Choice under Uncertainty

1 Introduction

- Uncertainty is a fact of life. We will study individual behavior with respect to choices involving uncertainty. We will also investigate financial institutions such as insurance markets and stock markets that can mitigate at least some of these risks.
  - We begin by studying the *expected utility* theory.
  - We then study the concept of *risk aversion* and discuss its measurement.
  - Finally, we fit the choice under uncertainty into the standard consumer theory.

2 Expected Utility Theory

2.1 Modelling Uncertainty

- We use **lotteries** to describe risky alternatives.

- Suppose first that the number of possible outcomes is finite.

  - Fix a set of outcomes $C = \{c_1, \ldots, c_N\}$.

  - Let p_n be the probability that outcome $c_n \in C$ occurs and suppose these probabilities are objectively known.

Definition 2.1 (Lottery)**.**

A **(simple) lottery** $L = (p_1, \ldots, p_n)$ is an assignment of probabilities to each outcome c_n, where $p_n \geq 0$ for all n and $\sum_n p_n = 1$.

- The collection of such lotteries can be written as

$$\mathcal{L} = \left\{ (p_1, \ldots, p_N) \,\middle|\, \sum_{n=1}^{N} p_n = 1,\ p_n \geq 0 \text{ for } n = 1, \ldots N \right\}.$$

- We can also think of a **compound lottery** $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$, which allows the outcomes of a lottery to be lotteries.

  – It is immediate to see that any compound lottery can be reduced to a simple lottery defined as above.

  – ex) $C = \{c_1, c_2\}$, $L_1 = (p, 1-p)$, $L_2 = (q, 1-q)$. Then,

$$(L_1, L_2; \alpha, 1-\alpha) = (\alpha p + (1-\alpha)\, q, \alpha\, (1-p) + (1-\alpha)\,(1-q))\,.$$

- Hence, we can only focus on simple lotteries.

- One special and important class of lotteries is **money lotteries**, whose outcomes are real numbers, i.e., $C = \mathbb{R}$.
  - A money lottery can be characterized by a cumulative distribution function F, where $F : \mathbb{R} \longrightarrow [0, 1]$ is nondecreasing.
  - $F(x)$ is the probability of receiving a prize less than or equal to x.
  - That is, if t is distributed according to F, then $F(x) = \text{Prob}\,(t \leq x)$.

2.2 Expected Utility

- If an individual has "reasonable preferences" about consumption in different circumstances, we will be able to use a utility function to describe these preferences just as we do in other contexts.

- However, the fact that we are considering choice under uncertainty adds some special structures to the choice problem, which we will see below.

- Historically, the study of individual behavior under uncertainty is originated from attempts to understand (and hopefully to win) games of chance.

  – One may think that the key determinant of behavior under uncertainty is the expected return of the gamble. However, people are generally reluctant to play fair games.

Example 1 (St. Petersburg Paradox). Consider the following gamble: you toss a coin repeatedly until the head comes up. If this happens in the nth toss, the gamble gives a monetary payoff of 2^n. What is the expected return of this game? How much would you pay to play this gamble?

- Bernoulli(1654-1705)'s solution to this paradox was to argue that people do not care directly about the dollar prizes of a game; rather, they respond to the "utility" these dollars provide. In Daniel Bernoulli's own words:

> The determination of the value of an item must not be based on the price, but rather on the utility it yields. There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.

- If we assume that the marginal utility of wealth declines as wealth increases, then the St. Petersberg game may converge to a finite **expected utility** value that player would be willing to pay for the right to play.
  - ex: $u(x) = \sqrt{x}$.

$$\text{Expected utility} = \frac{1}{2} \times \sqrt{2} + \frac{1}{4} \times \sqrt{4} + \frac{1}{8} \times \sqrt{8} + \cdots = 1 + \sqrt{2}.$$

Definition 2.2.

Let $\mathcal{L}$ be the set of lotteries. The utility function $U : \mathcal{L} \longrightarrow \mathbb{R}$ has an **expected utility form** if there exists an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N)$, we have

$$U(L) = p_1 u_1 + \cdots + p_N u_N.$$

- von Neumann and Morgenstern later showed in their book "The theory of games and economic behavior (1944)" that this hypothesis can be derived from more basic axioms of rational behavior and satisfies

$$L \succsim L' \Longleftrightarrow U(L) = \sum_{n=1}^{N} u_n p_n \geq U(L') = \sum_{n=1}^{N} u_n p'_n.$$

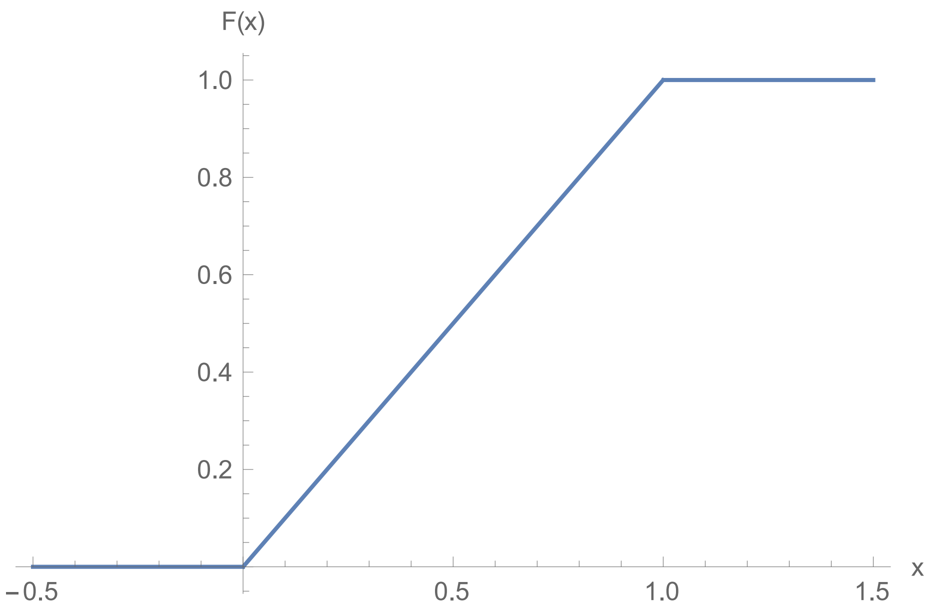
Example 2. Suppose a consumer has some wealth w and is considering investing some amount z in a risky asset. This asset could earn a rate of return r_b in the bad state, or it could earn a rate of return r_g in the good state. The bad state occurs with probability π and the good state occurs with probability $1 - \pi$. What is the expected utility if the consumer decides to invest z dollars?

3 Risk Aversion

- In many economic environments, individuals display aversion to risk.
- We formalize the notion of *risk aversion* and study some of its properties.
- We focus on money lotteries, i.e., risky alternatives whose outcomes are amounts of money. It is convenient to treat money as a continuous variable.
  - We have so far assumed a finite number of outcomes to derive the expected utility representation. How to extend this?

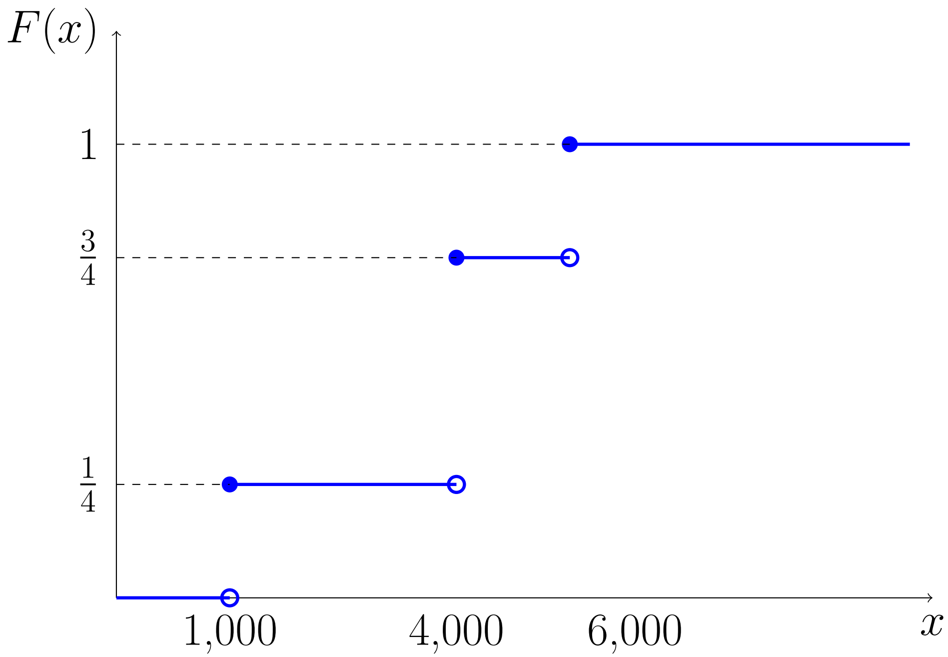
3.1 Expected utility framework on monetary outcomes

- We describe a monetary lottery by means of a *cumulative distribution functions* $F : \mathbb{R} \to [0, 1]$.
  - $F(x)$ is the probability that the realized payoff is less than or equal to x. That is, if t is distributed according to F, then $F(x) = \text{Prob}(t \leq x)$.
  - ex) Uniform distribution $U[0, 1]$



– ex) Discrete distribution:

$$\left.\begin{array}{l}\text{Prob(1,000 won)} = \frac{1}{4} \\ \text{Prob(4,000 won)} = \frac{1}{2} \\ \text{Prob(6,000 won)} = \frac{1}{4}\end{array}\right\} \longrightarrow F(x) = \begin{cases} 0 & \text{if } x < 1,000 \\ \frac{1}{4} & \text{if } 1,000 \le x < 4,000 \\ \frac{3}{4} & \text{if } 4,000 \le x < 6,000 \\ 1 & \text{if } x \ge 6,000. \end{cases}$$



- Consider a preference relation $\succsim$ on $\mathcal{L}$. It has an expected utility representation if

$$F \succsim F' \Longleftrightarrow U(F) \geq U(F'),$$

where

$$U(F) = \int_{-\infty}^{\infty} u(x)\, dF(x)$$

or

$$U(F) = \int_{-\infty}^{\infty} u(x)\, f(x)\, dx$$

if F is differentiable and $f = dF/dx$.

- Note that U is defined on lotteries whereas u is defined on sure amounts of money.

  – We call U the **von Neumann-Morgenstern expected utility function** and $u(\cdot)$ the **Bernoulli utility function.**

  – We assume that u is *(strictly) increasing* and *continuous*.

3.2 Attitude toward risk

Definition 3.1.

Let u be a utility function defined on money outcomes that represents $\succsim$.

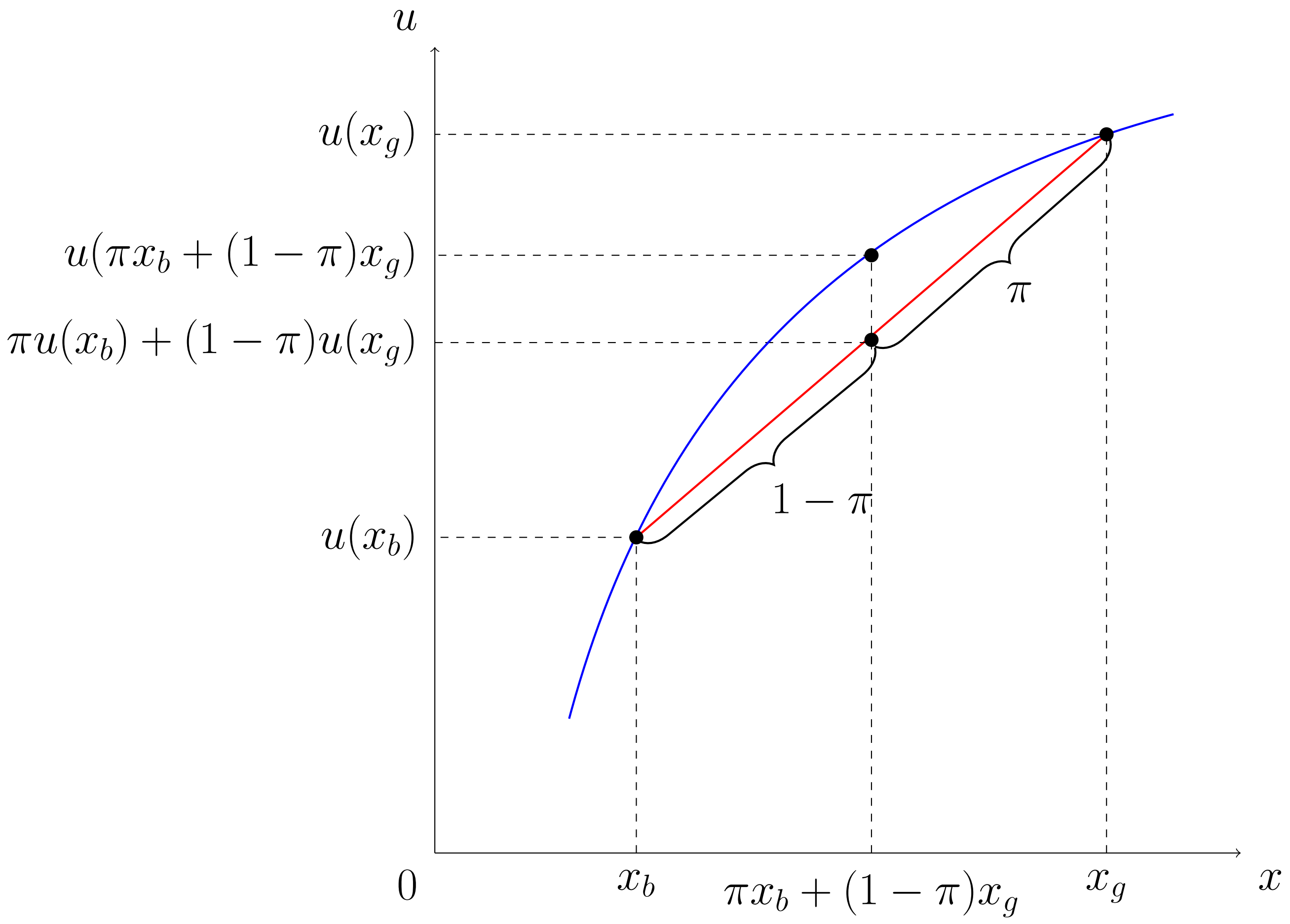
We say that $\succsim$ exhibits

$$\left(\begin{array}{l}\textbf{risk aversion}\\ \textbf{risk neutrality}\\ \textbf{risk loving}\end{array}\right) \Longleftrightarrow \int u(x)\, dF(x) \left(\begin{array}{c}<\\ =\\ >\end{array}\right) u\left(\int x\, dF(x)\right)$$

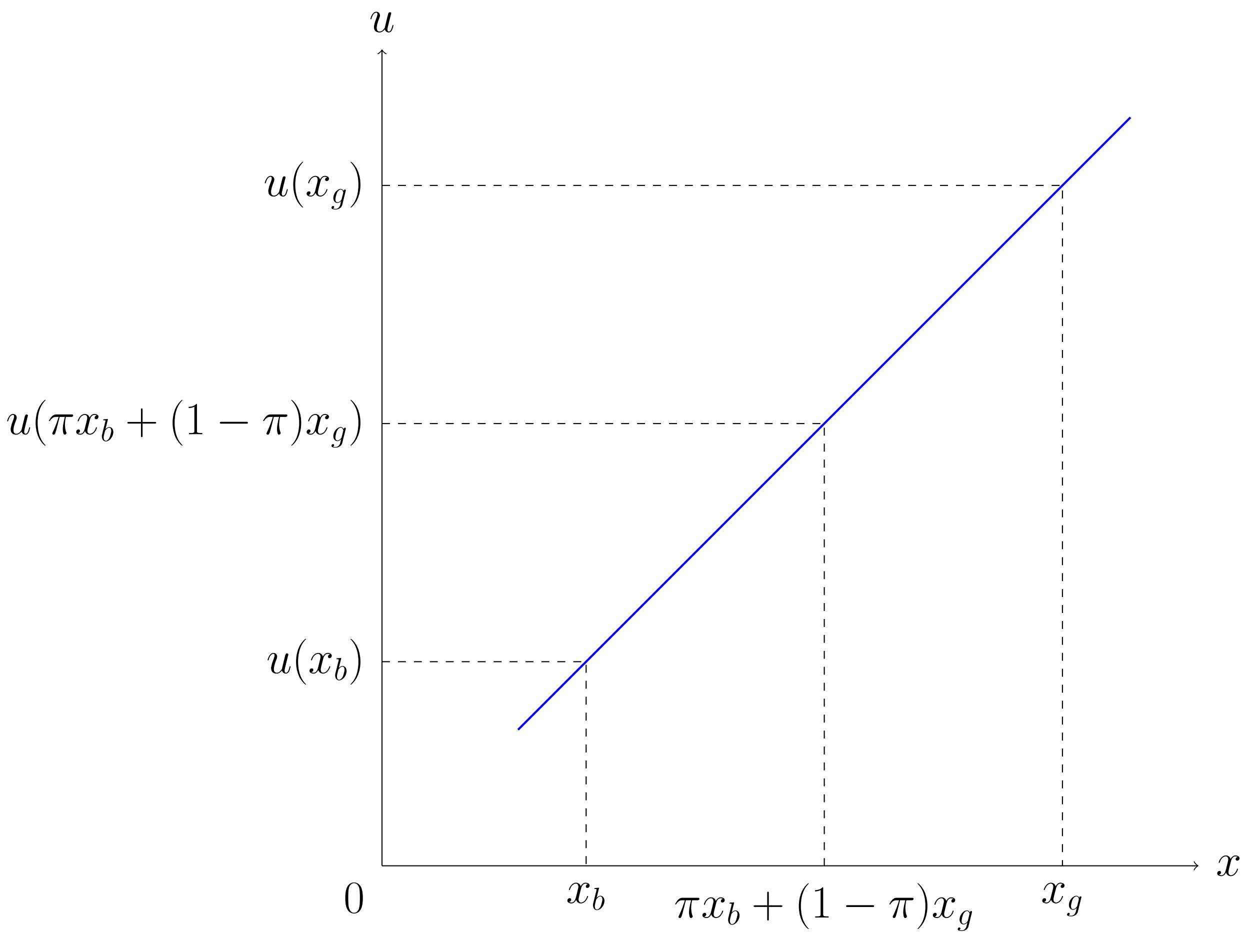
for all lotteries F.

- Equivalently, $\succsim$ exhibits risk aversion if $\mathbb{E}[u(X)] < u(\mathbb{E}[X])$.
- Notice that if $\succsim$ is risk averse (neutral, loving), then u is concave (linear, convex).

- Recall Example 2.
  - Risk aversion: $u(\pi x_b + (1-\pi)x_g) > \pi u(x_b) + (1-\pi)u(x_g)$



– Risk neutrality: $u(\pi x_b + (1-\pi)x_g) = \pi u(x_b) + (1-\pi)u(x_g)$



– Risk loving: $u(\pi x_b + (1-\pi)x_g) < \pi u(x_b) + (1-\pi)u(x_g)$

Example 3 (Continued from Example 2). Suppose the consumer is risk averse. Show that his expected utility is concave in z.

3.3 Certainty equivalent

- A risk averse individual prefers a sure thing to a fair gamble.

- Is there a smaller amount of certain wealth that would be viewed as *equivalent* to the gamble?

Definition 3.2.

The **certainty equivalent** (CE) of F is the amount of money for which the individual is indifferent between the gamble F and the certain amount CE; that is,

$$u\left(CE\right) = \int u\left(x\right) dF\left(x\right).$$

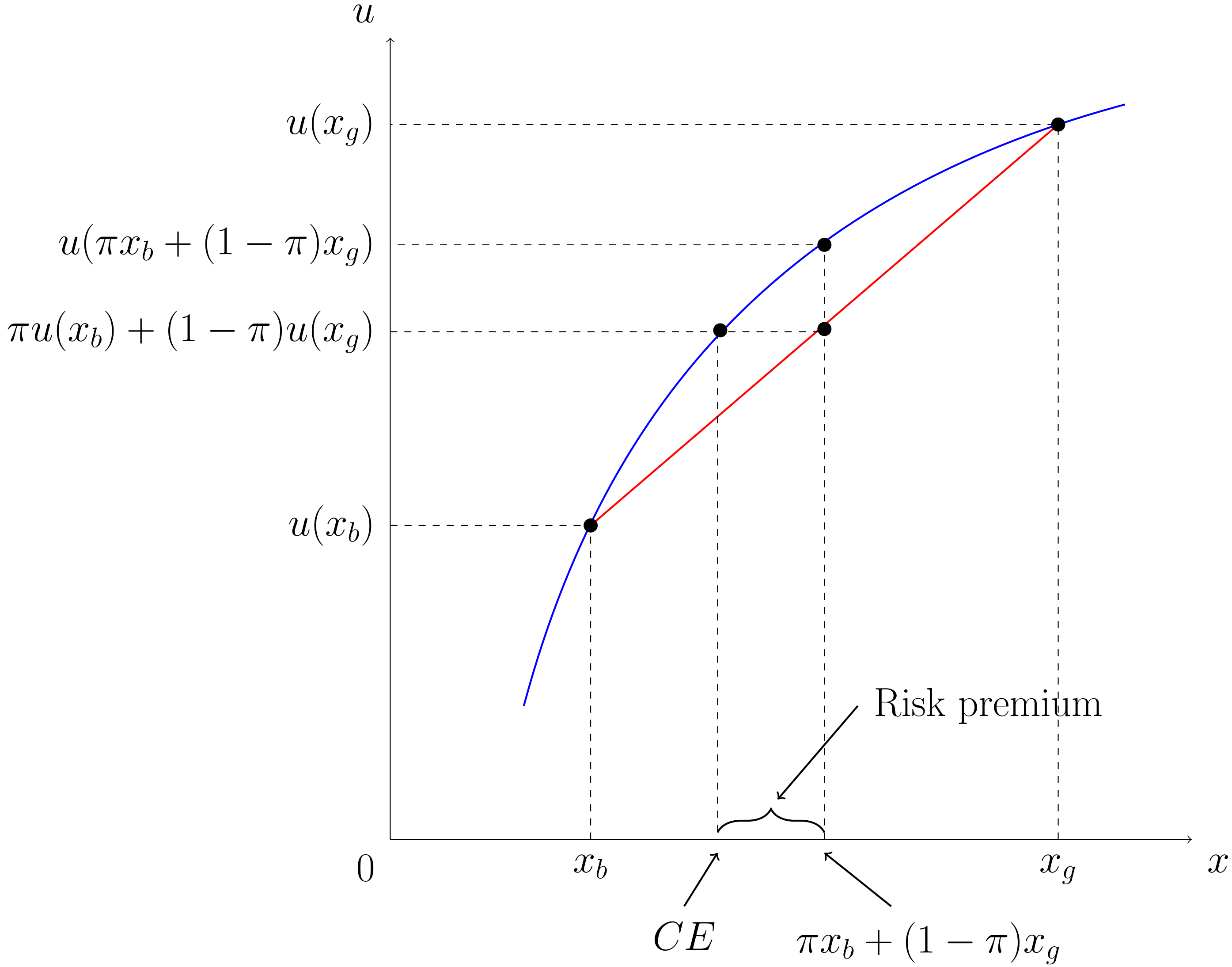


Figure 1: Certainty equivalent and risk premium in Example 2

- Figure 1 exhibits the geometric construction of CE. Note that

$$CE < \pi x_b + (1-\pi)x_g,$$

implying that some expected return is traded for certainty.

- The **risk premium** (RP) associated with F is the maximum amount of money an individual is prepared to pay to avoid the game; that is,

$$\mathbb{E}[u(X)] = u\big(\mathbb{E}[X] - RP\big)$$

Clearly, $RP = \mathbb{E}[X] - CE$.

- It turns out that $\succsim$ exhibits risk aversion if and only if

$$CE \leq \mathbb{E}[X].$$

EXAMPLE 4. In EXAMPLE 2, let $u(x) = \sqrt{x}$, $x_b = 100$, $x_g = 400$ and $\pi = 0.5$. Find the certainty equivalent. What is the risk premium?

4 Measurement of Risk Aversion

- It is convenient to have a measure of risk aversion.
  - Intuitively, the more concave the utility function, the more risk averse the consumer. Thus, the second derivative of u could be used to measure risk aversion.
  - However, this is not invariant to changes in the utility function.
  - E.g., Consumer with $v(\cdot) = 2u(\cdot)$ yields the same behavior as a consumer with $u(\cdot)$; but $v''(\cdot) \neq u''(\cdot)$.
  - If we normalize the second derivative by dividing by the first, we get a reasonable measure.

Definition 4.1 (Arrow-Pratt measure of absolute risk aversion)**.**

$$r_A(x,u) := -\frac{u''(x)}{u'(x)}$$

- Interpreting r_A:

$$-\frac{u''}{u'} = -\frac{du'/dx}{u'} = -\frac{du'/u'}{dx} = -\frac{\text{\% change in MU}}{\text{absolute change in } x}$$

- $r_A(x)$ is positive, negative, or zero as the agent is risk averse, risk loving, or risk neutral.

- Another interpretation: Consider an extension of EXAMPLE 2 in which the consumer can invest z_b for bad state and z_g for good state, separately.

  – Let $x_b = w + r_b z_b$ and $x_g = w + r_g z_g$. Draw indifferent curve:

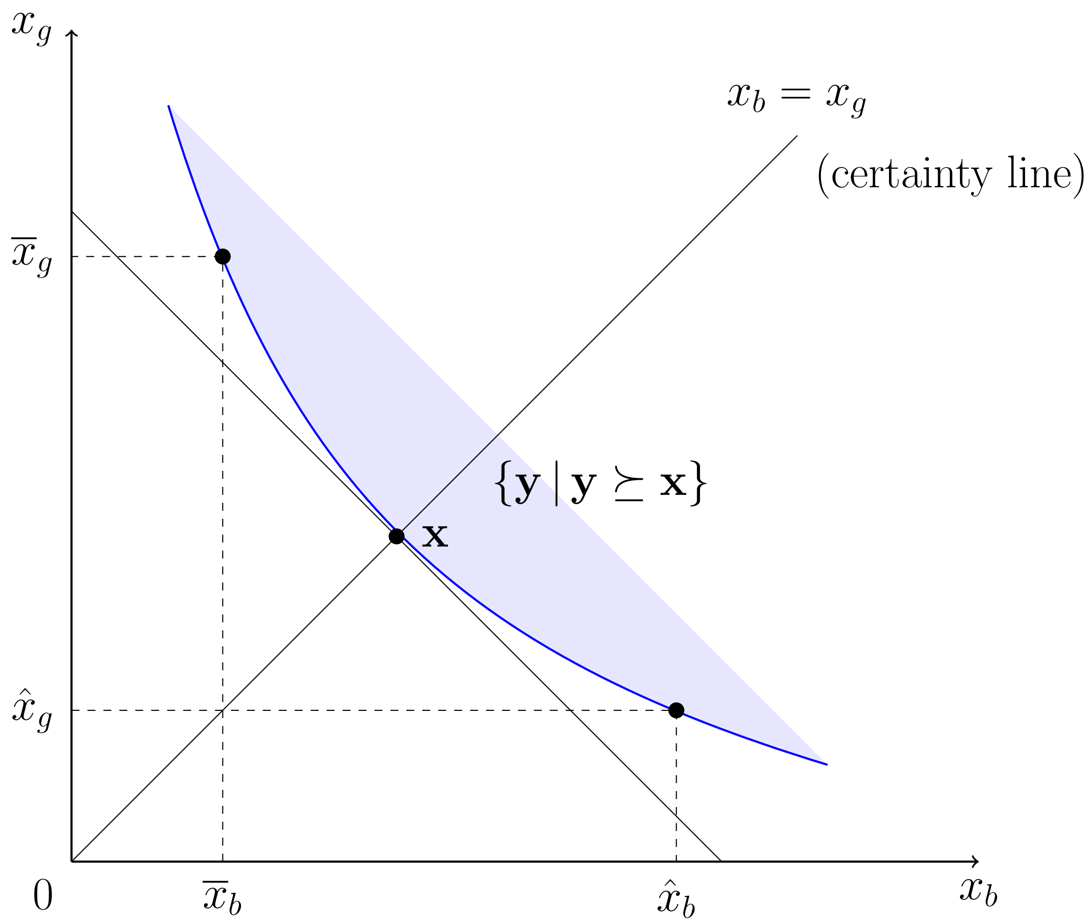
$$\pi u(x_b) + (1-\pi)u(x_g) \equiv U$$

  – By totally differentiating this, we have

$$\pi u'(x_b) + (1-\pi)u'(x_g)\frac{dx_g}{dx_b} = 0. \tag{1}$$

  – Hence, the *marginal rate of substitution* (MRS) is

$$\frac{dx_g}{dx_b} = -\frac{\pi}{1-\pi}\frac{u'(x_b)}{u'(x_g)}. \tag{2}$$



$$\left|\frac{dx_g}{dx_b}\right| \begin{pmatrix} = \\ < \\ > \end{pmatrix} \frac{\pi}{1-\pi} \text{ when } x_b \begin{pmatrix} = \\ > \\ < \end{pmatrix} x_g, \text{ showing that } u(\cdot) \text{ is concave.}$$

- Define the consumers' "preferred set" at $\mathbf{x}$ to be the set of all outcome the consumer will prefer to $\mathbf{x}$, i.e., $\{\mathbf{y} \,|\, \mathbf{y} \succeq \mathbf{x}\}$

- Suppose now we have two consumers, i and j.

  – It is natural to say that consumer i is (locally) more risk averse than consumer j if consumer i's preferred set at $\mathbf{x}$ is *contained* in j's preferred set at $\mathbf{x}$.

  – Consumer i's indifferent curve is "more curved" than consumer j's one at $\mathbf{x}$.

  – Differentiate (1) one more with respect to x_b,

$$\pi u''(x_b) + (1-\pi)u''(x_g)\left(\frac{dx_g}{dx_b}\right)\left(\frac{dx_g}{dx_b}\right) + (1-\pi)u'(x_g)\left(\frac{d^2x_g}{dx_b^2}\right) = 0$$

  – Using (2), we have

$$\frac{d^2x_g}{dx_b^2} = \frac{\pi}{(1-\pi)^2}\left[-\frac{u''(x)}{u'(x)}\right] \text{ when } x_b = x_g = x.$$

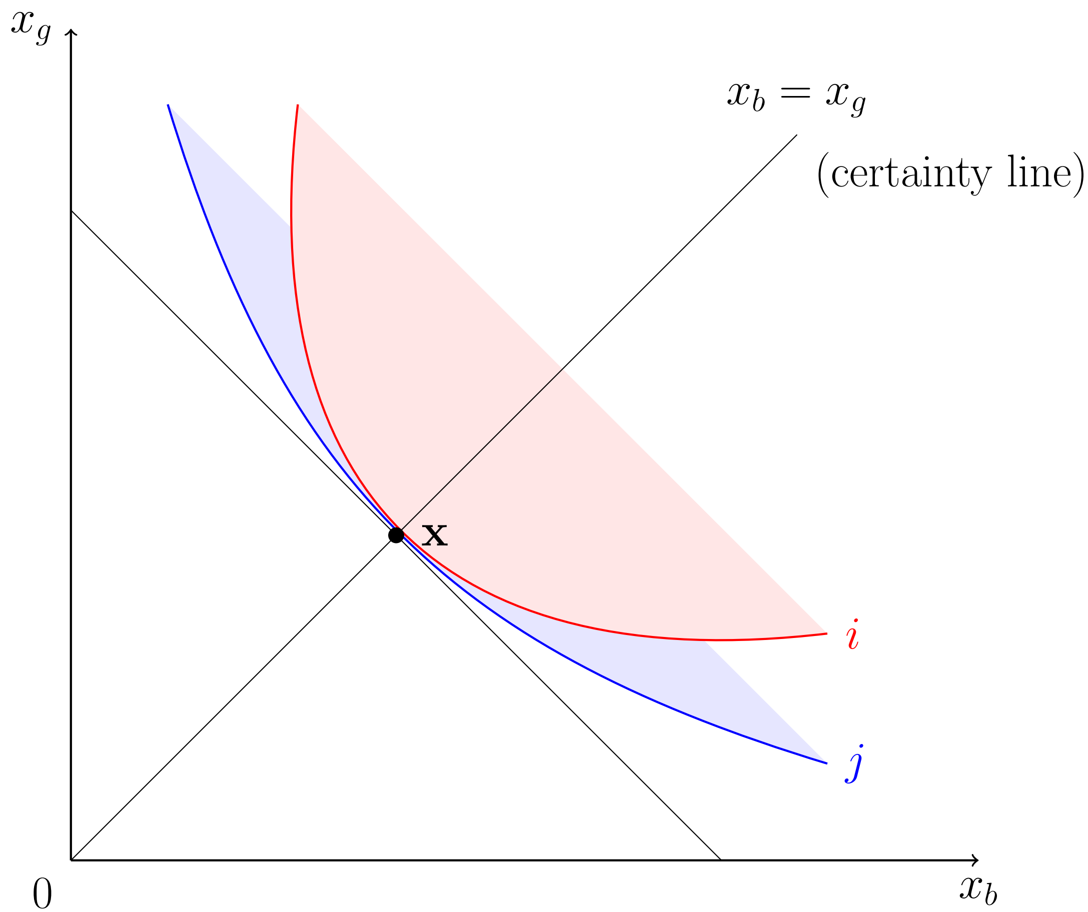


Figure 2: Arrow-Pratt measure of absolute risk aversion

- Given two utility functions $u_i(\cdot)$ and $u_j(\cdot)$, when can we say that $u_i(\cdot)$ is *more risk averse than* $u_j(\cdot)$?

(i) $r_A(x, u_i) \geq r_A(x, u_j)$ for all x. That is, consumer i has a higher degree of risk aversion than consumer j everywhere.

(ii) There exists an increasing concave function $\psi(\cdot)$ such that $u_i(x) = \psi(u_j(x))$ for all x. In other words, $u_i(\cdot)$ is "more concave" than $u_j(\cdot)$.

(iii) $CE_i \leq CE_j$ (or $RP_i \geq RP_j$, i.e., i would be willing to pay more to avoid a given risk than j would.)

- The Arrow-Pratt measure is a sensible interpretation of *local* risk aversion—i.e., one consumer is more risk averse than another *at* $\mathbf{x}$. Hence, it need not be the same at every level of wealth.

- Arrow has proposed a simple classification of Bernoulli utility functions according to how $r_A(x, u)$ varies with x. That is, $r_A(x)$ remains constant, decreases, or increases with x.

  – Constant absolute risk aversion (CARA): the same willingness to take risks.

    E.g., $u(x) = -\alpha e^{-ax} + \beta$, where $a > 0 \Rightarrow r_A(x) = a$.

  – DARA: less averse to taking risks at higher levels of wealth.

    E.g., $u(x) = \alpha + \beta \log x$, where $\beta > 0 \Rightarrow r_A(x) = \frac{1}{x}$.

  – IARA: more averse to taking risks at higher levels of wealth.

    E.g., $u(x) = -(b - x)^c$, where $x \leq b$ and $c > 1 \Rightarrow r_A(x) = \frac{c-1}{b-x}$

5 Applications

5.1 Contingent Commodity

- A contingent commodity is a good that is available only if a particular event (or state of nature) occurs. It specifies conditions under which each contingent becomes available.

- We now treat contingent commodities as different goods. People have preferences over different consumption plan, just like they have preferences over actual consumption.

- If we think of a contingent consumption plan as a consumption bundle, we are in the framework of standard consumer choice model.

- Suppose there are only two states, bad and good, which occur with probability π and $1-\pi$, respectively.

- Denote the consumption in state $i = b, g$ by c_i and

$$U(c_b, c_g) \equiv \pi u(c_b) + (1-\pi)u(c_g).$$

  – The marginal rate of substitution (MRS) at a point (c_b, c_g) is

$$MRS \equiv -\frac{d\,c_g}{d\,c_b} = \frac{\pi}{(1-\pi)}\frac{u'(c_b)}{u'(c_g)}. \tag{3}$$

  – Recall that if $\succsim$ is risk averse, then u is concave. This implies that the indifference curves will be bowed toward the origin (i.e., $\frac{d\,MRS}{d\,c_b} < 0$).

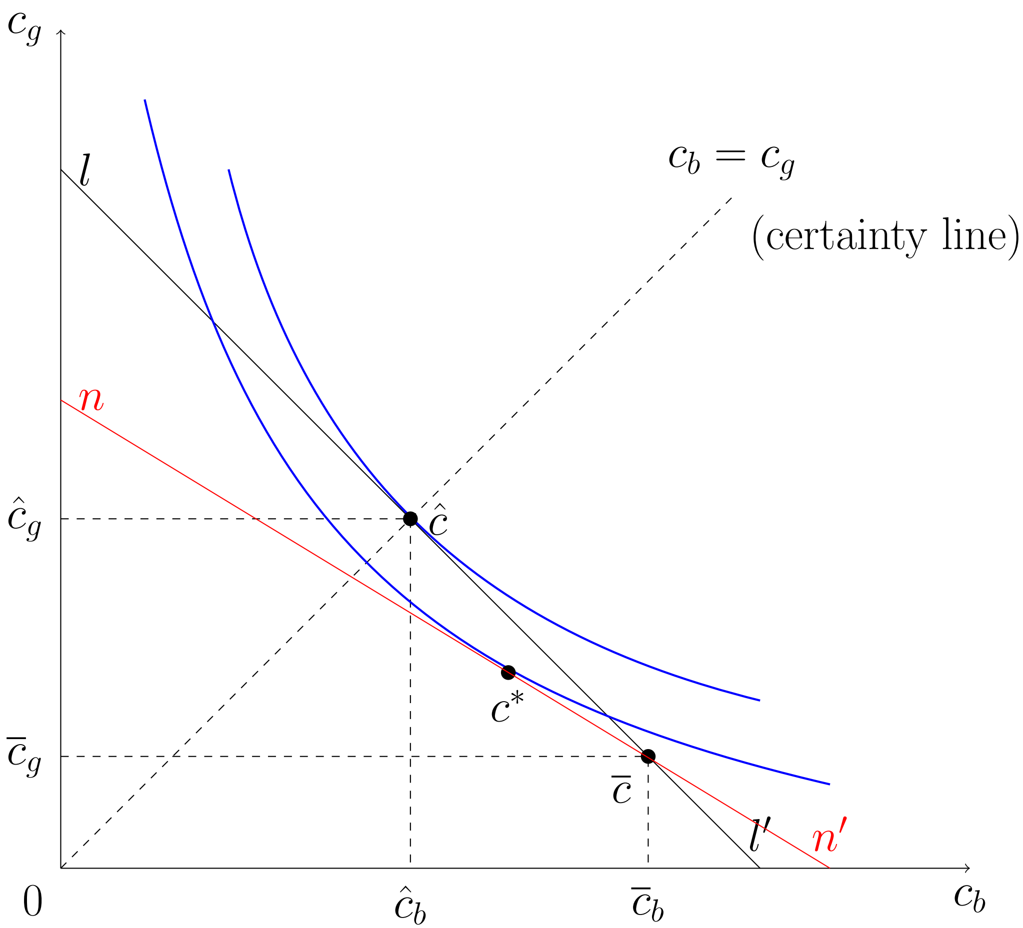


Figure 3: Contingent commodities

- A risk-averse individual would never accept a "fair gamble." That is, the individual would always prefer a sure consequence to any probabilistic mixture of consequences having the same expectation.

  – From (3), the slope of ll' is

$$\frac{dc_g}{dc_b} = -\frac{\pi}{1-\pi}.$$

  – The equation for ll' is $\pi c_b + (1-\pi)c_g = \pi\overline{c}_b + (1-\pi)\overline{c}_g = \hat{c}$.

  – In Figure 3, the line ll' through the point $(\hat{c}_b, \hat{c}_g)$ shows all c_b, c_g combinations having the same expectation.

  – Thus, the certainty of having outcome $\hat{c}$ is preferred to any other (c_b, c_g) with expectation $\hat{c}$.

- Suppose the individual is a price-taker in a market where contingent commodities c_b and c_g can be exchanged in accordance with the price ratio p_b/p_g, which is indicated in Figure 3 by the budget line nn'.
  - The point $\bar{c} = (\bar{c}_b, \bar{c}_g)$ represents the individual's initial endowment.
  - The equation for the budget line nn' is $p_b c_b + p_g c_g = p_b \bar{c}_b + p_g \bar{c}_g$.
  - Maximizing utility subject to the budget constraint leads to

$$\frac{\pi u'(c_b)}{(1-\pi)u'(c_g)} = \frac{p_b}{p_g} \quad \Leftrightarrow \quad \frac{\pi u'(c_b)}{p_b} = \frac{(1-\pi)u'(c_g)}{p_g},$$

    i.e., the marginal utility per dollar of income will be equal in each state.

- If $p_b/p_g = \pi/(1-\pi)$, the market is offering an opportunity to transact fair games.

- If $p_b/p_g \neq \pi/(1-\pi)$, the individual would accept some risk.

5.2 Insurance

- Consider a strictly risk averse individual who has a wealth w and faces damage $D < w$ with probability π.

- Suppose also that he can buy insurance; one unit of insurance costs P (premium) and pays 1 (coverage) if loss occurs.

- He can buy as many units of insurance as he wants as long as he can afford it.

  – If he buys α units of insurance, his wealth in good times is $w_g = w - P\alpha$ and $w_b = w - P\alpha - D + \alpha$ in bad times.

- How many units of insurance would he buy?

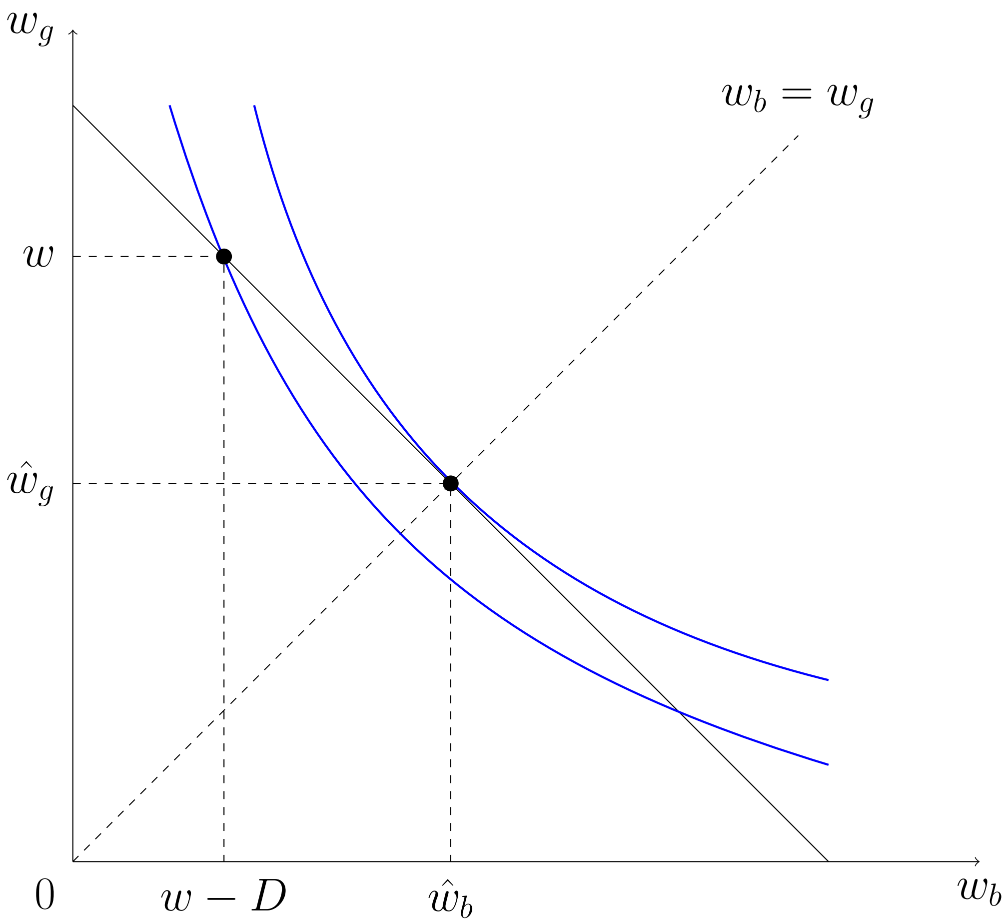


Figure 4: Insurance

- Find the "budget line." What is its slope?

  - Since the budget line goes through $(w_b, w_g) = (w - D, w)$ and $(w_b, w_g) = (w - P\alpha - D + \alpha, w - P\alpha)$, we get

$$w_g = -\frac{P}{1-P} w_b + \frac{w}{1-P} - \frac{P}{1-P} D.$$

- How is the slope of an indifference curve (i.e., MRS) related to π?

  - From $U(w_b, w_g) = \pi u(w_b) + (1-\pi)u(w_g)$, we get

$$MRS = -\frac{\pi\, u'(w_b)}{(1-\pi)\, u'(w_g)}.$$

  - Maximizing utility subject to the budget constraint leads to

$$\frac{\pi u'(w_b)}{(1-\pi)u'(w_g)} = \frac{P}{1-P}$$

- Show that the individual will fully insure if $P = \pi$, i.e., if the insurance is actuarially fair.

  – At the optimal α^*, from

$$\frac{\pi\, u'(w_b^*)}{(1-\pi)\, u'(w_g^*)} = \frac{\pi}{1-\pi}.$$

  where the equality holds since $P = \pi$. Since $u'(w_G^*) = u'(w_B^*)$ and $u'(\cdot)$ is strictly decreasing, we have $w_G^* = w_B^*$, i.e.,

$$w - P\alpha^* = w - P\alpha^* - D + \alpha^*.$$

  or, equivalently, $\alpha^* = D$.

5.3 Portfolio choice problem: Mean-variance analysis

- Suppose there are two lotteries L_1 and L_2.
  - The mean and the variance of L_1 and L_2 are $(\mu_1, 0)$ and (μ_2, σ_2^2), respectively, where $\mu_1 < \mu_2$.
  - L_1 is the *risk-free asset* and L_2 is a *risky asset*.
- Suppose there is an agent who wants invest in L_1 and/or L_2. How much this individual invests in L_1 and L_2?
  - Note that "risk aversion" means that an increase in expected return is a good and an increase in the variance of the return is a bad.

- Assume that the agent's preference can be described by the mean and variance.
  - If a utility function is *quadratic*, e.g., $u(x) = ax^2 + bx + c$,

$$\mathbb{E}[u(X)] = a\mathbb{E}[X^2] + b\mathbb{E}[X] + c = a\big(\mu^2 + \sigma^2\big) + b\mu + c.$$

  - If the risky assets are all normally distributed,

$$\mathbb{E}[u(X)] = \int_{-\infty}^{\infty} u(x) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

- Suppose the individual holds a fraction α of his wealth in L_1 and a fraction $(1-\alpha)$ of his wealth in L_2, and denote such a portfolio by L_α

  – The mean and the variance of L_α is

$$\mu_\alpha = \alpha\mu_1 + (1-\alpha)\mu_2 \quad \text{and} \quad \sigma_\alpha^2 = (1-\alpha)^2\sigma_2^2.$$

  – Since $\mu_\alpha = \alpha\mu_1 + (1-\alpha)\mu_2 = \mu_1 + (1-\alpha)(\mu_2 - \mu_1)$ and $\sigma_\alpha = (1-\alpha)\sigma_2$,

$$\mu_\alpha = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2}\sigma_\alpha.$$

- The portfolio problem becomes:

$$\max_{\mu_\alpha, \sigma_\alpha} u(\mu_\alpha, \sigma_\alpha) \quad s.t. \quad \mu_\alpha = \mu_1 + \frac{\mu_2 - \mu_1}{\sigma_2}\sigma_\alpha$$

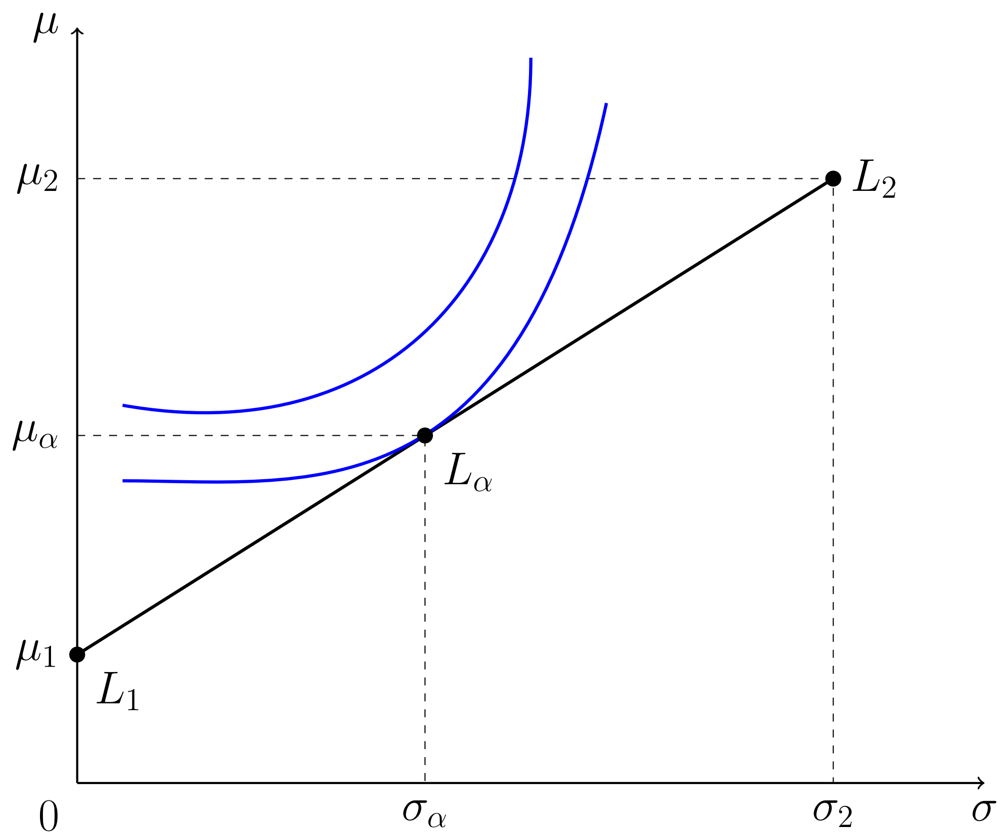


Figure 5: Mean-variance analysis

– At the optimal choice of risk and return, the slope of the indifference curve has to equal the slope of the budget line. This slope, $\frac{\mu_2-\mu_1}{\sigma_2}$, is interpreted as the *price of risk* since it measures how risk and return can be traded off.

EXAMPLE 5. Suppose the risk-free rate of return is 3% and a risky asset is available with a return of 6% and a standard deviation of 3%. What is the price of risk? Suppose you are willing to accept a standard deviation of 2%. What is the maximum rate of return you can achieve?

Hidden Information: Screening

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1 Introduction

- We focus on the basic static adverse selection problem.
- There is a principal facing one agent who has private information on his “type”
  - *Type* represesnets the agent's preference or intrinsic productivity.
- We first study how to solve such problems when the agent can be one of two types, a case that will give us the key insights from adverse selection models.
- General setup
  - An agent, informed party, is privately informed about his type.
  - A principal, uninformed party, designs a contract in order to screen different types of agent and maximize her payoff.
  - This is a problem of hidden information, often referred to as screening problem.

- **Examples**:

  - Price discrimination: monopolist selling to a buyer whose demand is privately known. (‘*second-degree price discrimination*’)

  - Credit rationing in financial markets: lender investing in a project whose profitability and/or risk is privately known.

  - Optimal income taxation: government designing the tax scheme for the people whose income generating ability is privately known.

  - Insurance contract: insurance company offering the insurance package to the insured whose risk is privately known.

  - Implicit labor contract: employer offering the wage contract to an employee whose productivity is privately known.

- In these examples, "contracts" correspond to the pricing scheme, investment decision, tax scheme, and wage contract.

- To describe the contractual situations in general, we adopt the following time line:

  - $t = 0$, the agent learns his type.
  - $t = 1$, the principal offers a contract to the agent.
  - $t = 2$, the agent accepts or rejects the contract. If rejects, the principal and agent get their outside utilities. If accepts, they go to $t = 3$.
  - $t = 3$, the transaction (or allocation) occurs following the contract

- We then ask what is the optimal contract in the principal's perspective (i.e., the contract which maximizes the principal's payoff).

2 A Model of Price Discrimination

- Consider a transaction between a buyer (agent) and a seller (principal).
- The buyer has utility function given by $u(q, T, \theta) = \theta v(q) - T$
  - q is the units purchased by the buyer and T is the amount paid to the seller.
  - The buyer's characteristic is represented by θ, which is only known to the buyer.
  - Assume that $\theta = \theta_L$ with probability $\beta \in (0, 1)$ and $\theta = \theta_H$ with probability $1 - \beta$, where $\theta_L < \theta_H$. Define $\Delta\theta := \theta_H - \theta_L$.
  - Outside utility of the buyer is fixed as $\overline{u} = 0$.
  - Technical assumptions: $v(0) = 0$, $v'(q) > 0$, and $v''(q) < 0$ for all q.
- The seller's preference is given by $\Pi = T - c\,q$, where c is the seller's production cost per unit (constant marginal cost).

3 Full Information Benchmark

- Suppose that the seller is perfectly informed about the buyer's type.

- The seller can treat each type of buyer separately and offer a type-dependent contract: (q_i, T_i) for type θ_i, $i = H, L$.

- The seller solves

$$\max_{(q_i,T_i)_{i=H,L}} T_i - c\,q_i \quad \text{subject to} \quad \theta_i v(q_i) - T_i \geq 0.$$

  - The constraint is called participation constraint or individual rationality (IR) constraint.

  - At the optimal solution, (IR) must be satisfied as an equality or *binding*.

- A solution to this problem is (q_i^F, T_i^F) given by

$$T_i^F = \theta_i v(q_i^F) \quad \text{and} \quad \theta_i v'(q_i^F) = c.$$

  - This quantity is efficient since it maximizes the total surplus:

$$q_i^F = \arg\max_q \; \theta_i v(q) - c\, q \;\text{ for }\; i = H, L.$$

  - With this solution, the seller takes all the surplus equal to

$$\beta(\theta_L v(q_L^F) - c\, q_L^F) + (1-\beta)(\theta_H v(q_H^F) - c\, q_H^F),$$

    while the buyer gets no surplus.

- This solution is called *first-best* or *perfect price discrimination*.

Example 1. Suppose $\theta_L = 4$, $\theta_H = 8$, $c = 2$, and $v(q) = \sqrt{q}$. Then, the first-best menu of contract is $B_H^F := (q_H^F, T_H^F) = (4, 16)$ and $B_L^F := (q_L^F, T_L^F) = (1, 4)$

- It can be implemented by
  - offering type-specific bundles B_H^F and B_L^F; or
  - type-specific membership fee, 8 for θ_H and 2 for θ_L, and uniform price $p = c = 2$.
- The seller's expected profit from is

$$\Pi^F := \beta(T_L^F - c\, q_L^F) + (1-\beta)(T_H^F - c\, q_H^F) = 8 - 6\beta.$$

4 Asymmetric Information

- Suppose from now on that the seller cannot observe the type of the buyer, facing the *adverse selection* problem.
  - The first-best contract above is no longer feasible.
  - The contract set is potentially large since the seller can offer any combination of quantity-payment pair $(q, T(q))$.
- Let us consider two simple functions among others.
  - Linear pricing
  - Two-part tariff

4.1 Linear pricing: $T(q) = Pq$

- The buyer pays a uniform price P for each unit he buys.

- Given this contract, the buyer of type θ_i chooses q_i to maximize

$$\theta_i v(q_i) - Pq_i, \qquad \text{where } i = L, H.$$

  - First-order condition: $\theta_i v'(q_i) = P$.

  - Demand function: $q_i = D_i(P)$, $i = L, H$.

  - Buyer's net surplus: $S_i(P) := \theta_i\, v(D_i(P)) - P\, D_i(P)$ for $i = L, H$.

  - Define

$$D(P) := \beta D_L(P) + (1 - \beta) D_H(P),$$

$$S(P) := \beta S_L(P) + (1 - \beta) S_H(P).$$

- With linear pricing, the seller solves

$$\max_P \ (P-c)D(P).$$

  - First-order condition:

$$P^L = c - \frac{D(P^L)}{D'(P^L)}.$$

  - The buyer obtains positive rents, i.e.,

$$S(P) > 0.$$

  - But, the buyer consumes inefficiently low quantities since

$$\theta_i v'(q_i^L) = P^L > c = \theta_i v'(q_i^F).$$

4.2 Two-part tariff: $T(q) = F + Pq$

- The seller charges a fixed fee (F) up-front, and a price P for each unit purchased.

- Note that for any given price P, the maximum fee the seller can charge up-front is $F = S_L(P)$ if he wants to serve both types.

- The seller chooses P to maximize

$$\beta\left[S_L(P) + (P - c)D_L(P)\right] + (1-\beta)\left[S_L(P) + (P - c)D_H(P)\right] = S_L(P) + (P-c)D(P).$$

  - First order condition:

$$P^T = c - \frac{D(P^T) + S_L'(P^T)}{D'(P^T)}.$$

  - Again, the quantities are inefficiently low: $q_i^T < q_i^F$ for $i = L, H$.

  - But, the inefficiency is reduced: $q_i^T > q_i^L$.

Example 2. Consider Example 1: $\theta_L = 4$, $\theta_H = 8$, $c = 2$, and $v(q) = \sqrt{q}$.

- Linear pricing:
  - The demand function of low- and high-type is

$$D_L(P) = \frac{4}{P^2} \text{ and } D_H(P) = \frac{16}{P^2}.$$

  - The seller's expected profit from P is

$$\Pi^L = \frac{(16 - 12\beta)(P - 2)}{P^2}$$

    so the optimal price is $P^L = 4$.

  - Both types of the buyer would earn positive rents (1 for θ_L and 4 for θ_H), and the seller would earn profit

$$\Pi^L = 2 - \frac{3}{2}\beta.$$

- Two-par tariff

  – When the seller serves both types:

$$\Pi = \underbrace{\frac{4}{P}}_{=S_L(P)} +(P-c)\underbrace{\left[\beta\frac{4}{P^2}+(1-\beta)\frac{16}{P^2}\right]}_{=D(P)} \quad\Rightarrow\quad P^T = \frac{4(4-3\beta)}{5-3\beta}$$

  and so the seller's profit is $\Pi^T_{\text{both}} = \frac{(5-3\beta)^2}{8-6\beta}$

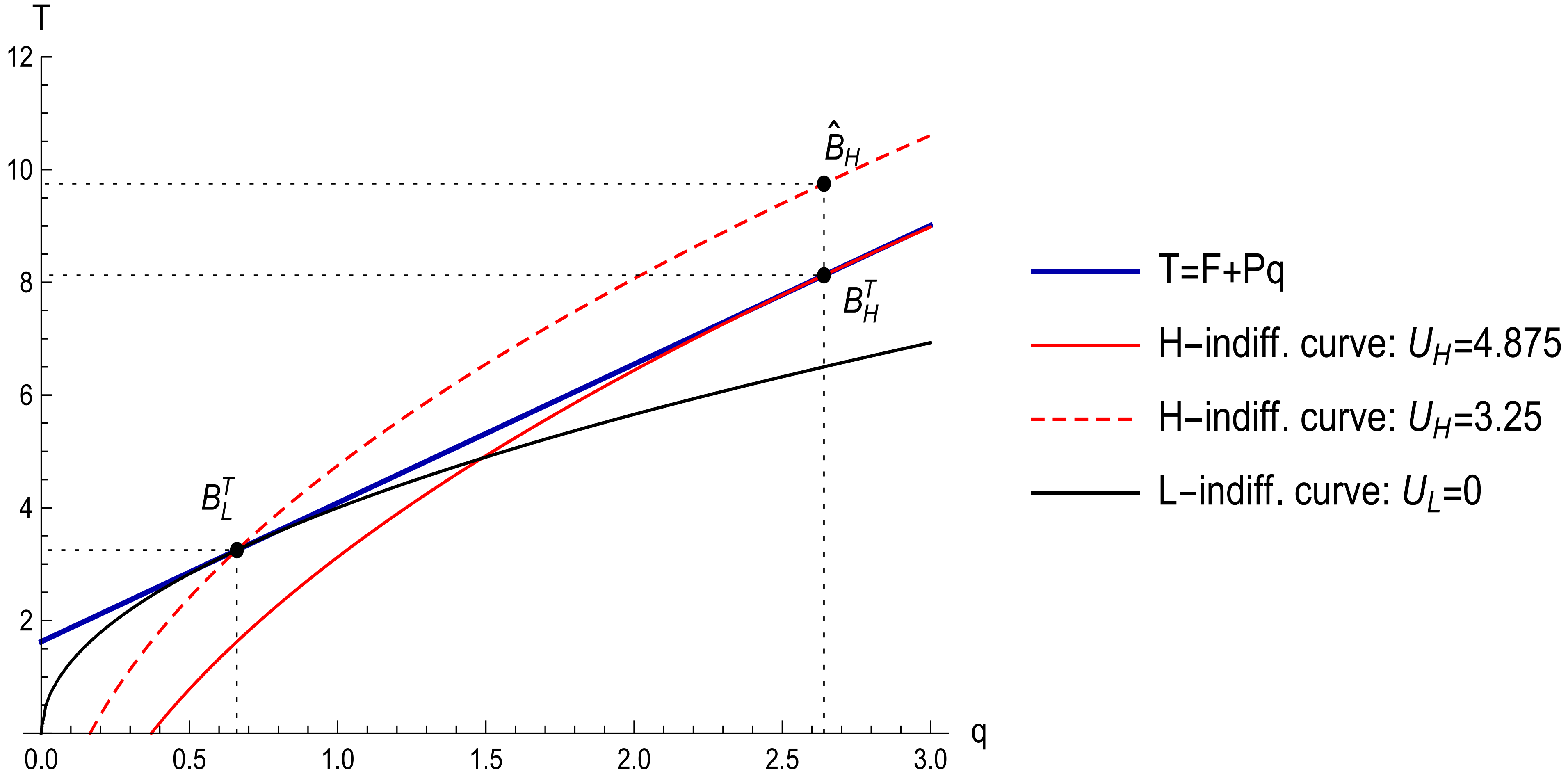
  – When the seller serves only H-type:

$$\Pi = (1-\beta)\left[\underbrace{\frac{16}{P}}_{=S_H(P)} +(P-c)\underbrace{\frac{16}{P^2}}_{=D_H(P)}\right] \quad\Rightarrow\quad P=2$$

  and so the seller's profit is $\Pi^T_H = 8-8\beta$.

  – $\Pi^T_{\text{both}} \geq \Pi^T_H$ if and only if $\beta \geq 0.727$. Otherwise, it is more profitable to exclude the low-type buyer.

- θ_H prefers B_H to B_L. The seller could raise the payment of θ_H to offer $\widehat{B}_H$.
  - Let $\beta = 0.8$. Then, $p = 2.56$ and $F = 1.625$.
  - $B_L^T := (q_L^T, T_L^T) = (0.66, 3.25)$ and $B_H^T := (q_H^T, T_H^T) = (2.64, 8.125)$.



5 Optimal Nonlinear Pricing

- Here, we look for the best pricing scheme among all possible ones. That is, we look for the *second-best* outcome.

- In general, the pricing scheme can be described as $(q, T(q))$, where the function $T(q)$ specifies how much the buyer has to pay for each quantity q.

- We do not restrict the function $T(\cdot)$ to be linear or affine as before.

- *Step 1: Write the seller's problem*

  - Consider any bundle $(q_i, T(q_i))$ and denote $T_i = T(q_i)$ for $i = H, L$.

$$\max_{\substack{(q_L, T_L) \\ (q_H, T_H)}} \beta(T_L - cq_L) + (1-\beta)(T_H - cq_H)$$

subject to

$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L \tag{IC_H}$$

$$\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H \tag{IC_L}$$

$$\theta_H v(q_H) - T_H \geq 0 \tag{IR_H}$$

$$\theta_L v(q_L) - T_L \geq 0. \tag{IR_L}$$

  - The first two constraints are called incentive compatibility constraint, which guarantees that each type selects the bundle that is designed for him.

- *Step 2:* (IR_H) *is automatically satisfied, provided that* (IC_H) *and* (IR_L) *are satisfied.*

  – Given that (IC_H) and (IR_L) hold,

$$\theta_H v(q_H) - T_H \underbrace{\geq}_{(IC_H)} \theta_H v(q_L) - T_L \geq \theta_L v(q_L) - T_L \underbrace{\geq}_{(IR_L)} 0,$$

  so (IR_H) is satisfied.

  – We can thus ignore (IR_H) constraint.

- *Step 3:* (IR_L) *must be binding at the optimal solution.*

  – Why?

$$T_L = \theta_L v(q_L). \tag{1}$$

- *Step 4:* (IC_H) *must be binding at the optimal solution.*

  – Why?

$$T_H = \theta_H v(q_H) - (\theta_H v(q_L) - T_L) = \theta_H v(q_H) - \underbrace{(\theta_H - \theta_L)\, v(q_L)}_{\text{information rent}}. \qquad (2)$$

- *Step 5: Given that* (IC_H) *is binding,* (IC_L) *is satisfied if and only if* $q_H \geq q_L$.

  – Given that (IC_H) is binding, (IC_L) constraint can be written as

$$\theta_L(v(q_L) - v(q_H)) \geq T_L - T_H = \theta_H(v(q_L) - v(q_H)),$$

  which will be satisfied if and only if $q_H \geq q_L$.

  – Thus, we can replace (IC_L) constraint by the constraint $q_H \geq q_L$.

- *Step 6: Eliminate T_L and T_H using (1) and (2) and solve the problem without any constraint.*

  – For a moment, ignore the constraint $q_H \geq q_L$, which will be verified later.

  – Substituting (1) and (2), the seller's problem is turned into

$$\max_{q_L, q_H} \beta\big(\theta_L v(q_L) - cq_L\big) + (1-\beta)\big(\theta_H v(q_H) - cq_H - (\theta_H - \theta_L)v(q_L)\big)$$

  – From the first-order condition, the optimal quantities, q_H^* and q_L^*, solve

$$\begin{aligned} \theta_H v'(q_H^*) &= c \\ \theta_L v'(q_L^*) &= \frac{c}{1 - \frac{(1-\beta)}{\beta}\frac{(\theta_H - \theta_L)}{\theta_L}} > c, \end{aligned} \tag{3}$$

  provided that $1 > \frac{1-\beta}{\beta}\frac{\theta_H - \theta_L}{\theta_L}$ (otherwise, $q_L^* = 0$).

  – This implies that $q_H^* = q_H^F > q_L^F > q_L^*$, as desired.

Example 3. Recall Example 1 again: $\theta_L = 4$, $\theta_H = 8$, $c = 2$ and $v(q) = \sqrt{q}$.

- Write the seller's optimization problem as:

$$\max_{\{q_H, q_L\}} \beta\Big(\underbrace{4\sqrt{q_L}}_{=T_L} - 2q_L\Big) + (1-\beta)\Big(\underbrace{8\sqrt{q_H} - 4\sqrt{q_L}}_{=T_H} - 2q_H\Big)$$

- Taking the derivative w.r.t. q_H and setting it equal to zero, we obtain $q_H^* = 4$.

- Taking the derivative w.r.t. q_L and setting it equal to zero, we obtain

$$q_L^* = \begin{cases} \left(1 - \frac{1-\beta}{\beta}\right)^2 & \text{if } \beta \geq \frac{1}{2} \\ 0 & \text{if } \beta < \frac{1}{2} \end{cases}$$

  That is, if the high type is more likely, the contract with shouting down of θ_L is more profitable to the seller.

- We have disregarded IC_L. So, we have to check if $q_H^* \geq q_L^*$:

$$q_H^* - q_L^* = \begin{cases} 4 - \left(1 - \frac{1-\beta}{\beta}\right)^2 & \text{if } \beta \geq \frac{1}{2} \\ 4 - 0 & \text{if } \beta < \frac{1}{2} \end{cases}$$

- The optimal payment rule follow from the binding conditions:

  - If $\beta \geq \frac{1}{2}$, then

$$T_H^* = 16 - 4 \times \left(1 - \frac{1-\beta}{\beta}\right) \text{ and } T_L^* = 4 \times \left(1 - \frac{1-\beta}{\beta}\right).$$

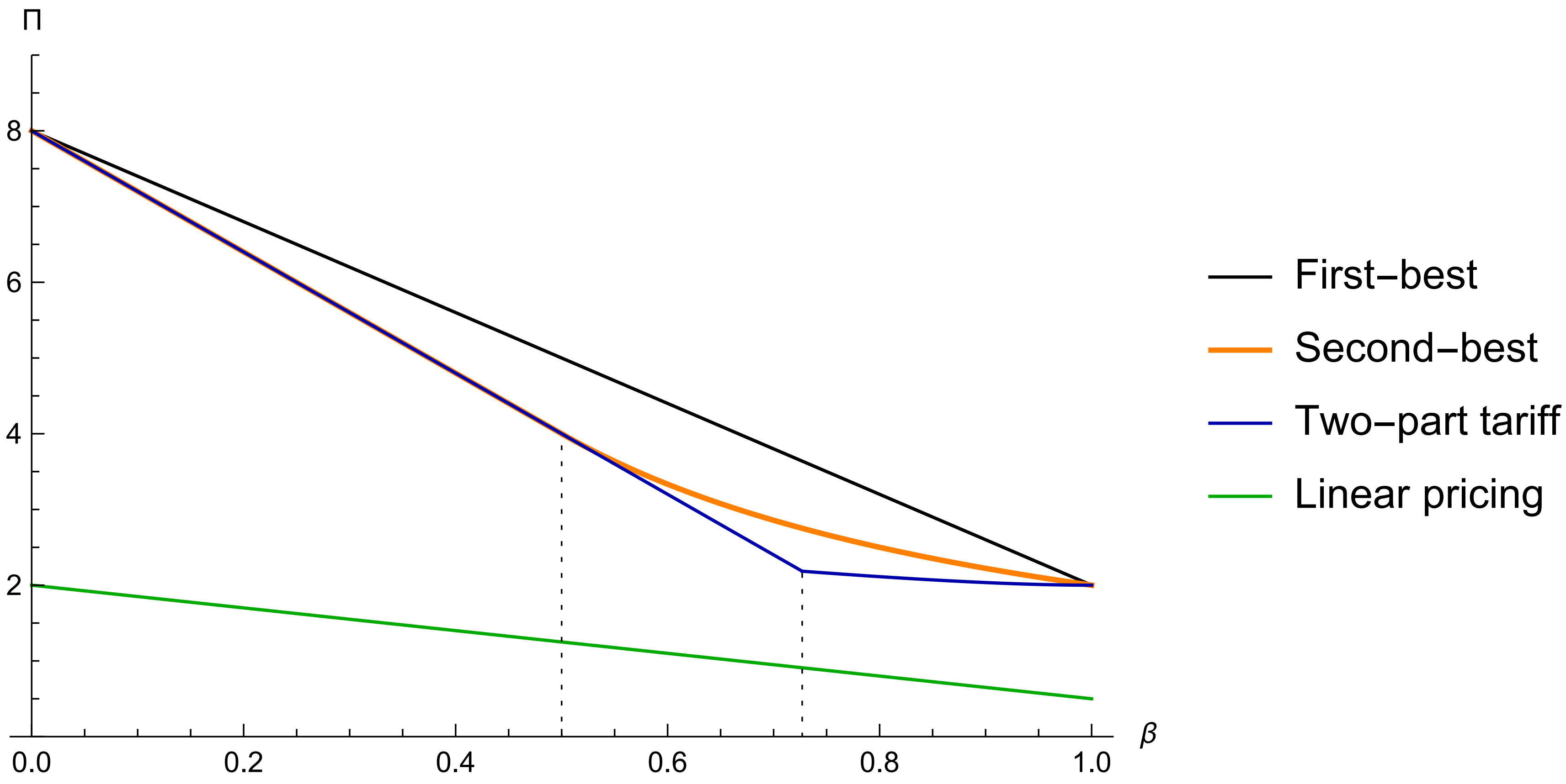
  - If $\beta < \frac{1}{2}$, then $T_H^* = 16$ and $T_L^* = 0$.

- Therefore, the menu of contracts we found is indeed optimal, maximizing the seller's expected payoff given the four constraints.

- The seller's expected payoff from the optimal menu is

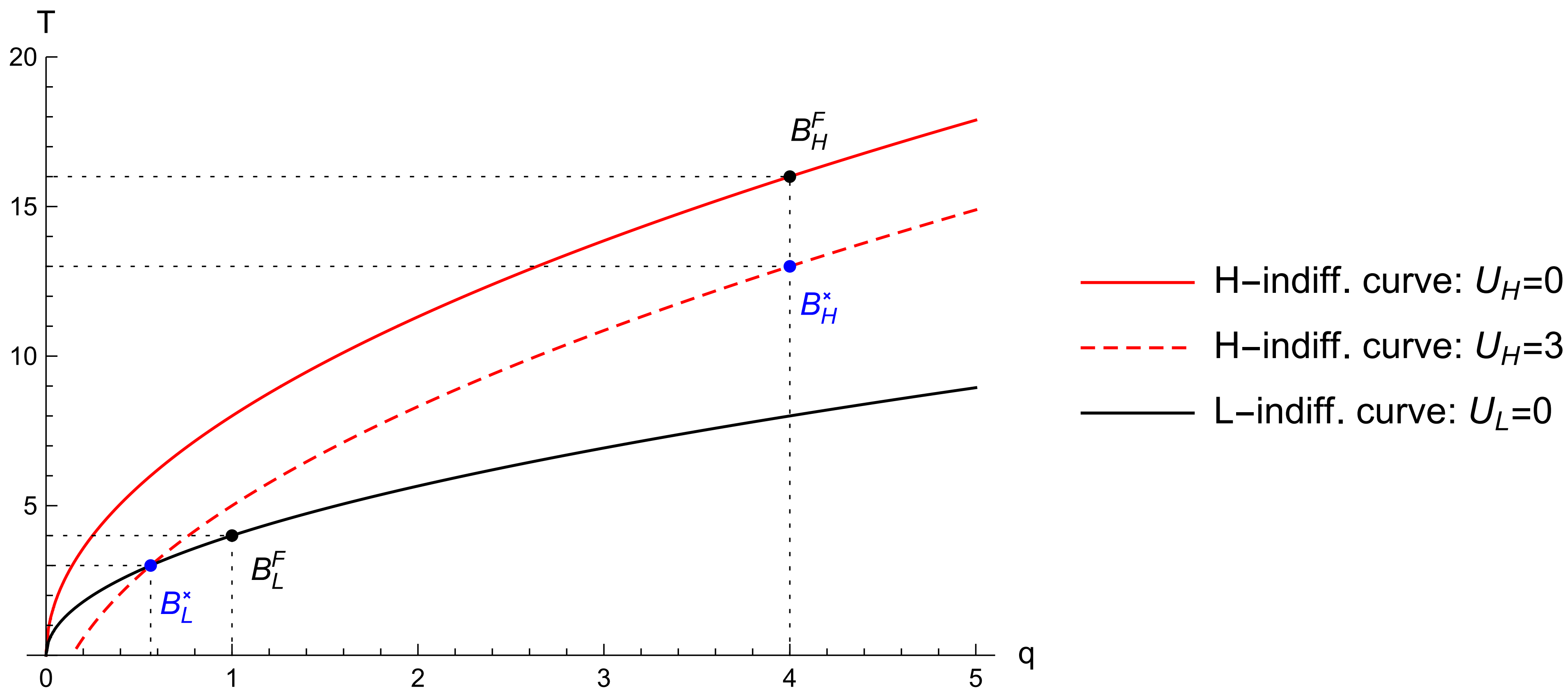
$$\Pi^* = \begin{cases} 4 - 2 \times \left(\frac{2\beta - 1}{\beta}\right) & \text{if } \beta \geq \frac{1}{2} \\ 8(1-\beta) & \text{if } \beta < \frac{1}{2} \end{cases}$$

- Profit comparison

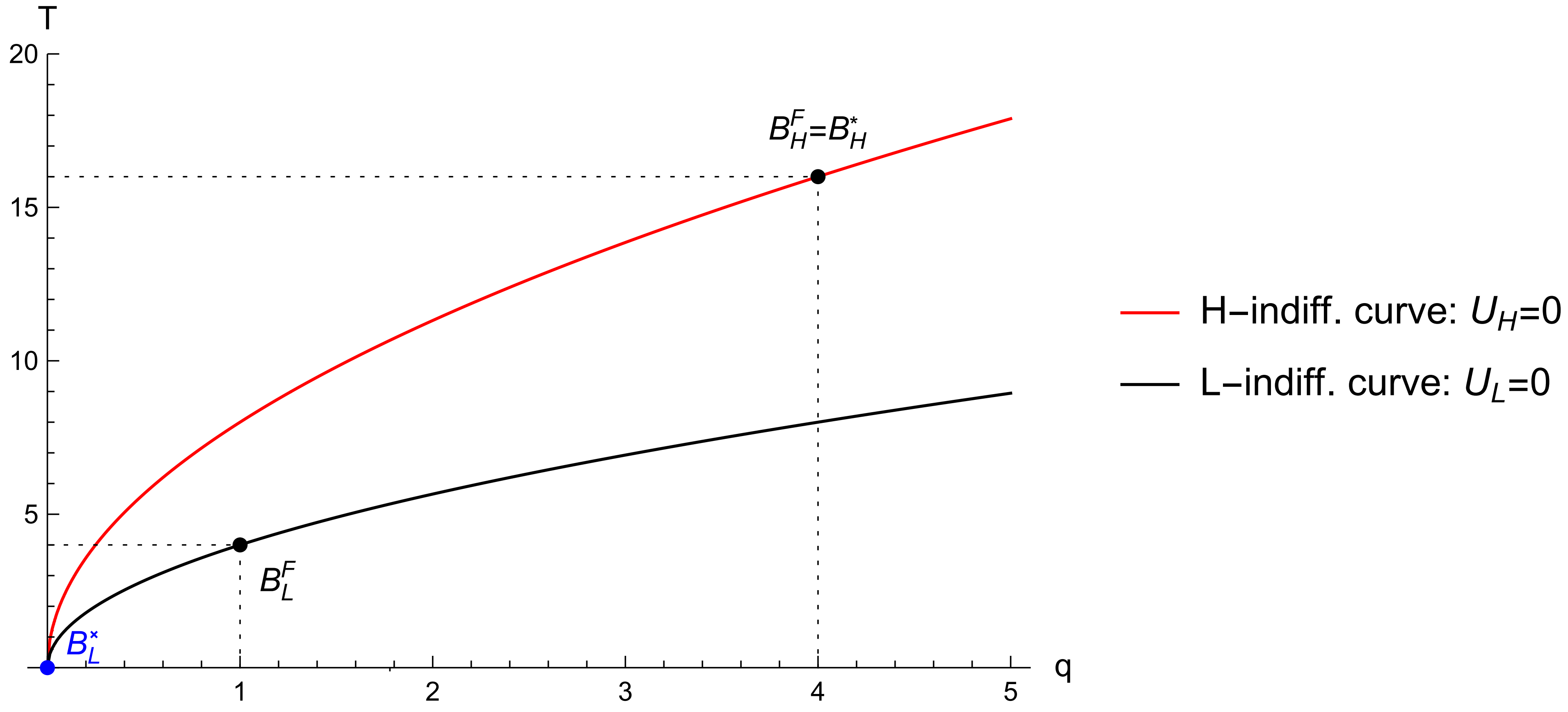


- First-best vs. second-best outcomes

  – Let $\beta = 0.8$. Then, $B_H^* := (q_H^*, T_H^*) = (4, 13)$ and $B_L^* := (q_L^*, T_L^*) = (0.56, 3)$.



– Let $\beta = 0.3$. Then, $B_H^* = (4, 16)$ and $B_L^* = (0, 0)$.



6 Applications

6.1 Regulation

- The public regulators are often subject to an informational disadvantage with respect to the regulated utility or natural monopoly.

- Consider a regulator who is concerned about a monopoly not putting enough effort to maintain its cost at an optimal level.

- The difficulty is that the regulator does not know the firm's intrinsic cost structure.

- The monopoly has an intrinsic cost $\theta \in \{\theta_L, \theta_H\}$ with $\Delta\theta = \theta_H - \theta_L > 0$.

  – The cost is θ_L with probability β and θ_H with $1 - \beta$.

  – The firm's cost of producing good is observable (and contractible) and given by $c = \theta - e$, where $e > 0$ is the cost-reducing effort etailing cost $\psi(e) = e^2/2$.

- The regulator tries to minimize its payment $P = c + s$ to the firm, where s is a "subsidy" to compensate for the cost-reducing effort.

  – The regulator can only observe c while both e and θ are unobservable.

  – Hence, the regulator can require the monopolist to exhibit some cost c, which will force the monopolist to exert effort $e = \theta - c$ if it is of type θ.

- Symmetric information benchmark: for each type $i = H, L$, the regulator solves

$$\min_{(e_i, s_i)} s_i + c_i \ (= s_i + \theta_i - e_i)$$

subject to

$$s_i - e_i^2/2 \geq 0. \tag{IR_i}$$

- Under the asymmetric information, letting $\Delta\theta := \theta_H - \theta_L$, the regulator solves

$$\min_{\substack{(e_L, s_L) \\ (e_H, s_H)}} \beta(s_L - e_L) + (1-\beta)(s_H - e_H)$$

subject to

$$s_L - e_L^2/2 \geq 0 \tag{IR_L}$$

$$s_H - e_H^2/2 \geq 0 \tag{IR_H}$$

$$s_L - e_L^2/2 \geq s_H - (e_H - \Delta\theta)^2/2 \tag{IC_L}$$

$$s_H - e_H^2/2 \geq s_L - (e_L + \Delta\theta)^2/2. \tag{IC_H}$$

6.2 Ex-ante contracting

- There are situations in which the agent can learn his type only after he signs a contract
  - e.g., an employee who is hired to work on a project may not know whether his experties is suited to the project until he starts working on it.
- Assume that the agent privately learns his type between $t = 2$ and $t = 3$.
  - It is at the ex-ante stage that the contracinting occurs.
  - In the origianl model, the contracting occurs at the interim stage in which the agent is already informed of his type.
- Analyze optimal ex-ante contract problem in our basic model of price discrimination.

- The seller's problem:

$$\max_{\substack{(q_L,T_L)\\(q_H,T_H)}} \beta(T_L - cq_L) + (1-\beta)(T_H - cq_H)$$

subject to

$$\beta(\theta_L v(q_L) - T_L) + (1-\beta)(\theta_H v(q_H) - T_H) \geq 0 \tag{IR}$$

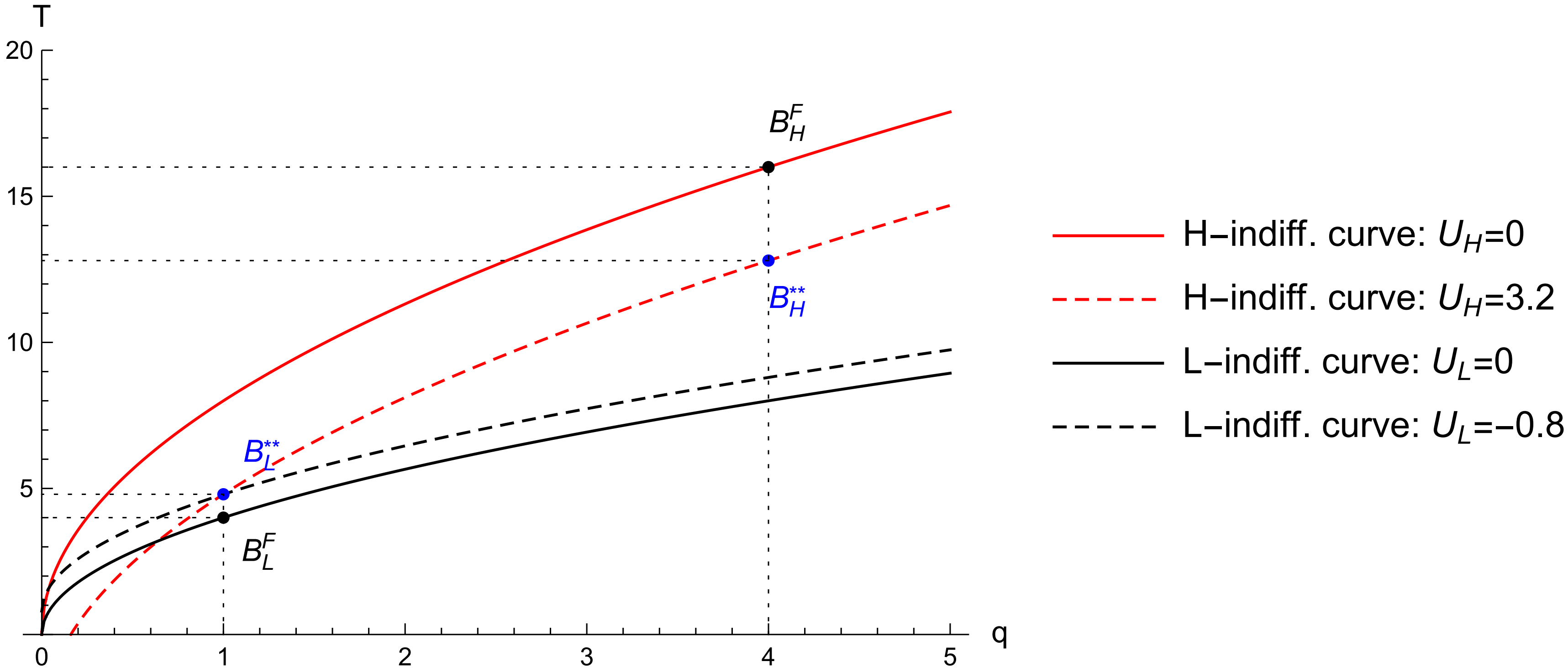
$$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L \tag{IC_H}$$

$$\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H \tag{IC_L}$$

  - (IR) is for the buyer to participate in the contract without knowing his type.
  - (IC_H) and (IC_L) are intact since the information the buyer learns in the post-contracting stage remains to be private.

- The first-best outcome is achievable!
  - Set $(q_L, q_H) = (q_L^F, q_H^F)$ and choose T_L and T_H to make (IR) and (IC_H) binding.
  - Then, (IC_L) is automatically satisfied since $q_H^F > q_L^F$ and (IC_H) is binding.

Example 4. Let $\beta = 0.8$, $\theta_L = 4$, $\theta_H = 8$, $c = 2$ and $v(q) = \sqrt{q}$. Then, $B_L^{**} = (q_L^{**}, T_L^{**}) = (1, 4.8)$ and $B_H^{**} = (q_H^{**}, T_H^{**}) = (4, 12.8)$



- Obervations
  - The seller pushes down L-type payoff below outside utility and guarantees H-type a payoff above outside utility.
  - The buyer without knowing his type breaks even on average and thus is willing to participate.
- Lessons
  - What generates a rent for the agent is the asymmetric information at the contracting stage.
  - It is not the lack of information itself but the asymmtery that causes the inefficiency in the previous section.

Hidden Action: Moral Hazard

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1 Introduction

- We have discussed **screening** problem
  - The uninformed party combats the problem of adverse selection by screening the other.
- We now discuss another class of asymmetric information problem.
  - Asymmetric information arises from imperfect monitoring of players' actions (or **hidden action**).

- Basic setup:
  - Agent takes an action that is not observable nor verifiable and thus cannot be contracted upon.
  - The agent's action is costly to himself but benefits the principal.
  - The agent's action generates a random outcome, which is both observable and verifiable.
  - Principal designs a contract based on the outcome.
- We refer this kind of setup as **moral hazard** problem.

- **Examples:**

  - In a financial market, the borrower might invest in a risky project with is undesirable form the lender's point of view, because the project can reduce the chance of getting back a loan from the borrower.

  - After buying an insurance for one asset, the insurant may have less incentive to take care against risks, so he may be more liable to lose things.

  - In most major companies, shareholders hire a CEO and delegates several management task. However, the CEO may often make a corporate decisions for her own interest, not for shareholders' interest.

- We are interested in
  - How to design a contract which gives agent an incentive to participate (*participation constraint*) and choose the "right" action (*incentive constraint*).
  - When and how the unobservability of agent's action distorts the contract away from the first-best.
- Time-line:
  - At date 0, principal offers a contract.
  - At date 1, agent accepts or rejects the contract. If rejects, both pricipal and agent get their outside utilities, zero. If accepts, they proceed.
  - At date 2, agent chooses an action.
  - At date 3, outcome is realized.
  - At date 4, the contract is executed.

2 Binary Model

- Suppose there is an employer (principal) and an employee (agent)

- Agent could shirk ($e = 0$ or low effort) or work hard ($e = 1$ or high effort), which is not observable by the principal.

  - The level of production (outcome), which is observable and verifiable, is stochastic, taking two values q_H with probability π_e and q_L with the remaining probability.

  - $q_H > q_L$ and $\pi_1 > \pi_0$.

	q_H	q_L
$e = 0$	π_0	$1 - \pi_0$
$e = 1$	π_1	$1 - \pi_1$

- Agent's utility when exerting e and paid t: $u(t) - c\, e$
  - Assume $u(0) = 0$, $u' > 0$ and $u'' < 0$.
  - The agent's outside option is u_0.
- Principal's utility: $S(q) - t$, where $q = q_H, q_L$ and $S(\cdot)$ is linear.
  - The principal can only offer a contract based on q.
  - Denote $t(q)$ with $t_H := t(q_H)$ and $t_L := t(q_L)$.
  - Let $S_H := S(q_H)$ and $S_L := S(q_L)$.

- If agent exerts e, then principal receives the expected utility

$$V_e = \pi_e(S_H - t_H) + (1 - \pi_e)(S_L - t_L).$$

- The principal induces a high effort ($e = 1$) if

$$\pi_1 u(t_H) + (1 - \pi_1)u(t_L) - c \geq \pi_0 u(t_H) + (1 - \pi_0)u(t_L)$$
$$\pi_1 u(t_H) + (1 - \pi_1)u(t_L) - c \geq u_0.$$

- The principal induces a low effort ($e = 0$) if

$$\pi_1 u(t_H) + (1 - \pi_1)u(t_L) - c \leq \pi_0 u(t_H) + (1 - \pi_0)u(t_L)$$
$$u_0 \leq \pi_0 u(t_H) + (1 - \pi_0)u(t_L)$$

3 First-Best Contract

- In this benchmark, assume that the effort level is observable and verifiable.

- If principal wants to induce e, then he solves

$$\max_{t_H, t_L} \quad \pi_e(S_H - t_H) + (1 - \pi_e)(S_L - t_L)$$

subject to

$$\pi_e u(t_H) + (1 - \pi_e)u(t_L) - c\, e \geq u_0. \tag{IR}$$

- Set up the Lagrangian function

$$\mathcal{L} = \pi_e(S_H - t_H) + (1-\pi_e)(S_L - t_L) + \lambda\big[\pi_e u(t_H) + (1-\pi_e)u(t_L) - c\,e - u_0\big].$$

  – From the first-order condition,

$$-\pi_e + \lambda \pi_e u'(t_H^F) = 0,$$
$$-(1-\pi_e) + \lambda(1-\pi_e)u'(t_L^F) = 0.$$

  – We thus have

$$\lambda = \frac{1}{u'(t_H^F)} = \frac{1}{u'(t_L^F)},$$

  implying that $t_H^F = t_L^F \equiv t^F$. By (IR), $t^F = u^{-1}(c\,e + u_0)$.

  – The risk-neutral principle offers a **full insurance** to the risk-averse agent and then extracts the full surplus.

- Principal prefers $e = 1$ if

$$\pi_1 S_H + (1-\pi_1) S_L - u^{-1}(c + u_0) \geq \pi_0 S_H + (1-\pi_0) S_L - u^{-1}(u_0)$$

or

$$\underbrace{(\pi_1 - \pi_0)(S_H - S_L)}_{\text{expected gain of effort}} \geq \underbrace{u^{-1}(c+u_0) - u^{-1}(u_0)}_{\text{cost of incuding effort}}$$

- Otherwise, principal prefers $e = 0$.

Example 1. Suppose that

$$\pi_0 = 0.4,\ \pi_1 = 0.8,\ q_L = 50,\ q_H = 250,\ u_0 = 9,\ S(q) = q \ \text{ and } \ u(t) = \sqrt{t}.$$

To induce $e = 0$, the principal offers $t_H = t_L = 81$.

$$V_0 = 0.4 \times 250 + 0.6 \times 50 - 81 = 49.$$

Let $c = 1$. To induce $e = 1$, the principal offers $t_H = t_L = 100$.

$$V_1 = 0.8 \times (250 - 100) + 0.2 \times (50 - 100) = 110 > V_0.$$

Hence, it is optima to indcue $e = 1$ for the principal.

Let $c = 4$. To induce $e = 1$, the principal offer $t_H = t_L = 169$.

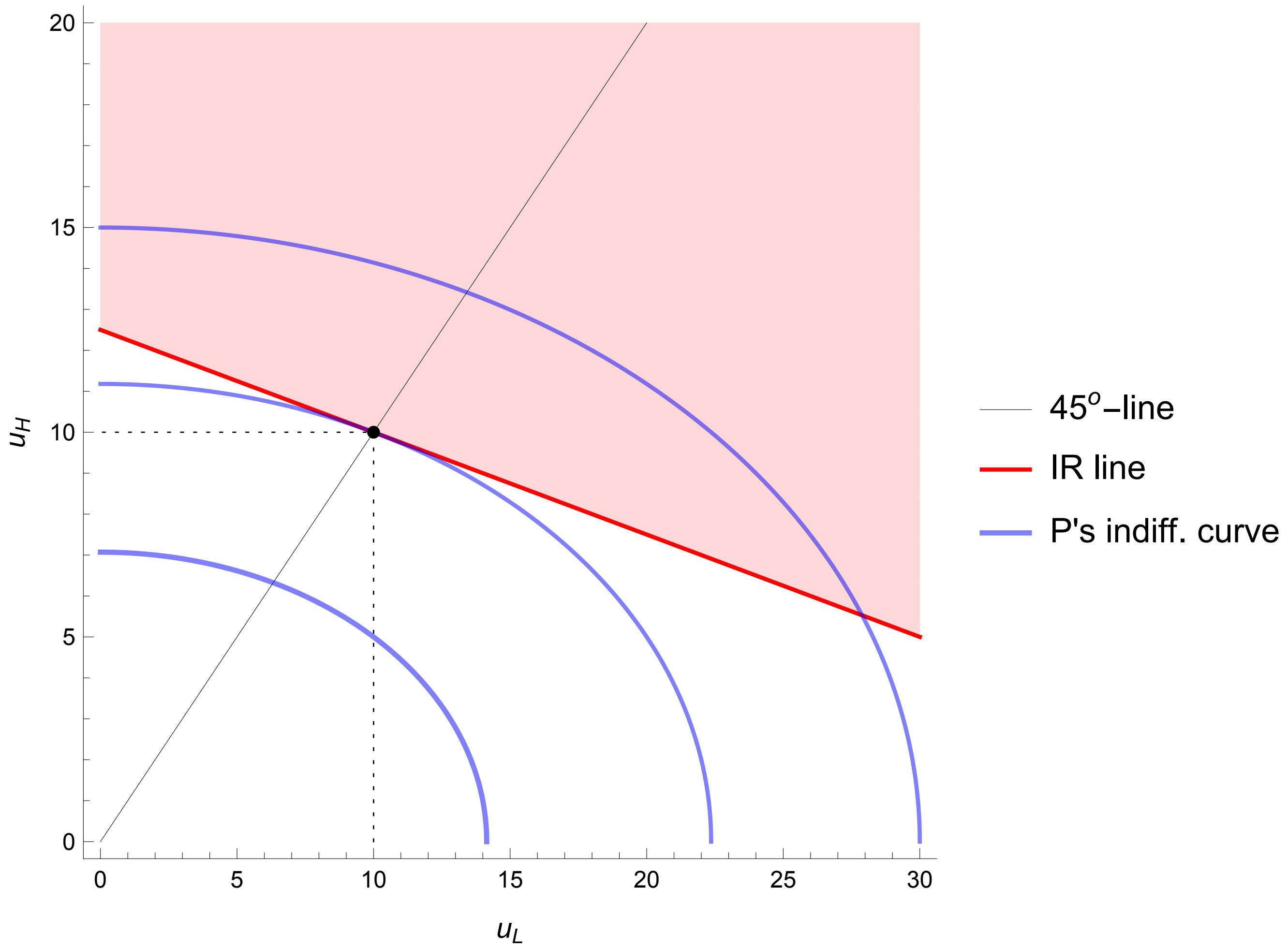
$$V_1 = 0.8 \times (250 - 169) + 0.2 \times (50 - 169) = 41 < V_0,$$

Inducing $e = 1$ is too costly, so $e = 0$ is optimal for the principal.

Denote $u_H = \sqrt{t_H}$, $u_L = \sqrt{t_L}$. Then,

$$V_1 = 0.8 \times (250 - u_H^2) + 0.2 \times (50 - u_L^2)$$

$$(\text{IR}): \; 0.8u_H + 0.2u_L - 1 \geq 9.$$



4 Second-Best Contract

- Assume that the effort exerted by the agent is unobservable.

- The principal's problem to induce $e = 1$ is

$$\max_{t_H, t_L} \; \pi_1(S_H - t_H) + (1 - \pi_1)(S_L - t_L)$$

subject to

$$\pi_1 u(t_H) + (1 - \pi_1)u(t_L) - c \geq \pi_0 u(t_H) + (1 - \pi_0)u(t_L) \tag{IC}$$

$$\pi_1 u(t_H) + (1 - \pi_1)u(t_L) - c \geq u_0 \tag{IR}$$

4.1 Optimal Incentive Scheme

- Set up the Lagrangian function

$$\mathcal{L} = \pi_1(S_H - t_H) + (1-\pi_1)(S_L - t_L) + \lambda\Big[(\pi_1 - \pi_0)\big(u(t_H) - u(t_L)\big) - c\Big] + \mu\Big[\pi_1 u(t_H) + (1-\pi_1)u(t_L) - c - u_0\Big].$$

- Two first-order conditions

$$\frac{\partial \mathcal{L}}{\partial t_H} = 0 \;\Rightarrow\; \frac{1}{u'(t_H^*)} = \mu + \lambda\left[1 - \frac{\pi_0}{\pi_1}\right], \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial t_L} = 0 \;\Rightarrow\; \frac{1}{u'(t_L^*)} = \mu + \lambda\left[1 - \frac{1-\pi_0}{(1-\pi_1)}\right]. \tag{2}$$

- Show that both λ and μ are positive so that (IC) and (IR) are both binding.

  - From (1) and (2),

$$\mu = \frac{\pi_1}{u'(t_H^*)} + \frac{1-\pi_1}{u'(t_L^*)} > 0.$$

  - If $\lambda = 0$, then (1) and (2) imply $t_H^* = t_L^*$, which violates (IC).

- From (IC) and (IR) as equality, we obtain

$$t_H^* = u^{-1}\left(u_0 + \frac{1-\pi_0}{\pi_1 - \pi_0}c\right) \quad \text{and} \quad t_L^* = u^{-1}\left(u_0 - \frac{\pi_0}{\pi_1 - \pi_0}c\right). \tag{3}$$

- Comparing with the first-best contract, we have

$$t_H^* > t^F > t_L^*.$$

implying that there is a **trade-off** incuding effort and providing insurace to the agent.

Example 2. Suppose that

$$\pi_0 = 0.4,\ \pi_1 = 0.8,\ q_L = 50,\ q_H = 250,\ c = 1,\ u_0 = 9,\ S(q) = q \text{ and } u(t) = \sqrt{t}.$$

Suppose (t_H, t_L) implements $e = 1$. If it was optimal, both constraints must bind:

$$0.8\sqrt{t_H} + 0.2\sqrt{t_L} - 1 = 0.4\sqrt{t_H} + 0.6\sqrt{t_L},$$
$$0.8\sqrt{t_H} + 0.2\sqrt{t_L} - 1 = 9.$$

Optimal contract is $t_H^* = 110.25$ and $t_L^* = 64$. Thus,

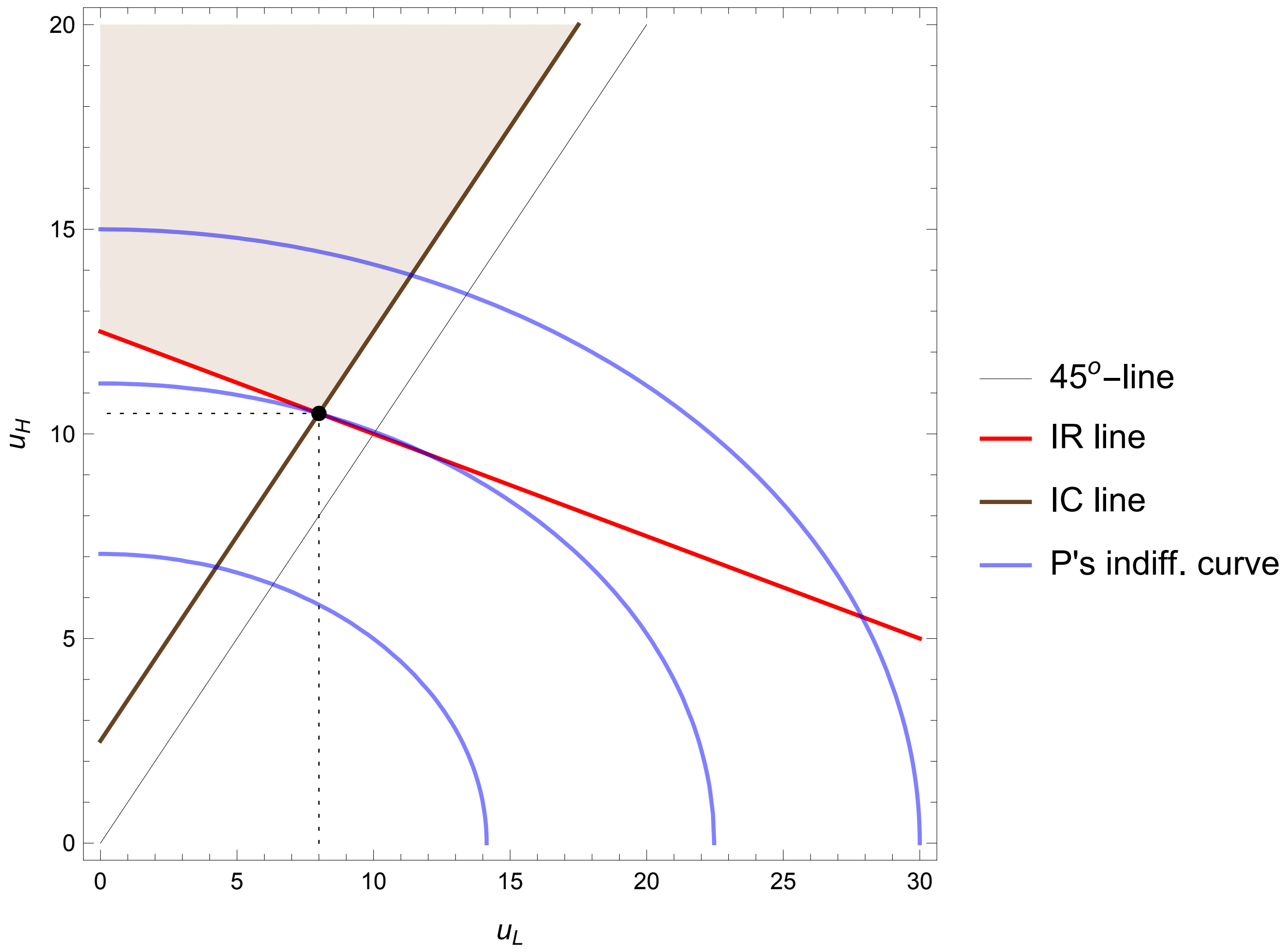
$$V_1^* = 0.8 \times (250 - 110.25) + 0.2 \times (50 - 64) = 109.$$

Suppose (t_H, t_L) implements $e = 0$. Then, $t_H = t_L = 81$ and

$$V_0 = 0.4 \times 250 + 0.6 \times 50 - 81 = 49.$$

Denote $u_H = \sqrt{t_H}$, $u_L = \sqrt{t_L}$. Then, $V_1 = 0.8 \times (250 - u_H^2) + 0.2 \times (50 - u_L^2)$ and

$$(\text{IC}): \ 0.8u_H + 0.2u_H - 1 \geq 0.4u_H + 0.6u_L, \quad (\text{IR}): \ 0.8u_H + 0.2u_L - 1 \geq 9.$$



4.2 Optimal Effort Policy

- The cost of inducing high effort under moral hazard is

$$C^* := \pi_1 t_H^* + (1 - \pi_1) t_L^* - u^{-1}(u_0).$$

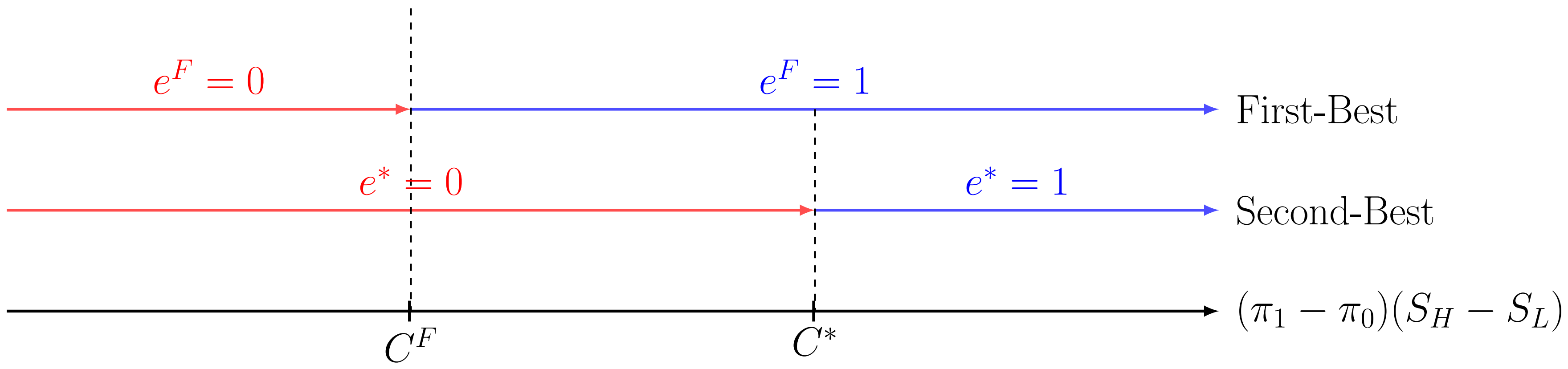
- $e = 1$ is optimal if $(\pi_1 - \pi_0)(S_H - S_L) \geq C^*$. Otherwise, $e = 0$ is optimal.

- The second-best cost of inducing a high effort is higher than the first-best cost:

$$\begin{aligned}
C^* &= \pi_1 t_H^* + (1 - \pi_1) t_L^* - u^{-1}(u_0) \\
&= \pi_1 u^{-1}\left(u_0 + \frac{1 - \pi_0}{\pi_1 - \pi_0} c\right) + (1 - \pi_1) u^{-1}\left(u_0 - \frac{\pi_0}{\pi_1 - \pi_0} c\right) - u^{-1}(u_0) \\
&> u^{-1}\left(\pi_1\left(u_0 + \frac{1 - \pi_0}{\pi_1 - \pi_0} c\right) + (1 - \pi_1)\left(u_0 - \frac{\pi_0}{\pi_1 - \pi_0} c\right)\right) - u^{-1}(u_0) \\
&= u^{-1}(c + u_0) - u^{-1}(c_0) =: C^F,
\end{aligned}$$

where the inequality holds due to the convexity of $u^{-1}(\cdot)$.

- The principal induces a positive effort from the agent **less often** than when effort is observable.



5 Extensions

5.1 Risk-neutral agent

- Suppose the agent is risk-neutral and let $u(t) = t$.
- In what follows, we show that the principal can achieve the **first-best** outcome.
- Because $u^{-1}(t) = t$, the optimal contract is immediate from (3)

$$t_H^* = u_0 + \frac{1-\pi_0}{\pi_1 - \pi_0}c \quad \text{and} \quad t_L^* = u_0 - \frac{\pi_0}{\pi_1 - \pi_0}c.$$

- As a consequence, the cost of inducing high effort in the second-best is

$$C^* = C^F.$$

- Observe that

  - The agent is rewarded if the outcome is high ($t_H^* > u_0$) while punished if the outcome is low ($t_L^* < u_0$).
  - The principal induces $e = 1$ if $(\pi_1 - \pi_0)(S_H - S_L) \geq c$.
  - The principal obtains the first-best surplus, $\pi_1 S_H + (1 - \pi_1)S_L - c$.
  - The agent's expected gain of exerting a high effort is $(\pi_1 - \pi_0)(t_H^* - t_L^*) = c$.

- **Intuition:** How to share risks is not a problem between two risk-neutral parities.

5.2 Limited Liability

- The agent is still risk-neutral ($u(t) = t$). Assume that the agent is protected by **limited liability** constraint that the transfer received by the agent should no less than $t_0 := u^{-1}(u_0)$.

- The principal's problem is

$$\max_{t_H, t_L} \pi_1(S_H - t_H) + (1 - \pi_1)(S_L - t_L)$$

subject to

$$\pi_1 t_H + (1 - \pi_1)t_L - c \geq \pi_0 t_H + (1 - \pi_0)t_L \tag{4}$$

$$\pi_1 t_H + (1 - \pi_1)t_L - c \geq u_0 \tag{5}$$

$$t_H \geq u_0 \tag{6}$$

$$t_L \geq u_0 \tag{7}$$

- Show that (4) and (7) are binding. Thus,

$$t_H^* = u_0 + \frac{c}{\pi_1 - \pi_0} \quad \text{and} \quad t_L^* = u_0.$$

- The agent's rent is greater than u_0,

$$\pi_1 t_H^* + (1-\pi_1)t_L^* - c = u_0 + \frac{\pi_0}{\pi_1 - \pi_0}c > u_0.$$

- The principal's payoff is

$$V_1^* := \pi_1(S_H - t_H^*) + (1-\pi_1)(S_L - t_L^*) = \pi_1 S_H + (1-\pi_1)S_L - \left(\frac{\pi_1}{\pi_1 - \pi_0}c + u_0\right).$$

- The principal would like to induce $e = 1$ if $V_1^* \geq V_0 := \pi_0 S_H + (1-\pi_0)S_L - u_0$

$$\underbrace{(\pi_1 - \pi_0)(S_H - S_L)}_{\text{expected gain of increasing effort}} \geq \underbrace{\frac{\pi_1}{\pi_1 - \pi_0}c = c + \overbrace{\frac{\pi_0}{\pi_1 - \pi_0}c}^{\text{agent's rent}}}_{\text{second-best cost of of increasing effort}}$$

and otherwise induce $e = 0$.

6 Application: Insurance Market

- Moral hazard is pervasive in insurance markets.

- Lets consider a risk-averse agent with utility function $u(\cdot)$ and initial wealth w.

  – Effort $e \in \{0, 1\}$ is a level of safety care and $c\,e$ is the cost of choosing e.

  – Given effort e, an accident occurs with probability $1 - \pi_e$ with damage d being incurred. Otherwise, no damage incurred.

- Agent's outside utility (without insurance)

  – $u_0(e = 1) := \pi_1 u(w) + (1 - \pi_1)u(w - d) - c$.

  – $u_0(e = 0) := \pi_0 u(w) + (1 - \pi_0)u(w - d)$.

  – We assume $u_0(e = 1) > u_0(e = 0)$, so that the agent would choose $e = 1$ (or due care) if there is no insurance.

- Assume that the insurance market is perfectly competitive.
  - One unit of insurance costs p and pays 1 if damage occurs.
  - If the agent buys α units of insurance, his wealth is either $w_H = w - p\alpha$ or $w_L = w - p\alpha - d + \alpha$.
- Suppose that $e = 1$ is chosen in equilibrium. That is,
  - the agent is willing to participate
  - the insurance firms cannot do better by inducing $e = 0$.
- Note that because of the perfect competition among insurance companies, the contract should maximize the agent's expected utility.

■ First-best benchmark

- Without moral hazard, the insurance company solves the following problem:

$$\max_{w_H, w_L} \ \pi_1 u(w_H) + (1 - \pi_1) u(w_L) - c$$

subject to

$$\pi_1 (w - w_H) + (1 - \pi_1)(w - d - w_L) \geq 0 \tag{8}$$

– Condition (8) means that the competition should not be so severe as to earn the insurance company a negative payoff.

– $p\alpha = w - w_H$ if accident doesn't occur, $p\alpha - \alpha = w - d - w_L$ if accident occurs.

- Perfect competitive implies that (8) is binding.
  - As in section 3, the solution of the above problem yields $w_H = w_L \equiv w^F$, where

$$w^F = w - d(1 - \pi_1).$$

- The agent is better off with insurance than without:

$$\begin{aligned} U^F := \pi_1 u(w^F) + (1-\pi_1)u(w^F) - c &= u\big(w - d(1-\pi_1)\big) - c \\ &= u\big(\pi_1 w + (1-\pi_1)(w-d)\big) - c \\ &\geq \pi_1 u(w) + (1-\pi_1)u(w-d) - c \\ &= u_0(e=1) \end{aligned}$$

  - The inequality holds since$u(\cdot)$ is a concave function.
  - The agent is willing to participate.

■ Second-best

- Under moral hazard, the insurance company solves the following problem:

$$\max_{w_H, w_L} \ \pi_1 u(w_H) + (1-\pi_1)u(w_L) - c$$

subject to

$$\pi_1 u(w_H) + (1-\pi_1)u(w_L) - c \geq \pi_0 u(w_H) + (1-\pi_0)u(w_L) \tag{9}$$

$$\pi_1(w - w_H) + (1-\pi_1)(w - d - w_L) \geq 0 \tag{10}$$

- Set up the Lagrangian function

$$\begin{aligned}\mathcal{L} = \pi_1 u(w_H) + (1-\pi_1)u(w_L) - c + \lambda\Big[(\pi_1 - \pi_0)(u(w_H) - u(w_L)) - c\Big] \\ + \mu\Big[\pi_1(w - w_H) + (1-\pi_1)(w - d - w_L)\Big]\end{aligned}$$

- Two first-order conditions

$$\frac{1}{u'(w_H^*)} = \frac{1}{\mu} + \frac{\lambda}{\mu}\frac{\pi_1 - \pi_0}{\pi_1} \quad \text{and} \quad \frac{1}{u'(w_L^*)} = \frac{1}{\mu} - \frac{\lambda}{\mu}\frac{\pi_1 - \pi_0}{(1-\pi_1)}.$$

  – Both λ and μ are positive, so that (9) and (10) are binding.

  – Let (w_H^*, w_L^*) be the solution obtained from binding constraints (9) and (10)

  – Let $U^* := \pi_1 u(w_H^*) + (1-\pi_1)u(w_L^*) - c$ denote the agent's expected utility.

- Note that $U^* \geq u_0$, i.e., the agent is willing to participate.

  – $(w_H, w_L) = (w, w-d)$ satisfies both (9) and (10).

  – Since (w_H^*, w_L^*) is the optimal solution, we must have $U^* \geq u_0$.

Example 3. Let $w = 100$, $d = 91$, $\pi_1 = 0.9$, $\pi_0 = 0.6$, $c = 1$ and $u(w) = \sqrt{w}$. Then,

$$u_0(e=1) = 0.9\sqrt{100} + 0.1\sqrt{9} - 1 = 8.3 > u_0(e=0) = 0.6\sqrt{100} + 0.4\sqrt{9} = 7.2$$

The first-best contract is

$$w^F = w - d(1 - \pi_1) = 90.9 \;\text{ amd }\; U^F = \sqrt{90.9} - 1 = 8.53.$$

The second-best contract from binding constraints (9) and (10):

$$0.3(\sqrt{w_H} - \sqrt{w_L}) = 1,$$

$$0.9(100 - w_H) + 0.1(9 - w_L) = 0.$$

We have $(w_H^*, w_L^*) = (96.33, 42.01)$ and $U^* = 8.48$.

Appendix: Lagrangian Method

- Let f and g_i, where $i = 1, \ldots, m$, be C^1 functions, both defined on the open and convex set $U \subset \mathbb{R}^n$.

- Consider the following maximization problem:

$$\max_x f(x_1, \ldots, x_n)$$

$$\text{subject to} \begin{cases} g_1(x_1, \ldots, x_n) \leq 0, \\ \qquad \ldots\ldots \\ g_m(x_1, \ldots, x_n) \leq 0. \end{cases}$$

- Set up the Lagrangian for this problem, which is defined on $U \times \mathbb{R}^m_+$, as follow:

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x).$$

- The first-order conditions and *complementary slackness conditions*:

$$Df(x^*) - \lambda_i^* Dg_i(x^*) = 0$$

$$\lambda_i \geq 0,\ \ g_i(x^*) \leq 0 \text{ and } \lambda_i^* g_i(x^*) = 0, \forall i = 1, \cdots, m.$$

  – Either the constraint is binding ($g_i(x^*) = 0$) or the associated multiplier is zero, or both.

  – If the multiplier is strictly positive, the constraint must be binding.

  – If the constraint is not binding, the multiplier must be zero.

Example 4. Consider the following utility maximization problem:

$$\max_{x_1,x_2}\ x_1^{\alpha}x_2^{\beta}\ \ s.t.\ \ p_1x_1+p_2x_2\le I,\ \ x_1\ge 0,\ \ x_2\ge 0.$$

Form the Lagrange function

$$L=x_1^{\alpha}x_2^{\beta}-\lambda_1(p_1x_1+p_2x_2-I)-\lambda_2(-x_1)-\lambda_3(-x_2)$$

The FOCs and complementary slackness conditions are

$$\frac{\partial L}{\partial x_1}=\alpha x_1^{\alpha-1}x_2^{\beta}-\lambda_1p_1=0,\ \ \frac{\partial L}{\partial x_2}=\beta x_1^{\alpha}x_2^{\beta-1}-\lambda_1p_2=0, \tag{11}$$

$$\lambda_1\ge 0,\ \ p_1x_1+p_2x_2-I\le 0,\ \ \lambda_1(p_1x_1+p_2x_2-I)=0,$$

$$\lambda_2\ge 0,\ \ -x_1\le 0,\ \ \lambda_2(-x_1)=0,\ \ \lambda_3\ge 0,\ \ -x_2\le 0,\ \ \lambda_3(-x_2)=0.$$

We must have $\lambda_1>0=\lambda_2=\lambda_3$ (why?). Hence $p_1x_1+p_2x_2-I=0$, and together with (11), this yields that

$$x_1^*=\frac{\alpha}{\alpha+\beta}\frac{I}{p_1}\quad\text{and}\quad x_2^*=\frac{\beta}{\alpha+\beta}\frac{I}{p_2}.$$