Lecture 5. Applied Interest Rate Analysis

1. Capital Budgeting

Capital budgeting refers to capital allocation among projects or investments. Capital budgeting problems arise in a firm where several proposed projects compete for funding but cannot all be funded because of a budget limitation.

- The projects are *independent* if the feasibility of one is not dependent on whether others are undertaken. You can select *any* combination from the independent projects so long as they are all funded.
- The projects are *interdependent* if the feasibility of one is dependent on whether others are undertaken. The capital budgeting of interdependent projects assumes that there are several *independent goals*, but each goal has *more than* one project. For independent goals, you can select any combination of goals so long as they can be funded. Among all possible projects, however, only one project is selected to implement the goal.

1.1. Independent Projects

Suppose that there are m independent projects. For each project i, let b_i be the net present value (NPV) of the project and let c_i denote the initial cost. The variable x_i is zero if the ith project is rejected and one if it is accepted. Let C be the budget.

- The capital budgeting problem solves

$$\max_{x_1,\ldots,x_m} \sum_{i=1}^{m} b_i x_i$$
$$\text{subject to} \sum_{i=1}^{m} c_i x_i \leq C.$$

Example 1. A company have three independent projects as follows:

Project	Initial outlay (c_i)	NPV (b_i)
1	\$100	\$200
2	\$20	\$30
3	\$150	\$200

The company can make available up to \$160 for these projects. The capital budget problem is stated as

$$\max_{x_1,x_2,x_3} 200x_1 + 30x_2 + 200x_3$$
$$\text{subject to} 100x_1 + 20x_2 + 150x_3 \leq 160,$$

where x_i equals to one if the ith project is selected and zero otherwise. The following worksheet shows all possible combinations of the projects:

	(1)	(2)	(3)	**(4)**	(5)	(6)	(7)
x_1	1	0	0	**1**	1	0	1
x_2	0	1	0	**1**	0	1	1
x_3	0	0	1	**0**	1	1	1
$\sum_{i=1}^{3} c_i x_i$	100	20	150	**120**	250	170	270
$\sum_{i=1}^{3} b_i x_i$	200	30	200	**230**			

Therefore, the optimal solution is $(x_1^*, x_2^*, x_3^*) = (1,1,0)$, or equivalently, selecting projects 1 and 2 for a total expenditure of \$120 and a total net present value of \$230.

1.2. Interdependent Projects

Suppose that there are m independent goals and goal i has n_i possible projects. Only one project is selected for any goal. The variable x_{ij} for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$ equals one if goal i is chosen and implemented by project j; otherwise, it is zero.

- The capital budget problem is

$$\begin{aligned} \max_{x_{ij}} & \sum_{i=1}^{m} \sum_{j=1}^{n_i} b_{ij} x_{ij} \\ \text{subject to} & \sum_{i=1}^{m} \sum_{j=1}^{n_i} c_{ij} x_{ij} \leq C \\ & \sum_{j=1}^{n_i} x_{ij} \leq 1 \text{ for } i = 1, \ldots, m. \end{aligned}$$

Example 2. Suppose a transportation authority has two independent goals. One goal is to construct a road between two cities, and can be implemented by (a) Project 11: 2 lanes or (b) Project 12: 4 lanes. The other goal is to improve a bridge, and can be implemented by (c) Project 21: Repair existing or (d) Project 22: Add lane. The following table shows the cost and the net present value for each of the projects:

Goal ($i = 1,2$)	Project ($j = 1,2$)	Initial outlay (c_{ij})	NPV (b_{ij})
Goal 1	Project 11: 2 lanes	$c_{11} = 200$	$b_{11} = 400$
	Project 12: 4 lanes	$c_{12} = 300$	$b_{12} = 500$
Goal 2	Project 21: Repair existing	$c_{21} = 50$	$b_{21} = 100$
	Project 22: Add lane	$c_{22} = 150$	$b_{22} = 150$

Given the budget constraint $C = \$400$, the capital budgeting problem is

$$\begin{aligned} \max_{x_{11}, x_{12}, x_{21}, x_{22}} & \; 400x_{11} + 500x_{12} + 100x_{21} + 150x_{22} \\ \text{subject to} & \; 200x_{11} + 300x_{12} + 50x_{21} + 150x_{22} \leq 400 \\ & \; x_{11} + x_{12} \leq 1 \\ & \; x_{21} + x_{22} \leq 1. \end{aligned}$$

The following worksheet shows all possible combinations of the projects:

	(1)	(2)	(3)	(4)	(5)	(6)	**(7)**	(8)
x_{11}	1	0	0	0	1	1	**0**	0
x_{12}	0	1	0	0	0	0	**1**	1
x_{21}	0	0	1	0	1	0	**1**	0
x_{22}	0	0	0	1	0	1	**0**	1
$\sum_{i=1}^{2}\sum_{j=1}^{2} c_{ij}x_{ij}$	200	300	50	150	250	350	**350**	450
$\sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij}x_{ij}$	400	500	100	150	500	550	**600**	

Therefore, the optimal solution is $(x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*) = (0,1,1,0)$, or equivalently, selecting projects 12 and 21 for a total expenditure of \$350 and a total net present value of \$600.

2. Optimal Portfolios

The "simplest" form of *optimal portfolios* refers to the *cash matching problem*. When there are a sequence of future monetary obligations, one wants to construct a portfolio by purchasing bonds of various maturities, and use the coupon payments and redemption values to meet the the obligations.

- Let $(y_1, y_2, \ldots, y_T)$ be a stream of future obligations. Suppose that there are total m candidate bonds for the portfolio. For the jth bond, denote the cash stream by $(c_{1j}, c_{2j}, \ldots, c_{Tj})$, and its price by p_j. The variable x_j represents the number of of bond j to be held in the portfolio. The cash matching problem is to find the x_j's of minimum cost which guarantee that the obligations can be met. The optimal portfolio solves the following minimization problem:

$$\begin{aligned} \min_{x_j} & \sum_{j=1}^{m} p_j x_j \\ \text{subject to} & \sum_{j=1}^{m} c_{tj} x_j \geq y_t \text{ for } t = 1,2,\ldots,T \\ & x_j \geq 0 \text{ for } j = 1,2,\ldots,m. \end{aligned}$$

Example 3. When matching the cash obligations over 3 periods, consider 4 bonds as follows:

	Period (t)			
	1	2	3	Price (p_j)
Obligation (y_t)	1,000,000	2,000,000	3,000,000	
Bond 1's cash flow (c_{t1})	100	100	1,100	1,090
Bond 2's cash flow (c_{t2})	70	70	1,070	950
Bond 3's cash flow (c_{t3})	50	1,050		1,000
Bond 4's cash flow (c_{t4})	1,000			980

The optimal portfolio is

$$\begin{aligned}
\min_{x_1,x_2,x_3,x_4} \quad & 1,090x_1 + 950x_2 + 1,000x_3 + 980x_4 \\
\text{subject to} \quad & 100x_1 + 70x_2 + 50x_3 + 1,000x_4 \geq 1,000,000 \\
& 100x_1 + 70x_2 + 1,050x_3 \geq 2,000,000 \\
& 1,100x_1 + 1,070x_2 \geq 3,000,000 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0 \\
& x_4 \geq 0.
\end{aligned}$$

The solution is found by any linear programming packages like `R`, etc.

```
> library(linprog)
> cvec <- c(1090, 950, 1000, 980)
> Amat <- rbind(c(100, 70, 50, 1000), c(100, 70, 1050, 0), c(1100, 1070, 0, 0))
> bvec <- c(1000000, 2000000, 3000000)
> solveLP(cvec, bvec, Amat, const.dir = rep(">=", length(bvec)))
Results of Linear Programming / Linear Optimization
Objective function (Minimum):  5084887
Iterations in phase 1:  3
Iterations in phase 2:  2
Solution opt
1    0.000
2 2803.738
3 1717.846
4 717.846
Basic Variables
       opt
2 2803.738
3 1717.846
4  717.846
Constraints
  actual dir  bvec free     dual dual.reg
1  1e+06  >= 1e+06    0 0.980000      Inf
2  2e+06  >= 2e+06    0 0.905714 15074766
3  3e+06  >= 3e+06    0 0.764486 11521429
All Variables (including slack variables)
          opt cvec        min.c   max.c      marg    marg.reg
1       0.000 1090    99.000000   77.00 60.493992     2727.27
2    2803.738  950 -1768.000000 1008.84        NA          NA
```

3	1717.846	1000	-1951.000000	3265.50	NA	NA
4	717.846	980	-1960.000000	3245.50	NA	NA
S 1	0.000	0	-0.980000	Inf	0.980000	Inf
S 2	0.000	0	-0.905714	Inf	0.905714	15074766.36
S 3	0.000	0	-0.764486	Inf	0.764486	11521428.57

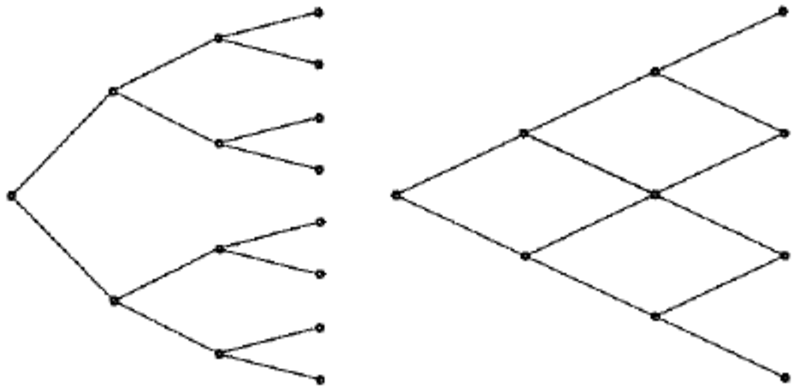
The optimal solution is $(x_1^*, x_2^*, x_3^*, x_4^*) = (0.000, 2803.738, 1717.846, 717.846)$ and the minimum total cost amounts to \$5,084,887.

3. Dynamic Cash Flow Processes

3.1. Representation of Dynamic Choice

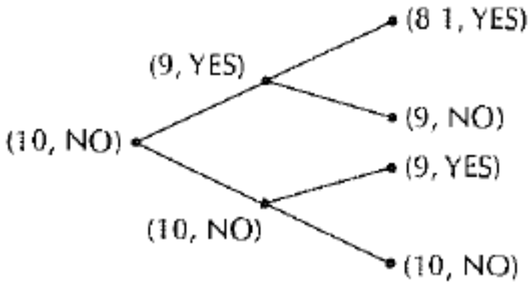
A *dynamic model* represents the possible choices at each period, and the effect that those choices have on future cash flows. Among others, a *tree* is the simplest way to represent dynamic choice.

Example 4. Binomial Tree and Binomial Lattice



(a) Binomial tree (b) Binomial lattice

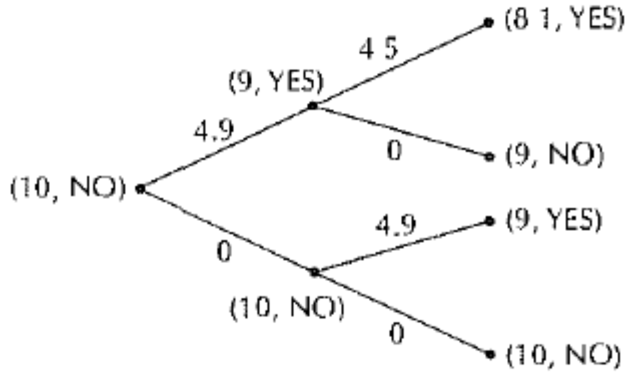
Example 5. Consider the management of the oil well. The well has initial reserves of 10 million barrels of oil. Each year it is possible to pump out 10% of the current reserves, but to do so a crew must be hired and paid. If a crew is already on hand since it was used in the previous year, the hiring expenses are avoided. The following binomial tree for the two years accounts for both the level of reserves and the status of a crew:



Example 6. Suppose that the cost of hiring a crew is \$100,000 and the profit from oil product is \$5.00 per barrel. Let x be the level of reserves in the well. The net profit for a year of production is given by

$$\begin{cases} 5\times 0.1\times x - 100,000 & \text{if a crew must be hired} \\ 5\times 0.1\times x & \text{if a crew is already on hand} \end{cases}.$$

Then, the cash flow corresponding to a decision is listed on the branch as follows:



3.2. Optimal Management

We set up the *optimal management plan* using the method of *running dynamic programming* in which the present value of the cash flow specified in each path is computed step by step, starting at the termination time and working back to the beginning. The optimal management plan corresponds to selecting the path which has the largest present value among all possible paths.

- Assign to the ith node at time t, denoted by (t,i), a value equal to the best running present value, denoted by $V_{t,i}$, that neglects all previous cash flows. Define $c^j_{t,i}$ to be the cash flow generated by moving from node (t,i) to node $(t+1,j)$ for $j=1,\ldots,n_i$. The best running present value is computed for each i as
$$V_{t,i} = \max_j (c^j_{t,i} + d_{t\to t+1}V_{t+1,j})$$
for $j = 1,2,\ldots,n_i$. This recursion procedure is continued, working backward until time zero is reached. The resulting V_0 is the optimal present value at time zero, and hence it is the overall best value. The optimal decisions and cash flows can be determined as a by-product of the dynamic programming procedure.

Example 7. Consider again the oil project. Assume further that each of the final nodes has no salvage value. Suppose $f_{0\to 1} = 6\%$ and $f_{1\to 2} = 7.5\%$. Thus, we know $d_{0\to 1} = 0.94$ and $d_{1\to 2} = 0.93$. At the node (9, YES) in time 1, it shows

$$V_{1,(9,YES)} = \max(4.5 + 0.93\times 0, 0 + 0.93\times 0) = 4.5,$$

so that the node (8.1, YES) is superior to the node (9, NO) in time 2. At the node (10, NO) in time 1, it shows

$$V_{1,(10,NO)} = \max(4.9 + 0.93\times 0, 0 + 0.93\times 0) = 4.9,$$

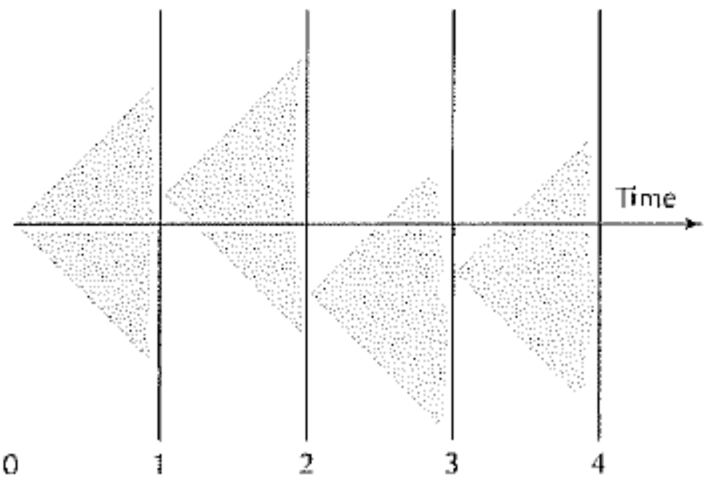
so that the node (9, YES) is superior to the node (10, NO) in time 2. Finally, the node (10, NO) in time 0 shows that the optimal present value is

$$V_{0,(10,NO)} = \max(4.9 + 0.94 \times 4.5, 0 + 0.94 \times 4.9) = 9.13,$$

so that the node (9, YES) is superior to the node (10, NO) in time 1. The optimal path is indicated as follows:

$$\begin{array}{ccccc}
 & & & & \textbf{(8.1, YES)} \\
 & & & \nearrow & \\
 & & \textbf{(9, YES)} & \rightarrow & \text{(9, NO)} \\
 & \nearrow & V_{1,(9,YES)} = 4.5 & & \\
\textbf{(10, NO)} & & & & \\
V_{0,(10,NO)} = 9.13 & \searrow & & & \\
 & & \text{(10, NO)} & \rightarrow & \text{(9, YES)} \\
 & & V_{1,(10,NO)} = 4.9 & \searrow & \\
 & & & & \text{(10, NO)}
\end{array}$$

Remark 8. The principle of running dynamic programming can be applied to a *continuous lattice* which has a continuum of nodes at any state and a continuum of possible decisions at any node.



Example 9. You are running a gold mine. Let x_t be the amount of gold remaining in year t. The cost to extract z_t ounces of gold in that year is given by cz_t^2/x_t where $z_t < x_t$. The price of gold, denoted by p, remains constant over time. The profit in year t is $\pi_t(z_t) = pz_t - cz_t^2/x_t$. You want to set the optimal management over 10 years. For simplicity, assume that $d_{t \to t+1}$ is fixed at δ for all t. Neglecting $x_0, x_1, \ldots, x_9$, one writes the optimal value in year 10 as

$$V_{10} = \max_{z_{10}} (\pi_{10}(z_{10})) = \max_{z_{10}} \left(pz_{10} - \frac{cz_{10}^2}{x_{10}} \right),$$

and finds z_{10}^* by setting $d\pi_{10}/dz_{10}$ equal to zero:

$$\frac{d\pi_{10}(z_{10})}{dz_{10}} = p - \frac{2cz_{10}}{x_{10}} = 0$$

or

$$z_{10}^* = \frac{px_{10}}{2c}.$$

So, the optimal value in year 10 is

$$V_{10} = \frac{p^2 x_{10}}{2c} - \frac{p^2 x_{10}}{4c} = \left(\frac{p^2}{4c}\right) x_{10} = K_{10} x_{10}.$$

Neglecting $x_0, x_1, \ldots, x_8$, one writes the optimal value in year 9 as

$$V_9 = \max_{z_9} \left(\pi_9(z_9)\right) = \max_{z_9} \left(pz_9 - \frac{cz_9^2}{x_9} + \delta V_{10} \right). \tag{1}$$

Since $x_{10} = x_9 - z_9$, the optimal value in year 10, seen in year 9, is stated as

$$V_{10} = \max_{z_{10}} \left(\pi_{10}(z_{10})\right) = \max_{z_{10}} \left(pz_{10} - \frac{cz_{10}^2}{x_9 - z_9} \right),$$

it shows

$$\begin{aligned} z_{10}^* &= \frac{p(x_9 - z_9)}{2c} \\ V_{10} &= K_{10}(x_9 - z_9). \end{aligned} \tag{2}$$

Using (2), one can restate (1) as

$$V_9 = \max_{z_9} \left(pz_9 - \frac{cz_9^2}{x_9} + \delta K_{10}(x_9 - z_9) \right)$$

and find z_9^* by setting dV_9/dz_9 equal to zero:

$$p - \frac{2cz_9}{x_9} - \delta K_{10} = 0$$

or

$$z_9^* = \frac{(p - \delta K_{10}) x_9}{2c}.$$

Hence, the optimal value in year 9 is

$$\begin{aligned} V_9 &= \frac{p(p-\delta K_{10})x_9}{2c} - \frac{c(p-\delta K_{10})^2 x_9}{4c^2} + \delta K_{10}\left(x_9 - \frac{(p-\delta K_{10})x_9}{2c} \right) \\ &= \left[\frac{(p-\delta K_{10})^2}{4c} + \delta K_{10} \right] x_9 \\ &= K_9 x_9. \end{aligned}$$

Continuing backward in this way, one obtains

$$
\begin{aligned}
V_t &= K_t x_t \\
K_t &= \frac{(p-\delta K_{t+1})^2}{4c} + \delta K_{t+1} \\
z_t^* &= \frac{(p-\delta K_{t+1})x_t}{2c} \\
x_{t+1} &= x_t - z_t^*
\end{aligned}
$$

for $t = 0, 1, \ldots, 9$. The last value, V_0, is the largest present value, and the optimal plan, z_t^*, is determined as a by-product of the dynamic programming. For instance, suppose that $x_0 = 50,000$, $\delta = 1/1.1$, $p = 400$, and $c = 500$. The following worksheet computes K_t, V_t, z_t^*, and x_t.

t	K_t	x_t (ounces)	z_t^* (ounces)	V_t (dollars)
0	215.52	50,000.00	10,281.14	**10,775,878.87**
1	213.81	39,718.86	8,252.35	8,492,485.52
2	211.45	31,466.51	6,631.72	6,653,720.63
3	208.17	24,834.79	5,337.67	5,169,853.63
4	203.58	19,497.13	4,304.87	3,969,229.91
5	197.13	15,192.26	3,480.90	2,994,782.60
6	187.96	11,711.36	2,823.61	2,201,320.59
7	174.79	8,887.75	2,298.94	1,553,489.17
8	155.47	6,588.81	1,879.12	1,024,364.02
9	126.28	4,709.69	1,541.35	594,744.36
10	80.00	3,168.34	1,267.33	253,466.97