Lecture 3. Fixed-Income Securities

1. Value Formulas

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1.1. Perpetual Annuities 영구채.

Definition 1. A *perpetual annuity*, or *perpetuity*, is a cash flow paying a fixed sum periodically forever.

- The present value of a perpetuity that pays an amount A every period is

$$\begin{aligned} PV &= \frac{A}{(1+r)} + \frac{A}{(1+r)^2} + \frac{A}{(1+r)^3} + \cdots \\ &= \frac{A}{(1+r)}\left[1 + \left(\frac{1}{1+r}\right) + \left(\frac{1}{1+r}\right)^2 + \cdots\right]. \end{aligned} \tag{1}$$

  Recalling

$$y = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x},$$

  one simplifies (1) to

$$PV = \frac{A}{1+r} \times \frac{1}{1-\frac{1}{1+r}} = \frac{A}{r}.$$

1.2. Finite-Life Streams

Definition 2. An *annuity* is a finite set of level sequential cash flows. An *ordinary annuity* has a first cash flow that occurs one period from now, while an *annuity due* has a first cash flow that occurs immediately.

- The present value of an ordinary annuity that pays an amount A each period for a total of T periods is

$$\begin{aligned} PV &= \frac{A}{(1+r)} + \frac{A}{(1+r)^2} + \cdots + \frac{A}{(1+r)^T} \\ &= \frac{A}{(1+r)}\left[1 + \left(\frac{1}{1+r}\right) + \cdots + \left(\frac{1}{1+r}\right)^{T-1}\right]. \end{aligned} \tag{2}$$

  Recalling

$$y = 1 + x + x^2 + \cdots + x^{T-1} = \frac{1-x^T}{1-x},$$

  one simplifies (2) to

$$PV = \frac{A}{(1+r)}\left[\frac{1-\left(\frac{1}{1+r}\right)^T}{1-\frac{1}{1+r}}\right] = \frac{A}{r}\left[1 - \frac{1}{(1+r)^T}\right].$$

2. Bond Details

A *bond* is an obligation by a bond issuer to pay money to a bond holder according to rules specified at the time the bond is issued. The issuer sells the bonds to raise capital immediately, and then is obligated to make the prescribed payments.

- A bond pays its *face value* (or *par value*) at the date of maturity, and the par values in general are even amounts such as \$1,000 or \$10,000. Most bonds pay periodic *coupon payment*. Since the last coupon date corresponds to the maturity date, the last payment to bond holders is equal to the face value plus the coupon value.

Example 3. Consider a bond with a coupon rate of 10% and coupons paid annually. The face value is \$1,000 and the bond has 5 periods to maturity.

	0	1	2	3	4	5
Coupons		\$100	\$100	\$100	\$100	\$100
Face Value						\$1,000

Remark 4. Although bonds offer a supposedly fixed-income stream, they are subject to default if the issuer has financial difficulties. To characterize the nature of this *credit risk*, bonds are rated by rating organizations such as Moody's and Standard & Poor's.

3. Yield

Definition 5. A bond's yield, or *yield to maturity* (YTM) is the interest rate at which the present value of the stream of payments is exactly equal to the current price.

- Suppose that a bond with face value F makes a coupon payment of C each period and there are T periods remaining. Given that the current price of the bond is P_0, then the YTM is "computed" as the value of λ such that

$C = F \cdot$ coupon rate

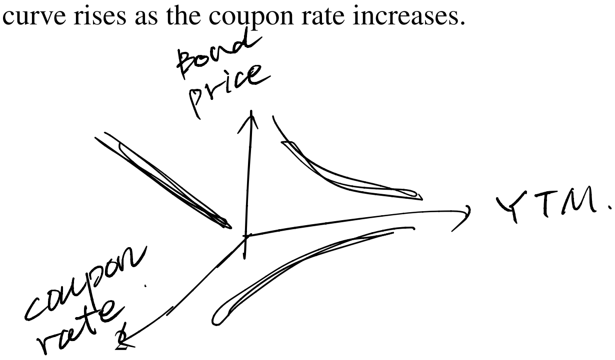
$$\begin{aligned} P_0 &= \sum_{t=1}^{T} \frac{C}{(1+\lambda)^t} + \frac{F}{(1+\lambda)^T} \\ &= \frac{C}{\lambda}\left[1 - \frac{1}{(1+\lambda)^T}\right] + \frac{F}{(1+\lambda)^T}. \end{aligned} \tag{3}$$

Remark 6. In most cases, one should rely on numerical methods to find solutions in three or higher-order polynomial equations such as (3).

3.1. *Price-Yield Curves*

- Price and YTM have an inverse relation; i.e., the price-yield curve is convex. When people say "the bond market went down," they mean that YTM went up.

- When YTM is equal to the coupon rate, the value of the bond is equal to the par value. With a fixed maturity date, the price-yield curve rises as the coupon rate increases.



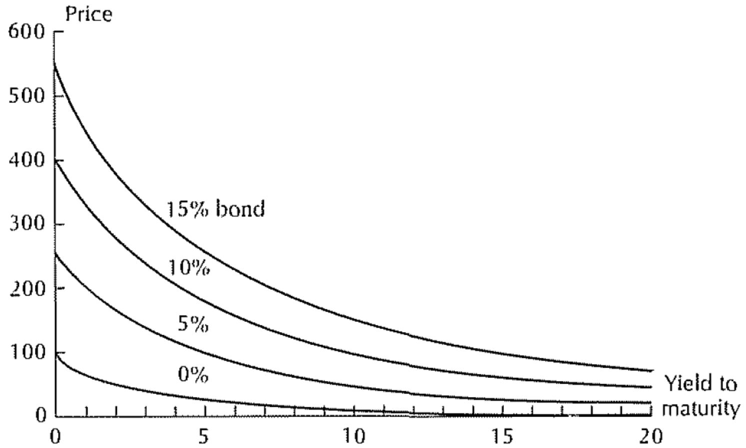


FIGURE 3.3 Price–yield curves and coupon rate All bonds shown have a maturity of 30 years and the coupon rates indicated on the respective curves Prices are expressed as a percentage of par

- As the maturity is increased, the price-yield curve becomes steeper, essentially pivoting about the par point. This indicates that *longer maturity implies greater sensitivity of price to YTM*.

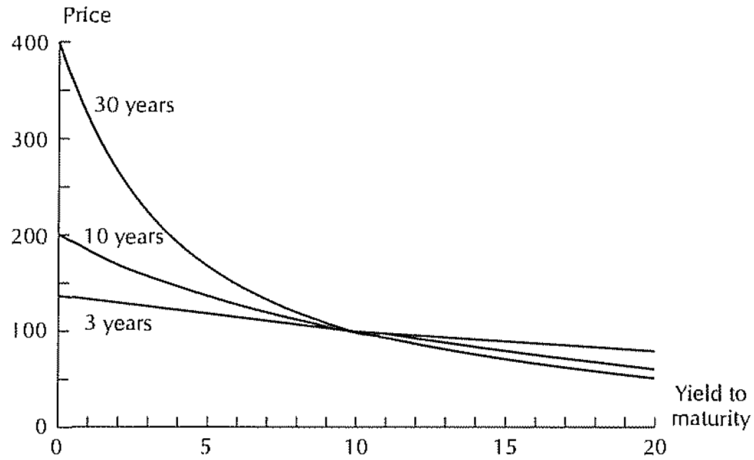


FIGURE 3.4 Price–yield curves and maturity. The price–yield curve is shown for three maturities All bonds have a 10% coupon

3.2. Interest Rate Risk

Bond holders are subject to the *interest rate risk,* in that the "future" changes in YTM lead to the "future" changes in bond prices.

- If you continue to hold the bond at maturity, you continue to receive the promised coupon payments and the face value at the maturity. These cash flows will not be affected by the YTM changes.

- If you plan to sell the bond before maturity, however, the future price at which you can sell will be governed by the price-yield curve at that moment.

Example 7. Bonds with long maturities have steeper price-yield curves than bonds with short maturities, thereby meaning that the prices of *long bonds* are more sensitive to YTM changes than those of *short bonds*.

TABLE 3 5
Prices of 9% Coupon Bonds

	Yield				
Time to maturity	**5%**	**8%**	**9%**	**10%**	**15%**
1 year	103 85	100 94	100 00	99 07	94 61
5 years	117 50	104 06	100 00	96 14	79 41
10 years	131 18	106 80	100 00	93 77	69 42
20 years	150 21	109 90	100 00	91 42	62 22
30 years	161 82	111 31	100 00	90 54	60 52

The prices of long-maturity bonds are more sensitive to yield changes than are the prices of bonds of short maturity

4. Duration

4.1. Macaulay Duration

Definition 8. Suppose that an asset makes one payment per period, with the payment in period t being C_t, and there are T periods remaining. The *Macaulay duration,* denoted by D, is defined as

$$D = \frac{PV(C_1)\times 1 + PV(C_2)\times 2 + \cdots + PV(C_T)\times T}{PV}$$
$$= \frac{\sum_{t=1}^{T}[C_t/(1+\lambda)^t]\,t}{PV},$$

where λ is the YTM and

$$PV = \sum_{t=1}^{T} PV(C_t) = \sum_{t=1}^{T} \frac{C_t}{(1+\lambda)^t}.$$

- By definition, the duration is a weighted average of the times that payments are made and the weighting coefficients are $PV(C_t)/PV$. When C_t are nonnegative, hence, it shows $1 \le D \le T$. For a zero-coupon bond, D equals to T, since $C_1 = \cdots = C_{T-1} = 0$ and $C_T > 0$. The duration of a coupon-bearing bond is strictly less than T.

Example 9. Consider a 7% bond with 6 years to maturity. Assume that the bond pays a payment per year and is selling at 8% yield.

T	C_t	$PV(C_t)$	$PV(C_t)/PV$	$[PV(C_t)/PV]\times t$
1	70	64.81	0.07	0.07
2	70	60.01	0.06	0.13
3	70	55.57	0.06	0.17
4	70	51.45	0.05	0.22
5	70	47.64	0.05	0.25
6	1,070	674.28	0.71	4.24
Sum		$PV = 953.77$	1.00	$D = 5.08$ (in years)

4.2. Duration and Sensitivity

Duration is a useful measure of interest rate risk, since it directly measures the sensitivity of price to changes in YTM.

Theorem 10. *The derivative of price P with respect to yield λ of a fixed-income security is given by*

$$\frac{dP}{d\lambda} = -D_M P, \tag{4}$$

where

$$D_M = \frac{D}{1+\lambda}$$

is called the modified duration.

Proof. Consider

$$\frac{dP}{d\lambda} = \sum_{t=1}^{T} \frac{dPV(C_t)}{d\lambda}. \tag{5}$$

Since the present value of C_t is given by

$$PV(C_t) = \frac{C_t}{(1+\lambda)^t},$$

it shows

$$\begin{aligned} \frac{dPV(C_t)}{d\lambda} &= -\frac{tC_t}{(1+\lambda)^{t+1}} \\ &= -\frac{t}{1+\lambda} PV(C_t). \end{aligned} \tag{6}$$

Using (6), one writes (5) as

$$\begin{aligned} \frac{dP}{d\lambda} &= -\sum_{t=1}^{T} \frac{tPV(C_t)}{1+\lambda} \\ &= -\frac{1}{1+\lambda} \sum_{t=1}^{T} tPV(C_t) \\ &= -\left(\frac{D}{1+\lambda}\right) P \\ &= -D_M P. \end{aligned}$$

□

- By writing (4) as

  $$\frac{(dP/P)}{d\lambda} = -D_M,$$

  one knows that D_M measures the *relative change* in price as λ changes. Given the approximation that $dP/d\lambda \approx \Delta P/\Delta\lambda$, furthermore, (4) can be used to estimate the *change in price* due to a small change in yield:

  $$\Delta P \approx -D_M P \Delta\lambda.$$

Remark 11. Modified duration leads to a straight-line approximation to the price-yield curve. Algebraically, this approximation is equivalent to a first-order Taylor expansion at a given point, and shall be improved by including a second-order term. This second order is known as *convexity*, which is the relative curvature at a given point on the price-yield curve.

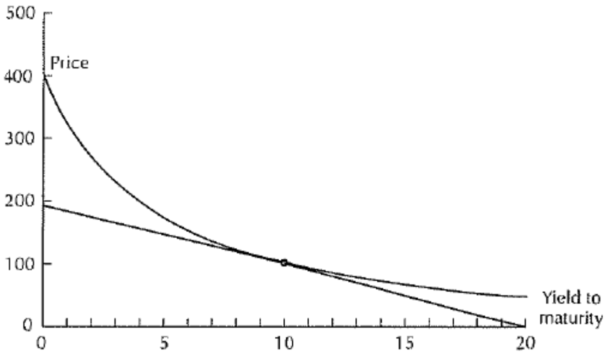


FIGURE 3 6 Price–yield curve and slope The slope of the line tangent to the curve at P is $-D_M P$

4.3. Duration of a Portfolio

Theorem 12. *Suppose there are N fixed-income securities with prices and durations of P_i and D_i respectively, $i = 1,\ldots,N$, all computed at a hypothetically* ***common*** *yield. The portfolio consisting of the aggregate of these securities has price P^* and duration D^*, given by*

$$\begin{aligned} P^* &= P_1 + \cdots + P_N \\ D^* &= w_1 D_1 + \cdots + w_N D_N, \end{aligned}$$

where $w_i = P_i / P^*$ for $i = 1,\ldots,N$.

Proof. Regardless of different maturities and coupon payments, the duration of each fixed-income security in a portfolio is represented as

$$D_i = \frac{\sum_{t=1}^{T} t PV(C_t^i)}{P_i}$$

for $i = 1,\ldots,N$, where C_t^i is the bond i's coupon payment made at time t. Then, it shows

$$\begin{aligned} P_1 D_1 + \cdots + P_N D_N &= \sum_{t=1}^{T} t PV(C_t^1) + \cdots + \sum_{t=1}^{T} t PV(C_t^N) \\ &= \sum_{t=1}^{T} t \left[PV(C_t^1) + \cdots + PV(C_t^N) \right] \\ &= \sum_{t=1}^{T} t \cdot PV(C_t^P), \end{aligned} \tag{7}$$

where C_t^P represents the aggregate coupon payment at time t, generated from the portfolio and defined as $C_t^P = \sum_{i=1}^{N} C_t^i$. Dividing both sides of (7) by $P^* = \sum_{i=1}^{N} P_i$, finally, one obtains

$$w_1 D_1 + \cdots + w_N D_N = \frac{\sum_{t=1}^{T} t \cdot PV(C_t^P)}{P^*} = D^*.$$

□

- The duration of a portfolio measures the interest rate sensitivity of the portfolio, just as normal duration measures it for a single bond. Thus, the change in the portfolio price in response to the YTM change is approximately as

$$\Delta P^* \approx -D_M^* P^* \Delta\lambda,$$

where $D_M^* = D^*/(1+\lambda)$.

Remark 13. In practice, the bonds composing a portfolio have "different" yields. In this case, an average of the different yields is used as a hypothetically single yield.

5. Immunization

Definition 14. *Immunization* refers to the structuring of a bond portfolio to protect against the interest rate risk, thereby making the portfolio value be "immunized" against interest rate changes.

- Suppose that you face a series of cash obligations and wish to acquire a portfolio to pay these obligations as they arise. A solution is to obtain a portfolio having a value equal to the present value of stream of obligations; i.e., you sell some of the portfolio whenever cash is needed to meet a particular obligation. If the yields change, however, both present values of the portfolio and the obligation stream change by amounts that differ from one another. Consequently, the portfolio will no longer be matched.

To solve the miss-matching problem, the immunization matches both *durations* and *present values*. If the portfolio duration matches the obligation duration, the present values of the portfolio and the obligation stream will respond "identically" to a change in yield. As a result, the portfolio will cover the obligation appropriately. The immunization procedure is widely used by pension funds, insurance companies, etc.

- Specifically, the immunized portfolio of m fixed-income securities is found by solving the following two equations

$$\begin{aligned} PV(\text{portfolio}) &= PV(\text{obligation}) \\ D_{\text{portfolio}} &= D_{\text{obligation}}. \end{aligned}$$

Example 15. Suppose that a firm has an obligation to pay \$1 million in 10 periods. For constructing a portfolio to meet this obligation, the firm is considering two bonds, Bond 1 and Bond 2, described in the following table:

	Coupon Payment	Maturity	Price	Yield
Bond 1	60	30	691.79	9%
Bond 2	110	10	1,128.35	9%

The durations of two bonds are $D_1 = 11.88$ and $D_2 = 6.73$, since

T	C_t	$PV(C_t)$	$PV(C_t)/PV$	$[PV(C_t)/PV] \times t$
1	60	55.05	0.08	0.08
2	60	50.50	0.07	0.15
		(skip)		
29	60	4.93	0.01	0.21
30	1060	79.89	0.12	3.46
		691.79		11.88
Sum		$PV_1 = 691.79$	1.00	$D_1 = 11.88$

and

T	C_t	$PV(C_t)$	$PV(C_t)/PV$	$[PV(C_t)/PV] \times t$
1	110	100.92	0.09	0.09
2	110	92.58	0.08	0.16
		(skip)		
9	110	50.65	0.05	0.41
10	1110	464.65	0.41	4.13
Sum		$PV_2 = 1,124.13$	1.00	$D_2 = 6.73$

The present value of the obligation is computed at 9% interest and is equal to

$$PV(\text{obligation}) = \frac{1,000,000}{(1.09)^{10}} = 422,410.81.$$

The duration of the obligation equals 10. Finally, one has a system of two equations,

$$\begin{aligned} n_1 \times 691.79 + n_2 \times 1,124.13 &= 422,410.81 \\ w_1 \times 11.88 + w_2 \times 6.73 &= 10, \end{aligned}$$

where n_i is the number of shares of bond i and the weights are defined as

$$w_1 = \frac{n_1 \times 691.79}{422,410.81} \text{ and } w_2 = \frac{n_2 \times 1,124.13}{422,410.81}.$$

Solving the system for n_i leads to $n_1 = 387.89$ and $n_2 = 137.06$, respectively.