Lecture 9. Models of Asset Dynamics

1. Multiplicative Model

Definition 1. If X is a random variable whose logarithm is normally distributed, then X has a *lognormal distribution*.

- For $\ln X \sim N(\mu, \sigma^2)$, it shows

$$\begin{aligned} E[X] &= \exp\left(\mu + \frac{\sigma^2}{2}\right) \\ Var[X] &= \exp\left(2(\mu+\sigma^2)\right) - \exp\left(2\mu+\sigma^2\right). \end{aligned}$$

Definition 2. The price at time k, denoted by $S(k)$, follows the *multiplicative model* if it has the form

$$S(k+1) = u(k)S(k) \tag{1}$$

for $k = 0, 1, \ldots$, where $u(k)$ is an i.i.d. random variable and reflects "shocks" to the price.

- Taking the logarithm of both sides of (1) yields

$$\ln S(k+1) = \ln S(k) + \ln u(k).$$

  Assume that $u(k)$ follows a lognormal distribution; i.e., $\ln u(k) \sim N(v, \sigma^2)$. The logarithm of next price is normally distributed as

$$\ln S(k+1) \sim N(\ln S(k) + v, \sigma^2)$$

  and the logarithm of gross return is normally distributed as

$$\ln\left(\frac{S(k+1)}{S(k)}\right) = \ln S(k+1) - \ln S(k) \sim N(v, \sigma^2).$$

By direct substitution, one obtains

$$S(k) = u(k-1)u(k-2)\cdots u(0)S(0)$$

or

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} \ln u(i).$$

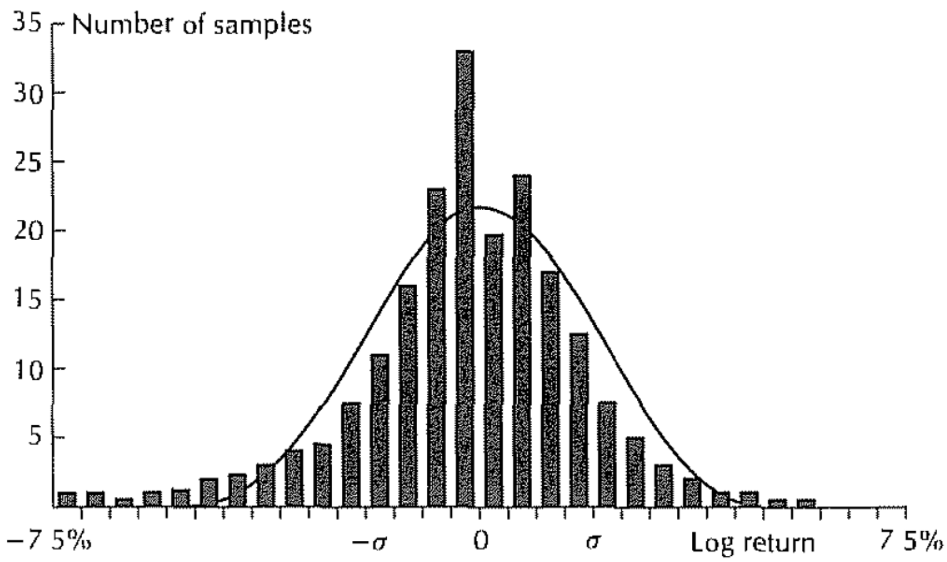
- Since the sum of normal random variables is a normal random variable, it shows

$$\sum_{i=0}^{k-1} \ln u(i) \sim N(kv, k\sigma^2)$$

  and

$$\ln S(k) \sim N(\ln S(0) + kv, k\sigma^2).$$

Remark 3. The distribution of the logarithm of historical gross return resembles a normal distribution. Relative to a true normal distribution, the historical distribution is characterized with *fat tails*; i.e., the observed distribution is larger in the tails than a normal distribution. The fat tails imply that large price changes tend to occur more frequently than would be predicted by a normal distribution of the same variance.



2. Random Walks and Brownian Motions

Definition 4. A *random walk* process z has the form

$$z(t_{k+1}) = z(t_k) + \varepsilon(t_k)\sqrt{\Delta t}$$
$$t_{k+1} = t_k + \Delta t$$

$t_{k+1} - t_k = \Delta t$.

for $k = 0, 1, \ldots$, where $\varepsilon(t_k) \sim i.i.d.N(0,1)$. An interval between any two successive periods has the length of Δt.

$E[\varepsilon(t_k)] = 0$

$Var[\varepsilon(t_k)] = 1 = \underbrace{E[\varepsilon^2(t_k)]}_{1} - \{\underbrace{E[\varepsilon(t_k)]}_{0}\}^2$

- The difference $z(t_k) - z(t_j)$ for $j < k$ is given by

$$\begin{aligned} z(t_k) - z(t_j) &= z(t_k) - z(t_{k-1}) + z(t_{k-1}) - z(t_{k-2}) + \cdots + z(t_{j+1}) - z(t_j) \\ &= \varepsilon(t_{k-1})\sqrt{\Delta t} + \varepsilon(t_{k-2})\sqrt{\Delta t} + \cdots + \varepsilon(t_j)\sqrt{\Delta t} \\ &= \sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}. \end{aligned}$$

- It shows

$$E[z(t_k) - z(t_j)] = E\left[\sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}\right] = 0$$

$\varepsilon(t_k) \sim N(0,1)$.

$E(\varepsilon(t_k)) = 0$

$E(\varepsilon^2(t_k)) = 1$.

and

$$
\begin{aligned}
Var[z(t_k) - z(t_j)] &= E[(z(t_k) - z(t_j))^2] \\
&= E\left[\left(\sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}\right)^2\right] \\
&= E\left[\sum_{i=j}^{k-1} \varepsilon(t_i)^2 \Delta t\right] \\
&= (k-j)\Delta t \\
&= t_k - t_j.
\end{aligned}
$$

$t_{k+1} = t_k + \Delta t$

$t_k - t_j$

$\to \Delta t$

$E[\sum \varepsilon^2(t_i)]\Delta t$

$\Delta t\, E[\varepsilon^2(t_j)] + E[\varepsilon^2(t_{j+1})] + \cdots + E[\varepsilon^2(t_{k-1})]$

Since the sum of normal random variables is also a normal random variable, one sees

$$z(t_k) - z(t_j) \sim N(0, t_k - t_j)$$

$(k-1) - j + 1$

$k - j$

for $j < k$.

Remark 5. If $t_{k_1} < t_{k_2} \leq t_{k_3} < t_{k_4}$, then $z(t_{k_2}) - z(t_{k_1})$ is uncorrelated with $z(t_{k_4}) - z(t_{k_3})$; i.e., the difference variable associated with two different time intervals are uncorrelated if the two intervals are non-overlapping.

Definition 6. A process z is a *Brownian motion* if it satisfies the following properties:

1. $z(t) - z(s) \sim N(0, t-s)$ for $s < t$. $\to$ $z(t) \sim N(0, t)$. $(\because z(0) = 0)$.
2. $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are uncorrelated for $0 \leq t_1 < t_2 \leq t_3 < t_4$.
3. $z(0) = 0$ with probability 1.

- These properties parallel the properties of the random walk. In fact, the Brownian motion is a *continuous* version of the random walk; i.e., as $\Delta t \to 0$, the random walk process becomes

$$dz = \varepsilon(t)\sqrt{dt}$$

which is an equation governing the Brownian motion. If $\varepsilon(t) \sim i.i.d.N(0,1)$, it shows

$$dz \sim i.i.d.N(0, dt).$$

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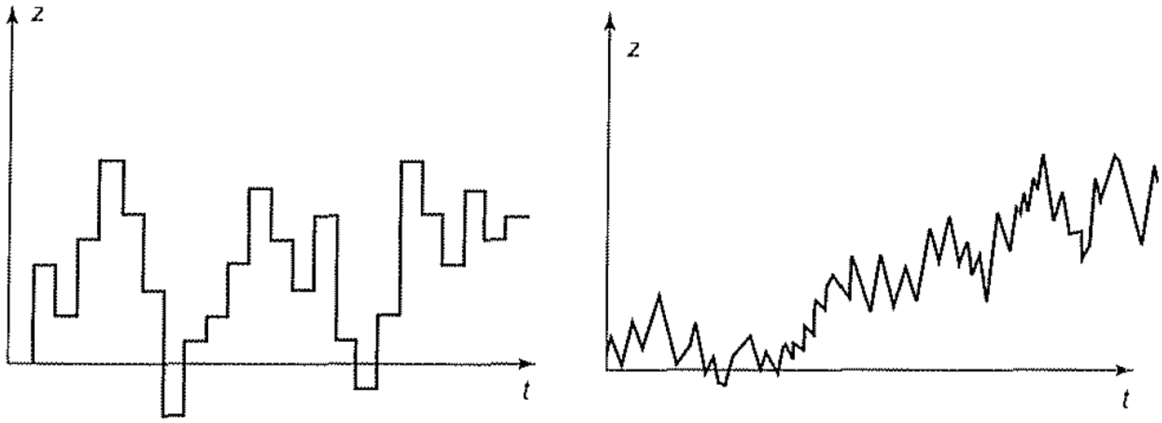
$$
\begin{aligned}
E[dz] &= E(\varepsilon(t))\sqrt{dt} = 0. \\
Var(dz) &= E[dz^2] - [E(dz)]^2 \\
&= E[dz^2] \\
&= E[\varepsilon^2(t)]\,dt \\
&= dt.
\end{aligned}
$$



3. Maclaurin Expansion

To expand a function $y = f(x)$ around x_0 means to transform f into a *polynomial* form in which the coefficients of the terms are expressed in terms of the evaluated derivatives $f^{'}(x_0), f^{''}(x_0), \ldots$ at the point of expansion x_0. Consider the following polynomial function of the nth degree:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

The derivatives are stated as follows:

$$\begin{aligned} f^{(1)}(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} \\ f^{(2)}(x) &= 2a_2 + 2 \cdot 3 a_3 x + \cdots + (n-1) n a_n x^{n-2} \\ &\vdots \\ f^{(n)}(x) &= 1 \times 2 \times 3 \times \cdots \times (n-1) \times n a_n. \end{aligned}$$

Evaluate the function f and its derivatives above at $x = 0$:

$$\begin{aligned} f(0) &= a_0 \\ f^{(1)}(0) &= a_1 \\ f^{(2)}(0) &= 2a_2 \\ &\vdots \\ f^{(n)}(x) &= 1 \times 2 \times 3 \times \cdots \times (n-1) \times n a_n. \end{aligned}$$

Definition 7. For a positive integer n, $n!$ (read n factorial) is the product of all of the positive integers less than or equal to n:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

- Note that $0!$ is defined as equal to 1.

Using factorials , one writes the evaluated derivatives as

$$\begin{aligned} a_0 &= \frac{f(0)}{0!} \\ a_1 &= \frac{f^{(1)}(0)}{1!} \\ a_2 &= \frac{f^{(2)}(0)}{2!} \\ &\vdots \\ a_n &= \frac{f^{(n)}(0)}{n!}. \end{aligned} \tag{2}$$

Substituting (2) into the given function f, we have the *Maclaurin expansion* around 0 as follows:

$$f(x)|_{x=0} = \frac{f(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

4. Taylor Expansion

4.1. Taylor Expansion for a Polynomial Function

Consider the following quadratic function

$$f(x) = a_0 + a_1 x + a_2 x^2.$$

Let $x = x_0 + \delta$, where δ represents the deviation from the fixed value x_0. Then, we have

$$\begin{aligned} f(x) &= f(x_0 + \delta) \\ &= a_0 + a_1(x_0 + \delta) + a_2(x_0 + \delta)^2. \end{aligned} \tag{3}$$

Since x_0 is fixed and δ is only variable in (3), $f(x)$ is a function of δ, say, $g(\delta)$; i.e.,

$$\begin{aligned} g(\delta) &= f(x_0 + \delta) \\ &= (a_0 + a_1 x_0 + a_2 x_0^2) + (a_1 + 2a_2 x_0)\delta + a_2 \delta^2 \end{aligned}$$

and the derivatives of $g(\delta)$ are

$$\begin{aligned} g^{'}(\delta) &= a_1 + 2a_2 x_0 + 2a_2 \delta \\ g^{''}(\delta) &= 2a_2. \end{aligned}$$

Using the Maclaurin expansion for $g(\delta)$ around $\delta = 0$, we have

$$g(\delta)|_{\delta=0} = \frac{g(0)}{0!} + \frac{g^{'}(0)}{1!}\delta + \frac{g^{''}(0)}{2!}\delta^2.$$

Since $x = x_0$ when $\delta = 0$, we write

$$g(0) = f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2$$
$$g^{'}(0) = f^{'}(x_0) = a_1 + 2a_2 x_0$$
$$g^{''}(0) = f^{''}(x_0) = 2a_2,$$

so that we obtain the *Taylor expansion* for $f(x)$ around $x = x_0$:

$$\begin{aligned} f(x)|_{x=x_0} &= \frac{g(0)}{0!} + \frac{g^{'}(0)}{1!}\delta + \frac{g^{''}(0)}{2!}\delta^2 \\ &= \frac{f(x_0)}{0!} + \frac{f^{'}(x_0)}{1!}(x - x_0) + \frac{f^{''}(x_0)}{2!}(x - x_0)^2 \\ &= (a_0 + a_1 x_0 + a_2 x_0^2) + (a_1 + 2a_2 x_0)(x - x_0) + a_2(x - x_0)^2 \\ &= a_2 x^2 + a_1 x + a_0. \end{aligned}$$

Theorem 8. *(Taylor's formula) For a given polynomial function $f \in C^{(n)}$ of the nth degree, we have the Taylor expansion around $x = x_0$ as follows:*

$$f(x)|_{x=x_0} = \frac{f(x_0)}{0!} + \frac{f^{'}(x_0)}{1!}(x - x_0) + \frac{f^{''}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{4}$$

- Let $\Delta x \equiv x - x_0$ and $\Delta f \equiv f(x) - f(x_0)$. Then, we have

$$\Delta f = f^{'}(x_0)\Delta x + \frac{f^{''}(x_0)}{2!}\Delta x^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}\Delta x^n.$$

4.2. Taylor Expansion for an Arbitrary Function

Theorem 9. *(Taylor's formula with remainder) In general, we have the Taylor expansion for an arbitrary function $f \in C^{(n)}$:*

$$\begin{aligned} f(x)|_{x=x_0} &= \left[\frac{f(x_0)}{0!} + \frac{f^{'}(x_0)}{1!}(x - x_0) + \frac{f^{''}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n\right] + R_n \\ &= P_n + R_n, \end{aligned}$$

where R_n is a remainder.

- Since $f(x)$ is an arbitrary function, *not necessarily* a polynomial function, it cannot be transformed *exactly* into the polynomial form shown in (4). P_n represents a polynomial approximation to $f(x)$ and R_n measures the error of the approximation. As we choose higher n, then R_n will be smaller, meaning that the approximation error will become smaller.

Linear approximation When $n = 1$, we have a *linear approximation* to $f(x)$ around $x = x_0$ as

$$f(x)|_{x=x_0} \approx f(x_0) + f^{'}(x_0)(x - x_0).$$

Quadratic approximation When $n = 2$, we have a *quadratic approximation* to $f(x)$ around $x = x_0$ as

$$f(x)|_{x=x_0} \approx f(x_0) + f^{'}(x_0)(x - x_0) + \frac{f^{''}(x_0)}{2}(x - x_0)^2.$$

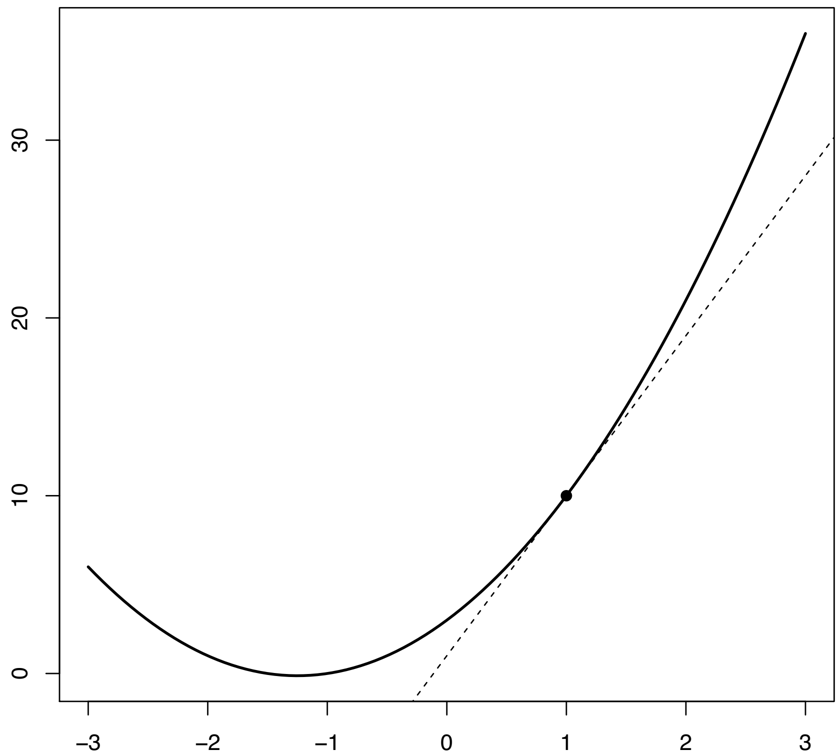
Remark 10. The quadratic approximation is more accurate than the linear approximation because R_1 is equal to or greater than R_2.

Example 11. Given the function

$$f(x) = 2x^2 + 5x + 3,$$

we have a linear approximation around $x = 1$ as follows:

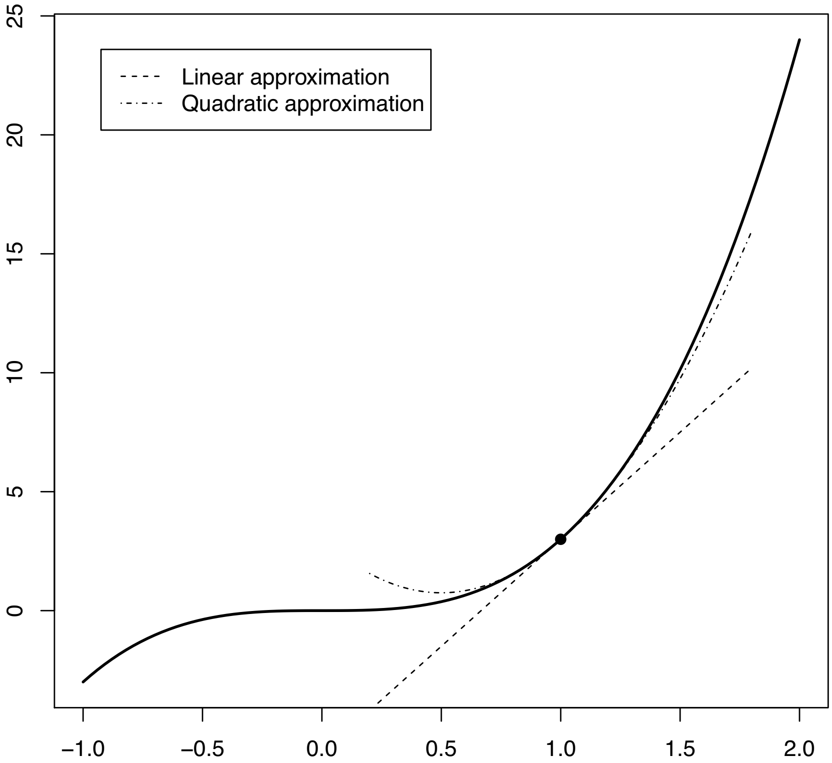
$$f(x)|_{x=1} \approx 10 + 9(x - 1) = 9x + 1.$$



- The linear approximation gives the exact value of $f(x)$ at the expansion point. Elsewhere around the expansion point, the approximation error R_1 is strictly positive.

Example 12. Consider the function $f(x) = 3x^3$. Around $x = 1$, a linear approximation and a quadratic approximation are respectively:

$$
\begin{aligned}
f(x)|_{x=1} &\approx 3+9(x-1) \\
f(x)|_{x=1} &\approx 3+9(x-1)+9(x-1)^2.
\end{aligned}
$$



4.3. Taylor Expansion for a Multivariate Function

Theorem 13. *Let f be an infinitely differentiable function in some open neighborhood around $(x,y) = (x_0,y_0)$. Then, we have the Taylor expansion for an arbitrary function f:*

$$
\begin{aligned}
f(x,y)|_{x=x_0,y=y_0} &= \frac{f(x_0,y_0)}{0!} + \frac{f_x(x_0,y_0)}{1!}(x-x_0) + \frac{f_y(x_0,y_0)}{1!}(y-y_0) \\
&\quad + \frac{1}{2!}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right] + \cdots.
\end{aligned}
$$

$\begin{pmatrix} f_x & f_y \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$

$(x-x_0 \quad y-y_0)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$

- Let $\Delta x \equiv x - x_0$, $\Delta y = y - y_0$ and $\Delta f \equiv f(x,y) - f(x_0,y_0)$. Then, we have

$$
\begin{aligned}
\Delta f &= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y \\
&\quad + \frac{1}{2!}\left[f_{xx}(x_0,y_0)\Delta x^2 + 2f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)\Delta y^2\right] + \cdots.
\end{aligned}
$$

$\Delta f = Df(x_0,y_0)h + \frac{1}{2}h^T H f(x_0,y_0)h$

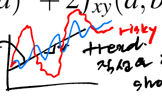
$P_2(x) = f(a) + Df(a)h + \frac{1}{2}h^T Hf(a)h \qquad h = x-a$

Linear approximation A *linear approximation* to $f(x)$ in a neighborhood of $(x,y) = \overset{x_0\ y_0}{(a,b)}$ is given by:

$$f(x,y)|_{x=a,y=b}^{x_0\ \ y_0} \approx f\overset{x_0 y_0}{(a,b)} + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Quadratic approximation A *quadratic approximation* to $f(x)$ in a neighborhood of $(x,y) = \overset{x_0\ y_0}{(a,b)}$ is given by:

$$\begin{aligned} f(x,y)|_{x=a,y=b} &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &+ \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]. \end{aligned}$$



5. Stock Price Process

5.1. Multiplicative Model in Continuous Time

A *continuous-time* multiplicative model is given by

$S(K+1) = u(K) \cdot S(K)$

$d \ln S(t) = vdt + 6dz$

$$d\ln S(t) = \overset{\text{trend}}{v}dt + \overset{\text{risk}}{\sigma}dz, \quad (5)$$

$\Rightarrow \ln S(K+1) - \ln S(K) = \ln u(K)$

price의 변화율의 영향 값 상승/하락 등

$\ln u(K) \sim N(v, 6^2)$

where z is a Brownian motion.

$\Rightarrow \ln S(K+1) \sim N(\cancel{\ln S(K)} + v, 6^2)$

$\ln S(K) \sim N(\ln S(0) + kv, k6^2)$

- By the fact $dz = \varepsilon(t)\sqrt{dt} \sim i.i.d.N(0,dt)$, it shows

$$d\ln S(t) \sim N(vdt, \sigma^2 dt).$$

$d\ln S(t) = vdt + 6dz$

$dz \sim iid\ N(0, dt)$

Integrating both sides of (5) yields

$\Rightarrow d\ln S(t) \sim N(v\,dt, 6^2 dt)$

$$\int_0^t d\ln S(t) = \int_0^t vdt + \int_0^t \sigma dz(t)$$

$\int_0^t d\ln S(t) = \int_0^t v\,dt + \int_0^t 6\,dz$

or

$z(0) = 0$

$$\ln S(t) = \ln S(0) + vt + \sigma z(t). \quad (6)$$

$z(t) - z(s) \sim N(0, t-s)$

Since the first property of the Brownian motion implies $z(t) \sim N(0,t)$, (6) leads to

$z(t) - z(0) \sim N(0, t)$

$$\ln S(t) \sim N(\ln S(0) + vt, \sigma^2 t).$$

$z(t) \sim N(0, t)$

Due to the fact that $S(t)$ has a lognormal distribution, one shows

$\ln X \sim N(\mu, 6^2)$

$$\begin{aligned} E[S(t)] &= \exp\left(\ln S(0) + vt + \frac{\sigma^2 t}{2}\right) \\ &= S(0)e^{\left(v+\frac{\sigma^2}{2}\right)t}. \end{aligned}$$

$E[X] = \exp(\mu + \frac{6^2}{2})$

$\ln X \sim N(\mu, 6^2)$

$\Rightarrow E(X) = e^{\mu + \frac{6^2}{2}}$

5.2. Standard Ito Form

$Var(X) = e^{2\mu + 26^2} - e^{2\mu + 6^2}$

Definition 14. An *Ito process* is defined by an equation of the form

$$dx(t) = a(x,t)dt + b(x,t)dz,$$

$dx(t) = a(x,t)\,dt + b(x,t)\,dz$

z : Brownian motion

$dz = \varepsilon(t)\sqrt{dt}$

where z is a Brownian motion, and the coefficients $a(x,t)$ and $b(x,t)$ may depend on x and t.

- A standard Ito form for the *instantaneous* rate of return, denoted by $dS(t)/S(t)$, is given by

순간수익률 확정적으로 trend 불확실성.

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz$$

or

$$dS(t) = \underbrace{\mu S(t)}_{a} dt + \underbrace{\sigma S(t)}_{b} dz.$$

Theorem 15. *(Ito's Lemma) Suppose that the random process x is defined by the Ito process*

$$dx(t) = a(x,t)dt + b(x,t)dz,$$

bivariate. (x,t).

where z is a Brownian motion. Suppose that the process $y(t)$ is defined by $y(t) = F(x,t)$. Then $y(t)$ satisfies the Ito equation

$$dy(t) = \underbrace{\left(F_x a(x,t) + F_t + \frac{1}{2}F_{xx}b(x,t)^2\right)}_{a} dt + \underbrace{F_x b(x,t)}_{b} dz.$$

새로운 arbitrary function 으로 ito eq.

Proof. Consider a continuous and differential function F of two variables x and t; i.e., $F(x,t) \in C^{(n)}$. If Δx and Δt are small changes in x and t and ΔF is the resulting small change in F, the Taylor expansion of ΔF is

$$\Delta F = F_x \Delta x + F_t \Delta t + \frac{1}{2}F_{xx}\underbrace{\Delta x^2}_{b^2dt} + F_{xt}\Delta x \Delta t + \frac{1}{2}F_{tt}\Delta t^2 + \cdots. \tag{7}$$

Suppose that a variable x follows the Ito's process

$$dx(t) = a(x,t)dt + b(x,t)dz,$$

which can be discretized to

discrete $dz = \varepsilon\sqrt{dt}$

$Var[\varepsilon] = E[\varepsilon^2] - (E(\varepsilon))^2 = 1$, $E(\varepsilon) = 0$

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}. \tag{8}$$

$\therefore E[\varepsilon^2] = 1$

$a^2\cancel{\Delta t^2} + b^2\underbrace{\varepsilon^2\Delta t}_{\to dt} + ab\varepsilon\cancel{\Delta t^{\frac{3}{2}}} \to 0$

From (8), one has

$\varepsilon \sim N(0,1)$

$$\Delta x^2 = b^2\varepsilon^2\Delta t + \text{terms of higher order in } \Delta t. \tag{9}$$

$\varepsilon^2\Delta t \sim N(\Delta t, \quad)$

$\Delta x^2 \to b^2 dt$

Since $\varepsilon \sim i.i.d.N(0,1)$, it shows

$E[\varepsilon^2\Delta t] = \Delta t\, E[\varepsilon^2] = \Delta t$

$$\varepsilon^2\Delta t \sim N(\Delta t, \Delta t^2),$$

$Var[\varepsilon^2\Delta t] = (\Delta t)^2 Var(\varepsilon^2)$

meaning that the typical size of $\varepsilon^2\Delta t$ is Δt; i.e., $\sigma[\varepsilon^2\Delta t] \sim O(\Delta t)$. Thus, as $\Delta t \to 0$, $\sigma[\varepsilon^2\Delta t]$ vanishes, so that $\varepsilon^2\Delta t$ becomes non-stochastic and converges to dt (i.e., the limit of its expected value). Consequently, it follows from (9) that Δx^2 becomes non-stochastic and equal to b^2dt as $\Delta t \to 0$. Taking limits as Δx and Δt tend to zero in (7), one obtains

$Var[\varepsilon^2] = E(\varepsilon^4) - \underbrace{(E(\varepsilon^2))^2}_{1}$

$$dF = F_x dx + F_t dt + \frac{1}{2}F_{xx}b(x,t)^2 dt. \tag{10}$$

$\sigma(\varepsilon^2\Delta t) \to 0$

as $\Delta t \to 0$

non stochastic.

ito's $\begin{cases} dx(t) = a\,dt + b\,dz \\ dF = (F_x a + F_t + \frac{1}{2} F_{xx} b^2)\,dt + F_x b\,dz \end{cases}$

Substituting for dx in (10) yields

stochastic process.

$$dF = \left(F_x a(x,t) + F_t + \frac{1}{2} F_{xx} b(x,t)^2 \right) dt + F_x b(x,t) dz.$$ //

$d F(x) = d \ln x$ $\qquad$ $dx(t) = a\,dt + b\,dz$

□

Example 16. With $F(x) = \ln x$ and $dS(t) = \underbrace{\mu S(t)}_{a} dt + \underbrace{\sigma S(t)}_{b} dz$, Ito's lemma shows

$x = S(t)$ //

$$\begin{aligned} d \ln S(t) &= \left(\frac{1}{S(t)} \underbrace{\mu S(t)}_{a} - \frac{1}{2} \frac{1}{S(t)^2} \underbrace{\sigma^2 S(t)^2}_{b^2} \right) dt + \underbrace{\frac{1}{S(t)}}_{F_x} \underbrace{\sigma S(t)}_{b} dz \\ &= \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz. \end{aligned} \tag{11}$$

$F_x = \frac{d \ln x}{dx} = \frac{1}{x}$

Integrating both sides of (11) yields

$$\ln S(t) = \ln S(0) + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma z(t).$$

$\int_0^t d \ln S(t) = \int_0^t (\mu - \cdots$

$\ln S(t) - \ln S(0)$.

Using $z(t) \sim N(0,t)$, one obtains

$\ln X \sim N(\mu, \sigma^2)$.

$$\ln S(t) \sim N \left(\ln S(0) + \left(\mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right)$$

$E(X) = e^{\mu + \frac{\sigma^2}{2}}$

$Var(x) = e^{(2\mu + \sigma^2)} - e^{2\mu + \sigma^2}$

and

$$\begin{aligned} E[S(t)] &= \exp \left(\ln S(0) + \left(\mu - \frac{\sigma^2}{2} \right) t + \frac{\sigma^2 t}{2} \right) \\ &= S(0) e^{\mu t}. \end{aligned}$$

$\ln X \sim N(\mu, \sigma^2)$

$E[X] = \exp(\mu + \frac{\sigma^2}{2})$

$Var[X] = \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2)$

6. Black-Scholes Option Formula

6.1. *Option Concepts and Values*

Definition 17. An *option* is the right, but not the obligation, to buy (or sell) an underlying asset under specified terms. An option that gives right to buy something is called a *call* option, while an option that gives the right to sell something is called a *put* option.

- If the option holder actually buys or sells the asset according to the terms of the option, the option holder is said to "exercise" the option. An *American option* allows exercise at any time before and including the expiration date. A *European option* allows exercise only on the expiration date. The *exercise price* (or *strike price*) is the price at which the asset can be purchased or sold upon the exercise of the option.

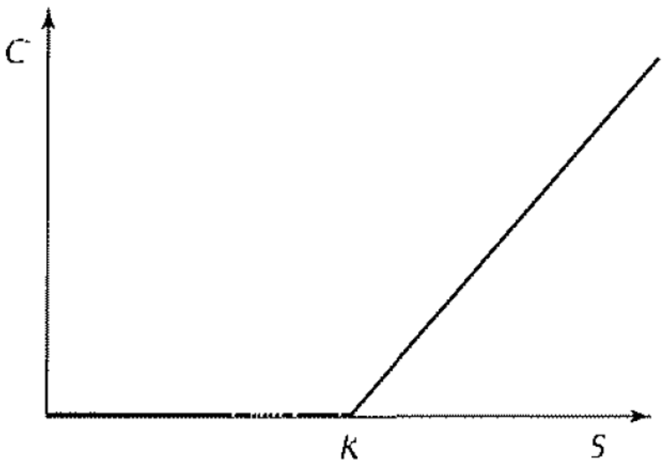
Consider a call option on a stock with a strike price of K. Suppose that on the expiration date the price of the underlying stock is S_T. Then, the value of the call option at expiration is

$$C = \begin{cases} S_T - K & \text{if } S_T > K \\ 0 & \text{otherwise,} \end{cases}$$

or

$$C = \max(0, S_T - K).$$

- A call option is *in the money*, *at the money*, or *out of the money*, depending on whether $S_T > K$, $S_T = K$, or $S_T < K$, respectively.



Remark 18. In a reverse sense, the value of a put option at expiration is given by

$$P = \begin{cases} K - S_T & \text{if } K > S_T \\ 0 & \text{otherwise,} \end{cases}$$

or

$$P = \max(0, K - S_T).$$

6.2. The Black-Scholes Differential Equation

The price S of an underlying asset follows

$$dS = \mu S dt + \sigma S dz$$

over a time interval $[0, T]$, where z is standard Brownian motion. The price B of a risk-free asset paying an interest rate of r over $[0, T]$ follows

$$dB = rBdt.$$

$\underbrace{\frac{dB}{B}}_{\text{순간수익률}} = \underbrace{rdt}_{\text{risk free.}}$

Consider a security that is derivative to S. Let $f(S,t)$ be the price of this derivative security at time t when the underlying asset price is S.

파생상품이기 때문에 underlying asset 의 가격에 영향을 받음.

- One forms a portfolio with the stock and the bond so that this portfolio exactly replicates the derivative security. Then, *no arbitrage opportunities* imply that the value of this portfolio must equal the value of the derivative security.

이 가정이 없으면 option price에 대해 진행이 안됨

$x \quad t$

Theorem 19. *(Black-Scholes differential equation) A derivative price $f(S,t)$ satisfies the partial differential equation*

$$f_t + f_S rS + \frac{1}{2} f_{SS} \sigma^2 S^2 = rf.$$

$dS = \mu S dt + \sigma S dz$

Proof. Ito's lemma leads to

$dy = \left(F_x a(x,t) + F_t + \frac{1}{2} F_{xx} b(x,t)^2\right) dt + F_x b(x,t)\, dz$

$$df = \left(f_S \mu S + f_t + \frac{1}{2} f_{SS} \sigma^2 S^2 \right) dt + f_S \sigma S dz. \tag{12}$$

Consider a portfolio of S and B

$$G(t) = x_t S(t) + y_t B(t),$$

where x_t is an amount of the stock and y_t is an amount of the bond. The instantaneous gain in value of the portfolio is

$$\begin{aligned} dG &= x_t dS + y_t dB \\ &= x_t(\mu S dt + \sigma S dz) + y_t rB dt \\ &= (x_t \mu S + y_t rB) dt + x_t \sigma S dz. \end{aligned} \tag{13}$$

One wants to select x_t and y_t so that $G(t)$ replicates $f(S,t)$ at each time t. This requires two conditions be met,

$$G = f \tag{14}$$

가격 같음.

$$dG = df. \tag{15}$$

수익도 같아야함.

The condition (14) implies

$$x_t S + y_t B = f$$

or

$$y_t = \frac{1}{B}(f - x_t S). \tag{16}$$

The condition (15) implies

$$x_t \sigma S dz = f_S \sigma S dz$$

dt의 계수를 맞추는 방식
dz의 계수를 맞추는 방법 (easy)

or

$$x_t = f_S. \tag{17}$$

Plugging (17) in (16) yields

$$y_t = \frac{1}{B}(f - f_S S). \tag{18}$$

After substituting (17) and (18) in (13), the condition (15) implies

$$\left(f_S \mu S + \frac{1}{\cancel{B}}(f - f_S S) r\cancel{B} \right) dt = \left(f_S \mu S + f_t + \frac{1}{2} f_{SS} \sigma^2 S^2 \right) dt$$

$rf - f_S r S = f_t + \frac{1}{2} f_{SS} \sigma^2 S^2$

or

$$rf = f_t + f_S rS + \frac{1}{2} f_{SS} \sigma^2 S^2.$$

$rf = f_t + f_S r S + \frac{1}{2} f_{SS} \sigma^2 S^2$

□

배당X.

Remark 20. The Black-Scholes differential equation establishes a condition that must hold for the price of any derivative dependent on a *non-dividend-paying* stock in the absence of arbitrage opportunities. Any price function $f(S,t)$ that is a solution to the Black-Scholes differential equation is the *theoretical price* of a derivative that could be traded. Black and Scholes (1973) find an analytic solution to the Black-Scholes equation for a European call option. Notice that it is usually impossible to find an analytic solution in most cases of derivatives.

Remark 21. The particular derivative that is obtained when the equation is solved depends on the *boundary conditions*. These specify the values of the derivative at the boundaries of possible values of S and t. In the case of a European call option, for instance, the boundary condition is $C(S,t) = \max(S-K,0)$ when $t = T$.

현재까지 Analytic solution이 나오는 것　　American은 없음.

증명 필요X.

Theorem 22. *(Black-Scholes call option formula) Consider a European call option with strike price K and expiration time T. If the underlying stock pays no dividends during the time $[0,T]$ and if interest is constant and continuously compounded at a rate r, the Black-Scholes analytic solution to $f(S,t) = C(S,t)$ is given by*

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and $N(x)$ denotes the standard cumulative normal probability distribution.

Proof. By the Black-Scholes differential equation, one knows

$$rC = C_t + C_S rS + \frac{1}{2} C_{SS}\sigma^2 S^2.$$

$f \rightarrow C$

The boundary condition for a European call option at expiration is

$$C(S,T) = \max(S-K,0).$$

B&S 의 천재성.

To solve the differential equation, Black and Scholes (1973) make the following substitution:

$$C(S,t) = e^{r(t-T)}y(u,s), \tag{19}$$

where

$$u = \frac{2}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)\left[\ln\left(\frac{S}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)(t-T)\right]$$
$$s = -\frac{2}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2 (t-T).$$

With this substitution, the differential equation becomes

$$\frac{\partial y}{\partial s} = \frac{\partial^2 y}{\partial u^2}, \tag{20}$$

열전도 방정식.

and the boundary condition becomes

$$y(u,0) = \begin{cases} c\left(\exp\left(\frac{uv^2/2}{r-v^2/2}\right) - 1\right) & \text{if } u \geq 0 \\ 0 & \text{if } u < 0. \end{cases}$$

Black and Scholes (1973) recognize that the differential equation (20) is the *heat-transfer equation* of physics and its solution is given by

$$y(u,s) = \frac{1}{\sqrt{2\pi}} \int_{-u/\sqrt{2s}}^{\infty} c\left(\exp\left(\frac{(u+q\sqrt{2s})v^2/2}{r-v^2/2}\right) - 1\right) e^{-q^2/2} dq. \tag{21}$$

Substituting (21) into (19) yields the Black-Scholes formula for a European call option. $\square$