Topology I – Homework for Chapter 3, 4

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Problem 1. Determine, for each of the following topologies on $\mathbb{R}$, which of the others it contains.

$\mathcal{T}_1 =$ Standard topology
$\mathcal{T}_2 =$ Finite complement topology
$\mathcal{T}_3 =$ Topology generated by the following basis;

$$B_3 = \{(a,b) \subset \mathbb{R} | a, b \in \mathbb{R}\} \cup \{(c,d) - K \in \mathbb{R} | c, d \in \mathbb{R},\ K = \{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}\}$$

$\mathcal{T}_4 =$ Topology generated by the following basis;
$B_4 = \{(a,b] \subset \mathbb{R} | a, b \in \mathbb{R}\} =$ basis of upper limit topology
$\mathcal{T}_5 =$ Topology generated by the following basis;
$B_5 = \{(-\infty, a) \subset \mathbb{R} | a \in \mathbb{R}\}$

Proof. $(\mathcal{T}_2 \subset \mathcal{T}_1)$ Since for $(a,b) \in \mathcal{T}_1$, $\mathbb{R} \setminus (a,b) = (-\infty, a] \cup [b, +\infty)$ is not finite, $(a,b) \notin \mathcal{T}_2$. For $U \in \mathcal{T}_2$, it is either $U = \emptyset \in \mathcal{T}_1$ or $\mathbb{R} \setminus U$ is finite. So we can put $\mathbb{R} \setminus U = \{x_1, x_2, \cdots, x_n\}$. Then $U = (-\infty, x_1) \cup \bigcup_{i=1}^{n-1} (x_i, x_{i+1}) \cup (x_n, \infty) \in \mathcal{T}_1$. Since it is an union of open intervals, $\mathcal{T}_2 \subset \mathcal{T}_1$.

$(\mathcal{T}_1 \subset \mathcal{T}_3)$ For basis element $(a,b) \in \mathcal{T}_1$ and $x \in (a,b)$, it is also a basis of $\mathcal{T}_3$. However, given the basis element $B_3 = (c,d) - K \in \mathcal{T}_3$, there is no open interval that contains 0 and lies in B_3. Thus, $\mathcal{T}_3$ is strictly finer than $\mathcal{T}_1$.

$(\mathcal{T}_1 \subset \mathcal{T}_4)$ For $(a,b) \in \mathcal{T}_1$, $(a,b) = \bigcup (a,x]$ for $x \in (a,b)$. It is arbitrary union of basis elements of $\mathcal{T}_4$, it is open in $\mathcal{T}_4$. But, for any basis element $(a,b]$ of $\mathcal{T}_4$, there is no basis element $(c,d) \in \mathcal{T}_1$ such that contains b and contained in $(a,b]$.

$(\mathcal{T}_5 \subset \mathcal{T}_1)$ Since for $(-\infty, a) \in \mathcal{T}_5$, $(-\infty, a) = \bigcup_{n \in \mathbb{N}} (-n, a) \in \mathcal{T}_1$. However, for $(a,b) \in \mathcal{T}_1$ and $x \in (a,b)$, there is no basis element $(-\infty, a) \in \mathcal{T}_5$ such that $x \in (-\infty, a) \subset (a,b)$. Thus, $\mathcal{T}_5 \subset \mathcal{T}_1$.

$(\mathcal{T}_2 \subset \mathcal{T}_3)$ Since we have proven that $\mathcal{T}_2 \subset \mathcal{T}_1$ and $\mathcal{T}_1 \subset \mathcal{T}_3$, it follows that $\mathcal{T}_2 \subset \mathcal{T}_3$.

$(\mathcal{T}_2 \subset \mathcal{T}_4)$ We have proven that $\mathcal{T}_2 \subset \mathcal{T}_1$ and $\mathcal{T}_1 \subset \mathcal{T}_4$. Thus, $\mathcal{T}_2 \subset \mathcal{T}_4$.

$(\mathcal{T}_2 \not\subset \mathcal{T}_5$ **and** $\mathcal{T}_5 \not\subset \mathcal{T}_2)$ For $U = (-\infty, x) \cup (x, +\infty) \in \mathcal{T}_2$, there is no basis element $(-\infty, a) \in \mathcal{T}_5$ containing $x+1$. Thus, $\mathcal{T}_2 \not\subset \mathcal{T}_5$. For $(-\infty, a) \in \mathcal{T}_5$, and $a - 1 \in (-\infty, a)$, there is no basis element $U \in \mathcal{T}_2$ such that $a - 1 \in U \subset (-\infty, a)$. If there exists, $a - 1 \in U \neq \emptyset$ and $\mathbb{R} \setminus U$ is finite. However, from $U \subset (-\infty, a)$, $[a, +\infty) \subset \mathbb{R} \setminus U$, which is a contradiction to finite set. Thus $\mathcal{T}_5 \not\subset \mathcal{T}_2$.

$(\mathcal{T}_3 \subset \mathcal{T}_4)$ For $(a,b) \in \mathcal{T}_3$, we have that $(a,b) \in \mathcal{T}_1 \subset \mathcal{T}_4$. For $(a,b) \setminus K \in \mathcal{T}_3$ such that $(a,b) \cap K \neq \emptyset$, we have four cases. If $a < 0 < 1 < b$, then we have that $(a,b) \setminus K = (a, 0] \cup \bigcup_{n \in \mathbb{N}} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, b) \in \mathcal{T}_4$. If $a < 0 < b < 1$, then $(a,b) \setminus K = (a, 0] \cup \bigcup_{n=m}^{\infty} (\frac{1}{n+1}, \frac{1}{n}) \cup (\frac{1}{m}, b) \in \mathcal{T}_4$ for smallest number $m \in \mathbb{N}$ such that $\frac{1}{m} < b$. If $0 < a < 1 < b$, then $(a,b) \setminus K = (a, \frac{1}{k}) \bigcup_{n=1}^{k-1} (\frac{1}{n+1}, \frac{1}{n}) \cup (1, b) \in \mathcal{T}_4$ for largest number $k \in \mathbb{N}$ such that $a < \frac{1}{k}$. If $0 < a < b < 1$, then $(a,b) \setminus K = (a, \frac{1}{k}) \cup \bigcup_{n=m}^{k-1} (\frac{1}{n+1}, \frac{1}{n}) \cup (\frac{1}{m}, b) \in \mathcal{T}_4$ for smallest number $m \in \mathbb{N}$ and largest number $k \in \mathbb{N}$ such that $a < \frac{1}{k}$, $m \in \mathbb{N}$.

For $(a, 0] \in \mathcal{T}_4$, there is no basis element B of $\mathcal{T}_3$ such that $0 \in B \subset (a, 0]$ because every basis element B containg 0 contains some positive number. Thus $\mathcal{T}_3 \subset \mathcal{T}_4$.

($\mathcal{T}_5 \subset \mathcal{T}_3$) We have proven that $\mathcal{T}_5 \subset \mathcal{T}_1$ and $\mathcal{T}_1 \subset \mathcal{T}_3$. Thus, $\mathcal{T}_5 \subset \mathcal{T}_3$.

($\mathcal{T}_5 \subset \mathcal{T}_4$) We have proven that $\mathcal{T}_5 \subset \mathcal{T}_1$ and $\mathcal{T}_1 \subset \mathcal{T}_4$. Thus, $\mathcal{T}_5 \subset \mathcal{T}_4$.

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Problem 2. Let $D = \{(x, x) \in \mathbb{R}^2 | x \in \mathbb{R}\}$ be the diagonal line in $\mathbb{R}^2$. Describe the topology D inherits as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$ where $\mathbb{R}_l$ is the real line equipped with the lower limit topology.

Proof. The basis of $\mathbb{R}_l \times \mathbb{R}_l$ is composed of the subsets $[a, b) \times (c, d)$. For $(x, x) \in D$, the intersection of any $[x, b) \times (c, d)$ with $x \in (c, d)$ with D is a left-closed, right-open interval with x as left endpoint, and these intersections form a basis for D that is homeomorphic to $\mathbb{R}_l$. The basis of $\mathbb{R}_l \times \mathbb{R}_l$ is composed of the subsets $[a, b) \times [c, d)$. The same argument as in the previous case gives us that D is homeomorphic to $\mathbb{R}_l$. ∎

Problem 3. Show that X is Hausdorff if and only if $D = \{(x, x) \in X \times X | x \in X\}$ is closed in $X \times X$.

Proof. Assume the diagonal D is closed, and let $x \neq y$ be two distinct points in X. Then (x, y) belongs to the open set $X \times Y \setminus D$, hence there is a basis element $U \times V$ ($U, V \subset X$ open) such that $(x, y) \in U \times V \subset X \times X \setminus D$. Since $(x, y) \in U \times V$, U is a neighborhood of x and V is a neighborhood of y. Moreover, if U and V had non-empty intersection, then $z \in U \cap V$ would give $(z, z) \in U \times V \cap D$, contradiction; so $U \cap V = \emptyset$. This proves X is Hausdorff.
Conversely, if X is Hausdorff, and $(x, y) \in X \times X \setminus D$, then $x \neq y$ so there exists neighborhoods U of x and V of y such that $U \cap V = \emptyset$; then $U \times V$ is a neighborhood of (x, y) in the product topology, and $U \times V$ is disjoint from D. So $(x, y) \in U \times V \subset X \times X \setminus D$, which proves that $X \times X \setminus D$ is open, i.e. D is closed. ∎

Problem 4. Fine non-Hausdorff space with non-closed compact subset.

Proof. Let $(\mathbb{R}, \mathcal{T})$ be the real numbers with finite complement topology, namely a set is closed if and only if it is finite; and a set is open if and only if its complement is finite. Consider the natural numbers as a subset of the real line, this is an infinite set, but it is clear that its complement is not finite, so it is neither open nor closed. Suppose that $\{U_i | i \in I\}$ is an open cover of $\mathbb{N}$. There is some $i_0 \in I$ such that $0 \in U_{i_0}$, and since U_{i_0} is open it means that it contains everything except finitely many points, in particular it must contain all the natural numbers, except maybe finitely many of them. For every $n \in N \setminus U_i$ we can find some U_{i_n}. We found, therefore, a finite subcover of this open cover, and so $\mathbb{N}$ is compact. ∎

Problem 5. Prove that X is disconnected if and only if there exists a continuous function from X onto two point set with the discrete topology $f : X \to \{a, b\}$.

Proof. Without loss of generality, let two point set be $\{0, 1\}$. Let X be disconnected. Then there exists two non-empty disjoint open subset A and B of X such that $X = A \cup B$. Define a mapping f of X onto $\{0, 1\}$ by setting $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. Open sets of $\{0, 1\}$ on discrete topology are $\emptyset$, $\{0\}$, $\{1\}$ and $\{0, 1\}$. By the definition of f, $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$, $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\{0, 1\}) = X$. Thus, we have shown that the inverse image under f of every open subset of $\{0, 1\}$ is open in X and therefore f is continuous.
Conversely, if there exists such a mapping then X is disconnected because of continuity. For if X were connected, the $\{0, 1\}$ would be connected. But this is impossible since every discrete space is disconnected. ∎

Problem 6. Show that for a continuous function $f : S^1 \to \mathbb{R}$, there exists a point $x \in S^1$ with $f(x) = f(-x)$.

Proof. Assume $f : S^1 \to \mathbb{R}$ is continuous. Let $g : S^1 \to \mathbb{R}$ be the map defined by $g(x) = f(x) - f(-x)$, which is also continuous. If $g(x) = 0$ for all $x \in S^1$ then we are done; otherwise, there exists $x \in S^1$ such that $g(x) \neq 0$. Without loss of generality, we can assume that $g(x) > 0$; and then $g(-x) = f(-x) - f(x) = -g(x) < 0$. Since S^1 is connected and g is continuous, the intermediate value theorem implies the existence of $y \in S^1$ such that $g(y) = 0$, i.e. $f(y) = f(-y)$. ■

Proof (Alternative proof)**.** If $f(x) \neq f(-x)$ for all $x \in S^1$, then setting $U = \{x \in S^1 | f(x) < f(-x)\}$ and $V = \{x \in S^1 | f(x) > f(-x)\}$, then U and V are open (by continuity of f) and disjoint, and $S^1 = U \cup V$, so by connectedness one of U and V must be all of S^1 and the other must be empty. This is impossible since $x \in U \Leftrightarrow -x \in V$. ■