Lecture 2. The Basic Theory of Interest

1. Principal and Interest

1.1. Simple and Compound Interests 단리, 복리

If you invest the *principal* of $A in a bank account paying interest at a rate of r per year, the amount A will grow to $(1+r)A$ one year from today. *Simple interest* is the interest you earned each period on the principal.

원금에 붙은 이자

	0	1	Future Values
Principal	A		A
Simple interest		rA	rA
Total			$(1+r)A$

이자에 관한 이자 (2년차부터 발생).

In year 2, the amount A invested today will grow to $(1+r)^2A$. The *compounding* effect refers to the effect of *interest on interest*.

t	0	1	2	Future Values
Principal	A			A
Simple interest for the 1st year		rA		rA
Simple interest for the 2nd year			rA	rA
Interest on interest			r^2A	r^2A
Total				$(1+2r+r^2)A=(1+r)^2A$

- In general, the amount A will grow to $(1+r)^TA$ in year T:

$$FV_T=(1+r)^TA.$$

1.2. Compounding at Various Intervals

With more than one compounding period per year, as most banks do currently, it is traditional to quote the interest rate on a yearly basis, but apply the appropriate proportion of that interest rate over each compounding period.

- Specifically, the future value formula is

$$FV_T=A\left(1+\frac{r}{m}\right)^{mT},$$

where r is the quoted (or nominal) annual interest rate, m is the number of compounding periods per year, and T is the number of years.

Example 1. Quarterly compounding at an interest rate of r per year

0	1Q	2Q	3Q	1
A	$[1+(r/4)]A$	$[1+(r/4)]^2A$	$[1+(r/4)]^3A$	$[1+(r/4)]^4A$

Remark 2. An *effective interest rate*, denoted by r^*, is defined as the equivalent yearly interest rate that would produce the same result after 1 year without compounding. Put differently, the effective interest rate is the number r^* satisfying

$$1+r^* = \left(1+\frac{r}{m}\right)^m.$$

1.3. Continuous Compounding

If the number of compounding periods per year becomes infinite, then the interest is said to compound *continuously*. Specifically, the future value is defined by

$$FV_T = \lim_{m\to\infty} A\left(1+\frac{r}{m}\right)^{mT}.$$

- Using the mathematical definition of e, i.e.,

$$e \equiv \lim_{w\to\infty}\left(1+\frac{1}{w}\right)^w,$$

  one shows that

$$\begin{aligned} FV_T &= A\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mT} \\ &= A\left[\lim_{w\to\infty}\left(1+\frac{1}{w}\right)^w\right]^{rT} \\ &= Ae^{rT}. \end{aligned}$$

1.4. Present Value

Present value (PV) refers to the value that should be assigned today to future money that is to be received at a later time.

Example 3. Consider two options such as (1) you will receive \$110 in 1 year and (2) you receive \$100 now and deposit it in a bank account for 1 year at 10% interest. The two options are identical after 1 year since you will receive \$110 in either case. This equivalence can be restated by saying that \$110 received in 1 year is equivalent to the receipt of \$100 today when the interest rate is 10%. In other words, *\$110 received in 1 year has a present value of \$100 today when the interest is 10%.*

- Given FV_T, r, and T, the formula for present value is

$$PV = \frac{FV_T}{(1+r)^T}.$$

  The term $1/(1+r)^T$ is called the *discount factor* by which the future value, FV_T, must be discounted to obtain the present value.

Remark 4. In the case of multiple compounding, the present value formula is

$$PV = \frac{FV_T}{\left(1+\frac{r}{m}\right)^{mT}}.$$

2. Present and Future Values of Cash Flow Streams

Consider a cash flow stream $(x_0, x_1, \ldots, x_T)$ and the interest rate r each period. The future value of the stream is computed as

$$FV_T = x_0(1+r)^T + x_1(1+r)^{T-1} + \cdots + x_T \tag{1}$$

and the present value is given by

$$PV = x_0 + \frac{x_1}{(1+r)} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_T}{(1+r)^T}. \tag{2}$$

- From (1) and (2), the present value and the future value are related by

$$PV = \frac{FV_T}{(1+r)^T}$$

  for cash flow streams.

Theorem 5. *The cash flow streams $x = (x_0, x_1, \ldots, x_T)$ and $y = (y_0, y_1, \ldots, y_T)$ are equivalent if and only if the present values of two streams, evaluated at the given interest rate of r, are equal.*

Proof. Let PV_x and PV_y be the present values of x and y, respectively. We have

$$\begin{aligned} x &\Leftrightarrow (PV_x, 0, \ldots, 0) \\ y &\Leftrightarrow (PV_y, 0, \ldots, 0). \end{aligned}$$

Thus, x and y are equivalent if and only if $PV_x = PV_y$. □

- The theorem implies that present value is the "only" number needed to characterize a cash flow stream, so that you can evaluate alternative cash flows simply by using their present values.

3. Internal Rate of Return

IRR을 구해서 r와 비교.
NPV : 주어진 R.

Definition 6. Let $(x_0, x_1, \ldots, x_T)$ be a cash flow stream. Then, the *internal rate of return* (IRR) of this stream is a number r satisfying the equation

$$0 = x_0 + \frac{x_1}{(1+r)} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_T}{(1+r)^T}.$$

find r : numerical analysis.

- Since it is a polynomial equation of degree T, the equation may have T solutions (IRRs) at maximum, some or all of which can be complex numbers.

IRR 문제점 1) 허근 2) 음수 3) 중근(multiple) ⇒ 의사결정을 못함.

Theorem 7. *Suppose that the cash flow stream $(x_0, x_1, \ldots, x_T)$ has $x_0 < 0$ and $x_k \geq 0$ for all k, $k = 1, 2, \ldots, T$, with at least one term being strictly positive. Then, there is a unique positive root to the equation*

$$0 = x_0 + cx_1 + c^2 x_2 + \cdots + c^T x_T.$$

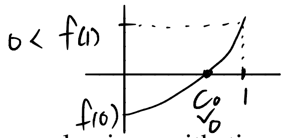
Furthermore, if $\sum_{k=0}^{T} x_k > 0$ *(meaning that the total amount returned exceeds the initial investment), then the corresponding internal rate of return*

$$r = \frac{1}{c} - 1$$

is positive.

조건 1) $x_0 < 0$ (초기 투자).
2) $x_k \geq 0$ $(k \geq 1)$
3) at least 1 term > 0.
4) $-x_0 < x_1 + \cdots + x_k$.

Proof. Write $f(c) = x_0 + cx_1 + c^2x_2 + \cdots + c^T x_T$. Since $x_0 < 0$, $f(0) < 0$. It shows that $f^{'}(c) = x_1 + 2cx_2 + \cdots + Tc^{T-1}x_T > 0$ for $c > 0$, since at least one of the cash flow terms is strictly positive. Similarly, $f^{''}(c) \geq 0$ for $c > 0$. Since the continuous $f(c)$ is strictly increasing for $c > 0$ and $f(0) < 0$, there exists a unique positive value of c_0 satisfying $f(c_0) = 0$. If $\sum_{k=0}^{T} x_k > 0$, then $f(1) > 0$, thereby meaning that the solution c_0 satisfying $f(c_0) = 0$ must be less than one. Therefore, $r_0 = 1/c_0 - 1 > 0$. □



4. Inflation and Real Interest Rate

Inflation means an increase in general prices with time, thus lowering the purchasing power of money. Denote an inflation rate by $f(> 0)$ and a current price by p_0. By definition, the price 1 year from now, p_1, will be equal to $(1 + f)p_0$. Since p_1 is greater than p_0, the value of money is decreasing. Put differently, the purchasing power of a dollar next year in terms of the purchasing power of today's dollar (i.e., p_0/p_1) is equal to $1/(1 + f)$.

Definition 8. *Real dollars* are defined relative to a given reference year, and represent the hypothetical dollars that continue to have the same purchasing power as dollars did in the reference year.

- The real dollars are in contrast to the *nominal* dollars that we really use in transactions, in the sense that they eliminate the influence of inflation.

Definition 9. The *real interest rate* r_0 is defined as the rate at which real dollars increase if left in a bank that pays the nominal rate r. Put differently, the real interest rate is the number r_0 satisfying

$$1 + r_0 = \frac{1 + r}{1 + f}. \tag{3}$$

- The equation (3) expresses that money in bank increases by $(1 + r)$, but its purchasing power is deflated by $1/(1 + f)$.

- Solving (3) for r_0, one has

$$r_0 = \frac{r - f}{1 + f}.$$

  For small values of f, the real interest rate is approximately equal to the nominal interest rate minus the inflation rate:

$$r_0 \approx r - f.$$