Lecture 6. The Generalized Roll Model

1. The Structural Model

Assume that the efficient price, m_t, is driven by both the trade direction indicator variable, q_t, and the public information shock, u_t: i.e.,

$$m_t = m_{t-1} + w_t$$
$$w_t = \lambda q_t + u_t$$

with $E[u_t] = 0$ and $Cov[q_t, u_{t-k}] = 0$ for $k = 1, 2, \ldots$ (i.e., the public information shock is non tradable).

- The coefficient $\lambda(> 0)$ suggests that q_t partially updates the efficient price from m_{t-1} to m_t. That is, the order flow at time t contains information. In comparison, the original Roll model is built on the idea that the order flow has no information content by assuming $m_t = m_{t-1} + u_t$.

Assume further (a) that the trade occurs at the ask if $q_t = 1$ (i.e., a buy order) and at the bid if $q_t = -1$ (i.e., a sell order) and (b) that a dealer charges the non-informational fixed cost of c per transaction.

- So, the transaction price is $p_t = m_t + cq_t$ and the price change is

$$\begin{aligned} \Delta p_t &= \Delta m_t + c(q_t - q_{t-1}) \\ &= \lambda q_t + u_t + c(q_t - q_{t-1}). \end{aligned} \tag{1.1}$$

2. The Statistical Representation

Proposition 2.1. *Assume that $E[q_t] = 0$, $E[q_t^2] = 1$, and $E[q_t q_{t-k}] = 0$ for $k = 1, 2, \ldots$. Then it shows*

$$E[\Delta p_t] = 0$$
$$Var[\Delta p_t] = c^2 + (c + \lambda)^2 + \sigma_u^2 \tag{2.1}$$
$$Cov[\Delta p_t, \Delta p_{t-k}] = \begin{cases} -c(\lambda + c) & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases} \tag{2.2}$$

Proof. See Reading 3. □

- From (2.1), and (2.2), the price change Δp_t can be statistically represented as a MA(1) process of the form

$$\Delta p_t = \varepsilon_t + \theta \varepsilon_{t-1}, \tag{2.3}$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$, which in turn suggests the following variance and autocovariance:

$$\gamma_0 = (1+\theta^2)\sigma_\varepsilon^2 \tag{2.4}$$

$$\gamma_k = \begin{cases} \theta\sigma_\varepsilon^2 & \text{for } k = 1 \\ 0 & \text{for } k \geq 2. \end{cases} \tag{2.5}$$

Remark 2.2. On the basis of the time-series data of $\{\Delta p_t\}$, one can estimate the MA parameters of θ and σ_ε^2 in (2.3). Notice that the two MA parameters do not suffice to identify the three structural model parameters of λ, c, and σ_u^2 in (1.1). That is, there is no unique solution for λ, c, and σ_u^2 in the equations of

$$c^2 + (c+\lambda)^2 + \sigma_u^2 = (1+\theta^2)\sigma_\varepsilon^2$$
$$-c(\lambda + c) = \theta\sigma_\varepsilon^2,$$

provided that θ and σ_ε^2 are given exogenously. So, one should set one of the structural parameters to zero for identification, and after then are able to compute the remaining structural parameters using the estimates of the two MA parameters.

With the statistical representation of price changes in (2.3), the limit of the price forecast f_{t+k}, denoted by f_t^*, is given by

$$\begin{aligned} f_t^* &= \lim_{k\to\infty} E[p_{t+k}|I_t] \\ &= \lim_{k\to\infty} E[p_{t+k-1} + \varepsilon_{t+k} + \theta\varepsilon_{t+k-1}|I_t] \\ &= \lim_{k\to\infty} E[p_{t+k-1}|I_t] \\ &\ \vdots \\ &= E[p_{t+1}|I_t] \\ &= E[p_t + \varepsilon_{t+1} + \theta\varepsilon_t|I_t] \\ &= p_t + \theta\varepsilon_t. \end{aligned} \tag{2.6}$$

Proposition 2.3. *A filtered state estimate refers to the expectation of an unobserved variable conditional on the history of observations. The filtered state estimate $E[m_t|I_t]$ is equal to the price forecast f_t^*, i.e.,*

$$E[m_t|I_t] = p_t + \theta\varepsilon_t.$$

Proof. See Reading 3. □

3. The Pricing Error

Definition 3.1. The *pricing error* is defined by $\sigma_s^2 \equiv Var[s_t]$ where $s_t = p_t - m_t$, and measures how closely the transaction price tracks the efficient price.

- If market is efficient, s_t should be constant over time but is not equal to zero for the presence of microstructure effects (e.g., transaction cost, price impact, etc). Put differently, the pricing error σ_s^2 should be close to zero in the efficient market.

Proposition 3.2. *The lower bound of σ_s^2 is $\underline{\sigma}_s^2 = \theta^2\sigma_\varepsilon^2$, provided that θ and σ_ε^2 are given exogenously.*

Proof. Since f_t^* is a linear projection of m_t onto I_t (i.e., $f_t^* = E[m_t|I_t]$) and $p_t \in I_t$, $p_t - f_t^*$ is orthogonal to $m_t - f_t^*$. Given that $p_t = m_t + cq_t$ and $f_t^* = p_t + \theta\varepsilon_t$, then it shows

$$\begin{aligned}
\sigma_s^2 &= Var[p_t - m_t] \\
&= Var\left[(p_t - f_t^*) - (m_t - f_t^*)\right] \\
&= Var[p_t - f_t^*] + Var[m_t - f_t^*] \\
&= Var[-\theta\varepsilon_t] + Var[m_t - p_t - \theta\varepsilon_t] \\
&= \theta^2\sigma_\varepsilon^2 + Var[-cq_t - \theta\varepsilon_t]. \qquad (3.1)
\end{aligned}$$

Suppose that $\sigma_u^2 = 0$; equivalently $u_t = 0$ for all t. In this case, the price change is

$$\begin{aligned}
\Delta p_t &= \lambda q_t + c(q_t - q_{t-1}) \\
&= (\lambda + c)q_t - cq_{t-1}. \qquad (3.2)
\end{aligned}$$

By comparing the structural model in (3.2) with its statistical representation, i.e., $\Delta p_t = \varepsilon_t + \theta\varepsilon_{t-1}$, one obtains

$$\begin{aligned}
(\lambda + c)q_t &= \varepsilon_t \\
-cq_{t-1} &= \theta\varepsilon_{t-1}. \qquad (3.3)
\end{aligned}$$

In particular, (3.3) implies that $Var[-cq_t - \theta\varepsilon_t] = 0$ and thus (3.1) reduces to

$$\sigma_s^2 = \theta^2\sigma_\varepsilon^2.$$

Since $Var[-cq_t - \theta\varepsilon_t]$ is nonnegative in (3.1), "any" restriction other than the restriction $\sigma_u^2 = 0$ yields $\sigma_s^2 > \theta^2\sigma_\varepsilon^2$. Thus, the lower bound for σ_s^2 is $\underline{\sigma}_s^2 = \theta^2\sigma_\varepsilon^2$. □

4. General Univariate Random-Walk Decomposition

Assume (a) that p_t is decomposed into the random walk component m_t (representing informational permanent effects) and the stationary component s_t (representing non-informational transient effects), i.e.,

$$p_t = m_t + s_t \qquad (4.1)$$

where $m_t = m_{t-1} + w_t$ and $w_t \sim WN(0, \sigma_w^2)$ and (b) s_t may be serially correlated and partially or completely correlated with w_t: i.e.,

$$\begin{aligned} s_t &= (A_0 w_t + A_1 w_{t-1} + A_2 w_{t-2} + \cdots) + (\eta_t + B_1 \eta_{t-1} + B_2 \eta_{t-2} + \cdots) \\ &= (A_0 + A_1 L + A_2 L^2 + \cdots) w_t + (1 + B_1 L + B_2 L^2 + \cdots) \eta_t \\ &= A(L) w_t + B(L) \eta_t, \end{aligned}$$

where $\eta_t \sim WN(0, \sigma_\eta^2)$ and $E[w_t \eta_{t-k}] = 0$ for all k.

- This random-walk decomposition model is more general than the generalized Roll model, as it requires fewer economic structures. Specifically, the current model require neither the assumption that q_t affects prices nor the assumption that the dealer charges c per transaction.

The price change in the structural model (4.1) is

$$\begin{aligned} \Delta p_t &= (1-L) p_t \\ &= (1-L) m_t + (1-L) s_t \\ &= w_t + (1-L)\left(A(L) w_t + B(L) \eta_t\right) \\ &= \left[1 + (1-L) A(L)\right] w_t + (1-L) B(L) \eta_t. \qquad (4.2) \end{aligned}$$

- Since w_t and η_t are stationary errors, Δp_t is also stationary. So, the Wold theorem implies that the stationary series of Δp_t is characterized with a moving average model of arbitrary order. That is to say, the statistical representation corresponding to (4.2) is

$$\begin{aligned} \Delta p_t &= \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots \qquad (4.3) \\ &= (1 + \theta_1 L + \theta_2 L^2 + \cdots) \varepsilon_t \\ &= \theta(L) \varepsilon_t, \end{aligned}$$

  where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$.

The limit of the price forecast f_{t+k} is derived as

$$\begin{aligned} f_t^* &\equiv \lim_{k \to \infty} E[p_{t+k} | I_t] \\ &= E[p_t + (p_{t+1} - p_t) + (p_{t+2} - p_{t+1}) + \cdots | I_t] \\ &= p_t + E[\Delta p_{t+1} | I_t] + E[\Delta p_{t+2} | I_t] + \cdots. \end{aligned}$$

- Using (4.3), one obtains

$$\begin{aligned} E[\Delta p_{t+k} | I_t] &= E[\varepsilon_{t+k} + \theta_1 \varepsilon_{t+k-1} + \theta_2 \varepsilon_{t+k-2} + \cdots + \theta_k \varepsilon_t + \theta_{k+1} \varepsilon_{t-1} + \cdots | I_t] \\ &= \theta_k \varepsilon_t + \theta_{k+1} \varepsilon_{t-1} + \cdots, \end{aligned}$$

which implies

$$
\begin{aligned}
f_t^* &= p_t + E[\Delta p_{t+1}|I_t] + E[\Delta p_{t+2}|I_t] + \cdots \\
&= p_t + (\theta_1\varepsilon_t + \theta_2\varepsilon_{t-1} + \cdots) + (\theta_2\varepsilon_t + \theta_3\varepsilon_{t-1} + \cdots) + \cdots \\
&= p_t + (\theta_1 + \theta_2 + \cdots)\varepsilon_t + (\theta_2 + \theta_3 + \cdots)\varepsilon_{t-1} + \cdots \\
&= p_t + \left(\sum_{j=1}^{\infty}\theta_j\right)\varepsilon_t + \left(\sum_{j=2}^{\infty}\theta_j\right)\varepsilon_{t-1} + \cdots . \qquad (4.4)
\end{aligned}
$$

Suppose that f_t^* is equal to m_t. As shall be shown later, this supposition is equivalent to the restriction $\eta_t = 0$ in the equation of s_t. If $f_t^* = m_t$, then (4.4) implies that

$$
\begin{aligned}
s_t &= p_t - m_t \\
&= p_t - \left[p_t + \left(\sum_{j=1}^{\infty}\theta_j\right)\varepsilon_t + \left(\sum_{j=2}^{\infty}\theta_j\right)\varepsilon_{t-1} + \cdots\right] \\
&= c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + \cdots , \qquad (4.5)
\end{aligned}
$$

where $c_i = -\sum_{j=i+1}^{\infty}\theta_j$ and

$$\sigma_s^2 = \sum_{i=0}^{\infty} c_i^2\sigma_\varepsilon^2. \qquad (4.6)$$

- Notice that σ_s^2 can be computed using MA estimates of θ_j and σ_ε^2, both of which are estimated from the statistical representation (4.3) on the basis of the time-series of $\{\Delta p_t\}$. Hasbrouck (1993) shows that (4.6) is the lower bound for the variance of s_t.

5. Identification in Random-Walk Decomposition

Suppose that $\eta_t = 0$ for all t; i.e., $s_t = A(L)w_t$. Equating the structural model (4.2) and its statistical representation (4.3) yields

$$[1 + (1-L)A(L)]\, w_t = \theta(L)\varepsilon_t, \qquad (5.1)$$

and evaluating both sides of (5.1) at $L = 1$ results in

$$w_t = \theta(1)\varepsilon_t. \qquad (5.2)$$

After plugging (5.2) in (5.1), one obtains

$$[1 + (1-L)A(L)]\, \theta(1) = \theta(L)$$

or

$$A(L) = (1-L)^{-1}[\theta(L) - \theta(1)]\theta(1)^{-1}.$$

- Using $(1-L)^{-1} = 1 + L + L^2 + \cdots$, one writes $A(L)$ as

$$\begin{aligned}
A(L) &= (1+L+L^2+\cdots)[\theta(L)-\theta(1)]\theta(1)^{-1} \\
&= [\theta(L)-\theta(1)]\theta(1)^{-1} + [\theta(L)-\theta(1)]\theta(1)^{-1}L + [\theta(L)-\theta(1)]\theta(1)^{-1}L^2 + \cdots \\
&= (\theta_1 L + \theta_2 L^2 + \cdots - \theta_1 - \theta_2 - \cdots)\theta(1)^{-1} \\
&\quad + (\theta_1 L + \theta_2 L^2 + \theta_3 L^3 + \cdots - \theta_1 - \theta_2 - \theta_3 - \cdots)\theta(1)^{-1}L \\
&\quad + (\theta_1 L + \theta_2 L^2 + \theta_3 L^3 + \theta_4 L^4 + \cdots - \theta_1 - \theta_2 - \theta_3 - \theta_4 - \cdots)\theta(1)^{-1}L^2 + \cdots \\
&= (-\theta_1 - \theta_2 - \cdots)\theta(1)^{-1} + (-\theta_2 - \theta_3 - \cdots)\theta(1)^{-1}L \\
&\quad + (-\theta_3 - \theta_4 - \cdots)\theta(1)^{-1}L^2 + \cdots.
\end{aligned} \tag{5.3}$$

Using (5.2) and (5.3), one shows

$$\begin{aligned}
s_t &= A(L)w_t \\
&= A(L)\theta(1)\varepsilon_t \\
&= (-\theta_1 - \theta_2 - \cdots)\varepsilon_t + (-\theta_2 - \theta_3 - \cdots)\varepsilon_{t-1} + (-\theta_3 - \theta_4 - \cdots)\varepsilon_{t-2} + \cdots.
\end{aligned}$$

Therefore, s_t under the restriction $\eta_t = 0$ is identical to s_t under the restriction $f_t^* = m_t$ in (4.5).