Economics of Information and Uncertainty

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Based on lecture by In-Koo Cho in spring 2021

Contents

Chapter 1

Introduction

1.1 Expected utility theory

Why needed? A decision is often made under uncertainty.

- Objective uncertainty: The value of a decision may depend upon the contingency, which is not observable at the time of decision but the probability of the event is known.
- Subjective uncertainty: The value of a decision may depend upon the decisions of the other players, which are not observable at the time of decision. The probability of the event is endogeneous.

We need a formal theory to evaluate a choice whose value is not a deterministic value, but a probability distribution over values.

History John von Neumann and Oskar Morgenstern developed the theory of games. They immediately recognized that a decision maker faces uncertainty. A decision maker does not know the actual value from his decision, but realizes the value only after he made the decision. The value of a decision is more like a probability distribution than a number. In order to model his decision problem, we need a formal way to evaluate a probability distribution.

They developed the expected utility theory, as a way to investigate the interactive decision problem. The expected utility theory appears in the appendix rather than in the main text of their classic book.[1]

In this edition, Harold Kuhn and Ariel Rubinstein contributed short essays in the front and the back of the book, illustrating the status of economic theory

[1] John von Neumann and Oskar Morgenstern [2007]: Theory of Games and Economic Behavior: 60th Anniversary Commemorative Edition (Princeton Classic Editions).

60 years after the first edition of the classic book is first published. Students are strong encouraged to read the short essay by Ariel Rubinstein in the postscript, as well as the original foreward by von Neumann and Morgenstern who foresaw the future of the game theory.

Reference This lecture is drawn from David M. Kreps [1988][2] which is probably the best reference for the choice under uncertainty.

1.2 Description

Let Z be a finite set of outcomes, or attributes, from which a decision maker generates utility. Let $p\colon Z \to [0,1]$ be the probability distribution over Z. That is, $p(z) \geq 0\ \forall z$, and $\sum_{z \in Z} p(z) = 1$.

Prof. Koh

We use lotteries to describe risky alternatives. Suppose first that the number of possible outcomes is finite. Fix a set of outcomes $C = \{c_1, \cdots, c_N\}$. Let p_n be the probability that outcome $c_n \in C$ occurs and suppose these probabilities are objectively known.

Definition 1 (Lottery). A (simple) lottery $L = (p_1, \cdots, p_n)$ is an assignment of probabilities to each outcome c_n, where $p_n \geq 0$ for all n and $\sum_n p_n = 1$.

The collection of such lotteries can be written as

$$\mathcal{L} = \left\{ (p_1, \ldots, p_N) \,\middle|\, \sum_{n=1}^{N} p_n = 1,\ p_n \geq 0 \text{ for } n = 1, \ldots, N \right\}.$$

We can also think of a **compound lottery** $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$, which allows the outcomes of a lottery to be lotteries.

It is immediate to see that any compound lottery can be reduced to a simple lottery defined as above.

Example. $C = \{c_1, c_2\}$, $L_1 = (p, 1-p)$, $L_2 = (q, 1-q)$. Then,

$$(L_1, L_2; \alpha, 1-\alpha) = (\alpha p + (1-\alpha) q, \alpha(1-p) + (1-\alpha)(1-q)).$$

Hence, we can only focus on simple lotteries. One special and important class of lotteries is money lotteries, whose outcomes are real numbers, i.e., $C = \mathbb{R}$. A money lottery can be characterized by a cumulative distribution function F, where $F : \mathbb{R} \to [0,1]$ is nondecreasing. $F(x)$ is the probability of receiving a

[2]David M. Kreps. (1988). *Notes on the Theory of Choice.* Westview press.

prize less than or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

If an individual has reasonable preferences about consumption in different circumstances, we will be able to use a utility function to describe these preferences just as we do in other contexts. However, the fact that we are considering choice under uncertainty adds some special structures to the choice problem, which we will see below. Historically, the study of individual behavior under uncertainty is originated from attempts to understand (and hopefully to win) games of chance. One may think that the key determinant of behavior under uncertainty is the expected return of the gamble. However, people are generally reluctant to play fair games.

> **Example** (St. Petersburg Paradox). Consider the following gamble: you toss a coin repeatedly until the head comes up. If this happens in the nth toss, the gamble gives a monetary payoff of 2^n. What is the expected return of this game? How much would you pay to play this gamble?

Lottery We call p a lottery, which is an object of the choice by the decision maker. Let P be the collection of all lotteries. Mathematically, P is the unit simplex in $\mathbb{R}^L$ where L is the number of elements in Z. For this reason, we sometimes write $\Delta(Z)$ in place of P to emphasize the relationship between Z and the probability distribution over Z.

Our goal is to explain how a decision maker chooses a particular lottery from the set of feasible lotteries. The main difference from the consumer theory is that the decision maker does not observe $z \in Z$ before he chooses a lottery. In order to examine the decision making process under uncertainty, we need to formulate how a decision maker choose a lottery, or formalize the preference ordering over P.

We have many theories for the decision making under uncertainty. The most prominet theory is the expected utility theory by John von Neumann and Oskar Morgenstern, which was developed as a part of developing the theory of games.

Expected utility Let $\succeq$ be an ordering over P, which represents the decision maker's preference over lotteries. If $p \succeq q$, then we say that p is preferred to q. The only difference from the conventional consumer theory is that p and q are probabilities, rather than attributes (or goods) which the decision maker draw utility.

> **Definition 2.** $\succeq$ is complete if $\forall p, q \in P$, $p \succeq q$ or $q \succeq p$. $\succeq$ is transitive if $\forall p, q, r \in P$, $p \succeq q$ and $q \succeq r$ imply $p \succeq r$. We say that $\succeq$ is a preference ordering if $\succeq$ is complete and transitive.

> **Axiom 1.** $\succeq$ is a preference ordering over P.

This axiom is hardly controversial, although experimental evidence shows that the ordering of a human being is often not complete or not transitive. Throughout this class, we maintain the assumption that $\succeq$ is complete and transitive.

Definition 3. $\forall p, q \in P$, $\forall a \in [0, 1]$, a composite lottery is $ap + (1 - a)q$.

If one interpret $a \in [0, 1]$ as a probability, one can interpret a composite lottery as a lottery over lotteries. One can interpret a as the amount of lottery a in the portfolio. A stock is a lottery, because the value of a stock depends upon the profitability and the market condition, but the decision maker does not observe the true state when he purchases a stock. A mutual fund is a composite lottery.

An important observation is that P is a convex set. Therefore, a composite lottery is an element of P.

The second axiom is called the substitution axiom, the independence axiom or the linearity axiom.

Axiom 2 (Substitution, Independence, Linearity)**.** $\forall p, q, r \in P$, $\forall a \in (0, 1]$, if $p \succeq q$, then $ap + (1 - a)r \succeq aq + (1 - a)r$.

The preference between two composite lotteries is determined by the preference between p and q, independently of r. In that sense, this axiom is called the independence axiom.

Note. If $p \succeq q$, then the preference between the two composite lotteries is independent of the size of a. This is the crucial feature of linear preferences, which this axiom implies.

As important as this assumption is for the expected utility theory, the linearity of the preference has been challenged by many experiments. In reponse, many alternative axioms were proposed. Still, the linearity of the expected utility allows us to use the mathematical expectation to formulate the optimization problem. For this reason, this axiom endures the challenges.

Axiom 3 (Continuity, Archimedian)**.** $\forall p, q, r \in P$, if $p \succeq q \succeq r$, then $\exists a, b \in (0, 1)$ such that $ap + (1 - a)r \succeq q \succeq bp + (1 - b)r$.

This axiom is called the continuity axiom or Archimedian axiom. A key implication is that the utility must be finite. Suppose that the utility from p is infinite. Then, it would be impossible to find $b \in (0, 1)$ to construct a composite lottery so that

$$q \succeq pb + (1 - b)r.$$

Similarly, if you assign $-\infty$ utility to lottery p (such as death with probability 1), then it would be impossible to construct a composite lottery in which the proportion of p is $a \in (0, 1)$ such that

$$ap + (1 - a)r \succeq q.$$

The fundamental theorem by von Neumann and Morgenstern is that we can represent any preference satisfying three axioms by the expected value of a utility.

Theorem 1. $\succeq$ satisfies three axiom if and only if there exists a utility function $u\colon Z \to \mathbb{R}$ such that $\forall p, q \in P$, $p \succeq q$ if and only if

$$\sum_{z \in Z} u(z)p(z) > \sum_{z \in Z} u(z)q(z).$$

Moreover, if u represents $\succeq$, then v represents $\succeq$ if and only if $\exists c > 0$, $\exists d \in \mathbb{R}$ such that $v(z) = cu(z) + d$.

Discussion

- The first part is the existing of utility function u, which measures how much utils the decision maker obtains by consuming $z \in Z$. This utility function is called von Neumann Morgenstern utility function. The probability distribution is used to calculate the (mathematical) expected value of u, which the name expected utility came from.
- We call $f(z) = cz + d$ where $c > 0$ and $z, d \in \mathbb{R}$ an affine function, which is a linear function with a constant term. The second part is the uniqueness result up to the affine transformation. That is, if u represents $\succeq$, its affine transformation represents the same preference.
- In the neoclassical consumer theory, we assumed the ordinal preference, which is represented by a utility function. An important result is that the utility function is unique up to monotonic transformation. If f is a strictly increasing function, and u represents the preference, then $v = f(u)$ represents the same preference. Since f preserves the order of the preference, but not the cardinal value of the utility, we regard this result as the mathematical formulation of the ordinal utility function.
- An affine function is a strictly increasing function, but not vice versa. An affine function does not preserve the cardinal value of the utility, but the uniqueness does not extend to all strictly increasing functions. In this sense, a von Neumann Morgenstern utility function is a cardinal utility.

1.3 Challenges

Experimental challenges Expected utility theory allows us to use the statistical method to formulate and solve the optimization problem to examine the decision making process under uncertainty. The probability enters the decision problem linearly, which simplifies the problem tremendously.

However, expected utility theory has been challenged by experimental data for a long time. Let us discuss three best known examples.

Allais's paradox

Many different versions of the same experiment have been conducted over time. We examine the one by Kahneman and Tversky. Consider the following experiments which consist of two parts.

In the first part, a group of subjects is asked to choose between two lotteries: A and B where

$$A = \begin{cases} 2500 & \text{with probability } 0.33 \\ 2400 & \text{with probability } 0.66 \\ 0 & \text{with probability } 0.01 \end{cases}$$

and

$$B = 2400 \text{ with probability } 1.$$

In the second part, the same group of subjects is asked to choose between two lotteries: C and D where

$$C = \begin{cases} 2500 & \text{with probability } 0.33 \\ 0 & \text{with probability } 0.67 \end{cases}$$

and

$$D = \begin{cases} 2400 & \text{with probability } 0.34 \\ 0 & \text{with probability } 0.66 \end{cases}.$$

At the end of the second round, the experimenter compares the choice by the subjects in the first and the second rounds. The focus is the consistency of the choice across different pairs of options.

82% choose B over A, while 83% chooses C over D, which means at least 65% ($\simeq 82\% \times 83\%$) chooses B over A and C over D.

A plausible heuristic explanation is that between A and B, \$2400 for sure would be better than a little bit of uncertainty, while between C and D, the difference between the size of the prize outweights the difference of the probability.

Whatever the reason might be, the behavior of a substantial portion of subjects is inconsistent with the prediction of the expected utility theory. The inconsistency is due to the violation of the independence axiom.

If the preference of a subject satisfies three axioms, we have

$$u\colon \{2500, 2400, 0\} \to \mathbb{R}.$$

Suppose that $u(2500)$, $u(2400)$, $u(0)$ are the von Neumann Morgenstern utilities of 2500, 2400 and 0 prizes.

If the subject chooses B over A, then

$$u(2400) > 0.33u(2500) + 0.66u(2400) + 0.01u(0)$$

which is equivalent to

$$0.33u(2500) - 0.34u(2400) + 0.1u(0) < 0. \tag{1.1}$$

If the same subject chooses C over D, then

$$0.33u(2500) - 0.34u(2400) + 0.1u(0) > 0. \tag{1.2}$$

But, (1.1) and (1.2) are inconsistent.

The outcome is an evidence of the violation of the independence axiom. Suppose that a subject has a well defined ordering between two lotteries:

$$X = \begin{cases} 2500 & \text{with probability } \frac{33}{34} \\ 0 & \text{with probability } \frac{1}{34} \end{cases}$$

and

$$Y = 2400\text{with probability } 1.$$

Independence axiom says that $Y \succeq X$ if and only if

$$0.34Y + 0.66 \cdot 2400 = B \succeq A = 0.34X + 0.66 \cdot 2400.$$

Similarly, $Y \succeq X$ if and only if

$$0.34Y + 0.66 \cdot 0 = D \succeq C = 0.34X + 0.66 \cdot 0.$$

If a subject chooses B over A but C over D, his preference must violate the independence axiom.

Framing effect (Tversky and Kahneman (1981))

Imagine that the U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume that the exact scientific estimate of the consequences of the program are as follows.

If program A is adopted, 200 people will be save. If program B is adopted, there is 2/3 probability that no one will be saved, and 1/3 probability that 600 people will be saved.

If program C is adopted, 400 people will die with certainty. If program D is adopted, there is 2/3 probability that 600 people will die, and 1/3 probability that no one will die.

- 72% of subjects say $B \preceq A$.
- 78% of subjects say $C \preceq D$.
- Thus, $.72 \times .78 \simeq 50\%$ of subjects say $B \preceq A$ and $C \preceq D$.
- But, $A = C$ and $B = D$.

Ellsberg paradox (1961)

I have an urn with 300 balls in it. Some of the balls are red, some blue and some yellow. All the balls are the same size and weight, and they are not distinguished in any way except in color. I am willing to tell you that precisely 100 of the balls are red. I am unwilling to say how many are blue and how many are yellow, except that, of course, the total number of blue and yellow is 200. I want to know your preferences between gambles based on the outcome of this random event. In all these gambles, you will either win \$1000 or you will win nothing.

Gambles 1

A: Get \$1000 if the ball drawn out is red, and \$0 if it is blue or yellow.

B: Get \$1000 if the ball drawn out is blue, and \$0 if it is red or yellow.

Gambles 2

C: Get \$1000 if the ball drawn out is blue or yellow, and \$0 if it is red.

D: Get \$1000 if the ball drawn out is red or yellow, and \$0 if it is blue.

A typical response is $A \succeq B$ and $C \succeq D$. You know the odd in A, but not in B. You know the odd in C but not in D. What is wrong with this observation? $A \succeq B$ if you think the number of blue balls is less than 100. If so, $D \succeq C$. We typically do not like ambiguous problems.

Chapter 2

Attitude Toward Risk

2.1 Attitude toward risk

Why useful? Despite experimental evidence against the axioms, the expected utility theory is widely used. We can describe and analyze the decision problem using the same mathematical tool to compute expectations. The vNM utility provides a convenient way of formulating the attitude toward risk.

Attitude toward risk In many economic environments, individuals display aversion to risk. We formalize the notion of risk aversion and study some of its properties.

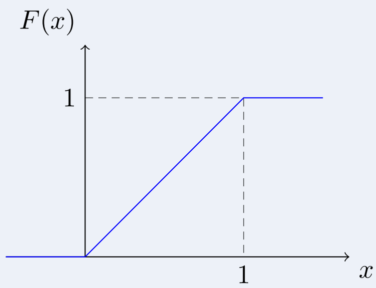
Utility on money We focus on money lotteries, i.e., risky alternatives whose outcomes are amounts of money. It is convenient to treat money as a continuous variable. We have so far assumed a finite number of outcomes to derive the expected utility representation. How to extend this?

It is convenient to assume that $X = [0, \infty)$ is money, and consider a lottery over X. Let u be the vNM utility over X. Any probability distribution over X can be represented by cumulative distribution functions (or cdf) $F\colon \mathbb{R} \to [0, 1]$ where $F(x) = P(x' \leq x)$.

We assume that F is differentiable and $f(x) = F'(x)$, which is known as the density funtion. If X is discrete, $f(x)$ corresponds to the probability of event $x \in X$.

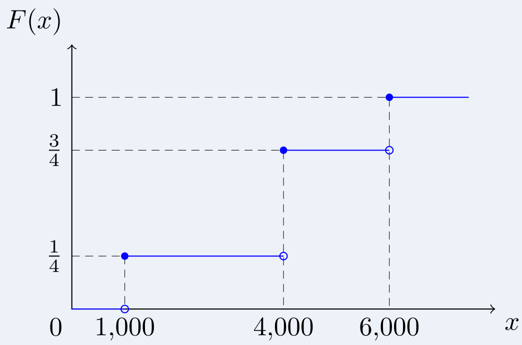
Expected utility framework on monetary outcomes We describe a monetary lottery by means of a cumulative distribution functions $F : \mathbb{R} \to [0, 1]$. $F(x)$ is the probability that the realized payoff is less that or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

Example (Uniform distribution)**.** cdf of uniform distribution $U[0,1]$ is



Example (Discrete distribution)**.**

$$\left.\begin{array}{l}\text{Prob}(1{,}000\text{ won}) = \frac{1}{4}\\ \text{Prob}(4{,}000\text{ won}) = \frac{1}{2}\\ \text{Prob}(6{,}000\text{ won}) = \frac{1}{4}\end{array}\right\} \to F(x) = \begin{cases} 0 & \text{if } x < 1{,}000 \\ \frac{1}{4} & \text{if } 1{,}000 \le x < 4{,}000 \\ \frac{3}{4} & \text{if } 4{,}000 \le x < 6{,}000 \\ 1 & \text{if } x \ge 6{,}000. \end{cases}$$



Expected utility Consider a preference relation $\succsim$ on $\mathcal{L}$. It has an expected utility representation if $F \succsim F' \Leftrightarrow U(F) \ge U(F')$, where

$$U(F) = \int_{-\infty}^{\infty} u(x)\, dF(x)$$

or

$$U(F) = \int_{-\infty}^{\infty} u(x) f(x)\, dx$$

if F is differentiable and $f = dF/dx$.

Note. U is defined on lotteries whereas u is defined on money.

To differentiate the two objects, we often call U the (von Neumann Morgenstern) expected utility function and $u(\cdot)$ the Bernoulli utility function or von Neumann Morgenstern utility of money.

We assume that u is (strictly) increasing, implying that the marginal utility of money is strictly positive, and twice continuously differentiable, for analytic convenience.

Attitude toward risk

Definition 4 (Attitude toward risk)**.** Let u be a utility function defined on money outcomes that represents $\succsim$. We say that $\succsim$ exhibits

$$\begin{pmatrix}\text{risk aversion}\\ \text{risk neutrality}\\ \text{risk loving}\end{pmatrix} \iff \int u(x)\,dF(x) \begin{pmatrix}<\\=\\>\end{pmatrix} u\left(\int x\,dF(x)\right)$$

for all lotteries F.

Equivalently, $\succsim$ exhibits risk aversion if $\mathbb{E}[u(X)] < u(\mathbb{E}[X])$. Notice that if $\succsim$ is risk averse (neutral, loving), then u is concave (linear, convex).

Risk averse decision maker Consider $X = \{x_g, x_b\}$ where $x_g > x_b$. Recall that u shows risk aversion if

$$u(\pi x_b + (1-\pi)x_g) > \pi u(x_b) + (1-\pi)u(x_g).$$

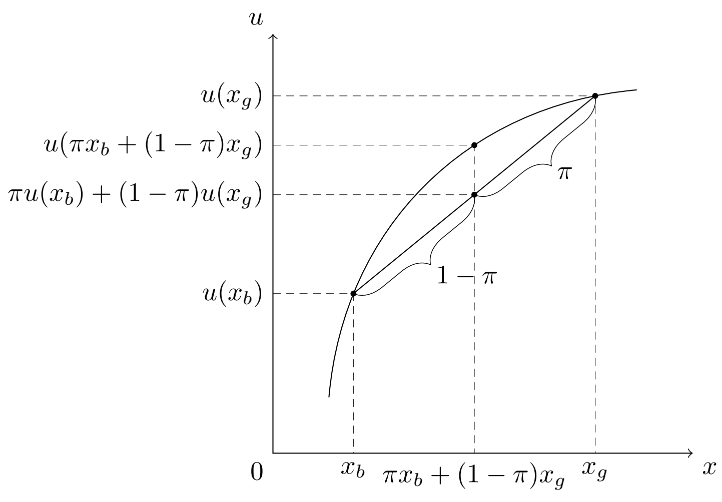


Figure 2.1: Risk aversion

Sure thing and fair gamble If u is concave, Jensen's inequality says

$$\int u(z)\,dF(z) = \mathbb{E}[u(z)] \leq u\left(\int z\,dF(z)\right) = u(\mathbb{E}[z]).$$

The left hand side is the expected utility from the bet whose return z is distributed according to F. The right hand side is the utility from money whose amount is equal to the expected value of the random variable.

Definition 5 (Sure thing)**.** By a sure thing, we mean a deterministic outcome z. A bet is a random variable. A fair bet is a random variable whose expected return is equal to the sure thing.

Risk averse and fair bet Let ε be a random variable whose expected value is 0. Given z^e, a fair bet to z^e is $z^e + \varepsilon$. Let $z^e = \mathbb{E}[z]$, and $\varepsilon = z - z^e$ whose distribution function is G. Then,

$$\int u(z^e + \varepsilon)\, dG(\varepsilon) = \mathbb{E}[u(z)] \leq u\left(\int z\, dF(z)\right) = u(\mathbb{E}[z]) = u(z^e).$$

We often say that u shows risk averse attitude if and only if the decision maker prefers a sure thing over a fair bet.

Risk neutral decision maker If a decision maker is risk neutral, then

$$u(\pi x_b + (1-\pi)x_g) = \pi u(x_b) + (1-\pi)u(x_g).$$

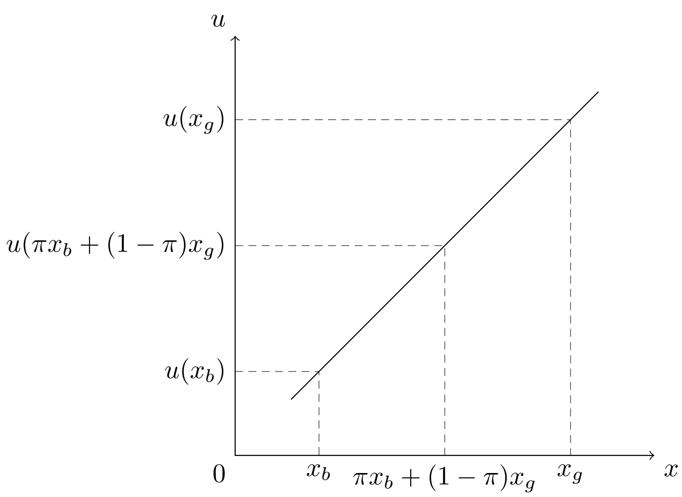


Figure 2.2: Risk neutrality

Risk loving A decision maker is risk loving if

$$u(\pi x_b + (1-\pi)x_g) < \pi u(x_b) + (1-\pi)u(x_g).$$

You may think that only a professional gambler might be risk loving. A policy with a good intention can turn a risk neutral decision maker into a risk loving decision maker.

Credit guarantee Suppose that a firm is a risk neutral decision maker whose Bernouille utility function (or vNM utility) is $u(z) = z$. The firm has a fixed cost D, but the return is a random variable R distributed over $[0, \infty)$. The profit of the firm is $u(R - D) = R - D$ which is a random variable. Note that if $R - D < 0$, then the firm loses money, which may lead to default.

It is not unusual that a government sometimes offers a rescue plan, by covering the loss in the bad state. Suppose that the government covers the loss whenever the firm incurs loss. The subsidy S is therefore, $S = -\min(R - D, 0)$. The firm's utility is now $R - D + S = \max(R - D, 0)$ which is a convex function. The intervention of the government changes the behavior of the firm from a risk neutral decision maker to a risk loving decision maker.

2.2 Certainty equivalent

Alternative ways The expected utility theory provides alternative ways to represent the attitude toward risk other than the shape of the vNM utility is one way. Let us discuss a couple of widely used methods.

Certainty equivalence A risk averse individual prefers a sure thing to a fair gamble. Is there a smaller amount of certain wealth that would be viewed as equivalent to the gamble?

> **Definition 6** (Certainty equivalent). The certainty equivalent (CE) of F is the amount of money for which the individual is indifferent between the gamble F and the certain amount CE; that is, $u(CE) = \int u(x)\, dF(x)$.

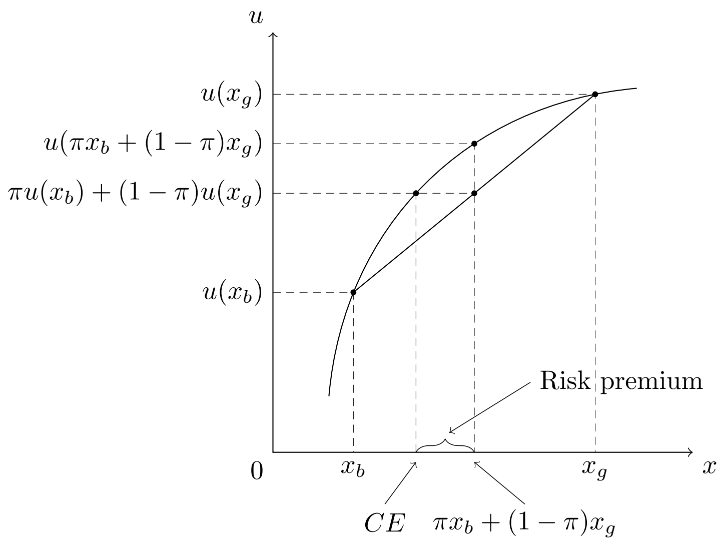


Figure 2.3: Certainty equivalent and risk premium

Note. u is concave if and only if $CE < \pi x_b + (1-\pi)x_g$.

If a risk averse decision maker is offered two options: CE and $\pi x_b+(1-\pi)x_g$, then he will accept the expected return.

This behavior provides an alternative way to represents the attitude toward risk.

Definition 7 (Risk premium). The risk premium (RP) associated with F is the maximum amount of money an individual is prepared to pay to avoid the game: $\mathbb{E}[u(X)] = u(\mathbb{E}[X] - RP)$. Clearly, $RP = \mathbb{E}[X] - CE$.

Theorem 2. u exhibits risk aversion if and only if $RP \geq 0$.

2.3 Measurement of Risk Aversion

We sometimes have to rank two decision makers according to their attitude toward risk by saying that a decision maker is more risk averse than the other. Intuitively, the more concave the utility function, the more risk averse the consumer. Thus, the second derivative of u is a natural candidate for the measure risk aversion.

Recall that vNM utility is invariant with respect to affine transformation. Thus, if we change u by $\alpha u + \beta$ for some $\alpha > 0$, the attitude toward risk does not change. The problem of u'' as the measure of the risk aversion is that it is not invariant with respect to the affine transformation.

As an example, consider a decision maker with $v(\cdot) = 2u(\cdot)$, who has the same preference over the bet as the decision maker with u. But, $v''(\cdot) = 2u''(\cdot) \neq u''(\cdot)$.

Definition 8 (Arrow-Pratt measure of absolute risk aversion).

$$r_A(x, u) := -\frac{u''(x)}{u'(x)}$$

The idea of constructing r_A is intuitive. We normalize the degree of concavity by u' so that the measure is invariant with respect to affine transformation. More precisely,

$$-\frac{u''}{u'} = -\frac{du'/dx}{u'} = -\frac{du'/u'}{dx} = -\frac{\%\text{ change in MU}}{\text{absolute change in } x}.$$

$r_A(x)$ is positive, negative, or zero as the agent is risk averse, risk loving, or risk neutral.

Another interpretation Let us consider two outcomes: bad outcome $x_b = w + r_b z$ and good outcome $x_g = w + r_g z$. Draw indifferent cure:

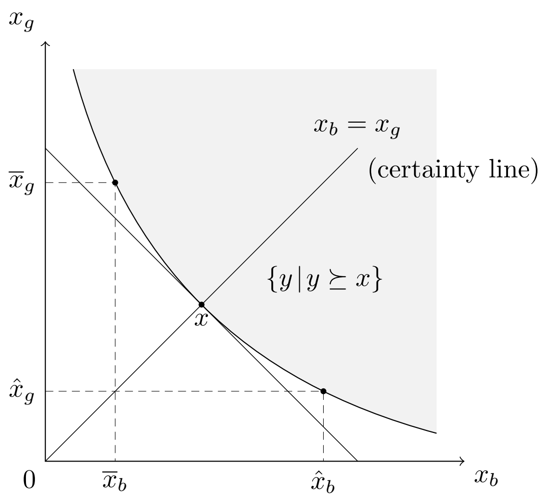
$$\pi u(x_b) + (1-\pi)u(x_g) \equiv \overline{u}.$$

By totally differentiating both sides, we implicitly derive the marginal rate of substitution

$$\pi u'(x_b) + (1-\pi)u'(x_g)\frac{dx_g}{dx_b} = 0. \tag{2.1}$$

Hence, the marginal rate of substitution (MRS) is

$$\frac{dx_g}{dx_b} = -\frac{\pi}{1-\pi}\frac{u'(x_b)}{u'(x_g)}. \tag{2.2}$$



$$\left|\frac{dx_g}{dx_b}\right| \begin{pmatrix} = \\ < \\ > \end{pmatrix} \frac{\pi}{1-\pi} \text{ when } x_b \begin{pmatrix} = \\ > \\ < \end{pmatrix} x_g, \text{ showing that } u(\cdot) \text{ is concave.}$$

Define the consumer's preferred set at x to be the set of all outcome the consumer will prefer to x, i.e., $\{y \,|\, y \succeq x\}$.

Suppose now we have two consumers, i and j. It is natural to say that consumer i is (locally) more risk averse than consumer j if consumer i 's preferred set at x is contained in j's one. Consumer i's indifferent curve is more curved than consumer j's one at x.

Differentiate (2.1) one more with respect to x_b,

$$\pi u''(x_b) + (1-\pi)u''(x_g)\left(\frac{dx_g}{dx_b}\right)\left(\frac{dx_g}{dx_b}\right) + (1-\pi)u'(x_g)\left(\frac{d^2 x_g}{dx_b^2}\right) = 0.$$

Using (2.2), we have

$$\frac{d^2 x_g}{dx_b^2} = \frac{\pi}{(1-\pi)^2}\left[-\frac{u''(x)}{u'(x)}\right] \text{ when } x_b = x_g = x.$$

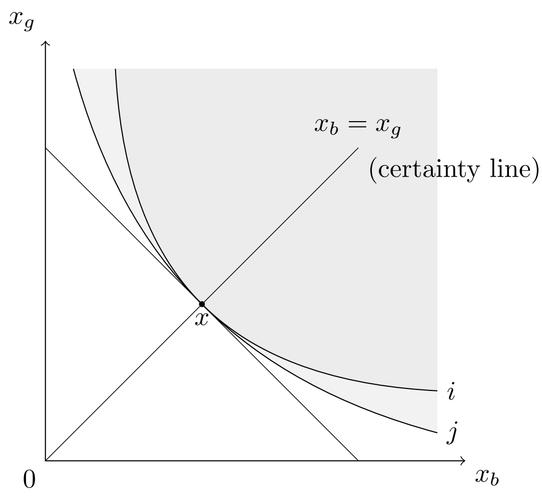


Figure 2.4: Arrow-Pratt measure of absolute risk aversion

Comparison of attitude toward risk Given two utility functions $u_i(\cdot)$ and $u_j(\cdot)$, when can we say that $u_i(\cdot)$ is more risk averse than $u_j(\cdot)$?

Theorem 3. TFAE.

1. Decision maker i is more risk averse than j.
2. $r_A(x, u_i) \geq r_A(x, u_j)$ for all x. That is, consumer i has a higher degree of risk aversion than consumer j everywhere.
3. There exists an increasing concave function $\psi(\cdot)$ such that $u_i(x) = \psi(u_j(x))$ for all x. In other words, $u_i(\cdot)$ is more concave than $u_j(\cdot)$.
4. $CE_i \leq CE_j$ (or $RP_i \geq RP_j$, i.e., i would be willing to pay more to avoid a given risk than j would.)

Chapter 3

Hidden Information

3.1 Economy with uncertainty

State contingent claim We have learned the equilibrium model under certainty, where a decision maker knows all characteristics of the goods, and the state at the time when he makes a decision. As we move from a model with certainty to uncertainty, we had to develop a new way of evaluating an object, a lottery, over the set of commodities. A fundamental question is whether the presence of uncertainty changes the equilibrium allocation of the competitive market.

Arrow and Debreu showed the condition under which the presence of uncertainty does not matter. We can apply exactly the same exercise as we learned from the model with certainty. More importantly, the first and the second welfare theorems continue to hold.

Complete market This condition is called the complete market hypothesis: each commodity has a market where it can be traded. In order to make the notion of complete market precise, Arrow invented the notion of contingent commodity.

We first state the model of competitive market satisfying the complete market hypothesis. We do so, because it provides an important benchmark against which an economy with incomplete market is examined, providing a fundamental insight into the role of uncertainty to the equilibrium outcome of the market.

Uncertainty By a state, we mean any factor that affects the decision of an economic agent. The quality of a product is a good example, which may or may not be known to the decision maker at the time of his decision. Let S be the set of states, and $s \in S$ be a generic element. Let us assume that S is finite.

Definition 9. The economy is subject to uncertainty, if a state is not revealed to a decision maker at the time of his decision making.

A lottery is one of the examples. The value of the lottery is a state, which is not revealed at the time when a decision maker purchases a lottery at a certain price.

Let us consider a finite exchange economy, which is populated by I consumers endowed with neoclassical utility function, L commodities and S states, each of which has $\#I$, $\#L$ and $\#S$ elements.

We start with the description of the initial endowment, which is a complete specification of endowment for all possible contingencies. Let $\omega = (\omega_\ell)_{\ell \in L} \in \mathbb{R}^{\#L}$ be the profile of commodities. Because his endowment is affected by state $s \in S$, we need to spell out the profile of endowments for all states in S. Thus, the endowment of agent i is

$$\omega_i = (\omega_{s,i}) \in \mathbb{R}^{\#L \times \#S}.$$

Commodity We differentiate a commodity by state. This is a fundamental innovation of Arrow and Debreu. That is, a commodity is differentiated by an attribute which is relevant to the decision of an economic agent, including a state. For example, an umbrella when it rains is a different commodity from an umbrella when it shines. A stock when the firm is generating a large profit is a different commodity form a stock when the same firm is bankrupted.

Contract Another important interpretation of a commodity by Arrow and Debreu is that a commodity is a contract which promises to deliver the same physical goods if such state arises.

Depending upon a model, it is more convenient to regard an umbrella in rain as a contract which promises to deliver one unit of physical umbrella when it rains. certainly, this contract carries a price. Similarly, a contract that promises to deliver one stock when the firm is generating a large profit is different from a contract that promises to deliver one stock when the firm is bankrupted. The two commodities would carry different prices.

We can measure the quantity of a commodity according to a basic unit, known as contigent commodity.

Definition 10. A contingent commodity (or Arrow security) is a contract to deliver one unit of a good if a particular state is realized.

As we expand the economy by incorporating uncertainty, we are expanding the notion of commodity from a profile of physical characteristics, For each of L commodities, we have to consider as many as $\#S$ contingent commodities. As a result, we have $\#L \times \#S$ commodities.

Let x_{ls} be the contract to deliver x_l units of l-th good if state s occurs.

The collection of commodities is now $\mathbb{R}^{\#L\times\#S}$. Let $\succeq_i$ be the preference over $\mathbb{R}^{\#L\times\#S}$.

Example. Suppose that π_s is the probability that state s is realized, and $u_i(x,s)$ is the utility of $x \in \mathbb{R}^{\#L}$ under state s. Let $x = (x_s)_{s\in S} \in \mathbb{R}^{\#L\times\#S}$ and $x' = (x'_s)_{s\in S} \in \mathbb{R}^{\#L\times\#S}$ be a pair of state contingent commodity bundles. Then, $x \succ x'$ if

$$\sum_{x\in S} u(x_s, s)\pi_s > \sum_{x\in S} u(x'_s, s)\pi_s.$$

It is important to know that the expected utility is one of many different ways to evaluate the state contingent commodity bundles.

Sequence of moves

1. Before a state $s \in S$ is realized, there is a market to trade commodities.
2. After trading commodities (or contracts), a state is realized.
3. The good is delivered according to the contract.
4. Goods are consumed, and utility is generated.

The first and third steps warrant a careful examination.

Forward market A commodity is trade in a market, before a state is realized. Thus, is is more convenient to interpret a commodity as a contingent contract, which promises to deliver the specific amount of goods if a state arises.

In the sense that the contingent contract is traded in a forward market, the contract is often called a forward contract.

Enforcement When the contract is traded, a buyer of the commodity pays money to the seller, and receives a piece paper with a promise on it. The good is not delivered, until a state is realized.

An important assumption is that the contract is enforced without any exception, or the seller is committed to carry out the contract. If the enforcement is not complete, or if the seller has a limited commitment, then the contract may not be traded, or will fetch a lower price than under the full commitment.

For example, a debt contract is a promise that a borrower will pay back the principal and the interest back to the lender by a specific time. In Arrow Debreu economy, a debt contract will be enforced without any exception.

Symmetric information At the time when the good is traded, no agent in the economy observes the state. Every decision maker faces uncertainty. In this sense, uncertainty is symmetric.

Complete market hypothesis A fundamental assumption of Arrow Debre economy is that every contingent commodity has a market where it can be traded. This assumption is called the complete market hypothesis. Because a forward contract is traded, we sometimes say that Arrow Debre economy assumes a complete set of forward markets.

With a complete set of markets, and with full commitment, we can follow exactly the same analysis as for the economy with certainty to establish the first and the second welfare theorems. Any failure of the fundamental welfare theorems can be traced back to the missing market.

3.2 Informational efficiency

Symmetric vs. asymmetric information

- An economy with uncertainty is subject to a state which is not revealed to an agent at the time of decision.
- If no agent observes a state, the economy is subject to uncertainty, but the uncertainty is symmetric.
- If an agent observe a state, but another agent does not observe the same state, asymmetric information exists.

Rational expectations Presence of asymmetric information does not necessarily lead to inefficient allocation, as the competitive market can aggregate dispersed information into the market clearing price. Fredrick von Hayek called this property informational efficiency of competitive market.[1]

Information aggregation Let us consider an exchange economy with uncertainty with two consumers with identical utility function:

$$u_i(x_{1,i}, x_{2,i}) = \beta \ln x_{1,i} + x_{2,i}$$

where $\beta \in \{1, 2\}$ with probability $P(\beta = 1) = 0.5$. Assuming that the second good is a numeraire, let p be the price of the first commodity. Each consumer has 1 unit of the first good as an initial endowment, which makes the aggregate supply of the first good 2 units.

[1] Friedrich Hayek. (September 1945). The use of knowledge in society. *The American economic review*, 35(4), 519-530.

Suppose that neither agent observes the actual realized value of $\beta \in \{1, 2\}$ at the time of deciding the demand. That is, uncertainty is symmetric.

The demand for the first good should equate the expected marginal utility of the first good to the market clearing price p:

$$\frac{1}{2}\left[\frac{1}{x_{1,i}} + \frac{2}{x_{1,i}}\right] = p$$

and the uniformed demand of consumer i would be

$$x_{1,i}(p) = \frac{3}{2p} \quad \forall i.$$

The market clearing price would be

$$x_{1,1}(p) + x_{1,2}(p) = 2$$

implying $p = \frac{3}{2}$.

If consumer i observes the true state $\beta \in \{1, 2\}$, then his demand is conditioned on β and p. A simple calculation shows consumer i's demand for good 1 at price p in state β is

$$x_{1,i}(p; \beta) = \frac{\beta}{p}.$$

The aggregate demand in state β is therefore

$$x_{1,1}(p; \beta) + x_{1,2}(p; \beta) = \frac{2\beta}{p}$$

and the market clearing price in state β is $p = \beta$.

Note. The market price is one-to-one correspondence to the underlying state. If a rational agent outside of the market observes the market price, he can infer the underlying state.

Asymmetric information Suppose that consumer 1 observes the actual state, while consumer 2 does not. Because consumer 1 observes the true state, his demand is conditioned on β:

$$x_{1,1}(p; \beta) = \frac{\beta}{p} \quad \forall \beta \in \{1, 2\}$$

while consumer 2's demand is independent of β:

$$x_{1,2}(p) = \frac{3}{2p}.$$

The market clearing price solves

$$\frac{\beta}{p} + \frac{3}{2p} = 2$$

implying

$$p = \frac{1}{2}\left[\beta + \frac{3}{2}\right].$$

Note. The market clearing price is one-to-one correspondence to the state.

Rational expectations If consumer 2 is rational, he can infer the true state from the market clearing price. Consumer 2's decision is not optimal, because his belief does not incorporate all information available.

If he is rational, he should be able to infer the underlying state, and should behave as if he observes the underlying state. As a result, the equilibrium price must be fully revealing. The market clearing price must be $p = \beta$.

Private information of consumer 1 is aggregated into the market clearing price so that all consumers in the economy can behave optimally for each state. This property is known as informational efficiency of competitive market. We can trace back the idea to Wealth of Nation by Adam Smith, but Frederick von Hayek is generally credited for the idea of informational efficiency.

Chapter 4

Lemon's Problem

4.1 Lemon's problem

Symmetric vs. asymmetric information If an agent observes the underlying state, while another agent does not, we say that asymmetric information exists. The economic impact of asymmetric information was first demonstrated by an example of used cars in Akerlof [1970].

At the time, the profession believes in the informational efficiency of the competitive market. He constructed two simple examples of markets with asymmetric information, which completely changed the way how we understand the role of asymmetric information.

Lemon's market By a lemon, we mean a used car which has a low quality but cannot be differentiated from a good quality car. A buyer can see a used car, but cannot tell whether is a good or bad quality used car, before paying for the car, if he chooses to buy one. The seller observes the true quality of the used car, before the car is put on the market. Asymmetric information exists in the market.

The state is $S = \{H, L\}$. If the state is H, the quality is ϕ_H and the outside option for the high quality used car is c_H. Similarly, if the state is L, the quality is ϕ_L and the outside option for the low quality used car is C_L which we normalized to be 0.

We assume that $\phi_H > \phi_L$ which is the utility from consuming the used car. A high quality car generates more utility than a lower quality car. Similarly, we assume $c_H > c_L = 0$. The outside option of a high quality used car is higher than that of a low quality used car. One can also interpret c_H and c_L as the cost for the good and bad used cars. In order to have a good used car, the seller should spend more money to keep the car in good condition.

Key assumptions A model with lemons problem satisfies the following three conditions. Let $\pi_H = P(s = H)$ be the probability that the true state is H.

1. Gains from trading: $\phi_H > c_H > \phi_L > c_L$.
2. Single crossing property: $\phi_H - c_H > \phi_L - c_L$.
3. Severe lemon's problem: $\pi_H \phi_H + (1 - \pi_H)\phi_L < c_H$.

Gains from trading If $\phi_H > c_H > \phi_L > c_L$, then the gain from trading is positive in each state. If the true state is H, the gain from trading is $\phi_H - c_H > 0$, and if the true state is L, the gain from trading is $\phi_L - c_L > 0$. Every agent in the economy knows that the gain from trading is always positive.

Single crossing property The gain from trading increases in a high state:

$$\phi_H - c_H > \phi_L - c_L.$$

In order to achieve an efficient allocation, it is necessary that trading occurs in the high state with a positive probability.

Severe lemon's problem In an economy with a complete set of markets, the used car should fetch a price equal to its utility ϕ_s. Before the state is revealed, the market clearing price, if one exists, must be equal to the average quality $\pi_H \phi_H + (1 - \pi_H)\phi_L$. If the market price is lower than c_H, then an owner of a high quality used car would not put the car on sale, because he can fetch a higher price from the outside source, or he cannot recover the cost c_H.

Lemon's market

Theorem 4. The market clearing price is ϕ_L, and only the low quality product is traded.

Proof. We show that ϕ_L is the only possible market clearing price. Suppose that p is the

- $p > \phi_H$ is not possible, because no consumer will buy a used car whose quality cannot exceed ϕ_H.
- $c_H \leq p \leq \phi_H$. Since $p \geq c_H > c_L$, all low quality sellers will put their low quality cars in the market. As a result, the average quality of a used car in the market cannot be more than $\pi_H \phi_H + (1-\pi_H)\phi_L$ which is strictly less than c_H by the last assumption. Thus, no high quality used car wil lbe on the market, which implies that the quality of the used car is exactly $\phi_L < c_H$ by the first assumption. Since $c_H \leq p$, no buyer will pay p to buy a used car with quality $\phi_L < c_H \leq p$. Hence, p cannot be an equilibrium price.

- $\phi_L < p < c_H$. Since $p < c_H$, only the low quality car will be in the market. No buyer is willing to pay a price more than ϕ_L. Thus, p cannot be an equilibrium price.
- $p < \phi_L$. Because buyers compete for a used car whose utility is ϕ_L, the market experiences excess demand.

If $p = \phi_L$, only the low quality used car will be on the market and a buyer is willing to pay for his utility for the car. □

Discussion

- The equilibrium allocation is evidently inefficient, because no high quality used car will be traded.
- A surprising part is that the high quality good is driven out of the market, even though the gain from trading is larger and every agent in the economy knows the existence of the positive gain from trading. The logic behind Grasham's law is exactly the same as the lemon's market.
- The nature of uncertainty should be noted. The quality of the used car determines the cost of the seller and the utility of the buyer. In this sense, the quality of the car is the common value of the two players. One of the key components of the lemon's problem is that the seller has private information about the common value.
- The lemon's problem arises in many different cases of asymmetric information over the common value components. Because the lemon's problem leads to an inefficient allocation, it has become a fundamental challenge for economist to find a way to alleviate the implications of the lemon's problem.

Continuous distribution The lemon's problem persists even if we admit more than two types. For example, suppose that the quality is distributed according to continuous density function f over interval $[\phi_l, \phi_h]$. Let $c(\phi)$ be the cost (or the outside option) of the used car with quality ϕ. Assume that $c(\phi)$ is a strictly increasing continuously differentiable function, and $c(\phi) < \phi$ so that there is gain from trading regardless of the quality of a used car. We assume that

$$c(\phi_h) > \mathbb{E}[\phi] = \int_{\phi_l}^{\phi_h} \phi f(\phi)\, d\phi$$

which implies that if the market clearing price is slightly higher than the average quality, the highest quality used car owner would not put his car on the market.

Second example of Akerlof

Theorem 5. If the market clearing price is determined according to the average quality of the products in the market, then the lemon's problem arises and the only equilibrium price is ϕ_l.

Proof. Let p be an equilibrium price. Since the average utility of the products determines the market clearing price,

$$p \leq \mathbb{E}[\phi] = \int_{\phi_l}^{\phi_h} \phi f(\phi)\, d\phi.$$

$c(\phi_h) > \mathbb{E}[\phi]$ and c is a continuous function. $\exists \varepsilon_1$ such that $\forall \phi \in (\phi_h - \varepsilon_1, \phi_h]$ will not put the product in the market since $c(\phi) > \mathbb{E}[\phi]$, where

$$c(\phi_h - \varepsilon_1) = \mathbb{E}[\phi].$$

Then, the aerage expected price cannot be higher than

$$p \leq \mathbb{E}[\phi \,|\, \phi \leq \phi_h - \varepsilon_1] = \int_{\phi_l}^{\phi_h - \varepsilon_1} \phi f(\phi \,|\, \phi \leq \phi_h - \varepsilon_1)\, d\phi.$$

If we iterate the same process for n rounds, we have ε_n so that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) = \mathbb{E}\left[\phi \,\middle|\, \phi \leq \phi_h - \sum_{k=1}^{n-1} \varepsilon_k\right].$$

By applying the same logic, we conclude that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) < \mathbb{E}\left[\phi \,\middle|\, \phi \leq \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

Since $c(\cdot)$ is continuous, $\exists \varepsilon_{n+1} > 0$ so that

$$c\left(\phi_h - \sum_{k=1}^{n+1} \varepsilon_k\right) < \mathbb{E}\left[\phi \,\middle|\, \phi \leq \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

This process continues as long as

$$\phi_h - \sum_{k=1}^{n+1} \varepsilon_k > \phi_l.$$

Thus, ϕ_l is the only equilibrium price. □

Discussion

- In case of a continuous distribution, the conclusion is even more pessimistic than what the discrete example says. In the first example of the discrete model, we expect that a low quality car will be traded at ϕ_l, whose mass is as much as $1 - \pi_h > 0$.

- In case of a continuous distribution, the mass of ϕ_l quality car is infinitesimal. With probability 1, no trading occurs. The market collapses.
- Most students probably have heard Grasham's law, saying that a low quality gold coin drives out a high quality gold coin. The underlying logic is identical with the lemon's problem.

4.2 Extension

Endogenous trigger From the first two examples, one might conclude that the lemon's problem arises because of the parameters of the models are assumed in a specific way. The next example is to show that the lemon's problem can be triggered endogenously through the optimization behavior of an agent.

Merger A corporate raider is trying to buy a firm. Let π be the profit of the firm under the present management, which is distributed uniformed over $[0, 1]$. Under the new management, the profit will be $1.5\pi\ \forall\pi \in [0, 1]$.

In the first period, the raider makes a tender offer p. By the end of the first round, the profit π is realized, and observed by the present manager, but not by the raider. Conditioned on π, the manager decides to weather to accept or reject the tender offer p.

If the management accepts the offer, the management receives p, and the firm will be under the new management appointed by the raider. The profit will be the new profit minus the cost of taking over the firm: $1.5\pi - p$. The management receives the tender offer p, by giving away the firm. If the management rejects the offer, the management receives π and the corporate raider receives 0.

Calculation of an equilibrium We can solve the problem backward, from the optimization problem of the management, who has to decide whether to accept or reject p, conditioned on the realized profit of π. The management accept p only if $p \geq \pi$.

We next calculate the expected return to the raider from purchasing the firm at price p. Since the raider does not observe π, he has to infer the value of π from the response of the management to his tender offer p. To purchase the firm, the management must accept the offer which happens only if $p \geq \pi$. Thus, the expected profit from purchasing the firm at p should be

Chapter 5

Primer of Information Economics

5.1 Review

Compete market

- Arrow-Debreu economy presumes a compete set of markets so that each commodity can be trade at a market clearing price.
- Without market, externality prevails and the first welfare theorem fails.
- All market failure can be traced back to the absence of a market.
- Inefficiency in the lemon's market can be explained by the absence of a market for information.

Market for information Creating a market for information is extremely difficult.

5.2 Definitions

5.3 Baseline model

5.4 Signaling

Chapter 6

Screening Problem

6.1 Screening

Firm To escape from the lemon's problem, or to prevent the lower productive workers from entering the employee pool, a firm uses a mechanism design to screen lower quality workers.

The firm has to rely on the difference of the marginal rate of substitution between the wage and the education to screen out one from another group.

Assumptions We maintain the same assumptions on the utility function of the worker, and the firm. Let us summarize the assumptions.

$$u(\theta, w, e) = w - c(e, \theta),\ c(0, \theta) = 0,\ \frac{\partial c(e, \theta)}{\partial e} > 0,\ \frac{\partial^2 c(e, \theta)}{\partial e^2} > 0,$$

and

$$\frac{\partial c(e, \theta)}{\partial \theta} < 0,\ \frac{\partial^2 c(e, \theta)}{\partial e \partial \theta} < 0.$$

Interpretation We continue to assume that the education is only to generate disutility of the workers. We can regard education as (unpleasant) task which must be completed in return for the job (and wage).

First best solution It is easy to see that if productivity θ_i is known to the firm, the firm has to pay for the productivity, without any unpleasant task in an efficient allocation.

Proposition 1. Suppose that the worker's ability is public information. Then,

$$(w_i^*, e_i^*) = (\theta_i, 0) \quad \forall i \in \{h, l\}.$$

and the firms obtain 0 profit.

Proof. Since θ_i is known to the firm, it is easy to see that the wage must be equal to the productivity. Since the only function of the unpleasant task is to generate disutility on the part of the worker, no unpleasant task should be imposed in an efficient allocation (the first best solution).

The difficult part is to show that the firms cannot entertain positive profit. Let us assume that there are two firms competing each other as the Bertrand competitor. The case of the multiple firms can be analyzed in the same way.

Let Π_k be the profit of firm k. Define $\Pi = Pi_1 + \Pi_2$. Suppose that $\Pi > 0$. Since all firms are identical, we can assume without loss of generality that

$$\Pi_1 \leq \frac{\Pi}{2}.$$

Suppose that firm 1 offers $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$ instead of (w_h^*, e_h^*) and (w_l^*, e_l^*). Since (w_l^*, e_l^*) is an equilibrium for low productive workers, the incentive compatibility condition of θ_l worker must satisfy:

$$w_h^* - c(\theta_l, e_h^*) \leq w_l^* - c(\theta_l, e_l^*).$$

By adding equal amount of ε on both sides, we know that $(w_l^* + \varepsilon, e_l^*)$ is also satisfying the incentive compatibility constraint:

$$(w_h^* + \varepsilon) - c(\theta_l, e_h^*) \leq (w_l^* + \varepsilon) - c(\theta_l, e_l^*).$$

By applying the same logic to θ_h worker, we also conclude that $(w_h^* + \varepsilon, e_h^*)$ is incentive compatible. Choose $\varepsilon > 0$ sufficiently small so that

$$\Pi - \varepsilon > \frac{\Pi}{2} > 0.$$

If firm 1 offers menu of contracts of $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$, then all θ_h workers will take $(w_l^* + \varepsilon, e_l^*)$, thus generating profit of $\Pi - \varepsilon$ for firm 1. By assumption,

$$\Pi - \varepsilon > \frac{\Pi}{2} \geq \Pi_1$$

which implies that Π_1 is no longer an equilibrium payoff. This is a contradiction to the hypothesis that Π_1 is an equilibrium profit. □

Asymmetric information Suppose that the workers observe their productivity, but no firms observe the productivity of workers. Akerlof indicated that the market is exposed to the lemon's problem.

In contrast to the signaling model of Spence [1973] in which the workers make move, Rothschild and Stiglitz [1976] demonstrated that the uniformed firms can design a menu of contract which allows the firm to escape from the lemon's problem.

Key concepts A contract is (w, e), which sepcifies wage w and the level $e \geq 0$ of task associated with the job. A menu of contracts is the list of state contingent contracts:

$$M = ((w_h, e_h), (w_l, e_l))$$

where (w_i, e_i) is supposed to be accepted by θ_i workers.

Definition 11 (Equilibrium). (M^1, M^2) is an equilibrium if $\forall i$, M^i is a menu of feasible contracts (satisfying incentive compatibility constraint), where

$$w_j^i - c^i(\theta_j, e_j^i) \geq 0 \quad \forall i \in \{1, 2\}, \forall j \in \{h, l\}$$

and M^i is a best response against $M^{i'}$ among all possible contracts of firm i $\forall i \neq i' \in \{1, 2\}$.

We will focus on symmetric equilibrium where the two firms offer identical menus: $M^1 = M^2$, thus dropping the superscript to simplify analysis.

Main conclusions Let us summarize the main findings of Rothschild and Stiglitz [1976].

Theorem 6.
1. In any equilibrium, firm's profit is 0.
2. No pooling equilibrium exists.
3. If a separating equilibrium exists, (w_l, e_l) and (w_h, e_h) satisfy

$$w_l = \theta_l,\ e_l = 0;\ w_h = \theta_h$$

and e_h is defined implicitly by the incentive compatibility constraint of θ_l worker:

$$\theta_h - c(\theta_l, e_h) = \theta_l - c(\theta_l, 0).$$

Proof. We prove the main conclusions in multiple steps, which reveals how the hidden information affects the incentive of the workers, and how the firm can exploit the worker's incentive to screen out different workers.

We follow the same logic as in the previous classes to show that the equilibrium profit of the firm must be 0.

Lemma 1. In any equilibrium, each firm receive 0 profit.

In Spence [1973], the single crossing property allows the high productivity worker to signal his productivity credibly to the firm, to fetch a wage equal to his true productivity. In Rothschild and Stiglitz [1976], the single crossing property of the worker's utility allows the firm to screen them out.

Lemma 2. No pooling equilibrium exists. That is, if $((w_h, e_h), (w_l, e_l))$ is an equilibrium menu, then $(w_h, e_h) \neq (w_l, e_l)$.

If $(w_h, e_h) \neq (w_l, e_l)$, we say it is a separating equilibrium. In a separating equilibrium, (w_j, e_j) is accepted only by θ_j worker $\forall j \in \{h, l\}$. Since the firm makes at least 0 profit, the wage must be equal to the productivity.

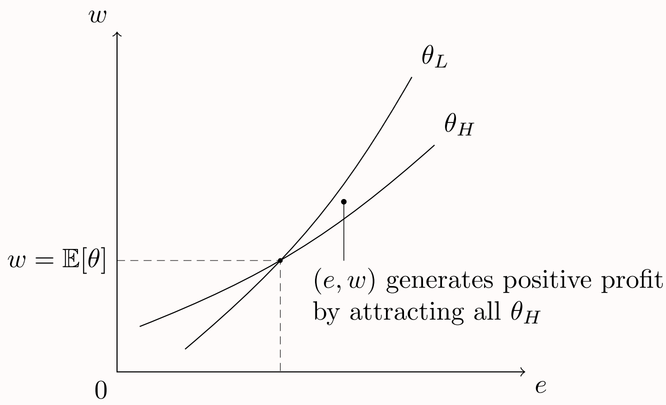


Figure 6.1: No pooling equilibrium exists

Lemma 3. In a separating equilibrium, $w_i = \theta_i \ \forall i \in \{h, l\}$.

In a separating equilibrium, the competitive pressure forces each firm to offer $e_l = 0$ so that the low productivity workers endure no unpleasant task. This result is similar to the property of the signaling equilibrium of Spence [1973] where the low productivity worker does not take any (unpleasant) education. The difference is that in Spence [1973], the decision by the worker is motivated completely by the negative payoff of taking education, while in Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer no task for low productivity workers.

Lemma 4. In a separating equilibrium, $e_l = 0$.

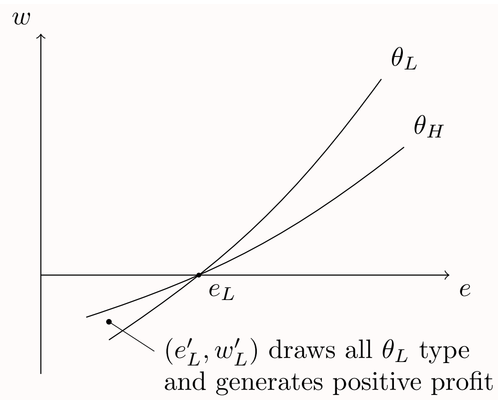


Figure 6.2: $e_L = 0$ in a separating equilibrium

To be a feasible menu, the incentive compatibility condition must hold in a separating equilibrium $((w_h, e_h), (w_l, e_l))$.

$$\theta_h - c(\theta_l, e_h) \leq \theta_l - c(\theta_l, e_l).$$

In Spence [1973], there are multiple signaling equilibria where the weak inequality holds strictly. Only in the Riley outcome, the weak inequality holds with inequality. In Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer a menu in which the incentive compatibility constraint is binding (i.e., the weak inequality holds with equality).

Lemma 5. In a separating equilibrium $((w_h, e_h), (w_l, e_l))$,

$$\theta_h - c(\theta_l, e_h) = \theta_l - c(\theta_l, e_l).$$

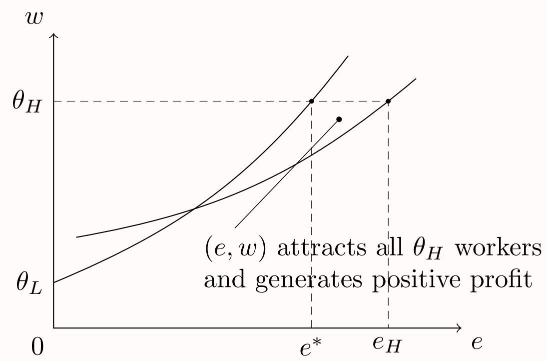


Figure 6.3: $e_H = e^*$ in a separating equilibrium

Existence A separating equilibrium may not exists in Rothschild and Stiglitz [1976]. Because no pooling equilibrium exists, no equilibrium exists if a separating equilibrium fails to exists in Rothschild and Stiglitz [1976].

The conditions under which no equilibrium exists in Rothschild and Stiglitz [1976] further reveals the close relationship between Spence [1973] and in Rothschild and Stiglitz [1976].

Let us consider the Riley outcome, which is the best possible signaling equilibrium for the workers. Because of the disutility of education, a pooling equilibrium can generate higher (ex ante) profit for workers.

As it turns out, Rothschild and Stiglitz [1976] fails to have an equilibrium, if and only if a pooling equilibrium generates higher (ex ante) profit than the Riley outcome in Spence [1973].

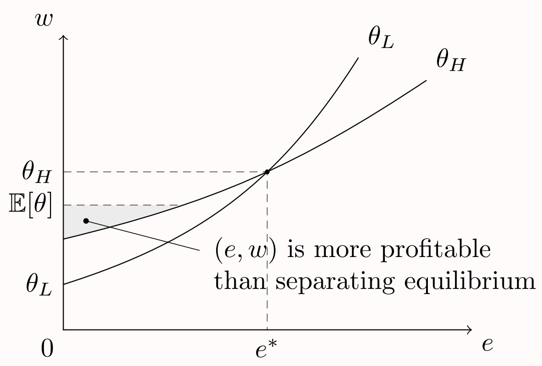


Figure 6.4: No separating equilibrium exists

□

Note. Rothschild and Stiglitz [1976] did not establish the existence of a separating equilibrium. Later studies modified the original model of Rothschild and Stiglitz [1976] to ensure the existence of an equilibrium for all parameter values.

6.2 Hidden action

Definition 12 (Moral hazard problem)**.** An incentive problem arising from hidden action is called the moral hazard problem.

Arrow [1963] observed the moral hazard problem in the health care industry, and first treated it as an incentive problem rather than a moral problem. His observation is ground breaking. If it is a moral problem, an economist has little to say about the remedy. If the problem arises as a consequence of incentive of a rational agent, an economist can propose an alternative rule of the game to guide the incentive to generate more efficient outcome.

Principal agent problem The moral hazard problem is a core concept in the contract literature, which is too extensive for us to cover in this class. Instead, we examine a textbook example of a principal agent problem, which is a classic case of contract subject to moral hazard problem.

Story Consider a risk neutral principal who has to write a contract to ask a risk averse agent to produce an object. A share cropping is an old example of the principal agent problem, where the landlord (the principal) rents out a plot of land so that a tenant farmer produce grain to generate profit.

The profit is then to

Useful observation The optimization problem is complex, because the principal is maximizing over the set of wage schedules instead of wages. We make two observations.

Lemma 6. If $(e^*, w^*(\pi))$ is an optimal solution, then the individual rationality constraint must be binding:

$$\int v(w^*(\pi))f(\pi|e^*)\,d\pi - g(e^*) = \underline{u}.$$

Proof. Suppose that

$$\int v(w^*(\pi))f(\pi|e^*)\,d\pi - g(e^*) > \underline{u}.$$

Then, the principal can offer slightly less wage, still satisfying

$$\int v(w^*(\pi) - \varepsilon)f(\pi|e^*)\,d\pi - g(e^*) > \underline{u}.$$

Since the utility of the principal is strictly decreasing in wage, the principal has higher expected payoff from $(e^*, w^*(\pi) - \varepsilon)$. We treat the individual rationality constraint as an equality constraint. For each e, we solve

$$\max_{(w(\pi))_\pi} \int (\pi - w(\pi))f(\pi|e)\,d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) = \underline{u}.$$

Then, we choose e^* to maximize objective function. □

First order condition

Chapter 7

Basic Auctions

Games with incomplete information The idea of screening different types of workers is essentially to design a program so that the agent is willing to reveal his private information.

7.1 Introduction

Monopolistic market is the trading protocol in which a single seller is trading against many buyers. The number of commodities available to the buyer is usually assumed to be no more than the number of buyers so that the buyer have to compete among themselves.

The monopolistic seller tries to exploit

7.2 Institutional variations

Why study auctions? Auction is an important institution through which a large amount of goods and services is traded. It is probably one of the oldest trading protocol, going back to Babylong according to Herodotus in allocating brides to grooms. As auctions are widely used for a broad range of economic environment, the auction has many institutional and informational variations.

Four basic auctions We start with four basic auctions among standard auctions.

- First price auction
- Dutch auction

- English auction
- Second price auction

Let us describe verbally the rule of each of four basic auctions.

Dutch auction The name comes from the fact that one of the main exports of Netherlands is tulip, which requires a quick transaction. In fact, the auction of fish often follows the same format, which also requires a quick transaction.

In front of bidders, a bulletin board displays a price. Starting from a price so high no one will ever buy the object, the displayed price drops continuously. When a bidder stops the clock, the auction stops. The bidder who stops the clock wins the object, paying the price displayed on the bulletin board.

7.3 Informational variations

7.4 Four basic auctions

Chapter 8

Open Auctions

8.1 Vickrey auction

8.2 Revenue

8.3 English auction

Chapter 9

Closed Auctions

9.1 Introduction

9.2 Symmetric equilibrium

9.3 2nd order condition

9.4 Uniqueness

Chapter 10

Mechanism Design

10.1 Revenue comparison

10.2 Mechanism design

Chapter 11

Optimal Auction

11.1 Revenue equivalence theorem

11.2 Optimal auction

Chapter 12

Public Goods

12.1 Public good

12.2 Grove Clarke scheme

Chapter 13

Efficient Mechanism

13.1 Efficient mechanism

13.2 Basic properties of VCG

Chapter 14

Search and Matching

14.1 Search

Friction Arrow Debreu economy is the foundation of modern microeconomics, where we establish the two fundamental welfare theorems.

Definition 13. By a friction, we mean any institutional or informational restriction imposed upon the optimization problem of an agent in the economy.

Asymmetric information is a friction in this sense, as the agent without private information cannot identify the true characteristics of the property, and there is no market for insurance against the unknown states. We often call asymmetric information an example of informational friction.

Search friction

Market with search friction

Theorem 7. $\forall \varepsilon > 0$, the unique equilibrium price is $p_s = b$ for every seller s.

Proof. Because every seller is identical, the equilibrium price $p = p_s \, \forall s$. We first show that

$$\underbrace{(0, \ldots, 0)}_{s}$$

is not an equilibrium. Suppose seller s increases the price by $\varepsilon/2$. If the buyer who is assign to seller s accept the price, his surplus is

$$b - \frac{\varepsilon}{2}$$

and if he rejects, his surplus cannot exceed $b - \varepsilon$ because of the vacancy cost. Thus, it is optimal for a buyer to accept $\varepsilon/2$. By the same reasoning, we conclude that for any $p < b$,

$$(\underbrace{p, \ldots, p}_{s})$$

is an equilibrium. Thus, the only equilibrium price is $p_s = b \, \forall s$. $\square$

This is a sharp contrast to the economy without any friction. Even if the supply exceeds the demand, it takes only an arbitrarily small search cost to shift the equilibrium from the competitive price 0 to the monopolistic price b.

Discussion

Chapter 15

Repeated Games with Perfect Monitoring

15.1 Introduction

15.2 Questions

15.3 Finitely repeated game

Chapter 16

Infinitely Repeated Game

16.1 Infinitely repeated game

We discuss three well known repeated game strategies.

Example (D forever)**.** Define

$$\sigma_i(h_t) = D \quad \forall h_t.$$

The pair $\sigma = (\sigma_1, \sigma_2)$ constitutes a subgame perfect equilibrium.

Proof. First, we show that σ is a Nash equilibrium of G^∞. In the equilibrium, each player receives average payoff of 1, since the outcome path

$$f(\sigma) = ((D, D), \dots, (D, D), \dots).$$

If player i deviates following any history along the equilibrium path, he cannot receives more than 1, because D is the strictly dominant strategy of the component game. Therefore, player i has no profitable deviation.

Second, we prove that σ induces a Nash equilibrium in every subgame $G|_{h_t}$. Note that following any history h_t, the continuation play of $G|_{h_t}$ is identical with the outcome of G. Following the same logic, we can show that σ induces a Nash equilibrium in $G|_{h_t}$. Thus, σ is a subgame perfect equilibrium. $\diamond$

We can generalize this result. Fix any normal form game G and a Nash equilibrium $s = (s_1, s_2)$ of G. Define a repeated game strategy for G^∞ as

$$\sigma_i(g_t) = s_i \quad \forall g_t.$$

Such a strategy is called the repetition of one shot Nash equilibrium.

Proposition 2. The repetition of one shot Nash equilibrium is a subgame perfect equilibrium in G^∞.

The repetition of one shot Nash equilibrium is an important benchmark, aginst which the virtue of a long term relationship is compared.

Example (Grim trigger)**.** Define $\sigma_i(h_1) = C$ and for $t \geq 2$,

$$\sigma_i(h_t) = \begin{cases} C & \text{if } h_t = ((C,C),\dots,(C,C)) \\ D & \text{otherwise} \end{cases}$$

as the grim trigger strategy.

The strategy triggers punishment D, if anyone plays D. Otherwise, the strategy dictates to play C. Once D starts, player i will play D forever. The punishment is grim.

The pair of grim trigger strategies induce outcome path

$$(C,C),\dots,(C,C),\dots$$

and the average payoff of each player is 3, which Pareto dominates the Nash equilibrium outcome of the component game (which is 1).

Proposition 3. The pair of grim trigger strategies is a subgame perfect equilibrium.

Definition 14. Let V be the set of feasible payoff vectors. $v \in V$ is individually rational, if $v_i \geq \underline{v}_i \; \forall i$.

The following theorem was discovered by a number of people, thus given the name Folk theorem.

Theorem 8 (Aumann and Shapley; Rubinstein)**.** $\forall v \in V$ which is individually rational, there exists a subgame perfect equilibrium σ such that $v = u(\sigma)$.

Proposition 4. The pair of tit-for-tat strategies constitutes a Nash equilibrium, but not a subgame perfect equilibrium in G^∞.

Proof. Along the equilibrium path,

$$(C,C),(C,C),\dots,(C,C),\dots$$

and the average payoff of each player is 3. □

Chapter 17

Repeated Games with Imperfect Monitoring Examples and Illustration

17.1 Introduction

17.2 Repeated game with imperfect monitoring

17.3 Example

Chapter 18

Repeated Moral Hazard Problem

18.1 Repeated principal agent problem

18.2 Results

Chapter 19

Nash Bargaining Problem

19.1 Plan

Competitive equilibrium In a competitive market, decisions are made in a decentralized manner. A consumer optimizes subject to the budget constraint and the price. A producer maximizes profit for a given price.

It is not obvious how the market clearing price is determined. We often use a fictitious auctioneer *Walrasian auctioneer* who construct the aggregate demand and supply from individual demand and supply curves. He then finds the intersection of the two curves and announces the market clearing price. Although it is a useful educational tool to explain the market clearing mechanism, the presence of the Walraisn auctioneer goes directly against the very spirit of the decentralized trading of the competitive market: the invisible hand.

Informational efficiency Hayek observed that the competitive market price aggregates dispersed information so that the individual agents in the economy can take and action in a decentralized manner, but can coordinate to achieve and efficient allocation.

The general equilibrium model of Arrow and Debreu formulates the first welfare theorem but remains vague about the price determination process and the information aggregation process.

Decentralized trading model We need to understand the mechanism that aggregates the private information of the individual agents into the market clearing price.

If the trading is done decentralized, it is not clear how there should a single market clearing price. While we analyze the information aggregation process, we can explain the law of single price.

Dynamic decentralized trading model We build a decentralized trading model from a smallest unit of trading: bargaining. To trade something, you need at least two players. Bargining is a trading protocol between two players. We introduce a matching process as we examined in the model of Peter Diamond so that the trading partner may change over time. We examine the prices at which trading occurs, and how the difference among different trading partners vanish, and converge to the competitive equilibrium price.

Plan

- Bargaining
  - Axiomatic model of Nash [1950]
  - Strategic model of Rubinstein [1982]
- Matching and bargaining

19.2 Bargaining

Trading institution between one seller and one buyer to determine the price and the delivery condition of good or service. Also called bilateral monopoly problem.

Finest unit of trading, which forms the foundation of the market. To trade goods, you need at least two people. The bargaining is a trading unit with two people: one seller vs. one buyer.

Old and widely used, and has many institutional variations. Difficult to formulate and anlyze.

19.3 (Nash) Divide a dollar

A game between two players.
$A_i = [0, 1]$

$$u_i(a_1, a_2) = \begin{cases} a_i & \text{if } a_1 + a_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Any division of 1 dollar can be sustained by a Nash equilibrium. We have a theory, which cannot make any useful prediction. This is often considered as a failure of the theory, revealing the difficulty of the problem, as the bargaining outcome is affected by the details of the rule of the game.

Why difficult? The bargaining problem is difficult, not because we need a fancy mathematical tool. The analysis must satisfy two conditions.

1. The rule of the bargaining must be reasonable, in which the two parties have comparable bargaining power and can influence the outcome, A dictatorial game is a form of bargaining, but is not considered a reasonable model of bargaining.

2. The model must make a sharp prediction to tell us what would the most likely outcome from the bargaining process. Divide a dollar game is a reasonable bargaining model. We cannot learn anything from Nash equilibrium, because every efficient division is a Nash equilibrium.

19.4 Nash bargaining problem

Approach by Nash [1950] In 1950, we did not have tools to tackle the difficulties of the bargaining problems. It is before John Nash invented (Nash) equilibrium concept. Instead, Nash chose to bypass the difficulties, but search for a way to say something useful regarding the bargaining process.

The major innovation of Nash's approach is to suppress the details of the procedure, treating it as a black box, and he focuses on the properties of the reasonable outcome.

A bargaining problem is regarded as a mapping from the data which consists of the preference of the players and the structure of the surplus to the division of the surplus.

He asked the following question: A reasonable bargaining outcome must satisfy a certain set of properties, what is the mathematical formula to calculate the outcome?

Bargaining solution Let S be the set of surplus and d be the disagreement point. S is the collection of all possible outcomes attainable, if agreement is reached. d is the pair of payoffs associated with disagreement.

Example. In case of the divide dollar,

$$\{(a_1, a_2) : a_i \geq 0, a_1 + a_2 \leq 1\}$$

and

$$d = (0, 0).$$

Definition 15. A bargaining problem is (S, d), where $S \subset \mathbb{R}^2$ is compact and convex and $\exists\, s \in S$ such that $s_i > d_i \ \forall i \in \{1, 2\}$.

We admit randomized contact which makes the set of all feasible utilities convex. $S \subset \mathbb{R}^2$ if and only if S is closed and bounded. If S is unbounded, then the bargaining can be meaningless, because each party can get what he wants without negotiation. The closeness of S is a technical condition to ensure the existence of a solution of the optimization problem.

The last condition ensures that the bargaining is not degenerate. If every feasible payoff vector is Pareto dominated by the disagreement outcome, there is no point of negotiation.

Let S be the set of all bargaining problems.

Definition 16. A bargaining solution $f(S, d) = (u_1, u_2)$ is the rule that specifies which outcome is determined:

$$f : \mathcal{S} \to \mathbb{R}^2.$$

Note. A bargaining solution is not conditioned on a particular bargaining problem. Instead, the way how a bargaining outcome is determined should be spelled out before a particular bargaining problem is selected.

Example (Dictatorial). Let

$$\overline{u}_1 = \arg\max_{u'_1} \{u'_1 \,|\, \exists u_2,\, (u'_1, u_2) \in S,\, u_2 \geq d_2\}$$

be the best outcome of player 1 in S.

$$f(S, d) = \{(\overline{u}_1, u_2) \,|\, (\overline{u}_1, u_2) \in S\}.$$

chooses the best possible outcome.

Example (Always disagree).

$$f(S, d) = d.$$

Discussion

- Dictatorial bargaining solution does not sound reasonable, because the lack of symmetry. By a bargaining situation, we refer to a situation where each party has some, if not equal, control over the outcome of the negotiation. Dictatorial solution does not allow any room for negotiation by player 2.
- If a negotiation always breaks downm the outcome is not efficient. Alluding to Coase theorem, such a bargaining rule should be replaced by another rule which generates more efficient outcome.

Axioms Let us spell out the properties which any reasonable bargaining solution must satisfy. John Nash call these properties axioms on the ground that they are evidently reasonable. Let us state four axioms, along with discussions.

A reasonable bargaining solution should be such that its outcome is affected by the units of the utils.

Axiom 4 (Invariance)**.** Consider the two bargaining problems, (S, d) and (S', d') where (S', d') is obtained by applying a positive affine transformation to (S, d): $\forall i$, $\exists \alpha_i \geq 0$ and $\beta_i \in \mathbb{R}$ such that

$$s_i' = \alpha_i s_i + \beta_i$$

f satisfies the invariance axiom if

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i \quad \forall i.$$

A reasonable bargaining solution should not produce an outcome which is Pareto dominated by another feasible outcome.

Axiom 5 (Pareto)**.** f satisfies the Pareto axiom if $\exists (t_1, t_2) \in S$ such that $t_i > s_i \; \forall i$ implies $f(S, d) \neq (s_1, s_2)$.

Definition 17. A bargaining problem (S, d) is symmetric if $(s_1, s_2) \in S$ implies that $(s_2, s_1) \in S$.

If the bargaining problem is symmetric, then the name of a player should not matter, implying that the two parties have equal bargaining power.

Axiom 6 (Symmetry)**.** f satisfies the symmetry axiom if for any symmetric bargaining problem (S, d), $f_1(S, d) = f_2(S, d)$.

The symmetry axiom does not require that the bargaining outcome must be equal for all players. The axiom applies only to a symmetric bargaining problem. The same axiom imposes no restriction on bargaining problems which are not symmetric.

The first three axioms (INV, PAR and SYM) are the restrictions on f over individual bargaining problems. The next axiom specifies how the solutions from two different bargaining problems should be related.

Axiom 7 (Independence of Irrelevant Alternatives)**.** Consider two bargaining problems, (S, d) and (T, d) with $T \subset S$. f satisfies the axiom of independence of irrelevant alternatives, if $f(S, d) \in T$ implies $f(T, d) = f(S, d)$.

Discussion

- While INV, PAR and SYM appear to be considered evidently reasonable, IIA need motivation, as it imposes a restriction on the relationship between the solutions of two bargaining problems.
- IIA is essentially identical with the weak axiom of choice. If a decision maker choose an object from T shich contains S, but the selected com-

modity bundle is in S, then the consumer should choose the same bundle when he is constrained to choose from S. The alternatives in $T \setminus S$ are irrelevant. We know that the weak axiom in this sense implies that the consumer behavior can be described as a consequence of utility maximization.

- IIA is reasonable, if we accept the view that a reasonable bargaining solution should be a solution of a social welfare function. Otherwise, it is not. The existence of a social welfare function is not always guaranteed.

- For me, IIA appears to be reasonable, but has room to be improved. Quite a few people followed up Nash [1950] proposing alternative set of axioms. Usually, INV, PAR and SYM are not touched, but IIA is replaced by something else.

Consistency and uniqueness We choose INV, PAR, SYM and IIA because they are considered reasonable. We did not consider whether four axioms are consistent with each other. If they are not consistent, no bargaining solution satisfying four axioms exists.

If a bargaining solution satisfying four axioms exists, we need to ask how many solutions satisfy the axioms. If too many solutions satisfy the axioms, there is a room to impose additional axioms.

Nash [1950] answer these two question rigorously and elegantly.

Nash bargaining solution Nash [1950] proposes a bargaining solution.

Definition 18. f^N is the Nash bargaining solution if

$$f^N(S, d) = \arg \max_{(s_1, s_2) \in S, s_i \geq d_i} (s_1 - d_1)(s_2 - d_2).$$

$s_i - d_i$ represents the gain from reaching agreement over the disagreement payoff. We call $s_i - d_i$ the Nash gain, and $(s_1 - d_1)(s_2 - d_s)$ the Nash product.

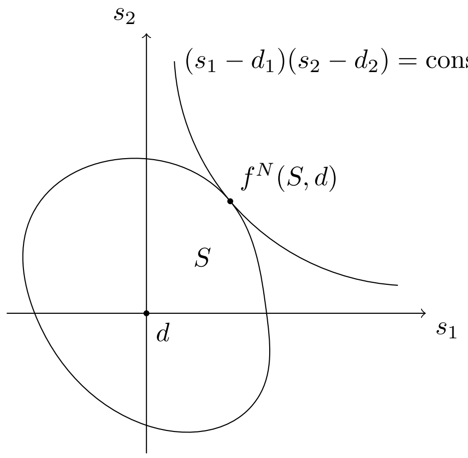


Figure 19.1: Nash bargaining solution (Osborne and Rubinstein [1990])

Note. The Nash product $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_s)$ is strictly quasi concave continuous function. Since S is convex and compact, the maximizer exists and is unique. Thus, the Nash bargaining solution is well defined.

Characterization The fundamental theorem of Nash [1950] is that the Nash bargaining solution is the only bargaining solution which satisfies INV, PAR, SYM and IIA.

Theorem 9. Bargaining solution f satisfies INV, PAR, SYM and IIA if and only if $f = f^N$.

Proof. We prove that the Nash bargaining solution satisfies INV, PAR, SYM and IIA.

INV Suppose that $s_i' = \alpha_i s_i + \beta_i$ and $d_i' = \alpha_i d_i + \beta_i$. Note

$$(s_1' - d_1')(s_2' - d_2') = \alpha_1 \alpha_2 (s_1 - d_1)(s_2 - d_2).$$

Thus, if (s_1^*, s_2^*) maximizes the right hand side, then $(\alpha_1 s_1^* + \beta_1, \alpha_2 s_2^* + \beta_2)$ maximizes the left hand side.

PAR Since $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_2)$ is a strictly increasing function of (s_1, s_2), an optimal solution must be at the Pareto frontier of S.

SYM Fix a symmetric problem (S, d) where $d_1 = d_2 = d$. Note

$$W(s_1, s_2) = (s_1 - d)(s_2 - d) = (s_2 - d)(s_1 - d),$$

If (s_1^*, s_2^*) maximizes the left hand side, then $(s_2^*, s_1^*) \in S$ maximizes the right hand side. It remains to show $s_1^* = s_2^*$.

Suppose that $s_1^* \neq s_2^*$. Since $W(s_1, s_2)$ is strictly quasi concave and S is convex, $\forall \lambda \in (0, 1)$,

$$\lambda(s_1^*, s_2^*) + (1 - \lambda)(s_2^*, s_1^*) \in S$$

and

$$W(\lambda(s_1^*, s_2^*) + (1 - \lambda)(s_2^*, s_1^*)) > W(s_1^*, s_2^*) = W(s_2^*, s_1^*)$$

which contradicts to the hypothesis that (s_1^*, s_2^*) maximizes W over S.

IIA Suppose that (s_1^*, s_2^*) maximizes $W(s_1, s_2)$ over T and

$$(s_1^*, s_2^*) \in S \subset T.$$

Since

$$W(s_1^*, s_2^*) \geq W(s_1, s_2) \quad \forall (s_1, s_2) \in T,$$

and $S \subset T$,

$$W(s_1^*, s_2^*) \geq W(s_1, s_2) \quad \forall (s_1, s_2) \in S.$$

Thus, $(s_1^*, s_2^*) \in S$ must maximize W over S.

The difficult part is to show that Nash bargaining solution is the only solution satisfying four axioms. □

Lemma 7. $\forall (s_1, s_2) \in S$, $s_1 + s_2 \leq 1$.

Proof. Suppose that $\exists (t_1, t_2) \in S$ such that $t_1 + t_2 > 1$. Recall that

$$\left(\frac{1}{2}, \frac{1}{2}\right) \in S$$

and S is convex. Thus, $\forall \lambda \in (0, 1)$,

$$(1 - \lambda)\left(\frac{1}{2}, \frac{1}{2}\right) + \lambda t \in S.$$

Note

$$\begin{aligned}
&\left[(1-\lambda)\frac{1}{2} + \lambda t_1\right]\left[(1-\lambda)\frac{1}{2} + \lambda t_2\right] \\
= &\left[\frac{1}{2} + \lambda\left(t_1 - \frac{1}{2}\right)\right]\left[\frac{1}{2} + \lambda\left(t_2 - \frac{1}{2}\right)\right] \\
= &\frac{1}{4} + \frac{\lambda}{2}(t_1 + t_2 - 1) + \lambda^2\left(t_1 - \frac{1}{2}\right)\left(t_2 - \frac{1}{2}\right) > \frac{1}{4}
\end{aligned}$$

for a sufficiently small $\lambda > 0$, because $t_1 + t_2 - 1 > 0$. But, this contradicts the fact that $f^N(S, 0) = (1/2, 1/2)$. □

Chapter 20

Alternating Offer Bargaining

20.1 Bargaining

20.2 Infinite horizon Rubinstein

20.3 Nash equilibrium

20.4 Subgame perfect equilibrium

Chapter 21

Dynamic Monopoly

21.1 Introduction

Monopoly market Monopolistic market is a trading protocol in which a single seller is facing many buyers. We teach the monopolistic market in the undergraduate class through a market with a linear demand, and show that the profit maximization by the monopolist leads to a market clearing price higher than the marginal production cost. The market clearing price in the monopolist market is above the marginal cost, and the equilibrium quantity is less than the competitive equilibrium quantity. The monopolist market leaves unrealized gains from trading, thus inefficient.

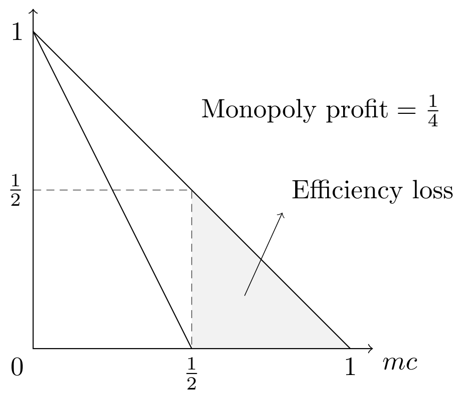


Figure 21.1: $MC = MR$

Monopolistic power The ability to charge a price higher than the marginal cost is the monopolistic power, measured by the difference between the market clearing price and the marginal production cost. A fundamental question is about the source of the monopolistic power.

Discussion The fact that a single seller controls the price of the good is not the source of the monopolistic power, as was first discovered by Ronald H. Coase. Understanding the foundation is important for the policy maker to develop a remedy to regulate the monopolistic market to recover the efficiency. Dynamic monopoly problem is one of a few economic problems, which are important for theory and policy.

21.2 Heuristics

Coase [1973] Ronald H. Coase examines a simple monopoly market in which the monopolist is selling out (commercial) land. The important characteristics of the commercial land is durability.

21.3 Rational expectations

Coasian dynamics

Lemma 8. If $\{p_t\}_{t=1}^{\infty}$ is an optimal pricing rule, then $q_t - q_{t-1} > 0$.

If $q_t = q_{t-1}$, then the monopolist did not make any sales in period t, wasting time. By skipping p_t and offering p_{t+1}, the monopolist can increase profit. Thus, $q_t - q_{t-1} > 0$.

Corollary (Coasian dynamics)**.** In any optimal pricing rule, $p_t > t_{t+1}$.

If $p_t \leq p_{t+1}$, then no consumer will purchase at p_{t+1}, since p_t is lower and offered earlier than p_{t+1}.

21.4 Subgame perfect equilibrium

Extensive form game We take advantage of the continuum of infinitesimal consumers, assuming that the action of a single buyer does not make any difference. A history of the game at the beginning of period t is $h_t = (p_1, \ldots, p_{t-1})$.

A strategy of the seller is $\sigma(h)t) = p_t$ and the buyer's strategy is summarized by the optimal decision rule: accept p_t if

$$v - p_t > \delta(v - \sigma(h_t, p_t)).$$

Thanks to the successive skimming property, we can write the critical type v_t

21.5 Examples

Let us consider a market demand function which is not continuously downward sloping. One half of population has reservation value $v = 3$ and the remaining half of the consumers has reservation value $v = 1$.

To calculate the static monopolistic profit maximizing solution, we cannot use the differential calculus to equate the marginal cost to the marginal revenue. We need to rely on basic reasoning.

- The seller will never charge less than 1.
- If above 1, the price must be 3.
- The seller never charges above 3.
- 3 will generate profit of 1.5 while 1 will generate 1.

The monopolist profit maximizing price is 3. the profit is 1.5, serving only the high reservation value consumers. Thus, the allocation is inefficient.

Gap. vs. No gap The lowest reservation value buyer is 1, which is above the marginal cost 0, in this example. If the lowest reservation value of the buyer is above the marginal cost, then we call it gap case.

In the model of Stokey, the lowest reservation value of a buyer is 0, which is equal to the marginal production. It is called no gap case.

We use the lowest reservation value of the buyer as the competitive benchmark.

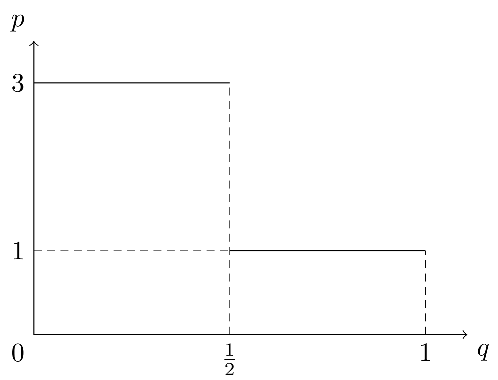


Figure 21.2: Market demand function

Dynamic monopoly problem Suppose that the monopolist opens the market until all buyers are served, and the monopolist has sufficient capacity to serve all buyers in the market. The time span of each period is Δ and discount rate is $r > 0$, and therefore, the discount factor is $\delta = e^{-\Delta r}$.

Preliminaries

Lemma 9. $\exists T < \infty$ such that all consumers are seved in a subgame perfect equilibrium.

Proof. Suppose that a positive mass, say $q > 0$, of consumers is never served. If so, the monopolist can offer ε so that

$$1 - \varepsilon > \delta(1 - 0).$$

Consumers with reservation value 1 will find it optimal to accept $\varepsilon > 0$ right away, because even if the monopolist offers 0 tomorrow, it is better to accept $\varepsilon > 0$ right away. Thus, the monopolist can recover $\varepsilon q > 0$ profit which wolud have been wasted. $\square$

This is the most important consequence of the gap case. Because the lowest reservation value of the buyer is strictly higher than the marginal production cost, the seller's profit is bounded away from 0.

Terminal round

Lemma 10. $p_T = 1$.

If T is the last round when every consumer is served, then $p_T \leq 1$. If $p_T < 1$, then the buyer will accept, because T is the last round. Thus, the equilibrium offer must be $p_T = 1$.

Penultimate round In T round, if any consumer is still active, the consumer must have reservation value 1. If some consumer has reservation value 3, then in period $T-1$, no reservation value 3 consumer has purchased, and therefore, the monopolist has wasted one round. We know in any equilibrium, the monopolist has to sell a positive amount to consumers. Thus, no consumer with reservation value 3 should be left in period T.

We conclude tat in period T, only the reservation value 1 consumers remain to be served at delivery price $p_T = 1$.

In the penultimate round $T-1$, the price will be higher than 1 so that only reservation value 3 consumers will be served. Thus, p_{T-1} must satisfy

$$3 - p_{T-1} \geq \delta(3 - p_T) = \delta(3 - 1) = 2\delta.$$

Thus, the highest price which reservation value 3 consumer is willing to accept is

$$p_{T-1} = 3 - 2\delta.$$

Since there are two types of consumers, $T = 2$.

Weak stationary equilibrium

Definition 19 (Weak stationary equilibrium)**.** A subgame perfect equilibrium is a stationary equilibrium, if $\forall h_t, \sigma(h_t)$ depends only upon the residual demand by the end of h_t. A subgame perfect equilibrium is a weakly stationary equilibrium, if $\forall h_t, \sigma(h_t)$ depends only upon the residual demand and the previous round's offer along h_t.

The economy is populated by a unit mass of consumers, each of whom has reservation value $v \in [\underline{v}, \overline{v}]$ and demand one unit of the good. Let

$$F(v) = P(v' \leq v)$$

and therefore,

$$q = 1 - F(p)$$

is the market demand if the seller makes a take-it-or-leave-it offer. As we change p, we can derive the demand, which we call the market demand.

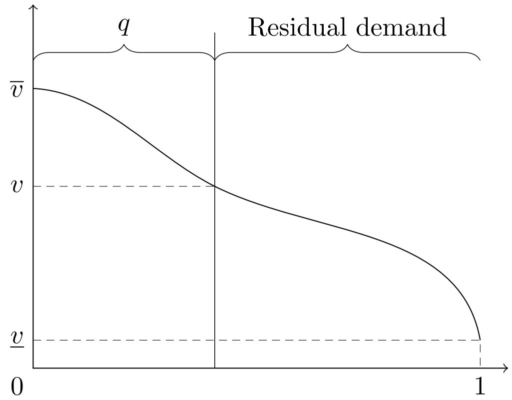


Figure 21.3: Residual demand curve

while we can admit more general demand curve, let us assume that $F(p)$ is strictly increasing continuous function. Thus, the market demand function is strictly decreasing. Thanks to the successive skimming property, a residual demand is parametrized by the mass of consumers who have been served. By residual demand q, we mean the portion of the aggregate demand without consumers with $v \geq 1 - q$.

Chapter 22

Bargaining under Uncertainty

22.1 Delay in Bargaining

22.2 Analysis

22.3 Delay and uncertainty

Chapter 23

Uncertainty and Delay

23.1 Uncertainty and delay

23.2 Stationarity

23.3 Gains from trading

23.4 Common value

Chapter 24

Search and Matching

24.1 Introduction

24.2 Overview

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Chapter 25

Synthetic Market

25.1 Introduction

25.2 Model

25.3 Analysis

25.4 Variations