Differential Equations – Final Exam

Junwoo Yang

June 19, 2020

1. Solve the given initial value problems.
   (a) $y'' + y = \sec t, \quad y(0) = 1, \quad y'(0) = 0.$
   (b) $y'' + 2y' + 3y = 3\delta(t - 2\pi) + u(t - \pi)\cos t, \quad y(0) = 0, \quad y'(0) = 1.$
2. Let y be the solution of $y' = y^2 - ty$, $y(0) = 1$. Estimate $y(1)$ using the Improved Euler method with the step size $h = 1$.
3. Find a second solution y_2 of the given differential equation
$$t^2y'' - 2t(t+1)y' + 2(t+1)y = 0, \quad t > 0, \quad y_1(t) = t$$
by using the method of the reduction of order.
4. For $x > 0$, find two (linearly independent) solutions of $x^2y'' + (x^2 + x)y' - y = 0$.
5. Find the general solution of
$$X' = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} X + \begin{pmatrix} -4\cos t \\ 3\cos t \end{pmatrix}.$$
6. For the following two problems
$$\text{(a)} \begin{cases} x' = y - x^2 \\ y' = (x - y^2)(1 - 4y) \end{cases} \qquad \text{and} \qquad \text{(b)} \begin{cases} x' = -4x + y + x^2 \\ y' = y - x - 2 \end{cases}$$
   (a) Find all critical points and classify the type of each critical point.
   (b) Draw the phase plane, sketch a few of the trajectories in the neighborhood of each critical point.
7. Find the solution of
$$\begin{cases} u_{xx} = u_t, & 0 < x < 20, \quad t > 0 \\ u(0,t) = 30, \quad u(20,t) = 10, & t > 0 \\ u(x,0) = x(20 - x)/10, & 0 \le x \le 20 \end{cases}.$$
8. If u_1 and u_2 are solutions of
$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, \quad t > 0 \\ u(x,0) = f(x), & 0 < x < L \\ u(0,t) = g_1(t), \quad u(L,t) = g_2(t), & t \ge 0 \end{cases},$$
then show that $u_1 = u_2$.