

## Lecture note 3: Stochastic calculus

**1 Exercises**

**Problem 1.1.** (10 points) Let  $(\mathcal{F}_t)_{t \geq 0}$  and  $(\mathcal{G}_t)_{t \geq 0}$  be two filtrations such that  $\mathcal{G}_t \subseteq \mathcal{F}_t$  for all  $t \geq 0$ . Suppose that  $M$  is a martingale with respect to  $(\mathcal{F}_t)_{t \geq 0}$  and is adapted to  $(\mathcal{G}_t)_{t \geq 0}$ . Show that  $M$  is a martingale with respect to filtration  $(\mathcal{G}_t)_{t \geq 0}$ .

**Problem 1.2.** (10 points) Let  $(B_t)_{t \geq 0}$  be a Brownian motion. Show that the following processes  $M$  are martingales with respect to the filtration generated by  $B$ .

- (i)  $M_t = B_t^2 - t$
- (ii)  $M_t = B_t^3 - 3 \int_0^t B_u du$

Hint: First, show these with respect to the filtration  $(\mathcal{F}_t^B)_{t \geq 0}$  generated by  $B$ .

**Problem 1.3.** (20 points) Let  $B = (B_t^{(1)}, B_t^{(2)})_{t \geq 0}$  be a two dimensional Brownian motion. Calculate the followings.

$$(i) \quad \mathbb{P}(B_t^{(1)} > \sqrt{3}B_s^{(2)})$$

$$(ii) \quad \mathbb{E}(e^{B_t^{(1)} + \int_0^s B_u^{(2)} du})$$

- (iii) Consider the filtration  $(\mathcal{F}_t)_{t \geq 0}$  generated by  $B$ . Show that the product  $(B_t^{(1)} B_t^{(2)})_{t \geq 0}$  is a martingale with respect to  $(\mathcal{F}_t)_{t \geq 0}$ .

- (iv) Show that the process

$$\left( t^2 B_t^{(1)} B_t^{(2)} - 2 \int_0^t s B_s^{(1)} B_s^{(2)} ds \right)_{t \geq 0}$$

is a martingale with respect to  $(\mathcal{F}_t)_{t \geq 0}$ .

**Problem 1.4.** (15 points) Let  $B$  be a Brownian motion. Show that the first order variation of  $B$  is  $V_T^1(B) = \infty$  almost surely for  $T \geq 0$ .

**Problem 1.5.** (10 points) Let  $B = (B_t^{(1)}, B_t^{(2)})_{t \geq 0}$  be a two dimensional Brownian motion. Define  $X_t = t B_t^{(1)}$  and  $Y = t^2 B_t^{(2)}$  for  $t \geq 0$ . Find the quadratic covariation  $\langle X, Y \rangle_T$ .

**Problem 1.6.** (15 points) Show that a right-continuous adapted process is progressively measurable.

**Problem 1.7.** Solve the following problems.

(i) (10 points) Show that the set

$$\Sigma := \{A \subseteq [0, \infty) \times \Omega : A \text{ is progressive}\}$$

is a  $\sigma$ -algebra.

(ii) (10 points) Let  $X$  be a stochastic process. Show that  $X$  is progressively measurable if and only if  $X$  is  $\Sigma$ -measurable.

## References

1.1. Since  $M$  is martingale w.r.t  $(F_t)_{t \geq 0}$ ,

$$E(M_t | F_s) = M_s \text{ for all } 0 \leq s < t$$

Take  $E(\cdot | g_s)$

$$E(E(M_t | F_s) | g_s) = E(M_s | g_s).$$

$$\Rightarrow E(M_t | g_s) = M_s \text{ for all } 0 \leq s < t.$$

( $\because g_s \subseteq F_s$ ,  $M$  is adapted to  $(g_t)_{t \geq 0}$ .)

This implies that  $M$  is martingale w.r.t  $(g_t)_{t \geq 0}$ .

1.2. (i).  $M_t = B_t^2 - t$

Claim 1.  $M_t$  is martingale w.r.t  $(F_t^B)_{t \geq 0}$

It is clear that  $M_t = B_t^2 - t$  is adapted to  $F_t^B$ .

For  $0 \leq s < t$ ,

$$\begin{aligned} E(M_t | F_s^B) &= E(B_t^2 - t | F_s^B) = E((B_t - B_s)^2 + 2B_t B_s - B_s^2 - t | F_s^B) \\ &= E((B_t - B_s)^2 | F_s^B) + 2E(B_t B_s | F_s^B) - E(B_s^2 | F_s^B) - t \\ &= E((B_t - B_s)^2) + 2E((B_t - B_s) B_s + B_s^2 | F_s^B) - B_s^2 - t \\ &\quad (\because B_t - B_s \perp F_s^B = \sigma(B_u | 0 \leq u \leq s), B_s^2 : F_s^B - \text{m'ble}). \\ &= \text{Var}(B_t - B_s) + 2E((B_t - B_s) B_s | F_s^B) + 2E(B_s^2 | F_s^B) - B_s^2 - t \\ &\quad (\because B_t - B_s \sim N(0, t-s)). \\ &= t-s + 2B_s E(B_t - B_s | F_s^B) + 2B_s^2 - B_s^2 - t \\ &= 2B_s E(B_t - B_s) + B_s^2 - s \\ &= B_s^2 - t. \quad (\because E(B_t - B_s) = 0) \end{aligned}$$

Claim 2.  $M$  is a martingale w.r.t  $(F_t^M)_{t \geq 0}$

Since  $M$  is a martingale w.r.t  $(F_t^B)_{t \geq 0}$ ,

$$E(M_t | F_s^B) = M_s \text{ for all } 0 \leq s < t$$

Since the natural filtration  $(F_t^M)_{t \geq 0}$  is a family of smallest  $\sigma$ -alg. which make  $M$  adapted,  $F_t^M \subseteq F_t^B$  for all  $t \in [0, \infty)$ .

Then take  $E(\cdot | F_s^M)$

$$E(ELM_t | \mathcal{F}_s^B) | \mathcal{F}_s^M = E(M_s | \mathcal{F}_s^M).$$

$$\Rightarrow ELM_t | \mathcal{F}_s^M = M_s \quad (\because \mathcal{F}_s^M \subseteq \mathcal{F}_s^B, M is adapted to (\mathcal{F}_t^M)_{t \geq 0}).$$

$\therefore M_t = B_t^3 - t$  is a martingale w.r.t  $(\mathcal{F}_t^M)_{t \geq 0}$ .  $\square$ .

1.2. (ii). By 1.2. (i) Claim 2, it suffices to show that w.r.t  $(\mathcal{F}_t^B)_{t \geq 0}$ .

$$M_t = B_t^3 - 3 \int_0^t B_u du$$

It is clear that  $B_t^3 - 3 \int_0^t B_u du$  is  $\mathcal{F}_t^B$ -m'ble.

For  $0 \leq s < t$ ,

$$E(B_t^3 - 3 \int_0^t B_u du | \mathcal{F}_s^B)$$

$$= E((B_t - B_s)^3 + 3B_t^2 B_s - 3B_t B_s^2 + B_s^3 | \mathcal{F}_s^B) - 3E(\int_0^s B_u du + \int_s^t B_u du | \mathcal{F}_s^B)$$

$$= E((B_t - B_s)^3) + E(3(B_t - B_s)^2 B_s + 6B_t B_s^2 - 3B_s^3 - 3B_t B_s^2 + B_s^3 | \mathcal{F}_s^B)$$

$$- 3 \int_0^s B_u du - 3E(\int_s^t B_u du)$$

$$(\because B_t - B_s \perp \mathcal{F}_s^B, \int_0^s B_u du : \mathcal{F}_s^B\text{-m'ble}, \int_s^t B_u du \perp \mathcal{F}_s^B)$$

$$= 3B_s E((B_t - B_s)^2) + E(3(B_t - B_s) B_s^2 | \mathcal{F}_s^B) + B_s^3 - 3 \int_0^s B_u du - 3 \int_s^t E(B_u) du$$

$$(\because E((B_t - B_s)^3) = 0, B_s^3 : \mathcal{F}_s^B\text{-m'ble}, E(\int_s^t B_u du) = \int_s^t E(B_u) du \text{ by Fubini's thm.})$$

$$= 3B_s(t-s) + B_s^3 - 3 \int_0^s B_u du$$

???

pf for  $E(L(B_t - B_s)^3)$

$$B := B_t - B_s \sim N(0, t-s)$$

$$\Rightarrow \phi_B(t) = E(e^{itB}) = e^{-\frac{1}{2}\sigma^2 t^2} \quad \text{where } \sigma^2 = t-s$$

$$\frac{d}{dt} \phi_B(t) = \phi'_B(t) = E(iB e^{itB}) = (-\sigma^2 t) e^{-\frac{1}{2}\sigma^2 t^2}$$

$$\phi''_B(t) = E(-B^2 e^{itB}) = [(-\sigma^2 t)^2 - \sigma^2] e^{-\frac{1}{2}\sigma^2 t^2}$$

$$\phi'''_B(t) = E(-iB^3 e^{itB}) = [(-\sigma^2 t)^3 - \sigma^2(-\sigma^2 t) - 2\sigma^2(-\sigma^2 t)] e^{-\frac{1}{2}\sigma^2 t^2}$$

$$\phi'''_B(0) = E(-iB^3) = 0. \Rightarrow E(B^3) = 0.$$

1.3. (i).  $B = (B_t^{(1)}, B_t^{(2)})_{t \geq 0}$ . : 2-dim'l B.M.

$$P(B_t^{(1)} > \sqrt{3} B_s^{(2)}) = P(B_t^{(1)} - \sqrt{3} B_s^{(2)} > 0).$$

Since  $B_t^{(1)} \sim N(0, t) \approx \sqrt{t} Z^{(1)}$ ,  $B_s^{(2)} \sim N(0, s) \approx \sqrt{s} Z^{(2)}$ ,

$$B_t^{(1)} \perp B_s^{(2)},$$

$$\Rightarrow P(\sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)} > 0).$$

$\sqrt{t} Z^{(1)} \perp \sqrt{3s} Z^{(2)} \Rightarrow \sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)}$  is normal.

$$E(\sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)}) = 0,$$

$$\text{Var}(\sqrt{t} Z^{(1)} - \sqrt{3s} Z^{(2)}) = \text{Var}(\sqrt{t} Z^{(1)}) + \text{Var}(\sqrt{3s} Z^{(2)}) = t + 3s.$$

$$\Rightarrow P(\sqrt{t+3s} Z > 0) = \frac{1}{2}.$$

$$1.3. (\text{ii}). E[e^{B_t^{(1)} + \int_0^s B_u^{(2)} du}] = E[e^{B_t^{(1)}}] \cdot E[e^{\int_0^s B_u^{(2)} du}] \quad (\because B_t^{(1)} \perp B_s^{(2)}).$$

Since  $B_t^{(1)} \sim N(0, t)$ ,  $\int_0^s B_u^{(2)} du$  : normal.

$$E(\int_0^s B_u^{(2)} du) = \int_0^s E(B_u^{(2)}) du = 0.$$

$$\begin{aligned} \text{Var}(\int_0^s B_u^{(2)} du) &= \int_0^s \int_0^s E(B_u^{(2)} B_v^{(2)}) du dv \\ &= \int_0^s \int_0^s \min(u, v) du dv = \frac{s^3}{2} \end{aligned}$$

$$E[e^{B_t^{(1)}}] \cdot E[e^{\int_0^s B_u^{(2)} du}] = e^{\frac{t}{2}} \cdot e^{\frac{s^3}{4}} = e^{\frac{t}{2} + \frac{s^3}{4}}$$

1.3. (iii). Let  $M_t = (B_t^{(1)}, B_t^{(2)})_{t \geq 0}$   $(F_t^B)_{t \geq 0}$ .

It is clear that  $B_t^{(1)} B_t^{(2)}$  is adapted to  $(F_t^B)_{t \geq 0}$ .

$$\begin{aligned} E(M_t | F_s^B) &= E(B_t^{(1)} B_t^{(2)} | F_s^B) \\ &= E((B_t^{(1)} - B_s^{(1)}) B_t^{(2)} | F_s^B) + E(B_s^{(1)} B_t^{(2)} | F_s^B) \\ &= E((B_t^{(1)} - B_s^{(1)})(B_t^{(2)} - B_s^{(2)}) | F_s^B) + E((B_t^{(1)} - B_s^{(1)}) B_s^{(2)} | F_s^B) + B_s^{(1)} E(B_t^{(2)} | F_s^B) \\ &= E((B_t^{(1)} - B_s^{(1)})(B_t^{(2)} - B_s^{(2)})) + B_s^{(2)} E(B_t^{(1)} - B_s^{(1)} | F_s^B) + B_s^{(1)} B_s^{(2)} \\ &= B_s^{(1)} B_s^{(2)} = M_s \quad \square. \end{aligned}$$

$$1.3. (IV). \left( t^2 B_t^{(1)} B_t^{(2)} - 2 \int_0^t u B_u^{(1)} B_u^{(2)} du \right) + 20.$$

$$E(t^2 B_t^{(1)} B_t^{(2)} | \mathcal{F}_s^B) = t^2 B_s^{(1)} B_s^{(2)} \text{ by 1.3. (iii).}$$

$$\begin{aligned} E(-2 \int_0^t u B_u^{(1)} B_u^{(2)} du | \mathcal{F}_s^B) &= E(-2 \int_0^s u B_u^{(1)} B_u^{(2)} du - 2 \int_s^t u B_u^{(1)} B_u^{(2)} du | \mathcal{F}_s^B) \\ &= -2 \int_0^s u B_u^{(1)} B_u^{(2)} du - 2 \int_s^t E(u B_u^{(1)} B_u^{(2)} du | \mathcal{F}_s^B) \\ &= -2 \int_0^s u B_u^{(1)} B_u^{(2)} du - 2(t-s) B_u^{(1)} B_u^{(2)} \end{aligned}$$