed, given two points $y_0, y_1 \in Y$, let $f_0, f_1\colon X \to Y$ be the constant maps taking the values $f_0(x) = y_0$ and $f_1(x) = y_1$ for all $x \in X$. Since Y is path-connected, there exists a path $g\colon I \to Y$ s.t. $g(0) = y_0$ and $g(1) = y_1$. We then consider the map $F\colon X \times I \to Y$ defined by $F(x, t) = g(t)$, which gives a homotopy between $F(x, 0) = g(0) = y_0 = f_0(x)$ and $F(x, 1) = g(1) = y_1 = f_1(x)$.

Thus, assuming X contractible and Y path-connected, any continuous map of X into Y is homotopic to a constant map, and any two constant maps are homotopic to each other. It follows that any two continuous maps from X to Y are homotopic to each other.

$\square$