Economics of Information and Uncertainty

Based on lectures by In-Koo Cho in spring 2021
Notes taken by Junwoo Yang, proofread by In-Koo Cho

Contents

Chapter 1

Introduction

1.1 Expected utility theory

Why needed?

A decision is often made under uncertainty.

- Objective uncertainty: The value of a decision may depend upon the contingency, which is not observable at the time of decision.
- Subjective uncertainty: The value of a decision may depend up on the decisions of the other players, which are not observable at the time of decision.

We need a formal theory to evaluate a choice whose value is not a deterministic value, but a probability distribution over values.

History

John von Neumann and Oskar Morgenstern developed the theory of games. They immediately recognized that a decision maker faces uncertainty. A decision maker does not know the actual value from his decision, but realizes the value only after he made the decision. The value of a decision is more like a probability distribution than a number. In order to model his decision problem, we need a formal way to evaluate a probability distribution.

They developed the expected utility theory, as a way to investigate the interactive decision problem. The expected utility theory appears in the appendix rather than in the main text of their classic book:

Reference

This lecture is drawn from

1.2 Description

Let Z be a finite set of outcomes, or attributes, from which a decision maker generates utility. Let

$$p : Z \to [0, 1]$$

be the probability distribution over Z. That is,

$$p(z) \geq 0 \quad \forall z,$$

and

$$\sum_{z \in Z} p(z) = 1.$$

Lottery

We use lotteries to describe risky alternatives. Suppose first that the number of possible outcomes is finite. Fix a set of outcomes $C = \{c_1, \cdots, c_N\}$. Let p_n be the probability that outcome $c_n \in C$ occurs and suppose these probabilities are objectively known.

> **Definition 1** (Lottery). A (simple) lottery $L = (p_1, \cdots, p_n)$ is an assignment of probabilities to each outcome c_n, where $p_n \geq 0$ for all n and $\sum_n p_n = 1$.

The collection of such lotteries can be written as

$$\mathcal{L} = \left\{ (p_1, \ldots, p_N) \,\middle|\, \sum_{n=1}^{N} p_n = 1,\ p_n \geq 0 \text{ for } n = 1, \ldots, N \right\}.$$

We can also think of a **compound lottery** $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$, which allows the outcomes of a lottery to be lotteries.

It is immediate to see that any compound lottery can be reduced to a simple lottery defined as above.

> **Example.** $C = \{c_1, c_2\}$, $L_1 = (p, 1-p)$, $L_2 = (q, 1-q)$. Then,
>
> $$(L_1, L_2; \alpha, 1-\alpha) = (\alpha p + (1-\alpha) q, \alpha(1-p) + (1-\alpha)(1-q)).$$

Hence, we can only focus on simple lotteries. One special and important class of lotteries is money lotteries, whose outcomes are real numbers, i.e., $C = \mathbb{R}$. A money lottery can be characterized by a cumulative distribution function F, where $F : \mathbb{R} \to [0, 1]$ is nondecreasing. $F(x)$ is the probability of receiving a prize less than or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

Expected utility

If an individual has reasonable preferences about consumption in different circumstances, we will be able to use a utility function to describe these preferences just as we do in other contexts. However, the fact that we are considering choice under uncertainty adds some special structures to the choice problem, which we will see below. Historically, the study of individual behavior under uncertainty is originated from attempts to understand (and hopefully to win) games of chance. One may think that the key determinant of behavior under uncertainty is the expected return of the gamble. However, people are generally reluctant to play fair games.

Example (St. Petersburg Paradox)**.** Consider the following gamble: you toss a coin repeatedly until the head comes up. If this happens in the nth toss, the gamble gives a monetary payoff of 2^n. What is the expected return of this game? How much would you pay to play this gamble?

Let $\succeq$ be an ordering over P, which represents the decision maker's preference over lotteries. If $p \succeq q$, then we say that p is referred to q. The only difference from the conventional consumer theory is that p and q are probabilities, rather than attributes (or goods) which the decision maker draw utility.

Definition 2. $\succeq$ is complete if $\forall p, q \in P$, $p \succeq q$ or $q \succeq p$. $\succeq$ is transitive if $\forall p, q, r \in P$, $p \succeq q$ and $q \succeq r$ imply $p \succeq r$. We say that $\succeq$ is a preference ordering if $\succeq$ is complete and transitive.

Axiom 1. $\succeq$ is a preference ordering over P.

This axiom is hardly controversial, although experimental evidence shows that the ordering of a human being is often not complete or not transitive. Throughout this class, we maintain the assumption that $\succeq$ is complete and transitive.

Definition 3. $\forall p, q \in P$, $\forall a \in [0, 1]$, a composite lottery is

$$ap + (1 - a)q$$

If one interpret $a \in [0, 1]$ as a probability, one can interpret a composite lottery as a lottery over lotteries. One can interpret a as the amount of lottery a in the portfolio. A stock is a lottery, because the value of a stock depends upon the profitability and the market condition, but the decision maker does not observe the true state when he purchases a stock. A mutual fund is a composite lottery.

An important observation is that P is a convex set. Therefore, a composite lottery is an element of P.

The second axiom is called the substitution axiom, the independence axiom or the linearity axiom.

Axiom 2. $\forall p, q, r \in P$, $\forall a \in (0, 1]$, if $p \succeq q$, then

$$ap + (1 - a)r \succeq aq + (1 - a)r.$$

The preference between two composite lotteries is determined by the preference between p and q, independently of r. In that sense, this axiom is called the independence axiom.

Note that if $p \succeq q$, then the preference between the two composite lotteries is independent of the size of a. This is the crucial feature of linear preferences, which this axiom implies.

As important as this assumption is for the expected utility theory, the linearity of the preference has been challenged by many experiments. In reponse, many alternative axioms were proposed. Still, the linearity of the expected utility allows us to use the mathematical expectation to formulate the optimization problem. For this reason, this axiom endures the challenges.

Axiom 3. $\forall p, q, r \in P$, if $p \succeq q \succeq r$, then $\exists a, b \in (0,1)$ such that

$$ap + (1-a)r \succeq q \succeq bp + (1-b)r.$$

This axiom is called the continuity axiom or Archimedian axiom. A key implication is that the utility must be finite. Suppose that the utility from p is infinite. Then, it would be impossible to find $b \in (0,1)$ to construct a composite lottery so that

$$q \succeq pb + (1-b)r.$$

Similarly, if you assign $-\infty$ utility to lottery p (such as death with probability 1), then it would be impossible to construct a composite lottery in which the proportion of p is $a \in (0,1)$ such that

$$ap + (1-a)r \succeq q.$$

The fundamental theorem by von Neumann and Morgenstern is that we can represent any preference satisfying three axioms by the expected value of a utility.

Theorem 1. $\succeq$ satisfies three axiom if and only if there exists a utility function

$$u : Z \to \mathbb{R}$$

such that $\forall p, q \in P$,

$$p \succeq q$$

if and only if

$$\sum_{z \in Z} u(z)p(z) > \sum_{z \in Z} u(z)q(z).$$

Moreover, if u represents $\succeq$, then v represents $\succeq$ if and only if $\exists c > 0$, $\exists d \in \mathbb{R}$ such that

$$v(z) = cu(z) + d.$$

The first

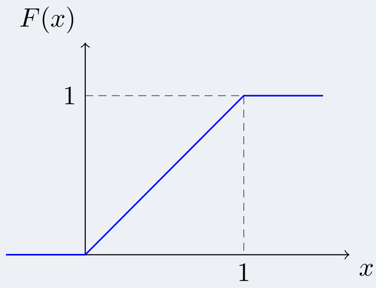
1.3 Risk Aversion

In many economic environments, individuals display aversion to risk. We formalize the notion of risk aversion and study some of its properties. We focus on money lotteries, i.e., risky alternatives whose outcomes are amounts of money. It is convenient to treat money as a continuous variable. We have so far assumed a finite number of outcomes to derive the expected utility representation. How to extend this?

1.3.1 Expected utility framework on monetary outcomes

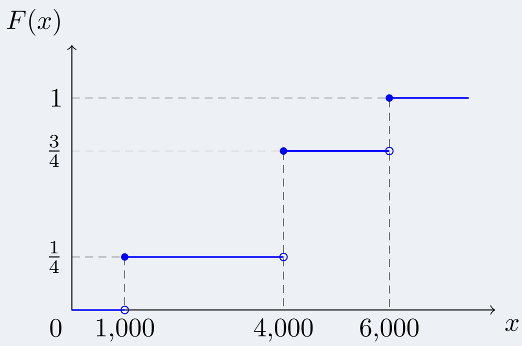
We describe a monetary lottery by means of a cumulative distribution functions $F : \mathbb{R} \to [0,1]$. $F(x)$ is the probability that the realized payoff is less that or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

Example. Uniform distribution $U[0,1]$



Example. Discrete distribution:

$$\left.\begin{array}{l}\text{Prob}(1{,}000 \text{ won}) = \frac{1}{4} \\ \text{Prob}(4{,}000 \text{ won}) = \frac{1}{2} \\ \text{Prob}(6{,}000 \text{ won}) = \frac{1}{4}\end{array}\right\} \to F(x) = \begin{cases} 0 & \text{if } x < 1{,}000 \\ \frac{1}{4} & \text{if } 1{,}000 \leq x < 4{,}000 \\ \frac{3}{4} & \text{if } 4{,}000 \leq x < 6{,}000 \\ 1 & \text{if } x \geq 6{,}000. \end{cases}$$



Consider a preference relation $\succsim$ on $\mathcal{L}$. It has an expected utility representation if

$$F \succsim F' \Leftrightarrow U(F) \geq U(F'),$$

where

$$U(F) = \int_{-\infty}^{\infty} u(x)\, dF(x)$$

or

$$U(F) = \int_{-\infty}^{\infty} u(x) f(x)\, dx$$

if F is differentiable and $f = dF/dx$.

Note that U is defined on lotteries whereas u is defined on money. To differentiate the two objects, we often call U the (von Neumann-Morgenstern) expected utility function and $u(\cdot)$ the Bernoulli utility function or von Neumann Morgenstern utility of money.

We assume that u is (strictly) increasing, implying that the marginal utility of money is strictly positive, and twice continuously differentiable, for analytic convenience.

1.3.2 Attitude toward risk

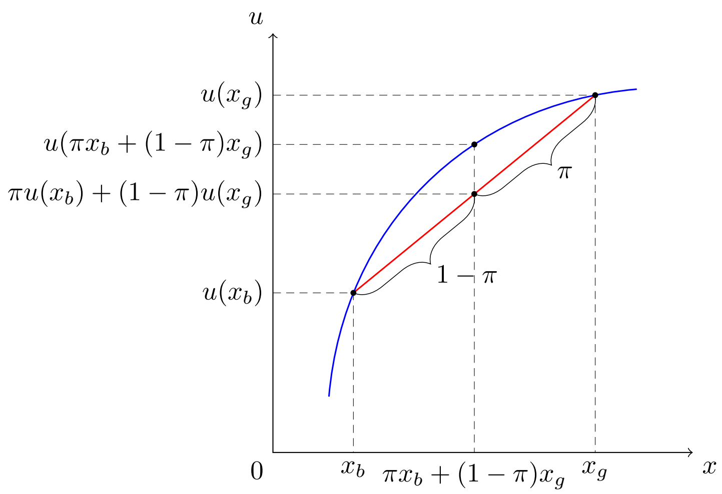
Definition 4. Let u be a utility function defined on money outcomes that represents $\succsim$. We say that $\succsim$ exhibits

$$\begin{pmatrix} \text{risk aversion} \\ \text{risk neutrality} \\ \text{risk loving} \end{pmatrix} \iff \int u(x)\, dF(x) \begin{pmatrix} < \\ = \\ > \end{pmatrix} u\left(\int x\, dF(x)\right)$$

for all lotteries F.

Equivalently, $\succsim$ exhibits risk aversion if $\mathbb{E}[u(X)] < u(\mathbb{E}[X])$. Notice that if $\succsim$ is risk averse (neutral, loving), then u is concave (linear, convex).

Consider $X = \{x_g, x_b\}$ where $x_g > x_b$. Recall that u shows risk aversion if $u(\pi x_b + (1-\pi)x_g) > \pi u(x_b) + (1-\pi)u(x_g)$.



If u is concave, Jensen's inequality says

$$\int u(z)\, dF(z) = \mathbb{E}[u(z)] \leq u\left(\int z\, dF(z)\right) = u(\mathbb{E}[z]).$$

The left hand side is the expected utility from the bet whose return z is distributed according to F. The right hand side is the utility from money whose amount is equal to the expected value of the random variable.

Definition 5. By a sure thing, we mean a deterministic outcome z. A bet is a random variable. A fair bet is a random variable whose expected return is equal to the sure thing.

Let ϵ be a random variable whose expected value is $0 : \mathbb{E}[\epsilon] = 0$. Given z^e, a fair bet to z^e is

$$z^e + \epsilon.$$

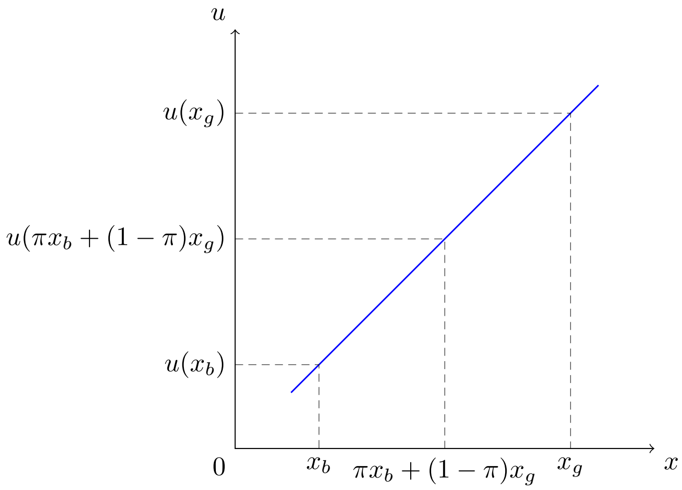
Let $z^e = \mathbb{E}[z]$, and $\epsilon = z - z^e$ whose distribution function is G. Then,

$$\int u(z^e + \epsilon)\, dG(\epsilon) = \mathbb{E}[u(z)] \leq u\left(\int z\, dF(z)\right) = u(\mathbb{E}[z]) = u(z^e).$$

We often say that u shows risk averse attitude if and only if the decision maker prefers a sure thing over a fair bet.

If a decision maker is risk neutral, then

$$u(\pi x_b + (1-\pi)x_g) = \pi u(x_b) + (1-\pi)u(x_g).$$



A decision maker is risk loving if

$$u(\pi x_b + (1-\pi)x_g) < \pi u(x_b) + (1-\pi)u(x_g).$$

You may think that only a professional gambler might be risk loving. A policy with a good intention can turn a risk neutral decision maker into a risk loving decision maker.

Suppose that a firm is a risk neutral decision maker whose Bernouille utility function (or vNM utility) is

$$u(z) = z.$$

The firm has a fixed cost D, but the return is a random variable R distributed over $[0, \infty)$. The profit of the firm is

$$u(R-D) = R - D$$

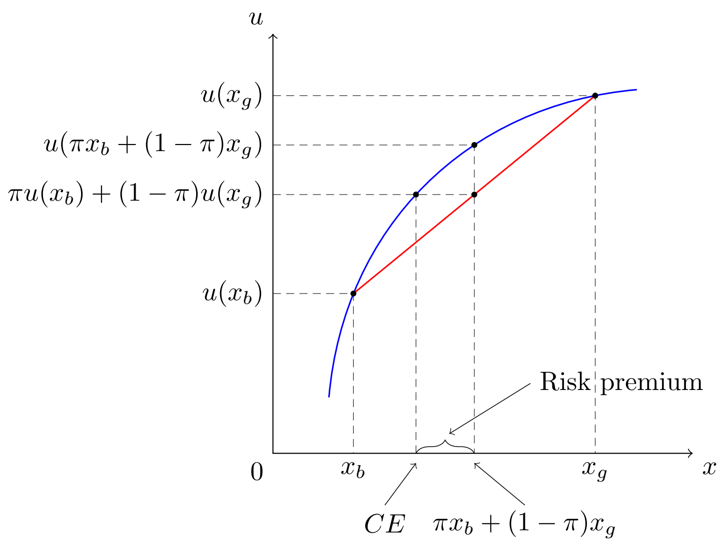
which is a random variable.

Certainty equivalent

A risk averse individual prefers a sure thing to a fair gamble. Is there a smaller amount of certain wealth that would be viewed as equivalent to the gamble?

Definition 6. The certainty equivalent (CE) of F is the amount of money for which the individual is indifferent between the gamble F and the certain amount CE; that is,

$$u(CE) = \int u(x)\, dF(x).$$



Note that u is concave if and only if

$$CE < \pi x_b + (1-\pi)x_g.$$

If a risk averse decision maker is offered two options: CE and $\pi x_b + (1-\pi)x_g$, then he will accept the expected return.

This behavior provides an alternative way to represents the attitude toward risk.

Definition 7. The risk premium (RP) associated with F is the maximum amount of money an individual is prepared to pay to avoid the game:

$$\mathbb{E}[u(X)] = u(\mathbb{E}[X] - RP)$$

Clearly, $RP = \mathbb{E}[X] - CE$.

Theorem 2. u exhibits risk aversion if and only if $RP \geq 0$.

1.4 Measurement of Risk Aversion

We sometimes have to rank two decision makers according to their attitude toward risk by saying that a decision maker is more risk averse than the other. Intuitively, the more concave the utility function, the more risk averse the consumer. Thus, the second derivative of u is a natural candidate for the measure risk aversion.

Recall that vNM utility is invariant with respect to affine transformation. Thus, if we change u by $\alpha u + \beta$ for some α, the attitude toward risk does not change. The problem of u'' as the measure of the risk aversion is that it is not invariant with respect to the affine transformation.

As an exam p le, consider a decision maker with $v(\cdot) = 2u(\cdot)$, who has the same preference over the bet as the decision maker with u. But, $v''(\cdot) = 2u''(\cdot) \neq u''(\cdot)$.

Definition 8 (Arrow-Pratt measure of absolute risk aversion).

$$r_A(x, u) := -\frac{u''(x)}{u'(x)}$$

The idea of constructing r_A is intuitive. We normalize the degree of concavity by u' so that the measure is invariant with respect to affine transformation. More precisely,

$$-\frac{u''}{u'} = -\frac{du'/dx}{u'} = -\frac{du'/u'}{dx} = -\frac{\text{\% change in MU}}{\text{absolute change in } x}.$$

$r_A(x)$ is positive, negative, or zero as the agent is risk averse, risk loving, or risk neutral.

Let us consider tow outcomes: bad outcome $x_b = w + r_b z$ and good outcome $x_g = w + r_g z$. Draw indifferent cure:

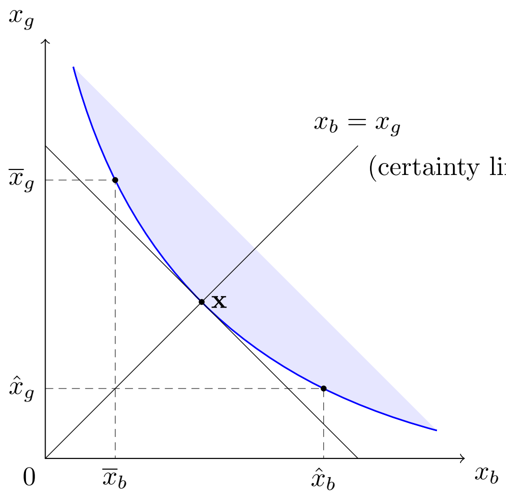
$$\pi u(x_b) + (1-\pi)u(x_g) = \overline{u}.$$

By totally differentiating both sides, we implicitly derive the marginal rate of substitution

$$\pi j'(x_b) + (1-\pi)u'(x_g)\frac{dx_g}{dx_b} = 0. \tag{1.1}$$

Hence, the marginal rate of substitution (MRS) is

$$\frac{dx_g}{dx_b} = -\frac{\pi}{1-\pi}\frac{u'(x_b)}{u'(x_g)}. \tag{1.2}$$



$$\left|\frac{dx_g}{dx_b}\right| \begin{pmatrix} = \\ < \\ > \end{pmatrix} \frac{\pi}{1-\pi} \text{ when } x_b \begin{pmatrix} = \\ > \\ < \end{pmatrix} x_g, \text{ showing that } u(\cdot) \text{ is concave.}$$

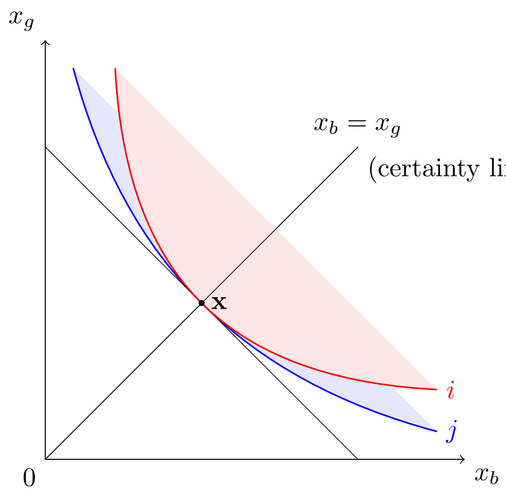
Define the consumers' preferred set at $\mathbf{x}$ to be the set of all outcome the consumer will prefer to $\mathbf{x}$, i.e., $\{\mathbf{y} \mid \mathbf{y} \succeq \mathbf{x}\}$.

Suppose now we have two consumers, i and j. It is natural to say that consumer i is (locally) more risk averse than consumer j if consumer i 's preferred set at $\mathbf{x}$ is contained in j's preferred set at $\mathbf{x}$. Consumer i's indifferent curve is "more curved" than consumer j's one at $\mathbf{x}$. Differentiate (1.1) one more with respect to x_b,

$$\pi u''(x_b) + (1-\pi)u''(x_g)\left(\frac{dx_g}{dx_b}\right)\left(\frac{dx_g}{dx_b}\right) + (1-\pi)u'(x_g)\left(\frac{d^2x_g}{dx_b^2}\right) = 0.$$

Using (1.2), we have

$$\frac{d^2x_g}{dx_b^2} = \frac{\pi}{(1-\pi)^2}\left[-\frac{u''(x)}{u'(x)}\right] \text{ when } x_b = x_g = x.$$



Given two utility functions $u_i(\cdot)$ and $u_j(\cdot)$, when can we say that $u_i(\cdot)$ is more risk averse than $u_j(\cdot)$?

Theorem 3.

1. $r_A(x, u_i) \geq r_A(x, u_j)$ for all x. That is, consumer i has a higher degree of risk aversion than consumer j everywhere.
2. There exists an increasing concave function $\psi(\cdot)$ such that $u_i(x) = \psi(u_j(x))$ for all x. In other words, $u_i(\cdot)$ is "more concave" than $u_j(\cdot)$.
3. $CE_i \leq CE_j$ (or $RP_i \geq RP_j$, i.e., i would be willing to pay more to avoid a given risk than j would.)

1.5 Applications

1.5.1 Contingent Commodity

A contingent commodity is a good that is available only if a particular event (or state of nature) occurs. It specifies conditions under which each contingent becomes available. We now treat contingent commodities as different goods. People have preferences over different consumption plan, just like they have preferences over actual consumption.

1.5.2 Insurance

Consider a strictly risk averse individual who has a wealth w and faces damage $D < w$ with probability π.

1.5.3 Portfolio choice problem: Mean-variance analysis

Suppose there are two lotteries L_1 and L_2.

Chapter 2

Hidden Information

2.1 Economy with uncertainty

state contingent claim

We have learned the equilibrium model under certainty, where a decision maker knows all characteristics of the goods, and the state at the time when he makes a decision. As we move from a model with certainty to uncertainty, we had to develop a new way of evaluating an object, a lottery, over the set of commodities. A fundamental question is whether the presence of uncertainty changes the equilibrium allocation of the competitive market.

Arrow and Debreu showed the condition under which the presence of uncertainty does not matter. We can apply exactly the same exercise as we learned from the model with certainty. More importantly, the first and the second welfare theorems continue to hold.

Complete market

This condition is called the complete market hypothesis: each commodity has a market where it can be traded. In order to make the notion of complete market precise, Arrow invented the notion of contingent commodity.

We first state the model of competitive market satisfying the complete market hypothesis. We do so, because it provides an important benchmark against which an economy with incomplete market is examined, providing a fundamental insight into the role of uncertainty to the equilibrium outcome of the market.

Uncertainty

By a state, we mean any factor that affects the decision of an economic agent. The quality of a product is a good example, which may or may not be known to the decision maker at the time of his decision. Let S be the set of states, and $s \in S$ be a generic element. Let us assume that S is finite.

> **Definition 9.** The economy is subject to uncertainty, if a state is not revealed to a decision maker at the time of his decision making.

A lottery is one of the examples. The value of the lottery is a state, which is not revealed at the time when a decision maker purchases a lottery at a certain price.

Let us consider a finite exchange economy, which is populated by I consumers endowed with neoclassical utility function, L commodities and S states, each of which has $\#I$, $\#L$ and $\#S$ elements.

We start with the description of the initial endowment, which is a complete specification of endowment for all possible contingencies. Let $\omega = (\omega_\ell)_{\ell \in L} \in \mathbb{R}^{\#L}$ be the profile of commodities. Because his endowment is affected by state $s \in S$, we need to spell out the profile of endowments for all states in S. Thus, the endowment of agent i is

$$\omega_i = (\omega_{s,i}) \in \mathbb{R}^{\#L \times \#S}.$$

Commodity

We differentiate a commodity by state. This is a fundamental innovation of Arrow and Debreu. That is, a commodity is differentiated by an attribute which is relevant to the decision of an economic agent, including a state. For example, an umbrella when it rains is a different commodity from an umbrella when it shines. A stock when the firm is generating a large profit is a different commodity form a stock when the same firm is bankrupted.

Contract

Another important interpretation of a commodity by Arrow and Debreu is that a commodity is a contract which promises to deliver the same physical goods if such state arises.

Depending upon a model, it is more convenient to regard an umbrella in rain as a contract which promises to deliver one unit of physical umbrella when it rains. certainly, this contract carries a price. Similarly, a contract that promises to deliver one stock when the firm is generating a large profit is different from a contract that promises to deliver one stock when the firm is bankrupted. The two commodities would carry different prices.

We can measure the quantity of a commodity according to a basic unit, known as contigent commodity.

Definition 10. A contingent commodity (or Arrow security) is a contract to deliver one unit of a good if a particular state is realized.

As we expand the economy by incorporating uncertainty, we are expanding the notion of commodity from a profile of physical characteristics, For each of L commodities, we have to consider as many as $\#S$ contingent commodities. As a result, we have $\#L \times \#S$ commodities.

Let x_{ls} be the contract to deliver x_l units of l-th good if state s occurs. The collection of commodities is now $\mathbb{R}^{\#L \times \#S}$. Let $\succeq_i$ be the preference over $\mathbb{R}^{\#L \times \#S}$.

Example. Suppose that π_s is the probability that state s is realized, and

$$u_i(x, s)$$

is the utility of $x \in \mathbb{R}^{\#L}$ under state s. Let $x = (x_s)_{s \in S} \in \mathbb{R}^{\#L \times \#S}$ and $x' = (x'_s)_{s \in S} \in \mathbb{R}^{\#L \times \#S}$ be a pair of state contingent commodity bundles.

$x \succ x'$ if

$$\sum_{x \in S} u(x_s, s)\pi_s > \sum_{x \in S} u(x'_s, s)\pi_s.$$

It is important to know that the expected utility is one of many different ways to evaluate the state contingent commodity bundles.

Sequence of moves

1. Before a state $s \in S$ is realized, there is a market to trade commodities.
2. After trading commodities (or contracts), a state is realized.
3. The good is delivered according to the contract.
4. Goods are consumed, and utility is generated.

The first and third steps warrant a careful examination.

Forward market

A commodity is trade in a market, before a state is realized. Thus, is is more convenient to interpret a commodity as a contingent contract, which promises to deliver the specific amount of goods if a state arises.

In the sense that the contingent contract is traded in a forward market, the contract is often called a forward contract.

Enforcement

When the contract is traded, a buyer of the commodity pays money to the seller, and receives a piece paper with a promise on it. The good is not delivered, until a state is realized.

An important assumption is that the contract is enforced without any exception, or the seller is committed to carry out the contract. If the enforcement is not complete, or if the seller has a limited commitment, then the contract may not be traded, or will fetch a lower price than under the full commitment.

For example, a debt contract is a promise that a borrower will pay back the principal and the interest back to the lender by a specific time. In Arrow Debreu economy, a debt contract will be enforced without any exception.

Symmetric information

At the time when the good is traded, no agent in the economy observes the state. Every decision maker faces uncertainty. In this sense, uncertainty is symmetric.

Complete market hypothesis

A fundamental assumption of Arrow Debre economy is that every contingent commodity has a market where it can be traded. This assumption is called the complete market hypothesis. Because a forward contract is traded, we sometimes say that Arrow Debre economy assumes a complete set of forward markets.

With a complete set of markets, and with full commitment, we can follow exactly the same analysis as for the economy with certainty to establish the first and the second welfare theorems. Any failure of the fundamental welfare theorems can be traced back to the missing market.

2.2 Informational efficiency

Symmetric vs. asymmetric information

- An economy with uncertainty is subject to a state which is not revealed to an agent at the time of decision.
- If no agent observes a state, the economy is subject to uncertainty, but the uncertainty is symmetric.
- If an agent observe a state, but another agent does not observe the same state, asymmetric information exists.

Rational expectations

Presence of asymmetric information does not necessarily lead to inefficient allocation, as the competitive market can aggregate dispersed information into the market clearing price. Fredrick von Hayek called this property informational efficiency of competitive market.

Information aggregation

Let us consider an exchange economy with uncertainty with two consumers with identical utility function:

$$u_i(x_{1,i}, x_{2,i}) = \beta \log x_{1,i} + x_{2,i}$$

where

2.3 Lemon's problem

Lemon's market

Theorem 4. The market clearing price is ϕ_L, and only the low quality product is traded.

Proof. We show that ϕ_L is the only possible market clearing price. Suppose that p is the

- $p > \phi_H$ is not possible, because no consumer will buy a used car whose quality cannot exceed ϕ_H.
- $c_H \leq p \leq \phi_H$. Since $p \geq c_H > c_L$, all low quality sellers will put their low quality cars in the market. As a result, the average quality of a

used car in the market cannot be more than $\pi_H\phi_H+(1-\pi_H)\phi_L$ which is strictly less than c_H by the last assumption. Thus, no high quality used car wil lbe on the market, which implies that the quality of the used car is exactly $\phi_L < c_H$ by the first assumption. Since $c_H \le p$, no buyer will pay p to buy a used car with quality $\phi_L < c_H \le p$. Hence, p cannot be an equilibrium price.

- $\phi_L < p < c_H$. Since $p < c_H$, only the low quality car will be in the market. No buyer is willing to pay a price more than ϕ_L. Thus, p cannot be an equilibrium price.
- $p < \phi_L$. Because buyers compete for a used car whose utility is ϕ_L, the market experiences excess demand.

If $p = \phi_L$, only the low quality used car will be on the market and a buyer is willing to pay for his utility for the car. □

Second example of Akerlof

Theorem 5. If the market clearing price is determined according to the average quality of the products in the market, then the lemon's problem arises and the only equilibrium price is ϕ_l.

Proof. Let p be an equilibrium price. Since the average utility of the products determines the market clearing price,

$$p \le \mathbb{E}[\phi] = \int_{\phi_l}^{\phi_h} \phi f(\phi)\, d\phi.$$

$c(\phi_h) > \mathbb{E}[\phi]$ and c is a continuous function. $\exists\varepsilon_1$ such that $\forall\phi \in (\phi_h-\varepsilon_1, \phi_h]$ will not put the product in the market since $c(\phi) > \mathbb{E}[\phi]$, where

$$c(\phi_h - \varepsilon_1) = \mathbb{E}[\phi].$$

Then, the aerage expected price cannot be higher than

$$p \le \mathbb{E}[\phi \,|\, \phi \le \phi_h - \varepsilon_1] = \int_{\phi_l}^{\phi_h-\varepsilon_1} \phi f(\phi \,|\, \phi \le \phi_h - \varepsilon_1)\, d\phi.$$

If we iterate the same process for n rounds, we have ε_n so that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) = \mathbb{E}\left[\phi \,|\, \phi \le \phi_h - \sum_{k=1}^{n-1} \varepsilon_k\right].$$

By applying the same logic, we conclude that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) < \mathbb{E}\left[\phi \,|\, \phi \le \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

Since $c(\cdot)$ is continuous, $\exists \varepsilon_{n+1} > 0$ so that

$$c\left(\phi_h - \sum_{k=1}^{n+1} \varepsilon_k\right) < \mathbb{E}\left[\phi \,|\, \phi \leq \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

This process continues as long as

$$\phi_h - \sum_{k=1}^{n+1} \varepsilon_k > \phi_l.$$

Thus, ϕ_l is the only equilibrium price. $\square$

Discussion

Chapter 3

Primer of Information Economics

3.1 Review

Compete market

- Arrow-Debreu economy presumes a compete set of markets so that each commodity can be trade at a market clearing price.
- Without market, externality prevails and the first welfare theorem fails.
- All market failure can be traced back to the absence of a market.
- Inefficiency in the lemon's market can be explained by the absence of a market for information.

Market for information

Creating a market for information is extremely difficult.

3.2 definitions

3.3 Baseline model

3.4 Signaling

3.5 Screening

Firm

To escape from the lemon's problem, or to prevent the lower productive workers from entering the employee pool, a firm uses a mechanism design to screen lower quality workers.

The firm has to rely on the difference of the marginal rate of substitution between the wage and the education to screen out one from another group.

Assumptions

We maintain the same assumptions on the utility function of the worker, and the firm. Let us summarize the assumptions.

$$u(\theta, w, e) = w - c(e, \theta),\ c(0, \theta) = 0,\ \frac{\partial c(e, \theta)}{\partial e} > 0,\ \frac{\partial^2 c(e, \theta)}{\partial e^2} > 0,$$

and

$$\frac{\partial c(e, \theta)}{\partial \theta} < 0,\ \frac{\partial^2 c(e, \theta)}{\partial e \partial \theta} < 0.$$

Interpretation

We continue to assume that the education is only to generate disutility of the workers. We can regard education as (unpleasant) task which must be completed in return for the job (and wage).

First best solution

It is easy to see that if productivity θ_i is known to the firm, the firm has to pay for the productivity, without any unpleasant task in an efficient allocation.

Proposition 1. Suppose that the worker's ability is public information. Then,

$$(w_i^*, e_i^*) = (\theta_i, 0) \quad \forall i \in \{h, l\}.$$

and the firms obtain 0 profit.

Proof. Since θ_i is known to the firm, it is easy to see that the wage must be equal to the productivity. Since the only function of the unpleasant task is to generate disutility on the part of the worker, no unpleasant task should be imposed in an efficient allocation (the first best solution).

The difficult part is to show that the firms cannot entertain positive profit. Let us assume that there are two firms competing each other as the Bertrand competitor. The case of the multiple firms can be analyzed in the same way.

Let Π_k be the profit of firm k. Define $\Pi = Pi_1 + \Pi_2$. Suppose that $\Pi > 0$. Since all firms are identical, we can assume without loss of generality that

$$\Pi_1 \leq \frac{\Pi}{2}.$$

Suppose that firm 1 offers $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$ instead of (w_h^*, e_h^*) and (w_l^*, e_l^*). Since (w_l^*, e_l^*) is an equilibrium for low productive workers, the incentive compatibility condition of θ_l worker must satisfy:

$$w_h^* - c(\theta_l, e_h^*) \leq w_l^* - c(\theta_l, e_l^*).$$

By adding equal amount of ε on both sides, we know that $(w_l^* + \varepsilon, e_l^*)$ is also satisfying the incentive compatibility constraint:

$$(w_h^* + \varepsilon) - c(\theta_l, e_h^*) \leq (w_l^* + \varepsilon) - c(\theta_l, e_l^*).$$

By applying the same logic to θ_h worker, we also conclude that $(w_h^* + \varepsilon, e_h^*)$ is incentive compatible. Choose $\varepsilon > 0$ sufficiently small so that

$$\Pi - \varepsilon > \frac{\Pi}{2} > 0.$$

If firm 1 offers menu of contracts of $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$, then all θ_h workers will take $(w_l^* + \varepsilon, e_l^*)$, thus generating profit of $\Pi - \varepsilon$ for firm 1. By assumption,

$$\Pi - \varepsilon > \frac{\Pi}{2} \geq \Pi_1$$

which implies that Π_1 is no longer an equilibrium payoff. This is a contradiction to the hypothesis that Π_1 is an equilibrium profit. □

Asymmetric information

Suppose that the workers observe their productivity, but no firms observe the productivity of workers. Akerlof indicated that the market is exposed to the lemon's problem.

In contrast to the signaling model of Spence [1973] in which the workers make move, Rothschild and Stiglitz [1976] demonstrated that the uniformed firms can design a menu of contract which allows the firm to escape from the lemon's problem.

Key concepts

A contract is (w, e), which sepcifies wage w and the level $e \geq 0$ of task associated with the job. A menu of contracts is the list of state contingent contracts:

$$M = ((w_h, e_h), (w_l, e_l))$$

where (w_i, e_i) is supposed to be accepted by θ_i workers.

Main conclusions

Let us summarize the main findings of Rothschild and Stiglitz [1976].

Theorem 6.

1. In any equilibrium, firm's profit is 0.
2. No pooling equilibrium exists.
3. If a separating equilibrium exists, (w_l, e_l) and (w_h, e_h) satisfy

$$w_l = \theta_l,\ e_l = 0;\ w_h = \theta_h$$

and e_h is defined implicitly by the incentive compatibility constraint of θ_l worker:

$$\theta_h - c(\theta_l, e_h) = \theta_l - c(\theta_l, 0).$$

Proof. We prove the main conclusions in multiple steps, which reveals how the hidden information affects the incentive of the workers, and how the firm can exploit the worker's incentive to screen out different workers.

1. We follow the same logic as in the previous classes to show that the equilibrium profit of the firm must be 0.

   In Spence [1973], the single crossing property allows the high productivity worker to signal his productivity credibly to the firm, to fetch a wage equal to his true productivity. In Rothschild and Stiglitz [1976], the single crossing property of the worker's utility allows the firm to screen them out.

2. If $(w_h, e_h) \neq (w_l, e_l)$, we say it is a separating equilibrium. In a separating equilibrium, (w_j, e_j) is accepted only by θ_j worker $\forall j \in \{h, l\}$. Since the firm makes at least 0 profit, the wage must be equal to the productivity.

3. In a separating equilibrium, the competitive pressure forces each firm to offer $e_l = 0$ so that the low productivity workers endure no unpleasant task. This result is similar to the property of the signaling equilibrium of Spence [1973] where the low productivity worker does not take any (unpleasant) education. The difference is that in Spence [1973], the decision by the worker is motivated completely by the negative payoff of taking education, while in Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer no task for low productivity workers.

   To be a feasible menu, the incentive compatibility condition must hold in a separating equilibrium $((w_h, e_h), (w_l, e_l))$.

$$\theta_h - c(\theta_l, e_h) \leq \theta_l - c(\theta_l, e_l).$$

   In Spence [1973], there are multiple signaling equilibria where the weak inequality holds strictly. Only in the Riley outcome, the weak inequality holds with inequality. In Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer a menu in which the incentive compatibility constraint is binding (i.e., the weak inequality holds with equality).

□

Note that Rothschild and Stiglitz [1976] did not establish the existence of a separating equilibrium. Later studies modified the original model of Rothschild and Stiglitz [1976] to ensure the existence of an equilibrium for all parameter values.

Definition 11 (Moral hazard problem)**.** An incentive problem arising from hidden action is called the moral hazard problem.

Arrow

3.6 Hidden action

Useful observation

The optimization problem is complex, because the principal is maximizing over the set of wage schedules instead of wages. We make two observations.

Lemma 1. If $(e^*, w^*(\pi))$ is an optimal solution, then the individual rationality constraint must be binding:

$$\int v(w^*(\pi))f(\pi|e^*)\,d\pi - g(e^*) = \underline{u}.$$

Proof. Suppose that

$$\int v(w^*(\pi))f(\pi|e^*)\,d\pi - g(e^*) > \underline{u}.$$

Then, the principal can offer slightly less wage, still satisfying

$$\int v(w^*(\pi) - \varepsilon)f(\pi|e^*)\,d\pi - g(e^*) > \underline{u}.$$

Since the utility of the principal is strictly decreasing in wage, the principal has higher expected payoff from $(e^*, w^*(\pi) - \varepsilon)$. We treat the individual rationality constraint as an equality constraint. For each e, we solve

$$\max_{(w(\pi))_\pi} \int (\pi - w(\pi))f(\pi|e)\,d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) = \underline{u}.$$

Then, we choose e^* to maximize objective function. $\square$

First order condition

Chapter 4

Hidden Information: Screening

4.1 Introduction

We focus on the basic static adverse selection problem. There is a principal facing one agent who has private information on his "type". Type represnets the agent's preference or intrinsic productivity. We first study how to solve such problems when the agent can be one of two types, a case that will give us the key insights from adverse selection models.

General setup

An agent, informed party, is privately informed about his type.

A principal, uninformed party, designs a contract in order to screen different types of agent and maximize her payoff.

This is a problem of hidden information, often referred to as screening problem.

4.2 A Model of Price Discrimination

Consider a transaction between a buyer (agent) and a seller (principal).

4.3 Full Information Benchmark

Suppose that the seller is perfectly informed about the buyer's type. The seller can treat each type of buyer separately and offer a type-dependent contract: (q_i, T_i) for type $\theta_i, i = H, L$.

4.4 Asymmetric Information

Suppose from now on that the seller cannot observe the type of the buyer, facing the *adverse selection* problem.

4.4.1 Linear pricing: $T(q) = Pq$

The buyer pays a uniform price P for each unit he buys.

4.4.2 Two-part tariff: $T(q) = F + Pq$

The seller charges a fixed fee (F) up-front, and a price P for each unit purchased. Note that for any given price P, the maximum fee the seller can charge up-front is $F = S_L(P)$ if he wants to serve both types. The seller chooses P to maximize

$$\beta[S_L(P)+(P-c)D_L(P)]+(1-\beta)[S_L(P)+(P-c)D_H(P)] = S_L(P)+(P-c)D(P).$$

4.5 Optimal Nonlinear Pricing

Here, we look for the best pricing scheme among all possible ones. That is, we look for the *second-best* outcome.

4.6 Applications

4.6.1 Regulation

The public regulators are often subject to an informational disadvantage with respect to the regulated utility or natural monopoly.

4.6.2 Ex-ante contracting

There are situations in which the agent can learn his type only after he signs a contract

Chapter 5

Hidden Action: Moral Hazard

5.1 Introduction

We have discussed screening problem. The uninformed party combats the problem of adverse selection by screening the other. We now discuss another class of asymmetric information problem. Asymmetric information arises from imperfect monitoring of players' actions (or hidden action).

5.2 Binary Model

Suppose there is an employer (principal) and an employee (agent). Agent could shirk ($e = 0$ or low effort) or work hard ($e = 1$ or high effort), which is not observable by the principal.

5.3 First-Best Contract

In this benchmark, assume that the effort level is observable and verifiable. If principal wants to induce e, then he solves

$$\max_{t_H, t_L} \pi_e(S_H - t_H) + (1 - \pi_e)(S_L - t_L)$$

subject to

$$\pi_e u(t_H) + (1 - \pi_e)u(t_L) - ce \geq u_0. \tag{IR}$$

Set up the Lagrangian function

$$\mathcal{L} = \pi_e(S_H - t_H) + (1 - \pi_e)(S_L - t_L) + \lambda[\pi_e u(t_H) + (1 - \pi_e)u(t_L) - ce - u_0].$$

From the first-order condition,

$$-\pi_e + \lambda \pi_e u'(t_H^F) = 0,$$
$$-(1 - \pi_e) + \lambda(1 - \pi_e)u'(t_L^F) = 0.$$

We thus have

$$\lambda = \frac{1}{u'(t_H^F)} = \frac{1}{u'(t_L^f)},$$

implying that $f_H^F = t_L^F = t^F$. By (IR), $t^F = u^{-1}(ce + u_0)$. The risk-neutral principle offers a full insurance to the risk-averse agent and then extracts the full surplus.

Principal prefers $e = 1$ if

$$\pi_1 S_H + (1-\pi_1)S_L - u^{-1}(c+u_0) \geq \pi_0 S_H + (1-\pi_0)S_L - u^{-1}(u_0)$$

or

$$\underbrace{(\pi_1 - \pi_0)(S_H - S_L)}_{\text{expected gain of effort}} \geq \underbrace{u^{-1}(c+u_0) - u^{-1}(u_0)}_{\text{cost of including effort}}.$$

Otherwise, principal prefers $e = 0$.

5.4 Second-Best Contract

Assume that the effort exerted by the agent is unobservable. The principal's problem to induce $e = 1$ is

$$\max_{t_H, t_L} \pi_1(S_H - t_H) + (1-\pi_1)(S_L - t_l)$$

subject to

$$\pi_1 u(t_H) + (1-\pi_1)u(t_L) - c \geq \pi_0 u(t_H) + (1-\pi_0)u(t_L) \tag{IC}$$
$$\pi_1 u(t_H) + (1-\pi_1)u(t_L) - c \geq u_0 \tag{IR}$$

5.4.1 Optimal Incentive Scheme

Set up the Lagrangian function

5.4.2 Optimal Effort Policy

The cost of inducing high effort under moral hazard is

$$C^* := \pi_1 t_H^* + (1-\pi_1)t_L^* - u^{-1}(u_0).$$

$e = 1$ is optimal if $(\pi_1 - \pi_0)(S_H - S_L) \geq C^*$. Otherwise, $e = 0$ is optimal. The second-best cost of inducing a high effort is higher than the first-best cost:

5.5 Extensions

5.5.1 Risk-neutral agent

Suppose the agent is risk-neutral and let $u(t) = t$. In what follows, we show that the principal can achieve the first-best outcome. Because $u^{-1}(t) = t$, the optimal contract is immediate from (3)

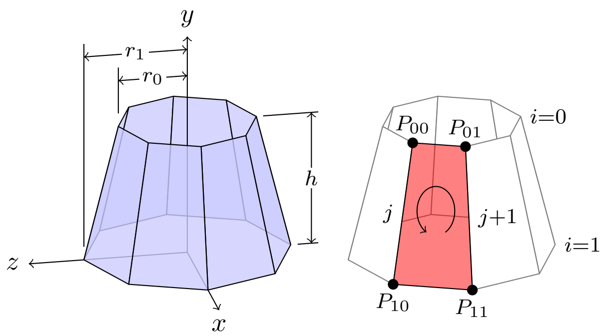
5.5.2 Limited Liability

The agent is still risk-neutral ($u(t) = t$). Assume that the agent is protected by limited liability constraint that the transfer received by the agent should no less than $t_0 \coloneqq u^{-1}(u_0)$. The principal's problem is

5.6 Application: Insurance Market

Moral hazard is pervasive in insurance markets. Lets consider a risk-averse agent with utility function $u(\cdot)$ and initial wealth w.

3d cone



Chapter 6

Nash Bargaining Problem

6.1 Plan

Competitive equilibrium

In a competitive market, decisions are made in a decentralized manner. A consumer optimizes subject to the budget constraint and the price. A producer maximizes profit for a given price.

It is not obvious how the market clearing price is determined. We often use a fictitious auctioneer *Walrasian auctioneer* who construct the aggregate demand and supply from individual demand and supply curves. He then finds the intersection of the two curves and announces the market clearing price. Although it is a useful educational tool to explain the market clearing mechanism, the presence of the Walraisn auctioneer goes directly against the very spirit of the decentralized trading of the competitive market: the invisible hand.

Informational efficiency

Hayek observed that the competitive market price aggregates dispersed information so that the individual agents in the economy can take and action in a decentralized manner, but can coordinate to achieve and efficient allocation.

The general equilibrium model of Arrow and Debreu formulates the first welfare theorem but remains vague about the price determination process and the information aggregation process.

Decentralized trading model

We need to understand the mechanism that aggregates the private information of the individual agents into the market clearing price.

If the trading is done decentralized, it is not clear how there should a single market clearing price. While we analyze the information aggregation process, we can explain the law of single price.

Dynamic decentralized trading model

We build a decentralized trading model from a smallest unit of trading: bargaining. To trade something, you need at least two players. Bargining is a trading protocol between two players. We introduce a matching process as we examined in the model of Peter Diamond so that the trading partner may change over time. We examine the prices at which trading occurs, and how the difference among different trading partners vanish, and converge to the competitive equilibrium price.

Plan

- Bargaining
  - Axiomatic model of Nash [1950]
  - Strategic model of Rubinstein [1982]
- Matching and bargaining

6.2 Bargaining

Trading institution between one seller and one buyer to determine the price and the delivery condition of good or service. Also called bilateral monopoly problem.

Finest unit of trading, which forms the foundation of the market. To trade goods, you need at least two people. The bargaining is a trading unit with two people: one seller vs. one buyer.

Old and widely used, and has many institutional variations. Difficult to formulate and anlyze.

6.3 (Nash) Divide a dollar

A game between two players.
$A_i = [0, 1]$

$$u_i(a_1, a_2) = \begin{cases} a_i & \text{if } a_1 + a_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Any division of 1 dollar can be sustained by a Nash equilibrium. We have a theory, which cannot make any useful prediction. This is often considered as a failure of the theory, revealing the difficulty of the problem, as the bargaining outcome is affected by the details of the rule of the game.

Why difficult?

The bargaining problem is difficult, not because we need a fancy mathematical tool. The analysis must satisfy two conditions.

1. The rule of the bargaining must be reasonable, in which the two parties have comparable bargaining power and can influence the outcome, A dictatorial game is a form of bargaining, but is not considered a reasonable model of bargaining.
2. The model must make a sharp prediction to tell us what would the most likely outcome from the bargaining process. Divide a dollar game is a reasonable bargaining model. We cannot learn anything from Nash equilibrium, because every efficient division is a Nash equilibrium.

6.4 Nash bargaining problem

Approach by Nash [1950]

In 1950, we did not have tools to tackle the difficulties of the bargaining problems. It is before John Nash invented (Nash) equilibrium concept. Instead, Nash chose to bypass the difficulties, but search for a way to say something useful regarding the bargaining process.

The major innovation of Nash's approach is to suppress the details of the procedure, treating it as a black box, and he focuses on the properties of the reasonable outcome.

A bargaining problem is regarded as a mapping from the data which consists of the preference of the players and the structure of the surplus to the division of the surplus.

He asked the following question: A reasonable bargaining outcome must satisfy a certain set of properties, what is the mathematical formula to calculate the outcome?

Bargaining solution

Let S be the set of surplus and d be the disagreement point. S is the collection of all possible outcomes attainable, if agreement is reached. d is the pair of payoffs associated with disagreement.

Example. In case of the divide dollar,

$$\{(a_1, a_2) : a_i \geq 0, a_1 + a_2 \leq 1\}$$

and

$$d = (0, 0).$$

Definition 12. A bargaining problem is (S, d), where $S \subset \mathbb{R}^2$ is compact and convex and $\exists s \in S$ such that $s_i > d_i \, \forall i \in \{1, 2\}$.

We admit randomized contact which makes the set of all feasible utilities convex. $S \subset \mathbb{R}^2$ if and only if S is closed and bounded. If S is unbounded, then the bargaining can be meaningless, because each party can get what he wants without negotiation. The closeness of S is a technical condition to ensure the existence of a solution of the optimization problem.

The last condition ensures that the bargaining is not degenerate. If every feasible payoff vector is Pareto dominated by the disagreement outcome, there is no point of negotiation.

Let S be the set of all bargaining problems.

Definition 13. A bargaining solution $f(S, d) = (u_1, u_2)$ is the rule that specifies which outcome is determined:

$$f : \mathcal{S} \to \mathbb{R}^2.$$

Note. A bargaining solution is not conditioned on a particular bargaining problem. Instead, the way how a bargaining outcome is determined should be spelled out before a particular bargaining problem is selected.

Example (Dictatorial)**.** Let

$$\overline{u}_1 = \arg\max_{u_1'}\{u_1' \mid \exists u_2,\, (u_1', u_2) \in S,\, u_2 \geq d_2\}$$

be the best outcome of player 1 in S.

$$f(S, d) = \{(\overline{u}_1, u_2) \mid (\overline{u}_1, u_2) \in S\}.$$

chooses the best possible outcome.

Example (Always disagree)**.**

$$f(S, d) = d.$$

Discussion

- Dictatorial bargaining solution does not sound reasonable, because the lack of symmetry. By a bargaining situation, we refer to a situation where each party has some, if not equal, control over the outcome of the negotiation. Dictatorial solution does not allow any room for negotiation by player 2.
- If a negotiation always breaks downm the outcome is not efficient. Alluding to Coase theorem, such a bargaining rule should be replaced by another rule which generates more efficient outcome.

Axioms

Let us spell out the properties which any reasonable bargaining solution must satisfy. John Nash call these properties axioms on the ground that they are evidently reasonable. Let us state four axioms, along with discussions.

A reasonable bargaining solution should be such that its outcome is affected by the units of the utils.

Axiom 4 (Invariance)**.** Consider the two bargaining problems, (S, d) and (S', d') where (S', d') is obtained by applying a positive affine transformation to $(S, d) : \forall i, \exists \alpha_i \geq 0$ and $\beta_i \in \mathbb{R}$ such that

$$s_i' = \alpha_i s_i + \beta_i$$

f satisfies the invariance axiom if

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i \quad \forall i.$$

A reasonable bargaining solution should not produce an outcome which is Pareto dominated by another feasible outcome.

Axiom 5 (Pareto). f satisfies the Pareto axiom if $\exists (t_1, t_2) \in S$ such that $t_i > s_i \, \forall i$ implies $f(S, d) \neq (s_1, s_2)$.

Definition 14. A bargaining problem (S, d) is symmetric if $(s_1, s_2) \in S$ implies that $(s_2, s_1) \in S$.

If the bargaining problem is symmetric, then the name of a player should not matter, implying that the two parties have equal bargaining power.

Axiom 6 (Symmetry). f satisfies the symmetry axiom if for any symmetric bargaining problem (S, d), $f_1(S, d) = f_2(S, d)$.

The symmetry axiom does not require that the bargaining outcome must be equal for all players. The axiom applies only to a symmetric bargaining problem. The same axiom imposes no restriction on bargaining problems which are not symmetric.

The first three axioms (INV, PAR and SYM) are the restrictions on f over individual bargaining problems. The next axiom specifies how the solutions from two different bargaining problems should be related.

Axiom 7 (Independence of Irrelevant Alternatives). Consider two bargaining problems, (S, d) and (T, d) with $T \subset S$. f satisfies the axiom of independence of irrelevant alternatives, if $f(S, d) \in T$ implies $f(T, d) = f(S, d)$.

Discussion

- While INV, PAR and SYM appear to be considered evidently reasonable, IIA need motivation, as it imposes a restriction on the relationship between the solutions of two bargaining problems.

- IIA is essentially identical with the weak axiom of choice. If a decision maker choose an object from T shich contains S, but the selected commodity bundle is in S, then the consumer should choose the same bundle when he is constrained to choose from S. The alternatives in $T \setminus S$ are irrelevant. We know that the weak axiom in this sense implies that the consumer behavior can be described as a consequence of utility maximization.

- IIA is reasonable, if we accept the view that a reasonable bargaining solution should be a solution of a social welfare function. Otherwise, it is not. The existence of a social welfare function is not always guaranteed.

- For me, IIA appears to be reasonable, but has room to be improved. Quite a few people followed up Nash [1950] proposing alternative set of axioms. Usually, INV, PAR and SYM are not touched, but IIA is replaced by something else.

Consistency and uniqueness

We choose INV, PAR, SYM and IIA because they are considered reasonable. We did not consider whether four axioms are consistent with each other. If they are not consistent, no bargaining solution satisfying four axioms exists.

If a bargaining solution satisfying four axioms exists, we need to ask how many solutions satisfy the axioms. If too many solutions satisfy the axioms, there is a room to impose additional axioms.

Nash [1950] answer these two question rigorously and elegantly.

Nash bargaining solution

Nash [1950] proposes a bargaining solution.

Definition 15. f^N is the Nash bargaining solution if

$$f^N(S,d) = \arg \max_{(s_1,s_2)\in S, s_i \geq d_i} (s_1 - d_1)(s_2 - d_2).$$

$s_i - d_i$ represents the gain from reaching agreement over the disagreement payoff. We call $s_i - d_i$ the Nash gain, and $(s_1 - d_1)(s_2 - d_s)$ the Nash product.

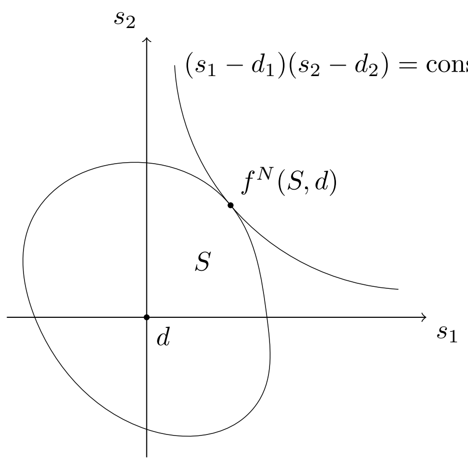


Figure 6.1: Nash bargaining solution (Osborne and Rubinstein [1990])

Note. The Nash product $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_s)$ is strictly quasi concave continuous function. Since S is convex and compact, the maximizer exists and is unique. Thus, the Nash bargaining solution is well defined.

Characterization

The fundamental theorem of Nash [1950] is that the Nash bargaining solution is the only bargaining solution which satisfies INV, PAR, SYM and IIA.

Theorem 7. Bargaining solution f satisfies INV, PAR, SYM and IIA if and only if $f = f^N$.

Proof. We prove that the Nash bargaining solution satisfies INV, PAR, SYM and IIA.

INV Suppose that $s'_i = \alpha_i s_i + \beta_i$ and $d'_i = \alpha_i d_i + \beta_i$. Note

$$(s'_1 - d'_1)(s'_2 - d'_2) = \alpha_1 \alpha_2 (s_1 - d_1)(s_2 - d_2).$$

Thus, if (s^*_1, s^*_2) maximizes the right hand side, then $(\alpha_1 s^*_1 + \beta_1, \alpha_2 s^*_2 + \beta_2)$ maximizes the left hand side.

PAR Since $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_2)$ is a strictly increasing function of (s_1, s_2), an optimal solution must be at the Pareto frontier of S.

SYM Fix a symmetric problem (S, d) where $d_1 = d_2 = d$. Note

$$W(s_1, s_2) = (s_1 - d)(s_2 - d) = (s_2 - d)(s_1 - d),$$

If (s^*_1, s^*_2) maximizes the left hand side, then $(s^*_2, s^*_1) \in S$ maximizes the right hand side. It remains to show $s^*_1 = s^*_2$.

Suppose that $s^*_1 \neq s^*_2$. Since $W(s_1, s_2)$ is strictly quasi concave and S is convex, $\forall \lambda \in (0, 1)$,

$$\lambda(s^*_1, s^*_2) + (1 - \lambda)(s^*_2, s^*_1) \in S$$

and

$$W(\lambda(s^*_1, s^*_2) + (1 - \lambda)(s^*_2, s^*_1)) > W(s^*_1, s^*_2) = W(s^*_2, s^*_1)$$

which contradicts to the hypothesis that (s^*_1, s^*_2) maximizes W over S.

IIA Suppose that (s^*_1, s^*_2) maximizes $W(s_1, s_2)$ over T and

$$(s^*_1, s^*_2) \in S \subset T.$$

Since

$$W(s^*_1, s^*_2) \geq W(s_1, s_2) \quad \forall (s_1, s_2) \in T,$$

and $S \subset T$,

$$W(s^*_1, s^*_2) \geq W(s_1, s_2) \quad \forall (s_1, s_2) \in S.$$

Thus, $(s^*_1, s^*_2) \in S$ must maximize W over S.

The difficult part is to show that Nash bargaining solution is the only solution satisfying four axioms. □

Lemma 2.

$$\forall (s_1, s_2) \in S,\ s_1 + s_2 \leq 1.$$

Proof. Suppose that $\exists (t_1, t_2) \in S$ such that $t_1 + t_2 > 1$. Recall that

$$\left(\frac{1}{2}, \frac{1}{2}\right) \in S$$

and S is convex. Thus, $\forall \lambda \in (0,1)$,

$$(1-\lambda)\left(\frac{1}{2}, \frac{1}{2}\right) + \lambda t \in S.$$

Note

$$\begin{aligned}
&\left[(1-\lambda)\frac{1}{2} + \lambda t_1\right]\left[(1-\lambda)\frac{1}{2} + \lambda t_2\right] \\
&= \left[\frac{1}{2} + \lambda\left(t_1 - \frac{1}{2}\right)\right]\left[\frac{1}{2} + \lambda\left(t_2 - \frac{1}{2}\right)\right] \\
&= \frac{1}{4} + \frac{\lambda}{2}(t_1 + t_2 - 1) + \lambda^2\left(t_1 - \frac{1}{2}\right)\left(t_2 - \frac{1}{2}\right) > \frac{1}{4}
\end{aligned}$$

for a sufficiently small $\lambda > 0$, because $t_1 + t_2 - 1 > 0$. But, this contradicts the fact that $f^N(S, 0) = (1/2, 1/2)$. □

Chapter 7

Dynamic Monopoly

7.1 Introduction

Monopoly market

7.2 Heuristics

Ronald H. Coase examines a simple monopoly market in which the monopolist is selling out (commercial) land. The important characteristics of the commercial land is durability.

7.3 Rational Expectations

Coasian dynamics

Lemma 3. If $\{p_t\}_{t=1}^{\infty}$ is an optimal pricing rule, then $q_t - q_{t-1} > 0$.

If $q_t = q_{t-1}$, then the monopolist did not make any sales in period t, wasting time. By skipping p_t and offering p_{t+1}, the monopolist can increase profit. Thus,$q_t - q_{t-1} > 0$.

Corollary (Coasian dynamics)**.** In any optimal pricing rule, $p_t > t_{t+1}$.

If $p_t \leq p_{t+1}$, then no consumer will purchase at p_{t+1}, since p_t is lower and offered earlier than p_{t+1}.

7.4 Subgame Perfect Equilibrium

7.5 Examples

Let us consider a market demand function which is not continuously downward sloping. One half of population has reservation value $v = 3$ and the remaining half of the consumers has reservation value $v = 1$.

To calculate the static monopolistic profit maximizing solution, we cannot use the differential calculus to equate the marginal cost to the marginal revenue. We need to rely on basic reasoning.

- The seller will never charge less than 1.
- If above 1, the price must be 3.
- The seller never charges above 3.
- 3 will generate profit of 1.5 while 1 will generate 1.

The monopolist profit maximizing price is 3. the profit is 1.5, serving only the high reservation value consumers. Thus, the allocation is inefficient.

Gap. vs. No gap

The lowest reservation value buyer is 1, which is above the marginal cost 0, in this example. If the lowest reservation value of the buyer is above the marginal cost, then we call it gap case.

In the model of Stokey, the lowest reservation value of a buyer is 0, which is equal to the marginal production. It is called no gap case.

We use the lowest reservation value of the buyer as the competitive benchmark.

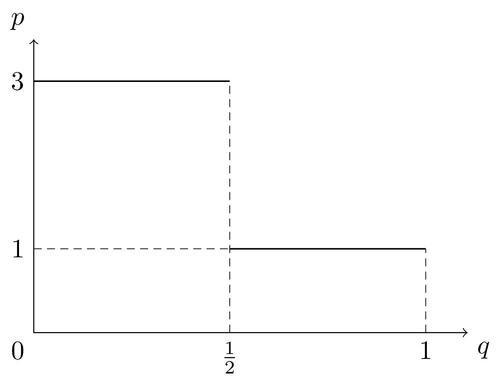


Figure 7.1: Market demand function

Dynamic monopoly problem

Suppose that the monopolist opens the market until all buyers are served, and the monopolist has sufficient capacity to serve all buyers in the market. The time span of each period is Δ and discount rate is $r > 0$, and therefore, the discount factor is $\delta = e^{-\Delta r}$.

Preliminaries

Lemma 4. $\exists T < \infty$ such that all consumers are seved in a subgame perfect equilibrium.

Proof. Suppose that a positive mass, say $q > 0$, of consumers is never served. If so, the monopolist can offer ε so that

$$1 - \varepsilon > \delta(1 - 0).$$

Consumers with reservation value 1 will find it optimal to accept $\varepsilon > 0$ right away, because even if the monopolist offers 0 tomorrow, it is better to accept $\varepsilon > 0$ right away. Thus, the monopolist can recover $\varepsilon q > 0$ profit which wolud have been wasted. □

This is the most important consequence of the gap case. Because the lowest reservation value of the buyer is strictly higher than the marginal production cost, the seller's profit is bounded away from 0.

Terminal round

Lemma 5.

$$p_T = 1$$

If T is the last round when every consumer is served, then $p_T \leq 1$. If $p_T <!$, then the buyer will accept, because T is the last round. Thus, the equilibrium offer must be $p_T = 1$.

Penultimate round

In T round, if any consumer is still active, the consumer must have reservation value 1. If some consumer has reservation value 3, then in period $T - 1$, no reservation value 3 consumer has purchased, and therefore, the monopolist has wasted one round. We know in any equilibrium, the monopolist has to sell a positive amount to consumers. Thus, no consumer with reservation value 3 should be left in period T.

We conclude tat in period T, only the reservation value 1 consumers remain to be served at delivery price $p_T = 1$.

In the penultimate round $T - 1$, the price will be higher than 1 so that only reservation value 3 consumers will be served. Thus, p_{T-1} must satisfy

$$3 - p_{T-1} \geq \delta(3 - p_T) = \delta(3 - 1) = 2\delta.$$

Thus, the highest price which reservation value 3 consumer is willing to accept is

$$p_{T-1} = 3 - 2\delta.$$

Since there are two types of consumers, $T = 2$.

Weak stationary equilibrium

Definition 16 (Weak stationary equilibrium)**.** A subgame perfect equilibrium is a stationary equilibrium, if $\forall h_t, \sigma(h_t)$ depends only upon the residual demand by the end of h_t. A subgame perfect equilibrium is a weakly stationary equilibrium, if $\forall h_t, \sigma(h_t)$ depends only upon the residual demand and the previous round's offer along h_t.

The economy is populated by a unit mass of consumers, each of whom has reservation value $v \in [\underline{v}, \overline{v}]$ and demand one unit of the good. Let

$$F(v) = P(v' \leq v)$$

and therefore,

$$q = 1 - F(p)$$

is the market demand if the seller makes a take-it-or-leave-it offer. As we change p, we can derive the demand, which we call the market demand.

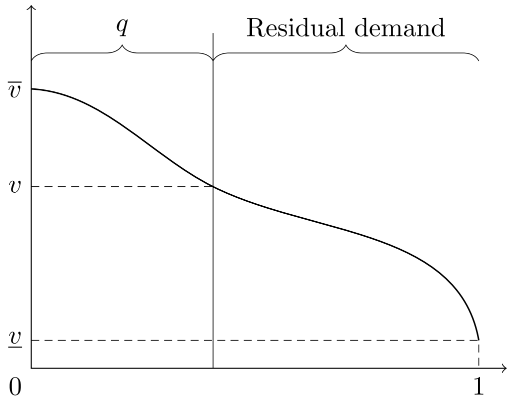


Figure 7.2: Residual demand curve