Economics
Social science

☆ 논쟁
Exante prediction 사전예측 ← mathematics.
Expost explanation 사후설명.
혼에이 X.



# Lecture 2. The Roll Model

Mkt noise

Autocorrelation
Random walk.

## 1. The Random-Walk Model of Security Prices

A time series $\{x_t\}$ is a set of "repeated" observations on the same random variables, ordered in time.

$x_t \frown x_{t-k}$

**Definition 1.1.** The $k$th-order *autocovariance* of $\{x_t\}$ is defined by

$$\gamma_k = Cov[x_t, x_{t-k}]$$

$k=0 \quad \gamma_0 = Cov(x_t, x_t) = Var(x_t)$

for $k = 0, 1, \ldots$. By definition, it satisfies that (a) $\gamma_0 = Var[x_t]$ and (b) $\gamma_{-k} = \gamma_k$.

**Definition 1.2.** The $k$th-order *autocorrelation* of $\{x_t\}$ is defined by

$\rho_0 = 1$
$\rho_1 = \ \ldots$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov[x_t, x_{t-k}]}{Var[x_t]}$$

$\{\rho_0, \rho_1, \ldots\}$ $=$

for $k = 0, 1, \ldots$. The collection of autocorrelations is called the *autocorrelation function* (ACF).

☆ AR model 차수 결정.

**Definition 1.3.** A time series $\{x_t\}$ is a *white noise process*, denoted by $x_t \sim WN(0, \sigma^2)$, if (a) $E[x_t] = 0$ for all $t$, (b) $Cov[x_t, x_{t-k}] = 0$ for all $t$ and $k \neq 0$, and (c) $Var[x_t] = \sigma^2$ for all $t$.

time invariant ↳ 과거의 movement가 현재 movement에 영향을 주지 않는다.

Let $p_t$ denote the transaction price at time $t$. The *random-walk model with drift* is given by

$$p_t = \mu + p_{t-1} + \varepsilon_t, \tag{1.1}$$

stochastic trend

where $\varepsilon_t$ is a white noise. The term of $\varepsilon_t$ arises from new information that bears on the security value.

unexpected shock. ⇒ WN로 표현.
expected event : 이미 가격에 반영.

Future cash flow
< E LN (expected return).
$V = \frac{D_1}{r-g}$ DCF

- One writes (1.1) as

$\log P_t - \log P_{t-1} = \log \frac{P_t}{P_{t-1}}$ return.

$$\Delta p_t = \mu + \varepsilon_t$$

$= P_t - P_{t-1}$

$t-1 \quad t$
$P_{t-1} \quad \varepsilon_t \quad P_t$

and shows

$$\begin{aligned} Cov[\Delta p_t, \Delta p_{t-k}] &= Cov[\mu + \varepsilon_t, \mu + \varepsilon_{t-k}] \\ &= Cov[\varepsilon_t, \varepsilon_{t-k}] \\ &= 0 \end{aligned}$$

Monthly : RW
Weekly, daily : random walk (X).
positive auto co-

⇒ return의 autocovariance 는 0.
(room-Box test).

for all $t$ and $k \neq 0$. So, the changes in transaction prices are "uncorrelated" in the random-walk model.

*Remark* 1.4. In actual microstructure samples, the first-order autocorrelations of short-run price changes are usually "negative." So, the random-walk model is no longer considered to be a valid description of short-term price dynamics.

tic

TIC by TIC data는 RW로 설명 불가.



Investment : 대부분 monthly → RW로 보는 무방.
mkt micro- : TIC trade by trade : RW 이용 x.
1st − negative

## 2. Martingales in Market Microstructure Analyses

**Definition 2.1.** The process $\{x_t\}$ is a *martingale* with respect to the process $\{z_t\}$ if $E[|x_t|] < \infty$ for all $t$ and $E[x_{t+1}|z_t, z_{t-1}, \ldots] = x_t$.

예측불가 value.

Suppose that the payoff of a security is a random variable $v$. Traders form beliefs based on a sequence of information sets $\Phi_1 \subset \Phi_2 \subset \Phi_3 \subset \cdots$. Then, the conditional expectation $m_t = E[v|\Phi_t]$ is a martingale with respect to the sequence of information sets $\{\Phi_t\}$: i.e.,

⇒ learning

$$E[m_{t+1}|\Phi_t, \Phi_{t-1}, \ldots] = m_t.$$

- When the conditioning information $\Phi_t$ is all public information, $m_t$ is called the *fundamental value* or the *efficient price* of the security.

## 3. The Roll Model

Mkt friction
ASK, BID에서 사고 팜
BID-ASK spread.
⇓
Mkt noise.

Roll (~~1994~~ 1984) specifies that the transaction price at time $t$, denoted by $p_t$, consists of (a) the *long-lasting* fundamental value, denoted by $m_t$, arising from information about future cash flows and (b) the *transient* value, denoted by $q_t c$, attributable to the market organization and trading process: i.e.,

일시적 영향

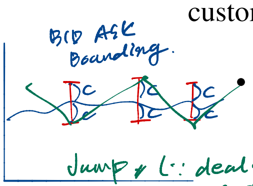
$q_t = 1$ buy $P_t = m_t + c$
$q_t = -1$ sell $P_t = m_t - c$
BID-ASK spread.

$$p_t = m_t + q_t c, \tag{3.1}$$

negative auto 음의요.

where $q_t$ is a trade direction indicator having one if the customer is buying and minus one if the customer is selling.

trading 프로토콜이 있기 때문에 value는 변하지 않아도 price는 움직인다.
even if (information shock이 발생되지 않아도)



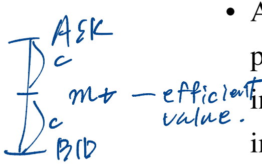

Jump ↑ (∵ dealer quoting).
discontinuous.

- Let $m_t$ denote $E[v|\Phi_t]$, where $v$ is the payoff from a security. Assume that $m_t$ follows a random walk,

$$m_t = m_{t-1} + \varepsilon_t, \tag{3.2}$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$.

dealer mkt




- Assume (a) that all trades are conducted through dealers who quote bid price, $b_t$, and ask price, $a_t$; i.e., a customer must buy (sell) at the dealer's ask (bid) price and (b) that dealers incur a cost of $c$ per trade and compete to the point where the cost is just covered. This implies that $b_t = m_t - c$ and $a_t = m_t + c$, so the bid-ask spread is $a_t - b_t = 2c$ which is "constant" over time.

compensation.

완전경쟁시장
expected 이익 = 0

1 round → dealer ⇒ +2c

From (3.1) and (3.2), the change in prices is written as

Buy $P_t = m_t + c$
Sell $P_t = m_t - c$

BID-ASK spread = 2c
constant.

$$\begin{aligned} \Delta p_t &= (m_t + q_t c) - (m_{t-1} + q_{t-1} c) \\ &= (m_t - m_{t-1}) + c(q_t - q_{t-1}) \\ &= \varepsilon_t + c(q_t - q_{t-1}). \end{aligned}$$

Def) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$ : filtered prob. space. $\mathbb{T} = \{0, 1, \cdots, T\}$.

$M : \Omega \times \mathbb{T} \to \mathbb{R}$ is a stochastic process

We say $M$ is a martingale

if ① $M$ is adapted. $M_t$ is $\mathcal{F}_t$-m'ble for $\forall t \in \mathbb{T}$

② $E^P(M_{t+1} \mid \mathcal{F}_t) = M_t$ for $t = (0, 1, 2, \cdots, T-1)$.

사람들이 정보가 없다는 것. 무작위로 사고 팔고. (주사위, 동전 던지기)

Assume (a) that buys and sells are equally likely: i.e., $E[q_t] = 0$ and $E[q_t^2] = 1$, (b) that buys and sells are serially independent: i.e., $E[q_t q_{t-k}] = 0$ for $k = 1, 2, \ldots,$ , and (c) that the customers buy or sell independently of $\varepsilon_t$: i.e., $E[q_\tau \varepsilon_s] = 0$ for all $\tau \neq s$.

$\tau > s$, $\tau < s$ 관계 없이 hold.

$q_t \left\langle \begin{matrix} 1 \\ -1 \end{matrix} \right.$ w/p 0.5

시장 efficient ⇒ shock이 영향을 미치지 X. public info.

symetric information ⇓ everybody unknown.

- Then, it shows that

$$
\begin{aligned}
E[\Delta p_t] &= E[\varepsilon_t + c(q_t - q_{t-1})] \\
&= E[\varepsilon_t] + c(E[q_t] - E[q_{t-1}]) \\
&= 0,
\end{aligned}
$$

$E[q_t] = 0$

$Var(q_t) = E(q_t^2) = 1.$

$$
\begin{aligned}
Var[\Delta p_t] &= E[(\Delta p_t)^2] \\
&= E[(\varepsilon_t + c(q_t - q_{t-1}))^2] \\
&= E[\varepsilon_t^2 + 2\varepsilon_t c\underbrace{(q_t - q_{t-1})}_{0} + c^2(q_t - q_{t-1})^2] \\
&= \sigma_\varepsilon^2 + 2c^2,
\end{aligned}
$$

$\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

$Var(\varepsilon_t) = E(\varepsilon_t^2) = \sigma_\varepsilon^2$

asymetric informed ⇒ 차이가 존재.

$$
\begin{aligned}
Cov[\Delta p_t, \Delta p_{t-1}] &= E[\Delta p_t \Delta p_{t-1}] \\
&= E[(\varepsilon_t + c(q_t - q_{t-1}))(\varepsilon_{t-1} + c(q_{t-1} - q_{t-2}))] \\
&= E[\varepsilon_t \varepsilon_{t-1}] + E[\varepsilon_t c(q_{t-1} - q_{t-2})] + E[c(q_t - q_{t-1})\varepsilon_{t-1}] \\
&\quad + E[c^2(q_t - q_{t-1})(q_{t-1} - q_{t-2})] \\
&= -c^2,
\end{aligned}
$$

$\parallel$ $E((\Delta p_t - E(\Delta p_t)) \times (\Delta p_{t-1} - E(\Delta p_{t-1}))$

Roll model – exante prediction

1st autocorrelation = negative.

and $Cov[\Delta p_t, \Delta p_{t-k}] = 0$ for all $k > 1$.

- In particular, the Roll model implies that the first-order covariance is "negative", which is consistent with the real data and the main contribution of the Roll model.

*Remark* 3.1. The Roll model can be used to estimate the bid-ask spread with $2\hat{c} = 2\sqrt{-\hat{\gamma}_1}$ in situations where the bid and ask data are not available.

*Remark* 3.2. Contrary to the Roll's assumptions, actual datasets show (a) that the bid-ask spread is time-varying, (b) that $\{q_t\}$ is serially correlated; i.e., buys tend to follow buys, and sells tend to follow sells, and (c) that a trade direction is positively associated with a shock.

In 실제 data. positive autoco-

order flow는 AR persistent

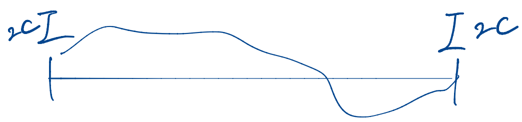
informed — shock 발생 시 position 정해지는

uninformed

Roll

: value는 random walk 따르고 있으면 mkt friction이 없기 때문에 (제약조건) BID-ASK.

⇒ negative autocovariance.




But real data X.

실제 data: BID Ask spread 변함.

order flow persistent

information assymetric 존재.