Midterm Exam (Take Home)

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Problem 1

In our course, we defined that a subset E of $\mathbb{R}^d$ is (Lebesgue) measurable if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O} - E) \leq \epsilon$.

(i) Show that E is measurable if and only if for any $\epsilon > 0$ there is a closed set F with $F \subset E$ and $m_*(E - F) \leq \epsilon$.

(ii) Describe the relationship between the original definition of the measurability and the exterior measure. By using this, discuss what would be the natural definition of the interior measure. Explain also the relationship between the measurability condition in (i) and the interior measure.

(iii) A set E in $\mathbb{R}^d$ is Carathéodory measurable if

$$m_*(A) = m_*(E \cap A) + m_*(E^c \cap A) \quad \text{for every } A \subset \mathbb{R}^d.$$

Prove that this notion is equivalent to the original definition of the measurability. What is its advantage and disadvantage?

Proof. (i) Suppose that E is measurable. Since the complement of a measurable set is measurable, E^c is measurable. Then for any $\epsilon > 0$, there is a open set $\mathcal{O}_\epsilon$ with $E^c \subset \mathcal{O}_\epsilon$ and $m_*(\mathcal{O}_\epsilon - E^c) \leq \epsilon$. The complement F_ϵ of $\mathcal{O}_\epsilon$ is closed. Now we can observe that $F_\epsilon \subset E$ and $\mathcal{O}_\epsilon - E^c = E - F_\epsilon$. Thus $m_*(\mathcal{O}_\epsilon - E^c) = m_*(E - F_\epsilon) \leq \epsilon$ for any $\epsilon > 0$. Conversely, let F_ϵ be closed set with $F \subset E$ and $m_*(E - F_\epsilon) \leq \epsilon$ for any $\epsilon > 0$. The complement $\mathcal{O}_\epsilon$ of F_ϵ is open. We can observe that $E^c \subset \mathcal{O}_\epsilon$ and $E - F_\epsilon = \mathcal{O}_\epsilon - E^c$. Then $m_*(E - F_\epsilon) = m_*(\mathcal{O}_\epsilon - E^c) \leq \epsilon$. This implies that E^c is measurable. Thus E is measurable.

(ii) The definition of exterior measure is as following.

Definition (Exterior measure). *If E is any subset of $\mathbb{R}^d$, the exterior measure of E is $m_*(E) = \inf \sum_{j=1}^\infty |Q_j|$, where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^\infty Q_j$ by closed cubes.*

The exterior measure can't guarantee that $m_*(E) = m_*(E_1) + m_*(E_2)$ for $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$. It is also said that an exterior measure cannot have countable additivity property. A family of Lebesgue measurable sets, however, always holds above identity. In other words, exterior measures can be defined for all subset of $\mathbb{R}^d$, but not Lebesgue measures. We say that E is Lebesgue measurable when there is an open set that makes the difference in exterior measure smaller than $\epsilon > 0$. Similarly from this perspective, the interior measure can be defined naturally as following.

Definition (Interior measure). *If E is any subset of $\mathbb{R}^d$, the interior measure of E is $m_*(E) = \sup \sum_{j=1}^\infty |Q_j|$, where the supremum is taken over all countable union $\bigcup_{j=1}^\infty Q_j \subset E$ by closed cubes.*

A Lebesgue measure can be derived from this definition. E is measurable if there is a closed $F \subset E$ makes the difference in measure less than ϵ. In other words, it is possible to derive the condition 1.1 which equivalent condition with original definition of measurability from the interior measure.

(iii) **Carathéodory $\Rightarrow$ Lebesgue** By definition of m_*, for any $\epsilon > 0$, there exists an open set $\mathcal{O}_\epsilon \supset E$ such that $m_*(\mathcal{O}_\epsilon) \le m_*(E) + \epsilon$. Since E is Carathéodory measurable, for any ϵ, $m_*(\mathcal{O}_\epsilon) = m_*(E \cap \mathcal{O}_\epsilon) + m_*(E^c \cap \mathcal{O}_\epsilon) = m_*(E) + m_*(\mathcal{O}_\epsilon - E) \le m_*(E) + \epsilon$. Thus we get $m_*(\mathcal{O}_\epsilon - E) \le \epsilon$, and this implies E is measurable.

Lebesgue $\Rightarrow$ Carathéodory We want to show that for measurable set E and all $A \in \mathbb{R}^d$, $m_*(A) = m_*(A \cap E) + m_*(A \cap E^c)$. By countable sub-additivity of m_*, it is clear that $m_*(A) \le m_*(A \cap E) + m_*(A \cap E^c)$. So, if we show the reverse inequality, we are done. Consider countable covering $\bigcup_{j=1}^\infty Q_j \supset A$ such that $\sum_{j=1}^\infty m_*(Q_j) < m_*(A) + \epsilon$ where Q_j is measurable. Since countable union of measurable sets is measurable, $Q_j \cup E$ and $Q_j \cup E^c$ are disjoint measurable sets. Then,

$$\begin{aligned}
m_*(A) + \epsilon > \sum_{j=1}^\infty m_*(Q_j) &= \sum_{j=1}^\infty m_*((Q_j \cap E) \cup (Q_j \cap E^c)) \\
&= \sum_{j=1}^\infty m_*(Q_j \cap E) + m_*(Q_j \cap E^c) \qquad (1.1) \\
&= \sum_{j=1}^\infty m_*(Q_j \cap E) + \sum_{j=1}^\infty m_*(Q_j \cap E^c) \\
&\ge m_*((\bigcup_{j=1}^\infty Q_j) \cap E) + m_*((\bigcup_{j=1}^\infty Q_j) \cap E^c) \qquad (1.2) \\
&\ge m_*(A \cap E) + m_*(A \cap E^c) \qquad (1.3)
\end{aligned}$$

while (**??**) is due to below theorem:

Theorem. *If $E_1, E_2, \cdots$ are disjoint measurable sets, and $E = \bigcup_{j=1}^\infty E_j$, then $m(E) = \sum_{j=1}^\infty m(E_j)$.*

(**??**) and (**??**) are due to countable sub-additivity and monotonicity, respectively. Take $\epsilon \to 0$ to get $m_*(A) \ge m_*(A \cap E) + m_*(A \cap E^c)$.

If E satisfies the Carathéodory condition, E can be used as a powerful tool to divide arbitrary set A into two sets. The obvious reason is that "divide, measure each parts, and add up" is so essential to make a measure useful. If E satisfies the Carathéodory condition, all sets are neatly divided into two, whereas if it does not hold, some sets will be divided into two odd-shaped subsets.

□

Problem 2

(i) Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of complex-valued functions on $\mathbb{R}^d$ such that $f_n \to f$ a.e. as $n \to \infty$. Also, assume that

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} |f_n| \le C$$

for a constant $C > 0$. Derive

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} (|f_n| - |f_n - f| - |f|) = 0,$$

and interpret it in terms of Fatou's lemma.

(ii) By inspecting the proof of the dominated convergence theorem, prove that if

$$\begin{cases} g_n, h_n, g, h \in L^1(\mathbb{R}^d), \\ g_n \to g \text{ a.e.} \quad \text{as } n \to \infty, \\ h_n \to h \text{ a.e.} \quad \text{as } n \to \infty, \\ |g_n| \le h_n \quad \text{for all } n \in \mathbb{N}, \\ \int_{\mathbb{R}^d} h_n \to \int_{\mathbb{R}^d} h \quad \text{as } n \to \infty, \end{cases}$$

then

$$\int_{\mathbb{R}^d} g_n \to \int_{\mathbb{R}^d} g \quad \text{as } n \to \infty.$$

(iii) Applying (i) and (ii), deduce that if $F_n, F \in L^1(\mathbb{R}^d)$ and $F_n \to F$ a.e. as $n \to \infty$, then

$$\int_{\mathbb{R}^d} |F_n - F| \to 0 \quad \text{if and only if} \quad \int_{\mathbb{R}^d} |F_n| \to \int_{\mathbb{R}^d} |F|$$

as $n \to \infty$.

Proof. (i) First to show that $\int_{\mathbb{R}^d} |f| \le C$, i.e., $f \in L^1$, we use Fatou's lemma.

Lemma (Fatou)**.** *Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then $\int f \le \liminf_{n\to\infty} \int f_n$.*

Since $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |f_n| \le C$, by Fatou's lemma,

$$\int_{\mathbb{R}^d} |f| = \int_{\mathbb{R}^d} \lim_{n\to\infty} |f_n| \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} |f_n| \le C.$$

By triangle inequality,

$$|f_n| = |f + f_n - f| \le |f| + |f_n - f|.$$

Let $g_n = |f_n| - |f_n - f| - |f| \le 0$. Then,

$$|g_n| = |f| + |f_n - f| - |f_n| \le |f| + |f_n| + |f| - |f_n| = 2|f|.$$

Thus $|g_n| \in L^1$, since $|f| \in L^1$. We are now ready to use the dominated convergence theorem.

Theorem (Lebesgue's dominated convergence theorem (LDCT))**.** *Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x, as n tends to infinity. If $|f_n(x)| \le g(x)$, where g is integrable, then $\int |f_n - f| \to 0$ as $n \to \infty$, and consequently $\int f_n \to \int f$ as $n \to \infty$.*

By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g_n = \int_{\mathbb{R}^d} \lim_{n\to\infty} g_n = \int_{\mathbb{R}^d} 0 = 0.$$

(ii) Since $|g_n| \le h_n$, $h_n + g_n$ is non-negative measurable function. Then, by Fatou's lemma,

$$\begin{aligned} \int h + \int g = \int (h+g) &= \int \lim_{n\to\infty} (h_n + g_n) \\ &\le \liminf_{n\to\infty} \int (h_n + g_n) = \liminf_{n\to\infty} (\int h_n + \int g_n) \\ &= \liminf_{n\to\infty} \int h_n + \liminf_{n\to\infty} \int g_n = \int h + \liminf_{n\to\infty} \int g_n. \end{aligned}$$

Thus, $\int g \le \liminf_{n\to\infty} \int g_n$. Since $h_n - g_n$ is also non-negative, similarly,

$$\begin{aligned}\int h - \int g = \int (h-g) &= \int \lim_{n\to\infty}(h_n - g_n)\\ &\le \liminf_{n\to\infty} \int (h_n - g_n) = \liminf_{n\to\infty}\left(\int h_n - \int g_n\right)\\ &= \liminf_{n\to\infty}\int h_n + \liminf_{n\to\infty}\int -g_n = \int h + \liminf_{n\to\infty}\int -g_n\\ &= \int h - \limsup_{n\to\infty}\int g_n.\end{aligned}$$

Thus, $\limsup_{n\to\infty}\int g_n \le \int g$. Hence,

$$\liminf_{n\to\infty}\int g_n \le \limsup_{n\to\infty}\int g_n \le \int g \le \liminf_{n\to\infty}\int g_n \le \limsup_{n\to\infty}\int g_n.$$

This means all of them have same value. Thus,

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} g_n = \int_{\mathbb{R}^d} g.$$

(iii) Let $G_n = |F_n - F| + |F|$. Then, $|F_n| \le G_n$.

($\Rightarrow$) Since $G_n = |F_n - F| + |F|$, $\int G_n \to \int |F|$ is clear. By (ii), letting $g_n = |F_n|$, $h_n = G_n$, $g = |F|$, $h = |F|$, then

$$\int_{\mathbb{R}^d} |F_n| \to \int_{\mathbb{R}^d} |F|.$$

($\Leftarrow$) There exists $N \in \mathbb{N}$ such that $\forall n \ge N$,

$$\int_{\mathbb{R}^d} |F_n| < \int_{\mathbb{R}^d} |F| + 1.$$

Let

$$C = \max\left\{\int_{\mathbb{R}^d} |F| + 1, \int_{\mathbb{R}^d} |F_1|, \cdots, \int_{\mathbb{R}^d} |F_{N-1}|\right\}.$$

Then $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |F_n| \le C$, and by using (i),

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} (|F_n| - |F_n - F| - |F|) = 0.$$

Since $F_n \to F \Rightarrow |F_n| \to |F|$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} |F_n - F| = 0.$$

$\square$

Problem 3

(i) Let f be any non-negative measurable function in $\mathbb{R}^d$. Employing Fubini's theorem, derive

$$\int_{\mathbb{R}^d} f(x)^p \,\mathrm{d}x = p\int_0^\infty t^{p-1} m(\{x \in \mathbb{R}^d : f(x) > t\})\,\mathrm{d}t$$

for any $p > 1$. Describe its relationship with Cavalieri's principle.

(ii) Let T be an element of the general linear group $GL(d, \mathbb{R})$ of degree d. Check that if f is a measurable function in $\mathbb{R}^d$, so is $f \circ T$. Also, prove that if $f \in L^1(\mathbb{R}^d)$, then $f \circ T \in L^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f = |\det T| \int_{\mathbb{R}^d} f \circ T. \tag{3.1}$$

(iii) A map $G : \mathbb{R}^d \to \mathbb{R}^d$ is called a $C^1(\mathbb{R}^d)$-diffeomorphism if G is injective and its derivative $DG(x)$ is invertible for all $x \in \mathbb{R}^d$. Discuss how the identity in (ii) becomes if T is replaced with a $C^1(\mathbb{R}^d)$-diffeomorphism G and justify why your answer should holds.

Proof. (i)

$$\begin{aligned}\int_0^\infty pt^{p-1} m(\{x \in \mathbb{R}^d \colon f(x) > t\})\,\mathrm{d}t &= \int_0^\infty \int_{\mathbb{R}^d} pt^{p-1} \mathbb{I}_{\{f(x)>t\}}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}^d} \int_0^{f(x)} pt^{p-1}\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)^p\,\mathrm{d}x\end{aligned}$$

where the second equality is due to Fubini's theorem.

Theorem (Fubini's theorem). *Suppose $f(x,y)$ is integrable on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:*

(i) The slice f^y is integrable on $\mathbb{R}^{d_1}$.

(ii) The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)\,\mathrm{d}x$ is integrable on $\mathbb{R}^{d_2}$.

(iii) $\int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f(x,y)\,\mathrm{d}x)\,\mathrm{d}y = \int_{\mathbb{R}^d} f$.

Principle (Cavalieri's principle (2-dimensional case)). *Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.*

(ii) Since linear map on finite dimensional space is continuous, T^{-1} is continuous mapping. By below property, T^{-1} is measurable.

Property. *If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and Φ is continuous, then $\Phi \circ f$ is measurable.*

For $E \in \mathcal{B}_{\mathbb{R}^d}$, $(f \circ T)^{-1}(E) = T^{-1}(f^{-1}(E)) \in L(\mathbb{R}^n)$. Thus $f \circ T$ is Lebesgue measurable. If $|f|$ is measurable, $|f \circ T|$ is measurable. If (**??**) is true for $T, S \in GL(d, \mathbb{R})$, it is true for $T \circ S$, since

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= |\det T| \int f \circ T(x)\,\mathrm{d}x = |\det T||\det S| \int f \circ T \circ S(x)\,\mathrm{d}x \\ &= |\det(T \circ S)| \int f \circ T \circ S(x)\,\mathrm{d}x.\end{aligned}$$

Every $T \in GL(d, \mathbb{R})$ can be written as the product of finitely many transformations of the types T_1, T_2, and T_3 given below. Hence it suffices to prove (**??**) when T is of the types T_1, T_2, and T_3.

$$\begin{aligned}T_1(x_1, x_2, \cdots, x_n) &= (x_1, \cdots, cx_j, \cdots, x_n) \\ T_2(x_1, x_2, \cdots, x_n) &= (x_1, \cdots, x_j + cx_k, \cdots, x_n) \\ T_3(x_1, x_2, \cdots, x_j, \cdots, x_k, \cdots, x_n) &= (x_1, x_2, \cdots, x_k, \cdots, x_j, \cdots, x_n)\end{aligned}$$

This is, however, simple consequence of the Fubini-Tonelli theorem.

(iii) It is clear by following theorem.

Theorem. *Let $G \colon \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ be C^1-diffeomorphism, $G = (g_1, g_2, \cdots, g_n)$. Then,*

(i) If f is Lebesgue measurable on $G(\Omega)$, $f \circ G$ is Lebesgue measurable on Ω.

(ii) If $f \geq 0$ or $f \in L^1(G(\Omega), m)$, then $\int_{G(\Omega)} f(x)\,\mathrm{d}x = \int_\Omega f \circ G(x) |\det DxG|\,\mathrm{d}x$.

□