Lecture Notes on Economics of Information and Uncertainty

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February 6, 2021

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Chapter 1

Choice under Uncertainty

1.1 Introduction

Uncertainty is a fact of life. We will study individual behavior with respect to choices involving uncertainty. We will also investigate financial institutions such as insurance markets and stock markets that can mitigate at least some of these risks.

We begin by studying the expected utility theory. We then study the concept of risk aversion and discuss its measurement. Finally, we fit the choice under uncertainty into the standard consumer theory.

1.2 Expected Utility Theory

1.2.1 Modelling Uncertainty

We use lotteries to describe risky alternatives. Suppose first that the number of possible outcomes is finite. Fix a set of outcomes $C = \{c_1, \cdots, c_N\}$. Let p_n be the probability that outcome $c_n \in C$ occurs and suppose these probabilities are objectively known.

Definition 1.2.1 (Lottery)**.** A (simple) lottery $L = (p_1, \cdots, p_n)$ is an assignment of probabilities to each outcome c_n , where $p_n \geq 0$ for all n and $\sum_n p_n = 1$.

The collection of such lotteries can be written as

$$\mathcal{L} = \left\{ (p_1, \ldots, p_N) \,\middle|\, \sum_{n=1}^{N} p_n = 1,\ p_n \geq 0 \text{ for } n = 1, \ldots, N \right\}.$$

We can also think of a **compound lottery** $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$, which allows the outcomes of a lottery to be lotteries.

It is immediate to see that any compound lottery can be reduced to a simple lottery defined as above.

Example. $C = \{c_1, c_2\}$, $L_1 = (p, 1-p)$, $L_2 = (q, 1-q)$. Then,

$$(L_1, L_2; \alpha, 1-\alpha) = (\alpha p + (1-\alpha)q, \alpha(1-p) + (1-\alpha)(1-q)).$$

Hence, we can only focus on simple lotteries. One special and important class of lotteries is money lotteries, whose outcomes are real numbers, i.e., $C = \mathbb{R}$. A money lottery can be characterized by a cumulative distribution function F, where $F : \mathbb{R} \to [0, 1]$ is nondecreasing. $F(x)$ is the probability of receiving a prize less than or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

1.2.2 Expected Utility

If an individual has "reasonable preferences" about consumption in different circumstances, we will be able to use a utility function to describe these preferences just as we do in other contexts. However, the fact that we are considering choice under uncertainty adds some special structures to the choice problem, which we will see below. Historically, the study of individual behavior under uncertainty is originated from attempts to understand (and hopefully to win) games of chance. One may think that the key determinant of behavior under uncertainty is the expected return of the gamble. However, people are generally reluctant to play fair games.

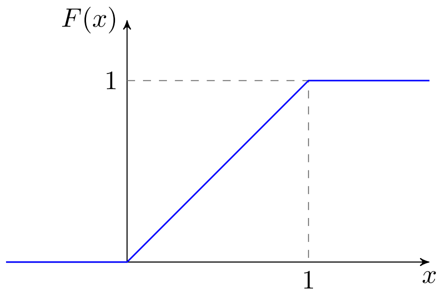
1.3 Risk Aversion

In many economic environments, individuals display aversion to risk. We formalize the notion of risk aversion and study some of its properties. We focus on money lotteries, i.e., risky alternatives whose outcomes are amounts of money. It is convenient to treat money as a continuous variable. We have so far assumed a finite number of outcomes to derive the expected utility representation. How to extend this?

1.3.1 Expected utility framework on monetary outcomes

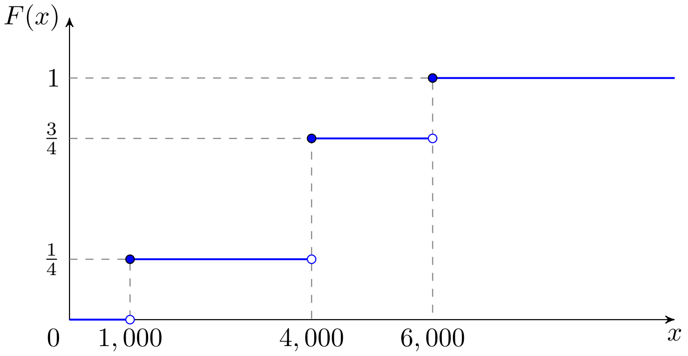
We describe a monetary lottery by means of a cumulative distribution functions $F : \mathbb{R} \to [0, 1]$. $F(x)$ is the probability that the realized payoff is less that or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

Example. Uniform distribution $U[0, 1]$



Example. Discrete distribution:

$$\left.\begin{array}{l}\text{Prob(1,000 won)} = \frac{1}{4} \\ \text{Prob(4,000 won)} = \frac{1}{2} \\ \text{Prob(6,000 won)} = \frac{1}{4}\end{array}\right\} \to F(x) = \begin{cases} 0 & \text{if } x < 1,000 \\ \frac{1}{4} & \text{if } 1,000 \le x < 4,000 \\ \frac{3}{4} & \text{if } 4,000 \le x < 6,000 \\ 1 & \text{if } x \ge 6,000. \end{cases}$$



Consider a preference relation $\succsim$ on $\mathcal{L}$. It has an expected utility representation if

$$F \succsim F' \Leftrightarrow U(F) \ge U(F'),$$

where

$$U(F) = \int_{-\infty}^{\infty} u(x)\, dF(x)$$

or

$$U(F) = \int_{-\infty}^{\infty} u(x) f(x)\, dx$$

if F is differentiable and $f = dF/dx$.

Note that U is defined on lotteries whereas u is defined on money. To differentiate the two objects, we often call U the (von Neumann-Morgenstern) expected utility function and $u(\cdot)$ the Bernoulli utility function or von Neumann Morgenstern utility of money.

We assume that u is (strictly) increasing, implying that the marginal utility of money is strictly positive, and twice continuously differentiable, for analytic convenience.

1.3.2 Attitude toward risk

Definition 1.3.1. Let u be a utility function defined on money outcomes that represents $\succsim$. We say that $\succsim$ exhibits

$$\begin{pmatrix} \text{risk aversion} \\ \text{risk neutrality} \\ \text{risk loving} \end{pmatrix} \Longleftrightarrow \int u(x)\, dF(x) \begin{pmatrix} < \\ = \\ > \end{pmatrix} u\left(\int x\, dF(x)\right)$$

for all lotteries F.

Equivalently, $\succsim$ exhibits risk aversion if $\mathbb{E}[u(X)] < u(\mathbb{E}[X])$. Notice that if $\succsim$ is risk averse (neutral, loving), then u is concave (linear, convex).

Consider $X = \{x_g, x_b\}$ where $x_g > x_b$. Recall that u shows risk aversion if $u(\pi x_b + (1-\pi)x_g) > \pi u(x_b) + (1-\pi)u(x_g)$.

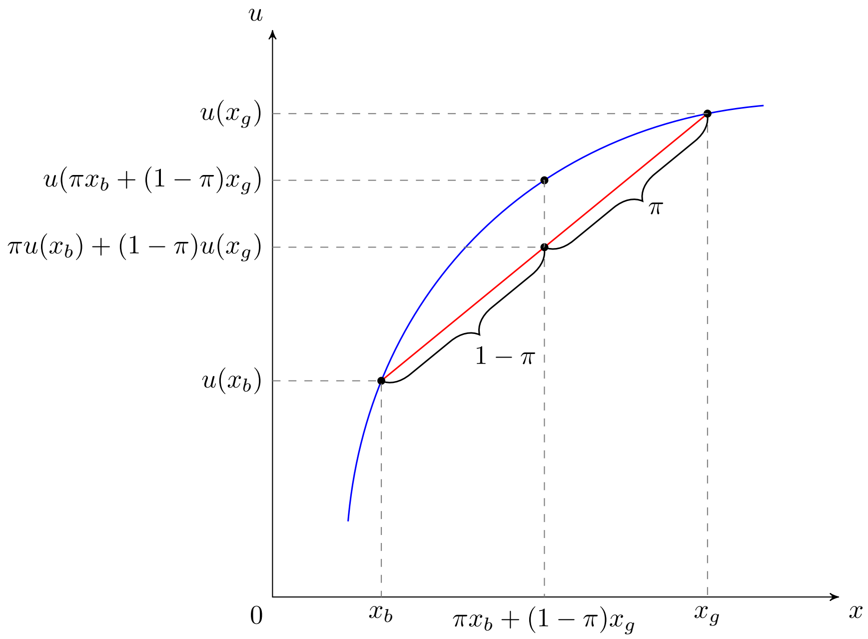


Figure 1.1: Risk aversion

If u is concave, Jensen's inequality says

$$\int u(z)\, dF(z) = \mathbb{E}[u(z)] \leq u\left(\int z\, dF(z)\right) = u(\mathbb{E}[z]).$$

The left hand side is the expected utility from the bet whose return z is distributed according to F. The right hand side is the utility from money whose amount is equal to the expected value of the random variable.

Definition 1.3.2. By a sure thing, we mean a deterministic outcome z. A bet is a random variable. A fair bet is a random variable whose expected return is equal to the sure thing.

Let ϵ be a random variable whose expected value is $0 : \mathbb{E}[\epsilon] = 0$. Given z^e, a fair bet to z^e is

$$z^e + \epsilon.$$

Let $z^e = \mathbb{E}[z]$, and $\epsilon = z - z^e$ whose distribution function is G. Then,

$$\int u(z^e + \epsilon)\, dG(\epsilon) = \mathbb{E}[u(z)] \leq u\left(\int z\, dF(z)\right) = u(\mathbb{E}[z]) = u(z^e).$$

We often say that u shows risk averse attitude if and only if the decision maker prefers a sure thing over a fair bet.

If a decision maker is risk neutral, then

$$u(\pi x_b + (1-\pi)x_g) = \pi u(x_b) + (1-\pi)u(x_g).$$

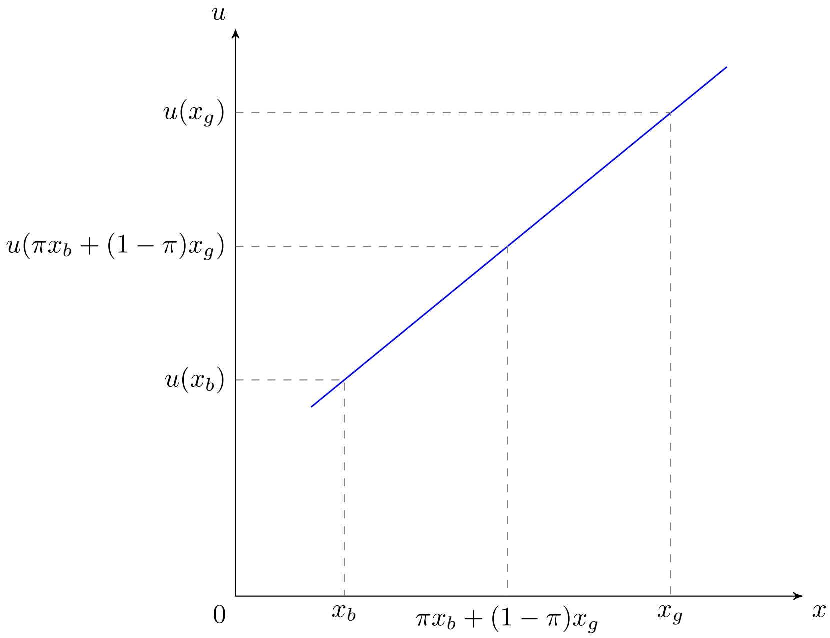


Figure 1.2: Risk neutrality

A decision maker is risk loving if

$$u(\pi x_b + (1-\pi)x_g) < \pi u(x_b) + (1-\pi)u(x_g).$$

You may think that only a professional gambler might be risk loving. A policy with a good intention can turn a risk neutral decision maker into a risk loving decision maker.

Suppose that a firm is a risk neutral decision maker whose Bernouille utility function (or vNM utility) is

$$u(z) = z.$$

The firm has a fixed cost D, but the return is a random variable R distributed over $[0, \infty)$. The profit of the firm is

$$u(R - D) = R - D$$

which is a random variable.

1.3.3 Certainty equivalent

A risk averse individual prefers a sure thing to a fair gamble. Is there a smaller amount of certain wealth that would be viewed as equivalent to the g amble?

1.4 Measurement of Risk Aversion

It is convenient to have a measure of risk aversion.

1.5 Applications

1.5.1 Contingent Commodity

A contingent commodity is a good that is available only if a particular event (or state of nature) occurs. It specifies conditions under which each contingent becomes available. We now treat contingent commodities as different goods. People have preferences over different consumption plan, just like they have preferences over actual consumption.

1.5.2 Insurance

Consider a strictly risk averse individual who has a wealth w and faces damage $D < w$ with probability π.

1.5.3 Portfolio choice problem: Mean-variance analysis

Suppose there are two lotteries L_1 and L_2.

Chapter 2

Hidden Information: Screening

2.1 Introduction

We focus on the basic static adverse selection problem. There is a principal facing one agent who has private information on his “type”. Type represnets the agent’s preference or intrinsic productivity. We first study how to solve such problems when the agent can be one of two types, a case that will give us the key insights from adverse selection models.

General setup

An agent, informed party, is privately informed about his type.

A principal, uninformed party, designs a contract in order to screen different types of agent and maximize her payoff.

This is a problem of hidden information, often referred to as screening problem.

2.2 A Model of Price Discrimination

Consider a transaction between a buyer (agent) and a seller (principal).

2.3 Full Information Benchmark

Suppose that the seller is perfectly informed about the buyer’s type. The seller can treat each type of buyer separately and offer a type-dependent contract: (q_i, T_i) for type $\theta_i, i = H, L$.

2.4 Asymmetric Information

Suppose from now on that the seller cannot observe the type of the buyer, facing the *adverse selection* problem.

2.4.1 Linear pricing: $T(q) = Pq$

The buyer pays a uniform price P for each unit he buys.

2.4.2 Two-part tariff: $T(q) = F + Pq$

The seller charges a fixed fee (F) up-front, and a price P for each unit purchased. Note that for any given price P, the maximum fee the seller can charge up-front is $F = S_L(P)$ if he wants to serve both types. The seller chooses P to maximize

$$\beta[S_L(P)+(P-c)D_L(P)]+(1-\beta)[S_L(P)+(P-c)D_H(P)] = S_L(P)+(P-c)D(P).$$

2.5 Optimal Nonlinear Pricing

Here, we look for the best pricing scheme among all possible ones. That is, we look for the *second-best* outcome.

2.6 Applications

2.6.1 Regulation

The public regulators are often subject to an informational disadvantage with respect to the regulated utility or natural monopoly.

2.6.2 Ex-ante contracting

There are situations in which the agent can learn his type only after he signs a contract

Chapter 3

Hidden Action: Moral Hazard

3.1 Introduction

We have discussed screening problem. The uninformed party combats the problem of adverse selection by screening the other. We now discuss another class of asymmetric information problem. Asymmetric information arises from imperfect monitoring of players' actions (or hidden action).

3.2 Binary Model

Suppose there is an employer (principal) and an employee (agent). Agent could shirk ($e = 0$ or low effort) or work hard ($e = 1$ or high effort), which is not observable by the principal.

3.3 First-Best Contract

In this benchmark, assume that the effort level is observable and verifiable. If principal wants to induce e, then he solves

$$\max_{t_H, t_L} \pi_e(S_H - t_H) + (1 - \pi_e)(S_L - t_L)$$

subject to

$$\pi_e u(t_H) + (1 - \pi_e)u(t_L) - ce \geq u_0. \tag{IR}$$

Set up the Lagrangian function

$$\mathcal{L} = \pi_e(S_H - t_H) + (1 - \pi_e)(S_L - t_L) + \lambda[\pi_e u(t_H) + (1 - \pi_e)u(t_L) - ce - u_0].$$

From the first-order condition,

$$\begin{aligned} -\pi_e + \lambda \pi_e u'(t_H^F) &= 0, \\ -(1 - \pi_e) + \lambda(1 - \pi_e)u'(t_L^F) &= 0. \end{aligned}$$

We thus have

$$\lambda = \frac{1}{u'(t_H^F)} = \frac{1}{u'(t_L^f)},$$

implying that $f_H^F = t_L^F = t^F$. By (IR), $t^F = u^{-1}(ce + u_0)$. The risk-neutral principle offers a full insurance to the risk-averse agent and then extracts the full surplus.

Principal prefers $e = 1$ if

$$\pi_1 S_H + (1 - \pi_1)S_L - u^{-1}(c + u_0) \geq \pi_0 S_H + (1 - \pi_0)S_L - u^{-1}(u_0)$$

or

$$\underbrace{(\pi_1 - \pi_0)(S_H - S_L)}_{\text{expected gain of effort}} \geq \underbrace{u^{-1}(c + u_0) - u^{-1}(u_0)}_{\text{cost of including effort}}.$$

Otherwise, principal prefers $e = 0$.

3.4 Second-Best Contract

Assume that the effort exerted by the agent is unobservable. The principal's problem to induce $e = 1$ is

$$\max_{t_H, t_L} \pi_1(S_H - t_H) + (1 - \pi_1)(S_L - t_l)$$

subject to

$$\pi_1 u(t_H) + (1 - \pi_1)u(t_L) - c \geq \pi_0 u(t_H) + (1 - \pi_0)u(t_L) \tag{IC}$$
$$\pi_1 u(t_H) + (1 - \pi_1)u(t_L) - c \geq u_0 \tag{IR}$$

3.4.1 Optimal Incentive Scheme

Set up the Lagrangian function

3.4.2 Optimal Effort Policy

The cost of inducing high effort under moral hazard is

$$C^* := \pi_1 t_H^* + (1 - \pi_1)t_L^* - u^{-1}(u_0).$$

$e = 1$ is optimal if $(\pi_1 - \pi_0)(S_H - S_L) \geq C^*$. Otherwise, $e = 0$ is optimal. The second-best cost of inducing a high effort is higher than the first-best cost:

3.5 Extensions

3.5.1 Risk-neutral agent

Suppose the agent is risk-neutral and let $u(t) = t$. In what follows, we show that the principal can achieve the first-best outcome. Because $u^{-1}(t) = t$, the optimal contract is immediate from (3)

3.5.2 Limited Liability

The agent is still risk-neutral ($u(t) = t$). Assume that the agent is protected by limited liability constraint that the transfer received by the agent should no less than $t_0 \coloneqq u^{-1}(u_0)$. The principal's problem is

3.6 Application: Insurance Market

Moral hazard is pervasive in insurance markets. Lets consider a risk-averse agent with utility function $u(\cdot)$ and initial wealth w.

A short note on a somewhat more systematic construction of such plots

The function you want to plot seems to have two branches with local maxima. We are interested in constructing something that resembles that with little effort. The basic observation is that the plot looks like a superpostion of a cosine and some line with constant negative slope, and some part omitted, as sketched in Figure 1.1.

This suggests an ansatz of the form

$$f(x) \;=\; a\,\cos[b\,(x-x_0)]+c\,x+d\;. \tag{3.1}$$

With all these preparations one only needs to adjust the parameters in (3.1) to change the appearance of the plot without losing the features.

The man who passes the sentence should swing the sword.