

## Prob. 1.1.

Claim 1).  $tB_t \in H^2 = \{g: \text{prog. m'ble} \mid E \int_0^T g_u^2 du < \infty\}$ .

(i)  $(tB_t)_{0 \leq t \leq T}$ : adapted + conti  $\Rightarrow$  prog m'ble.

$$\begin{aligned} (\text{ii}). E \int_0^T (tB_t)^2 dt &= E \int_0^T t^2 B_t^2 dt \\ &= \int_0^T t^2 E(B_t^2) dt = \int_0^T t^3 dt = \frac{T^4}{4} < \infty \end{aligned}$$

$\therefore \int_0^T tB_t dB_t$  is well-defined.

Claim 2).  $E(\int_0^T tB_t dB_t) = 0$ .

Since  $tB_t \in H^2$ ,  $(\int_0^t uB_u dB_u)_{0 \leq t \leq T}$ : martingale.

$$\therefore E(\int_0^T tB_t dB_t) = E(\int_0^0 tB_t dB_t) = 0.$$

We calculate

$$\begin{aligned} \text{Var}(\int_0^T tB_t dB_t) &= E((\int_0^T tB_t dB_t)^2) - (E(\int_0^T tB_t dB_t))^2 \\ &= E(\int_0^T t^2 B_t^2 dt) \quad (\text{by Itô isometry and claim 2.}) \\ &= \int_0^T t^2 E(B_t^2) dt = \int_0^T t^3 dt = \frac{T^4}{4} \quad \square. \end{aligned}$$

## Prob. 1.2 (i).

Since  $f: [0, T] \rightarrow \mathbb{R}$  is Borel function,

for  $A \in B(\mathbb{R})$ ,  $f^{-1}(A) \in B([0, T])$ .

$f|_{[0, t]}: [0, t] \rightarrow \mathbb{R}$ ,  $u \mapsto f_{uw}$  for  $t \in [0, T]$ ,  $u \in [0, t]$

$(f|_{[0, t]})^{-1}(A) = (f^{-1}(A) \cap [0, t]) \in B([0, t])$  for  $A \in B(\mathbb{R})$ .

$\tilde{F}: \Omega \times [0, T] \rightarrow \mathbb{R}$ ,  $(w, t) \mapsto f_{wt}$  for  $w \in \Omega$ ,  $t \in [0, T]$

$\tilde{F}^{-1}(A) \in \Omega \times B([0, T])$ .

$\tilde{F}|_{[0, t]}: \Omega \times [0, t] \rightarrow \mathbb{R}$ ,  $(w, u) \mapsto f_{uw}$  for  $w \in \Omega$ ,  $t \in [0, T]$ ,  $u \in [0, t]$

$\therefore (\tilde{F}|_{[0, t]})^{-1}(A) = \Omega \times (\tilde{F}^{-1}(A) \cap [0, t]) \in \Omega \times B([0, t])$

$$\subseteq \mathcal{F}_t \times B([0, t])$$

$\therefore \tilde{F}$  is  $\mathcal{F}_t \times B([0, t])$ -measurable,

namely, progressively measurable.

### Prob. 1.2 (ii)

$(\int_0^t f(u) dB_u)_{0 \leq t \leq T}$  is Gaussian process.

$\Leftrightarrow (\int_0^{t_1} f(u) dB_u, \dots, \int_0^{t_n} f(u) dB_u)$  is jointly normal  $\forall t_1, \dots, t_n \in [0, T]$

$\Leftrightarrow \sum_{i=1}^n \lambda_i \int_0^{t_i} f(u) dB_u$  is normal  $\forall \lambda_i \in \mathbb{R}$ .

$\Leftrightarrow \int_0^T \left( \sum_{i=1}^n \lambda_i f(u) \mathbf{1}_{[0, t_i]}(u) \right) dB_u$  is normal  $\forall \lambda_i \in \mathbb{R}$ .

$$\int_0^T \left( \sum_{i=1}^n \lambda_i f(u) \mathbf{1}_{[0, t_i]}(u) \right) dB_u$$

$$= \int_0^T ((\lambda_1 + \dots + \lambda_n) f(u) \mathbf{1}_{[0, t_1]} + (\lambda_2 + \dots + \lambda_n) f(u) \mathbf{1}_{[t_1, t_2]} + \dots + \lambda_n f(u) \mathbf{1}_{[t_{n-1}, t_n]}) dB_u$$

$$\Rightarrow \int_0^T \sum_{i=0}^{n-1} c_i f(u) \mathbf{1}_{[t_i, t_{i+1}]} dB_u \text{ where } t_i = \frac{iT}{n}, c_i \in \mathbb{R}$$

$\therefore$  It suffices to show that  $\int_0^T f(t) dB_t$  is normal  
where  $\int_0^T f^2(t) dt < \infty$ .

As a stochastic process,  $(f(t))_{0 \leq t \leq T}$  is progressively measurable by (i).

Since  $f$  is prog. m'ble and  $E \int_0^T f^2 dt = \int_0^T f^2 dt < \infty$ ,  $f \in H^2$ .

Thus,  $f$  can be approximated by a seq. of simple processes.

$(f^{(n)})_{n \geq 1}$  s.t.  $E \int_0^T (f^{(n)} - f)^2 dt \rightarrow 0$ .

$f^{(n)} = \sum_{i=0}^{n-1} f_{t_i} \mathbf{1}_{[t_i, t_{i+1}]}$  where  $\{0 = t_0 \leq t_1 \leq \dots \leq t_n = T\}$ .

Note that  $\int_0^T f(t) dB_t = \lim_{n \rightarrow \infty} \int_0^T f^{(n)}(t) dB_t$ ,

$\int_0^T f^{(n)}(t) dB_t = \sum_{i=0}^{n-1} f_{t_i} (B_{t_{i+1}} - B_{t_i})$  is normal.

By the fact that the limit of normal distributions is normal,  
 $\int_0^T f(t) dB_t$  is normal.  $\square$ .

### Prob. 1.2 (iii)

Since  $t$  is Borel function, as a stochastic process,  $t$  is prog. m'ble.

Also,  $E \int_0^T t^2 dt = \int_0^T t^2 dt = \frac{T^3}{3} < \infty$ . Thus  $t \in H^2$  and

$(\int_0^t u dB_u)_{0 \leq t \leq T}$  is martingale.

$\therefore E(\int_0^T t dB_t) = E(\int_0^0 t dB_t) = 0$ .

$Var(\int_0^T t dB_t) = E((\int_0^T t dB_t)^2) = E(\int_0^T t^2 dt) = \int_0^T t^2 dt = \frac{T^3}{3}$

$\curvearrowleft$  Ito isometry. ( $t \in H^2$ ).

By (ii),  $\int_0^T t dB_t$  is normal.

$\therefore E(e^{\int_0^T t dB_t}) = e^{E(\int_0^T t dB_t) + \frac{1}{2} Var(\int_0^T t dB_t)} = e^{\frac{T^3}{6}}$   $\square$

### Prob. 1.3. (i)

Since  $f$  is a Borel function with  $\int_0^T f^2(t) dt < \infty$ , By Prob. 1.2,  $f \in H^2$  and  $\int_0^t f(u) dB_u$  is normal for  $0 \leq t \leq T$ .  
 $(\int_0^t f(u) dB_u)_{0 \leq t \leq T}$  is martingale.

$$\begin{aligned} E(M_t | F_s) &= E(e^{\int_0^t f(u) dB_u - \frac{1}{2} \int_0^t f^2(u) du} | F_s) \text{ for } 0 \leq s < t \\ &= e^{-\frac{1}{2} \int_0^t f^2(u) du} E(e^{\int_0^t f(u) dB_u} | F_s) \text{ for } 0 \leq s < t \\ &= e^{-\frac{1}{2} \int_0^t f^2(u) du} E(e^{\int_0^s f(u) dB_u} \cdot e^{\int_s^t f(u) dB_u} | F_s) \\ &= e^{\int_0^s f(u) dB_u - \frac{1}{2} \int_0^s f^2(u) du} E(e^{\int_s^t f(u) dB_u}) \quad \xrightarrow{*} \\ &= e^{\int_0^s f(u) dB_u - \frac{1}{2} \int_0^s f^2(u) du} \cdot e^{\frac{1}{2} \int_s^t f^2(u) du} (\because \int_s^t f(u) dB_u \text{ is normal}). \\ &= e^{\int_0^s f(u) dB_u - \frac{1}{2} \int_0^s f^2(u) du} = M_s. \end{aligned}$$

(\*). Since  $f \in H^2$ ,  $f$  can be approximated by a seq. of simple processes.

$$\int_0^t f(u) dB_u = \lim_{n \rightarrow \infty} \int_0^t f^{(n)}(u) dB_u = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(u_i) (B_{u_{i+1}} - B_{u_i})$$

(where  $u_0 = 0$ ,  $u_n = t$ ).

$$e^{\int_0^s f(u) dB_u} = \lim_{n \rightarrow \infty} e^{\sum_{i=0}^{n-1} f(u_i) (B_{u_{i+1}} - B_{u_i})} \quad \text{where } u_0 = 0, u_n = s$$

(Then  $B_{u_{i+1}} - B_{u_i} \in F_s$   $\forall i$ , each step  $n$ ).

$$e^{\int_s^t f(u) dB_u} = \lim_{n \rightarrow \infty} e^{\sum_{i=0}^{n-1} f(u_i) (B_{u_{i+1}} - B_{u_i})} \quad \text{where } u_0 = s, u_n = t$$

(Then  $B_{u_{i+1}} - B_{u_i} \perp F_s$   $\forall i$ , each step  $n$ ).

$$\therefore E(e^{\int_0^s f(u) dB_u} \cdot e^{\int_s^t f(u) dB_u} | F_s) = e^{\int_0^s f(u) dB_u} E(e^{\int_s^t f(u) dB_u})$$

$$M_t = e^{\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds}, \quad 0 \leq t \leq T$$

### Prob. 1.3. (ii)

$\theta \in H^2_{loc} \Rightarrow (\int_0^t \theta(u) dB_u)_{0 \leq t \leq T}$  is local martingale.

There exists seq. of stopping times  $(\tau_n)_{n \geq 1}$  s.t.  $\tau_n \leq \tau_{n+1}$ ,  $\lim_{n \rightarrow \infty} \tau_n = T$ ,  $(\int_0^{\tau_n} \theta(u) dB_u)_{0 \leq t \leq T}$  is a martingale.

WTS  $(M_t \wedge \tau_n)_{0 \leq t \leq T}$  is a martingale.

$$\begin{aligned} E(M_t \wedge \tau_n | F_s) &\xrightarrow{\text{by Prob. 1.3. (i)}} \\ &= E(e^{\int_0^{\tau_n} \theta(u) dB_u - \frac{1}{2} \int_0^{\tau_n} \theta^2(u) du} | F_s) = e^{\int_0^{\tau_n} \theta(u) dB_u - \frac{1}{2} \int_0^{\tau_n} \theta^2(u) du} \\ &= M_{s \wedge \tau_n} \quad \square. \end{aligned}$$

### Prob. 1.4. (i)

$$B = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0}, (F_t^B)$$

$(B_t^{(1)} B_t^{(2)}, t B_t^{(3)}, 0)$  is continuous and adapted. Thus progressively m'ble.

$$\begin{aligned} E \int_0^T |(B_t^{(1)} B_t^{(2)}, t B_t^{(3)}, 0)|^2 dt &= E \int_0^T (B_t^{(1)^2} B_t^{(2)^2} + t^2 B_t^{(3)^2}) dt \\ &= \int_0^T (E(B_t^{(1)^2}) E(B_t^{(2)^2}) + t^2 E(B_t^{(3)^2})) dt \quad (\because B^{(1)} \perp B^{(2)}) \\ &= \int_0^T t^2 + t^3 dt = \frac{T^3}{3} + \frac{T^4}{4} < \infty \end{aligned}$$

$$\therefore (B_t^{(1)} B_t^{(2)}, t B_t^{(3)}, 0) \in H^2(\Omega \times [0, T], \mathbb{R}^3, F, (F_t^B), P).$$

We know that for  $g \in H^2$ ,  $(\int_0^t g u d B_u)_{0 \leq t \leq T}$  is martingale.

Thus,  $(\int_0^t (B_s^{(1)} B_s^{(2)}, u B_s^{(3)}, 0) \cdot (d B_s^{(1)}, d B_s^{(2)}, d B_s^{(3)}))_{0 \leq t \leq T}$  is mart.  $\forall T > 0$ .

Hence,  $(\int_0^t B_s^{(1)} B_s^{(2)} d B_s^{(1)} + \int_0^t u B_s^{(3)} d B_s^{(3)})_{t \geq 0}$  is martingale w.r.t  $(F_t^B)_{t \geq 0}$ .

### Prob. 1.4. (ii)

$$E \left( \int_0^T B_s^{(1)} B_s^{(2)} d B_s^{(1)} + \int_0^T s B_s^{(3)} d B_s^{(3)} \right) = E \left( \int_0^T B_s^{(1)} B_s^{(2)} d B_s^{(1)} + \int_0^T s B_s^{(3)} d B_s^{(3)} \right) = 0.$$

$$\begin{aligned} \text{Var} \left( \int_0^T B_s^{(1)} B_s^{(2)} d B_s^{(1)} + \int_0^T s B_s^{(3)} d B_s^{(3)} \right) &= \text{Var} \left( \int_0^T (B_s^{(1)} B_s^{(2)}, s B_s^{(3)}, 0) d B_s \right) \\ &= E \left( \left( \int_0^T (B_s^{(1)} B_s^{(2)}, s B_s^{(3)}, 0) d B_s \right)^2 \right) \\ &= E \left( \int_0^T (B_s^{(1)^2} B_s^{(2)^2} + s^2 B_s^{(3)^2}) ds \right) \quad (\because \text{Itô isometry}) \\ &= \int_0^T E(B_s^{(1)^2}) E(B_s^{(2)^2}) + s^2 E(B_s^{(3)^2}) ds \\ &= \int_0^T s^2 + s^3 ds = \frac{T^3}{3} + \frac{T^4}{4} \end{aligned}$$

### Prob. 1.4. (iii)

We know that  $g \in H^2 \Rightarrow ((\int_0^t g u d B_u)^2 - \int_0^t |g u|^2 du)_{0 \leq t \leq T}$  is martingale.

$$\text{Let } h = (1, -2, B_s^{(1)})$$

Since  $h$  is continuous and adapted,  $h$  is progressively m'ble.

$$E \int_0^T |h|^2 ds = E \int_0^T 1 + B_s^{(1)^2} ds = \int_0^T 1 + s ds = \frac{1}{2} T + \frac{T^2}{2} < \infty.$$

$\therefore h \in H^2$  for each  $T > 0 \Rightarrow ((\int_0^t h u d B_u)^2 - \int_0^t |h u|^2 du)_{t \geq 0}$  is martingale.

$$\Rightarrow ((\int_0^t 1 \cdot d B_s + \int_0^t -2 \cdot d B_s + \int_0^t B_s^{(1)} d B_s^{(1)})^2 - \int_0^t (1 + B_s^{(1)^2}) ds)_{t \geq 0}$$

$$\Rightarrow ((B_t^{(1)} - 2 B_t^{(2)} + \int_0^t B_s^{(1)} d B_s^{(1)})^2 - \int_0^t B_s^{(1)^2} ds - \frac{1}{2} T)_{t \geq 0} \text{ is martingale. } \square$$

### Prob. 1.5. (i).

Since  $g, h \in \mathcal{H}^2$ ,  $g+h \in \mathcal{H}^2$

Apply the Itô isometry to  $\int_0^T (gt + ht) dB_t$

$$\Rightarrow E\left(\left(\int_0^T (gt + ht) dB_t\right)^2\right) = E\left(\int_0^T (gt + ht)^2 dt\right)$$

$$(LHS) = E\left(\left(\int_0^T gt dB_t + \int_0^T ht dB_t\right)^2\right)$$

$$= E\left(\left(\int_0^T gt dB_t\right)^2\right) + E\left(\left(\int_0^T ht dB_t\right)^2\right) + 2E\left(\int_0^T gt dB_t \int_0^T ht dB_t\right)$$

$$= E\left(\int_0^T g^2 dt\right) + E\left(\int_0^T h^2 dt\right) + 2E\left(\int_0^T gt dB_t \int_0^T ht dB_t\right)$$

$$(RHS) = E\left(\int_0^T (g^2 + 2gt \cdot ht + h^2) dt\right)$$

$$= E\left(\int_0^T g^2 dt\right) + E\left(\int_0^T h^2 dt\right) + 2E\left(\int_0^T gt \cdot ht dt\right)$$

$$\text{Hence, } E\left(\int_0^T gt dB_t \int_0^T ht dB_t\right) = E\left(\int_0^T gt \cdot ht dt\right)$$

### Prob 1.5. (ii).

$$\text{Let } B = (B_{\frac{T}{2}}^{(1)}, B_T^{(1)} - B_{\frac{T}{2}}^{(1)})_{t \geq 0}$$

Note  $B_{\frac{T}{2}}^{(1)} \perp B_T^{(1)} - B_{\frac{T}{2}}^{(1)}$  ( $\because$  by def. of B.M.). and

$$B_T^{(1)} - B_{\frac{T}{2}}^{(1)} \sim N(0, \frac{T}{2}) \sim B_{\frac{T}{2}}^{(2)} \Rightarrow \text{one-dim'l B.M.}$$

Thus  $B = (B_{\frac{T}{2}}^{(1)}, B_T^{(1)} - B_{\frac{T}{2}}^{(1)})$  is two-dim'l B.M.

$$\begin{aligned} E\left(B_{\frac{T}{2}}^{(1)} \int_0^T (B_T^{(2)})^2 dB_t\right) &= E\left(\int_0^T (I, D) dB_t \int_0^T (B_T^{(2)})^2 dB_t\right) \\ &= E\left(\int_0^T (I, D) \cdot (B_T^{(2)})^2 dt\right) \\ &= E\left(\int_0^T B_T^{(2)} dt\right) = \int_0^T E(B_T^{(2)}) dt = \int_0^T t dt = \frac{T^2}{2} \end{aligned}$$

$$E(B_T^{(1)} B_T^{(2)} \int_0^T e^{B_T^{(2)}} dB_t)$$

$$= E\left(\int_0^T (D, B_T^{(1)}) dB_t \cdot \int_0^T (e^{B_T^{(2)}}, 0) dB_t\right). \quad (\text{where } B = (B_T^{(1)}, B_T^{(2)}, B_T^{(2)})_{t \geq 0})$$

$$= E\left(\int_0^T (D, B_T^{(1)}) \cdot (e^{B_T^{(2)}}, 0) dt\right) = 0.$$

$$(\text{or } = E\left(\int_0^T (B_T^{(2)}, 0) dB_t \int_0^T (e^{B_T^{(2)}}, 0) dB_t\right))$$

$$= E\left(\int_0^T B_T^{(2)} e^{B_T^{(2)}} dt\right)$$

$$= \int_0^T E(B_T^{(2)} e^{B_T^{(2)}}) dt$$

$$= \int_0^T E(B_T^{(2)} - B_{\frac{T}{2}}^{(2)}) e^{B_T^{(2)}} + B_{\frac{T}{2}}^{(2)} e^{B_T^{(2)}} dt$$

$$= \int_0^T E(B_T^{(2)} - B_{\frac{T}{2}}^{(2)}) E(e^{B_T^{(2)}}) + E(B_T^{(2)} e^{B_T^{(2)}}) dt$$

$$= \int_0^T t e^{\frac{t}{2}} dt = [t \cdot 2e^{\frac{t}{2}}]_0^T - \int_0^T 2e^{\frac{t}{2}} dt$$

$$= 2Te^{\frac{T}{2}} - 4e^{\frac{T}{2}}$$

$(\because B_T^{(2)} \sim FZ, E(Z e^{uz}) = u e^{\frac{u^2}{2}})$

### Prob. 1.6.

$$X_t = -t^2 + \sin(B_t^2) = \int_0^t -2s ds + \int_0^t \cos(B_s^2) 2B_s dB_s$$

$$Y_t = \int_0^t B_s ds + \int_0^t s B_s^2 dB_s$$

$$\langle X \rangle_t = \int_0^t \cos^2(B_s^2) 4B_s^2 ds$$

$$\langle Y \rangle_t = \int_0^t s^2 B_s^4 ds$$

$$\langle X, Y \rangle_t = \int_0^t 2s B_s^3 \cos(B_s^2) ds$$

### Prob. 1.7.

$$d\langle X, Y \rangle_t = dX_t \cdot dY_t$$

$$= (b_t^X dt + \delta_t^X dB_t)(b_t^Y dt + \delta_t^Y dB_t)$$

$$= (b_t^X dt + \delta_t^{X,(d)} dB_t^{(d)} + \dots + \delta_t^{X,(d)} dB_t^{(d)}) (b_t^Y dt + \delta_t^{Y,(d)} dB_t^{(d)} + \dots + \delta_t^{Y,(d)} dB_t^{(d)})$$

$$= \delta_t^{X,(d)} \delta_t^{Y,(d)} dB_t^{(d)^2} + \dots + \delta_t^{X,(d)} \delta_t^{Y,(d)} dB_t^{(d)^2}$$

$$(\because (dt)^2 = dt dB_t^{(d)} = dB_t^{(d)} dB_t^{(d)} = 0.)$$

$$= (\delta_t^{X,(d)} \delta_t^{Y,(d)} + \dots + \delta_t^{X,(d)} \delta_t^{Y,(d)}) dt \quad (\because dB_t^{(d)^2} \approx dt).$$

$$= (\delta_t^X \cdot \delta_t^Y) dt$$

$$\therefore \langle X, Y \rangle_t = \int_0^t \delta_u^X \cdot \delta_u^Y du$$

### Prob. 1.8.(i)

$$Y_t = C + \int_0^t b_s ds + \int_0^t \delta_s dB_s = 0 \quad 0 \leq t \leq T.$$

$$Y_0 = C + \int_0^0 b_s ds + \int_0^0 \delta_s dB_s = C = 0. \Rightarrow C = 0.$$

$$\langle Y \rangle_t = \int_0^t \|\delta_s\|^2 ds = \langle D \rangle_t = 0. \Rightarrow E \int_0^t \|\delta_s\|^2 ds = 0 \quad \forall t \in [0, T]$$

$$\Rightarrow \|\delta_s\|^2 = 0 \Leftrightarrow \delta_s = 0. \text{ as on } \Omega \times [0, T]$$

Since  $\delta$  is progr. mble,  $E \int_0^T \|\delta_s\|^2 ds = 0 < \infty \Rightarrow \delta_s \in \mathcal{H}^2$

By Itô isometry,  $E \int_0^T \|\delta_s\|^2 ds = E \left( \int_0^T \delta_s dB_s \right)^2 = 0. \Rightarrow \int_0^T \delta_s dB_s = 0$

Now we get  $Y_t = \int_0^t b_s ds = 0.$

For any  $w \in \Omega$ ,  $Y_{t(w)} = \int_0^{t(w)} b_s(w) ds = 0 \Rightarrow b_s(w) = 0 \text{ a.e. } t \in [0, T]$

$\Rightarrow E \int_0^T |b_t| dt = E(D) = 0. \Rightarrow |b| = 0 \Rightarrow b = 0 \text{ a.s. on } \Omega \times [0, T]$

Prob. 1.8.(ii)

$$\begin{aligned}\text{Suppose } X_t &= X_0 + \int_0^t b_s ds + \int_0^t b'_s dB_s \\ &= X'_0 + \int_0^t b'_s ds + \int_0^t b''_s dB_s.\end{aligned}$$

$$\text{Then } X_t - X'_t = (X_0 - X'_0) + \int_0^t (b_s - b'_s) ds + \int_0^t (b'_s - b''_s) dB_s = 0.$$

By Prob. 1.8.(i),  $X_0 - X'_0 = 0$ ,  $b_s - b'_s = 0$ ,  $b'_s - b''_s = 0$  a.s. on  $\mathcal{S} \times [0, T]$

$$\text{Thus } X_0 = X'_0, \int_0^t b_s ds = \int_0^t b'_s ds, \int_0^t b'_s dB_s = \int_0^t b''_s dB_s.$$

Hence, the decomposition is unique.  $\square$ .