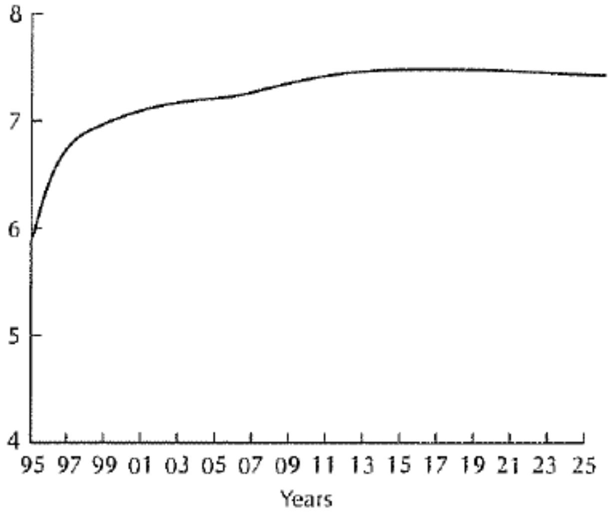
Lecture 4. The Term Structure of Interest Rates

1. The Yield Curve

A *yield curve* displays the yield as a function of time to maturity. In practice, the plot is constructed by plotting the yields of various bonds of a given quality class.

Example 1. Yield curve for government securities in June 1995



2. Spot Rates

Definition 2. A *spot rate,* denoted by s_t, is defined by the rate of interest, expressed in *yearly* terms, charged for money held from time 0 (or present time) until time t.

- Suppose a bank promises to pay a rate of s_1 for a one-year deposit and s_2 for a two-year deposit. If you invest \$1 in the one-year deposit, then, your money in year one is equal to $\$(1+s_1)$. If you invest the same amount in the two-year deposit, the money will grow to $\$(1+s_2)^2$ in year two.

Definition 3. Given the spot rate s_t in time t, the corresponding *discount factor* d_t in time t is defined by

$$d_t = \frac{1}{(1+s_t)^t}$$

for $t = 1, \ldots, T$.

- Given the discount factors, the present value is calculated as

$$PV = x_0 + d_1 x_1 + \cdots + d_T x_T$$

for the cash flow stream $(x_0, x_1, \ldots, x_T)$.

Example 4. Using spot rates, one computes the price of an 8% bond maturing 5 years as follows:

Year	1	2	3	4	5	
Cash Flow	80	80	80	80	1,080	
$s_t(\%)$	5.571	6.088	6.555	6.978	7.361	
d_t	0.9472	0.8885	0.8266	0.7635	0.7011	
Present Value	75.78	71.08	66.13	61.08	757.17	$P = 1,1031.23$

In practice, a series of s_t can be computed from the prices of coupon-bearing bonds by beginning with short maturities and working forward toward longer maturities.

- s_1 is immediately determined by taking the 1-year T bill rate. Suppose that a 2-year bond has price P, makes coupon payment of C, and has a face value F. Then, it shows

$$P = \frac{C}{(1+s_1)} + \frac{C+F}{(1+s_2)^2},$$

  so that one can solve this equation for s_2. By repeating this process forward, $s_3, s_4, \ldots$ are determined adequately.

Example 5. Using the spot rates at various maturities, one can construct a *spot rate curve.* The shape of a spot curve is referred to as a *term structure*.

Years	Spot Rate
1	5 571
2	6 088
3	6 555
4	6 978
5	7 361
6	7 707
7	8 020
8	8 304
9	8 561
10	8 793
11	9 003
12	9 193
13	9 365
14	9 520
15	9 661
16	9 789
17	9 904
18	10 008
19	10 103
20	10 188

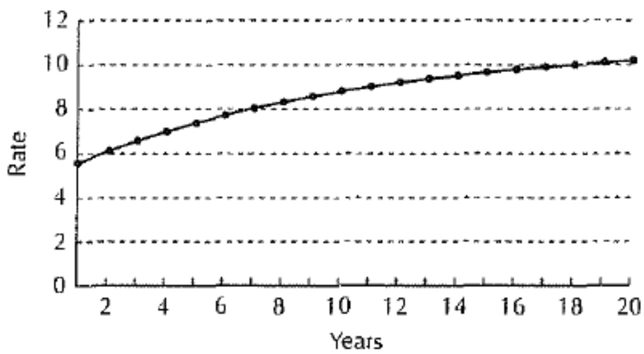


FIGURE 4.2 Spot rate curve The yearly rate of interest depends on the length of time funds are held

3. Forward Rates

Definition 6. A *forward rate* between times t_1 and t_2 with $t_1 < t_2$, denoted by $f_{t_1 \to t_2}$, is defined by the rate of interest charged for borrowing money at time t_1 which is to be repaid (with interest) at time t_2.

- Consider you are investing \$1 for 2 years. As the first option, you can invest \$1 in a 2-year account paying s_2. Alternatively, you may invest \$1 in a 1-year account paying s_1 and simultaneously enter a forward contract paying $f_{1\to2}$ for the period from year 1 to year 2.

Time	0	1	2
Option 1	1		$(1+s_2)^2$
Option 2	1	$(1+s_1)$	$(1+s_1)(1+f_{1\to2})$

No arbitrage in equilibrium suggests that the two options should produce the same amount at the end of year 2:

$$(1+s_2)^2 = (1+s_1)(1+f_{1\to2})$$

or

$$f_{1\to2} = \frac{(1+s_2)^2}{(1+s_1)} - 1.$$

In general, the relationship between the spot rates and the forward rate, for $j > i$, is stated as

$$(1+s_j)^j = (1+s_i)^i(1+f_{i\to j})^{j-i}.$$

The term $(1+s_j)^j$ is the factor by which money grows if it is directly invested for j years. The term $(1+s_i)^i(1+f_{i\to j})^{j-i}$ is the factor by which money grows if it is invested first for i years and then in a forward contract (arranged today) between years i and j.

- So, the forward rate is computed as

$$f_{i\to j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{1/(j-i)} - 1$$

given s_i and s_j.

4. Term Structure Explanations

4.1. Expectations Theory

Denote the expected future one-year spot rate in year 1 by $s_{1\to2}$. An investor can invest either in (a) a two-year loan or (b) two successive one-year loans:

Time	0	1	2
Option 1	1		$(1+s_2)^2$
Option 2	1	$(1+s_1)$	$(1+s_1)(1+s_{1\to2})$

- In equilibrium the expected payoffs from these two strategies must be equal. Since $(1+s_2)^2 = (1+s_1)(1+f_{1\to2})$, this implies that the forward rate, $f_{1\to2}$, must equal the expected spot rate, $s_{1\to2}$. In general, the expected future k-year spot rate in year j, denoted by $s_{j\to j+k}$, equals to $f_{j\to j+k}$.

Theorem 7. *The expectations theory implies that the only reason for an upward-sloping term structure (i.e., $s_2 > s_1$) is that investors expect the future one-year spot rate to rise (i.e., $s_{1\to2} > s_1$).*

Proof. Write $(1+s_2)^2 = (1+s_1)(1+f_{1\to2})$ as

$$\frac{1+s_2}{1+s_1} = \frac{1+f_{1\to2}}{1+s_2}.$$

If $s_2 > s_1$, then $(1+f_{1\to2})/(1+s_2) > 1$, or equivalently, $f_{1\to2} > s_2$, so that we know $f_{1\to2} > s_1$. Since $f_{1\to2} = s_{1\to2}$, $s_2 > s_1$ implies that $s_{1\to2} > s_1$. The converse is likewise true. □

4.2. Liquidity Preference

The *liquidity preference* explanation assumes that investors prefer short-term bonds over long-term bonds, because long-term bonds are more sensitive to interest rate changes than are short-term bonds.

- Given stronger demand for short-term bonds, higher rates must be offered for long-term bonds; otherwise, investors are hardly induced to hold long-term bonds. As a result, the theory predicts that $s_{t_1} < s_{t_2}$ for $t_1 < t_2$, or equivalently speaking, an upward slopping term structure.

4.3. Market Segmentation

In the *market segmentation* explanation, it is argued that the market for fixed-income securities is segmented by maturity dates, thereby implying that the group of investors competing for long-term bonds is different form the group competing for short-term bonds.

- Short and long rates can be determined independently; that is, each of them is determined by supply and demand in its own segment. This means that any shapes–upward, downward, etc.–are available for the term structure.

5. Expectations Dynamics

Definition 8. The *discount factor between periods i and j* is defined by

$$d_{i\to j} = \left[\frac{1}{1+f_{i\to j}}\right]^{j-i}.$$

- The discount factor $d_{i\to j}$ is used to discount cash received at time j back to an equivalent amount of cash at time i. To see the logic behind $d_{i\to j}$, consider the following table:

Time	0	$\cdots$	i	$\cdots$	j
Cash flow					x_j
$PV_0(x_j)$	$x_j d_j$				
$PV_i(x_j)$			$x_j d_j (1+s_i)^i$		

From the relation $x_j d_j (1+s_i)^i = x_j d_{i\to j}$ where $d_j = (1+s_j)^{-j}$, we have

$$d_{i\to j} = \frac{(1+s_i)^i}{(1+s_j)^j} = \left[\frac{1}{1+f_{i\to j}}\right]^{j-i}.$$

Theorem 9. *Suppose there are two discount factors, $d_{i\to j}$ and $d_{j\to k}$, where $i < j < k$. Then, two discount factors satisfy the compounding rule: that is,*

$$d_{i\to k} = d_{i\to j} \cdot d_{j\to k}.$$

In words, the discount of cash in time k back to time i is achieved by first discounting the cash from time k to j and then discounting it from time j to time i.

Proof. Using the definition, it is trivial to show

$$d_{i\to j} \times d_{j\to k} = \frac{(1+s_i)^i}{(1+s_j)^j} \times \frac{(1+s_j)^j}{(1+s_k)^k} = \frac{(1+s_i)^i}{(1+s_k)^k} = d_{i\to k}.$$

□

6. Running Present Value

Suppose that $(x_0, x_1, x_2, \ldots, x_T)$ is a cash flow stream starting at time zero (i.e., present) and ending at time T. Define $PV(t)$ as the present value, computed at time t, of the remainder of the cash flow stream at time t, which is $(x_t, x_{t+1}, \ldots, x_T)$.

- The present value at time zero is represented as

$$\begin{aligned} PV(0) &= x_0 + d_1 x_1 + d_2 x_2 + d_3 x_3 + \cdots + d_T x_T \\ &= x_0 + d_1 \left[x_1 + \frac{d_2}{d_1} x_2 + \frac{d_3}{d_1} x_3 + \cdots + \frac{d_T}{d_1} x_T \right]. \end{aligned}$$

  The terms d_t/d_1, $t = 2, \ldots, T$, are the discount factors between periods 1 and t, since

$$\frac{d_t}{d_1} = \frac{(1+s_1)}{(1+s_t)^t} = d_{1\to t}.$$

  So, one writes

$$\begin{aligned} PV(0) &= x_0 + d_1 \left[x_1 + d_{1\to 2} x_2 + d_{1\to 3} x_3 + \cdots + d_{1\to T} x_T \right] \\ &= x_0 + d_1 PV(1). \end{aligned}$$

A *running present value* method calculates present value in a recursive manner starting with the final cash flow and working backward to the present.

- In general, the present value at time t is given by

$$\begin{aligned} PV(t) &= x_t + d_{t\to t+1} x_{t+1} + d_{t\to t+2} x_{t+2} + d_{t\to t+3} x_{t+3} + \cdots + d_{t\to T} x_T \\ &= x_t + d_{t\to t+1} \left[x_{t+1} + \frac{d_{t\to t+2}}{d_{t\to t+1}} x_{t+2} + \frac{d_{t\to t+3}}{d_{t\to t+1}} x_{t+2} + \cdots + \frac{d_{t\to T}}{d_{t\to t+1}} x_T \right] \\ &= x_t + d_{t\to t+1} \left[x_{t+1} + d_{t+1\to t+2} x_{t+2} + d_{t+1\to t+3} x_{t+3} + \cdots + d_{t+1\to T} x_T \right] \\ &= x_t + d_{t\to t+1} PV(t+1). \end{aligned}$$

It states that the present value at time t is the sum of the current cash flow and a one-period discount of the next present value.

The running present value method is initiated by $PV(T) = x_T$. Then, we calculate $PV(T-1)$ as $x_{T-1} + d_{T-1\to T}PV(T)$, and so forth until $PV(0)$ is found.

Example 10. Running Present Value

t	0	1	2	3
x_t	25	30	35	40
$f_{t\to t+1}$	6.00	6.90	7.50	
$d_{t\to t+1}$	0.94	0.94	0.93	
$PV(t)$	117.03	97.55	72.21	40.00

$$PV(3) = x_3 = 40.00$$

$$PV(2) = x_2 + d_{2\to 3}PV(3) = 35 + 0.93 \times 40.00 = 72.21$$

$$PV(1) = x_1 + d_{1\to 2}PV(2) = 30 + 0.94 \times 72.21 = 97.55$$

$$PV(0) = x_0 + d_{0\to 1}PV(1) = 25 + 0.94 \times 97.55 = 117.03$$

7. Quasi-Modified Duration

Suppose that the spot rates $s_1, s_2, \ldots, s_T$ all change together by an amount λ; i.e., new spot rates are $s_1+\lambda, s_2+\lambda, \ldots, s_T+\lambda$. The *quasi-modified duration* can measure the relative sensitivity of price with this hypothetical *instantaneous* change.

- We have defined the modified duration as a measure of the relative price sensitivity with respect to a change in YTM. In fact, the YTM change is a special case of the parallel shift of the spot curve, since if the spot rate curve were flat (i.e., $s = s_1 = \cdots = s_T$), all spot rates would be equal to the common value of YTM. In this spirit, the quasi-modified duration generalizes the modified duration.

Consider a cash flow stream $(x_0, x_1, \ldots, x_T)$. The price associated with the new spot rates is given by

$$\begin{aligned} P(\lambda) &= x_0 + x_1 d_1 + x_2 d_2 + \cdots + x_T d_T \\ &= x_0 + \frac{x_1}{(1+(s_1+\lambda))} + \frac{x_2}{(1+(s_2+\lambda))^2} + \cdots + \frac{x_T}{(1+(s_T+\lambda))^T} \\ &= x_0 + \sum_{t=1}^{T} x_t \left(1+(s_t+\lambda)\right)^{-t}. \end{aligned}$$

- The first derivative of $P(\lambda)$ with respect to λ is

$$\frac{dP(\lambda)}{d\lambda} = -\sum_{t=1}^{T} x_t t \left(1+(s_t+\lambda)\right)^{-(t+1)},$$

so that we have

$$\frac{dP(0)}{d\lambda} \equiv \left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=0} = -\sum_{t=1}^{T} x_t t\,(1+s_t)^{-(t+1)}.$$

Definition 11. Under compounding one time per period, the *quasi-modified duration* of a cash flow stream $(x_0, x_1, \ldots, x_T)$ is given by

$$\begin{aligned} D_Q &= -\frac{1}{P(0)} \times \frac{dP(0)}{d\lambda} \\ &= \frac{\sum_{t=1}^{T} x_t t\,(1+s_t)^{-(t+1)}}{x_0 + \sum_{t=1}^{T} x_t\,(1+s_t)^{-t}}. \end{aligned}$$

- By definition, D_Q is a measure of the relative price change in response to a parallel shift in the spot rate curve. The change in price due to the instantaneous change in the spot rates can be estimated as

$$dP(0) = -D_Q P(0) d\lambda.$$

Remark 12. The term structure of interest rates produces a new, more robust method for portfolio immunization. This new method does not depend on selecting bonds with a common yield.

8. Immunization Using the Quasi-Modified Duration

Suppose that a firm has an obligation to pay \$1 million in 5 periods. For constructing a portfolio to meet this obligation, the firm is considering two bonds, B_1 and B_2. B_1 is a 12-year 5% bond and B_2 is a 5-year 10% bond.

- Under compounding one time per period, the following worksheets show how to compute the quasi-modified durations of the bonds:

Period	s_t	d_t	x_t	$d_t x_t$	$-x_t t\,(1+s_t)^{-(t+1)}$
1	7.67	0.93	50	46.44	-43.13
2	8.27	0.85	50	42.65	-78.79
3	8.81	0.78	50	38.81	-107.01
4	9.31	0.70	50	35.02	-128.15
5	9.75	0.63	50	31.40	-143.06
6	10.16	0.56	50	27.98	-152.39
7	10.52	0.50	50	24.82	-157.23
8	10.85	0.44	50	21.93	-158.28
9	11.15	0.39	50	19.31	-156.36
10	11.42	0.34	50	16.96	-152.19
11	11.67	0.30	50	14.85	-146.26
12	11.89	0.26	1050	272.71	-2,924.72
Total				$P(0) = 592.88$	$dP(0)/d\lambda = -4{,}347.57$

$$D_{Q,1} = -\frac{1}{P(0)} \times \frac{dP(0)}{d\lambda} = \frac{4{,}347.57}{592.88} = 7.33.$$

Period	s_t	d_t	x_t	$d_t x_t$	$-x_t t\,(1+s_t)^{-(t+1)}$
1	7.67	0.93	100	92.88	-86.26
2	8.27	0.85	100	85.31	-157.58
3	8.81	0.78	100	77.62	-214.02
4	9.31	0.70	100	70.04	-256.31
5	9.75	0.63	1,100	690.83	-3,147.28
Total				$P(0) = 1,016.68$	$dP(0)/d\lambda = -3,861.45$

$$D_{Q,2} = -\frac{1}{P(0)} \times \frac{dP(0)}{d\lambda} = \frac{3,861.45}{1,016.68} = 3.80.$$

The present value of the obligation in period 5 is

$$P(0) = \frac{1,000,000}{(1+s_5)^5} = \frac{1,000,000}{(1.0975)^5} = 628,025.61$$

and its quasi-modified duration is computed as

$$\begin{aligned} D_{Q,debt} &= -\frac{1}{P(0)} \times \frac{dP(0)}{d\lambda} \\ &= -\left(\frac{(1+s_5)^5}{1,000,000}\right) \times \left(-1,000,000 \times 5 \times (1+s_5)^{-6}\right) \\ &= \frac{5}{(1+s_5)} \\ &= 4.56. \end{aligned}$$

Finally, we have a system of two equations as follows:

$$\begin{aligned} n_1 \times 592.88 + n_2 \times 1,016.68 &= 628,025.61 \\ w_1 \times 7.33 + w_2 \times 3.80 &= 4.56, \end{aligned}$$

where n_i is the number of shares of bond i and the weights are defined as

$$w_1 = \frac{n_1 \times 592.88}{628,025.61} \text{ and } w_2 = \frac{n_2 \times 1,016.68}{628,025.61}.$$

Solving the system for n_i, we get $n_1 = 228.07$ and $n_2 = 484.72$.