Economics of Information and Uncertainty

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Based on a lecture by In-Koo Cho in spring 2021

Contents

Chapter 1

Introduction

1.1 Expected utility theory

Lecture 1. introduction Wed, Mar 3

Why needed? A decision is often made under uncertainty.

- Objective uncertainty: The value of a decision may depend upon the contingency, which is not observable at the time of decision but the probability of the event is known.
- Subjective uncertainty: The value of a decision may depend upon the decisions of the other players, which are not observable at the time of decision. The probability of the event is endogenous.

We need a formal theory to evaluate a choice whose value is not a deterministic value, but a probability distribution over values.

History John von Neumann and Oskar Morgenstern developed the theory of games. They immediately recognized that a decision maker faces uncertainty. A decision maker does not know the actual value from his decision, but realizes the value only after he made the decision. The value of a decision is more like a probability distribution than a number. In order to model his decision problem, we need a formal way to evaluate a probability distribution.

They developed the expected utility theory, as a way to investigate the interactive decision problem. The expected utility theory appears in the appendix rather than in the main text of their classic book.[1]

In this edition, Harold Kuhn and Ariel Rubinstein contributed short essays in the front and the back of the book, illustrating the status of economic theory 60 years after the first edition of the classic book is first published. Students are strong encouraged to read the short essay by Ariel Rubinstein in the postscript, as well as the original foreword by von Neumann and Morgenstern who foresaw the future of the game theory.

Reference This lecture is drawn from David M. Kreps [1988][2] which is probably the best reference for the choice under uncertainty.

[1]John von Neumann and Oskar Morgenstern [2007]: Theory of Games and Economic Behavior: 60th Anniversary Commemorative Edition (Princeton Classic Editions).

[2]David M. Kreps. (1988). *Notes on the Theory of Choice.* Westview press.

Description Let Z be a finite set of outcomes, or attributes, from which a decision maker generates utility. Let $p\colon Z \to [0,1]$ be the probability distribution over Z. That is, $p(z) \geq 0\ \forall z$, and $\sum_{z\in Z} p(z) = 1$.

We use lotteries to describe risky alternatives. Suppose first that the number of possible outcomes is finite. Fix a set of outcomes $C = \{c_1, \cdots, c_N\}$. Let p_n be the probability that outcome $c_n \in C$ occurs and suppose these probabilities are objectively known.

Definition 1. A ***(simple) lottery*** $L = (p_1, \cdots, p_n)$ is an assignment of probabilities to each outcome c_n, where $p_n \geq 0$ for all n and $\sum_n p_n = 1$.

The collection of such lotteries can be written as

$$\mathcal{L} = \left\{ (p_1, \ldots, p_N) \,\middle|\, \sum_{n=1}^{N} p_n = 1,\ p_n \geq 0 \text{ for } n = 1, \ldots, N \right\}.$$

We can also think of a compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$, which allows the outcomes of a lottery to be lotteries. It is immediate to see that any compound lottery can be reduced to a simple lottery defined as above.

Example. $C = \{c_1, c_2\}$, $L_1 = (p, 1-p)$, $L_2 = (q, 1-q)$. Then,

$$(L_1, L_2; \alpha, 1-\alpha) = (\alpha p + (1-\alpha)q, \alpha(1-p) + (1-\alpha)(1-q)).$$

Hence, we can only focus on simple lotteries. One special and important class of lotteries is money lotteries, whose outcomes are real numbers, i.e., $C = \mathbb{R}$. A money lottery can be characterized by a cumulative distribution function F, where $F : \mathbb{R} \to [0,1]$ is nondecreasing. $F(x)$ is the probability of receiving a prize less than or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

If an individual has reasonable preferences about consumption in different circumstances, we will be able to use a utility function to describe these preferences just as we do in other contexts. However, the fact that we are considering choice under uncertainty adds some special structures to the choice problem, which we will see below. Historically, the study of individual behavior under uncertainty is originated from attempts to understand (and hopefully to win) games of chance. One may think that the key determinant of behavior under uncertainty is the expected return of the gamble. However, people are generally reluctant to play fair games.

Example (St. Petersburg Paradox)**.** Consider the following gamble: you toss a coin repeatedly until the head comes up. If this happens in the nth toss, the gamble gives a monetary payoff of 2^n. What is the expected return of this game? How much would you pay to play this gamble?

Lottery We call p a lottery, which is an object of the choice by the decision maker. Let P be the collection of all lotteries. Mathematically, P is the unit simplex in $\mathbb{R}^L$ where L is the number of elements in Z. For this reason, we sometimes write $\Delta(Z)$ in place of P to emphasize the relationship between Z and the probability distribution over Z.

Our goal is to explain how a decision maker chooses a particular lottery from the set of feasible lotteries. The main difference from the consumer theory is that the decision maker does not observe $z \in Z$ before he chooses a lottery. In order to examine the decision making process under uncertainty, we need to formulate how a decision maker choose a lottery, or formalize the preference ordering over P.

We have many theories for the decision making under uncertainty. The most prominent theory is the expected utility theory by John von Neumann and Oskar Morgenstern, which was developed as a part of developing the theory of games.

Expected utility Let $\succeq$ be an ordering over P, which represents the decision maker's preference over lotteries. If $p \succeq q$, then we say that p is preferred to q. The only difference from the conventional consumer theory is that p and q are probabilities, rather than attributes (or goods) which the decision maker draw utility.

Definition 2. $\succeq$ is ***complete*** if $\forall p, q \in P$, $p \succeq q$ or $q \succeq p$, and ***transitive*** if $\forall p, q, r \in P$, $p \succeq q$ and $q \succeq r$ imply $p \succeq r$. We say that $\succeq$ is a ***preference ordering*** if $\succeq$ is complete and transitive.

Axiom 1. $\succeq$ is a preference ordering over P.

This axiom is hardly controversial, although experimental evidence shows that the ordering of a human being is often not complete or not transitive. Throughout this class, we maintain the assumption that $\succeq$ is complete and transitive.

Definition 3. $\forall p, q \in P$, $\forall a \in [0, 1]$, a ***composite lottery*** is $ap + (1 - a)q$.

If one interpret $a \in [0, 1]$ as a probability, one can interpret a composite lottery as a lottery over lotteries. One can interpret a as the amount of lottery a in the portfolio. A stock is a lottery, because the value of a stock depends upon the profitability and the market condition, but the decision maker does not observe the true state when he purchases a stock. A mutual fund is a composite lottery. An important observation is that P is a convex set. Therefore, a composite lottery is an element of P.

The second axiom is called the substitution axiom, the independence axiom or the linearity axiom.

Axiom 2 (Substitution, Independence, Linearity)**.** $\forall p, q, r \in P$, $\forall a \in (0, 1]$, if $p \succeq q$, then $ap + (1 - a)r \succeq aq + (1 - a)r$.

The preference between two composite lotteries is determined by the preference between p and q, independently of r. In that sense, this axiom is called the independence axiom.

Note. If $p \succeq q$, then the preference between the two composite lotteries is independent of the size of a. This is the crucial feature of linear preferences, which this axiom implies.

As important as this assumption is for the expected utility theory, the linearity of the preference has been challenged by many experiments. In response, many alternative axioms were proposed. Still, the linearity of the expected utility allows us to use the mathematical expectation to formulate the optimization problem. For this reason, this axiom endures the challenges.

Axiom 3 (Continuity, Archimedian)**.** $\forall p, q, r \in P$, if $p \succeq q \succeq r$, then $\exists a, b \in (0,1)$ such that $ap + (1-a)r \succeq q \succeq bp + (1-b)r$.

This axiom is called the continuity axiom or Archimedian axiom. A key implication is that the utility must be finite. Suppose that the utility from p is infinite. Then, it would be impossible to find $b \in (0,1)$ to construct a composite lottery so that

$$q \succeq pb + (1-b)r.$$

Similarly, if you assign $-\infty$ utility to lottery p (such as death with probability 1), then it would be impossible to construct a composite lottery in which the proportion of p is $a \in (0,1)$ such that

$$ap + (1-a)r \succeq q.$$

The fundamental theorem by von Neumann and Morgenstern is that we can represent any preference satisfying three axioms by the expected value of a utility.

Theorem 1 (Expected utility theorem)**.** $\succeq$ satisfies three axiom if and only if there exists a utility function $u\colon Z \to \mathbb{R}$ such that $\forall p, q \in P$, $p \succeq q$ if and only if

$$U(p) = \sum_{z \in Z} u(z)p(z) > \sum_{z \in Z} u(z)q(z) = U(q).$$

Moreover, if u represents $\succeq$, then v represents $\succeq$ if and only if $\exists c > 0$, $\exists d \in \mathbb{R}$ such that $v(z) = cu(z) + d$.

The function U is often called a von Neumann-Morgenstern (vNM) expected utility function.

Discussion

- The first part is the existing of utility function u, which measures how much utils the decision maker obtains by consuming $z \in Z$. This utility function is called von Neumann Morgenstern utility function. The probability distribution is used to calculate the (mathematical) expected value of u, which the name expected utility came from.
- We call $f(z) = cz + d$ where $c > 0$ and $z, d \in \mathbb{R}$ an affine function, which is a linear function with a constant term. The second part is the uniqueness

result up to the affine transformation. That is, if u represents $\succeq$, its affine transformation represents the same preference.

- In the neoclassical consumer theory, we assumed the ordinal preference, which is represented by a utility function. An important result is that the utility function is unique up to monotonic transformation. If f is a strictly increasing function, and u represents the preference, then $v = f(u)$ represents the same preference. Since f preserves the order of the preference, but not the cardinal value of the utility, we regard this result as the mathematical formulation of the ordinal utility function.
- An affine function is a strictly increasing function, but not vice versa. An affine function does not preserve the cardinal value of the utility, but the uniqueness does not extend to all strictly increasing functions. In this sense, a von Neumann Morgenstern utility function is a cardinal utility.

1.2 Challenges

Experimental challenges Expected utility theory allows us to use the statistical method to formulate and solve the optimization problem to examine the decision making process under uncertainty. The probability enters the decision problem linearly, which simplifies the problem tremendously.

However, expected utility theory has been challenged by experimental data for a long time. Let us discuss three best known examples.

Allais's paradox

Many different versions of the same experiment have been conducted over time. We examine the one by Kahneman and Tversky. Consider the following experiments which consist of two parts.

In the first part, a group of subjects is asked to choose between two lotteries: A and B where

$$A = \begin{cases} 2500 & \text{with probability } 0.33 \\ 2400 & \text{with probability } 0.66 \\ 0 & \text{with probability } 0.01 \end{cases}$$

and

$$B = 2400 \text{ with probability } 1.$$

In the second part, the same group of subjects is asked to choose between two lotteries: C and D where

$$C = \begin{cases} 2500 & \text{with probability } 0.33 \\ 0 & \text{with probability } 0.67 \end{cases}$$

and

$$D = \begin{cases} 2400 & \text{with probability } 0.34 \\ 0 & \text{with probability } 0.66 \end{cases}.$$

At the end of the second round, the experimenter compares the choice by the subjects in the first and the second rounds. The focus is the consistency of the choice across different pairs of options.

82% choose B over A, while 83% chooses C over D, which means at least 65% ($\simeq 82\% \times 83\%$) chooses B over A and C over D.

A plausible heuristic explanation is that between A and B, \$2400 for sure would be better than a little bit of uncertainty, while between C and D, the difference between the size of the prize outweights the difference of the probability.

Whatever the reason might be, the behavior of a substantial portion of subjects is inconsistent with the prediction of the expected utility theory. The inconsistency is due to the violation of the independence axiom.

If the preference of a subject satisfies three axioms, we have

$$u\colon \{2500, 2400, 0\} \to \mathbb{R}.$$

Suppose that $u(2500)$, $u(2400)$, $u(0)$ are the von Neumann Morgenstern utilities of 2500, 2400 and 0 prizes.

If the subject chooses B over A, then

$$u(2400) > 0.33u(2500) + 0.66u(2400) + 0.01u(0)$$

which is equivalent to

$$0.33u(2500) - 0.34u(2400) + 0.1u(0) < 0. \tag{1.1}$$

If the same subject chooses C over D, then

$$0.33u(2500) - 0.34u(2400) + 0.1u(0) > 0. \tag{1.2}$$

But, (1.1) and (1.2) are inconsistent.

The outcome is an evidence of the violation of the independence axiom. Suppose that a subject has a well defined ordering between two lotteries:

$$X = \begin{cases} 2500 & \text{with probability } \frac{33}{34} \\ 0 & \text{with probability } \frac{1}{34} \end{cases}$$

and

$$Y = 2400\text{with probability } 1.$$

Independence axiom says that $Y \succeq X$ if and only if

$$0.34Y + 0.66 \cdot 2400 = B \succeq A = 0.34X + 0.66 \cdot 2400.$$

Similarly, $Y \succeq X$ if and only if

$$0.34Y + 0.66 \cdot 0 = D \succeq C = 0.34X + 0.66 \cdot 0.$$

If a subject chooses B over A but C over D, his preference must violate the independence axiom.

Framing effect (Tversky and Kahneman (1981))

Imagine that the U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume that the exact scientific estimate of the consequences of the program are as follows.

If program A is adopted, 200 people will be save. If program B is adopted, there is 2/3 probability that no one will be saved, and 1/3 probability that 600 people will be saved.

If program C is adopted, 400 people will die with certainty. If program D is adopted, there is 2/3 probability that 600 people will die, and 1/3 probability that no one will die.

- 72% of subjects say $B \preceq A$.
- 78% of subjects say $C \preceq D$.
- Thus, $.72 \times .78 \simeq 50\%$ of subjects say $B \preceq A$ and $C \preceq D$.
- But, $A = C$ and $B = D$.

Ellsberg paradox (1961)

I have an urn with 300 balls in it. Some of the balls are red, some blue and some yellow. All the balls are the same size and weight, and they are not distinguished in any way except in color. I am willing to tell you that precisely 100 of the balls are red. I am unwilling to say how many are blue and how many are yellow, except that, of course, the total number of blue and yellow is 200. I want to know your preferences between gambles based on the outcome of this random event. In all these gambles, you will either win \$1000 or you will win nothing.

Gambles 1

A: Get \$1000 if the ball drawn out is red, and \$0 if it is blue or yellow.
B: Get \$1000 if the ball drawn out is blue, and \$0 if it is red or yellow.

Gambles 2

C: Get \$1000 if the ball drawn out is blue or yellow, and \$0 if it is red.
D: Get \$1000 if the ball drawn out is red or yellow, and \$0 if it is blue.

A typical response is $A \succeq B$ and $C \succeq D$. You know the odd in A, but not in B. You know the odd in C but not in D. What is wrong with this observation? $A \succeq B$ if you think the number of blue balls is less than 100. If so, $D \succeq C$. We typically do not like ambiguous problems.

1.3 Attitude toward risk

Lecture 2. attitude Mon, Mar 8

Why useful? Despite experimental evidence against the axioms, the expected utility theory is widely used. We can describe and analyze the decision problem using the same mathematical tool to compute expectations. The vNM utility provides a convenient way of formulating the attitude toward risk.

Attitude toward risk In many economic environments, individuals display aversion to risk. We formalize the notion of risk aversion and study some of its properties.

Utility on money We focus on money lotteries, i.e., risky alternatives whose outcomes are amounts of money. It is convenient to treat money as a continuous variable. We have so far assumed a finite number of outcomes to derive the expected utility representation. How to extend this?

It is convenient to assume that $X = [0, \infty)$ is money, and consider a lottery over X. Let u be the vNM utility over X. Any probability distribution over X can be represented by cumulative distribution functions (or cdf) $F\colon \mathbb{R} \to [0, 1]$ where $F(x) = P(x' \leq x)$.

We assume that F is differentiable and $f(x) = F'(x)$, which is known as the density function. If X is discrete, $f(x)$ corresponds to the probability of event $x \in X$.

Expected utility framework on monetary outcomes We describe a monetary lottery by means of a cumulative distribution functions $F : \mathbb{R} \to [0, 1]$. $F(x)$ is the probability that the realized payoff is less that or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

Expected utility Consider a preference relation $\succsim$ on $\mathcal{L}$. It has an expected utility representation if $F \succsim F' \Leftrightarrow U(F) \geq U(F')$, where

$$U(F) = \int_{-\infty}^{\infty} u(x)\, dF(x)$$

or

$$U(F) = \int_{-\infty}^{\infty} u(x) f(x)\, dx$$

if F is differentiable and $f = dF/dx$.

> **Note.** U is defined on lotteries whereas u is defined on money.

To differentiate the two objects, we often call U the (von Neumann Morgenstern) expected utility function and $u(\cdot)$ the Bernoulli utility function or von Neumann Morgenstern utility of money.

We assume that u is (strictly) increasing, implying that the marginal utility of money is strictly positive, and twice continuously differentiable, for analytic convenience.

Definition 4 (Attitude toward risk)**.** Let u be a utility function defined on money outcomes that represents $\succsim$. We say that $\succsim$ exhibits

$$\begin{array}{c}\textbf{\textit{risk aversion}}\\ \textbf{\textit{risk neutrality}}\\ \textbf{\textit{risk loving}}\end{array} \iff \int u(x)\, dF(x) \begin{array}{c}<\\=\\>\end{array} u\left(\int x\, dF(x)\right)$$

for all lotteries F.

Equivalently, $\succsim$ exhibits risk aversion if $\mathbb{E}[u(X)] < u(\mathbb{E}[X])$. Notice that if $\succsim$ is risk averse (neutral, loving), then u is concave (linear, convex).

Risk averse decision maker Consider $X = \{x_g, x_b\}$ where $x_g > x_b$. Recall that u shows risk aversion if

$$u(\pi x_b + (1-\pi)x_g) > \pi u(x_b) + (1-\pi)u(x_g).$$

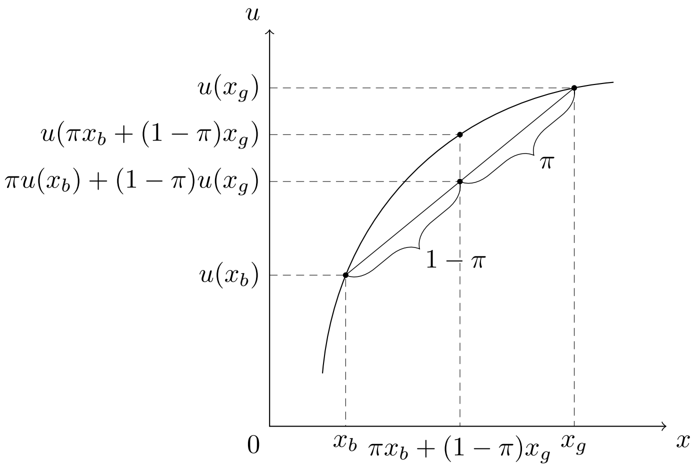


Figure 1.1: Risk aversion

Sure thing and fair gamble If u is concave, Jensen's inequality says

$$\int u(z)\,dF(z) = \mathbb{E}[u(z)] \le u\left(\int z\,dF(z)\right) = u(\mathbb{E}[z]).$$

The left hand side is the expected utility from the bet whose return z is distributed according to F. The right hand side is the utility from money whose amount is equal to the expected value of the random variable.

Definition 5. By a ***sure thing***, we mean a deterministic outcome z. A ***bet*** is a random variable and a ***fair bet*** is a random variable whose expected return is equal to the sure thing.

Risk averse and fair bet Let ε be a random variable whose expected value is 0. Given z^e, a fair bet to z^e is $z^e + \varepsilon$. Let $z^e = \mathbb{E}[z]$, and $\varepsilon = z - z^e$ whose distribution function is G. Then,

$$\int u(z^e + \varepsilon)\,dG(\varepsilon) = \mathbb{E}[u(z)] \le u\left(\int z\,dF(z)\right) = u(\mathbb{E}[z]) = u(z^e).$$

We often say that u shows risk averse attitude if and only if the decision maker prefers a sure thing over a fair bet.

Risk neutral decision maker If a decision maker is risk neutral, then

$$u(\pi x_b + (1-\pi)x_g) = \pi u(x_b) + (1-\pi)u(x_g).$$

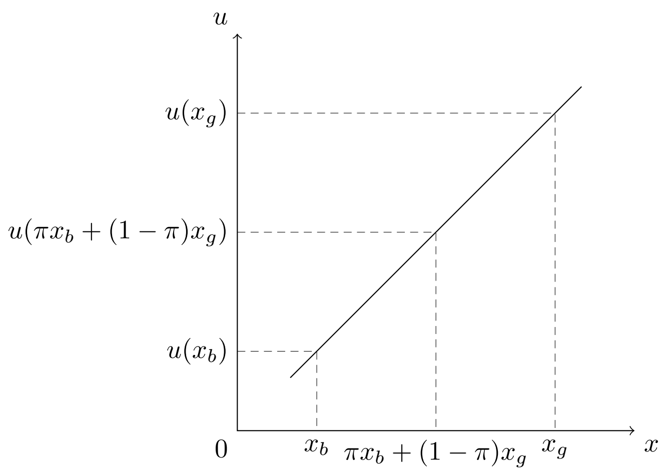


Figure 1.2: Risk neutrality

Risk loving A decision maker is risk loving if

$$u(\pi x_b + (1-\pi)x_g) < \pi u(x_b) + (1-\pi)u(x_g).$$

You may think that only a professional gambler might be risk loving. A policy with a good intention can turn a risk neutral decision maker into a risk loving decision maker.

Credit guarantee Suppose that a firm is a risk neutral decision maker whose Bernouille utility function (or vNM utility) is $u(z) = z$. The firm has a fixed cost D, but the return is a random variable R distributed over $[0, \infty)$. The profit of the firm is $u(R - D) = R - D$ which is a random variable. Note that if $R - D < 0$, then the firm loses money, which may lead to default.

It is not unusual that a government sometimes offers a rescue plan, by covering the loss in the bad state. Suppose that the government covers the loss whenever the firm incurs loss. The subsidy S is therefore, $S = -\min(R - D, 0)$. The firm's utility is now $R - D + S = \max(R - D, 0)$ which is a convex function. The intervention of the government changes the behavior of the firm from a risk neutral decision maker to a risk loving decision maker.

Alternative ways The expected utility theory provides alternative ways to represent the attitude toward risk other than the shape of the vNM utility is one way. Let us discuss a couple of widely used methods.

Certainty equivalence A risk averse individual prefers a sure thing to a fair gamble. Is there a smaller amount of certain wealth that would be viewed as equivalent to the gamble?

> **Definition 6.** The ***certainty equivalent*** of F is the amount of money for which the individual is indifferent between the gamble F and the certain amount; that is, $u(CE) = \int u(x)\, dF(x)$.

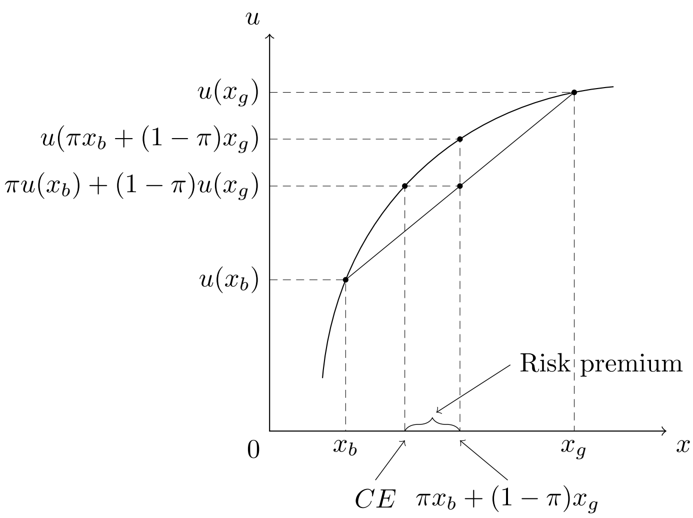


Figure 1.3: Certainty equivalent and risk premium

Note. u is concave if and only if $CE < \pi x_b + (1-\pi)x_g$.

If a risk averse decision maker is offered two options: CE and $\pi x_b + (1-\pi)x_g$, then he will accept the expected return.

This behavior provides an alternative way to represents the attitude toward risk.

Definition 7. The ***risk premium*** associated with F is the maximum amount of money that an individual is prepared to pay to avoid the game, i.e. $\mathbb{E}[u(X)] = u(\mathbb{E}[X] - RP)$. Clearly, $RP = \mathbb{E}[X] - CE$.

Proposition 1. TFAE:

(1) u exhibits risk aversion.

(2) u is concave.

(3) $CE \leq \mathbb{E}[X]$ (i.e., $RP \geq 0$).

Proof. (2) $\Rightarrow$ (1) If u is concave, then Jensen's inequality immediately implies

$$\int u(x)\,dF(x) \leq u\left(\int x\,dF(x)\right). \tag{1.3}$$

(1) $\Rightarrow$ (3) By (1.3), we have

$$u(CE) = \int u(x)\,dF(x) \leq u\left(\int x\,dF(x)\right) = u(\mathbb{E}[X]),$$

so, $CE \leq \mathbb{E}[X]$.

(3) ⇒ (2) Suppose that (2) does not hold. Then, there must exist x, y, and $\lambda \in (0,1)$ such that $u(\lambda x + (1-\lambda)y) < \lambda u(x) + (1-\lambda)u(y)$. Now, consider a binary distribution $F(\cdot)$ according to which x is drawn with probability λ while y is drawn with $(1-\lambda)$. Then,

$$u(\mathbb{E}[X]) = u(\lambda x + (1-\lambda)y) < \lambda u(x) + (1-\lambda)u(y) = u(CE).$$

Thus, $\mathbb{E}[X] < CE$. $\lightning$

■

Measurement of risk aversion We sometimes have to rank two decision makers according to their attitude toward risk by saying that a decision maker is more risk averse than the other. Intuitively, the more concave the utility function, the more risk averse the consumer. Thus, the second derivative of u is a natural candidate for the measure risk aversion.

Recall that vNM utility is invariant with respect to affine transformation. Thus, if we change u by $\alpha u + \beta$ for some $\alpha > 0$, the attitude toward risk does not change. The problem of u'' as the measure of the risk aversion is that it is not invariant with respect to the affine transformation.

As an example, consider a decision maker with $v(\cdot) = 2u(\cdot)$, who has the same preference over the bet as the decision maker with u. But, $v''(\cdot) = 2u''(\cdot) \neq u''(\cdot)$.

Definition 8 (Arrow–Pratt measure of absolute risk aversion)**.**

$$r_A(x,u) := -\frac{u''(x)}{u'(x)}$$

The idea of constructing r_A is intuitive. We normalize the degree of concavity by u' so that the measure is invariant with respect to affine transformation. More precisely,

$$-\frac{u''}{u'} = -\frac{du'/dx}{u'} = -\frac{du'/u'}{dx} = -\frac{\%\text{ change in MU}}{\text{absolute change in } x}.$$

$r_A(x)$ is positive, negative, or zero as the agent is risk averse, risk loving, or risk neutral.

Another interpretation Let us consider two outcomes: bad outcome $x_b = w + r_b z$ and good outcome $x_g = w + r_g z$. Draw indifferent cure:

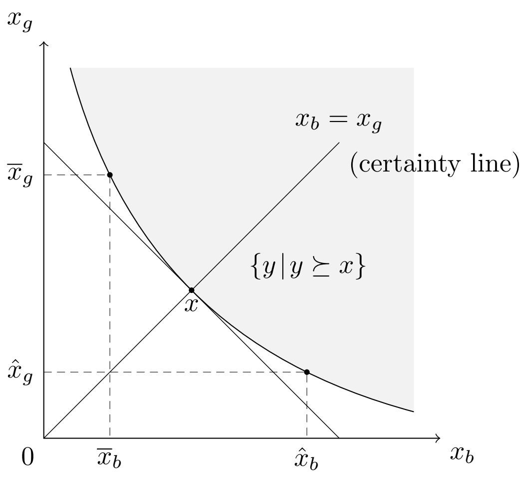
$$\pi u(x_b) + (1-\pi)u(x_g) \equiv \overline{u}.$$

By totally differentiating both sides, we implicitly derive the marginal rate of substitution

$$\pi u'(x_b) + (1-\pi)u'(x_g)\frac{dx_g}{dx_b} = 0. \tag{1.4}$$

Hence, the marginal rate of substitution (MRS) is

$$\frac{dx_g}{dx_b} = -\frac{\pi}{1-\pi}\frac{u'(x_b)}{u'(x_g)}. \tag{1.5}$$



$$\left|\frac{dx_g}{dx_b}\right| \begin{pmatrix} = \\ < \\ > \end{pmatrix} \frac{\pi}{1-\pi} \text{ when } x_b \begin{pmatrix} = \\ > \\ < \end{pmatrix} x_g, \text{ showing that } u(\cdot) \text{ is concave.}$$

Define the consumer's preferred set at x to be the set of all outcome the consumer will prefer to x, i.e., $\{y \,|\, y \succeq x\}$.

Suppose now we have two consumers, i and j. It is natural to say that consumer i is (locally) more risk averse than consumer j if consumer i 's preferred set at x is contained in j's one. Consumer i's indifferent curve is more curved than consumer j's one at x.

Differentiate (1.4) one more with respect to x_b,

$$\pi u''(x_b) + (1-\pi)u''(x_g)\left(\frac{dx_g}{dx_b}\right)\left(\frac{dx_g}{dx_b}\right) + (1-\pi)u'(x_g)\left(\frac{d^2x_g}{dx_b^2}\right) = 0.$$

Using (1.5), we have

$$\frac{d^2x_g}{dx_b^2} = \frac{\pi}{(1-\pi)^2}\left[-\frac{u''(x)}{u'(x)}\right] \text{ when } x_b = x_g = x.$$

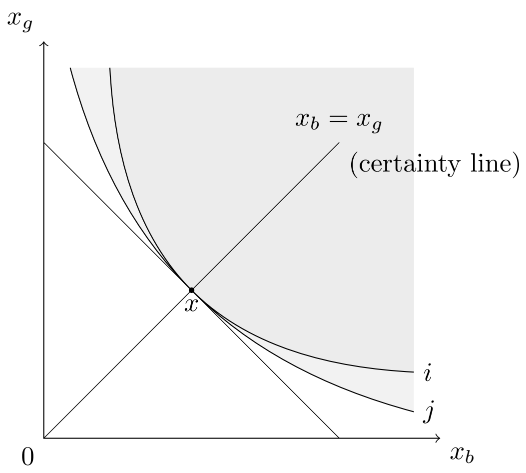


Figure 1.4: Arrow–Pratt measure of absolute risk aversion

Comparison of attitude toward risk In the cases of Proposition 2 below, we say that $u_i(\cdot)$ is more risk averse than $u_j(\cdot)$.

Proposition 2. Given two utility functions $u_i(\cdot)$ and $u_j(\cdot)$, TFAE:

(1) $r_A(x, u_i) \geq r_A(x, u_j)$ for all x. That is, consumer i has a higher degree of risk aversion than consumer j everywhere.

(2) There exists an increasing concave function $\psi(\cdot)$ such that $u_i(x) = \psi(u_j(x))$ for all x. In other words, $u_i(\cdot)$ is more concave than $u_j(\cdot)$.

(3) $CE_i \leq CE_j$ (or $RP_i \geq RP_j$, i.e., i would be willing to pay more to avoid a given risk than j would.)

Proof. (1) $\Leftrightarrow$ (2) Note that we always have $u_2(x) = \psi(u_1(x))$ for some increasing function ψ (this is true because u_1 and u_2 are ordinary identical). Differentiating, we get $u_2'(x) = \psi'(u_1(x))u_1'(x)$ and so $\log u_2'(x) = \log \psi'(u_1(x)) + \log u_1'(x)$. Differentiating again, we have

$$\frac{u_2''(x)}{u_2'(x)} = \frac{\psi''(u_1(x))}{\psi'(u_1(x))}u_1'(x) + \frac{u_1''(x)}{u_1'(x)}$$
$$\Leftrightarrow r_A(x, u_2) = r_A(x, u_1) - \frac{\psi''(u_1(x))}{\psi'(u_1(x))}u_1'(x).$$

Thus, $r_A(x, u_2) \geq r_A(x, u_1)$ $\forall x$ if and only if $\psi''(u_1) \leq 0$ $\forall u_1$.

(2) $\Rightarrow$ (3) By Jensen's inequality,

$$\begin{aligned} u_i(CE_i) &= \mathbb{E}[u_i(X)] = \mathbb{E}[\psi(u_j(X))] \\ &\leq \psi(\mathbb{E}[u_j(X)]) = \psi(u_j(CE_j)) = u_i(CE_j). \end{aligned}$$

Thus, $CE_i < CE_j$.

■

Chapter 2

Hidden Information

2.1 Economy with uncertainty

Lecture 3. information Wed, Mar 10

State contingent claim We have learned the equilibrium model under certainty, where a decision maker knows all characteristics of the goods, and the state at the time when he makes a decision. As we move from a model with certainty to uncertainty, we had to develop a new way of evaluating an object, a lottery, over the set of commodities. A fundamental question is whether the presence of uncertainty changes the equilibrium allocation of the competitive market.

Arrow and Debreu showed the condition under which the presence of uncertainty does not matter. We can apply exactly the same exercise as we learned from the model with certainty. More importantly, the first and the second welfare theorems continue to hold.

Complete market This condition is called the complete market hypothesis: each commodity has a market where it can be traded. In order to make the notion of complete market precise, Arrow invented the notion of contingent commodity.

We first state the model of competitive market satisfying the complete market hypothesis. We do so, because it provides an important benchmark against which an economy with incomplete market is examined, providing a fundamental insight into the role of uncertainty to the equilibrium outcome of the market.

Uncertainty By a state, we mean any factor that affects the decision of an economic agent. The quality of a product is a good example, which may or may not be known to the decision maker at the time of his decision. Let S be the set of states, and $s \in S$ be a generic element. Let us assume that S is finite.

> **Definition 9.** The economy is subject to ***uncertainty*** if a state is not revealed to a decision maker at the time of his decision making.

A lottery is one of the examples. The value of the lottery is a state, which is not revealed at the time when a decision maker purchases a lottery at a certain price.

Let us consider a finite exchange economy, which is populated by I consumers endowed with neoclassical utility function, L commodities and S states, each of which has $\#I$, $\#L$ and $\#S$ elements.

We start with the description of the initial endowment, which is a complete specification of endowment for all possible contingencies. Let $\omega = (\omega_\ell)_{\ell \in L} \in \mathbb{R}^{\#L}$ be the profile of commodities. Because his endowment is affected by state $s \in S$, we need to spell out the profile of endowments for all states in S. Thus, the endowment of agent i is

$$\omega_i = (\omega_{s,i}) \in \mathbb{R}^{\#L \times \#S}.$$

Commodity We differentiate a commodity by state. This is a fundamental innovation of Arrow and Debreu. That is, a commodity is differentiated by an attribute which is relevant to the decision of an economic agent, including a state. For example, an umbrella when it rains is a different commodity from an umbrella when it shines. A stock when the firm is generating a large profit is a different commodity form a stock when the same firm is bankrupted.

Contract Another important interpretation of a commodity by Arrow and Debreu is that a commodity is a contract which promises to deliver the same physical goods if such state arises.

Depending upon a model, it is more convenient to regard an umbrella in rain as a contract which promises to deliver one unit of physical umbrella when it rains. Certainly, this contract carries a price. Similarly, a contract that promises to deliver one stock when the firm is generating a large profit is different from a contract that promises to deliver one stock when the firm is bankrupted. The two commodities would carry different prices.

We can measure the quantity of a commodity according to a basic unit, known as contingent commodity.

Definition 10. A ***contingent commodity*** (or ***Arrow security***) is a contract to deliver one unit of a good if a particular state is realized.

As we expand the economy by incorporating uncertainty, we are expanding the notion of commodity from a profile of physical characteristics to profile of physical characteristics and states. For each of L commodities, we have to consider as many as $\#S$ contingent commodities. As a result, we have $\#L \times \#S$ commodities.

Let x_{ls} be the contract to deliver x_l units of l-th good if state s occurs. The collection of commodities is now $\mathbb{R}^{\#L \times \#S}$. Let $\succeq_i$ be the preference over $\mathbb{R}^{\#L \times \#S}$.

Example. Suppose that π_s is the probability that state s is realized, and $u_i(x, s)$ is the utility of $x \in \mathbb{R}^{\#L}$ under state s. Let $x = (x_s)_{s \in S} \in \mathbb{R}^{\#L \times \#S}$ and $x' = (x'_s)_{s \in S} \in \mathbb{R}^{\#L \times \#S}$ be a pair of state contingent commodity bundles. Then, $x \succ x'$ if

$$\sum_{x \in S} u(x_s, s)\pi_s > \sum_{x \in S} u(x'_s, s)\pi_s.$$

It is important to know that the expected utility is one of many different ways to evaluate the state contingent commodity bundles.

Sequence of moves

(1) Before a state $s \in S$ is realized, there is a market to trade commodities.

(2) After trading commodities (or contracts), a state is realized.

(3) The good is delivered according to the contract.

(4) Goods are consumed, and utility is generated.

The first and third steps warrant a careful examination.

Forward market A commodity is trade in a market, before a state is realized. Thus, is is more convenient to interpret a commodity as a contingent contract, which promises to deliver the specific amount of goods if a state arises.

In the sense that the contingent contract is traded in a forward market, the contract is often called a forward contract.

Enforcement When the contract is traded, a buyer of the commodity pays money to the seller, and receives a piece paper with a promise on it. The good is not delivered, until a state is realized.

An important assumption is that the contract is enforced without any exception, or the seller is committed to carry out the contract. If the enforcement is not complete, or if the seller has a limited commitment, then the contract may not be traded, or will fetch a lower price than under the full commitment.

For example, a debt contract is a promise that a borrower will pay back the principal and the interest back to the lender by a specific time. In Arrow Debreu economy, a debt contract will be enforced without any exception.

Symmetric information At the time when the good is traded, no agent in the economy observes the state. Every decision maker faces uncertainty. In this sense, uncertainty is symmetric.

Complete market hypothesis A fundamental assumption of Arrow Debreu economy is that every contingent commodity has a market where it can be traded. This assumption is called the complete market hypothesis. Because a forward contract is traded, we sometimes say that Arrow Debreu economy assumes a complete set of forward markets.

With a complete set of forward markets, and with full commitment, we can follow exactly the same analysis as for the economy with certainty to establish the first and the second welfare theorems. Any failure of the fundamental welfare theorems can be traced back to the missing market.

2.2 Informational efficiency

Symmetric vs. asymmetric information

- An economy with uncertainty is subject to a state which is not revealed to an agent at the time of decision.
- If no agent observes a state, the economy is subject to uncertainty, but the uncertainty is symmetric.
- If an agent observe a state, but another agent does not observe the same state, asymmetric information exists.

Rational expectations Presence of asymmetric information does not necessarily lead to inefficient allocation, as the competitive market can aggregate dispersed information into the market clearing price. Fredrick von Hayek called this property informational efficiency of competitive market.[1]

Information aggregation Let us consider an exchange economy with uncertainty with two consumers with identical utility function:

$$u_i(x_{1,i}, x_{2,i}) = \beta \ln x_{1,i} + x_{2,i}$$

where $\beta \in \{1, 2\}$ with probability $P(\beta = 1) = 0.5$. Assuming that the second good is a numeraire, let p be the price of the first commodity. Each consumer has 1 unit of the first good as an initial endowment, which makes the aggregate supply of the first good 2 units.

Symmetric uncertainty Suppose that neither agent observes the actual realized value of $\beta \in \{1, 2\}$ at the time of deciding the demand. That is, uncertainty is symmetric.

The demand for the first good should equate the expected marginal utility of the first good to the market clearing price p:

$$\frac{1}{2}\left[\frac{1}{x_{1,i}} + \frac{2}{x_{1,i}}\right] = p$$

and the uniformed demand of consumer i would be

$$x_{1,i}(p) = \frac{3}{2p} \quad \forall i.$$

The market clearing price would be $x_{1,1}(p) + x_{1,2}(p) = 2$ implying $p = \frac{3}{2}$.

Full information If consumer i observes the true state $\beta \in \{1, 2\}$, then his demand is conditioned on β and p. A simple calculation shows consumer i's demand for good 1 at price p in state β is

$$x_{1,i}(p; \beta) = \frac{\beta}{p}.$$

The aggregate demand in state β is therefore

$$x_{1,1}(p; \beta) + x_{1,2}(p; \beta) = \frac{2\beta}{p}$$

and the market clearing price in state β is $p = \beta$.

[1] Friedrich Hayek. (September 1945). The use of knowledge in society. *The American economic review*, 35(4), 519-530.

> **Note.** The market price is one-to-one correspondence to the underlying state. If a rational agent outside of the market observes the market price, he can infer the underlying state.

Asymmetric information Suppose that consumer 1 observes the actual state, while consumer 2 does not. Because consumer 1 observes the true state, his demand is conditioned on β:

$$x_{1,1}(p;\beta) = \frac{\beta}{p} \quad \forall \beta \in \{1,2\}$$

while consumer 2's demand is independent of β:

$$x_{1,2}(p) = \frac{3}{2p}.$$

The market clearing price solves

$$\frac{\beta}{p} + \frac{3}{2p} = 2$$

implying

$$p = \frac{1}{2}\left[\beta + \frac{3}{2}\right].$$

> **Note.** The market clearing price is one-to-one correspondence to the state.

Rational expectations If consumer 2 is rational, he can infer the true state from the market clearing price. Consumer 2's decision is not optimal, because his belief does not incorporate all information available.

If he is rational, he should be able to infer the underlying state, and should behave as if he observes the underlying state. As a result, the equilibrium price must be fully revealing. The market clearing price must be $p = \beta$.

Private information of consumer 1 is aggregated into the market clearing price so that all consumers in the economy can behave optimally for each state. This property is known as informational efficiency of competitive market. We can trace back the idea to Wealth of Nation by Adam Smith, but Frederick von Hayek is generally credited for the idea of informational efficiency.

Arrow Debreu Symmetric uncertainty. It sounds strong assumption in decentralized economy. extend to asymmetric uncertainty.

Hayek Positive. Informational efficiency of competitive market. Even if asymmetric information exists, competitive market should remain efficient.

Akerlof His example opens up possibility that asymmetric information can lead to inefficiency, more dramatically complete collapse of market. That prompted us to investigate how we can recover efficiency of market in a decentralized manner.

2.3 Lemon's problem

Lecture 4. lemon Mon, Mar 15

Symmetric vs. asymmetric information If an agent observes the underlying state, while another agent does not, we say that asymmetric information exists. The economic impact of asymmetric information was first demonstrated by an example of used cars in Akerlof [1970].

At the time, the profession believes in the informational efficiency of the competitive market. He constructed two simple examples of markets with asymmetric information, which completely changed the way how we understand the role of asymmetric information.

Lemon's market By a lemon, we mean a used car which has a low quality but cannot be differentiated from a good quality car. A buyer can see a used car, but cannot tell whether is a good or bad quality used car, before paying for the car, if he chooses to buy one. The seller observes the true quality of the used car, before the car is put on the market. Asymmetric information exists in the market.

The state is $S = \{H, L\}$. If the state is H, the quality is ϕ_H and the outside option for the high quality used car is c_H. Similarly, if the state is L, the quality is ϕ_L and the outside option for the low quality used car is c_L which we normalized to be 0.

We assume that $\phi_H > \phi_L$ which is the utility from consuming the used car. A high quality car generates more utility than a lower quality car. Similarly, we assume $c_H > c_L = 0$. The outside option of a high quality used car is higher than that of a low quality used car. One can also interpret c_H and c_L as the cost for the good and bad used cars. In order to have a good used car, the seller should spend more money to keep the car in good condition.

Key assumptions A model with lemon's problem satisfies the following three conditions. Let $\pi_H = P(s = H)$ be the probability that the true state is H. This π_H is not the proportion of high quality car in the used car market for sale but one among all used car being used by current owner.

(1) Gains from trading: $\phi_H > c_H > \phi_L > c_L$.

(2) Single crossing property: $\phi_H - c_H > \phi_L - c_L$.

(3) Severe lemon's problem: $\pi_H \phi_H + (1 - \pi_H)\phi_L < c_H$.

Gains from trading If $\phi_H > c_H > \phi_L > c_L$, then the gain from trading is positive in each state. If the true state is H, the gain from trading is $\phi_H - c_H > 0$, and if the true state is L, the gain from trading is $\phi_L - c_L > 0$. Every agent in the economy knows that the gain from trading is always positive. Gains from trading is common knowledge. By moving used car from seller to buyer, society can realize that positive gain from trading.

Single crossing property The gain from trading increases in a high state: $\phi_H - c_H > \phi_L - c_L$. In order to achieve an efficient allocation, it is necessary that trading occurs in the high state with a positive probability. In efficiency allocation, high quality used car must be traded. If not, allocation can not be efficient.

Severe lemon's problem In an economy with a complete set of markets, the used car should fetch a price equal to its utility ϕ_s. Before the state is revealed, the market clearing price, if one exists, must be equal to the average quality $\pi_H \phi_H + (1-\pi_H)\phi_L$. If the market price is lower than c_H, then an owner of a high quality used car would not put the car on sale, because he can fetch a higher price from the outside source, or he cannot recover the cost c_H.

Lemon's market

Theorem 2. The market clearing price is ϕ_L, and only the low quality product is traded.

Proof. We show that ϕ_L is the only possible market clearing price. Suppose that p is the

$p > \phi_H$ is not possible, because no consumer will buy a used car whose quality cannot exceed ϕ_H. $\lightning$

$c_H \leq p \leq \phi_H$. Since $p \geq c_H > c_L$, all low quality sellers will put their low quality cars in the market. As a result, the average quality of a used car in the market cannot be more than $\pi_H \phi_H + (1-\pi_H)\phi_L$ which is strictly less than c_H by the last assumption. Thus, no high quality used car will be on the market, which implies that the quality of the used car is exactly $\phi_L < c_H$ by the first assumption. Since $c_H \leq p$, no buyer will pay p to buy a used car with quality $\phi_L < c_H \leq p$. Hence, p cannot be an equilibrium price. $\lightning$

$\phi_L < p < c_H$. Since $p < c_H$, only the low quality car will be in the market. No buyer is willing to pay a price more than ϕ_L. Thus, p cannot be an equilibrium price. $\lightning$

$p < \phi_L$. Because buyers compete for a used car whose utility is ϕ_L, the market experiences excess demand. $\lightning$

If $p = \phi_L$, only the low quality used car will be on the market and a buyer is willing to pay for his utility for the car. ■

Discussion

- The equilibrium allocation is evidently inefficient, because no high quality used car will be traded.
- A surprising part is that the high quality good is driven out of the market, even though the gain from trading is larger and every agent in the economy knows the existence of the positive gain from trading. The logic behind Gresham's law is exactly the same as the lemon's market.
- The nature of uncertainty should be noted. The quality of the used car determines the cost of the seller and the utility of the buyer. In this sense, the quality of the car is the common value of the two players. One of the key components of the lemon's problem is that the seller has private information about the common value.

- The lemon's problem arises in many different cases of asymmetric information over the common value components. Because the lemon's problem leads to an inefficient allocation, it has become a fundamental challenge for economist to find a way to alleviate the implications of the lemon's problem.

Continuous distribution The lemon's problem persists even if we admit more than two types. For example, suppose that the quality is distributed according to continuous density function f over interval $[\phi_l, \phi_h]$. Let $c(\phi)$ be the cost (or the outside option) of the used car with quality ϕ. Assume that $c(\phi)$ is a strictly increasing continuously differentiable function, and $c(\phi) < \phi$ so that there is gain from trading regardless of the quality of a used car. We assume that

$$c(\phi_h) > \mathbb{E}[\phi] = \int_{\phi_l}^{\phi_h} \phi f(\phi)\, d\phi$$

which implies that if the market clearing price is slightly higher than the average quality, the highest quality used car owner would not put his car on the market.

Second example of Akerlof

Theorem 3. If the market clearing price is determined according to the average quality of the products in the market, then the lemon's problem arises and the only equilibrium price is ϕ_l.

Proof. Let p be an equilibrium price. Since the average utility of the products determines the market clearing price,

$$p \leq \mathbb{E}[\phi] = \int_{\phi_l}^{\phi_h} \phi f(\phi)\, d\phi.$$

$c(\phi_h) > \mathbb{E}[\phi]$ and c is a continuous function. $\exists \varepsilon_1$ such that $\forall \phi \in (\phi_h - \varepsilon_1, \phi_h]$ will not put the product in the market since $c(\phi) > \mathbb{E}[\phi]$, where

$$c(\phi_h - \varepsilon_1) = \mathbb{E}[\phi].$$

Then, the average expected price cannot be higher than

$$p \leq \mathbb{E}[\phi \,|\, \phi \leq \phi_h - \varepsilon_1] = \int_{\phi_l}^{\phi_h - \varepsilon_1} \phi f(\phi \,|\, \phi \leq \phi_h - \varepsilon_1)\, d\phi.$$

If we iterate the same process for n rounds, we have ε_n so that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) = \mathbb{E}\left[\phi \,|\, \phi \leq \phi_h - \sum_{k=1}^{n-1} \varepsilon_k\right].$$

By applying the same logic, we conclude that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) < \mathbb{E}\left[\phi \,|\, \phi \leq \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

Since $c(\cdot)$ is continuous, $\exists \varepsilon_{n+1} > 0$ so that

$$c\left(\phi_h - \sum_{k=1}^{n+1} \varepsilon_k\right) < \mathbb{E}\left[\phi \,|\, \phi \leq \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

This process continues as long as

$$\phi_h - \sum_{k=1}^{n+1} \varepsilon_k > \phi_l.$$

Thus, ϕ_l is the only equilibrium price. ■

Discussion

- In case of a continuous distribution, the conclusion is even more pessimistic than what the discrete example says. In the first example of the discrete model, we expect that a low quality car will be traded at ϕ_l, whose mass is as much as $1 - \pi_h > 0$.
- In case of a continuous distribution, the mass of ϕ_l quality car is infinitesimal. With probability 1, no trading occurs. The market collapses.
- Most students probably have heard Gresham's law, saying that a low quality gold coin drives out a high quality gold coin. The underlying logic is identical with the lemon's problem.

2.4 Extension

Endogenous trigger From the first two examples, one might conclude that the lemon's problem arises because of the parameters of the models are assumed in a specific way. The next example is to show that the lemon's problem can be triggered endogenously through the optimization behavior of an agent.

Merger A corporate raider is trying to buy a firm. Let π be the profit of the firm under the present management, which is distributed uniformed over $[0, 1]$. Under the new management, the profit will be 1.5π $\forall \pi \in [0, 1]$.

In the first period, the raider makes a tender offer p. By the end of the first round, the profit π is realized, and observed by the present manager, but not by the raider. Conditioned on π, the manager decides to weather to accept or reject the tender offer p.

If the management accepts the offer, the management receives p, and the firm will be under the new management appointed by the raider. The profit will be the new profit minus the cost of taking over the firm: $1.5\pi - p$. The management receives the tender offer p, by giving away the firm. If the management rejects the offer, the management receives π and the corporate raider receives 0.

Calculation of an equilibrium We can solve the problem backward, from the optimization problem of the management, who has to decide whether to accept or reject p, conditioned on the realized profit of π. The management accept p only if $p \geq \pi$.

We next calculate the expected return to the raider from purchasing the firm at price p. Since the raider does not observe π, he has to infer the value of π from the response of the management to his tender offer p. To purchase the firm, the management must accept the offer which happens only if $p \geq \pi$. Thus, the expected profit from purchasing the firm at p should be

$$\mathbb{E}\left[\frac{3}{2}\pi \,|\, p \geq \pi\right].$$

The net is then

$$\mathbb{E}\left[\frac{3}{2}\pi \,|\, p \geq \pi\right] - p = \frac{3}{2}\frac{1}{2}p - p = -\frac{p}{4} \leq 0.$$

Therefore, the best response of the raider is to offer 0, at which no trading will occur with a positive probability.

Discussion

- In an efficient allocation, the firm must be under the new management who can increase the profit by 50 percent.
- The challenge of the raider is to infer the profit of the firm, which the raider can obtain only if the management agrees to accept the offer.
- In this case, a lemon is a firm with a low profit. Only the firm with a lower profit than p is willing to accept the offer. The raider is served lemons endogenously by the decision of the present management.
- Knowing the response of the management, the corporate raider would not make any meaningful offer. As a result, the social gain under the new management is not realized.

Asymmetric information on common value component The heart of the lemon's problem is the presence of asymmetric information on common value component.

In the original model of Akerlof, the seller observes the quality of the car, but the buyer dose not. The quality affects the payoff of both the seller and the buyer. Similarly, in the model of merger, the management observes the profit of the firm, but not the corporate raider. The profit affects the payoff of both parties.

The outcome is often a complete collapse of the transaction, hence inefficient allocation. In order to alleviate the lemon's problem, we need to understand the mechanism how asymmetric information on common value component undermines otherwise mutually beneficial transaction.

Two fundamental mechanisms are the absence of screening (lemons) and the inability of credibly signaling (the quality of good used cars).

Buyers cannot screen out bad cars At any price $p > c_l$, an owner of a lemon is willing to put his car in the market. Since $c_h > c_l$, it is inevitable that the low quality cars enter the market, if $p \geq c_h$ so that the high quality car owner has incentive to put his car in the market.

Without screening out low quality cars, a buyer cannot keep low quality cars from entering the market. As a result, the proportion of bad used cars in the market becomes significant, pushing down the quality and leading to the lemon's problem.

Sellers cannot signal good cars In the used car market of Akerlof, a owner of a good used car does not have any mechanism to convince a buyer of the quality of his car. Even if an owner can talk to a buyer, a seller does not have an instrument to credibly signal the quality of the car, because an owner of a bad quality car has incentive to imitate the owner of a good car.

Unless a good car can separate itself away from a bad quality car, a good car cannot fetch a higher price than a bad quality car. If good and bad quality cars fetch the same price, an owner of a good quality car will receive a price below what is worth of his car. If the market price is below his cost of maintaining high quality car, he will pull out his car out of the market so that the lemon's problem arises.

Information economics George A. Akerlof [1970][2] is important, as he pointed out the presence of asymmetric information can hinder a competitive market from achieving an efficient allocation (failure of the first welfare theorem). In response, economists have pursued two possible ways to alleviate the lemon's problem.

Screening model A buyer who does not observe the true quality of a car wants to use an instrument to screen out bad quality cars from good quality cars. Insurance company uses the past information of clients to sort out clients according to the risk, offering different types of contracts. Michael Rothschild and Joseph E. Stiglitz [1976][3] is a seminal paper on this topic.

Signaling model A seller with a good quality car wants to signal the good quality of his car so that his car can be separated away from the rest of cars. Because the low quality car owners have the incentive to imitate, the seller with a high quality car must rely on an instrument, which can prevent an owner of a low quality car from imitating. A good example would be a generous warranty for the car, which incurs prohibitively large cost if the car is bad, but only a modest cost, if the car is good. A. Michael Spence [1973][4] started this line of research.

[2] George A. Akerlof [1970]: The Market for Lemons: Quality Uncertainty and the Market Mechanism. Quarterly Journal of Economics, Vol. 84, No. 3, pp. 488-500

[3] Michael Rothschild and Joseph E. Stiglitz [1976] Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information, Quarterly Journal of Economics, Vol. 90, No. 4, pp. 629-649

[4] A. Michael Spence [1973]: Job Market Signaling, Quarterly Journal of Economics, Vol. 87, No. 3, pp. 355-374

Information economics

- Akerlof [1970], Rothschild and Stiglitz [1976] and Spence [1973] opened up a new area of economics which examines the economic implications of information and uncertainty.
- Their pioneering work accompanied by development in game theory in early 1980's, which allows us to investigated the problem rigorously by equilibrium in games with incomplete information.
- This development ushered the game theory into the main stream of economics, which is often called the games with incomplete information revolution.

Chapter 3

Primer of Information Economics

Lecture 5. signaling Wed, Mar 17

Complete market

- Arrow-Debreu economy presumes a compete set of markets so that each commodity can be trade at a market clearing price.
- Without market, externality prevails and the first welfare theorem fails.
- All market failure can be traced back to the absence of a market.
- Inefficiency in the lemon's market can be explained by the absence of a market for information.

Market for information Creating a market for information is extremely difficult.

- The value of information is not concave, because information is often not divisible. For example, unless you have a complete set of service manual, you do not have information about how to maintain an airplane.
- Observation and verification by a third party is difficult, or even illegal. Observation of the medical record of an agent requires a complex legal process. Verifying an information by a third party is impossible in an international relationship, because the international organization often lacks the power of enforcement.

The goal of information economics is to examine and understand various implications of information.

Two sources of asymmetric information Asymmetric information arises from two sources.

- Hidden information. An agent has information about the state, while others do not observe the same state.
- Hidden action. The action of an agent is not observe by the other agent, whose payoff is affected by the action of the first agent.

A natural question would be whether there are other sources which we need to consider. The seminal work by Roger Myerson [1983] demonstrated that we can focus on these two sources without loss of generality. That is, within the context of Bayesian decision making, any decision problem involving asymmetric information can be reduced to the problem with hidden information or hidden action or the combination of both.

Hidden information

> **Definition 11.** An incentive problem arising from hidden information is called ***adverse selection problem***.

The term originates from insurance industry. If an insurance company offers a generous automobile insurance coverage, it was discovered that the same coverage attracts risky drivers, rendering the pool of clients adversely selected.

In a certain sense, the lemon's market of Akerlof is subject to the adverse selection problem. In order to induce the high quality car to be on the market, the market price must be at least $c_H > c_L$, which provides incentive for lower quality cars to enter the market, driving down the average quality of the car.

Hidden action

> **Definition 12.** An incentive problem arising from hidden action is called ***moral hazard problem***.

This term also originates from insurance industry. When an insurance company offers a generous coverage against automobile accidents, the driver tends to drive more recklessly, paying less attention than otherwise.

An insurance company cannot monitor how carefully a driver drives, and paying attention generates disutility to the driver. Because the driver knows the inability of the insurance company to monitor the driver, he has incentive to pay less attention, if the same coverage is offered against accident caused by careless driving or by chance despite careful driving.

Moral? Moral hazard problem is an incentive problem, and has little to do moral. Economists began to regard the moral hazard problem as an incentive problem rather than as a moral problem, following a series of exchanges between Kenneth Arrow[1][2] and Mark Pauly[3].

Why important? If one regards adverse selection and moral hazard as a consequence of bad moral behavior, economists have little to offer. On the other hand, if we regard them as a consequence of an incentive problem of a rational agent, economists can design the environment of the decision maker in such a way to guide his behavior in a socially beneficial way.

[1] Kenneth Arrow [1963]: Uncertainty and the Welfare Economics of Medical Care. American Economic Review, Vol. 53, No. 5, pp 941-973

[2] Kenneth Arrow [1968]: The Economics of Moral Hazard: Further Comment. American Economic Review, Vol. 58, No. 3, Part 1, pp. 537-539

[3] Mark Pauly [1968]: Moral Hazard: Comment, American Economic Review, Vol. 58, No. 3, Part 1, pp. 531-537

Arrow's observation changes what was considered simply a bad behavior to what can be fixed or alleviated by a suitable design of the institution. His insight lays the foundation of the incentive problem.

3.1 Baseline model

Labor market Let us use the labor market model with asymmetric information as the laboratory to examine the consequence of adverse selection and its remedy.

Asymmetric information Let $\theta \in \{\theta_l, \theta_h\}$ be the productivity of the worker with $\theta_h > \theta_l$. The probability that $\theta = \theta_h$ is π. There are firms with want to hire the worker. Because of competitive pressure, a firm has to pay the expected productivity based upon the information of the firm, as wage to hire a worker.

If a firm observes the productivity of the worker, the market wage will be $\theta \in \{\theta_l, \theta_h\}$ depending upon the productivity, but no firm observes the productivity of a worker at the time when the firm hires the worker. The wage w is then

$$w = \mathbb{E}[\theta \,|\, w \text{ is accepted by the worker}].$$

Lemon's problem Let us assume that a worker will not accept an offer, unless the wage is at least as much as his productivity:

$$w \geq \theta.$$

Thus, if w is accepted, the expected productivity of workers is

$$\mathbb{E}[\theta \,|\, \theta \leq w] \leq w$$

and the equality holds if and only if $w = \theta_l$. High productivity workers are driven out of the market.

How to alleviate? We will consider two approaches to alleviate the lemon's problem, depending upon which side of the market makes a move.

- Signaling. The worker, who is informed of the underlying state, makes a move, before he is on the market: the informed party moves first. Because his move is conditioned on the state which he observed, the firm should be able to infer what the worker knows (i.e., his productivity) from the signal.
- Screening. The firm, who is not informed of the underlying state, makes a move, before the firm is on the market: the uninformed party moves first. The firm offers a menu of contracts, which can separate workers with high ability from those with low ability.

3.2 Signaling

Signaling model of Spence [1973] We need to spell out what the signaling is, and how the signaling affects the payoff of each agent. Before a worker looks for a job, he decides how much education he will take. Let e be the amount of education: $0 \leq e < \infty$. The worker's utility is determined by three elements:

- θ. His productivity, or the true state
- w. Wage
- e. Education

Let $u(\theta, w, e)$ be the utility function of a worker in state θ, wage w, and education e. We need to impose a structure to $u(\theta, w, e)$ by specifying how (θ, w, e) determines the utility.

Utility of a worker For simplicity, we assume a quasi linear function:

$$u(\theta, w, e) = w - c(e, \theta)$$

where c is the disutility of education, conditioned on state θ. $\forall e \geq 0$, $\forall \theta$,

- $c(0, \theta) = 0$. No education, no disutility.
- $\partial c(e, \theta)/\partial e > 0$. Marginal disutility of education is positive.
- $\partial^2 c(e, \theta)/\partial e^2 > 0$. Marginal disutility of education is increasing.
- $\partial^2 c(e, \theta)/\partial e \partial \theta < 0$. Marginal disutility of education is decreasing with respect to productivity.

Two key features The first three conditions are easily motivated.

- The only role of education is to generate disutility for the worker. It is an extreme assumption, because education is an important tool to enhance the skill and the human capital of a worker. While restrictive and unrealistic, it strengthens the main conclusion. It is easy to see that if good education improves the productivity, a worker would like to endure disutility of education in return for higher wage in the future. Spence demonstrated that even if education does not improve the productivity, a high ability worker may have incentive to take education only to separate from the lower ability workers.
- The last condition $\partial^2 c(e, \theta)/\partial e \partial \theta < 0$ is known as single crossing property or (Spence-)Mirrlees condition, which warrants additional discussion.

Single crossing property In general, the single crossing property is the monotonicity of the marginal utility with respect to the state.

> **Definition 13.** We say that $u(\theta, e, w)$ satisfies the ***single crossing property*** if $\partial u(\theta, e, w)/\partial e$ is strictly monotonic with respect to θ, and the ***weak single crossing property*** if it is weakly monotonic.

In applications, the marginal (dis)utility of signal (education) can be strictly increasing or decreasing, depending upon the sign of the marginal utility itself. In case of the labor market signaling model, it is strictly decreasing, because the sign of the marginal utility is negative.

- The level of education is the signal for the productivity, because the decision to take education is conditioned on the productivity, and the firm knows the link, if not the productivity.
- The signaling is costly, because the marginal utility of education is strictly negative. In some models, the signaling is costless like cheap talk.
- The single crossing property was first invented by James Mirrlees to characterize the optimal taxation scheme through the first order condition, hence his name in the condition. The single crossing property is a sufficient condition under which the first order condition implies the local maximum.
- In the labor market signaling model, Spence used the same condition to allow the high productivity worker to separate from the low productivity worker. The single crossing property is an essential component for the construction of a separating equilibrium (or signaling equilibrium) in which different types of workers choose different levels of educations so that a firm can infer the productivity of a worker from the education.

Signaling equilibrium Let e_θ be the level of education selected by worker with productivity $\theta \in \{\theta_l, \theta_h\}$.

> **Definition 14.** (e_l, e_h) is a ***signaling equilibrium*** (or ***separating equilibrium***) if $u(\theta_l, e_l, \theta_l) \geq u(\theta_l, e_h, \theta_h)$ and $u(\theta_h, e_h, \theta_h) \geq u(\theta_h, e_l, \theta_l)$.

Discussion

- The original approach by Spence is not game theoretic, and remains vague about what the strategy space is, and what the solution concept is. Let us stick to the original approach, to appreciate his insight.
- A signaling equilibrium is a pair of different education levels, where neither type of workers has incentive to imitate the education level of the other type.
- Because different types of workers choose different levels of education, $\mathbb{E}[\theta \,|\, e_\theta] = \theta$ so that each worker's wage is exactly the productivity of the worker.
- The inequality says that in the signaling equilibrium, type θ worker must have a right incentive to choose the equilibrium education level e_θ instead of choosing the education level of the other type of worker. For this reason, we call the inequality the incentive compatibility constraint.

Why important?

- If a signaling equilibrium exists, the high productivity worker can fetch the wage he deserves, escaping from the trap of the lemon's problem. Thus, the high quality worker may enter the market, and the gains from trading may occurs.

- Signaling model of Spence is the first example to show how we can escape from the lemon's problem, without the third party (i.e., government) intervention. If an outcome is not efficient, the intervention of the government is justified. The insight of Spence is significant, because the lemon's problem can be alleviated without the government's intervention.

Existence

Theorem 4. Let $e_l = 0$, and define e_h implicitly as an education level where the incentive compatibility constraint of θ_l worker is satisfies:

$$u(\theta_l, e_l, \theta_l) \geq u(\theta_l, e_h, \theta_h).$$

Then (e_l, e_h) is a signaling equilibrium.

Proof. We need to verify the incentive compatibility constraint of each type of workers. By the construction of (e_l, e_h), the incentive compatibility constraint of the low productivity worker is satisfied. By the single crossing property, if

$$u(\theta_l, e_l, \theta_l) \geq u(\theta_l, e_h, \theta_h),$$

then

$$u(\theta_h, e_h, \theta_h) \geq u(\theta_h, e_l, \theta_l),$$

and therefore, the incentive compatibility condition of the high productivity worker is satisfied. ■

Define e_h^* implicitly $u(\theta_l, e_l, \theta_l) = u(\theta_l, e_h^*, \theta_h)$ as the education level that binds the incentive compatibility constraint. (e_l, e_h^*) is the most efficient signaling equilibrium among all signaling equilibrium, which is called the Riley outcome.

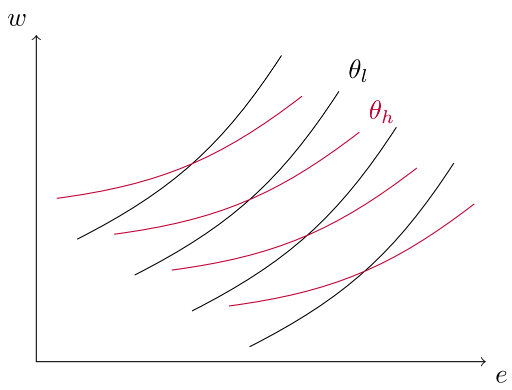


Figure 3.1: Indifference curves of θ_l and θ_h

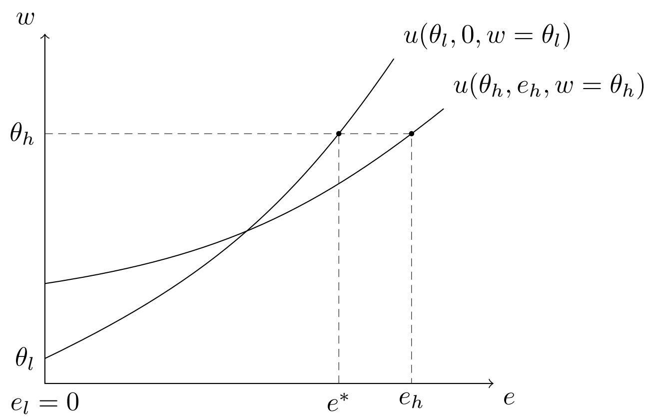


Figure 3.2: Signaling equilibrium in application

Efficient allocation If Spence shows how we can alleviate lemon's problem without the intervention of the government, is the allocation efficient?

- Yes, because both high and low productivity workers are hired by the firm, receiving the same wage as in the complete information about their productivity.
- No, because the incentive compatibility constraint must be satisfied. To do so, the positive amount of eduction must be taken.

First and second best solution If asymmetric information is present, the incentive compatibility constraint must be satisfied in order to reveal the private information about the state truthfully, incurring social cost.

> **Definition 15.** An allocation is the ***first best solution*** if it is a Pareto efficient allocation, and the ***second best solution*** if it is a Pareto efficient allocation subjecting to the incentive compatibility constraint.

The allocation of the Riley equilibrium is not the first best solution, but the second best solution. Typically, in models with asymmetric information, the first best solution is not feasible, unless the allocation satisfies the incentive compatibility constraint.

3.3 Screening

Firm To escape from the lemon's problem, or to prevent the lower productive workers from entering the employee pool, a firm uses a mechanism design to screen lower quality workers.

Lecture 6. screening Mon, Mar 22

The firm has to rely on the difference of the marginal rate of substitution between the wage and the education to screen out one from another group.

Assumptions We maintain the same assumptions on the utility function of the worker, and the firm. Let us summarize the assumptions.

$$u(\theta, w, e) = w - c(e, \theta),\ c(0, \theta) = 0,\ \frac{\partial c(e, \theta)}{\partial e} > 0,\ \frac{\partial^2 c(e, \theta)}{\partial e^2} > 0,$$

and

$$\frac{\partial c(e, \theta)}{\partial \theta} < 0,\ \frac{\partial^2 c(e, \theta)}{\partial e \partial \theta} < 0.$$

Interpretation We continue to assume that the education is only to generate disutility of the workers. We can regard education as (unpleasant) task which must be completed in return for the job (and wage).

First best solution It is easy to see that if productivity θ_i is known to the firm, the firm has to pay for the productivity, without any unpleasant task in an efficient allocation.

Proposition 3. Suppose that the worker's ability is public information. Then,

$$(w_i^*, e_i^*) = (\theta_i, 0) \quad \forall i \in \{h, l\},$$

and the firms obtain 0 profit.

Proof. Since θ_i is known to the firm, it is easy to see that the wage must be equal to the productivity. Since the only function of the unpleasant task is to generate disutility on the part of the worker, no unpleasant task should be imposed in an efficient allocation (the first best solution).

The difficult part is to show that the firms cannot entertain positive profit. Let us assume that there are two firms competing each other as the Bertrand competitor. The case of the multiple firms can be analyzed in the same way.

Let Π_k be the profit of firm k. Define $\Pi = \Pi_1 + \Pi_2$. Suppose that $\Pi > 0$. Since all firms are identical, we can assume without loss of generality that

$$\Pi_1 \leq \frac{\Pi}{2}.$$

Suppose that firm 1 offers $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$ instead of (w_h^*, e_h^*) and (w_l^*, e_l^*). Since (w_l^*, e_l^*) is an equilibrium for low productive workers, the incentive compatibility condition of θ_l worker must satisfy:

$$w_h^* - c(\theta_l, e_h^*) \leq w_l^* - c(\theta_l, e_l^*).$$

By adding equal amount of ε on both sides, we know that $(w_l^* + \varepsilon, e_l^*)$ is also satisfying the incentive compatibility constraint:

$$(w_h^* + \varepsilon) - c(\theta_l, e_h^*) \leq (w_l^* + \varepsilon) - c(\theta_l, e_l^*).$$

By applying the same logic to θ_h worker, we also conclude that $(w_h^* + \varepsilon, e_h^*)$

is incentive compatible. Choose $\varepsilon > 0$ sufficiently small so that

$$\Pi - \varepsilon > \frac{\Pi}{2} > 0.$$

If firm 1 offers menu of contracts of $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$, then all θ_h workers will take $(w_h^* + \varepsilon, e_h^*)$ and all θ_l workers will take $(w_l^* + \varepsilon, e_l^*)$, thus generating profit of $\Pi - \varepsilon$ for firm 1. By assumption,

$$\Pi - \varepsilon > \frac{\Pi}{2} \geq \Pi_1$$

which implies that Π_1 is no longer an equilibrium payoff. This is a contradiction to the hypothesis that Π_1 is an equilibrium profit. $\lightning$

Asymmetric information Suppose that the workers observe their productivity, but no firms observe the productivity of workers. Akerlof indicated that the market is exposed to the lemon's problem.

In contrast to the signaling model of Spence [1973] in which the workers make move, Rothschild and Stiglitz [1976] demonstrated that the uniformed firms can design a menu of contract which allows the firm to escape from the lemon's problem.

Key concepts A contract is (w, e), which specifies wage w and the level $e \geq 0$ of task associated with the job. A menu of contracts is the list of state contingent contracts:

$$M = ((w_h, e_h), (w_l, e_l))$$

where (w_i, e_i) is supposed to be accepted by θ_i workers.

Incentive compatibility constraint A menu of contracts is feasible if the menu satisfies the incentive compatibility constraint of θ_l worker

$$w_h - c(\theta_l, e_h) \leq w_l - c(\theta_l, e_l)$$

and the incentive compatibility constraint of θ_h worker

$$w_l - c(\theta_h, e_l) \leq w_h - c(\theta_h, e_h).$$

Thanks to the single crossing property, whenever the first inequality holds, the second inequality holds. By the incentive compatibility constraint, we usually mean the first constraint, if we assume the single crossing property.

Time line The economy is populated by a continuum of infinitesimal workers and a finite number of (say, two) of identical risk neutral firms. A state is realized so that the productivity of π portion of workers is θ_h and the remaining portion of worker has productivity θ_l where $\theta_h > \theta_l$.

(1) Workers observe their productivity, but no firm observe the productivity.

(2) Each firm offers a menu of contracts.

(3) Each workers chooses a contract from a menu of contracts from a particular firm. If a worker does not choose a contract, he receives 0.

(4) Payoff is realized.

Equilibrium concept Let $M^i = ((w_h^i, e_h^i), (w_l^i, e_l^i))$ be a menu of contracts offered by firm i. Given (M^1, M^2), let q_j^i be the proportion of θ_j workers who accept w_j^i. Since the proportion of θ_j worker is π_j, $q_j^i \pi_j$ mass of type θ_j workers receives wage w_j^i. Let

$$\mathcal{U}^i(M^1, M^2) = \sum_{j' \neq j \in \{h,l\}} [(q_j^i \pi_j \theta_j + (1 - q_{j'}^i)\pi_{j'}\theta_j) - (q_j^i \pi_j + (1 - q_{j'}^i)\pi_{j'})w_j^i]$$

be the expected payoff of firm $i \in \{1, 2\}$.

Note.

$$\begin{aligned}&\mathcal{U}^1(M^1, M^2)\\ &\qquad = (q_h^1 \pi_h + (1 - q_l^1)\pi_l)(\theta_h - w_h^1) + (q_l^1 \pi_l + (1 - q_h^1)\pi_h)(\theta_l - w_l^1).\end{aligned}$$

Note. The firm's payoff may be negative, if a high wage contract draws too many low productivity workers.

Definition 16. (M^1, M^2) is an ***equilibrium*** if $\forall i$, M^i is a menu of feasible contracts (satisfying incentive compatibility constraint), where

$$w_j^i - c^i(\theta_j, e_j^i) \geq 0 \quad \forall i \in \{1, 2\}, \forall j \in \{h, l\}$$

and M^i is a best response against $M^{i'}$ among all possible contracts of firm i $\forall i \neq i' \in \{1, 2\}$.

We will focus on symmetric equilibrium where the two firms offer identical menus: $M^1 = M^2$, thus dropping the superscript to simplify analysis.

Main conclusions Let us summarize the main findings of Rothschild and Stiglitz [1976].

Theorem 5. (1) In any equilibrium, firm's profit is 0.

(2) No pooling equilibrium exists. That is, if $((w_h, e_h), (w_l, e_l))$ is an equilibrium menu, then $(w_h, e_h) \neq (w_l, e_l)$.

(3) If a separating equilibrium exists, (w_l, e_l) and (w_h, e_h) satisfy

$$w_l = \theta_l,\ e_l = 0;\ w_h = \theta_h$$

and e_h is defined implicitly by the incentive compatibility constraint of θ_l worker:

$$\theta_h - c(\theta_l, e_h) = \theta_l - c(\theta_l, 0).$$

(4) A separating equilibrium may not exists.

Proof. We prove the main conclusions in multiple steps, which reveals how the hidden information affects the incentive of the workers, and how the firm can exploit the worker's incentive to screen out different workers.

(1) We follow the same logic as in Proposition 3 to show that the equilibrium profit of the firm must be 0.

(2) In Spence [1973], the single crossing property allows the high productivity worker to signal his productivity credibly to the firm, to fetch a wage equal to his true productivity. In Rothschild and Stiglitz [1976], the single crossing property of the worker's utility allows the firm to screen them out.

Suppose that pooling equilibrium exists. The wage for both type of workers would be expected marginal productivity. A firm can propose another contract (e', w') in Figure 3.3 such that it completely screen out low ability workers but attract only the high ability worker. Thus the firm has a deviation strategy that can generate positive profit.

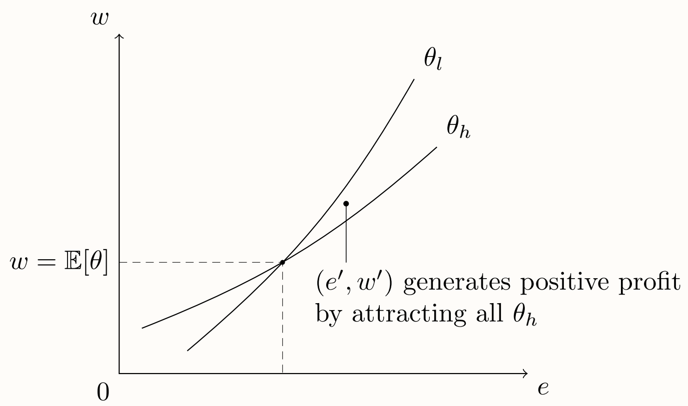


Figure 3.3: No pooling equilibrium exists

(3) If $(w_h, e_h) \neq (w_l, e_l)$, we say it is a separating equilibrium. In a separating equilibrium, (w_j, e_j) is accepted only by θ_j worker $\forall j \in \{h, l\}$. Since the firm makes at least 0 profit, the wage must be equal to the productivity.

In a separating equilibrium, the competitive pressure forces each firm to offer $e_l = 0$ so that the low productivity workers endure no unpleasant task. This result is similar to the property of the signaling equilibrium of Spence [1973] where the low productivity worker does not take any (unpleasant) education. The difference is that in Spence [1973], the decision by the worker is motivated completely by the negative payoff of taking education, while in Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer no task for low productivity workers.

We can find a profitable deviation by a firm. If I offer (w_h, e_h) and (w_l', e_l') instead (w_h, e_h) and (w_l, e_l), low ability workers will accept the new offer and in return I can pay them less wage. Note that it is still incentive compatible. In other word, I can let them work for me at low wage than before in return for less task which is completely useless for me because task is only used to generate disutility for the workers. So by reducing the task I can lower the wage and still offer attractable to low ability workers. In such a way I can have more profit. Therefore in equilibrium e_l should be 0.

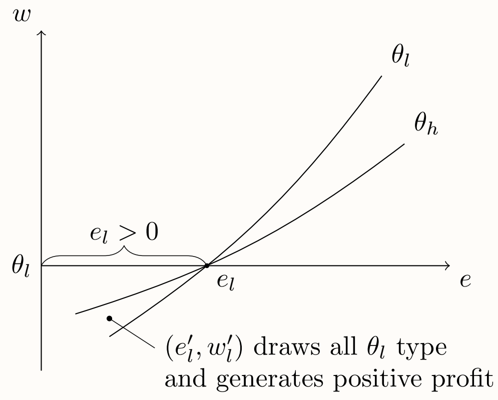


Figure 3.4: $e_l = 0$ in a separating equilibrium

To be a feasible menu, the incentive compatibility condition must hold in a separating equilibrium $((w_h, e_h), (w_l, e_l))$.

$$\theta_h - c(\theta_l, e_h) \leq \theta_l - c(\theta_l, e_l).$$

In Spence [1973], there are multiple signaling equilibria where the weak inequality holds strictly. Only in the Riley outcome, the weak inequality holds with equality. In Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer a menu in which the incentive compatibility constraint is binding (i.e., the weak inequality holds with equality).

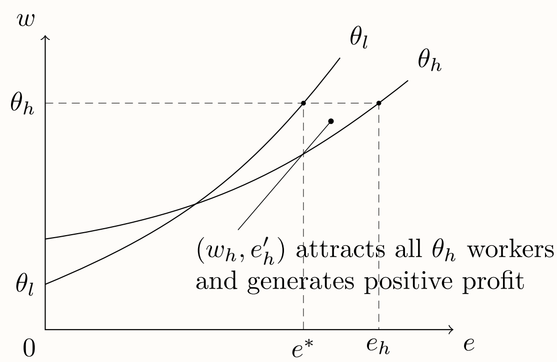


Figure 3.5: $e_h = e^*$ in a separating equilibrium

Suppose that $M = ((\theta_h, e_h), (\theta_l, 0))$ and $e_h > e^*$. Then, in return for lower task, a firm can offer lower wage for high ability workers. But still high ability workers will be happier in a new contract even though he receive less than marginal productivity because it is more compensated by less task. So, the firm can attract high ability workers at the lower wage which improve profit. Thus, if incentive compatibility constraint satisfied strictly, then firms can always find out profitable deviation. Hence in equilibrium, incentive constraint must be satisfied with equality.

(4) A separating equilibrium may not exists in Rothschild and Stiglitz [1976]. Because no pooling equilibrium exists, no equilibrium exists if a separating equilibrium fails to exists in Rothschild and Stiglitz [1976].

The conditions under which no equilibrium exists in Rothschild and Stiglitz [1976] further reveals the close relationship between Spence [1973] and in Rothschild and Stiglitz [1976].

Let us consider the Riley outcome, which is the best possible signaling equilibrium for the workers. Because of the disutility of education, a pooling equilibrium can generate higher (ex ante) profit for workers.

As it turns out, Rothschild and Stiglitz [1976] fails to have an equilibrium, if and only if a pooling equilibrium generates higher (ex ante) profit than the Riley outcome in Spence [1973].

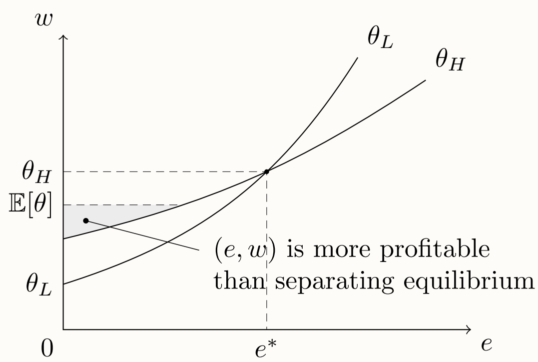


Figure 3.6: No separating equilibrium exists

■

Note. Rothschild and Stiglitz [1976] did not establish the existence of a separating equilibrium. Later studies modified the original model of Rothschild and Stiglitz [1976] to ensure the existence of an equilibrium for all parameter values.

3.4 Hidden action

Arrow [1963] observed the moral hazard problem in the health care industry, and first treated it as an incentive problem rather than a moral problem. His observation is ground breaking. If it is a moral problem, an economist has little to say about the remedy. If the problem arises as a consequence of incentive of a rational agent, an economist can propose an alternative rule of the game to guide the incentive to generate more efficient outcome.

Principal agent problem The moral hazard problem is a core concept in the contract literature, which is too extensive for us to cover in this class. Instead, we examine a textbook example of a principal agent problem, which is a classic case of contract subject to moral hazard problem.

Story Consider a risk neutral principal who has to write a contract to ask a risk averse agent to produce an object. A share cropping is an old example of the principal agent problem, where the landlord (the principal) rents out a plot of land so that a tenant farmer produce grain to generate profit.

The profit is then to shared between the two parties so that the principal can receive the rent. The contract stipulates the wage conditioned on the profit of the agent.

If there exists one to one correspondence between profit and effort of agent, by observing the profit, landlord knows exactly how much effort agent exercise. In such a case, it is called perfect monitoring and there is no room for hidden action.

The profit is an increasing function of the effort of the agent plus noise. The profit is a noisy signal of the effort level. The principal can only imperfectly monitor the agent's effort level because of the noise. Because agent's action (i.e., effort) is hidden from the principal, the moral hazard problem arises.

Informational structure Let π and e be the profit and the effort. The distribution of π is given by the density unction $f(\pi|e)$. We make two assumptions.

- Imperfect monitoring: $\forall e,\ f(\pi|e) > 0$.
- Stochastic increasing: if $e > e'$, then $f(\pi|e)$ first order stochastically dominates $f(\pi|e')$: $\forall \pi$

$$\int_{-\infty}^{\pi} f(x|e)\,dx < \int_{-\infty}^{\pi} f(x|e')\,dx.$$

Because the moral hazard problem arises whenever the monitoring of the agent's action is imperfect, we often call the class of moral hazard problems the models with imperfect monitoring.

Principal A contract is the wage schedule as a function of profit: $w = w(\pi)$. The principal is risk neutral, whose expected payoff is

$$E(\pi - w(\pi)|e) = \int_{\pi} (\pi - w(\pi)) f(\pi|e)\,d\pi.$$

Agent Agent's utility function is $u(w,e)$ where w is wage and e is effort.

- Risk averse: $u_w(w,e) > 0$ and $u_{ww}(w,e) < 0$.
- Increasing marginal disutility of effort: $u_e(w,e) < 0$ and $u_{ee}(w,e) < 0$.

In many applications, we use additively separable utility

$$u(w,e) = v(w) - g(e)$$

with $v' > 0$, $v'' < 0$, $g' > 0$ and $g'' > 0$.

First best solution Suppose that the monitoring is perfect so that the principal is not subject to the moral hazard problem.

The principal is risk neutral while the agent is risk averse. In the first best solution, the principal should take all the risk of the agent so that the agent's income stream must be constant.

Because the principal can monitor the effort e, he can include the level of effort in the contract along with the wage schedule. Thus, his optimization problem is

$$\max_{e,(w(\pi))_\pi} \int (\pi - w(\pi)) f(\pi|e)\, d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi)) f(\pi|e)\, d\pi - g(e) \geq \underline{u}$$

for some utility level $\underline{u}$, which is typically the outside option of the agent. We usually normalize $\underline{u} = 0$.

Useful observation The optimization problem is complex, because the principal is maximizing over the set of wage schedules instead of wages. We make two observations.

Lemma 1. If $(e^*, w^*(\pi))$ is an optimal solution, then the individual rationality constraint must be binding:

$$\int v(w^*(\pi)) f(\pi|e^*)\, d\pi - g(e^*) = \underline{u}.$$

Proof. Suppose that

$$\int v(w^*(\pi)) f(\pi|e^*)\, d\pi - g(e^*) > \underline{u}.$$

Then, the principal can offer slightly less wage, still satisfying

$$\int v(w^*(\pi) - \varepsilon) f(\pi|e^*)\, d\pi - g(e^*) > \underline{u}.$$

Since the utility of the principal is strictly decreasing in wage, the principal has higher expected payoff from $(e^*, w^*(\pi) - \varepsilon)$. ↯

We treat the individual rationality constraint as an equality constraint. For each e, we solve

$$\max_{(w(\pi))_\pi} \int (\pi - w(\pi)) f(\pi|e)\, d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi)) f(\pi|e)\, d\pi - g(e) = \underline{u}.$$

Then, we choose e^* to maximize objective function.

First order condition For fixed e, consider the Lagrangian

$$\mathcal{L} = \int (\pi - w(\pi)) f(\pi|e)\, d\pi + \lambda \left[\int v(w(\pi)) f(\pi|e)\, d\pi - g(e) - \underline{u} \right].$$

The first order condition with respect to $w(\pi)$ for each realized value of π is

$$f(\pi|e) - \lambda v'(w(\pi)) f(\pi|e) = 0.$$

After canceling out $f(\pi|e) \neq 0$ from both sides, we have

$$v'(w(\pi)) = \frac{1}{\lambda} \quad \forall \pi.$$

Since $v'' < 0$, v' is invertible. In any optimal value of $w(\pi)$ given e,

$$w(\pi) = [v']^{-1} \left(\frac{1}{\lambda} \right).$$

Note. The right hand side is independent of π.

Full coverage insurance Thus, the wage conditioned on π must be constant with respect to π, for any given e. In the first best solution w^*,

$$w^* = [v']^{-1} \left(\frac{1}{\lambda} \right)$$

so that the agent receives constant stream of wage regardless of the output level. The risk neutral principal offers a full insurance to the agent, in return for his effort e^* so that $v(w^*) - g(e^*) = 0$.

Discussion

- The first best solution relies critically on the assumption that the principal can perfectly monitor the effort level e^*.
- If the agent knows that his effort level is not perfectly monitored, then he has effort level is not perfectly monitored, then he has incentive to reduce his effort because his income w^* is independent of his effort, which is the moral hazard problem.

- The first best solution is infeasible if the monitoring is imperfect.
- To prevent the moral hazard problem, the principal has to design the contract to link the output to the effort level, exposing the agent to the risk. Because the output is stochastically increasing with respect to effort, lowering effort level leads to lower profit and therefore, lower wage on average.
- Instead of offering a full insurance against risk, the principal has to offer a partial insurance, if the moral hazard problem is present.

Hidden action If the principal has no way to get information about e, then there is no hope to have a contract between the principal and the agent. On the other hand, if the principal can perfectly monitor e, no moral hazard problem arises and the first best solution is feasible.

By hidden action, we mean that the principal can imperfectly monitor e so that the first best solution is not feasible. The moral hazard problem becomes relevant, as we can design a contract to guide the incentive of the agent to a socially desirable outcome, if the first best solution is not feasible.

Incentive compatibility constraint The central idea is to design the wage schedule $w(\pi)$ and the effort e so that it is better for the agent to follow the (implicitly) assigned effort level than to disobey the instruction:

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) \geq \int v(w(\pi))f(\pi|e')\,d\pi - g(e') \quad \forall e'.$$

The left hand side is the expected utility if the agent follows the contract $(w(\pi), e)$, while the right hand side is the expected payoff if the agent chooses e', given wage schedule $w(\pi)$.

This constraint is called the incentive compatibility constraint. Let us call $(w(\pi), e)$ a feasible contract if $(w(\pi), e)$ satisfies the incentive compatibility constraint.

Optimization problem of the principal The optimization problem under imperfect monitoring is similar to the optimization problem under perfect monitoring, but has the additional constraint of incentive compatibility:

$$\max_{e,(w(\pi))_\pi} \int (\pi - w(\pi))f(\pi|e)\,d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) \geq \underline{u}$$

and the incentive compatibility

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) \geq \int v(w(\pi))f(\pi|e')\,d\pi - g(e') \quad \forall e'.$$

Second best solution Since $e' = e$ must be the best response of the agent under $w(\pi)$, the first order condition of the utility maximization must hold if $e' = e$:

$$\int v(w(\pi)) \frac{\partial f(\pi|e)}{\partial e} \, d\pi - g'(e) = 0.$$

The optimal solution must satisfy the incentive compatibility constraint, thus generating lower profit for the principal. The optimal solution subject to the incentive compatibility constraint is called the second best solution.

Chapter 4

Auction Theory

Lecture 7.
basicauction
Wed, Mar 24

Review: Games with incomplete information

- The idea of screening different types of workers is essentially to design a program so that the agent is willing to reveal his private information.
- This idea of designing a program to extract private information of an agent is known as the mechanism design problem, which is one of the most important topics as it has a broad range of important applications.
- Instead of introducing to the mechanism design literature directly, let us use a monopolistic market with incomplete information as the laboratory to explore the area of mechanism design, while analyzing yet another extremely important topic of auctions.

4.1 Introduction

Monopolistic market is the trading protocol in which a single seller is trading against many buyers. The number of commodities available to the buyer is usually assumed to be no more than the number of buyers so that the buyer have to compete among themselves.

The monopolistic seller tries to exploit the competitive pressure among buyers to extract larger than the competitive equilibrium surplus.

By an auction, we mean a monopolistic trading protocol. A classic example would be a textbook model of the monopolistic market in which a single seller is facing a continuum of infinitesimal buyer, and sells the infinitely divisible goods by posting a single price.

Why study auctions? Auction is an important institution through which a large amount of goods and services is traded. It is probably one of the oldest trading protocol, going back to Babylon according to Herodotus in allocating brides to grooms. As auctions are widely used for a broad range of economic environment, the auction has many institutional and informational variations.

We study an auction to understand how the price is formed, which is intentionally left vague in Arrow Debreu economy. As the game theoretic tools develop, we can rigorously analyze the equilibrium outcome of the monopolistic trading protocol, and examine how the parameters of the model such as preference, technology and budget affect the delivery price of the good.

4.1.1 Institutional variations

Institutional assumptions Let us assume that there is a single seller who has one unit of a good for sale. We can also consider multiple units for sale, which is an important topic to study. For this class, we focus on the case of the single object for sale.

The seller has no role in the auction, other than placing the good for sale. The seller is risk neutral. We normalize the valuation of the seller to be 0. If the good is sold with probability π and the delivery price is p, then his expected payoff is πp.

Standard auctions There are N buyers, whom we often call bidders. Let v_i be the valuation of buyer i. The buyer has a linear preference, thus risk neutral. If he pays p, his surplus is $v_i - p$. Throughout this class, we focus on the class of standard auctions.

> **Definition 17.** An auction is a ***standard auction*** if a buyer with the highest bid wins the object.

Four basic auctions We start with four basic auctions among standard auctions.

- First price auction
- Dutch auction
- English auction
- Second price auction

Let us describe verbally the rule of each of four basic auctions.

First price auction

- Each bidder place a bid simultaneously. Let b_i be the bid.
- The highest bidder wins the object.
- The winner pays his bid.

Because each bidder does not observe the actions of other bidders, this auction is often called closed auction. The class of auctions in which the winner pays his bid is called *pay your bid auction.*

Dutch auction The name comes from the fact that one of the main exports of Netherlands is tulip, which requires a quick transaction. In fact, the auction of fish often follows the same format, which also requires a quick transaction.

In front of bidders, a bulletin board displays a price. Starting from a price so high no one will ever buy the object, the displayed price drops continuously. When a bidder stops the clock, the auction stops. The bidder who stops the clock wins the object, paying the price displayed on the bulletin board.

Difference Because the price drops, this class of auctions is called the *descending bid auction*. An interesting feature of this auction compared to the first price (sealed bid) auction is that a losing bid is not observed, even after the auction is over. Even though a bid is sealed, an economist can access the record of losing bids, if a good is auctioned according to the first price auction. In the Dutch auction, the auction stops immediately when a bidder stops the clock. Because losing bidders have no chance to take an action, an economist cannot access the record of what corresponds to losing bids.

English auction Auction houses like Christie's or Sothby's sell paintings and other goods through an auction, known as English auction.

The seller places a good on the table, and the auctioneer calls the starting bid. If a bidder responds to the bid by a signal (or through a proxy), the auctioneer increases the bid. The bidding continues until no one responds to the call of the auctioneer. Then, the good is delivered to the last bidder.

Because the bidding is done openly, and the price is increasing as the auction progresses, this class of auctions is called the *open ascending bid auction*.

Button auction The English auction in Christie's or Sothby's is quite complex to analyze. Instead, we examine a simple version of the open ascending bid auction, called the button auction or the Japanese auction.

The auction room has a bulletin board, which displays a price. Each bidder is placed in a cubicle so that he cannot see what other bidders are doing. Each cubicle has a button. A bidder places the button, if he wants to remain in the auction, competing for the good. If two or more bidders are pressing the button, the price in the bulletin board is increasing over time. As the price increases, bidders may drop out. As soon as only one bidder is left pressing the button, the auction ends. The last bidder wins the object, paying the price displayed on the bulletin board.

Simplifying features

- We do not allow re-entry. We assume that when a bidder stops pressing the button, he leaves the auction once and for all. In reality, re-entry is common, but the analysis of a model with re-entry of bidders is difficult.
- For now, we assume that each bidder does not observe whether others are pressing the button or not, because each bidder is placed in a cubicle. This assumption is mainly for simplifying the analysis, even though the same assumption eliminates the feature of open auction from the button auction.
- After describing and analyzing the basic model where each bidder is placed in a cubicle, we investigate an open auction version of the button auction, where each bidder observes at which price other bidders quit.
- Depending upon the informational structure of the auction, the openness of the button auction makes an important difference to the strategic behavior of bidders.

Second price auction Remember how the first price (sealed bid) auction is conducted. The second price auction is almost identical with the first price auction, except that the winner pays the second highest bid (thus, the highest losing bid), as the name suggests.

This auction format is rarely used, as it is. The main reason is that we consider the bid as the signal of bidder's willingness to pay. If the winner pays less than his bid, we tend to be suspicious of the rule of the auction.

Vickrey auction This auction was invented by William Vickrey. For his contribution, the second price auction is called *Vickrey auction*.

As the ensuing analysis reveals, this auction format has an important property, and its equilibrium outcome serves as the benchmark for the outcomes of other auctions. For that reason, the auction theory class usually starts with the second price auction.

Even though its original form is rarely used in reality, its equivalent form is widely used. We have yet to explain what we mean by being equivalent and to point out what the equivalent auction is.

4.1.2 Informational variations

Informational assumptions

- Private value. When a bidder attends the auction, the bidder knows his own valuation of the object, but not others. To generate utility, the bidder need to compete for the product, raising his bid. But, to obtain positive surplus, he would not pay more than what the good is worth to him.

- Common value. If the government auctions off the right to explore mineral or oil buried underground, the bidders do not know the value of the product, but observe only the samples which are noisy signal of the value of the product. The competitive pressure pushes the bid upward. Because the bidder does not know the true value, but knows only the estimated value based upon his information, it is possible that the winner may end up paying too much for the object. This phenomenon is known the winner's curse.

Inter-related utility In many auctions, the object has two component. A house is a good example. A house provides valuable service as a shelter. Its private value is the present discounted value of the service provided by the house. But, a house has an investment value, for which the bidder has only a noisy information.

We start with the private value model, spending most time analyzing the private value models. We later examine classic cases of (pure) common value models.

Definition 18. The basic institutional and informational assumptions for the private value auctions are known ***independent private value (IPV) models***.

A1 Private value: Bidder i observes his reservation value v_i drawn from $[\underline{v}_i, \overline{v}_i]$ according to cumulative differentiable distribution function $F_i(v_i)$. The seller's reservation value is normalized to 0.

A2 $\forall i \neq j$, $F_i \perp\!\!\!\perp F_j$: v_i and v_j are independent. $F_j(v_j|v_i) = F_j(v_j)$.

A3 Symmetry: $F_1 = \cdots = F_n = F$, $\underline{v}_i = 0$ and $\overline{v}_i = 1$.

A4 Risk neutral: The bidders and the seller are risk neutral. The expected payoff of a bidder can be represented as $p_i v_i - x_i$ where p_i is the probability of winning the object, and x_i is the expected payment.

A5 No reserve price: If the highest bid is higher than the seller's reservation value (which is normalized to 0), the good is sold.

A6 No entry fee: A losing bidder receives 0.

Discussion

- The private value assumption is a substantive assumption. Observing his reservation value at the start of the auction is the defining characteristics of the private value model.

- Independence and symmetry is imposed largely for the simplicity of exposition. If the private value is correlated, we have a number of interesting questions which we will not cover at this point. The symmetry is mainly for the simplicity, but also justify the selection of symmetric Nash equilibrium. We normalize $\underline{v} = 0$ and $\overline{v} = 1$.

Risk neutral players Risk neutral bidders and seller are substantive assumptions. Because the bidder's objective is summarized into the probability of winning and the expected payment, we greatly simplify the analysis. If the bidder is risk averse, the bidder is also concerned of the second moment of the event that he is winning, which changes the nature of bidding behavior substantially.

Note. We use the separated expected payoff $p_i v_i - x_i$, not $p_i(v_i - x_i)$ because even if the bidders lose the object, they pay some money like entry fee in some auction.

Reserve price and entry fee No reserve price and no entry fee assumptions are institutional assumptions. These are the instruments which the seller can use to control the competitive pressure and to raise additional revenue. By raising the reserve price, the seller increases the probability of no sales, but hopefully insure himself against unusually low winning price. The reserve price are sometimes public and sometimes private. If reserve price exists, English auction would start at certain price.

Entry fee generates revenue. If a bidder has low reservation value, he would find it not worthwhile attending the auction paying the entry fee because his chance of winning the object is too low. As a result, fewer bidders will participate which lower the competitive pressure among bidders and the revenue of the seller. The optimal ways of setting reserve price and entry fee were studied and well understood.

4.1.3 Four basic auctions

Let us formulate the four basic auctions as a game with incomplete information (Bayesian game) within the framework of IPV models. A strategy σ_i of bidder i is a function (bidding function) from his reservation value to a bid. Formally, $\sigma_i\colon [0,1] \to \mathbb{R}$ where $\sigma_i(v_i) = b_i$ is called a bid of bidder i. Let Σ_i be the strategy space of bidder i. To complete the description of each one of four basic auctions, we have to describe the payoff function:

$$\mathcal{U}_i\colon \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{R}.$$

It would be more convenient to describe the state contingent payoff

$$u_i(b_1, \cdots, b_n, v)$$

conditioned on the underlying state $v = (v_1, \cdots, v_n)$ and a profile $(b_1, \cdots, b_n)$ of bids.

Definition 19. The ***ex ante expected utility*** (payoff function) is unconditional expectation at the point before bidder i observe v_i:

$$\mathcal{U}_i(\sigma_1, \cdots, \sigma_n) = E_v[u_i(\sigma_1(v_1), \cdots, \sigma_n(v_n), v_1, \cdots, v_n)].$$

Definition 20. The ***interim expected utility*** is conditional expectation in the interim stage that bidder i observed v_i but not v_{-i}:

$$U_i(b_i, v_i) = E_{v-i}[u_i(\sigma_{-i}(v_{-i}), b_i, v_1, \cdots, v_n)|v_i]$$

where

$$\sigma_{-i}(v_{-i}) = (\sigma_1(v_1), \cdots, \sigma_{i-1}(v_{i-1}), \sigma_{i+1}(v_{i+1}), \cdots, \sigma_n(v_n))$$

is the profile of strategies except for bidder i's strategy.

A bidder makes a decision conditioned on v_i, but not on v_{-i}, because he only observes his valuation but not others. Therefore, the interim expected utility function is very much useful in describing the decision by bidder i.

At the seller's view, however, the ex ante expected utility is maybe more relevant since he can't observe v. On the other hand, the ex post is another stage that all $v_1, \cdots, v_n$ are revealed, in which the decision already made but the efficiency can be improved through resale.

First price auction A bidding function specifies the number (i.e., bid) submitted by bidder i conditioned on v_i, which is formally $\sigma_i\colon [0,1] \to \mathbb{R}$. The delivery rule of the auction is that the highest bidder wins the object, paying his bid. The state contingent payoff function of bidder i is then

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

where k is the number of the bidders with the highest bids. We assume no entry fee so that a losing bidder's payoff is 0. We also assume that if there are multiple winners, the good is allocated randomly among multiple winners with an equal probability.

Dutch auction A bidding function specifies the the price at which bidder i conditioned on v_i raises his hand to stop the clock, which is formally $\sigma_i : [0, 1] \to \mathbb{R}$. The delivery rule of the auction is that the bidder who first raises his hand (or stops the clock first) wins the object, paying his bid. The state contingent payoff function of bidder i is then

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

where k is the number of the bidders with the highest bids.

> **Note.** The mathematical specification of the strategy and the payoff of the first price and the Dutch auctions are identical. The only difference is the name of a strategy.

Strategic equivalence Because a rational player is free from framing effect, his decision should not be affected by the name of a strategy. His decision is completely determined by the strategy and the payoff function. If two games have the same strategy and payoff function, the outcomes played by rational players must be identical.

> **Definition 21.** If two games have the same normal form game, then the two games are ***strategically equivalent***.

Because the first and the Dutch auctions are strategically equivalent, we only examine the first price auction.

Second price auction As the name suggests, the delivery rule of the second price auction is that the highest bidder wins the object, but pays the second highest bid. Thus, the state contingent utility of bidder i is

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - \max_{j \neq i} b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - \max_{j \neq i} b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j. \end{cases}$$

Because the payoff function is different from the first price auction, the second price auction is a different game from the first price auction. Despite the formal different, the ensuing analysis will show that the equilibrium outcomes of these two auctions are quite similar.

English auction Let us consider the button auction version of English auction. The strategy of bidder i is to choose a price conditioned on v_i at which he drops out. Given b_i, the delivery rule of the button auction is identical with the second price auction. Thus, the state contingent utility of bidder i is

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - \max_{j \neq i} b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - \max_{j \neq i} b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j. \end{cases}$$

The second price auction and the English auction are strategically equivalent.

Closed and open auctions Among four basic auctions, the first and the Dutch auctions are strategically equivalent, and so are the second and the English auctions. We will examine only the first and the second price auctions.

Because the first price auction does not show the bids of other bidders, it is called a closed auction. In (some versions) of English auction, a bidder can see what other bidders are doing. In this sense, the second price auction is often called the open auction. We start with the analysis of the second price auction.

4.2 Open Auctions

Lecture 8. openauction Mon, Mar 29

4.2.1 Vickrey auction

Second price auction

Definition 22. $\sigma_i(v_i) = v_i$ is the ***truthful bidding strategy*** of bidder i.

Definition 23. σ_i is a ***dominant strategy*** if σ_i is a best response to any σ_{-i}:

$$\mathcal{U}_i(\sigma_i, \sigma_{-i}) \geq \mathcal{U}_i(\sigma_i', \sigma_{-i}) \quad \forall \sigma_i', \forall \sigma_{-i}.$$

Dominance solvable

Theorem 6. The truthful bidding strategy is a dominant strategy.

Proof. Since every player is ex ante identical, let us consider the optimization problem of bidder 1. Let

$$z = \max(\sigma_2(v_2), \cdots, \sigma_n(v_n)).$$

Let $A(b, z, v)$ be the utility of bidder 1, if his reservation value is v, bid b and the highest competing bid is z:

$$A(b, z, v) = \begin{cases} v - z & \text{if } b > z \\ \frac{1}{k}(v - z) & \text{if } b = z \\ 0 & \text{if } b < z \end{cases}$$

where k is the number of the highest bidders.

We show that for any z, $b = v$ is a best response. First, if $v \geq z$, winning is better than losing. By winning the object, the surplus is $v - z \geq 0$. If $v > z$, bidder 1 can win by bidding $b = v$. Second, if $v < z$, losing is better than winning. He can do so by bidding $b = v$. Thus,

$$A(v, z, v) \geq A(b, z, v) \quad \forall b, z,$$

implying that telling the truth is best response ex post. Even after bidder 1 realized what the other players' reservation values and bids are, bidder 1 will not regret. In that sense, ex post best response is much more stronger property than interim best response. Thus, we can show telling the truth is also an interim best response.

In Bayesian game, bidder 1 can't observe σ_{-1}, but instead has belief that z follows the distribution $G(z)$. Then, the interim expected payoff is

$$U_1(b, v) = \int A(b, z, v)\, dG(z).$$

The interim expected payoff of truthful bidding strategy $\sigma_1(v) = v$ is

$$U_1(v, v) = \int A(v, z, v)\, dG(z) \geq \int A(b, z, v)\, dG(z) = U_1(b, v) \quad \forall b,$$

implying that the truthful bidding strategy is a best response against any profile of bidding strategies of other bidders in the interim stage. ■

This result has a profound implication. Each bidder has private information about the valuation of the object. If a social planner wants to allocate a single object efficiently, he has to deliver the good to the bidder with the highest reservation value.

If the reservation value is private information, eliciting the information from the bidder is a major challenge, because the bidder has incentive to report his valuation to his advantage, and need not report truthfully.

The fundamental contribution of Vickrey is that if the utility function of the bidder is quasi-linear (or linear with respect to money), then we can design a trading protocol under which each agent has incentive to tell the truth, because truthful bid is a best response. The second price auction, thus named Vickrey auction, is an example that an economist can design a market in which each bidder is willing to tell the truth voluntarily.

Discussion

- The proof reveals that the truthful bid remains an optimal response, even after z is observed by the bidder, which is called the ex post stage. The truthful bid is optimal in the interim stage, as well as ex post stage.

- If each bidder uses a dominant strategy, then he bids according to his reservation value. Thus, the good is delivered to the bidder with the highest reservation value, if $\max(v_1, \cdots, v_n) \geq 0$ so that the gain from trading exists. The allocation is ex post efficient. Even after the reservation values are realized, we know good is delivered to the bidder with the highest reservation value.

- The activity of resale market is one of indicator of ex post efficiency. If the allocation is ex post efficient, there if no reason for resale, therefore resale market should be very inactive. On the other hand, if the allocation is not ex post efficient, that means, after good is delivered to somebody, everybody realize the reservation values, there may be reason for retrading. So, active resale market is sometimes used as the evidence of failure of ex post efficient.

- The ability of the second price auction to induce each bidder to reveal his reservation value voluntarily, to achieve an efficient allocation makes the second price auction and its equivalent auction, the English auction, an important benchmark for designing the market.

Nash equilibrium Suppose that $\sigma_i^*(v_i) = v_i$ is a truthful bidding strategy. Since σ_i^* is a best response against any profile of bidding strategies of other players, it is also a best response against a profile of truthful bidding strategy.

Theorem 7. A profile of truthful bidding strategies is a Nash equilibrium.

Other Nash equilibria The second price auction has a continuum of Nash equilibria.

Example. $\forall(v_1, \cdots, v_n)$, $(\sigma_1(v_1) = 1, \sigma_2(v_2) = 0, \cdots, \sigma_n(v_n) = 0)$ is a Nash equilibrium, but not efficient.

Proof. To show that the profile constitutes a Nash equilibrium, note that the delivery price is 0, since the second highest bid is 0 and bidder 1 gets the object at the price of 0. Since bidder 1 wins the object at the price of 0 with probability 1, his strategy is a best response.

We need show that $\sigma_{j\neq 1}$ is a best response. Observe that bidder j's equilibrium payoff is 0. To win the object, bidder j has to bid at least 1, because $\sigma_1(v_1) = 1$. If bidder j bids higher than bidder 1, the second highest bid is 1 which is the bid of bidder 1. Thus, the payoff from winning the object is $v_j - 1 \leq 0$. Thus, any deviation from $\sigma_j(v_j) = 0$ cannot generate a positive payoff. Thus, σ_j is a best response. $\diamond$

All other Nash equilibria than the one with the truthful bidding strategy shares common properties.

- The equilibrium strategy of some player must be dominated by the truthful bidding strategy, because the truthful bidding strategy is a dominant strategy. If we apply the elimination of dominated strategies, all Nash equilibria except for the truthful bidding strategy equilibrium are eliminated.

- The equilibrium allocation is not efficient with a positive probability, in contrast to the truthful bidding strategy Nash equilibrium, which always induces an efficient allocation ex post. It is due to the fact that the Nash equilibrium strategy is not symmetric so that the highest bid may not come from someone with the highest reservation value.

From now on, whenever we refer to the equilibrium outcome of the second price auction, we mean the outcome induced by the truthful bidding strategy of each bidder, which is efficient ex post.

4.2.2 Revenue

Auction is a monopolistic trading protocol. If the monopolist has a choice over the protocol, the expected revenue from an auction rule is a key figure to guide the choice of the monopolist.

In the second price auction, the winner is the bidder with the highest reservation value, but pays the second highest bid which is equal to the second highest reservation value. We need to calculate the expected value of the second highest reservation value out of $\{v_1, \cdots, v_n\}$.

Order statistics Let $v_{(k)}$ be the k-th highest reservation value among $v_1, \cdots$, v_n drawn from the same distribution F. We call it k-th order statistic. If we need to highlight the number of samples, we sometimes write $v_{(k:n)}$ in place of $v_{(k)}$. Since the bidder with the highest reservation value wins, and pays the second highest reservation value, $v_{(1)}$ and $v_{(2)}$ are the two random variables of interest. Let $F_{(k)}$ be the cumulative distribution function of k-th order statistic. Our goal is to calculate $F_{(k)}$ and the expected value of $v_{(k)}$ for $k \in \{1, 2\}$.

First order statistics

$$F_{(1)}(v) = P(v_{(1)} \leq v) = F^n(v)$$

Therefore, the density function is

$$f_{(1)}(v) = nF^{n-1}(v)f(v).$$

Note that

$$\lim_{n\to\infty} F^n(v) \begin{cases} 1 & \text{if } v \geq 1 \\ 0 & \text{if } v < 1. \end{cases}$$

Thus, $v_{(1)}$ converges weakly (converges in distribution) to 1.

Second order statistics

$$\begin{aligned} F_{(2)}(v) &= P(v_{(2)} \leq v) \\ &= P(v_{(1)} \leq v) + P(v_{(2)} \leq v < v_{(1)}) \\ &= F^n(v) + nF^{n-1}(v)(1 - F(v)) \\ &= nF^{n-1}(v) - (n-1)F^n(v) \\ f_{(2)}(v) &= n(n-1)F^{n-2}(v)(1 - F(v))f(v) \end{aligned}$$

Example. Suppose that $F(v)$ is the uniform distribution over $[0,1]$. Then,

$$\begin{aligned}
f_{(1)}(v) &= nv^{n-1} \\
E(v_{(1)}) &= \int_0^1 v f_{(1)}(v)\,dv = \frac{n}{n+1} \\
f_{(2)}(v) &= n(n-1)v^{n-2}(1-v) \\
E(v_{(2)}) &= \int_0^1 n(n-1)v^{n-1}(1-v)\,dv = n(n-1)\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{n-1}{n+1}.
\end{aligned}$$

4.2.3 English auction

When we illustrate the English auction, we presented two different versions of the auction.

- Basic version. Each bidder is placed in a cubicle so that he cannot observe the actions of other bidders. In particular, he cannot observe whether other bidders are still in the auction or have dropped out at a specific price.
- Open version. Each bidder can observe what others do.

The first version of the English auction is strategically equivalent to the second price auction. Let us examine the second version, which is indeed an open auction, in the sense that the behavior of every bidder is opened to others.

Open ascending bid auction In an open version of the English auction, the strategy space of each bidder is larger than the strategy space of the basic version of the English auction.

As the auction progresses, bidder i can observe the price at which another bidder drops out, and based upon the information, he can change the price at which he drops out. His strategy is now to choose a price to drop out, conditioned on who dropped out at what prices. Since every bidder is ex ante identical, the name of the bidder is not relevant. We can focus on the sequence of prices at which bidder have dropped out.

Let $h_k = \{p_{n-1}, \cdots, p_{k+1}\}$ with $h_{n-1} = \emptyset$ be a history of prices at which $n-k$ bidders have dropped out. Among $n-1$ competing bidders, one of the competitors has dropped out at price p_{n-1}, and p_{k+1} is the latest observation of the price at which one of $k+1$ competing bidders has dropped out. Conditioned on h_k, bidder i has k competing bidders.

Let H_k be the set of all possible h_k. Bidder i's strategy conditioned on h_k

$$\sigma_{i,k} \colon [0,1] \times H_k \to \mathbb{R}$$

specifies the price at which bidder i drops out. Bidder i's strategy is

$$\sigma_i = (\sigma_{i,n-1}, \cdots, \sigma_{i,1}).$$

Dominance solvable

Definition 24. A game is ***dominance solvable*** if the game has a unique strategy profile which survives the repeated elimination of dominated strategies.

If a game has a dominant strategy like the second price auction, the game has a unique strategy profile after one round of elimination of dominated strategies. We now allow the elimination process can continue more than a single round. Let us consider the truthful bidding strategy in the open ascending bid auction:

$$b_{i,k}(v_i, h_k) = v_i \quad \forall v_i, \, \forall h_k, \, \forall k \in \{1, \cdots, n-1\}.$$

That is, bidder i drops out at his reservation value following any history.

Theorem 8. The open ascending bid auction is dominance solvable. The only strategy surviving the repeated elimination of dominated strategies is the truthful bidding strategy following every history.

Proof. Fix h_1. If bidder i has a single competing bidder in the open ascending bid auction, he has the same decision problem as in the second price auction. His dominant strategy is to drop out at his reservation value.

Suppose that we have shown that dropping at at his reservation value is a dominant strategy for $h_1, \cdots, h_{k-1}$. Fix h_k so that k competing bidders are left. If $j \leq k-1$ bidders quit, then bidder i will play again $k-j$ bidders, where the truthful bid is shown to be a dominant strategy. If k bidders quit simultaneously, then he will be the winner.

Following the same logic as in the proof to show that the truthful bid is a dominant strategy in the second price auction, we can show that conditioned on being a winner against k bidders, it is a dominant strategy to bid his reservation value. Thus, conditioned on h_k, it is a dominant strategy to drop out at his reservation value. ∎

The crucial assumption is the private value model. Because each bidder observes the valuation of the object, the observation of other player's action does not change his valuation.

This is not the case in the common value model, in which a bidder does not know the true value, but has an estimate of the underlying value. By observing the price at which other bidders drop out, he can use the information to update his estimate of the value, and adjust his bid. The equilibrium outcome of the basic version of the English auction differs from the outcome of the open version of the English auction.

Discussion Because we have to apply the elimination process from the end of the game, we have to apply repeatedly the elimination of dominated strategies. In the end, we select a truthful bidding strategy, which leads to the efficient allocation.

The key assumption is that the valuation of the object does not change, conditioned on history. This assumption holds in the private value model, but not in the common value model which we will discuss later.

4.3 Closed Auctions

Lecture 9. closedauction Wed, Mar 31

First price auction In the second price auction, the truthful bidding strategy is a dominant strategy. In an open ascending bid auction, it is the only strategy that survives the repeated elimination of dominated strategies.

In a sharp contrast, the same strategy does not survive the repeated elimination of dominated strategies.

Proposition 4. The truthful bidding strategy does not survive the repeated elimination of dominated strategies in the first price auction.

Proof. The payoff from the truthful bidding strategy is always 0. If he wins, the payment is equal to his reservation value so that the surplus is 0. If he loses, the surplus is 0.

If bidder i places a bid $v_i - \varepsilon$, then with a positive probability $v_i - \varepsilon$ is the highest bid and his expected payoff is bounded away from 0. A strategy which admits a bid above the reservation value is a dominated strategy. With probability $F^{n-1}(v_i - \varepsilon)$, all reservation value of other bidders is less than $v_i - \varepsilon$ and therefore, $v_i - \varepsilon$ will be a winning bid to generate a positive surplus for bidder i. Thus, the truthful bid is dominated by $v_i - \varepsilon$, after we eliminate all bidding strategies which admits a bid above the reservation value. ■

The first price auction does not have a dominant strategy or is not dominance solvable. The calculation of a Nash equilibrium is more involved than for the second price auction.

Definition 25. A Nash equilibrium $(\sigma_1^*, \cdots, \sigma_n^*)$ is a ***symmetric Nash equilibrium*** if $\sigma_1^* = \cdots = \sigma_n^*$.

Bidders are ex ante symmetric: Until a bidder observes his reservation value v_i (which differs across different bidders), each bidder has the same type of utility function and the distribution over this reservation value is identical.

For this reason, a symmetric Nash equilibrium has been the focal point of analysis in the literature.

Interim probability In games with incomplete information, like the auctions we are examining now, the information is revealed as the game progresses. In a typical model, a bidder makes a decision in a interim stage, when bidder i observes v_i , but does not observes the valuation of others, v_{-i}.

A related, and important, concept is ex ante and ex post. Before any bidder observes his own reservation value, we call the stage ex ante. Naturally, the ex ante expected utility must be the right way to calculate the return from his choice.

The ex post stage is essentially the time when the game is over, so that each bidder observes the reservation value of every player. It is too late to make a decision at the ex post stage. Ex post stage is still relevant for the efficiency of allocation, and possibility of re-trading.

Because the decision is made in the interim stage, it is convenient to write down the relevant probability conditioned on the information of bidder i, which include his reservation value and his bid.

Because we focus on the symmetric Nash equilibrium, we will drop subscript i from the variable. For example, instead of v_i and b_i , we write v and b.

Notation Let $\hat{Q}_i(b_i)$ be the probability that a bidder wins the object if he places bid b_i. Formally,

$$\hat{Q}_i(b_i) = P\left(b_i \geq \max_{j \neq i} \sigma_j(v_j)\right).$$

> **Note.** $\hat{Q}$ does not depend on bidder i's reservation value.

Since bidder i does not observe v_{-i}, $\hat{Q}$ depends only upon b_i, given σ_{-i}. Then, the (interim) expected payoff of bidder i with reservation value v_i is

$$\Pi_i(v_i, b_i) = (v_i - b_i)\hat{Q}_i(b_i).$$

Since we consider a symmetric Nash equilibrium, we drop subscript i and write

$$\Pi(v, b) = (v - b)\hat{Q}(b).$$

Symmetric Nash equilibrium

> **Theorem 9.** Define
>
> $$\sigma^*(v) = v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{n-1} dx.$$
>
> $(\sigma^*, \cdots, \sigma^*)$ is the unique symmetric Nash equilibrium.

Since $F(x) > 0$ for $x > 0$, $\sigma^*(v) < v$ for $v > 0$. In equilibrium, bidder places a bid less than his reservation value. The difference is called bid shading.

> **Note.**
>
> $$\frac{F(x)}{F(v)} = P(v_j \leq x \,|\, v_j \leq v).$$

Thus,

$$\begin{aligned}\left[\frac{F(x)}{F(v)}\right]^{n-1} &= P\left(\max_{j \neq i} v_j \leq x \,\middle|\, \max_{j \neq i} v_j < v\right) \\ &= P(v_{(1:n-1)} \,|\, v_{(1:n-1)} \leq v)\end{aligned}$$

The symmetric equilibrium strategy has two alternative representations, which reveal different properties of the equilibrium strategy. By integrating by part,

$$\begin{aligned} v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{n-1} dx &= v - x \left[\frac{F(x)}{F(v)}\right]^{n-1}\Bigg|_0^v + \int_0^v x\, d\left[\frac{F(x)}{F(v)}\right]^{n-1} \\ &= \int_0^v x\, d\left[\frac{F(x)}{F(v)}\right]^{n-1} \\ &= \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-1}(v)}\, dx \end{aligned}$$

Note.

$$\int_0^v x\, d\left[\frac{F(x)}{F(v)}\right]^{n-1}$$

is the expected value of a random variable over $[0, v]$ whose distribution is

$$\left[\frac{F(x)}{F(v)}\right]^{n-1}$$

which is the distribution function of the highest reservation value among $n-1$ competing bidders, conditioned on the event that the highest reservation value does not exceed v. Recall that we write the first order statistics among $n-1$ samples as $v_{(1:n-1)}$.

Thus, the integration can be written as

$$E(v_{(1:n-1)}) \,|\, v_{(1:n-1)} \leq v).$$

For later reference, let us collect three different formula of the symmetric Nash equilibrium strategy.

$$\begin{aligned} \sigma^*(v) &= v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{n-1} dx \\ &= E(v_{(1:n-1)}) \,|\, v_{(1:n-1)} \leq v) \\ &= \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-1}(v)}\, dx. \end{aligned}$$

The first formula is useful to show the existence and the size of the bid shading. The second expression provides an economic interpretation of the equilibrium bid.

The last expression is used to prove the theorem. In particular, the last formula is used to show that the bidding function is strictly increasing with respect to v.

First and second price auctions Despite different rules of auctions, the two auctions share important properties.

- Since the bidding function is strictly increasing with respect to v, and every bidder uses the same strategy, the bidder with the highest reservation value wins the object with probability 1.

- The expected payoff of the bidder whose reservation value is the lowest (i.e., $v = 0$) is 0.

These are two main conditions for the revenue equivalence theorem which is the fundamental result in the auction theory. Roughly speaking, in the frame work of IPV, any Nash equilibrium in any auction which satisfies these two conditions generates the expected revenue for the seller equal to $E(v_{(2:n)})$ (that is the expected revenue of the second price auction).

If you compare the first and the second price auctions, the first price auction appears to have an advantage in generating higher revenue for the seller, because the winner has to pay the highest bid, instead of the second highest bid. Since the bidders are rational, the bidders bid less aggressively in the first price auction than in the second price auction. In the second price auction, a bidder is willing to bid up to his reservation value, but in the first price auction, his bid is strictly less than his reservation value. The revenue equivalence theorem says that the impacts of these two factors are perfectly cancelled out in any Nash equilibrium satisfying the two properties.

Example. Suppose that v is uniformly distributed over [0, 1]. We know that

$$E(v_{(2:n)}) = \frac{n-1}{n+1}.$$

$$\sigma^*(v) = v - \int_0^v \left(\frac{x}{v}\right)^{n-1} dx = v - \frac{1}{v^{n-1}}\frac{v^n}{n} = v - \frac{1}{n}v.$$

The amount of bid shading is $\frac{v}{n}$. As the number of bidders increases, a bidder is compelled to place a bid closer to his reservation value.

Since the winner pays his bid, the seller's expected revenue from the first price auction is

$$\begin{aligned} E(\sigma^*(v_{(1:n)})) &= E\left(\frac{n-1}{n}v_{(1:n)}\right) \\ &= \frac{n-1}{n}E(v_{(1:n)}) = \frac{n-1}{n}\frac{n}{n+1} = \frac{n-1}{n+1} \end{aligned}$$

which is exactly the same as the expected revenue from the second price auction.

Now we prove Theorem 9.

Proof (Theorem 9)**.** First, we show that $(\sigma^*, \cdots, \sigma^*)$ is the symmetric Nash equilibrium.

Step 1. $\sigma^*(v)$ is strictly increasing in $v > 0$. We use the third formula of $\sigma^*(v)$, in combination with the first formula.

$$\sigma^*(v) = \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-1}(v)} dx.$$

By differentiating the right hand side with respect to v, we have

$$\frac{d\sigma^*(v)}{dv} = \frac{(n-1)(v-\sigma^*(v))f(v)}{F(v)} > 0$$

since

$$v - \sigma^*(v) = \int_0^v \left(\frac{x}{v}\right)^{n-1} dx > 0 \quad \forall v > 0.$$

Since $\sigma^*(v)$ is strictly increasing, the inverse function of $\sigma^*(v)$ exists. Let

$$\varphi^*(b) = (\sigma^*)^{-1}(b)$$

be the inverse function of σ^*. Recall that $\hat{Q}(b)$ is the interim probability that bidder i wins the object if he bids b.

Step 2. $\hat{Q}(b) = F^{n-1}(\varphi^*(b))$

$$P(b \geq \sigma^*(v)) = P(\varphi^*(b) \geq v) = F(\varphi^*(b)).$$

Since the strategy of other players is independent,

$$\begin{aligned}\hat{Q}(v) &= P(b \geq \max_{j\neq i} \sigma^*(v_j)) \prod_{j\neq i} P(b \geq \sigma^*(v)) \\ &= \prod_{j\neq i} F(\varphi^*(b)) = F^{n-1}(\varphi^*(b)).\end{aligned}$$

We know that the interim expected utility of a bidder with reservation value v when he bids b is

$$\Pi(v,b) = (v-b)F^{n-1}(\varphi^*(b)).$$

First order condition.

$$\begin{aligned}\frac{\partial \Pi(v,b)}{\partial b} &= (n-1)(v-b)F^{n-2}(\varphi^*(b))f(\varphi^*(b))(\varphi^*)'(b) \\ &\qquad - F(\varphi^*(b))^{n-1} = 0\end{aligned} \tag{4.1}$$

if $b = \sigma^*(v)$. Because

$$(\varphi^*)'(b) = \frac{1}{\sigma^{*\prime}(v)}, \quad \varphi^*(\sigma^*(v)) = v,$$

we have

$$\frac{\partial \Pi(v,b)}{\partial b} = -F^{n-1}(v) + (n-1)(v-\sigma^*(v))F^{n-2}(v)f(v)\frac{1}{\sigma^{*\prime}(v)} = 0.$$

By collecting the terms, we have

$$(\sigma^*)'(v) = \frac{(n-1)(v - \sigma^*(v))F^{n-2}(v)f(v)}{F^{n-1}(v)}$$

which is the first order differential equation of σ^*.

Second order condition. Before we proceed, we must check whether the first order condition identifies a local maximum. To do so, we prove that the second order condition of the maximization is satisfied. This task is trickier than usual, because the density function is continuous but may not be differentiable. We need to show

$$\Pi_b(v, b) = \frac{\partial \Pi(v, b)}{\partial b}$$

is a decreasing function in a neighborhood of $b = \sigma^*(v)$, even if f is not differentiable. Recall that (4.1). We need to show that in a small neighborhood of $b = \sigma^*(v)$, if $b' > \sigma^*(v)$, then

$$\frac{\partial \Pi(v, b')}{\partial b} < 0.$$

Note

$$\frac{\partial^2 \Pi(v, b)}{\partial b \partial v} = (n-1)F^{n-2}(\varphi^*(b))f(\varphi^*(b))(\varphi^*)'(b) > 0.$$

Since σ^* is differentiable and strictly increasing, $\forall b' > \sigma^*(v)$ in a small neighborhood of $b = \sigma^*(v)$, $\exists v' > v$ so that $b' = \sigma^*(v')$. In particular, $b' = \sigma^*(v') > \sigma^*(v) = b$. Note that

$$\frac{\partial \Pi(v, \sigma^*(v))}{\partial b} = \frac{\partial \Pi(v', \sigma^*(v'))}{\partial b} = 0.$$

Since $\frac{\partial \Pi(v,b)}{\partial b}$ is strictly increasing in v, and $v' > v$,

$$0 = \frac{\partial \Pi(v', \sigma^*(v'))}{\partial b} > \frac{\partial \Pi(v, \sigma^*(v'))}{\partial b} = \frac{\partial \Pi(v, b')}{\partial b}$$

so that

$$0 = \frac{\partial \Pi(v, \sigma^*(v))}{\partial b} > \frac{\partial \Pi(v, b')}{\partial b}$$

for $b' > \sigma^*(v)$ as desired. By following the same logic, we have

$$\frac{\partial \Pi(v, \sigma^*(v))}{\partial b} < \frac{\partial \Pi(v, b')}{\partial b}$$

for $b' < \sigma^*(v)$.

Solving differential equation. From the first order condition, we have a differential equation.

$$(\sigma^*)'(v) = \frac{(n-1)(v - \sigma^*(v))F^{n-2}(v)f(v)}{F^{n-1}(v)}.$$

Moving terms, we have

$$(\sigma^*)'(v)F^{n-1}(v) + (n-1)\sigma^*(v)F^{n-2}(v)f(v) = (n-1)vF^{n-2}(v)f(v).$$

Note that the right hand side is the derivation of $[\sigma^*(v)F^{n-2}(v)]$.

Therefore,

$$[\sigma^*(v)F^{n-2}(v)]' = (n-1)vF^{n-2}(v)f(v).$$

By the fundamental law of calculus,

$$\sigma^*(v)F^{n-2}(v) - \sigma^*(0)F^{n-2}(0) = \int_0^v (n-1)xF^{n-2}(x)f(x)\,dx.$$

By assumption, $F(0) = 0$. Therefore,

$$\sigma^*(v) = \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-2}(v)}\,dx$$

which is the third formula of the symmetric Nash equilibrium strategy.

Uniqueness. It takes more work to show that the symmetric Nash equilibrium is unique. Because the method we used in the proof is replicated in different models, we will go through the proof.

We show that in any symmetric Nash equilibrium $(\sigma, \cdots, \sigma)$ the following three properties must hold.

A. $\sigma(0) = 0$.

B. σ is differentiable over $(0, 1)$.

C. σ is strictly increasing.

We first show that the three properties imply that σ must be the third formula of the symmetric Nash equilibrium strategy. Then, we establish three properties in several steps.

Proposition 5. If σ satisfies A, B, and C, then $\sigma(v) = \sigma^*(v)$.

Proof. By A and C, $\sigma(0) = 0$ and is strictly increasing over $[0, 1]$. Since σ is strictly increasing, it is invertible. Define $\varphi(b) = \sigma^{-1}(b)$. In a symmetric equilibrium in which the equilibrium strategy is strictly increasing, the probability of winning the object is

$$P\left(v \geq \max_{j \neq i} v_j\right) = F^{n-1}(v).$$

Thus, the interim expected equilibrium payoff is

$$\Pi(v, \sigma(v)) = (v - \sigma(v))F^{n-1}(v).$$

Since σ is an equilibrium strategy, any deviation should not be profitable. In particular, fix any $v, z \in [0, 1]$ with $v > z$. In equilibrium, reservation value v bidder should not have incentive to imitate the bid of reservation value z, and vice versa. Thus,

$$\begin{aligned}(v - \sigma(v))F^{n-1}(v) &\geq (v - \sigma(z))F^{n-1}(z)\\ (z - \sigma(z))F^{n-1}(z) &\geq (z - \sigma(v))F^{n-1}(v).\end{aligned}$$

From the first constraint,

$$v[F^{n-1}(v) - F^{n-1}(z)] \geq \sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z)$$

and from the second constraint,

$$\sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z) \geq z[F^{n-1}(v) - F^{n-1}(z)].$$

Combining the two inequality, and using $v > z$, we have

$$\begin{aligned}v\frac{F^{n-1}(v) - F^{n-1}(z)}{v - z} \geq \frac{\sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z)}{v - z}&\\ \geq z\frac{F^{n-1}(v) - F^{n-1}(z)}{v - z}&.\end{aligned}$$

Because these two inequalities must hold any $v - z > 0$, we can let $v - z \to 0$ to have

$$\sigma(v)F^{n-1}(v)]' = v[F^{n-1}(v)]' = (n - 1)vF^{n-2}(v)f(v).$$

By the fundamental law of calculus,

$$\sigma(v)F^{n-1}(v) - \sigma(0)F^{n-1}(0) = \int_0^v (n - 1)xF^{n-2}(x)f(x)\,dx.$$

Since $\sigma(0) = F(0) = 0$,

$$\sigma(v) = \int_0^v \frac{(n - 1)xF^{n-2}(x)f(x)}{F^{n-1}(v)}\,dx$$

which is exactly the third formula of $\sigma^*(v)$. ■

We establish A, B, and C in several steps.

Step 1. $\exists v_0$ such that $\forall v \leq v_0$, $\sigma(v) = 0$ if $Q(\sigma(v)) > 0$. Note that

$$\Pi(v, b) = (v - b)\hat{Q}(b)$$

is a strictly increasing function of v if $\hat{Q}(b) > 0$, and $\Pi(0, b) \leq 0$, $\forall b$. Define

$$v_0 = \sup\{\hat{v} \mid \sup_b \Pi(\hat{v}, b) \leq 0\}.$$

By the definition of v_0, $\forall v < v_0$, $\Pi(v, b) < 0$ if $\hat{Q}(b) > 0$.

Step 2. $\sigma(v)$ is a weakly increasing function of $v \geq v_0$. Fix any $v, z \geq v_0$. Without loss of generality, assume $v > z \geq v_0$. By the definition of v_0, $v - \sigma(v) > 0$ and $z - \sigma(z) > 0$. By the incentive compatibility constraint,

$$(v - \sigma(v))\hat{Q}(\sigma(v)) \geq (v - \sigma(z))\hat{Q}(\sigma(z))$$
$$(z - \sigma(z))\hat{Q}(\sigma(z)) \geq (z - \sigma(v))\hat{Q}(\sigma(v)).$$

Adding up both sides, and simplifying the terms, we have

$$(v - z)(\hat{Q}(\sigma(v)) - \hat{Q}(\sigma(z))) \geq 0.$$

Since $v - z > 0$, we have

$$\hat{Q}(\sigma(v)) - \hat{Q}(\sigma(z)) \geq 0$$

which is short of proving that $\hat{Q}(b)$ is weakly increasing, because we have yet to show that $\sigma(v)$ is weakly increasing.

By the definition of v_0, bidder with reservation value v receives strictly positive payoff. Thus,

$$\hat{Q}(\sigma(v)) > 0$$

which implies

$$\frac{\hat{Q}(\sigma(z))}{\hat{Q}(\sigma(v))} \leq 1.$$

From the incentive compatibility constraint of bidder with reservation value z, we know

$$\frac{\hat{Q}(\sigma(z))}{\hat{Q}(\sigma(v))} \geq \frac{z - \sigma(v)}{z - \sigma(z)}.$$

Therefore, $\sigma(v) \geq \sigma(z)$ from which we conclude that $\hat{Q}(b)$ is weakly increasing. The following is a useful corollary.

Corollary. $\hat{Q}(\sigma(v))$ is weakly increasing in v.

Step 3. $\sigma(v)$ is strictly increasing over $(v_0, 1]$. Suppose that $\sigma(v)$ is weakly increasing, but not strictly increasing around the neighborhood of some reservation value:

$$\exists\, z, w \in (v_0, 1] \text{ so that } \sigma(z) = \sigma(w).$$

Since σ is weakly increasing, $\forall z' \in (z, w)$,

$$\sigma(z) = \sigma(z')\sigma(w).$$

Recall that $\hat{Q}(b) = [P(\sigma(v) \leq b)]^{n-1}$. The probability inside of the bracket is the probability that b can beat bidder j whose is using strategy $\sigma(v)$. Since all bidders are ex ante identical, and there are $n-1$ competing bidders, we multiply the probability $n-1$ times. Since $\sigma(v)$ is flat over interval $[z, w]$,

$$\lim_{\varepsilon \to 0} \hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z) - \varepsilon) > 0$$

which should have been 0, if σ is strictly increasing.

In particular, if there are multiple winners, the object is distributed according to random allocation rule. Thus,

$$\lim_{\varepsilon \to 0} \hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z)) > 0.$$

At $v = z$, bidder i's equilibrium payoff is

$$(z - \sigma(z))\hat{Q}(\sigma(z)).$$

If he increases his bid by $\varepsilon > 0$, the expected payoff is

$$(z - \sigma(z) - \varepsilon)\hat{Q}(\sigma(z) + \varepsilon).$$

Note that

$$\begin{aligned}(z - \sigma(z) - \varepsilon)\hat{Q}(\sigma(z) + \varepsilon) - (z - \sigma(z))\hat{Q}(\sigma(z)) \\ = (z - \sigma(z))[\hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z))] \\ - \varepsilon[\hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z))].\end{aligned}$$

Since $z > v_0$, $z - \sigma(z) > 0$. As $\varepsilon \to 0$, the first term is bounded away from 0, while the second term vanishes. Thus, for a sufficiently small $\varepsilon > 0$, a slight increase of the bid from $\sigma(z)$ increases the profit, which contradicts the hypothesis that $\sigma(z)$ is an equilibrium bid. The following is a useful corollary.

Corollary. $\sigma(v) > 0 \ \forall v > v_0$.

Step 4. $v_0 = 0$ and $\lim_{v \to 0} \sigma(v) = 0$. We now know

$$\sigma(v) > 0 \quad \text{and} \quad \hat{Q}(b(v)) > 0 \quad \forall v > v_0.$$

Note that if a bidder place a bid equal to $\varepsilon > 0$, his chance of winning the object is at least $F^{n-1}(v_0 + \varepsilon)$. $\forall v \leq v_0$, a bidder can generate at least

$$(v - \varepsilon)F^{n-1}(v_0 + \varepsilon) \quad \forall \varepsilon > 0.$$

By the definition of v_0, the expected payoff of bidder with reservation value $v \leq v_0$ is 0. Thus,

$$0 \leq \lim_{\varepsilon \to 0}(v-\varepsilon)F^{n-1}(v_0+\varepsilon) = vF^{n-1}(v_0) \leq \Pi(v,\sigma(v)) = 0.$$

Thus, $v_0 = 0$. We know $\forall v > v_0$, $v - \sigma(v) > 0$. If

$$\lim_{v \to 0} \sigma(v) > 0,$$

then there is a small $v > 0$ where $v - \sigma(v) < 0$. Since $\forall v > v_0 = 0$, $\hat{Q}(v) > 0$, $v - \sigma(v) < 0$ implies that $\Pi(v, \sigma(v)) < 0$, which is a contradiction to the hypothesis that σ is an equilibrium strategy. $\lightning$

Step 5. σ is a differentiable and

$$\sigma'(v) = \frac{(n-1)(v-\sigma(v))f(v)}{F(v)}.$$

Since $\sigma(v)$ is strictly increasing over $[0,1]$,

$$\hat{Q}(\sigma(v)) = F^{n-1}(v).$$

Since the incentive compatibility constraint must hold $\forall v, z \in (0,1)$ with $v > z > 0$,

$$\begin{aligned}(v-\sigma(v))F^{n-1}(v) &\geq (v-\sigma(z))F^{n-1}(z)\\(z-\sigma(z))F^{n-1}(z) &\geq (z-\sigma(v))F^{n-1}(v)\end{aligned}$$

which implies that

$$\begin{aligned}v(F^{n-1}(v) - F^{n-1}(z) &\geq \sigma(v)F^{n-1}(v) - \sigma(z))F^{n-1}(z)\\ \sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z) &\geq z(F^{n-1}(v) - F^{n-1}(z)).\end{aligned}$$

Hence,

$$\begin{aligned} v\frac{F^{n-1}(v) - F^{n-1}(z)}{v-z} &\geq \frac{\sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z)}{v-z}\\ &\geq z\frac{F^{n-1}(v) - F^{n-1}(z)}{v-z}.\end{aligned}$$

As $v - z \to 0$, the first and the last term converge to the same limit:

$$v[F^{n-1}(v)]'.$$

Then,

$$(\sigma(v)F^{n-1}(v))' = v(F^{n-1}(v))'$$

from which the conclusion follows.

$\blacksquare$

Chapter 5

Mechanism Design

5.1 Revenue comparison

Lecture 10. mechanism Mon, Apr 5

Revenue equivalence theorem

- Auction is a monopolistic trading protocol. The expected revenue generated by an auction is an important factor for a monopolist to use the format.
- Many institutional and informational variations of the auction make it challenging to compare the equilibrium payoff of the monopolist.
- It is useful to have a benchmarket for the revenue comparison by identifying a set of conditions under which a broad set of auctions generates the same expected revenue of the monopolist: revenue equivalence theorem.
- The importance of the revenue equivalence theorem is not in the fact that we prove that a broad class of auctions generates the same expected revenue, but in the conditions under which the revenue equivalence can be established. Based upon the revenue equivalence theorem, we can infer the source of the difference in the revenue of two different auctions.

Theorem 10 (Revenue equivalence theorem)**.** Consider IPV. Fix any Nash equilibrium of any auction, which satisfies the two properties.

(1) the bidder with the highest reservation value wins the object with probability 1.

(2) the expected payoff the lowest reservation value is 0.

The expected revenue of the seller from such a Nash equilibrium is $E(v_{(2:n)})$ which is the expected revenue from the second price auction.

The first condition implies that the outcome must be efficient, because the good is delivered to a bidder with the highest reservation value with probability 1. The second condition implies that the auction should not have a positive entry fee.

Discussion

- Strong. The theorem applies to a broad class of IPV models, imposing little restriction on the institutional details of the auction.
- Weak. The conditions of the theorem are the properties of an endogenous variable (equilibrium strategy) rather than the primitives of the model.

5.2 Mechanism design

Revelation principle The proof appears to be an impossible task.

We need to calculate a Nash equilibrium from each IPV model, checking whether the Nash equilibrium satisfies the two properties and then, check the seller's revenue from the Nash equilibrium is equal to the revenue from the second price auction.

The paper by Myerson [1981][1] shows otherwise, opening up a new and important field of mechanism design.

Hurwicz triangle An auction is a social institution to allocate a good to bidders. The bidders communicate with the auction through their strategies. Based upon the strategies, the auction decides the probability of winning the object and the expected payment of each bidder.

Let us consider an abstract general model of auction, beyond the framework of IPV. Let

$$\sigma_i : [0,1] \rightarrow A_i$$

be a strategy of bidder. A_i is the space of feasible actions, which can be the price (or bid) or any signal. We impose no restriction on what A_i should be. Depending upon the rule of an auction, A_i can be highly complex. The complexity of A_i poses the major challenge in proving the revenue equivalence theorem.

The auction (or the auctioneer) does not observe $(v_1, \cdots, v_n)$ but only observes the realized value of the strategy of each player

$$\sigma(v) = (\sigma_1(v_1), \cdots, \sigma_n(v_n)).$$

Based upon the data, the auction decides

$$(\hat{Q}_i(\sigma(v)), \hat{p}_i(\sigma(v)))$$

where $\hat{Q}_i(\sigma(v))$ is the probability that the good is delivered to bidder i and $\hat{p}_i(\sigma(v))$ is the expected payment by bidder i. Let us call

$$(\hat{Q}, \hat{p}) = \{(\hat{Q}_i(\sigma(v)), \hat{p}_i(\sigma(v)))\}_{i=1}^n$$

the allocation rule.

Since $\hat{Q}_i(\sigma(v))$ is a probability,

$$\sum_{i=1}^n \hat{Q}_i(\sigma(v)) \leq 1$$

[1] Roger B. Myerson [1981] "Optimal Auction Design" Mathematics of Operations Research, Vol 6, No 1, pp 58-73

but does not have to be equal to 1, because the good may not be sold with a positive probability.

Similarly, the auctioneer may impose entry fee, or some sort of fee so that bidder i may have to pay even if he does not receive the good. Thus, it is entirely possible that for some realization of $(v_1, \cdots, v_n)$, $\hat{p}_i(\sigma(v)) > 0$ but $\hat{Q}_i(\sigma(v)) = 0$.

Given allocation rule $(\hat{Q}, \hat{p})$, each bidder chooses $\hat{\sigma}_i$ conditioned on his valuation v_i, maximizing his interim expected utility against the profile $\hat{\sigma}_{-i}$ of other bidders' strategies: $\forall i$, $\forall v_i \in [0, 1]$,

$$\begin{aligned} &E[\hat{Q}_i(\sigma(v))v_i - \hat{p}_i(\hat{\sigma}(v)) \,|\, v_i] \\ &\qquad \geq E[\hat{Q}_i(\hat{\sigma}_{-i}(v_{-i}), b_i)v_i - \hat{p}_i(\hat{\sigma}_{-i}(v_{-i}), b_i) \,|\, v_i] \quad \forall b_i. \end{aligned}$$

Note. The deviation b_i from the equilibrium strategy $\hat{\sigma}_i(v_i)$ can be conditioned on v_i.

As it is stated, the calculation of a Nash equilibrium is practically an impossible task, because the action space A_i can be extremely complicated. The fundamental contribution by Myerson [1981] is to show that we can construct another Bayesian game which has a Nash equilibrium with the same allocation as the Nash equilibrium of the original game, while the action space is significantly simpler.

Revelation game Given allocation rule $(\hat{Q}, \hat{p})$, and Nash equilibrium $\hat{\sigma}$ of an auction, let us consider a revelation game, in which the action space is the reservation values of bidder i:

$$\sigma_i \colon [0, 1] \to [0, 1].$$

In this game, each bidder reveals (not necessarily truthfully) his reservation value, thus the name of the game.

The allocation rule of the revelation game is defined as $Q(v) = \hat{Q}(\hat{\sigma}(v))$ and $p(v) = \hat{p}(\hat{\sigma}(v))$ $\forall v = (v_1, \cdots, v_n) \in [0, 1]^n$.

Theorem 11 (Revelation principle)**.** The constructed revelation game has a Nash equilibrium in which the truthful revelation

$$\sigma_i(v_i) = v_i \quad \forall i, \forall v_i$$

is a Nash equilibrium strategy. The allocation of the truthful Nash equilibrium is the same as the allocation of the Nash equilibrium of the original game $\forall v$.

Proof. Since $\hat{\sigma}$ is a Nash equilibrium of the original game with allocation rule $(\hat{Q}, \hat{p})$,

$$\begin{aligned} &E[\hat{Q}_i(\sigma(v))v_i - \hat{p}_i(\hat{\sigma}(v)) \,|\, v_i] \\ &\qquad \geq E[\hat{Q}_i(\hat{\sigma}_{-i}(v_{-i}), b_i)v_i - \hat{p}_i(\hat{\sigma}_{-i}(v_{-i}), b_i) \,|\, v_i] \quad \forall b_i. \end{aligned}$$

By the definition of new allocation rule (Q, p), we can write the same equilibrium condition as

$$E[Q_i(v)v_i - p_i(v) \,|\, v_i] \geq E[Q_i(v_{-i}, b_i)v_i - p_i(v_{-i}, b_i) \,|\, v_i] \quad \forall b_i.$$

Interpreting b_i as a different reservation value than v_i, we conclude that the inequality is precisely the incentive compatibility constraint of the truth telling Nash equilibrium. By the construction of allocation rule (Q, p),

$$Q(v) = \hat{Q}(\hat{\sigma}(v)) \quad \text{and} \quad p(v) = \hat{p}(\hat{\sigma}(v)) \quad \forall v,$$

thus inducing the same allocation as the Nash equilibrium of the original game. ∎

Discussion

- This result is known as the revelation principle. For any Nash equilibrium of any auctions, we can construct a revelation game which has a truth telling Nash equilibrium with the same allocation. In that sense, we can consider a revelation game and a truthful Nash equilibrium without loss of generality, where allocation rule (Q, p) satisfies the incentive compatibility constraint:

$$E[Q_i(v)v_i - p_i(v) \,|\, v_i] \geq E[Q_i(v_{-i}, v_i')v_i - p_i(v_{-i}, v_i') \,|\, v_i].$$

- The revelation principle shows that the complexity of the action space of different auctions is irrelevant for the analysis of the Nash equilibrium. The revelation game is not a social institution, but an analytic tool we construct to investigate the social institution of interest. For this reason, we sometimes refers to the revelation game associated with an auction (which is a social institution) the auction mechanism.

- In the revelation game, the profile of reservation values directly determines the allocation, while in the original game, the allocation is determined by the profile of reservations values, indirectly through the auction rule. First, the bidders reports the actions to the auctioneer conditioned on the realized value of reservation value, who then determine the allocation based on the actions, not the reservation values. For this reason, we often call the revelation game the direct mechanism.

- We saw the same incentive compatibility constraint while we examine the adverse selection problem. The uninformed party has to design a contract to elicit the truthful information. In this sense, we transfer the auction into an adverse selection problem, which turns out to be much more manageable than the original problem.

- Allocation rule (Q, p) satisfying the incentive compatibility constraint will be the focus of analysis. Since the design of (Q, p) determines the economic properties of the mechanism, our exercise is often called mechanism design problem.

5.3 Revenue equivalence theorem

Lecture 11. optimalauction Wed, Apr 7

Interim expectation Let us define the interim values of (Q, p).

$$Q_i(v_i) = \int Q_i(v_{-i}, v_i)\, dF_{-i}(v_{-i})$$
$$p_i(v_i) = \int p_i(v_{-i}, v_i)\, dF_{-i}(v_{-i})$$

If bidder with reservation value v_i reports truthfully, his interim expected payoff conditioned on v_i is

$$\Pi_i(v_i) = Q_i(v_i)v_i - p_i(v_i).$$

Increasing probability of winning Since the incentive compatibility constraint must satisfied, $\forall v_i, v_i'$ with $v_i > v_i'$,

$$\Pi_i(v_i) = Q_i(v_i)v_i - p_i(v_i) \geq Q_i(v_i')v_i - p_i(v_i')$$
$$\Pi_i(v_i') = Q_i(v_i')v_i' - p_i(v_i) \geq Q_i(v_i)v_i' - p_i(v_i)$$

By arranging terms, we have

$$Q_i(v_i)[v_i - v_i'] \geq \Pi_i(v_i) - \Pi_i(v_i') \geq Q_i(v_i')[v_i - v_i'].$$

Since $v_i > v_i'$,

$$Q_i(v_i) \geq Q_i(v_i')$$

which is the first important implication of the incentive compatibility constraint.

For the seller, a buyer with a high reservation value is more valuable than the one with a low valuation. Since the reservation value is a private information, the seller needs to provide an incentive for the buyer with a high reservation value to report his reservation value truthfully. To do so, the probability of winning should be increasing with respect to the reported reservation value. In a certain sense, the buyer is extracting informational rent from the seller.

Increasing equilibrium payoff The second important implication of the incentive compatibility constraint is that

$$\Pi_i(v_i) \geq \Pi_i(v_i')$$

or the equilibrium payoff must be increasing with respect to the reservation value.

Since $v_i > v_i'$, we can write the incentive compatibility constraint as

$$Q_i(v_i) \geq \frac{\Pi_i(v_i) - \Pi_i(v_i')}{v_i - v_i'} \geq Q_i(v_i').$$

Since Π_i is increasing, it is almost everywhere differentiable. Whenever it is differentiable,

$$\Pi_i'(v_i) = Q_i(v_i).$$

Interim payoff and probability of winning By the fundamental law of calculus,

$$\Pi_i(v_i) = \Pi_i(0) + \int_0^{v_i} Q_i(x)\,dx.$$

This equation is the third, and probably the most important, implication of the incentive compatibility constraint.

Recall that

$$\Pi_i(v_i) = Q_i(v_i)v_i - p_i(v_i).$$

In principle, the seller has to manipulate two functions of Q_i and p_i to control the incentive of the bidder. If (Q, p) satisfies the incentive compatibility constraint, $\Pi_i(v_i)$ is the function of $\Pi_i(0)$ and Q_i. Instead of controlling two functions, the seller controls one number $\Pi_i(0)$ and one function Q_i subject to the incentive constraint (Q_i is increasing), which is significantly simpler problem.

The expected payment is then derived according to

$$p_i(v_i) = Q_i(v_i)v_i - \int_0^{v_i} Q_i(x)\,dx - \Pi_i(0).$$

Revenue equivalence While the revenue equivalence theorem is a statement about the expected revenue of the seller, we can infer the equilibrium payment of each bidder in a Nash equilibrium which satisfies the two conditions of the revenue equivalence theorem.

Since the bidder with the highest reservation value wins the object with probability 1, $Q_i(v_i)$ is exactly the probability that

$$v_i \geq \max_{j \neq i} v_j$$

and therefore,

$$Q_i(v_i) = F^{n-1}(v_i).$$

Since the bidder with reservation value 0 receives 0,

$$\Pi_i(0) = 0.$$

Hence, in any Nash equilibrium satisfying the two conditions of the revenue equivalence theorem, the interim expected payment of bidder with reservation value v_i is

$$\Pi_i(v_i) = F^{n-1}(v_i)v_i - \int_0^{v_i} F^{n-1}(x)\,dx.$$

For example, in the first price and the second price auction, the expected payment of a bidder is the same.

Discussion

- We should take this statement with a grain of salt. The same conclusion remains silent about the second moment of the actual payment (which is a random variable). Because the bidder is risk neutral, his objective function is the expected payoff. Thus, from the view point of a risk neutral bidder, the expected payment of the two auctions is the same.

- If a bidder is risk averse, then his expected payoff is affected by the mean but also by the variance of the distribution of the payment. The first and the second price auctions induce the equipment payment with the same mean but different variance. (More later)

Back to proof

Proof (Theorem 10). Recall that

$$p_i(v_i) = Q_i(v_i) - \int_0^{v_i} Q_i(x)\,dx - \Pi_i(0).$$

Since the seller does not observe the reservation value of bidder i, the seller's expected revenue from bidder i is

$$\int_0^1 p_i(v_i)\,dF_i(v_i)$$
$$= \int_0^1 Q_i(v_i)v_i f(v_i)\,dv_i - \int_0^1 \left[\int_0^{v_i} Q_i(x)\,dx\right] f(v_i)\,dv_i - \Pi_i(0).$$

By integrating by parts,

$$\begin{aligned}\int_0^1 \left[\int_0^{v_i} Q_i(x)\,dx\right] f(v_i)\,dv_i &= \int_0^{v_i} Q_i(x)\,dxF(v_i)\Big|_0^1 - \int_0^1 Q_i(v_i)F(v_i)\,dv_i \\ &= \int_0^1 Q_i(v_i)\,dx - \int_0^1 Q_i(v_i)F(v_i)\,dv_i \\ &= \int_0^1 Q_i(v_i)(1-F(v_i))\,dx.\end{aligned}$$

After substitution, the expected revenue from bidder i is

$$\int_0^1 Q_i(v_i)v_i f(v_i)\,dv_i - \int_0^1 (1-F(v_i))Q_i(v_i)\,dv_i - \Pi_i(0)$$
$$= \int_0^1 \left[v - \frac{1-F(v)}{f(v)}\right] f(v)Q_i(v)\,dv - \Pi_i(0).$$

If the good is sold, the seller's revenue is

$$\sum_{i=1}^n \left[\int_0^1 \left[v - \frac{1-F(v)}{f(v)}\right] f(v)Q_i(v)\,dv - \Pi_i(0)\right].$$

The good is not sold with probability

$$1 - \sum_{i=1}^n \int_0^1 Q_i(v)f(v)\,dv.$$

If the seller's value of the good is v_s, the seller's revenue is

$$
\begin{aligned}
\Pi_s &= v_s \left[1 - \sum_{i=1}^{n} \int_0^1 Q_i(v) f(v)\, dv\right] \\
&\quad + \sum_{i=1}^{n} \left[\int_0^1 \left[v - \frac{1 - F(v)}{f(v)}\right] f(v) Q_i(v)\, dv - \Pi_i(0)\right] \\
&= v_s + \sum_{i=1}^{n} \left[\int_0^1 \left[v - \frac{1 - F(v)}{f(v)} - v_s\right] f(v) Q_i(v)\, dv\right] - \sum_{i=1}^{n} \Pi_i(0).
\end{aligned}
$$

By the first condition,

$$Q_i(v) = F^{n-1}(v) \quad \forall i$$

and by the second condition,

$$\Pi_i(0) = 0 \quad \forall i.$$

As we normalize $v_s = 0$,

$$
\begin{aligned}
\Pi_s &= v_s + n \left[\int_0^1 \left[v - \frac{1 - F(v)}{f(v)} - v_s\right] f(v) F^{n-1}(v)\, dv\right] \\
&= \int_0^1 \left[v - \frac{1 - F(v)}{f(v)}\right] n f(v) F^{n-1}(v)\, dv \\
&= \int_0^1 \left[v - \frac{1 - F(v)}{f(v)}\right] dF^n(v) \\
&= \int_0^1 \left[v - \frac{1 - F(v)}{f(v)}\right] dF_{(1:n)}(v).
\end{aligned}
$$

It remains to show that

$$
\begin{aligned}
\Pi_s &= E(v_{(2:n)}) \\
&= \int_0^1 v\, dF_{(2:n)}(v) \\
&= \int_0^1 v\, d(F^n(v) + n(1 - F(v))F^{n-1}(v)) \\
&= \int_0^1 v\, d(F_{(1:n)}(v) + n(1 - F(v))F^{n-1}(v)).
\end{aligned}
$$

Note

$$
\begin{aligned}
&\int_0^1 \left[v - \frac{1-F(v)}{f(v)}\right] dF_{(1:n)}(v) \\
&= \int_0^1 v\, dF_{(1:n)}(v) - \int_0^1 \frac{1-F(v)}{f(v)}\, dF_{(1:n)}(v) \\
&= \int_0^1 v\, dF_{(1:n)}(v) - \int_0^1 \frac{1-F(v)}{f(v)}\, dF^n(v) \\
&= \int_0^1 v\, dF_{(1:n)}(v) - \int_0^1 \frac{1-F(v)}{f(v)} nF^{n-1}(v) f(v)\, dv \\
&= vF_{(1:n)}(v)\big|_0^1 - \int_0^1 F_{(1:n)}(v)\, dv - \int_0^1 \frac{1-F(v)}{f(v)} nF^{n-1}(v) f(v)\, dv \\
&= \int_0^1 1 - (F_{(1:n)}(v) + n(1-F(v))F^{n-1}(v))\, dv \\
&= \int_0^1 1 - F_{(2:n)}(v)\, dv \\
&= v[1 - F_{(2:n)}(v)]\big|_0^1 - \int_0^1 v\, d(1 - F_{(2:n)}(v)) \\
&= \int_0^1 v\, dF_{(2:n)}(v).
\end{aligned}
$$

■

Price dispersion We now know that the first and the second price auctions are very similar. Both are efficient, and the expected revenue of the seller is the same. A risk neutral seller would treat the two auction the same. What if the seller is risk averse? Which auction does a risk averse seller prefer? Because the two auctions generate the same expected revenue, we need to check the variance of the revenue of each auction.

The variance is a measure of how a random variable is spread out. An alternative, but closely related, way to compare the dispersion of a random variable is the mean preserving spread.

Mean preserving spread

Definition 26. Y is a ***mean preserving spread*** of X if

$$E[Y|X = x] = x \quad \forall x.$$

If Y is a mean preserving spread of X, then Y can be represented as

$$Y = X + \varepsilon$$

where $E[\varepsilon|X] = 0$. While X and Y have the same mean, the variance of Y must be larger than the variance of X.

Dispersion comparison Let σ'' and σ' be the symmetric Nash equilibrium strategies of the second and the first price auctions. The expected revenue from the second price auction is $E[\sigma''(v_{(2:n)})]$ while the expected revenue from the first price auction is $E[\sigma'(v_{(1:n)})]$.

Proposition 6. The seller's revenue from the second price auction is the mean preserving spread of the seller's revenue from the first price auction:

$$E[\sigma''(v_{(2:n)})\,|\,\sigma'(v_{(1:n)}) = b] = b.$$

Proof.

$$\begin{aligned}
E[\sigma''(v_{(2:n)})\,|\,\sigma'(v_{(1:n)}) = b] &= E[\sigma''(v_{(2:n)})\,|\,v_{(1:n)} = (\sigma')^{-1}(b)] && (5.1)\\
&= E[v_{(2:n)}\,|\,v_{(1:n)} = (\sigma')^{-1}(b)] && (5.2)\\
&= E[v_{(1:n-1)}\,|\,v_{(1:n-1)} < (\sigma')^{-1}(b)] && (5.3)\\
&= \sigma'((\sigma')^{-1}(b)) && (5.4)\\
&= b.
\end{aligned}$$

(5.1) follows from the fact that σ' is strictly increasing and therefore, invertible. (5.2) is implied by the fact that the truthful bid is the equilibrium strategy in the second price auction.

(5.3) is the trickiest. Consider n samples of the reservation value, ranked from the highest to the lowest. The highest value of the reservation value is fixed to $(\sigma')^{-1}(b)$. The remaining $n-1$ samples remain random, which must be smaller than $(\sigma')^{-1}(b)$.

Because the highest value of the sample is fixed, the second highest value of n sample must be equal to the highest value of the remaining $n-1$ samples: $v_{(2:n)} = v_{(1:n-1)}$ if $v_{(1:n)}$ is fixed.

Similarly, the event that the highest value among n samples is fixed to $(\sigma')^{-1}$ is the same event that the second highest among n samples, or the highest among remaining $n-1$ samples, must be less than $(\sigma')^{-1}(b)$.

To understand (5.4), remember the second formula of the equilibrium strategy of the first price auction

$$\sigma'(v) = E[v_{(1:n-1)}\,|\,v_{(1:n-1)} \leq v].$$

If $v = (\sigma')^{-1}(b)$, we have (5.4). ∎

5.4 Optimal auction

Because an auction is a monopolistic trading protocol, the monopolist would like to implement a trading protocol which maximizes his expected profit. The exercise to find the most profitable monopolistic trading mechanism is known as the optimal auction design problem.

Similar This exercise is closely related but fundamentally different from the revenue equivalence problem. As in the proof of the revenue equivalence theorem, we confront the task of comparing the seller's revenue among all feasible monopolistic trading protocols. Thanks to the revelation principle, we focus on the set of the allocation rules (Q, p) satisfying the incentive compatibility constraint, and identify the most profitable allocation rule among incentive compatible allocation rules.

Different The objective function of the optimal auction is the expected profit of the monopolist, not the social welfare. Any Nash equilibrium satisfying the two conditions of the revenue equivalence theorem induces an efficient allocation. Because the objective of the monopolist generally differs from the objective of the society, an optimal auction is typically inefficient.

Textbook example In the intermediate microeconomics, we learn the monopolistic market in which a single seller is setting a product to a large number of buyers. A classic example is Cournot's mineral water example.

Monopoly market Consider a monopolistic seller of mineral water which is coming out of a spring. We normalize the marginal cost of production to be 0. A unit mass of a continuum of infinitesimal buyers is in the market, who values one unit of mineral water $v \in [0, 1]$. The valuation of buyers is distributed uniformly over $[0, 1]$.

Because the size of individual buyers is infinitesimal, the aggregate market demand function is

$$D(p) = 1 - p$$

over $\in [0, 1]$.

We can then proceed to compute the profit maximizing quantity of mineral water as 0.5, which gives market clearing price 0.5 and the profit of 0.25. Because the marginal production cost is 0, and the lowest reservation value of consumers is 0, the efficient allocation requires the firm to produce 1 unit, and charge 0 to mineral water. The market outcome is inefficient, because consumers with reservation value less than 0.5 cannot trade, even though there is a positive gain from trading.

What we do not teach undergraduates is to point out this is a particular form of a monopolistic market, and to wars students that it is too hasty to conclude that any monopolistic market leads to inefficient allocation. As we already find out, if the monopolist allocate the good according to the second or the first price auction, the allocation must be efficient.

Post price mechanism If one looks into the trading protocol, the rule of the game is not necessarily most realistic. The monopolist chooses a price, and post the price. We call the trading protocol the post price mechanism.

One may wonder whether the monopolist can implement more elaborate way to discriminate buyers to elicit information about the reservation value. For example, if a monopolist can observe the reservation value of a buyer, then he might be able to extract all gain from trading surplus by setting the price to be equal to the reservation price of a buyer. This practice is known the first order price discrimination, which leads to efficient allocation, while the seller receives the entire gain from trading.

Price discrimination When we teach students first order price discrimination, we tend to remain vague about whether such a practice is feasible, if the reservation value is private information of a buyer, or if the monopolist has only limited control over the secondary market to prevent the initial buyer from reselling the product to someone else.

In the end, we have to ask whether the monopolist can design a trading protocol in which he can generate higher profit than what he could have received from the post price mechanism. Myerson [1981] answered the question formally and elegantly.

Optimal auction We know that any incentive compatible allocation rule (Q, p) generates the expected surplus of the seller

$$\Pi_s = v_s + \left[\sum_{i=1}^{n} \int_0^1 \left[v_i - \frac{1 - F(v_i)}{f(v_i)} - v_s\right] Q_i(v_i) f(v_i)\, dv_i - \Pi_i(0)\right].$$

Optimization problem The optimization problem of the seller is

$$\max_{(Q,p)} \Pi_s$$

subject to (Q, p) satisfying the incentive compatibility constraint and interim individual rationality of bidder i:

$$\Pi_i(v_i) \geq 0 \quad \forall v_i \in [0, 1].$$

Since

$$p_i(v_i) = Q_i(v_i)v - \int_0^v Q_i(x)\, dx - \Pi_i(0),$$

the seller's problem is

$$\max_{(Q,p)} \Pi_s$$

subject to Q satisfying the incentive compatibility constraint, and $(\Pi_i(0))_{i=1}^n$ satisfying the interim individual rationality.

In order to maximize the profit, the seller should set $\Pi_i(0)$ as small as possible. To satisfy the interim individual rationality of $v_i = 0$,

$$\Pi_i(0) = 0 \quad \forall i.$$

Then, the seller's optimization problem is reduced to

$$\max_{(Q,p)} v_s + \sum_{i=1}^{n} \left[\int_0^1 \left[v_i - \frac{1 - F(v_i)}{f(v_i)} - v_s\right] \sum_{i=1}^{n} Q_i(v_i)\right] f(v_i)\, dv$$

subject to $Q = (Q_1, \cdots, Q_n)$ satisfying the incentive compatibility constraint.

Relaxed problem One of the implications of the incentive compatibility constraint on Q_i is that Q_i must be increasing. Let us consider a relaxed problem

$$\max_{(Q,p)} v_s + \sum_{i=1}^{n} \left[\int_0^1 \left[v_i - \frac{1 - F(v_i)}{f(v_i)} - v_s\right] Q_i(v_i) f(v_i)\, dv_i\right]$$

subject to Q_i is increasing $\forall i$.

We then show that the solution of the relaxed problem is indeed the solution of the maximization problem.

Challenge A natural candidate of optimal Q_i is

$$Q_i(v) = \begin{cases} 1 & \text{if } v - \frac{1-F(v)}{f(v)} - v_s \geq 0 \\ 0 & \text{if } v - \frac{1-F(v)}{f(v)} - v_s < 0 \end{cases}.$$

The problem is that the constructed Q_i is not increasing unless

$$v - \frac{1-F(v)}{f(v)} - v_s$$

is an increasing function of v.

Note that

$$\frac{1-F(v)}{f(v)}$$

is an inverse function of hazard rate. Thus, if the distribution function has the decreasing hazard rate property, then

$$v - \frac{1-F(v)}{f(v)} - v_s$$

is an increasing function of v, which Myerson [1981] called the regular case, as this condition is satisfied by many well know distributions such as uniform and Gaussian distributions.

Regular case Suppose that

$$v - \frac{1-F(v)}{f(v)} - v_s$$

is an increasing function. Then, we can find a unique $v^* \in [0, 1]$ so that

$$v - \frac{1-F(v)}{f(v)} - v_s \geq 0 \quad \Longleftrightarrow \quad v \geq v^*.$$

Define

$$Q_i(v) = \begin{cases} 1 & \text{if } v \geq v^* \\ 0 & \text{if } v < v^* < 0 \end{cases}.$$

Clearly, Q_i maximizes the objective function. The remaining step is to show that Q_i is incentive compatible. To this end, we calculate the interim expected payment of bidder i.

Incentive compatible Recall that $\Pi_i(0) = 0$ and

$$p_i(v_i) = Q_i(v_i)v_i - \int_0^{v_i} Q_i(x)\,dx.$$

A simple calculation shows

$$p_i(v_i) = \begin{cases} v^* & \text{if } v_i \geq v^* \\ 0 & \text{if } v_i < v^* \end{cases}.$$

The interim payoff is

$$\Pi_i(v_i) = \begin{cases} v_i - v^* & \text{if } v_i \geq v^* \\ 0 & \text{if } v_i < v^* \end{cases}.$$

Now, we are ready to show that $Q_i(v_i)$ we have constructed satisfies the incentive compatibility constraint.

Suppose that $v_i < v^*$. By reporting truthfully, the bidder receives 0. If $v_i' \neq v_i$ but $v_i' < v^*$, then reporting v_i' instead of v_i does not change the interim payoff. If $v_i' > v^*$, then the interim payoff is

$$v_i - v^* < 0$$

because $v_i' > v^*$ triggers the good is delivered to the bidder at price of v^*. Thus, if $v_i < v^*$, it is optimal for bidder i to report truthfully.

Suppose that $v_i \geq v^*$. By reporting truthfully, the bidder receives

$$v_i - v^* \geq 0.$$

If $v_i' \neq v_i$ and $v_i' \geq v^*$, reporting v_i' instead of v_i does not change the interim payoff of the bidder. If $v_i' < v^*$, he receives 0 instead of $v_i - v^* \geq 0$. Thus, the truthful reporting is an optimal strategy.

Second price auction with reserve price Let us examine how the optimal auction operates in the context of the Counot's mineral water example. Each bidder comes to the shop, and reports his reservation value. If his reservation value is above the threshold v^*, then the buyer will receive one unit of the mineral water, paying v^*. Otherwise, he will not get any mineral water, and his expected payoff is 0.

The monopolist is essentially running the second price auction with reserve price v^* against the buyer. Since one buyer is served each time whenever the reported price is above v^*, the delivery price is always v^* whenever the good is sold. For this reason, the optimal auction is sometimes called the second price auction with reserve price.

To see how the optimal auction is implement, let us assume that F is the uniform distribution over $[0, 1]$ and $v_s = 0$. To calculate v^*,

$$v - \frac{1 - F(v)}{f(v)} - v_s = v - (1 - v) = 0$$

at $v = v^*$. Thus, $v^* = \frac{1}{2}$, which is exactly the same price as the post price mechanism. That is, the maximum profit the monopolist can generate in any incentive compatible trading protocol is exactly the monopolist profit from the post price mechanism.

Chapter 6

Public Goods

6.1 Public good

Lecture 12. public Mon, Apr 12

Murder of C. Genovese In March 1964, 38 people witnessed the brutal murder of Catherine Genovese for 30 minutes, but no one did not call police, which could have prevented her murder.

Explanation One possible explanation is based upon the preference of the residents in the city, who become socially indifferent. While a different preference can explain this particular episode, the same approach does not explain the mechanism between the preference and the outcome. Instead, the outcome is a direct consequence of the preference. The same approach does not provide a useful policy implications to prevent further episode of the same kind. If no one reports to police because of their preference, the same behavior should be repeated under the same circumstance, and the third party (i.e., the government) has little room to intervene.

A simple and clear explanation is not necessarily the best explanation.

Rational model The economic approach is to describe the outcome as a consequence of a rational decision by 38 witnesses. One might be dismayed to see economists regard the outcome as a consequence of a deliberate decision, claiming that economists are without any sympathy toward the victim.

This view toward rational approach does not appreciate the virtue. By describing the behavior of 38 witnesses as a rational decision, we can understand why and when such an outcome arises. By understanding the mechanism between the preference and the outcome, we can propose a remedy which can improve the social welfare, possibly reducing the probability of repeating the same episode.

Free rider problem We regard the decision problem of 38 witnesses as a decision to provide a public good, while bearing the cost. By the nature of the public, everyone in the economy can benefit from the public good, once it is provided. A rational agent thus has an incentive to wait until someone else provides the public good so that he can have the service of the public good for free. This incentive problem is known as the free rider problem.

Questions

- Incentive to provide public good. The free rider problem is an example of an incentive problem of hidden information (adverse selection problem). Unless we address the adverse selection problem, the free rider problem prevents any possibility of achieving efficient allocation, in the same way as the lemon's problem does.
- Efficient provision of public good. Adverse selection hinders achieving efficient allocation. We need to identify the condition under which we can achieve efficient allocation.
- Budget balancing. To provide incentive, the policy maker has to design a scheme by providing side payment to the agents. The important question is whether the policy maker can implement the scheme while satisfying the budget constraint.

Back to the example We first analyze a simple model to show how the incentive of each individual can lead to inefficient provision of the public good, and what we can do about it.

Consider an economy populated by n players, with public good (called the public safety) from which every player receives $v > 0$, if exists. To provide the public good, at least one player must call the policy, which incur cost $c > 0$ where $v - c > 0$.

Let $\sigma_i \in \{0, 1\}$ be a strategy of player i where

$$\sigma_i = \begin{cases} 0 & \text{if no report} \\ 1 & \text{if report.} \end{cases}$$

Let u_i be the payoff function.

$$u_i(\sigma_i, \sigma_{-i}) = \begin{cases} v - c & \text{if } \sigma_i = 1 \\ v & \text{if } \sigma_i = 0,\ \exists j \neq i,\ \sigma_j = 1 \\ 0 & \text{if } \sigma_j = 0\ \forall j \in \{1, \cdots, n\} \end{cases}.$$

In order to provide the public good, someone has to contact police. Given that the public good is provided, each agent prefers someone else to report instead of himself.

Equilibrium Let us consider an equilibrium where the public good is provided:

$$(\sigma_1, \sigma_2, \cdots, \sigma_n) = (1, 0, \cdots, 0).$$

For $i \geq 2$, player i receives v, because public good is provided, which is the maximum payoff he can achieve. Thus, $\sigma_i = 0$ is a best response.

Player 1 receives $v - c > 0$. If he switches to $\sigma = 0$, then public good is not provided and receives 0. Thus, $\sigma_1 = 1$ is a best response.

Discussion

- This equilibrium does not reveal the mechanism through which the incentive to report is hindered by the incentive to free ride.
- All players are ex ante identical, as in the bidders in IPV model of auctions. Thus, it might be more reasonable to focus on a symmetric Nash equilibrium.

Symmetric Nash equilibrium Suppose that

$$\underbrace{(\sigma, \cdots, \sigma)}_{n}$$

is a Nash equilibrium. It is easy to show that σ must be a mixed strategy, and its proof is left as an exercise. Let $p = P(\sigma = 1)$ be the probability that a player reports to the policy. To be a mixed strategy Nash equilibrium, a player must be indifferent between reporting to the policy to get $v - c$ and not reporting to the policy, which entails two possible outcomes: someone else reports so that the player receive v, and no one else reports so that he receives 0. The probability of each event is determined by the equilibrium strategy of others.

The equilibrium condition is

$$\begin{aligned} v - c &= 0 \cdot P(\text{no one else reports}) + v \cdot P(\text{at least one person reports}) \\ &= 0 \cdot P(\text{no one else reports}) + v \cdot (1 - P(\text{no one else calls})). \end{aligned}$$

Thus,

$$\frac{c}{v} = P(\text{no one else calls}).$$

Since each player reports with probability p, the probability that no one among $n-1$ players calls the police is $(1-p)^{n-1}$. Solving

$$\frac{c}{v} = (1-p)^{n-1},$$

we have

$$p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$$

which is the equilibrium probability that a player calls the policy.

Comparative static Recall that

$$p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}.$$

As n increases, p decreases. As the number of neighbors increases, the probability that each resident calls the policy decreases. Since calling the policy incurs costs, he would rather let others call. In a symmetric equilibrium, someone will call with a positive probability.

Probability of report An important question is about the probability that the public good is provided in the society. In order to have public safety, at least one person has to call the police. To calculate this probability, let us calculate that no one in the society call the policy. Since

$$
\begin{aligned}
P(\text{No one calls}) &= P(\text{Player } i \text{ does not call}) \times P(\text{No other player } j \neq i \text{ calls}) \\
&= (1-p)\frac{c}{v} = \left(1 - \left[1 - \left(\frac{c}{v}^{\frac{1}{n-1}}\right)\right] \frac{c}{v} = \left(\frac{c}{v}\right)^{\frac{n}{n-1}},\right.
\end{aligned}
$$

$$0 < \lim_{n\to\infty} P(\text{No one calls}) = \frac{c}{v} < 1.$$

As the cost c of calling the police decreases, the probability that no one calls decreases. Thus, a sensible policy to reduce the same kind of tragic episode is to reduce the cost of calling the police.

6.2 Grove Clarke scheme

Financing public good Suppose that the social planner plans to implement a public project in an economy with n agent. The social cost of the public project is c. The utility of agent i is a quasi linear function:

$$u_i(x_i, m_i) = x_i \mathbb{I}_X - m_i$$

where x_i is the private valuation of the public project, and m_i is money. We do not assume that $x_i \geq 0$. It is possible that $x_i < 0$ if agent i receives disutility from the public project. X is the event that the public project is implemented where

$$\mathbb{I}_X = \begin{cases} 1 & \text{if the public project is implemented} \\ 0 & \text{if the public project is not implemented.} \end{cases}$$

If $m_i > 0$, then we interpret m_i as tax, and if $m_i \leq 0$, then we regard m_i as subsidy to agent i.

Efficient allocation The social welfare is measured in terms of the net gain from the public project:

$$\mathcal{W} = \sum_{i=1}^{n} x_i - c.$$

In an efficient allocation, the public project is implemented if and only if

$$\sum_{i=1}^{n} x_i \geq c.$$

The challenge is that x_i is a private information of agent i, who may not have incentive to report truthfully.

> **Example.** Suppose that $n = 10$, $x_1 = \cdots = x_{10} = 0.1$, and $c = 100$. Since the cost is larger than the sum of the valuations, it is efficient not to implement the project. Suppose that the social planner asks each agent to

report his valuation zi , and implement the project if

$$\sum_{i=1}^{n} z_i \geq c.$$

Since each agent receives a positive surplus from the project, every agent has incentive to exaggerate his valuation so that the project is implemented.

Eliciting truth In order to implement an efficient allocation, it is essential to elicit truth from each agent. To do so, the social planner devises a scheme, or a mechanism, which decides whether to implement the project, and the side payment, conditioned on the reported types. The side payment is needed in order to control the incentive of the agent to elicit truthful reporting.

T. Groves [1973][1] and E. Clarke [1971][2] discovered a mechanism in which truthful reporting is a dominant strategy. The mechanism is named as Groves Clarke scheme.

Groves Clarke scheme Let (Q, M) be a mechanism where $Q(z_1, \cdots, z_n)$ is the probability of implementing the project and

$$M(z, \cdots, z_n) = (M_i(z_1, \cdots, z_n))_{i=1}^n$$

specifies the amount of side payment (tax or subsidy) conditioned on the profile of reported valuations.

Definition 27. (Q, M) is the ***Groves Clarke scheme*** if

$$Q(z_1, \cdots, z_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^n z_i \geq c \\ 0 & \text{if } \sum_{i=1}^n z_i < c \end{cases}$$

and

$$M_i(z_1, \cdots, z_n) = c - \sum_{j \neq i} z_j \quad \forall i.$$

If $z_j \geq 0 \ \forall j$, then the Groves Clarke scheme offers subsidy (negative tax) to each agent. The total amount of transfer payment from the social planner to the agents is

$$\sum_{i=1}^{n} \sum_{j \neq i} z_j = -nc + (n-1) \sum_{i=1}^{n} z_i$$

which can be a large positive number. It is possible that the social planner runs deficit under the Groves Clarke scheme.

Incentive compatibility

Theorem 12. Given the Groves Clarke scheme, truthful reporting is a dominant strategy.

[1] T. Groves [1973]: "Incentives in Teams," Econometrica, 41: 617-631
[2] E. Clarke [1971]: "Multipart Pricing of Public Goods," Public Choice, 8: 19-33

Proof. Let z_{-i} be the profile of reported valuations by players other than player i. If player i report z_i, his payoff is

$$\begin{cases} x_i + \sum_{j\neq i} z_j - c & \text{if } z_i + \sum_{j\neq i} z_j \geq c \\ 0 & \text{if } z_i + \sum_{j\neq i} z_j < c. \end{cases}$$

We consider two cases.

First, suppose that $x_i+\sum_{j\neq i} z_j - c \geq 0$. By reporting truthfully ($z_i = x_i$), agent i can generate a positive surplus. If $z_i > x_i$, the project is implemented, but the payoff remains the same. If $z_i < x_i$, then the payoff depends upon whether the project is implemented or not. If it is implemented, the payoff of agent i remains $x_i + \sum_{j\neq i} z_j - c$. If z_i is so small that $z_i + \sum_{j\neq i} z_j - c < 0$, then the project will not be implemented and his payoff is 0, which is less than what he would have received by reporting his valuation truthfully.

Second, suppose that $x_i + \sum_{j\neq i} z_j - c < 0$. By reporting truthfully, agent i prevents the project from being implemented and receives 0. If $z_i < x_i$, his payoff remains 0. If $z_i > x_i$, and the project is not implemented, the payoff of agent i remains the same as what he would have received by reporting his type truthfully. If the project is implemented because z_i is so large that $z_i + \sum_{j\neq i} z_j - c < 0$, agent i receives $x_i + \sum_{j\neq i} z_j - c < 0$ which is less than what he would have received by reporting truthfully.

■

The social planner can induces the agents to tell the truth by providing a suitable amount of side payment as a dominant strategy. That is, regardless of the strategies of other players, it is always optimal to report truthfully.

Individually rational

Definition 28. A mechanism is ***interim individually rational*** if the interim expected payoff of an agent in the equilibrium is not negative, and ***ex post individually rational*** if the ex post expected payoff of an agent in the equilibrium is not negative.

Clearly, ex post individual rationality is stronger than interim individual rationality.

Proposition 7. The Grove Clarke scheme is ex post individually rational.

Suppose the valuation is truthfully reported. If the project is not implemented, the payoff is 0. The project is implemented only if

$$\sum_{i=1}^{n} x_i - c = x_i + \sum_{j\neq i} x_j - c \geq 0.$$

Note that the second term is the payoff of agent i, which is the same across all players.

Discussion The Groves Clarke scheme satisfies three important desirable properties.

- Incentive compatible. The truthful reporting is a dominant strategy.
- Efficient. The public implemented only if the surplus from the project is non-negative.
- Ex post individually rational. Even after the valuation of every agent is known, no agent receives less than 0.

However, the same scheme may generate a budget deficit, to finance the side payment to induce each player to report truthfully.

Generalization The proof to show that the truthful reporting is a dominant strategy under the Grove Clarke scheme is very much in line with the proof of the same property in the second price auction. This is not a coincidence.

We generalize the insight of Vickrey and Groves and Clarke to develop a mechanism with the following desirable properties.

- Incentive compatible. The truthful reporting is a dominant strategy.
- Efficient. The allocation of the mechanism is ex post efficient.
- Interim individually rational. Conditioned on the private information, each agent is willing to participate in the scheme.
- Budget balancing. The mechanism is self-sustaining without injection of fund from the outside, possibly generating positive surplus, but not running deficit.

We relax the individual rationality from the ex post to the interim individual rationality, but keep other properties. Clearly, the Vickrey auction satisfies all these properties. The goal is to identify conditions under which we can construct a mechanism with all four properties. The key mechanism for this line of research is known as Vickrey Clarke Grove mechanism or VCG mechanism.

Chapter 7

Efficient Mechanism

7.1 Efficient mechanism

Lecture 13. efficient Wed, Apr 14

Efficient allocations Let $X_i = [\alpha_i, \omega_i]$ be the space of the valuation of agent i, and

$$X = \prod_{i=1}^{N} X_i.$$

Allocation rule $Q\colon X \to \Delta^N$ is a probability distribution which specifies the probability that agent i wins the object. $Q_i(x)$ is the probability that agent i wins the object, which may depend upon the entire profile of all valuations.

Definition 29. An allocation rule Q^* is ***efficient*** if

$$\sum_{i=1}^{N} Q_j^*(x) x_i \geq \sum_{i=1}^{N} Q_j(x) x_i$$

for any allocation rule Q.

Because we assume the transferable utility, the transfer payment does not affect the efficiency of an allocation.

Efficient mechanism Given an efficient Q^*, define the social welfare in the efficient allocation as

$$W(x) = \sum_{i=1}^{N} Q_j^*(x) x_i$$

and

$$W_{-i}(x) = \sum_{j \neq i} Q^*(x) x_j.$$

In order to achieve an efficient allocation, it is necessary to provide incentive for an agent to reveal his private valuation truthfully. A typical way of providing proper incentive is to design a side payment conditioned on the reported valuation.

Definition 30. (Q^*, M^v) is the ***Vickrey–Clark–Groves (VCG) mechanism*** if Q^* is an efficient allocation mechanism and $M^v \in \mathbb{R}^n$ is a side payment scheme

$$M_i^v(x) = W(\alpha_i, x_{-i}) - W_{-i}(x)$$

that is the smallest marginal contribution of agent i by participating in the mechanism against x_{-i}.

As the name suggests, VCG mechanism is a generalization of the second price auction in the sense that we can write down the allocation of the second price auction as VCG mechanism.

Second price auction Consider the IPV framework. Suppose that $\alpha_i = 0$ $\forall i$. The good is delivered to the bidder with the highest reservation value so that

$$Q_i^*(x) = \begin{cases} 1 & \text{if } x_i \geq \max_{j \neq i} x_j \\ 0 & \text{if } x_i < \max_{j \neq i} x_j \end{cases}.$$

Note that among $x = (x_1, \cdots, x_n)$, the good is delivered to the bidder with the highest reservation value. Therefore, if bidder i has the highest reservation value

$$x_i \geq \max_{j \neq i} x_j,$$

then the sum of the surplus among bidders other than bidder i must be 0, because none of them receives the good with a positive probability. On the other hand, if bidder i is not the bidder with the highest reservation value

$$x_i < \max_{j \neq i} x_j,$$

then the good is delivered to some bidder $j \neq i$ who has the highest reservation value among bidders except for bidder i. Thus, the sum of the surplus in the efficient allocation among bidders except for bidder i must be

$$W_{-i}(x) = \max_{j \neq i} x_j.$$

Summarizing the observations, we have

$$W_{-i}(x) = \begin{cases} 0 & \text{if } x_i \geq \max_{j \neq i} x_j \\ \max_{j \neq i} x_j & \text{if } x_i < \max_{j \neq i} x_j \end{cases}.$$

Let us calculate the efficient allocation for $(0, x_{-i})$. Since bidder i has the lowest possible valuation among n bidders, he will not receive the good for sure. Instead, the good is delivered to the highest valuation buyer among bidders except i who has valuation 0. Therefore, the total surplus generated in the efficient allocation should be the highest reservation value among bidders except for bidder i:

$$W(0, x_{-i}) = \max_{j \neq i} x_j$$

In the VCG mechanism,

$$M_i^v(x) = W(0, x_{-i}) - W_{-i}(x)$$

we conclude that

$$M_i^v(x) = W(0, x_{-i}) - W_{-i}(x) = \begin{cases} \max_{j\neq i} x_j & \text{if } x_i \geq \max_{j\neq i} x_j \\ 0 & \text{if } x_i < \max_{j\neq i} x_j \end{cases}.$$

That is, only the bidder with the highest reservation value is paying positive amount of transfer payment, which is equal to the second highest reservation value.

7.2 Basic properties of VCG

Incentive compatibility We know that the truthful bidding strategy is a dominant strategy of each bidder. That is, the truthful reporting strategy is incentive compatible, regardless of other players' strategies. VCG mechanism maintains the same property.

Theorem 13. Fix a VCG mechanism (Q^*, M^v). Consider a revelation game, in which each agent is report his valuation, and the VCG mechanism decides the allocation. The truthful reporting is a dominant strategy of the revelation game.

Proof. Given z_{-i}, suppose that player i reports z_i, where z_j may not be equal to x_j $\forall j$. The expected payoff of player i is

$$\begin{aligned} &Q_i^*(z_i, z_{-i})x_i - M_i^v(z_i, z_{-i}) \\ &= Q_i^*(z_i, z_{-i})x_i - W(\alpha_i, z_{-i}) + W_{-i}(z_i, z_{-i}) \\ &= \left[Q_i^*(z_i, z_{-i})x_i + \sum_{j\neq i} Q_j^*(z_i, z_{-i})z_j\right] - W(\alpha_i, z_{-i}) \end{aligned}$$

where the first equality follows from the definition of $M_i^v(z_i, z_{-i})$ and the second equality follows from the definition of $W_{-i}(z_i, z_{-i})$.

Notice that $W(\alpha_i, z_{-i})$ is independent of player i's strategy. We show that the term inside of the bracket is maximized if $x_i = z_i$. Recall the definition of $Q^*(x_i, z_{-i})$.

$$Q_i^*(x_i, z_{-i})x_i + \sum_{j\neq i} Q_j^*(z_i, z_{-i})z_j \geq Q_i^*(z_i, z_{-i})x_i + \sum_{j\neq i} Q_j^*(z_i, z_{-i})z_j$$

Thus,

$$Q_i^*(x_i, z_{-i})x_i + M_i^v(x_i, z_{-i}) \geq Q_i^*(z_i, z_{-i})x_i - M_i^v(z_i, z_{-i}) \quad \forall z_{-i}, z_i$$

which implies that the truthful reporting is a best response against any z_{-i} . Therefore, the truthful reporting is a dominant strategy. ■

Individual rationality From the proof, we conclude that the (ex post) expected payoff of player i is $W(x) - W(\alpha_i, x_{-i})$ if each bidder reports his valuation truthfully.

In the second price auction, the expected payoff of each bidder is not negative. A losing bidder receives 0, while the winning bidder receives a positive payoff. To define the individual rationality, let us define the ex post expected utility

$$\mathcal{U}_i^v(x) = W(x) - W(\alpha_i, x_{-i}).$$

Definition 31. If the ex post expected equilibrium payoff is non-negative $\forall x$, then the allocation mechanism is ***ex post individually rational***. If the interim expected equilibrium payoff is non-negative $\forall x$, then the allocation mechanism is ***interim individually rational***.

Clearly, if a mechanism is ex post individually rational, then it is interim individually rational.

Proposition 8. A VCG mechanism is ex post individually rational.

Proof. It suffices to show that $W(x)$ is an increasing function of x, or equivalently, if $x_i > x_i'$,

$$W(x_i, x_{-i}) \geq W(x_i', x_{-i}) \quad \forall x_{-i}.$$

By the definition of W,

$$W(x_i', x_{-i}) = Q_i^*(x_i', x_{-i})x_i' + \sum_{j \neq i} Q_j^*(x_i', x_{-i})x_j.$$

Since $Q_i^*(\cdot)$ is positive, and $x_i > x_i'$,

$$\begin{aligned}
W(x_i', x_{-i}) &= Q_i^*(x_i', x_{-i})x_i' + \sum_{j \neq i} Q_j^*(x_i', x_{-i})x_j \\
&\geq Q_i^*(x_i', x_{-i})x_i + \sum_{j \neq i} Q_j^*(x_i', x_{-i})x_j \\
&\geq Q_i^*(x_i, x_{-i})x_i + \sum_{j \neq i} Q_j^*(x_i, x_{-i})x_j \\
&= W(x_i, x_{-i}).
\end{aligned}$$

The second inequality follows from the definition of $Q^*(x_i, x_{-i})$. ■

Four properties We now know that a VCG mechanism has three fundamental properties:

- Ex post efficiency
- Incentive compatible
- (Interim) individual rationality

One of the major challenge in financing the public project is to balance the budget, or not to run deficit. In the second price auction (which we know is a special case of the VCG mechanism), the seller generates a positive revenue. In the Grove's scheme, we know that the government has to run deficit.

Budget balance An important question is whether we can have an ex post efficient, incentive compatible and individually rational mechanism, which does not run deficit. The answer depends upon whether or not the VCG mechanism runs deficit.

Proposition 9. Among all mechanisms for allocating a single object that are ex post efficient, incentive compatible and (interim) individually rational, the VCG mechanism maximizes the expected payment from each agent.

Proof. Suppose that (Q^*, M) is an ex post efficient, incentive compatible and (interim) individually rational mechanism. By incentive compatibility and (interim) individual rationality,

$$\mathcal{U}_i(\alpha_i) \geq 0 \quad \text{and} \quad \mathcal{U}_i^v(\alpha_i) = 0.$$

Define

$$c_i = \mathcal{U}_i(\alpha_i) - \mathcal{U}_i^v(\alpha_i) = \mathcal{U}_i(\alpha_i) \geq 0.$$

Note that we are focusing on the efficient allocation Q^*. By Myerson [1983], we know in any incentive compatible allocation rule, the interim expected payoff is

$$\mathcal{U}_i(x_i) = \mathcal{U}_i(\alpha_i) + \int_{\alpha_i}^{x_i} Q_i^*(z)\, dz = c_i + \mathcal{U}_i^v(x_i) \geq \mathcal{U}_i^v(x_i).$$

Note that

$$E_{x_{-i}} M_i(x) = Q_i^*(x_i)x_i - \mathcal{U}_i(x_i) \leq Q_i^*(x_i)x_i - \mathcal{U}_i^v(x_i).$$

Therefore, the individual payment in the VCG is the largest among all ex post efficient, incentive compatible and (interim) individual rational mechanisms. ■

If the VCG mechanism does not run deficit, we have an ex post efficient, incentive compatible and individually rational mechanism, which does not run deficit. If the VCG mechanism runs deficit, the proposition says that all other ex post efficient, incentive compatible and individually rational mechanism must run deficit.

Let us define the ex post budget balance.

Definition 32. (Q^*, M) satisfies ***ex post budget balance*** if

$$\sum_{i=1}^{n} M_i(x) = 0 \quad \forall x.$$

If a social planner is trying to meet the budget constraint, the VCG mechanism offers the best chance to do so.

Proposition 10. There exists an ex post efficient, incentive compatible, (interim) individual rational and ex post budget balancing mechanism if and only if the VCG mechanism generates a non-negative surplus.

Proof. We have already proved that the "only if" part of the proposition, which shows that the VCG mechanism generates the largest surplus among all ex post efficient, incentive compatible, (interim) individual rational and ex post budget balancing mechanisms. We only prove the "if" part of the proposition The VCG ■

The VCG mechanism is ex post efficient, incentive compatible, (interim) individual rational and ex post budget balancing. The tricky part is to allocate the surplus among agents while maintaining these properties. To do so, we consider a different mechanism called AGV mechanism which was invented by (Arrow), d'Aspremont and Gérard-Varet.

Definition 33. (Q^*, M^A) is the ***AGV mechanism*** if Q^* is an efficient allocation rule and

$$M_i^A(x) = \frac{1}{n-1}\sum_{j\neq i} E_{x_{-j}} W_{-j}(x_j, x_{-j}) - E_{x_{-i}} W_{-i}(x_i, x_{-i}).$$

Let us examine the properties of the AVG mechanism. By the definition,

$$\sum_{i=1}^{N} M_i^A(x) = 0 \quad \forall x$$

implying that the AGV mechanism is ex post budget balancing.

Incentive compatible To show that the AGV mechanism is incentive compatible, note that the interim expected utility of player i with true type x_i when he reports z_i is

$$\begin{aligned}
&E_{x_{-i}} Q_i^*(z_i, x_{-i}) x_i - M_i^A(z_i, x_{-i}) \\
&\qquad = E_{x_{-i}}[Q_i^*(z_i, x_{-i}) x_i - W_{-i}(z_i, x_{-i})] - \frac{1}{n-1}\sum_{j\neq i} E_{x_{-j}} W_{-j}(x_j, x_{-j}).
\end{aligned}$$

The second term is independent of z_i. By the definition of Q_i^*, the first term is maximized if $z_i = x_i$. Therefore, the truthful reporting is a best response.

Individual rationality The AVG mechanism may not be individually rational. We shall build a new mechanism over the AVG mechanism to ensure the individual rationality, while maintaining the other properties.

Recall that the interim expected payoff in the VCG mechanism is

$$\mathcal{U}_i^v(x_i) = E_{x_{-i}} W(x_i, x_{-i}) - W(\alpha_i, x_{-i}) = E_{x_{-i}} W(x_i, x_{-i}) - c_i^v.$$

Since the AVG mechanism is incentive compatible, the difference between the interim expected payoff from the AVG mechanism and that from the VCG mechanism is a constant. Thus, we know $\exists c_i^A$ such that

$$\mathcal{U}_i^A(x_i) = E_{x_{-i}} W(x_i, x_{-i}) - c_i^A.$$

By the hypothesis of the proposition, the VCG mechanism generates surplus:

$$E \sum_{i=1}^{n} M_i^v(x) \geq 0.$$

Since

$$E \sum_{i=1}^{n} M_i^A(x) = 0,$$

$$E \sum_{i=1}^{n} M_i^v(x) \geq E \sum_{i=1}^{n} M_i^A(x) = 0.$$

Therefore,

$$\sum_{i=1}^{n} c_i^v \geq \sum_{i=1}^{n} c_i^A.$$

Define

$$d_i = c_i^A - c_i^V \quad \forall i > 1$$

and

$$d_1 = -\sum_{i=1}^{n} d_i.$$

Note. The AVG mechanism may not be (interim) individually rational.

Define

$$\overline{M}_i(x) = M_i^A(x) + d_i \quad \forall i.$$

By the construction, $\forall x$,

$$\sum_{i=1}^{n} \overline{M}_i(x) = \sum_{i=1}^{n} M_i^A(x) + \sum_{i=1}^{n} d_i = 0.$$

Since $\overline{M}_i$ is obtained by adding a lump sum transfer to $M_i^A(x)$, and since (Q^*, M^A) is incentive compatible, $(Q^*, \overline{M})$ remains incentive compatible.

It remains to prove that $(Q^*, \overline{M})$ is (interim) individually rational. For $i > 1$, consider

$$\begin{aligned} \overline{\mathcal{U}}_i(x_i) &= \mathcal{U}_i^A(x_i) + d_i = \mathcal{U}_i^A(x_i) + c_i^A - c_i^V \\ &= E_{x_{-i}} W(x_i, x_{-i}) - c_i^V = \mathcal{U}_i^V(x_i) \geq 0 \end{aligned}$$

since the VCG mechanism is (interim) individually rational. For $i = 1$, recall that

$$d_1 = -\sum_{i=2}^{n} d_i = \sum_{i=2}^{n} (c_i^V - c_i^A) \geq c_1^A - c_1^V.$$

The first equality follows from

$$\sum_{i=1}^{n} d_i = 0,$$

and the last inequality follows from

$$\sum_{i=1}^{n} c_i^V \geq \sum_{i=1}^{n} c_i^A.$$

We have

$$\overline{\mathcal{U}}_1(x_i) = \mathcal{U}_i^A(x_i) + d_i \geq \mathcal{U}_i^A(x_i) + c_1^A - c_1^V = \mathcal{U}_1^V(x_1) \geq 0$$

since the VCG mechanism is individually rational.

Chapter 8

Search and Matching

Friction Arrow Debreu economy is the foundation of modern microeconomics, where we establish the two fundamental welfare theorems.

Lecture 14. search Mon, Apr 19

Definition 34. By a ***friction***, we mean any institutional or informational restriction imposed upon the optimization problem of an agent in the economy.

Asymmetric information is a friction in this sense, as the agent without private information cannot identify the true characteristics of the property, and there is no market for insurance against the unknown states. We often call asymmetric information an example of informational friction.

Search friction Another important friction is the search friction. In Arrow Debreu economy, a seller can find a buyer who is willing to trade and vice versa. In reality, however, it takes some resource to identify a suitable partner for trading. Is the conclusion of Arrow Debreu robust against the presence of small search friction?

In a seminal paper, Peter Diamond[1] constructed a simple trading model with a small search cost, where a small search cost can make a big difference. The prediction of Arrow Debreu economy, which is built on frictionless economy, could be very sensitive to the presence of search cost. His example is often called the curse of Diamond.

Example. Consider an economy populated by S ex ante identical sellers. and B buyers. We assume that

$$B < S < \infty.$$

Each seller has goods for sale, whose production cost is normalized to 0. A seller is not subject to the capacity constraint. Each buyer demands a single unit of the product, whose valuation is $b > 0$.

[1] Peter A. Diamond [1971]: "A Model of Price Adjustment", Journal of Economic Theory, 3,156-168

Competitive benchmark Let us consider frictionless economy. Because $S > B$, the excess supply pushes the equilibrium price down to the marginal production cost of the seller. Therefore, the market clearing price must be equal to 0.

A crucial assumption is that a seller and a buyer are matched immediately without any cost to trade. A fundamental question is whether the same conclusion continues to hold if the matching process is not frictionless.

Matching and bargaining

- Matching. At the beginning of period $t \geq 1$, out of B buyers, $S < B$ buyers are selected, and matched to a seller. One seller is matched to a one buyer, and vice versa.
- Bargaining.
  - Seller $s \in S$ offers p_s to the buyer matched to seller s.
  - If the buyer accepts the offer p_s, the good is delivered immediately for consumption. The buyer's payoff is $b - p_s$, and the seller's payoff is p_s for each period. Upon completion of the trading, the buyer leaves the market permanently and is replaced by identical buyer by the end of period t.
  - If the buyer rejects the offer p_s , the buyer returns to the mass, and waits for the matching in period $t + 1$, paying a small vacancy cost $\varepsilon > 0$ at the end of period t.

Payoff Let us assume that each player maximizes the long run payoff. If a buyer agrees on price at k period after he enters the market, his payoff is

$$-k\varepsilon + (b - p_s).$$

Let (p_k, π_k) be the pair of the price and the probability of reaching agreement time $k \geq 1$. The seller's payoff is

$$\lim_{T\to\infty} E\left(\frac{1}{T}\right)\sum_{k=1}^{T} p_k \pi_k.$$

If the limit does not exists, we take lim inf.

Discussion

- The conclusion is very robust against the details of the specification of time preference. For simplicity, we assume that all agents do not discount the future payoff. We can easily modify the model by assuming that agents are discounting the future income stream.
- $\varepsilon > 0$ is often called the vacancy cost, in the sense that the buyer does not have the good, and keep the position vacant. In order to explore other stores, the buyer has to pay $\varepsilon > 0$. $\varepsilon > 0$ quantifies the amount of friction the buyer faces.
- If $\varepsilon = 0$, it is easy to see that $p_s = 0$ is the equilibrium price, as the seller is subject to Bertrand competition among themselves.

Market with search friction

Theorem 14. $\forall \varepsilon > 0$, the unique equilibrium price is $p_s = b$ for every seller s.

Proof. Because every seller is identical, the equilibrium price $p = p_s \,\forall s$. We first show that

$$\underbrace{(0, \dots, 0)}_{s}$$

is not an equilibrium. Suppose seller s increases the price by $\varepsilon/2$. If the buyer who is assign to seller s accept the price, his surplus is

$$b - \frac{\varepsilon}{2}$$

and if he rejects, his surplus cannot exceed $b - \varepsilon$ because of the vacancy cost. Thus, it is optimal for a buyer to accept $\varepsilon/2$. By the same reasoning, we conclude that for any $p < b$,

$$\underbrace{(p, \dots, p)}_{s}$$

is an equilibrium. Thus, the only equilibrium price is $p_s = b \,\forall s$. ■

This is a sharp contrast to the economy without any friction. Even if the supply exceeds the demand, it takes only an arbitrarily small search cost to shift the equilibrium from the competitive price 0 to the monopolistic price b.

Discussion

- Because we are subject to some form of search friction, Diamond [1971] opened up a large and active literature on search and matching. One line of research is to identify conditions under which we can escape from the curse of Diamond.

- One feature of Diamond [1971] is that the seller makes a take-it-or-leave-it offer to the buyer. Because the initial move in the ultimatum game extracts the entire gain from trading, it is reasonable to investigate whether this particular bargaining protocol is responsible for the curse of Diamond. In ensuing investigations, we often assume the random proposer model, in which the seller or the buyer is selected to be the proposer of the ultimatum game with an equal probability.

Decentralized trading models Matching and search model is ideal platform to formulate the decentralized trading protocol, which the competitive market is built on. When we teach undergraduate students about how to find the market clearing price, we often refer to a fictitious player called Walrasian auctioneer, who collect individual excess demand curve to find the market clearing price. However, the very idea of centralized information processing protocol is against the very spirit of decentralized trading model.

The natural question is therefore whether a decentralized trading model can approximate the competitive market, as the friction disappears. We list three fundamental questions.

Law of single price In the competitive market, there is a single price for each commodity to clear the market. If the trading occurs in decentralized manner, it is not clear how the delivery prices between different pairs of buyers and sellers can converge to a single price.

Information aggregation The fundamental contribution of F. Hayek[2] is to observe that the competitive market price aggregates the dispersed information to achieve an efficient allocation, which is first illustrated by Adam Smith[3] through an example of the pricing decision by a local baker.

First welfare theorem The efficient allocation is a solution of the central planner. The first welfare theorem says that the competitive equilibrium is efficient, but remains silent about the actual process in which the decentralized trading process leads to an efficient allocation, without any centralized information aggregation facility.

[2]Fredrick Hayek [1945]: "The Use of Knowledge in Society", American Economic Review, 35:4, 519-530

[3]Adam Smith [1776]: "An Inquiry into the Nature and Causes of the Wealth of Nations" Liberty Classics

Chapter 9

Repeated Games

9.1 Repeated Games with Perfect Monitoring

Lecture 15. repeatedgame Wed, Apr 21

Introduction

Long term relation

- Economic activities occur over time among the same group of people.
  - Employer vs. employee
  - Landlord vs. share cropper
  - Partners of a joint project
- In contrast to the matching models in which the interaction between two partners occurs just once, the interactions are repeated over time in the examples.
- The dynamic interactions influence the incentive of the agents, possibly inducing more socially desirable outcome than one can achieve in one shot interaction.
- The analytic tool is the repeated game.

Prisoner's dilemma Let consider the prisoner's dilemma game.

$$\begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 3,3 & 0,4 \\ 4,0 & 1,1 \end{pmatrix} \end{array}$$

Instead of meeting once, the two players meet multiple times.

Finite vs. Infinite repeated games If the two player interact $T < \infty$ rounds, then the game is called a finitely repeated game. If they interact infinitely many rounds, then the game is an infinitely repeated game.

If the game is repeated finitely many times, we often write G^T for the game obtained by repeating G T times. In order to differentiate the two games, we call G the component game and G^T a finitely repeated game of G. Similarly, if G is repeated infinitely many times, we write G^∞ for the infinitely repeated game of G.

Questions

Research Questions One might ask why an infinitely repeated game is relevant, because we all die in a finite number of periods? The reason to favor an infinitely repeated game over a finitely repeated game is analytic convenience and prediction.

Depending upon the nature of the problem, it is more convenient to analyze an infinitely repeated game than a finitely repeated game. (We will examine such an example such as a dynamic moral hazard problem.)

Finitely repeated prisoner's dilemma More importantly, the empirical evidence indicates that if the long term relationship is expected to last long time, we behave as if we perceive the relationship lasting forever. To explain a human behavior coherently, an infinitely repeated game is sometimes better.

Suppose that the two players play the prisoner's dilemma game for $T < \infty$ periods. What would be the theoretical prediction of the finitely repeated game? To answer this question, we need to describe at least informally key components of the repeated game.

Long run vs. short run incentive If the game is repeated, a player has to consider the payoffs over many periods and must consider the fact that the outcome of the future may depend upon his choice today. For example, if you play D instead of C while your partner is playing C, then you will receive 4 instead of 3. If the game is played just once, it is the end of the story so that you have incentive to play D. In fact, D is a strictly dominant strategy in the prisoner's dilemma. Any rational agent should play D, eliminating any possibility of cooperation between two rational agents.

Shadow of Future If the game is repeated, however, the threat of punishment in the future might deter a rational player from playing D. The consideration of future punishment is often called the shadow of future, which hopefully induces a rational player to play C. The logic is simple and intuitive. If I am nice to my partner by playing C, he will be nice to me. If I doublecross my partner by playing D when he is playing C, he will be angry and will switch to D in the future. If I follow the agreement by playing C, then my average payoff would be 3. If I deviates, then the payoff stream would be

$$4 + 1 + 1 + \cdots$$

whose average is less than 3.

Two Elements To implement the idea of shadow of future, two conditions must be met.

- Future is important for the decision maker. If the decision maker discount future significantly, the main objective is the immediate payoff, and the reduction of payoff in the future is not important.
- Punishment must be credible. As Selten [1974] pointed out, the threat must be credible, in the sense that it is optimal for the punisher to carry out the punishment. In many cases, however, punishing someone else is

not pleasant, and incurs considerable cost. Thus, it is not obvious whether a punisher is willing to carry out punishment.

Convention

- The cooperation requires that the decision makers are patient, who does not discount the future payoff significantly. All exercise presumes a high discount factor, or assumes that the decision maker does not discount future payoff.
- We focus on subgame perfect equilibrium, which is a Nash equilibrium that can be sustained by credible threat.

Finitely repeated game

Fix a large $T < \infty$, and suppose that

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 3,3 & 0,4 \\ 4,0 & 1,1 \end{pmatrix} \end{array}$$

is repeated T times.

History and strategy Let $a_t = (a_{1,t}, a_{2,t})$ be the pair of actions selected by each player. At the beginning of period t, the two players observe

$$h_t = (a_1, \cdots, a_{t-1})$$

which is called history at t. Based upon h_t, player i chooses $a_{i,t} \in \{C, D\}$. A strategy of player i in period t is $\sigma_{i,t}(h_t) \in \{C, D\}$ which maps a history to an action. Let $\sigma_i = (\sigma_{i,1}, \cdots, \sigma_{i,T})$ be a repeated game strategy of player i.

Nash equilibrium A pair (σ_1, σ_2) of repeated game strategies induces an outcome path

$$f(\sigma_1, \sigma_2) = (a_1, \cdots, a_T).$$

The payoff of player i is the sum of the payoffs over $t = 1, \cdots, T$:

$$\mathcal{U}_i(\sigma_1, \sigma_2) = \sum_{t=1}^{T} u_i(a_t).$$

(σ_1^*, σ_2^*) is a Nash equilibrium if

$$\mathcal{U}_i(\sigma_1^*, \sigma_2^*) \geq \mathcal{U}_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i, \forall i.$$

Subgame perfect equilibrium Given h_t, define $G|_{h_t}$ as the continuation game following h_t. We can define strategy and payoff of $G|_{h_t}$ by restricting the game to the part which follows h_t. We define a Nash equilibrium of $G|_{h_t}$ accordingly. (σ_1^*, σ_2^*) is a subgame perfect equilibrium if it induced a Nash equilibrium in every subgame.

Because G^T is a finite horizon game, we can invoke Kuhn's algorithm to compute a subgame perfect equilibrium.

- In period T, the decision problem is identical with the one shot problem, because there is no future. Both players choose D, following any history h_T.
- No matter what player i chooses in period $T-1$, the other player will choose D in period T. If player i plays C instead of D, player i cannot recover the loss from not playing D, be cause player $j \neq i$ will plays D in the next round. Thus, the best response of player i is to choose D in period $T-1$.
- Repeat the same argument for $t = T-2, \cdots, 1$.
- The unique subgame perfect equilibrium is $\sigma_1(h_t) = \sigma_2(h_t) = D \ \forall h_t, \forall t$.

No matter how large $< \infty$ is, the two players have no chance to cooperate in a finitely repeated prisoner's dilemma game.

Discussion While the result is striking, we can derive a stronger result. We know that any subgame perfect equilibrium is a Nash equilibrium

$$\mathrm{SPE}(G^T) \subset \mathrm{NE}(G^T).$$

A finitely repeated prisoner's dilemma game admit multiple Nash equilibria. But, in any Nash equilibrium (σ_1^*, σ_2^*), the outcome path must be the repetition of (D, D):

$$f(\sigma_1^*, \sigma_2^*) = ((D, D), \cdots, (D, D)).$$

Nash equilibrium We cannot use Kuhn's algorithm to examine the properties of a Nash equilibrium, because a Nash equilibrium does not have to induce a best response off the equilibrium path.

- In the last period, each party will choose D, because D strictly dominates C, following any history.
- Thus, along a Nash equilibrium, the last period's outcome must be (D, D).
- Note that
$$u_i(D, D) = \min_j \max_i u_i(a_i, a_j) = 1.$$
and that D is the minmax strategy which guarantees the maximum payoff against the opponent whose goal is to minimize player i's payoff.

Suppose there is $t < T$ when player i plays C along the equilibrium path. Let t^* be the last time when player i plays C along the equilibrium path:

$$\sigma_{1,t}(\cdot) = \sigma_{2,t}(\cdot) = D \quad \forall t = t^* + 1, \cdots, T.$$

From period t^*, player i's payoff is at most

$$u_i(C, x) + \underbrace{1 + \cdots + 1}_{T - t^*}$$

where x is player j's action in period t^* along the equilibrium path.

Let us consider a deviation strategy by player i. Up to period $t^* - 1$, follow the equilibrium strategy. In period t^*, play D and stick to D until the end of the game. The payoff from the deviation is

$$u_i(D, x) + \underbrace{u_i(D, x_{t^*+1}) + \cdots + u_i(D, x_T)}_{T-t^*}.$$

Clearly, $u_i(D, x) > u_i(C, x)$. Since D is the minmax strategy and 1 is the minmax payoff

$$u_i(D, x_t) \geq 1 \quad \forall t = t^* + 1, \cdots, T.$$

Thus, the deviation payoff is strictly larger than the equilibrium payoff, contradicting the hypothesis that the original path is induced by a Nash equilibrium.

Multiple Nash equilibria This logic only proves that all Nash equilibria must induce (D, D) always along the path, leaving the possibility that multiple Nash equilibria exist in this game. It does not say that each player has to play D following every history in a Nash equilibrium.

Discussion

- The theory makes a sharp prediction that no cooperation is possible in a finitely repeated prisoner's dilemma game, no matter how long the length of repetition might be.
- Contrary to the theoretical prediction, empirical and experimental evidences indicate that the agents cooperate at least with a positive probability if the prisoner's dilemma game is repeated many times.
- The sharp theoretical prediction is inconsistent with the actual observations, thus the game is considered a paradox. The discrepancy between actual observations and theory reveals that the finite repeated prisoner's dilemma game might miss something which should have been included to generate an equilibrium outcome with cooperation.

The finitely repeated prisoner's dilemma is one of several paradoxes in the theory of games, which generated active important research to resolve the paradox. Let us discuss a few ideas.

- Remove the terminal round. The reason why the shadow of future does not work is because the threat is not credible in the last round. By eliminating the last round, we can hope for constructing a credible threat to sustain cooperating. We will examine the infinitely repeated game.
- Common knowledge of rationality. The underlying assumption is that the rationality of the agent is common knowledge. In order to support the subgame perfect equilibrium, it is necessary that the players are rational, but not sufficient. We also need that the rationality of players is common knowledge. What if a player is not sure about the rationality of the other player? We will examine the games with incomplete information.

- Complexity of strategies. How many strategies do you have to consider to identify the best response? What if the agent is not capable of handling the computational problem? The idea of bounded rationality in repeated games is important, and has broad range of applications of computer science.

9.2 Infinitely Repeated Game

Lecture 16. infinitelyrepeated Mon, Apr 26

Resolution The paradoxical prediction of a finitely repeated game generates many ideas to reconcile the theoretical prediction with the empirical evidence. One of those ideas is the infinitely repeated game.

The paradox is a consequence of the backward induction, which is triggered by the presence of the final period. By repeating the game indefinitely, we can get rid of the final period, and hope for constructing a subgame perfect equilibrium which is consistent with empirical and experimental evidence.

Infinitely repeated game Let G^∞ be the game obtained by repeating component game G infinitely many times. We define a history and a strategy exactly in the same way as in the finitely repeated games. The only difference is how to evaluate the sequence of outcomes induced by a pair of strategies $\sigma = (\sigma_1, \sigma_2)$

$$f(\sigma) = (a_1, a_2, \cdots, a_t, \cdots)$$

because the infinite sum may not be well defined.

Limit of mean criterion We continue to assume that the players are infinitely patient and evaluate the outcome stream according to the mean payoff, whenever it is well defined.

$$\mathcal{U}_i(\sigma) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} u_i(a_t).$$

If the limit of the right hand side is not well defined,

$$\mathcal{U}_i(\sigma) = \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} u_i(a_t)$$

which is the smallest limit point. But, in this class, we only examine the cases where the limit of the average is well defined.

Back to the prisoner's dilemma game Let us consider the infinitely repeated prisoner's dilemma game

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 3,3 & 0,4 \\ 4,0 & 1,1 \end{pmatrix} \end{array}$$

We know that if the game is repeated finitely many times, then the only average payoff vector sustained by an equilibrium (subgame perfect equilibrium or Nash equilibrium) is (1, 1). The reasoning of the shadow of future requires credible threat, which does not survive the backward induction triggered by the existence of the final round.

Question The fundamental question is whether we can construct a subgame perfect equilibrium whose equilibrium payoff Pareto dominates the equilibrium payoff of the component game.

This question was first raised during the cold war when two superpowers are competing to build nuclear arsenal. One can regard C as the decision to eliminate all nuclear arsenal, and D as the decision to build nuclear capabilities.

Because the nuclear weapon is expensive, and potential threat for the survival of human race, many scholars are asking whether the two super powers can agree to reduce the nuclear arsenal.

If there is a third party which can enforce the contract, the answer is obvious. Let the two parties sign a contract, which will be enforced by the third party. However, in the international relationship, the third party which can enforce the contract generally does not exist, especially the contract is between two super powers.

Self-enforcing contract The agreement between the two parties must be self-enforcing in the sense that the contract is designed in such a way that it is optimal for each party to follow the contract rather than deviate from the contract. This is precisely what Nash equilibrium requires.

Aumann and Maschler answer the question formally in 1960's. In the infinitely repeated prisoner's dilemma game, there exists a Nash equilibrium in which both parties play C in every period. Later, this result was strengthened to say that in the infinitely repeated prisoner's dilemma game, there exists a Nash equilibrium in which both parties play C in every period.

Formal analysis Aumann and Maschler first demonstrated that it is possible that two rational players can achieve socially efficient outcome without an intervention by the third party. Their result opened up the field of infinitely repeated game.

Let us formulate the insight of Aumann and Maschler by constructing a subgame perfect equilibrium (or a Nash equilibrium) which sustains playing (C, C) always as an equilibrium outcome.

Three examples We discuss three well known repeated game strategies.

Example (D forever)**.** Define

$$\sigma_i(h_t) = D \quad \forall h_t.$$

The pair $\sigma = (\sigma_1, \sigma_2)$ constitutes a subgame perfect equilibrium.

Proof. First, we show that σ is a Nash equilibrium of G^∞. In the equilibrium, each player receives average payoff of 1, since the outcome path

$$f(\sigma) = ((D, D), \ldots, (D, D), \ldots).$$

If player i deviates following any history along the equilibrium path, he cannot receives more than 1, because D is the strictly dominant strategy of the component game. Therefore, player i has no profitable deviation.

Second, we prove that σ induces a Nash equilibrium in every subgame $G|_{h_t}$. Note that following any history h_t, the continuation play of $G|_{h_t}$ is identical with the outcome of G. Following the same logic, we can show that σ induces a Nash equilibrium in $G|_{h_t}$. Thus, σ is a subgame perfect equilibrium. $\diamond$

Generalization We can generalize this result. Fix any normal form game G and a Nash equilibrium $s = (s_1, s_2)$ of G. Define a repeated game strategy for G^∞ as

$$\sigma_i(h_t) = s_i \quad \forall h_t.$$

Such a strategy is called the repetition of one shot Nash equilibrium.

Proposition 11. The repetition of one shot Nash equilibrium is a subgame perfect equilibrium in G^∞.

The repetition of one shot Nash equilibrium is an important benchmark, against which the virtue of a long term relationship is compared.

Example (Grim trigger). Define $\sigma_i(h_1) = C$ and for $t \geq 2$,

$$\sigma_i(h_t) = \begin{cases} C & \text{if } h_t = ((C,C), \ldots, (C,C)) \\ D & \text{otherwise} \end{cases}$$

as the grim trigger strategy.

The strategy triggers punishment D, if anyone plays D. Otherwise, the strategy dictates to play C. Once D starts, player i will play D forever. The punishment is grim.

The pair of grim trigger strategies induce outcome path

$$(C,C), \ldots, (C,C), \ldots$$

and the average payoff of each player is 3, which Pareto dominates the Nash equilibrium outcome of the component game (which is 1).

Proposition 12. The pair of grim trigger strategies is a subgame perfect equilibrium.

Proof. First, we show that the pair of grim trigger strategies constitutes a Nash equilibrium. Recall that the equilibrium payoff is 3. If player i deviates from the equilibrium path, his payoff stream is at best

$$4, 1, 1, \cdots, 1, \cdots$$

because against D, the best possible payoff is 1. Thus, the long run average payoff from any deviation is 1, which is less than 3. Next, we show that the pair induces a Nash equilibrium in any subgame $G|_{h_t}$. We consider two cases. Suppose that

$$h_t = ((C,C), \cdots, (C,C)).$$

Then, the continuation play of the grim trigger strategy is exactly the same as the equilibrium path of the whole game. By applying the same logic, we conclude that the pair of grim trigger strategies induces a Nash equilibrium in this subgame.

Suppose that h_t is any other history which includes D by some player in the past. Then, the continuation play in $G|_{h_t}$ is

$$(D, D), (D, D), \cdots, (D, D)$$

and the grim trigger strategy is exactly the same as D forever strategy in $G|_{h_t}$. Because D forever strategy is a Nash equilibrium, we conclude that the pair of grim trigger strategies induces a Nash equilibrium in $G|_{h_t}$. ■

Discussion

- Grim trigger strategy shows that a credible punishment can sustain cooperation between two rational players.
- The same idea is used by Aumann and Maschler to show that it is possible that two rational players cooperate without a third party intervention.
- Two key assumption are
  - Players are patient so that the future punishment is relevant. In our model, we assume that the player does discount future payoff or is infinitely patient.
  - If any deviation occurs, the deviation can be detected immediately so that a credible punishment can be implemented.

Individually rational payoff vectors The idea of the grim trigger can be generalized. I would like to mention the basic form. Let

$$G = (I, A_1, A_2, u_1, u_2)$$

be the component game and σ^* be a Nash equilibrium of G. Let $u^* = (u_1^*, u_2^*)$ be the Nash equilibrium payoff vector.

Define $u(a) = (u_1(a_1, a_2), u_2(a_1, a_2))$ $\forall a \in A = A_1 \times A_2$.

$$V = \mathrm{co}(\{u(a) \colon a \in A\}) \subset \mathbb{R}^2$$

is the collection of all payoff vectors which can be represented as a convex combinations of $u(a)$, or the convex hull of $u(A)$. That is, $\forall v \in V$, a probability distribution $\exists \pi$ over A such that

$$v = \sum_{a \in A} \pi(a) u(a).$$

We call V the set of feasible payoff vectors.

Folk theorem

Theorem 15 (Folk theorem)**.** If $v_i > u_i^*$ $\forall i = 1, 2$, there exists a subgame perfect equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*)$ such that $v_i = u_i(\sigma_i^*)$ $\forall i$.

Proof. $\exists \pi \in \Delta(A)$ such that

$$v = \sum_a \pi(a)u(a).$$

Let $\pi_t(a)$ be the empirical probability distribution over A which converges to π. In particular, we can assign $a \in A$ in each period so that the empirical probability distribution at t is $\pi_t(a)$. We say that the state of the game follows the norm, if the sequence of outcome at t leads to $\pi_t(p)$.

Define $\sigma_i(h_t) = a_i$ if the state follows the norm where $a_i \in A_i$ is action assigned by the norm to realize empirical distribution $\pi_t(a)$. If the state does not follow the form at any $t' < t$, then $\sigma_i(h_t) = \sigma_i^*$ which is the Nash equilibrium of the component game.

The constructed strategy is essentially the grim trigger strategy. As long as every player follows the norm, player i follows the norm. If not, every player switches to the Nash equilibrium of the component game.

If we follow the equilibrium strategy, player i receives v_i by the construction. If he deviates, then the deviation triggers the punishment by switching to the repetition of one shot Nash equilibrium, which generates long run average payoff u_i^*. Since $v_i > u_i^*$, no player has incentive to deviate from the proposed strategy. Moreover, the repetition of one shot Nash equilibrium is a subgame perfect equilibrium, which makes the punishment credible. ∎

This is a version of Folk Theorem, which states that any payoff vector that Pareto dominates one shot Nash equilibrium payoff vector can be sustained by a subgame perfect equilibrium in G^∞

Discussion

- The punishment is the one shot Nash equilibrium. The reversal to one shot Nash equilibrium is a credible punishment to sustain any payoff vector that Pareto dominates the Nash equilibrium.
- In general, the set of subgame perfect equilibrium payoff vectors in G^∞ is larger than those which Pareto dominates one shot Nash equilibrium.

Minmax Minmax value is the worst possible payoff of player i against all other players who are trying to push player i's payoff down:

$$\underline{v}_i = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i}).$$

Note that for each selection of actions of other players, player i chooes the best response. In general, a_{-i} and a_i are mixed strategies. For now, we only consider pure strategies. In any equilibrium, player i's payoff cannot be lower than $\underline{v}_i$. In that sense, $\underline{v}_i$ is called the security level payoff of player i.

Definition 35. Let V be the set of feasible payoff vectors. $v \in V$ is ***individually rational*** if $v_i \geq \underline{v}_i$ $\forall i$.

Folk theorem The following theorem was discovered by a number of people, thus given the name Folk theorem.

Theorem 16 (Aumann and Shapley; Rubinstein). $\forall v \in V$ which is individually rational, there exists a subgame perfect equilibrium σ such that $v = u(\sigma)$.

Because $(\underline{v}_1, \underline{v}_2)$ is not a Nash equilibrium of the component game, the actual proof is quite involved. The most difficult part is to construct a punishment which is credible, because the punisher may suffer while punishing someone else. In order to provide incentive for the punisher to carry out punishment, we have to construct a hierarchy of punishments.

Prisoner's dilemma Consider

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 3,3 & 0,4 \\ 4,0 & 1,1 \end{pmatrix}. \end{array}$$

$\underline{v}_i = 1$. Thus, $(\underline{v}_1, \underline{v}_2) = (1, 1)$, which coincides with the Nash equilibrium of the component game.

Individually rational payoff vectors

Discussion Our exercise has been about how to construct a credible threat to sustain a socially desirable outcome. Interestingly, some of most widely used strategy is not a subgame perfect equilibrium strategy.

Example (Tit-for-tat). Let $a_{i,t}$ be the action by player i in period t. Define $\sigma_i(h_1) = C$ and $\forall t \geq 2$,

$$\sigma_i(h_t) = a_{j,t-1}.$$

Player i starts with C, and choose an action in period t which is the same action as what the other player chooses in period $t - 1$. If you were nice, I will be nice; if not, I will not be. Thus, the name Tit-for-tat for this strategy.

Proposition 13. The pair of tit-for-tat strategies constitutes a Nash equilibrium, but not a subgame perfect equilibrium in G^∞.

Proof. Along the equilibrium path,

$$(C, C), (C, C), \ldots, (C, C), \ldots$$

and the average payoff of each player is 3. Because the other player responds by imitating player i, the average payoff of playing D is no more than $(4+1)/2 = 2.5$. If he choose D, the largest payoff is 4, but in the next found, the opponent chooses D, against which the short run best response is playing D, generating 1. Thus, player i has no profitable deviation.

To see that the pair of tit-for-tat is not a subgame perfect equilibrium strategy, consider $h_2 = (C, D)$, which is the history not realized with a

> positive probability in the equilibrium. The tit-for-tat dictates player 1 to choose D in the second round, which gives an average payoff strictly less than 3. Instead, player 1 can keep playing C which will eventually lead the other part to choose C, generating average payoff in the long run. ■

Discussion If tit-for-tat is not a subgame perfect equilibrium, why is it a popular strategy often used in the international relation?

- The strategy is simple. To implement the strategy, each player only has to remember the immediate past move of the opponent. Grim trigger is simple in a certain sense, but still requires to remember when D occurs in the long past.
- The strategy is forgiving. If you tend to make a mistake, the grim trigger strategy is a terrible strategy, which eventually leads to (D, D) outcome with probability 1. Tit-for-tat forgets D in the long past. If the opponent was nice yesterday, you are willing to be nice, regardless of the history. In the international relationship, there is no friend or enemy that lasts forever.

9.3 Repeated Games with Imperfect Monitoring

Lecture 17. imperfectmonitor Wed, Apr 28

Introduction

Long term relation

- Economic activities occur over time among the same group of people.
  - Employer vs. employee
  - Landlord vs. share cropper
  - Partners of a joint project
- In contrast to the matching models in which the interaction between two partners occurs just once, the interactions are repeated over time in the examples.
- The dynamic interactions influence the incentive of the agents, possibly inducing more socially desirable outcome than one can achieve in one shot interaction.
- The analytic tool is the repeated game.

Monitoring

- If the other party deviates from the agreement or the norm, the deviation must be detected in order to implement the punishment.
- In the examples mentioned above, the monitoring is not perfect.

- Employer vs. employee. An employee is paid according to the performance, which increases stochastically with respect to effort. Low effort will lead to bad performance with higher probability. Despite much effort, the outcome may be disappointing. From low performance, the inference of the level of effort is not perfect.
- Landlord vs. share cropper
- Partners. The imperfect monitoring applies both sides.

Dynamic moral hazard problem The incentive problem arising from imperfect monitoring is the moral hazard problem. We already know that the presence of moral hazard problem prevents the social planner from implementing the first best solution.

If the moral hazard problem is repeated over periods, many opportunities to observe the action of the other party might reveal the action and could alleviate the moral hazard problem.

The repeated game with imperfect monitoring is a natural framework to answer these questions. We often refers to the game as the dynamic moral hazard problem, if only one side has the problem of the imperfect monitoring (employer vs. employee). If both parties are subject to imperfect monitoring (partners), we call the game the dynamic partnership problem.

Repeated game with imperfect monitoring

Details Repeated games with imperfect monitoring are important, and remain an active research agenda. Instead of examining the general case, I will discuss a couple of well known examples under two assumptions:

- The agents are perfectly patient.
- The outcome of the unknown action is publicly observable.

No discounting

- Facing noisy signal, the principal needs to collect many data to infer accurately the actions by the agent.
- Collecting many data takes time. If the principal discounts the future payoff, then he has to balance the cost and the benefit of collecting additional data.
- By assuming perfectly patient principal, we focus on the information extraction problem.

Public monitoring

- The principal might receive a signal about the agent's behavior privately, or publicly.
- We only examine the case with public outcomes. The repeated game with imperfect private monitoring is a difficult problem, and active research is still in progress.

Examples

Principal agent problem There are 2 players: the principal and the agent. The agent is involved in a project. Let c be the outcome of the project, which is either 1 or 0. Let $c = 1$ denote success, and $c = 0$ denote failure.

Agent The agent controls the probability of success of each project

$$P(c = 1\colon a) = a,$$

which is interpreted as the effort exercised by the agent for the k-th project. Naturally, $a \in [0, 1]$. The principal does not observe a but only observes outcome of projects c. The principal understand the stochastic relationship between the outcome and effort. Thus, the principal imperfectly monitor the agent's actions. Conditioned on the outcomes c of the projects, the principal decides compensation $w(c) \in \mathbb{R}_+$ for the agent according to

$$w\colon \{0, 1\} \to \mathbb{R}_+.$$

Principal The principal is risk neutral. Given c and transfer payment $w(c)$ conditioned on c, his payoff function is

$$u_p(c, w(c)) = \lambda c - w(c)$$

where $\lambda > 0$ which represents the marginal utility.

Agent We assume that the agent is risk averse, whose utility function

$$u_a(a, w)\colon [0, 1] \times \mathbb{R}_+ \to \mathbb{R}$$

is differentiable, and strictly concave: $\forall \mu \in (0, 1)$

$$u_a(\mu a + (1 - \mu)a', \mu w + (1 - \mu)w') > \mu u_a(a, w) + (1 - \mu)u_a(a', w').$$

Given a particular utility level u_a^*, define

$$B_a(u_a^*) = \{(a, w)\colon u_a(a, w) \geq u_a^*\}$$

as the (weakly) better-than-set. Since u_a is concave, $B_a(u_a^*)$ is convex. Without loss of generality, assume that

$$u_a(0, 0) = u_p(0, 0) = 0.$$

One vs. two sided moral hazard problem In the principal agent problem, the principal can only imperfectly monitor the effort of the agent. The agent can perfectly monitor the principal's action. In this sense, the moral hazard problem is one sided.

If the two parties form a partnership, then the moral hazard problem exist on both sides. If so, the problem is called the partnership game, or the double moral hazard problem.

Noisy Prisoner's Dilemma Let us consider the prisoner's dilemma

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 3,3 & 0,4 \\ 4,0 & 1,1 \end{pmatrix}. \end{array}$$

Suppose that even if C is selected, D can be played with probability $\varepsilon > 0$, while if D is played, C can be played with probability $\varepsilon > 0$. Define $\pi(a_1, a_2|a_1', a_2')$ as the probability distribution over the outcome conditioned on the actual actions (a_1', a_2'). In this example,

$$\begin{aligned} &\pi(C, C\,|\,C, C) = (1-\varepsilon)^2, \\ &\pi(C, D\,|\,C, C) = \pi(D, C\,|\,C, C) = \varepsilon(1-\varepsilon) \\ &\pi(D, D\,|\,C, C) = \varepsilon^2 \end{aligned}$$

Since $\pi(D, D\,|\,D, D) = (1-\varepsilon)^2$, the posterior probability of (C, C) conditioned on (D, D) is not zero. Thus, even if (D, D) is observed, one cannot conclude that the players are not cooperating.

Dilemma

- Observing that the opponent's action is D, you have to punish the opponent to provide proper incentive to cooperate.
- If you punish the opponent whenever you observe D action of the opponent, you may punish him too often, discouraging him to cooperate.

Questions If $\varepsilon \to 0$, does the equilibrium outcome converge to the equilibrium outcome of the (repeated) prisoner's dilemma?

Team Problem Two players agree to work together.

- The outcome can be either good (G) or bad (B). If the outcome is G, the value of the outcome is 8, which will be divided equally between the two players. If the outcome is B, then the value is 0.
- The agent can work (W) or shirk (S). If both work, then G occurs with probability $\frac{9}{16}$ and B occurs with probability $\frac{7}{16}$. If only one works and the other shirks, G occurs with probability $\frac{3}{8}$. If no one works (or both shirk), the G occurs with probability $\frac{1}{4}$.
- Work (W) costs 1 unit of utils, while shirk (S) costs 0.

Payoff function

$$\begin{aligned} u_1(W, W) &= 4 \cdot \frac{9}{16} - 1 = \frac{5}{4} \\ u_1(W, S) &= 4 \cdot \frac{7}{16} - 1 = \frac{3}{4} \\ u_1(S, W) &= 4 \cdot \frac{7}{16} = \frac{7}{4} \\ u_1(S, S) &= 4 \cdot \frac{1}{4} = 1 \end{aligned}$$

Similarly,

$$u_2(W,W) = \frac{5}{4},\ u_2(W,S) = \frac{7}{4},\ u_2(S,W) = \frac{3}{4},\ u_2(S,S) = 1.$$

Normal form game

$$\begin{array}{cc} & \begin{array}{cc} W & S \end{array} \\ \begin{array}{c} W \\ S \end{array} & \begin{pmatrix} \frac{5}{4}, \frac{5}{4} & \frac{3}{4}, \frac{7}{4} \\ \frac{7}{4}, \frac{3}{4} & 1, 1 \end{pmatrix} \end{array}$$

The resulting decision problem is a prisoner's dilemma game.

Questions

- If the game played once, the Nash equilibrium is to shirk, which is inefficient.
- We are interested in achieving an outcome which Pareto dominates one shot Nash equilibrium. What if the game is repeated infinitely many times?
- To achieve an outcome other than one shot Nash equilibrium, we need to construct a credible threat, which can be triggered when a deviation is detect. By the nature of the game, the detection of deviation is not perfect.
- Suppose that both players agree to W , hoping to achieve $\frac{5}{4}$ on average. To prevent the other party from deviating to S, the contract must include a punishment when S is detected.
- Compared to W, S decreases the probability of success from $\frac{7}{16}$ to $\frac{3}{8}$ assuming that player 1 is working. Because only public outcome is G or B, it is hard to detect the change of probability of G outcome from $\frac{7}{16}$ to $\frac{3}{8}$.
- If the players are infinitely patient, then one can use the sample average of success to infer the probability that the other party is playing W . Given that player 1 works, player 1 can conclude that if the sample average of success dropps significantly from $\frac{7}{16}$, he can conclude that the other party is not working with probability 1.
- The challenge is that the inference procedure requires a large number of data. Thus, if the players are impatient, this sort of inference procedure does not work.
- Another challenge is that one may punish a honest partner with a bad luck. Even if player 2 works according to the contract, the probability of success if $\frac{7}{16} < 1$. If he is punished because of a bad outcome, he does not have an incentive to work.
- The construction of an equilibrium in the repeated game with imperfect monitoring and with discounting requires a fundamentally different idea, which was discovered by Dilip Abreu in early 1980's.

Dynamic collusion We know that the incentive of the members undermines the cohesion of cartel. Given that the other parties reduce the production according to the agreement to raise the market price, each member of the cartel has incentive to increase production. The incentive problem in the cartel is captured by the prisoner's dilemma game.

Historically, the railroad between New York and Chicago has been controlled by large railroad companies. Not surprising, the fare has gone through ups and downs, indicating a successful enforcement of cartel, followed by its dissolution. An interesting question is whether a cartel can be formed under imperfect monitoring, which opens up the possibility of cheating. If it can, the next question is to calculate the most collusive outcome among the cartel members.

The classic paper by Edward Green and Robert Porter [1984] answered the question formally, which practically opened up the research on the repeated game with imperfect monitoring. Let us describe a simplified version of their model.

Model Consider a Cournot duopoly market, in which two firms with identical production technology operate by selecting the production quantity $q_i \geq 0$ $i = 1, 2$. The production cost is 0. The market price in period t conditioned on the output $q_{i,t}$ of firm $i = 1, 2$ is

$$p_t = 1 - (q_{1,t} + q_{2,t}) + \varepsilon_t$$

where ε_t is an i.i.d. white noise. For a moment, let us ignore the possibility that the market price can be negative due to $\varepsilon_t < 0$. Each firm's objective function is the expected profit

$$\Pi_i(q_{q,t}, q_{2,t}) = E[q_{i,t}(1 - q_{1,t} - q_{2,t} + \varepsilon_t)].$$

Repeated game Suppose that the game is repeated. At the end of period t, firm 1 and firm 2 observe the realized market price p_t. Thus, the public history at the beginning of period t

$$h_t^p = (p_1, \cdots, p_{t-1})$$

is a sequence of realized prices up to $t - 1$ period. Conditioned on h_t^p, firm i chooses $q_{i,t}$, thus firm i's strategy is a public strategy. Let σ_i be a strategy of firm i in the repeated game.

$$f(\sigma_1, \sigma_2) = (f_1(\sigma_1, \sigma_2), \cdots, f_t(\sigma_1, \sigma_2), \cdots)$$

is the sequence of quantity choices induced by (σ_1, σ_2).

Firm i's objective function in the repeated game is the expected discounted average payoff

$$\mathcal{U}_i(\cdot) = E(1 - \delta) \sum_{t=1}^{\infty} \Pi_i(f_t(\sigma_1, \sigma_2))\delta^{t-1}.$$

Challenge Given $q_{1,t}$, and p_t, firm 1 has to infer $q_{2,t}$. Low p_t is caused by

- $q_{2,t}$ is large. If so, firm 1 has to initiate the punishment.

- ε_t becomes negative, even if $q_{2,t}$ is equal to the agreed value. If so, no punishment is needed.
- Because of discounting, a firm cannot spend too much time to collect price data to infer the other firm's production level.

Results If the discount factor is sufficiently close to 1, then there is a threshold strategy to sustain an outcome which Pareto dominates one shot Nash equilibrium.

There is a threshold price p^* such that if $p_t > p^*$, then both firms produce an agreed amount. If $p_t < p^*$, then both firms switch to one shot Nash equilibrium outcome for a certain number of rounds to wipe out any gain from cheating. After the punishment periods, both firms return to the agreed output level.

The outcome path looks like the data. The price remains high for a while, before it collapses to the competitive level. After some period, the price recovers.

This is not the most collusive outcome, because the punishment is not the most severe. The calculation of the most collusive outcome requires a different approach.

9.4 Repeated Moral Hazard Problem

Lecture 18.
imperfectmonitor2
Mon, May 10

Repeated principal agent problem

A classic problem of moral hazard problem is the principal agent problem, in which the principal cannot perfectly monitor the effort of the agent which determine the outcome of the project stochastically. Let me describe a canonical model of a principal agent problem.

Principal agent problem There are 2 players: the principal and the agent. The agent is involved in K different projects. Let c_k be the outcome of the k-th project, which is either 1 or 0. Let $c_k = 1$ denote success, and $c_k = 0$ denote failure.

Agent The agent controls the probability of success of each project

$$P(c_k = 1 : a_k) = a_k,$$

which is interpreted as the effort exercised by the agent for the k-th project. Naturally, $a_k \in [0, 1]$. The principal does not observe $a = (a_1, \cdot, c_K)$ but only observes outcome of projects $c = (c_1, \cdots, c_K)$. The principal understand the stochastic relationship between the outcome and effort. Thus, the principal imperfectly monitor the agent's actions. Conditioned on the outcomes $c = (c_1, \cdots, c_K)$ of projects, the principal decides compensation $w(c) \in \mathbb{R}_+$ for the agent according to

$$w\colon \{0, 1\}^K \to \mathbb{R}_+.$$

Principal the principal is risk neutral. Given $c = (c_1, \cdots, c_K)$ and transfer payment $w(c)$ conditioned on c, his payoff function is

$$u_p(c, w(c)) = \sum_{k=1}^{K} \lambda_k c_k - w(c)$$

where $\lambda_k > 0$ which represents the marginal utility from the k-th project. By normalizing the principal's utility level, we make

$$\sum_{k=1}^{K} \lambda_k = 1.$$

This normalization is made only to simplify notation.

Agent We assume that the agent is risk averse, whose utility function

$$u_a(a, w) \colon [0,1]^K \times \mathbb{R}_+ \to \mathbb{R}$$

is differentiable, and strictly concave: $\forall \mu \in (0,1)$

$$u_a(\mu a + (1-\mu)a', \mu w + (1-\mu)w' > \mu u_a(a, w) + (1-\mu)u_a(a', w').$$

We also assume that for every $(a, w) \in \mathbb{R}^{K+1}$, $\partial u_a(a,w)/\partial a_k < 0$ for each $k = 1, \cdots, K$ and $\partial u_a(a,w)/\partial w > 0$. Given a particular utility level u_a^*, define

$$B_a(u_a^*) = \{(a, w) : u_a(a, w) \geq u_a^*\}$$

as the (weakly) better-than-set. Since u_a is concave, $B_a(u_a^*)$ is convex. Without loss of generality, assume that

$$u_a(0,0) = u_p(0,0) = 0.$$

First best solution Given the choice $a \in [0,1]^K$ of efforts by the agent, and the compensation scheme w for each $c \in \{0,1\}^K$ by the principle, we can compute the expected payoffs of the agent and the principle. We define the first best solutions.

> **Definition 36.** (a, w) is ***Pareto efficient*** (or simply ***efficient***) if there is no (a', w') that improves the expected payoffs of one player without reducing the other's expected payoff.

Constant compensation Since the agent is risk averse, in any efficient outcome, the principal completely eliminates the risk of the agent by offering the same compensation w^* for each realization $c \in \{0,1\}^K$. We call such compensation scheme a constant compensation. In this case, we will represent the scheme by $w^* \in \mathbb{R}_+$ instead of $w(c) \in \mathbb{R}_+^{2^K}$.

We can write an efficient outcome by the pair $(a^*, w^*) \in \mathbb{R}^{K+1}$ of the agent's effort and the constant compensation by the principal. In any Pareto efficient outcome (a^*, w^*), the agent's payoff is

$$v_a(a^*, w^*) = u_a(a^*, w^*)$$

and the principal's payoff is

$$v_p(a^*, w^*) = \sum_{k=1}^{K} \lambda_k a_k^* - w^*.$$

Notice that v_p and v_a is defined over the efforts and the constant compensation level w^* instead of general compensation schemes.

Individual rationality We assume that the players can quit the game if the expected payoff from the game is less than 0. A payoff vector is (strictly) individually rational if both the principal and the agent receive non-negative (positive) payoff. By using only constant compensation scheme, we can still support any individually rational payoff vector. In this sense, the constant compensation scheme is not restrictive.

Proposition 14. For any individually rational payoff vector (v_a', v_p'), there exists a pair of efforts $a^* \in [0, 1]$ and a constant compensation $w^* \in \mathbb{R}_+$ independent of the outcome of the project so that $v_a' = v_a(a^*, w^*)$ and $v_p' = v_p(a^*, w^*)$.

Inefficiency Because the principal is risk neutral, while the agent is risk averse, the principal must take care of all risk of the agent in the efficient allocation by offering a constant wage. Given the constant wage, the agent has no incentive to exercise effort which is costly. Thus, the best response of the agent to the constant wage is to set $a = 0$. The presence of moral hazard problem makes the first best solution infeasible.

Repeated Principal Agent problem As in the repeated prisoner's dilemma game, we ask whether the principal and the agent can sustain an expected payoff vector which Pareto dominates the one shot principal agent problem. We call the repeated game a repeated moral hazard problem, if only one party (agent) is subject to imperfect monitoring problem, while the other party (principal) can be monitored perfectly.

To sustain an outcome other than one shot Nash equilibrium, we need a punishment, which is triggered if the agent does not exercise the effort according to the contract. Because the principal cannot monitor the effort of the agent perfectly, our challenge is to design a punishment, which can punish a lazy agent without punishing excessively hard working but unlucky agent.

Public history The infinitely repeated principal agent model is obtained by repeating the above game infinitely many times. Let

$$h_p^t = (c^1, w^1(c^1), \cdots, c^t, w^t(c^t))$$

be a sequence of outcomes of projects and the principal's compensation to the agent. We call h_p^t a public history since both players observe this sequence.

Private history Let H_p^t be the set of all public histories at period $t+1$, and define $H_p = \bigcup_{t\geq 1} H_p^t$. In addition to the public history, the agent also knows the sequence of efforts exercised in each period. By a private history of the agent in period $t+1$, we mean the sequence $h_a^t = (a^1, c^1, w^1(c^1), \cdots, a^t, c^t, w^t(c^t))$. Let H_a^t be the set of all private histories at period $t+1$, and $H_a = \bigcup_{t\geq 1} H_a^t$.

Strategy The agent's strategy is

$$f_a \colon H_a \to [0,1]^K$$

and the principal's strategy is

$$f_p \colon H_p \to \mathbb{R}^{2^K}.$$

If a strategy depends only upon a public history, then the strategy is called a public strategy. We assume that the players use public strategies for analytic convenience. par Let F_a and F_p be the set of repeated game strategies of the agent and the principal, respectively, and define $F = F_a \times F_p$. Given a pair $f \in F$, let $\sigma^t(f)$ be the probability distribution over the efforts a by the agent, the outcomes c of the project and the compensation w by the principal, and $\sigma(f) = \{\sigma^t(f)\}$.

We assume that the players are infinitely patient in the sense that they do not discount the future expected payoff. The agent evaluates f according to

$$\pi_a(f) = \liminf_{T\to\infty} E\frac{1}{T}\sum_{t=1}^{T} u_a(a^t, w^t)$$

and the principal's long run expected payoff is

$$\pi_a(f) = \liminf_{T\to\infty} E\frac{1}{T}\sum_{t=1}^{T} u_p(c^t, w^t(c^t))$$

where E is the unconditional expectation operator determined by $\{\sigma^t(f)\}$.

By the repeated (principal agent) game, we mean

$$G^\infty = \langle F_a, F_p, \pi_a, \pi_p \rangle.$$

We defined a Nash equilibrium of G^∞ in a usual fashion.

Results

Caveat Students are reminded that we assume that the principal and the agent are perfectly patient. The principal is willing to collect as much as data as possible to infer the actions of the agent.

The goal of the lecture is narrow. We show that the repeated interactions provide opportunities for the principal to collect sufficient number of observations to infer the agent's effort reasonally accurately so that the principal can provide incentive to the agent.

If the decision makes is impatient, the same logic does not apply. We need to develop a new approach to analyze the repeated games with imperfect monitoring.

Results Let us state the main result of this paper.

Theorem 17. Fix a strictly individually rational payoff vector $v^* = (v_a^*, v_p^*)$ which is not Pareto efficient. There exists a Nash equilibrium $\varphi = (\varphi_a, \varphi_p)$ for G^∞; and $\pi_a(\varphi_a, \varphi_p) = v_a^*$, and $\pi_p(\varphi_a, \varphi_p) = v_p^*$.

Proof (Sketch of proof)**.** Using geometric intuition, we informally explain the key idea of the proof for the simplest case where the agent is engaged in a single project ($K = 1$). In this case, the principal's utility function is

$$u_p(c, w(c)) = c - w(c)$$

so that

$$v_p(a, w) = a - w.$$

Fix an individually rational payoff vector (v_a^*, v_p^*) which is not Pareto efficient. Let

$$\theta^* = \frac{dw}{da}\bigg|_{v_a = v_a^*}$$

be the marginal rate of substitution at (a^*, w^*) of the agent.

Notice that (a^*, w^*) is Pareto efficient if $\theta^* = 1$. Since the marginal rate of substitution is an increasing function of the effort (a, w) along the indifference curve of the agent, for any inefficient payoff vector (v_a^*, v_p^*), we can choose a pair (a^*, w^*) and $\theta^* < 1$. Since (v_a^*, v_p^*) is an individually rational payoff vector, $w^* > 0$ and $a^* > 0$. ■

Even though we may not achieve the Pareto frontier, we can approximate the frontier by some Nash equilibrium of the repeated game.

Summary statistics Let h_p^T be the public history at the beginning of period T. The core of the construction is to design an estimator for the average effort level of the agent from the public data.

We represent the summary statistics as a function of the average success rate and the average compensation. Given h_p^T for $T \geq 1$, define

$$x^T = \sum_{t=1}^{T} c^t/T \quad \text{and} \quad y^T = \sum_{t=1}^{T} w^t/T$$

with $x^o = y^o = 0$, which are the average success rate and the average wage.

Construction We start with the agent's strategy. Given

$$h_p^T = (c^1, w^1, \cdots, c^T, w^T),$$

the agent's strategy dictates

$$\varphi_a(h_p^T) = \begin{cases} a^* & \text{if } x^T - y^T - (a^* - w^*) \leq 0 \\ 0 & \text{if } x^T - y^T - (a^* - w^*) > 0. \end{cases}$$

One can see that φ_a dictates the agent to play a^* if the average payoff of the principal is less than his target payoff $v_p^* = a^* - w^*$; otherwise, it dictates the agent not to work.

The equilibrium linear strategy of the principal is constructed along the same line of idea. For each public history $h_p^T = (c^1, w^1, \cdots, c^T, w^T)$, define

$$\varphi_p(h_p^T) = \begin{cases} w^* & \text{if } y^T - \theta^* x^T - (w^* - \theta^* a^*) \le 0 \\ 0 & \text{if } y^T - \theta^* x^T - (w^* - \theta^* a^*) > 0. \end{cases}$$

Heuristics Roughly speaking, φ_p dictates the principal to pay w^* if the estimated average payoff of the agent is less than the target v_a^*; otherwise, he will pay nothing to the agent.

This interpretation is at best rough, even misleading. Because the agent's utility function is concave with respect to the compensation and the effort, the average compensation and the average success rate do not estimate the average payoff the agent after T rounds.

Law of large numbers As the principal accumulates more information, (Y^T, x^T) becomes an accurate estimator of the target wage w^* and the target effort level a^*. Because the principal's objective is to maximize the long run average payoff, he is willing to take $T \to \infty$ to obtain a better estimator to control the incentive. The same logic applies to the team problem, as long as the both parties do not discount the future payoff.

Discounting If the players are impatient so that they discount the future payoff, it is not obvious whether the law of large numbers can be applied to derive the same result. The same question was raised in early 1980's.

The law of large numbers allows the principal to estimate the agent's effort accurately in the long run. However, with discounting, the principal has to worry about how quickly he can construct an accurate estimator.

The answer is positive for the repeated principal agent problem (one sided moral hazard), but negative for the double moral hazard problem. The repeated game with imperfect monitoring with discounting is an important topic, which generates active research past decades.

Chapter 10

Nash Bargaining Problem

Lecture 19. Nash50-under Wed, May 12

Competitive equilibrium In a competitive market, decisions are made in a decentralized manner. A consumer optimizes subject to the budget constraint and the price. A producer maximizes profit for a given price.

It is not obvious how the market clearing price is determined. We often use a fictitious auctioneer *Walrasian auctioneer* who construct the aggregate demand and supply from individual demand and supply curves. He then finds the intersection of the two curves and announces the market clearing price. Although it is a useful educational tool to explain the market clearing mechanism, the presence of the Walrasian auctioneer goes directly against the very spirit of the decentralized trading of the competitive market: the invisible hand.

Informational efficiency Hayek observed that the competitive market price aggregates dispersed information so that the individual agents in the economy can take and action in a decentralized manner, but can coordinate to achieve and efficient allocation.

The general equilibrium model of Arrow and Debreu formulates the first welfare theorem but remains vague about the price determination process and the information aggregation process.

Decentralized trading model We need to understand the mechanism that aggregates the private information of the individual agents into the market clearing price.

If the trading is done decentralized, it is not clear how there should a single market clearing price. While we analyze the information aggregation process, we can explain the law of single price.

Dynamic decentralized trading model We build a decentralized trading model from a smallest unit of trading: bargaining. To trade something, you need at least two players. Bargining is a trading protocol between two players. We introduce a matching process as we examined in the model of Peter Diamond so that the trading partner may change over time. We examine the prices at which trading occurs, and how the difference among different trading partners vanish, and converge to the competitive equilibrium price.

Plan

- Bargaining
  - Axiomatic model of Nash [1950]
  - Strategic model of Rubinstein [1982]
- Matching and bargaining

Bargaining

- Trading institution between one seller and one buyer to determine the price and the delivery condition of good or service. Also called bilateral monopoly problem.
- Finest unit of trading, which forms the foundation of the market. To trade goods, you need at least two people. The bargaining is a trading unit with two people: one seller vs. one buyer.
- Old and widely used, and has many institutional variations.
- Difficult to formulate and analyze.

> **Example** (Divide a dollar game)**.** A game between two players. $A_i = [0, 1]$
>
> $$u_i(a_1, a_2) = \begin{cases} a_i & \text{if } a_1 + a_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$
>
> Any division of 1 dollar can be sustained by a Nash equilibrium. We have a theory, which cannot make any useful prediction. This is often considered as a failure of the theory, revealing the difficulty of the problem, as the bargaining outcome is affected by the details of the rule of the game.

Why difficult? The bargaining problem is difficult, not because we need a fancy mathematical tool. The analysis must satisfy two conditions.

- The rule of the bargaining must be reasonable, in which the two parties have comparable bargaining power and can influence the outcome, A dictatorial game is a form of bargaining, but is not considered a reasonable model of bargaining.
- The model must make a sharp prediction to tell us what would the most likely outcome from the bargaining process. Divide a dollar game is a reasonable bargaining model. We cannot learn anything from Nash equilibrium, because every efficient division is a Nash equilibrium.

Approach by Nash [1950] In 1950, we did not have tools to tackle the difficulties of the bargaining problems. It is before John Nash invented (Nash) equilibrium concept. Instead, Nash chose to bypass the difficulties, but search for a way to say something useful regarding the bargaining process.

The major innovation of Nash's approach is to suppress the details of the procedure, treating it as a black box, and he focuses on the properties of the reasonable outcome.

A bargaining problem is regarded as a mapping from the data which consists of the preference of the players and the structure of the surplus to the division of the surplus.

He asked the following question: A reasonable bargaining outcome must satisfy a certain set of properties, what is the mathematical formula to calculate the outcome?

Bargaining solution Let S be the set of surplus and d be the disagreement point. S is the collection of all possible outcomes attainable, if agreement is reached. d is the pair of payoffs associated with disagreement.

Example. In case of the divide dollar,

$$\{(a_1, a_2) : a_i \geq 0, a_1 + a_2 \leq 1\}$$

and $d = (0, 0)$.

Definition 37. A ***bargaining problem*** is (S, d), where $S \subset \mathbb{R}^2$ is compact and convex and $\exists s \in S$ such that $s_i > d_i\ \forall i \in \{1, 2\}$.

We admit randomized contact which makes the set of all feasible utilities convex. $S \subset \mathbb{R}^2$ if and only if S is closed and bounded. If S is unbounded, then the bargaining can be meaningless, because each party can get what he wants without negotiation. The closeness of S is a technical condition to ensure the existence of a solution of the optimization problem.

The last condition ensures that the bargaining is not degenerate. If every feasible payoff vector is Pareto dominated by the disagreement outcome, there is no point of negotiation.

Let S be the set of all bargaining problems.

Definition 38. A ***bargaining solution*** $f(S, d) = (u_1, u_2)$ is the rule that specifies which outcome is determined: $f : \mathcal{S} \to \mathbb{R}^2$.

Note. A bargaining solution is not conditioned on a particular bargaining problem. Instead, the way how a bargaining outcome is determined should be spelled out before a particular bargaining problem is selected.

Example (Dictatorial)**.** Let

$$\overline{u}_1 = \arg\max_{u_1'} \{u_1' \mid \exists u_2,\ (u_1', u_2) \in S,\ u_2 \geq d_2\}$$

be the best outcome of player 1 in S.

$$f(S, d) = \{(\overline{u}_1, u_2) \mid (\overline{u}_1, u_2) \in S\}.$$

chooses the best possible outcome.

Example (Always disagree). $f(S, d) = d$.

Discussion

- Dictatorial bargaining solution does not sound reasonable, because the lack of symmetry. By a bargaining situation, we refer to a situation where each party has some, if not equal, control over the outcome of the negotiation. Dictatorial solution does not allow any room for negotiation by player 2.
- If a negotiation always breaks down the outcome is not efficient. Alluding to Coase theorem, such a bargaining rule should be replaced by another rule which generates more efficient outcome.

Axioms Let us spell out the properties which any reasonable bargaining solution must satisfy. John Nash call these properties axioms on the ground that they are evidently reasonable. Let us state four axioms, along with discussions.

A reasonable bargaining solution should be such that its outcome is affected by the units of the utils.

Axiom 4. Consider the two bargaining problems, (S, d) and (S', d') where (S', d') is obtained by applying a positive affine transformation to (S, d): $\forall i$, $\exists \alpha_i \geq 0$ and $\beta_i \in \mathbb{R}$ such that $s'_i = \alpha_i s_i + \beta_i$. Then, f satisfies the ***invariance*** axiom if

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i \quad \forall i.$$

A reasonable bargaining solution should not produce an outcome which is Pareto dominated by another feasible outcome.

Axiom 5. f satisfies the ***Pareto*** axiom if $\exists (t_1, t_2) \in S$ such that $t_i > s_i\ \forall i$ implies $f(S, d) \neq (s_1, s_2)$.

Definition 39. A bargaining problem (S, d) is ***symmetric*** if $(s_1, s_2) \in S$ implies that $(s_2, s_1) \in S$.

If the bargaining problem is symmetric, then the name of a player should not matter, implying that the two parties have equal bargaining power.

Axiom 6. f satisfies the ***symmetry*** axiom if for any symmetric bargaining problem (S, d), $f_1(S, d) = f_2(S, d)$.

The symmetry axiom does not require that the bargaining outcome must be equal for all players. The axiom applies only to a symmetric bargaining problem. The same axiom imposes no restriction on bargaining problems which are not symmetric.

The first three axioms (INV, PAR and SYM) are the restrictions on f over individual bargaining problems. The next axiom specifies how the solutions from two different bargaining problems should be related.

Axiom 7. Consider two bargaining problems, (S, d) and (T, d) with $T \subset S$. f satisfies the axiom of ***independence of irrelevant alternatives*** if $f(S, d) \in T$ implies $f(T, d) = f(S, d)$.

Discussion

- While INV, PAR and SYM appear to be considered evidently reasonable, IIA need motivation, as it imposes a restriction on the relationship between the solutions of two bargaining problems.

- IIA is essentially identical with the weak axiom of choice. If a decision maker choose an object from T which contains S, but the selected commodity bundle is in S, then the consumer should choose the same bundle when he is constrained to choose from S. The alternatives in $T \setminus S$ are irrelevant. We know that the weak axiom in this sense implies that the consumer behavior can be described as a consequence of utility maximization.

- IIA is reasonable, if we accept the view that a reasonable bargaining solution should be a solution of a social welfare function. Otherwise, it is not. The existence of a social welfare function is not always guaranteed.

- For me, IIA appears to be reasonable, but has room to be improved. Quite a few people followed up Nash [1950] proposing alternative set of axioms. Usually, INV, PAR and SYM are not touched, but IIA is replaced by something else.

Consistency and uniqueness We choose INV, PAR, SYM and IIA because they are considered reasonable. We did not consider whether four axioms are consistent with each other. If they are not consistent, no bargaining solution satisfying four axioms exists.

If a bargaining solution satisfying four axioms exists, we need to ask how many solutions satisfy the axioms. If too many solutions satisfy the axioms, there is a room to impose additional axioms.

Nash [1950] answer these two question rigorously and elegantly.

Nash bargaining solution Nash [1950] proposes a bargaining solution.

Definition 40. f^N is the ***Nash bargaining solution*** if

$$f^N(S, d) = \underset{(s_1, s_2) \in S, s_i \geq d_i}{\arg\max} (s_1 - d_1)(s_2 - d_2).$$

$s_i - d_i$ represents the gain from reaching agreement over the disagreement payoff. We call $s_i - d_i$ the Nash gain, and $(s_1 - d_1)(s_2 - d_s)$ the Nash product.

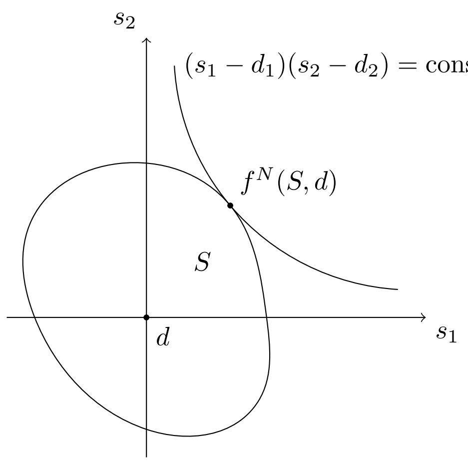


Figure 10.1: Nash bargaining solution (Osborne and Rubinstein [1990])

Note. The Nash product $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_s)$ is strictly quasi concave continuous function. Since S is convex and compact, the maximizer exists and is unique. Thus, the Nash bargaining solution is well defined.

Lecture 20. Nash50-under2 Mon, May 17

Characterization The fundamental theorem of Nash [1950] is that the Nash bargaining solution is the only bargaining solution which satisfies INV, PAR, SYM and IIA.

Theorem 18. Bargaining solution f satisfies INV, PAR, SYM and IIA if and only if $f = f^N$.

Proof. ($\Leftarrow$) We prove that the Nash bargaining solution satisfies INV, PAR, SYM and IIA.

INV Suppose that $s'_i = \alpha_i s_i + \beta_i$ and $d'_i = \alpha_i d_i + \beta_i$. Note

$$(s'_1 - d'_1)(s'_2 - d'_2) = \alpha_1 \alpha_2 (s_1 - d_1)(s_2 - d_2).$$

Thus, if (s_1^*, s_2^*) maximizes the right hand side, then $(\alpha_1 s_1^* + \beta_1, \alpha_2 s_2^* + \beta_2)$ maximizes the left hand side.

PAR Since $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_2)$ is a strictly increasing function of (s_1, s_2), an optimal solution must be at the Pareto frontier of S.

SYM Fix a symmetric problem (S, d) where $d_1 = d_2 = d$. Note

$$W(s_1, s_2) = (s_1 - d)(s_2 - d) = (s_2 - d)(s_1 - d),$$

If (s_1^*, s_2^*) maximizes the left hand side, then $(s_2^*, s_1^*) \in S$ maximizes the right hand side. It remains to show $s_1^* = s_2^*$.

Suppose that $s_1^* \neq s_2^*$. Since $W(s_1, s_2)$ is strictly quasi concave and S is convex, $\forall \lambda \in (0,1)$,

$$\lambda(s_1^*, s_2^*) + (1-\lambda)(s_2^*, s_1^*) \in S$$

and

$$W(\lambda(s_1^*, s_2^*) + (1-\lambda)(s_2^*, s_1^*)) > W(s_1^*, s_2^*) = W(s_2^*, s_1^*)$$

which contradicts to the hypothesis that (s_1^*, s_2^*) maximizes W over S.

IIA Suppose that (s_1^*, s_2^*) maximizes $W(s_1, s_2)$ over T and

$$(s_1^*, s_2^*) \in S \subset T.$$

Since

$$W(s_1^*, s_2^*) \geq W(s_1, s_2) \quad \forall (s_1, s_2) \in T,$$

and $S \subset T$,

$$W(s_1^*, s_2^*) \geq W(s_1, s_2) \quad \forall (s_1, s_2) \in S.$$

Thus, $(s_1^*, s_2^*) \in S$ must maximize W over S.

($\Rightarrow$) The difficult part is to show that Nash bargaining solution is the only solution satisfying four axioms. Given (S', d), choose (z_1, z_2) satisfying $z_i > d_i$ and

$$(z_1 - d_1)(z_2 - d_2) = \max_{(s_1, s_2) \in S, s_i > d_i} (s_1 - d_1)(s_2 - d_2)$$

or equivalently, $(z_1 z_2) = f^N(S', d)$. We can find $\alpha_i \geq 0$ and β_i such that

$$\frac{1}{2} = \alpha_i z_i + \beta_i$$
$$0 = \alpha_i d_i + \beta_i$$

where

$$a_i = \frac{1}{2(z_i - d_i)} > 0, \quad \beta_i = -\frac{d_i}{2(z_i - d_i)}.$$

Let $(S, 0)$ be the bargaining problem obtained by applying the positive affine transformation to (S', d).

Note that the positive affine transformation

$$s_i = \alpha_i s_i' + \beta_i \quad \forall i$$

is invertible. Thus, we can find a positive affine transformation which maps

$$\left(\frac{1}{2}, \frac{1}{2}\right) \mapsto (z_1, z_2), \quad (0,0) \mapsto (d_1, d_2)$$

by transfering bargaining problem $(S, 0)$ to (S', d). We know that f^N satisfies INV. Thus, $f^N(S, 0) = (\frac{1}{2}, \frac{1}{2})$ or equivalently,

$$s_1 s_2 \leq \frac{1}{4} \quad \forall (s_1, s_2) \in S.$$

We show that if f satisfies PAR, SYM, and IIA, then

$$f(S, 0) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Since f satisfies INV, and (S', d) is obtained by an affine transformation,

$$f(S', d) = (z_1, z_2) = f^N(S', d).$$

The following is a supporting hyperplane.

Lemma 2. $\forall (s_1, s_2) \in S$, $s_1 + s_2 \leq 1$.

Proof. Suppose that $\exists (t_1, t_2) \in S$ such that $t_1 + t_2 > 1$. Recall that

$$\left(\frac{1}{2}, \frac{1}{2}\right) \in S$$

and S is convex. Thus, $\forall \lambda \in (0, 1)$,

$$(1 - \lambda)\left(\frac{1}{2}, \frac{1}{2}\right) + \lambda t \in S.$$

Note

$$\begin{aligned} &\left[(1-\lambda)\frac{1}{2} + \lambda t_1\right]\left[(1-\lambda)\frac{1}{2} + \lambda t_2\right] \\ =\ &\left[\frac{1}{2} + \lambda\left(t_1 - \frac{1}{2}\right)\right]\left[\frac{1}{2} + \lambda\left(t_2 - \frac{1}{2}\right)\right] \\ =\ &\frac{1}{4} + \frac{\lambda}{2}(t_1 + t_2 - 1) + \lambda^2\left(t_1 - \frac{1}{2}\right)\left(t_2 - \frac{1}{2}\right) > \frac{1}{4} \end{aligned}$$

for a sufficiently small $\lambda > 0$, because $t_1 + t_2 - 1 > 0$. But, this contradicts the fact that $f^N(S, 0) = (1/2, 1/2)$. ■

Since S is a compact set and $S \subset \{s \,|\, s_1 + s_2 \leq 1\}$, we can construct a symmetric bargaining problem $(T, 0)$ where $S \subset T$. Fix $\omega_1, \omega_2 > 0$.

Define

$$T = \{(s_1, s_2)| - \omega_1 \leq s_1 + s_2 \leq 1,\ -\omega_2 \leq s_2 - s_1 \leq \omega_2\}.$$

Choose $\omega_1, \omega_2 > 0$ sufficiently large so that $S \subset T$. By the construction of T, $(t_1, t_2) \in T$ if and only if $(t_2, t_1) \in T$. Since f satisfies PAR,

$$f(T, 0) \in \{(t_1, t_2) \in T \,|\, t_1 + t_2 = 1\}.$$

Since f satisfies SYM,

$$f(T,0) \in \{(t_1,t_2) \in T \,|\, t_1 = t_2\}.$$

The only point that satisfies the two conditions is

$$f(T,0) = \left(\frac{1}{2}, \frac{1}{2}\right) \in T.$$

Since $f(T,0) \in S$, IIA implies

$$f(S,0) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

as desired. ■

None of four axioms is redundant. It is an excellent exercise to construct a bargaining solution which violates exactly one of the four axioms.

Example (INV)**.** If f^U chooses an outcome according to the utilitarian social welfare function while scaling the weight non-linearly, f^U violates INV. An example would be

$$f^U(S,d) = \underset{s_1>d_1, s_2>d_2}{\arg\max} \; (\sqrt{s_1 - d_1} + \sqrt{s_2 - d_2}).$$

Example (PAR)**.** $f(S,d) = (d_1,d_2)$ obviously violated PAR. But, INV, SYM, and IIA follow directly from the definitions.

Example (SYM)**.** If $\alpha \in (0,1)$,

$$f^\alpha(S,d) = \underset{s_1>d_1, s_2>d_2}{\arg\max} \; (s_1 - d_1)^\alpha (s_2 - d_2)^{1-\alpha}$$

violates SYM, because f^α treats two parties differently, even if the bargaining problem is symmetric. f^α satisfies INV, IIA, and PAR. f^α is called asymmetric Nash bargaining solution.

Example (IIA, Kalai Smordinsky solution)**.** Let us consider Kalai-Smordinsky solution. For any bargaining problem (S,d), let $\overline{s}_i$ be the maximum utility player i gets in

$$\{s \in S \,|\, s_i \geq d_i, \; i \in \{1,2\}\}$$

for $i \in \{1,2\}$, which is the best possible outcome for player i among the agreement outcomes which Pareto dominates disagreement outcome d.

Connect $\overline{s} = (\overline{s}_1, \overline{s}_2)$ and $d = (d_1,d_2)$. (Note that $d \in S$, but $\overline{s}$ may not be an element of S.) Locate a point $s^* = (s_1^*, s_2^*)$ along the line segment and the Pareto frontier of S. Formally s^* is defined as follows. Define

$$\lambda^* = \underset{0\leq\lambda\leq 1}{\arg\max}\{\lambda \,|\, \lambda\overline{s} + (1-\lambda)d \in S\}$$

and

$$s^* = \lambda^* \overline{s} + (1-\lambda^*)d.$$

Consider the solution

$$f^{KS}(S,d) = (s_1^*, s_2^*)$$

which is called Kalai Smordinsky solution.

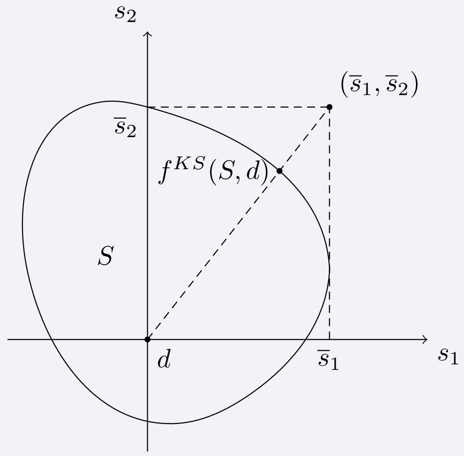


Figure 10.2: Kalai Smordinsky solution (Osborne and Rubinstein [1990])

Fix $d = (0,0)$. Consider T as the convex hull of (1,0), (0,1), (0,0). Then,

$$f^{KS}(T,0) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Suppose that $0 < \varepsilon < \frac{1}{2}$ and consider

$$S = \{s \in T \,|\, s_1 \leq 1-\varepsilon\}$$

which is a subset of T obtained by removing all payoff vectors in T in which player 1 receives more than $1-\varepsilon$. Note that $(\frac{1}{2}, \frac{1}{2}) \in S$. But,

$$f^{KS}(S,0) = \left(\frac{1-\varepsilon}{2-\varepsilon}, \frac{1}{2-\varepsilon}\right) \neq f^{KS}(T,0) \in S.$$

Contribution

- Nash's paper is the first rigorous analysis of the bargaining problem.
- Suppressing the institutional details, and focusing on the outcome, Nash was able to identify a solution concept for a class of bargaining problems. Nash's approach is called the axiomatic approach.

Nash program

- On the other hand, suppressing the institutional details obscures how the rule or the protocol of the bargaining affects the bargaining outcome.

- Nash asserted the need for the non-cooperative approach or strategic approach where we spell out the details of the bargaining protocol such as the order of the moves, the information and the delivery rule and invoke the non-cooperative solution concept (i.e., Nash equilibrium) to identify a solution.
- This program is called Nash program, which requires us to develop a new language to describe a strategic situation, called the extensive form games.

Chapter 11

Alternating Offer Bargaining

11.1 Bargaining

Lecture 21. rubinstein Mon, May 24

Ultimatum game Let us consider the ultimatum game. We know that in the unique subgame perfect equilibrium, the first mover (seller) receive the entire surplus from the transaction with probability 1, while the second mover (buyer).

Alternating offer model Suppose that the ultimatum game is followed by another ultimatum game. If the buyer rejects the offer from the seller, the buyer can start a new round of an ultimatum game, with the buyer as the first mover.

Subgame perfect equilibrium Let us calculate the subgame perfect equilibrium by invoking the backward induction process.

- In the second ultimatum game, the buyer will extract the entire gain from surplus, as the first mover in the second ultimatum game.
- At the end of the first ultimatum game, let us consider the decision problem of the buyer, conditioned on the seller's offer $p \in [0, 1]$. (Note that p does not have to be the equilibrium offer from the seller in the first ultimatum game.)
- The perfection requires that the buyer accept p if the surplus from accepting p (which is $1 - p$) is not smaller than the payoff from the continuation game which is 1: $1 - p \geq 1$. Thus, $p = 0$.
- Since any demand $p > 0$ will be rejected with probability 1, the only offer of the seller is $p = 0$.

Discounting Let us modify the two rounds of the ultimatum game by assuming that each agent discount future payoff by $\delta < 1$. If $\delta < 1$ is close to 1, the players are patient because they treat tomorrow's payoff almost as valuable as today's payoff. If $\delta > 0$ is close 0, then the players are impatient.

The main change is the decision problem of the buyer at the end of the first ultimatum game, conditioned on the seller's demand p. The buyer accept p only if $1 - p \geq \delta \cdot 1$.

Because the payoff from the second ultimatum game is realized tomorrow, the buyer has to discount 1 by δ. Thus, the seller can make a positive surplus only if $p \leq 1 - \delta$.

The highest price the seller can make is $p = 1 - \delta$ which is the equilibrium demand in the unique subgame perfect equilibrium.

Discussion

- Two factors determine the equilibrium share from the bargaining: timing of the move and timing preference.
- The proposer in the second ultimatum game (the buyer) can extract the entire surplus.
- To utilize the bargaining position in the second round, the buyer has to be patient.
- The more patient the second mover is, the larger the equilibrium share is.
- The equilibrium outcome is determined by the bargaining position and the patience.

Finite vs. infinite horizon Let us consider T periods extension of the game, where T is even. In the odd numbered round, the seller proposes and the buyer responds by accepting or rejecting. Accepting the demand concludes the game, and the payoff is realized. If the demand is rejected, then the game moves to another round of the negotiation with the role of the player switched.

As T becomes large, the advantage of being the proposer in the last round of negotiation vanishes. Our interest is how the patience and the bargaining position interact to determine the subgame perfect equilibrium offer.

The game is known as the finite horizon alternating offer bargaining, or the finite horizon Rubinstein's model.

Rubinstein's bargaining

- Two players bargaining over how to split a surplus of size 1, making an offer one after the other until they reach an agreement.
- Stationary time preference, discounting tomorrow's utility by $\delta \in (0, 1)$. If both parties agree with price p at time t, the seller receives $p\delta^{t-1}$ and the buyer $(1 - p)\delta^{t-1}$ where $\delta \in (0, 1)$.
- The seller starts with an offer, followed by the move by the buyer, who will accept or reject the offer. The acceptance leads to an immediate conclusion of the bargaining and the payoff is realized.
- Rejection leads to the next round, starting with an offer from the buyer.

Finite horizon We apply the backward induction process, and derive the formula as a function of T. We know that in round T, the buyer demands the whole surplus by offering $p_T = 0$ which is accepted by the seller with probability 1.

We also know that in round $T-1$, the seller demand $1 - p_{T-1} = \delta$ or $p_{T-1} = 1 - \delta$ which is accepted by the buyer with probability 1.

In $T-2$, the buyer offers p_{T-2} so that $p_{T-2} = \delta(1-\delta)$ because the seller's payoff in the continuation game is $1-\delta$ which is realized in the next round. The buyer's surplus is then

$$1 - p_{T-2} = 1 - \delta(1-\delta)$$

In $T-3$, the seller demand p_{T-3} so that

$$1 - p_{T-3} = \delta(1 - \delta(1-\delta))$$

because the buyer's surplus in the continuation game is $1 - \delta(1-\delta)$ which is realized in the next round. Thus,

$$p_{T-3} = 1 - \delta(1 - \delta(1-\delta)) = 1 - \delta + \delta^2 - \delta^3.$$

By repeating the same reasoning, we conclude that the seller's initial demand is

$$p_1 = 1 - \delta + \delta^2 - \cdots + (-\delta)^{T-1}$$

which is the equilibrium offer in the unique subgame perfect equilibrium of the finite horizon alternating offer bargaining problem. As $T \to \infty$,

$$p_1 \to \frac{1}{1+\delta}.$$

Discussion

- In the equilibrium, the seller who makes the first offer takes a little bit more than one half of the surplus, because $1/(1+\delta) > 0.5$.
- Compared to what he could have got in one round of ultimatum game, this is a small amount, which indicates that the long horizon of bargaining undermines the bargaining power of the first mover.
- On the other hand, the seller receives more than one half, because he exploits the impatience of the buyer. If the buyer is extremely impatient $\delta \simeq 0$, then the seller extract virtually all surplus. If the buyer is extremely patient, the seller has little room to exploit the buyer's patience and must be content with one half of the surplus.

Finite vs. infinite horizon While a finite horizon models sounds reasonable, it has an important restriction. The terminal round is exogenously fixed. The exogenously given terminal round makes sense in some context. We have deadlines, which cannot be extended. Lawyers have to negotiate before the trial begins. The trial date is almost impossible to change once it is set.

On other hand, many negotiation has no exogenously given deadline. The two parties negotiate until they reach agreement. An infinite horizon model is a right model to investigate a bargaining without exogenously given terminal date.

11.2 Infinite horizon Rubinstein

Infinite horizon The main difference is then that we cannot apply the backward induction process as we know it from the games with a finite horizon. Yet, the notion of subgame perfect equilibrium continues to apply here. That is, starting from any decision node, the continuation game is a subgame, and we would like to have an equilibrium that induces a Nash equilibrium in every subgame.

How to solve an infinite horizon problem? Note that the game starting from period 1 and the subgame starting from period 3 are virtually identical. We only have to scale up the payoff of the subgame.

We can exploit the stationarity property of the game to compute a subgame perfect equilibrium.

Guess a stationary equilibrium in which the seller always demands p whenever she has a chance to move, and the buyer always offers q when he has a chance to move.

Recall that the demand and the offer should be accepted by the other party immediately in a subgame perfect equilibrium. Also, each party should do so while maximizing his objective function.

A key equation

$$\begin{aligned} 1-p &= \delta(1-q) \\ q &= \delta p \\ p &= \frac{1}{1+\delta} \\ q &= \frac{\delta}{1+\delta}. \end{aligned}$$

Construction of a subgame perfect equilibrium The seller's strategy is to demand p always, and to accept any price $q' \geq q$. The buyer's strategy is to offer q always, and to accept any price $p' \leq p$. We can show that this pair of strategies form a subgame perfect equilibrium.

Infinite horizon Rubinstein [1982] We will examine Rubinstein [1982] rigorously in coming weeks, who shows that the pair of strategies we constructed is the only subgame perfect equilibrium, under a general condition.

Game

- Two players: seller vs. buyer
- One unit of surplus to divide.
- Two players negotiate the delivery price $p \in [0, 1]$.
- Seller demands price $p \in [0, 1]$ for sale, and buyer offers price for purchase.
- The outcome of the game is (p, t): Delivery price p and time of agreement t.

Extensive form A history at t is a sequence of rejected prices:

$$h_t = (p_1, \cdots, p_{t-1}).$$

Conditioned on h_t, a player makes a move, which depends upon whether t is odd or even. If t is odd, a strategy of the seller is

$$\sigma_s(h_t) = p_t \in [0, 1]$$

and a strategy of the buyer is

$$\sigma_b(h_t, p_t) \in \{A, R\}$$

where A means to accept p_t and R means to reject p_t. If t is even, a strategy of the buyer is

$$\sigma_b(h_t) = p_t \in [0, 1]$$

and a strategy of the seller is

$$\sigma_s(h_t, p_t) \in \{A, R\}$$

where A means to accept p_t and R means to reject p_t.

Let (σ_s, σ_b) be the pair of strategies of the seller and the buyer. Let Σ_s and Σ_b be the strategy spaces of the seller and the buyer. Define $f(\sigma_s, \sigma_b) = (p, t)$ as the outcome induced by (σ_s, σ_b). The game may not end, if both parties reject the offers always. In such case, a terminal node is not reached: $t = \infty$.

Payoff vector assigned to terminal node (p, t) is $(u_s(p, t), u_b(p, t))$ where

$$u_s(p, t) = p\delta^{t-1}, \quad u_b(p, t) = (1 - p)\delta^{t-1}$$

if $t < \infty$. We define

$$u_s(p, \infty) = u_b(p, \infty) = 0.$$

Define the payoff function

$$\mathcal{U}_s(\sigma_s, \sigma_b) = u_s(f(\sigma_s, \sigma_b)), \quad \mathcal{U}_b(\sigma_s, \sigma_b) = u_b(f(\sigma_s, \sigma_b)).$$

Definition 41. (σ_s^*, σ_b^*) is a ***Nash equilibrium*** if

$$\mathcal{U}_s(\sigma_s^*, \sigma_b^*) \geq \mathcal{U}_s(\sigma_s, \sigma_b^*) \quad \forall \sigma_s \in \Sigma_s$$

and

$$\mathcal{U}_b(\sigma_s^*, \sigma_b^*) \geq \mathcal{U}_b(\sigma_s^*, \sigma_b) \quad \forall \sigma_b \in \Sigma_b.$$

Subgame Let G be the bargaining game. Given h_t, let $G|_{h_t}$ be the bargaining game that follows h_t. We define strategies and payoff vectors by restricting those in G to $G|_{h_t}$. Given σ_s and σ_b, let $\sigma_s|_{h_t}$ and $\sigma_b|_{h_t}$ be the strategies in $G|_{h_t}$ induced by σ_s and σ_b, respectively.

Definition 42. (σ_s^*, σ_b^*) is a ***subgame perfect equilibrium*** if $(\sigma_s^*|_{h_t}, \sigma_b^*|_{h_t})$ is a Nash equilibrium in $G|_{h_t}$ for all h_t.

11.3 Nash equilibrium

Lecture 22. rubinstein2 Wed, May 26

Nash equilibrium is not restrictive at all. This result is reminiscent to the set of Nash equilibria of a static bargaining model like Nash demand game.

Theorem 19. $\forall(p^*, t^*)$, there exists a Nash equilibrium (σ_s^*, σ_b^*) such that $f(\sigma_s^*, \sigma_b^*) = (p^*, t^*)$.

Proof. Without loss of generality, let us assume that t is odd. Construct (σ_s^*, σ_b^*) as follows.

$$\sigma_s^*(h_t) = \begin{cases} 1 & \text{if } t^* \neq t \text{ is odd} \\ p^* & \text{if } t^* = t \text{ is odd} \\ R & \forall p \in [0,1] \end{cases}$$

$$\sigma_b^*(h_t) = \begin{cases} R & \forall p \in [0,1] \text{ if } t^* \neq t \text{ is odd or } p \neq p^* \text{ and } t = t^* \\ A & p = p^* \text{ and } t^* = t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

The equilibrium payoff of the seller is $p^*\delta^{t^*-1}$ and the buyer's payoff is $(1-p^*)\delta^{t^*-1}$. Because the buyer only offers 0, the seller cannot improve his payoff by accepting an offer from the buyer. If the seller offers any other price, the buyer rejects. Thus, the seller cannot get a positive payoff from any deviation.

Similarly, the buyer cannot improve his payoff against σ_s^*. Thus, the pair of strategies constitutes a Nash equilibrium. ■

11.4 Subgame perfect equilibrium

Theorem 20 (Existence of a subgame perfect equilibrium)**.** We construct a subgame perfect equilibrium. Conpute p and q satisfying

$$1 - p = \delta(1-q)$$
$$q = \delta p$$

whose solution is

$$p = \frac{1}{1+\delta}, \quad q = \frac{\delta}{1+\delta}.$$

If t is odd $\sigma_s(h_t) = p$ and

$$\sigma_b(h_t, p_t) = \begin{cases} A & \text{if } p_t \leq p \\ R & \text{if } p_t > p. \end{cases}$$

If t is even $\sigma_b(h_t) = q$ and

$$\sigma_s(h_t, p_t) = \begin{cases} A & \text{if } p_t \geq q \\ R & \text{if } p_t < q. \end{cases}$$

Proof. p and q are selected so that the seller is indifferent between accepting q today and offering p which is accepted tomorrow. Similarly, the buyer is indifferent between accepting p to generate $1-p$ surplus today and offering q which generates $1-q$ surplus tomorrow.

Nash We first show that the pair of strategies constitutes a Nash equilibrium. The expected payoff from (σ_s, σ_b) of the seller and the buyer is p and $1-p$, respectively, If the seller offers $p' < q$, the offer will be accepted, and the seller receives less than p. If the seller offer $p' > p$, then the offer is rejected and the best outcome from the continuation game is to accept q in the next round. Since $q = \delta p < p$, the seller cannot receive more than p.

Applying thee same logic to the buyer, we conclude that the pair of strategies constitutes a Nash equilibrium.

Stationarity Observe that the structure of the game is stationary. That is, the subgame starting from round 3, following any history, is identical with the original game. For any h_t with t even, $G|_{h_t}$ is isomorphic to the original game. Thus, if (σ_s, σ_b) is a Nash equilibrium in G, then $(\sigma_s|_{h_t}, \sigma_b|_{h_t})$ is a Nash equilibrium in $G|_{h_t}$. By applying the same logic to h_t with t odd, we conclude that (σ_s, σ_b) induces a Nash equilibrium in every subgame, and therefore is a subgame perfect equilibrium.

Equilibrium payoff Let SPE(G) be the set of subgame perfect equilibria of G. We have shown that SPE$(G) \neq \emptyset$. Since G and $G|_{h_t}$ are isomorphic where h_t is a history with t even, SPE(G) is isomorphic to SPE$(G|_{h_t})$, which also proves SPE$(G|_{h_t}) \neq \emptyset$. $\mathcal{U}_s(\text{SPE}(G))$ is the set of expected payoffs from subgame perfect equilibria of G. We know that $\forall h_t$ with t even,

$$\emptyset \neq \mathcal{U}_s(\text{SPE}(G)) = \mathcal{U}_s(\text{SPE}(G|_{h_t})) \subset \mathbb{R}.$$

Thus,

$$M_s = \sup \mathcal{U}_s(\text{SPE}(g)) \quad \text{and} \quad m_s = \inf \mathcal{U}_s(\text{SPE}(G))$$

are well defined. If the seller starts the bargaining, his subgame perfect equilibrium payoff cannot exceed M_s and cannot be below m_s.

Similarly, for history h_t with t odd, $G|_{h_t}$ is the subgame which starts with the offer from the buyer. Since a subgame perfect equilibrium exists,

$$\emptyset \neq \mathcal{U}_b(\text{SPE}(G|_{h_t})) \subset \mathbb{R}.$$

Thus,

$$M_b = \sup \mathcal{U}_s(\text{SPE}(G|_{h_t})) \quad \text{and} \quad m_s = \inf \mathcal{U}_s(\text{SPE}(G|_{h_t}))$$

are well defined, independently of h_t with h odd. If the buyers starts the bargaining, his subgame perfect equilibrium payoff cannot be higher than M_b and cannot be lower than m_b.

In the subgame perfect equilibrium we constructed, the equilibrium payoff of the seller is $\frac{1}{1+\delta}$. Since the seller starts the game, the seller's payoff is larger than the buyer's payoff $\frac{\delta}{1+\delta}$. Therefore,

$$m_s \leq \frac{1}{1+\delta} \leq M_s.$$

Similarly, if the buyer starts the game, his equilibrium payoff is $\frac{1}{1+\delta}$ and

$$m_b \leq \frac{1}{1+\delta} \leq M_b.$$

■

Theorem 21 (Unique equilibrium payoff)**.** $m_s = M_s$ and $m_b = M_b$.

Proof (Shaked and Sutton (1984))**.** It suffices to show the following inequalities.

- $m_b \geq 1 - \delta M_s$
- $M_s \leq 1 - \delta m_b$
- $m_s \geq 1 - \delta M_b$
- $M_b \leq 1 - \delta m_s$

Note that the first two inequalities are mirror images of the last two inequalities. Suppose that we have proved the first two inequalities. Then,

$$M_s \leq 1 - \delta m_b \leq 1 - \delta(1 - \delta M_s)$$

which implies

$$(1 - \delta^2)M_s \leq 1 - \delta$$

or equivalently,

$$M_s \leq \frac{1}{1+\delta}.$$

Since $M_s \geq \frac{1}{1+\delta}$, we have $M_s = \frac{1}{1+\delta}$ from which

$$m_s = \frac{1}{1+\delta} = M_s$$

follows. The symmetric argument proves that

$$m_b = \frac{1}{1+\delta} = M_b.$$

We do not know yet, whether $\mathcal{U}_s(\text{SPE}(G))$ is a close set. Consequently, we cannot assume at this moment that there is a subgame perfect equilibrium σ^* such that $\mathcal{U}_s(\sigma^*) = M_s$ or $\mathcal{U}_s(\sigma^*) = m_s$. This is the property which will derive rather than assume.

We prove $m_b \geq 1 - \delta M_s$. Suppose that the buyer proposes to buy at price $q > \delta M_s$. Because the best possible outcome in the next round in the continuation game is M_s, q will be accepted by the seller with probability 1. If so, the buyer can reduce the offer by $\varepsilon > 0$ so that $q - \varepsilon > \delta M_s$ and can do better. Thus, q cannot be an equilibrium offer.

Hence, the buyer's payoff $1 - q < 1 - \delta M_s$ cannot be an equilibrium payoff. An equilibrium payoff of the buyer must not be smaller than $1 - \delta M_s$. Thus, $1 - \delta M_s$ is a lower bound of the set of the equilibrium payoff. Since m_s is the largest lower bound of $\mathcal{U}_b$, $1 - \delta M_s \leq m_b$ from which the desired conclusion follows.

We prove $M_s \leq 1 - \delta m_b$. Suppose that the seller demand p such that $1 - p < \delta m_b$. Recall that m_b is a lower bound of the subgame perfect equilibrium payoff. Thus, if the strict inequality holds, the buyer has to reject the offer. Thus, if $p > 1 - \delta m_b$, then the seller's payoff p cannot be sustained by a subgame perfect equilibriu. If there is a subgame perfect equilibrium payoff of the seller, then it cannot be large than $1 - \delta m_b$, which implies that $1 - \delta m_b$ is an upper bound of $\mathcal{U}_s(\text{SPE}(G))$. Since M_s is the smallest upper bound of $\mathcal{U}_s(\text{SPE}(G))$, $1 - \delta m_b \geq M_s$ as desired. ■

We have shown that the equilibrium payoff is unique. We have to show that the equilibrium strategy is unique.

Theorem 22 (Unique pure strategy equilibrium)**.** The equilibrium strategy is unique.

Proof. In an equilibrium, the seller's offer must satisfy

$$1 - p \geq \delta \frac{1}{1+\delta}$$

because the continuation game payoff is exactly $\frac{1}{1+\delta}$. If

$$1 - p > \delta \frac{1}{1+\delta},$$

then p cannot be an equilibrium demand, because the seller can increase his demand slightly and the buyer still accepts the increased demand. In an equilibrium,

$$1 - p = \delta \frac{1}{1+\delta}$$

must hold. Therefore, $p = \frac{1}{1+\delta}$. Applying the same logic, we prove $q = \frac{\delta}{1+\delta}$. ■

Mixed strategy The subgame perfect equilibrium is a pure strategy equilibrium. What if we admit mixed strategy? Does there exist an equilibrium in mixed strategies?

The equilibrium payoff remains unique even if mixed strategies are allowed.

The same proof applies. If the continuation game payoff is unique, the equilibrium demand or offer must be deterministic.

The only possibility is that the acceptance rule might involved randomization, because in an equilibrium,

$$1 - p = \delta \frac{1}{1+\delta}$$

implies that the receiver is indifferent between accepting and rejecting the offer. However, if p is accepted with probability $\theta < 1$, then the seller's expected payoff is θp. If so, the seller can demand

$$\frac{1+\theta}{2} p < p$$

instead of p. Since

$$1 - \frac{1+\theta}{2} p < \delta \frac{1}{1+\delta},$$

the offer is accepted with probability 1 by the buyer, and the seller's payoff is

$$\frac{1+\theta}{2} p > \theta p.$$

Thus, in an equilibrium, the offer must be accepted with probability 1.

Comparative static analysis Discount factor is a function of discount rate (interest rate) r and the time span of each round Δ:

$$\delta = e^{-\Delta r}.$$

As $\Delta \to 0$, the two parties interact more frequently. As a result, the first mover advantage vanishes:

$$\frac{1}{1+\delta} \to \frac{1}{2}.$$

Nash program Let consider the Nash bargaining solution. Since the two parties split surplus of size 1, the bargaing set is

$$S = \{(s_1, s_2) : s_1 \geq 0,\ s_2 \geq 0,\ s_1 + s_2 \leq 1\}$$

and disagreement point is $(0, 0)$. Because the bargaining problem is symmetric, the Nash bargaining solution is the equal split of the surplus: $(0.5, 0.5)$.

Nash [1950, 1953]

- Rubinstein [1982] builds the extensive form by carefully specifying the conditions for the preference, and spelling out the bargaining rule. The rule is reasonable, because the alternating offer bargaining model is widely used.
- Rubinstein [1982] solves the game to generate a sharp prediction out of the model so that we can tell how the rule of game affects the outcome, and how the time preference influences the outcome.
- In this sense, Rubinstein [1982] is considered as the first successful model to address the Nash program.

Chapter 12

Dynamic Monopoly

12.1 Introduction

Lecture 23. dynamicmonopoly Mon, May 31

Monopoly market Monopolistic market is a trading protocol in which a single seller is facing many buyers. We teach the monopolistic market in the undergraduate class through a market with a linear demand, and show that the profit maximization by the monopolist leads to a market clearing price higher than the marginal production cost. The market clearing price in the monopolist market is above the marginal cost, and the equilibrium quantity is less than the competitive equilibrium quantity. The monopolist market leaves unrealized gains from trading, thus inefficient.

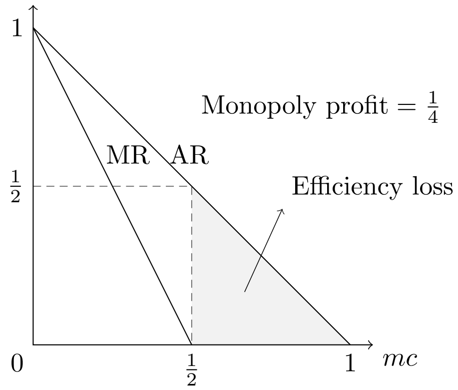


Figure 12.1: $MC = MR$

Monopolistic power The ability to charge a price higher than the marginal cost is the monopolistic power, measured by the difference between the market clearing price and the marginal production cost. A fundamental question is about the source of the monopolistic power.

Discussion The fact that a single seller controls the price of the good is not the source of the monopolistic power, as was first discovered by Ronald H. Coase. Understanding the foundation is important for the policy maker to develop a remedy to regulate the monopolistic market to recover the efficiency. Dynamic monopoly problem is one of a few economic problems, which are important for theory and policy.

12.2 Heuristics

Coase [1973] Ronald H. Coase examines a simple monopoly market in which the monopolist is selling out (commercial) land. The important characteristics of the commercial land is durability.

- After you own a piece of commercial land, the property continue to generate utils, possibly without any decreasing tendency. This feature differentiates commercial land from other consumable products which generate utils just one period.
- A unit of commercial land does not perish. If the seller cannot sell the goods today, she has the same commercial land in the next round and in the future indefinitely. This feature differentiates commercial land from perishable products such as fish which can last one period.

Re-examination of static monopoly problem Suppose that the (inverse) market demand for the durable good is $p = 1 - q$ and the marginal production cost is normalized to 0. The monopolist choose $q_1 = 0.5$ to generate profit 0.25, which is the largest profit among all incentive compatible trading mechanism according to Myerson. Since the market price p_1 is larger than the marginal cost, we say that the monopolistic market power exists.

Unrealized gain from trading Because $p_1 = 0.5 > 0$, the buyers with reservation value less than 0.5 is not served, and the gain from trading is not realized. Thus, the allocation is not efficient. A fundamental assumption is that the monopolist is committed to walk away even though the monopolist is aware of the possibility of making more profit from further trading.

Question Ronald Coase asked what if the monopolist cannot commit himself to walk away from the remaining consumers whose reservation value is less than 0.5. Answering this seemingly obvious question revolutionize the way how the monopolist is regulated, and how we understand inefficiency of bargaining process.

Dynamic monopoly Instead of one shot trading, the monopolist can trade as long as there are consumers whose valuations are higher than the marginal production cost. After serving $q_1 = 0.5$ consumers, the monopolist has a residual demand with one half unit mass of consumers whose reservation value is distributed uniformly between 0 and 0.5.

The monopolist charges the monopolistic price $p_2 = 0.25$ and serve $q_2 = 0.25$ against the residual demand. After serving $q1 + q2 = 0.75$ consumers, the monopolist is left with another residual demand.

Repeating the same logic, the monopolist continues to charge lower prices $p_t > p_{t+1} \to 0$ as long as the residual demand remains positive.

The consumers can follow the reasoning of the monopolist, and are not willing to accept any price significantly higher than the marginal production cost 0, because the consumers know that the monopolist will charge 0 sooner or later.

In order to induce a consumer to accept the offer, the monopolistic seller cannot charge any price higher than 0. The monopolistic power has vanished.

Discussion

- What we described is not very accurate, but only heuristic description of the consequence of the lack of commitment by the monopolist.
- Ronald H. Coase conjectured that if the monopolist can move quickly, then the low price must be offered quickly to serve the consumers with lower reservation value. Foreseeing the monopolist's move, no consumer is willing to accept any price significantly higher than the marginal production cost. It is known as Coase conjecture.
- The price must decrease over time to serve consumers with lower reservation value. The price dynamics is called Coasian dynamics.
- Coase conjecture requires a careful comparison between the speed at which the monopolist decreases and the time preference of the consumers. It took more than 10 years before the conjecture was proved formally.

Implications

- The Coase conjecture identifies the foundation of the monopolistic market power. It is not the fact that there is a single seller, but the commitment power of the seller that provides the market power.
- In order to recover the efficiency, the remedy should be to undermine the commitment power of the monopolist. Instead of letting the monopolist walk away from the market, the regulator should force the monopolist to continue to trade until all the gain from trading is realized.
- The classic case against the monopolist entails a remedy along this line: the monopolist is forced to sell the goods.
- By the same token, the monopolist wants to keep the monopolistic power by making the good less durable. Instead of transferring the ownership, the rental agreement will let the service provided by the good not durable, so that the market demand curve does not shrink in each period. One reason why many software companies move to subscription service from the traditional sales of the software can be explained in this vein.

12.3 Rational expectations

Rational expectations equilibrium The first formal analaysis of Coase conjecture is done by Bulow and Stokey, independently. Let us examine the model of Stokey. The monopolist is facing a unit mass of consumers with reservation value v which is uniformly distributed over [0, 1]. Thus, the (inverse) market demand function is $p = 1 - q$. We normalize the production cost of the monopolist to be 0.

The monopolist charges p_t in period t, and a consumer decides to accept or not to accept p_t. If a consumer with reservation value v accepts p_t, then the good is delivered immediately and the consumer receives $v - p_t$ in period t. Because the good generates a flow of utility permanently, the consumer leaves the market, immediately after purchasing the good.

Suppose that a consumer with reservation value v finds it optimal to purchase the good at p_t than to purchase the good tomorrow at $p_t + 1$:

$$v - p_t \geq \delta(v - p_{t+1})$$

where $\delta \in (0,1)$ is the discount factor. We follow the convention that

$$\delta = e^{-\Delta r}$$

where Δ is the time span of each period, and r is the discount rate (or interest rate). Note that as the monopolist make offer more frequently, $\Delta \to 0$, which implies $\delta \to 1$.

Successive skimming Define v_t as the critical type

$$v_t - p_t = \delta(v_t - p_{t+1})$$

who is indifferent between accepting today and tomorrow. Then, $\forall v \geq v_t$,

$$v - p_t \geq \delta(v - p_{t+1}).$$

In each period, all the consumers whose reservation value is higher than the critical type accepts the offer. This property is known as the successive skimming property.

Define $q_t = 1 - v_t$ which represents the mass of consumers who were served by the end of period t. We can parametrize the residual demand by qt with $q_0 = 0$. If $q = 0$, $D(0)$ is the original inverse demand function where the valuation is distributed uniformly over [0, 1].

If $q = 1$, every consumer has been served and the market is cleared. If $q \in (0,1)$, the $D(q)$ represents the residual demand in which the consumers with reservation value $v \in [0, 1 - q]$ remain in the market, and the valuation is uniformly distributed.

Optimal pricing rule Let $P(q)$ be the price offered by the monopolist when the residual demand is $D(q)$. The optimization problem of the monopolist for q is

$$R(q) = \max_{q' \geq q} P(q)(q' - q) + \delta R(q')$$

where $R(q)$ is the value function of the residual demand function $D(q)$. The monopolist's problem is to decides how much sales he will make at price $P(q)$ where q' is determined by the critical type

$$(1 - q') - P(q) = \delta(1 - q' - P(q')).$$

With initial condition $q_0 = 0$, we can compute optimal pricing sequence

$$p_t = P(q_{t-1}) \quad \forall t \geq 1.$$

Coasian dynamics

> **Lemma 3.** If $\{p_t\}_{t=1}^{\infty}$ is an optimal pricing rule, then $q_t - q_{t-1} > 0$.

If $q_t = q_{t-1}$, then the monopolist did not make any sales in period t, wasting time. By skipping p_t and offering p_{t+1}, the monopolist can increase profit. Thus, $q_t - q_{t-1} > 0$.

> **Corollary** (Coasian dynamics)**.** In any optimal pricing rule, $p_t > t_{t+1}$.

If $p_t \leq p_{t+1}$, then no consumer will purchase at p_{t+1}, since p_t is lower and offered earlier than p_{t+1}.

Linear strategy Stokey [1983] considers a special class of strategy where

$$P(q) = \beta(1 - q)$$

called the linear strategy, in the sense that the price offered by the seller is a linear function of the size of the residual demand.

We can solve the dynamic programming problem to find

$$\beta = 1 - \delta + \frac{\delta\sqrt{1-\delta}}{1 + \sqrt{1-\delta}}.$$

The initial price of the seller is therefore

$$p_1 = P(0) = \beta = 1 - \delta + \frac{\delta\sqrt{1-\delta}}{1 + \sqrt{1-\delta}}.$$

with converges to 0, as $\delta \to 1$, thus confirming the Coase conjecture.

Room to improve

- Stokey shows that there exists a rational expectations equilibrium in which the Coase conjecture holds, leaving the possibility of other rational expectations equilibria with different properties.
- The dynamic monopoly problem is a game with almost perfect information. Thus, we need some sort of perfection requirement to the solution. We need to analyze subgame perfect equilibria instead of rational expectations equilibria.
- We need to address the issue of multiplicity of equilibria.

12.4 Subgame perfect equilibrium

Extensive form game We take advantage of the continuum of infinitesimal consumers, assuming that the action of a single buyer does not make any difference. A history of the game at the beginning of period t is $h_t = (p_1, \ldots, p_{t-1})$.

A strategy of the seller is $\sigma(h)t) = p_t$ and the buyer's strategy is summarized by the optimal decision rule: accept p_t if

$$v - p_t > \delta(v - \sigma(h_t, p_t)).$$

Thanks to the successive skimming property, we can write the critical type v_t

$$v_t - p_t = \delta(v_t - \sigma(h_t, p_t))$$

and $q_t = 1 - v_t$ which defines the size of the residual demand following h_t.

The seller's objective function is

$$\sum_{t=1}^{\infty}(q_{t-1} - q_t)p_t\delta^{t-1}$$

where $p_t = \sigma(h_t)$. The optimization problem of the seller at the beginning of the game is

$$\max_{\sigma}\sum_{t=1}^{\infty}(q_{t-1} - q_t)p_t\delta^{t-1}$$

where $p_t = \sigma(h_t)$, $(1 - q_t) - p_t = \delta(1 - q_t - \sigma(h_t, p_t))$.

Stationary strategy We focus on a class of equilibria in which the seller's pricing rule depends only on the residual demand, instead of the entire history. Suppose that following two different histories h, and the seller's offer is p. Then, buyer with reservation value v accepts p if

$$v - p > \delta(v - \sigma(h, p)).$$

Note that $\sigma(h, p)$ is the expected price from the seller in the next round. Instead of a general pricing rule, the buyer expect $\sigma(h, p) = P(q)$. That is, if q mass of consumers has been served, the monopolist will offer $P(q)$.

The history of plays is summarized into the residual demand. Instead of tracking down the entire history, we represent the monopolist's strategy as a function of the residual demand.

The equilibrium strategy of Stokey belongs to this class, where the Coase conjecture was first formally proved. As we look for the stationary subgame perfect equilibrium instead of the stationary rational expectations equilibrium, we encounter the existence problem if the market demand function is not linear.

12.5 Example

Lecture 24. dynamicmonopoly2 Wed, Jun 2

Let us consider a market demand function which is not continuously downward sloping. One half of population has reservation value $v = 3$ and the remaining half of the consumers has reservation value $v = 1$.

To calculate the static monopolistic profit maximizing solution, we cannot use the differential calculus to equate the marginal cost to the marginal revenue. We need to rely on basic reasoning.

- The seller will never charge less than 1.

- If above 1, the price must be 3.
- The seller never charges above 3.
- 3 will generate profit of 1.5 while 1 will generate 1.

The monopolist profit maximizing price is 3. the profit is 1.5, serving only the high reservation value consumers. Thus, the allocation is inefficient.

Gap. vs. No gap The lowest reservation value buyer is 1, which is above the marginal cost 0, in this example. If the lowest reservation value of the buyer is above the marginal cost, then we call it gap case. In the model of Stokey, the lowest reservation value of a buyer is 0, which is equal to the marginal production. It is called no gap case. We use the lowest reservation value of the buyer as the competitive benchmark.

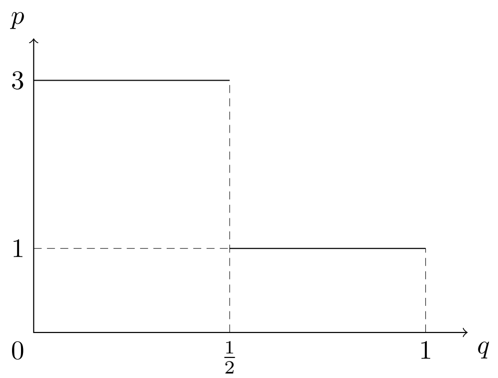


Figure 12.2: Market demand function

Dynamic monopoly problem Suppose that the monopolist opens the market until all buyers are served, and the monopolist has sufficient capacity to serve all buyers in the market. The time span of each period is Δ and discount rate is $r > 0$, and therefore, the discount factor is $\delta = e^{-\Delta r}$.

Preliminaries

Lemma 4. $\exists T < \infty$ such that all consumers are seved in a subgame perfect equilibrium.

Proof. Suppose that a positive mass, say $q > 0$, of consumers is never served. If so, the monopolist can offer ε so that

$$1 - \varepsilon > \delta(1 - 0).$$

Consumers with reservation value 1 will find it optimal to accept $\varepsilon > 0$ right away, because even if the monopolist offers 0 tomorrow, it is better to accept $\varepsilon > 0$ right away. Thus, the monopolist can recover $\varepsilon q > 0$ profit which wolud have been wasted. ■

This is the most important consequence of the gap case. Because the lowest reservation value of the buyer is strictly higher than the marginal production cost, the seller's profit is bounded away from 0.

Terminal round

Lemma 5. $p_T = 1$.

If T is the last round when every consumer is served, then $p_T \leq 1$. If $p_T < 1$, then the buyer will accept, because T is the last round. Thus, the equilibrium offer must be $p_T = 1$.

Penultimate round In T round, if any consumer is still active, the consumer must have reservation value 1. If some consumer has reservation value 3, then in period $T-1$, no reservation value 3 consumer has purchased, and therefore, the monopolist has wasted one round. We know in any equilibrium, the monopolist has to sell a positive amount to consumers. Thus, no consumer with reservation value 3 should be left in period T.

We conclude tat in period T, only the reservation value 1 consumers remain to be served at delivery price $p_T = 1$.

In the penultimate round $T-1$, the price will be higher than 1 so that only reservation value 3 consumers will be served. Thus, p_{T-1} must satisfy

$$3 - p_{T-1} \geq \delta(3 - p_T) = \delta(3-1) = 2\delta.$$

Thus, the highest price which reservation value 3 consumer is willing to accept is

$$p_{T-1} = 3 - 2\delta.$$

Since there are two types of consumers, $T = 2$.

SPE We have shown that in any subgame perfect equilibrium:

$$p_1 = 3 - 2\delta \; quad \text{and} \quad p_2 = 1$$

so that the market is cleared by the end of period 2. Along the equilibrium path, the pricing rule depends only upon the residual demand. Yet, we cannot conclude that the subgame perfect equilibrium strategy is stationary, because we have not checked whether the same property continues to hold off the equilibrium path.

Off the path In period 1, the seller is supposed to charge $p_1 = 3 - 2\delta$. If so, then $p_2 = 3 - 2\delta$. We have to describe what the seller would do, if $p_1 \neq 3 - 2\delta$ and have to show that the seller's strategy off the equilibrium path is optimal.

Different cases

- If $p_1 \leq 1$, then every consumer purchases and the market is cleared.
- If $1 < p_1 < 3 - 2\delta$, then all valuation 3 buyer purchases who expect that the monopolist will charge 1 in the next round. If all type 3 buyers purchase in the previous round, it is indeed optimal for the monopolist to charge 1 to serve the remaining type 1 buyer to clear the market.

- If $p_1 > 3-2\delta^2$, no type 3 buyer will purchase, expecting $3-2\delta$ in the next round. If no buyer purchases in the previous round, the residual demand remains the aggregate demand. It is optimal the monopolist charges $3-2\delta$.

Rational expectations An important observation is that when a buyer makes a decision, he has an expectation about what the seller will do in the next round. The expectation must be rational in the sense that following the decision by the buyer, the seller's optimal decision is exactly what the buyer predicts.

Difficulty case The remaining case is when $2-3\delta < p_1 \leq 3-2\delta^2$. We only consider the decision problem of a buyer with reservation value 3, because $p_1 > 1$. If a buyer with reservation value 3 rejects p_1, then he should expects that the price in the next round is 1 so that $3-p_1 < \delta(3-1)$. But, if he rejects, the residual demand is exactly the aggregate demand. The optimal strategy of the monopolist is to charge $3-2\delta$, not 1.

If a buyer with reservation value 3 accepts p_1, then he expects the next round price is $3-2\delta$ so that

$$3-p_1 > \delta(3-(3-2\delta)) = 2\delta^2.$$

But, if he accepts, the residual demand is 1 and the optimal pricing rule of the seller is to charge 1, not $3-2\delta$.

Discussion

- The example shows that a subgame perfect equilibrium fails to exist in pure strategies. Along the equilibrium path of the rational expectations equilibrium, the seller's strategy depends only on the residual demand.
- In a stationary equilibrium, the seller's pricing rule depends only on the residual demand $p_t = P(q_{t-1})$ where q_{t-1} is the mass of consumers served by the end of period $t-1$, and $P(q_{t-1})$ is the price the seller makes, conditioned on the residual demand q_{t-1}.
- The same example show that a stationary equilibrium does not exist.

Mixed strategy equilibrium We need to construct a mixed strategy equilibrium. In doing so, we need to allow the pricing rule of the seller to be more flexible.

Note that whether or not $3-2\delta$ or 1 is an optimal pricing depends upon the size of the residual demand. If q is close to 0, then $3-2\delta$ is optimal. If q is close to 1, 1 is optimal. A simple calculation shows that if $q = 0.25$ (one half of type 3 buyers served), then the monopolist is indifferent between $3-2\delta$ (followed by 1) or 1 (to clear the market right away).

Fix $p_1 \in (3-2\delta, 3-2\delta^2)$. Choose a probability $\alpha \in [0,1]$ such that

$$3-p_1 = \delta(3-[\alpha(3-2\delta)+(1-\alpha)1]).$$

Note that α depends upon p_1.

Following $p_1 \in (3-2\delta, 3-2\delta^2)$, the monopolist randomizes between $3-2\delta$ and 1 with probability α to $3-2\delta$ so that type 3 buyers are indifferent between accepting and rejecting p_1.

Conditioned on $p_1 \in (3-2\delta, 3-2\delta^2)$, one half of type 3 buyers (that is, 1/4 of consumers) accepts p_1 and the remaining half of type 3 buyers rejects p_1 so that the monopolist is indifferent between $3-2\delta$ and 1. Thus, the randomized pricing between $3-2\delta$ and 1 with probability α to $3-2\delta$ is an optimal strategy.

Discussion We observe the following properties which hold in general.

- The constructed subgame perfect equilibrium is the unique SPE of this game.
- SPE is not stationary, because the strategy off the equilibrium path must depend upon the residual demand and the offer in the previous round.
- Even though the seller has to use a mixed strategy off the equilibrium path, the seller's pricing rule along the equilibrium path is deterministic.

Definition 43. A subgame perfect equilibrium is a ***stationary equilibrium*** if $\forall h_t$, $\sigma(h_t)$ depends only upon the residual demand by the end of h_t, and a ***weakly stationary equilibrium*** if $\forall h_t$, $\sigma(h_t)$ depends only upon the residual demand and the previous round's offer along h_t.

12.6 General model

The economy is populated by a unit mass of consumers, each of whom has reservation value $v \in [\underline{v}, \overline{v}]$ and demand one unit of the good. Let

$$F(v) = P(v' \leq v)$$

and therefore,

$$q = 1 - F(p)$$

is the market demand if the seller makes a take-it-or-leave-it offer. As we change p, we can derive the demand, which we call the market demand.

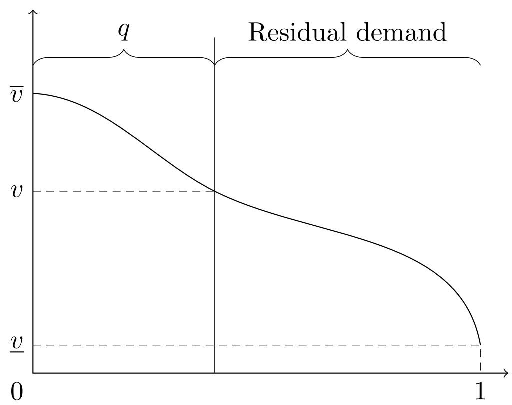


Figure 12.3: Residual demand curve

while we can admit more general demand curve, let us assume that $F(p)$ is strictly increasing continuous function. Thus, the market demand function is strictly decreasing. Thanks to the successive skimming property, a residual demand is parametrized by the mass of consumers who have been served. By residual demand q, we mean the portion of the aggregate demand without consumers with $v \geq 1 - q$.

Coase conjecture Suppose that $F'(\underline{v}) > 0$ and $\underline{v} > 0$ (a.k.a., gap case). Generically, there is a unique weakly stationary subgame perfect equilibrium. Let $T(\Delta)$ be the number of periods to clear the market, where Δ is the time span of each period.

- $T(\Delta) < \infty \ \forall \Delta > 0$
- (No delay) $\lim_{\Delta \to 0} T(\Delta)\Delta = 0$
- (Coase conjecture) $\lim_{\Delta \to 0} p_1 = \underline{v}$ which is an equivalent to no delay result.

Discussion

- Δ is the institutional parameter, which is determined by the rule of the game.
- As we have shown in the example, the market must be cleared in a finite number of rounds, if there is a gap. $F'(\underline{v}) > 0$ and $\underline{v} > 0$ imply that around $\underline{v}$ consumer, the monopolist can guarantee a strictly positive profit. If $T(\Delta) = \infty$, then the monopolist foregoes the positive profit, which is not possible in the optimal pricing rule.
- $T(\Delta)$ is the number of rounds, and $T(\Delta)\Delta$ is the amount of real time needed to clear the market. The insight of Coase is that as the monopolist can make offers more frequently, the total number of rounds may increase but the total amount of time to clear the market vanishes. The proof relies on the (weak) stationarity of the strategy.

In the initial round, $\overline{v}$ will accept p_1. To ensure that p_1 is acceptable for $\overline{v}$ consumer, it is necessary that

$$\overline{v} - p_1 \geq \max_{t \geq 2} \delta^{t-1}(\overline{v} - p_t) \geq \delta^{T(\Delta)-1}(\overline{v} - \underline{v})$$

because $p_{T(\Delta)} = \underline{v}$. Note

$$\delta^{T(\Delta)-1}(\overline{v} - \underline{v}) = e^{-\Delta T(\Delta) + \Delta}(\overline{v} - \underline{v}).$$

Therefore,

$$\overline{v} - p_1 \geq e^{-\Delta T(\Delta) + \Delta}(\overline{v} - \underline{v})$$

from which we can show that

$$\lim_{\Delta \to 0} p_1 = \underline{v}$$

if and only if

$$\lim_{\Delta \to 0} T(\Delta)\Delta = 0.$$

For this reason, we refer to the no delay result as the Coase conjecture as well.

Foundation of monopolist power

- The monopolist can make one offer every Δ unit of time. After making an offer, the monopolist cannot make another offer within Δ unit of time, and is committed not to make an offer.
- We use Δ as a measure of commitment. If $\Delta = \infty$, the model is reduced to a static model of monopolistic market, because the monopolist can make exactly one offer, and the consumer knows it.
- $p_1 - \underline{v}$ is the measure of monopolistic market power. The Coase conjecture says that as $\Delta \to 0$, $p_1 - \underline{v} \to 0$.
- The foundation of the monopolistic power is the power of commitment which is quantified by $\Delta > 0$.

Monopolistic power

- Transfer of the ownership of a durable good creates the competitor, because the new owner can sell the good in the secondary market. The monopolistic power relies on the control of the secondary market.
- Traditionally, the monopolist refuses to sell the durable goods. Instead, they offer lease or rental service to prevent the secondary market from emerging.
- Software companies change their business model from the sales of software to the service. As a service, the monopolist can make the software *not durable* so that the consumer has to return to the market after the service contract (or rental contract) expires. The residual demand curve does not shrink.

Policy implications

- The regulation of the monopolist must be targeted to undermine the commitment power.
- The policy remedies against AT&T, IBM, Xerox are based on the same principle by forcing the monopolist to provide the buyers an option to purchase.

Chapter 13

Bargaining under Uncertainty

Lecture 25.
bargainuncertain
Mon, Jun 7

13.1 Delay in Bargaining

13.2 Analysis

13.3 Delay and uncertainty

Chapter 14

Uncertainty and Delay

Lecture 26.
bargainuncertain2
Wed, Jun 9

14.1 Uncertainty and delay

14.2 Stationarity

14.3 Gains from trading

14.4 Common value

Chapter 15

Search and Matching

15.1 Introduction

Lecture 27.
search
Mon, Jun 14

15.2 Overview

15.3 Modeling features

Chapter 16

Synthetic Market

16.1 Introduction

Lecture 28.
RW85
Wed, Jun 16

16.2 Model

16.3 Analysis

16.4 Variations