

Lecture 8. Limit Order Markets

1. Introduction

A limit order market (LOM) features unclear distinction between liquidity suppliers and demanders, in that traders can decide whether to supply liquidity with limit orders or consume it with market orders. So, theoretical LOM models concentrate on explaining the trader's problem of order choice between limit and market orders.

Remark 1.1. The previous models for dealers might be applied to LOMs by considering that a limit order is a dealer quote by another name. Those models, however, require a sharp division between liquidity suppliers (i.e., dealers) and demanders (i.e., traders), which is inconsistent with the reality of LOMs.

Symmetric ↘

2. The Cohen, Maier, Schwartz, and Whitcomb (1981) Model CMSW

information

trade (X)

trader's portfolio optimum level.

asymmetric X

Suppose that a trader corrects his portfolio imbalance by either (a) submitting a market order, (b) submitting a limit order, or (c) doing nothing. There is a security with payoff $X \sim N(\mu_X, \sigma_X^2)$.

The trader initially holds n shares and has exponential utility $U(W) = -e^{-\alpha W}$, where $W \sim N(\mu_W, \sigma_W^2)$: that is,

$$E[U(W)] = -\exp\left(-\alpha\mu_W + \frac{\alpha^2\sigma_W^2}{2}\right).$$

α is a constant that represents the degree of risk preference ($\alpha > 0$ for risk aversion, $\alpha = 0$ for risk-neutrality, or $\alpha < 0$ for risk-seeking).

2.1. Limit Order Submission (Buy) 구매 주문

The expected utility of placing a limit buy order at a price L , denoted by $E[U_{Limit}(L)]$, is

① 정상 주문 여기서

$$E[U_{Limit}(L)] = P_{Hit}(L)E[U_{Hit}(L)] + (1 - P_{Hit}(L))E[U_{Base}],$$

② 제작 대상 주문

where $P_{Hit}(L)$ is the execution probability of the limit buy order, $E[U_{Hit}(L)]$ is the expected utility when the limit buy order is hit, and $E[U_{Base}]$ is the expected utility when the limit buy order is not hit.

구매 vs 판매

- non-execution risk 제작되지 않은 때
- When the limit buy order is hit, his wealth is $W = (n+1)X - L$. Since $W \sim N((n+1)\mu_X - L, (n+1)^2\sigma_X^2)$, it shows 가장 가격이 예상보다 높을 때.

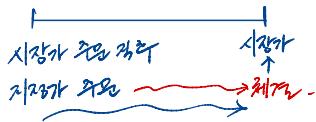
favorable price
제작 가능
non-execution risk

$$E[U_{Hit}(L)] = -\exp\left(-\alpha((n+1)\mu_X - L) + \frac{\alpha^2(n+1)^2\sigma_X^2}{2}\right).$$

- When the limit buy order is not hit, his wealth is $W = nX$. Since $W \sim N(n\mu_X, n^2\sigma_X^2)$, it shows

$$E[U_{Base}] = -\exp\left(-\alpha n\mu_X + \frac{\alpha^2 n^2 \sigma_X^2}{2}\right).$$

Definition 2.1. A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$ if its probability density function is



$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

↑
arrival process
modeling the order flow.

- For an exponentially distributed X , the cumulative distribution function is

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

limit buy (L) 317212902
seller의 청탁에
C < L 때는 주문이 왔다.

order book ~ book.

Let C denote the minimum reservation price across all sellers in the market. The limit buy order priced at L will be executed if $C \leq L$, since there is at least one seller who submits a market order to sell at price L .

- Assume that C is exponentially distributed with density $f(c) = \lambda \exp(-\lambda(c - \theta))$ for $c - \theta \geq 0$: i.e., $F(c) = \Pr(C \leq c) = 1 - \exp(-\lambda(c - \theta))$. Then, the execution probability of the limit buy order is

$$P_{Hit}(L) = \Pr(C \leq L) = 1 - \exp(-\lambda(L - \theta)) \quad (2.2)$$

for $L \geq \theta$.

Finally, one computes the expected utility of a limit buy order strategy in (2.1) using the exponential execution probability in (2.2): i.e.,

$$\begin{aligned} E[U_{Limit}(L)] &= P_{Hit}(L)E[U_{Hit}(L)] + (1 - P_{Hit}(L))E[U_{Base}] \\ &= (1 - \exp(-\lambda(L - \theta))) \times \left(-\exp\left(-\alpha((n+1)\mu_X - L) + \frac{\alpha^2(n+1)^2\sigma_X^2}{2}\right) \right) \\ &\quad + \exp(-\lambda(L - \theta)) \times \left(-\exp\left(-\alpha n \mu_X + \frac{\alpha^2 n^2 \sigma_X^2}{2}\right) \right). \end{aligned} \quad (2.3)$$

short selling

One finds the optimal L^* to maximize (2.3) via a numerical method.

Example 2.2. Suppose that $\alpha = 1$, $\mu_X = 1$, $\sigma_X^2 = 1$, $\lambda = 1$, and $\theta = 0$. One depicts the expected utility in (2.3) as a function of buy limit price $L \in [0, 2]$ for initial holdings $n = 0, -1$, and -2 . For a trader shorting one unit ($n = -1$), for instance, it shows that $E[U_{Limit}(L)]$ is maximized at $L^* = 0.75$. In this case, it shows

$$E[U_{Hit}(0.75)] = -\exp(0.75) = -2.12$$

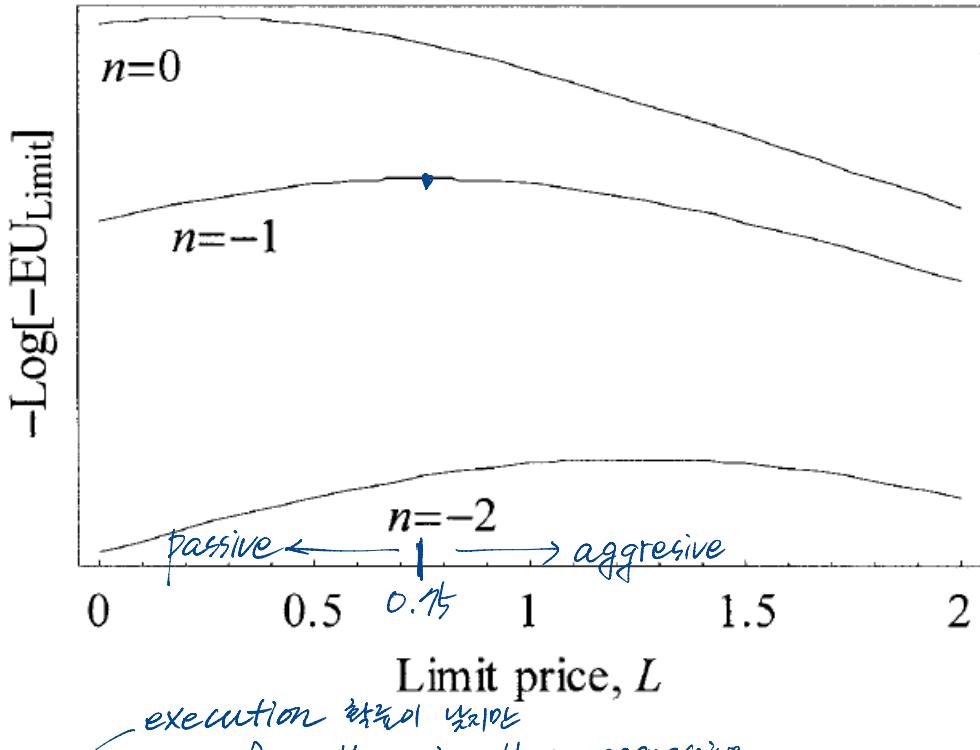
$$E[U_{Base}] = -\exp\left(1 + \frac{1}{2}\right) = -4.48 \quad \text{doing nothing yet}$$

$$P_{Hit}(0.75) = 1 - \exp(-(0.75)) = 0.53$$

$$E[U_{Limit}(0.75)] = 0.53 \times (-2.12) + 0.47 \times (-4.48) = -3.23$$

↑ $L^* = 0.75$ 일 때 $E[U_{Limit}(L)]$ 최대화.
utility가 더 좋다.

Since $E[U_{Limit}(0.75)] > E[U_{Base}]$, submitting a limit buy order at $L^* = 0.75$ is preferable to doing nothing when $n = -1$.



Remark 2.3. The concavity reflects the trade-off in expected utilities arising between (a) the use of a passive limit buy order (with a low bid price) that is less likely to be executed but is more profitable if executed and (b) the use of an aggressive limit buy order (with a high bid price) that is more likely to be executed but is less profitable if executed.

execution risk vs. less favorable price.

class notes
2.2.1

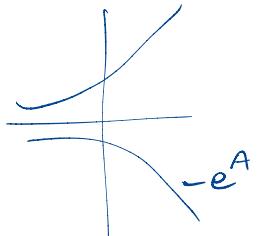
2.2. Market Order Submission

Ask! Suppose that a current ask price is A (quoted by a limit order seller). If the trader submits a market buy order, he purchases with certainty, paying A , and consequently his wealth is $W = (n+1)X - A$. Since $W \sim N((n+1)\mu_X - A, (n+1)^2\sigma_X^2)$, it shows

$$E[U_{Market}(A)] = -\exp\left(-\alpha((n+1)\mu_X - A) + \frac{\alpha^2(n+1)^2\sigma_X^2}{2}\right).$$

Example 2.4. With $\alpha = 1$, $\mu_X = 1$, $\sigma_X^2 = 1$, $\lambda = 1$, $\theta = 0$, and $n = -1$, it shows

$$E[U_{Market}(A)] = -\exp(A),$$



so $E[U_{Market}(A)] = E[U_{Base}]$ for $A = 1.5$. That is, the trader is indifferent between submitting a market buy order and doing nothing when $A = 1.5$. If $A < 1.5$, then $E[U_{Market}(A)] > E[U_{Base}]$, or equivalently submitting a market buy order is preferable to doing nothing.

*market buy order
vs
doing nothing*

2.3. Choice decision between limit and market orders

Suppose that $\alpha = 1$, $\mu_X = 1$, $\sigma_X^2 = 1$, $\lambda = 1$, $\theta = 0$, and $n = -1$. Turning to a comparison of market and limit buy order strategies, one finds that the value of A such that $E[U_{Market}(A)] = E[U_{Limit}(0.75)]$ by solving

$$-\exp(A) = -3.23.$$

Since $A = \ln(3.23) = 1.17$, one concludes that the trader (with $n = -1$) is indifferent between submitting a limit buy order at $L = 0.75$ and submitting a market buy order when the ask quote is priced at $A = 1.17$.

- When the ask price, approaching from above, hits 1.17, then the trader who plans to submit a limit buy order at $L = 0.75$ is pulled to the ask and switches to submitting a market buy order. CMSW refer to it as 'gravitational pull'. Overall, the CMSW model predicts that as the bid-ask spread becomes narrower, an incoming order is more likely to be a market order.

bid-ask spread ↗ Ask price ↗ Market order ↗ Market order
 ↘ Ask price ↘ Market order ↘ Market order

2.4. Limitation

A limit order placed at time t is executed only if market orders arriving at time $t+k$ execute "all preceding" limit orders in the book. So, execution probability depends on (a) the current status of price-time priority when the limit order is placed and (b) incoming market/limit orders after the limit order placement.

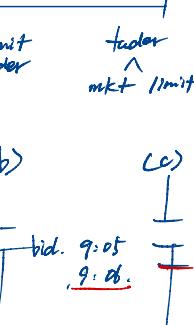
1. booker market order time priority. 2. trader execution prob order time priority. 3. CMSW exogenous market order PE incoming market order.

- Incoming orders change the price-time priority of existing limit orders in the book as follows: (a) the price priority decreases if a subsequent limit order is placed at a more favorable price, (b) the time priority increases if a subsequent market order executes against the existing orders at the same price, and (c) the price priority increases if a subsequent market order executes against the existing orders at better prices. When submitting a limit order, therefore, a trader should take into account how his order affects the incentives of subsequent traders to submit either market or limit orders.

general equilibrium

el 1993 : 9/2

Remark 2.5. In the CMSW model, the choice decision is "exogenous" in the sense that it is immune to the choice decisions of "other" traders. In reality, however, a trader's own choice is "endogenously" made, since his choice depends on the limit order execution probability, which in turn depends on his conjecture about the choices by other traders who arrive after him. The dependence between the trader's own choice and the choices made by others in the past and in the future motivates Parlour (1998) and Foucault (1999).

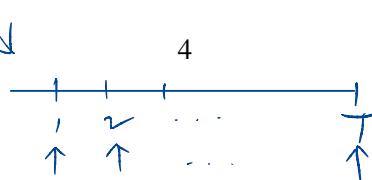


3. The Parlour (1998) Model

3.1. Traders

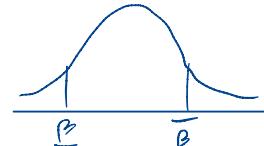
day 1 on 거래일 1일
day 2 on 거래일 2일

A security is traded in day 1 and pays off V on day 2. On day 1, at each of the times $t = 1, \dots, T$, a trader randomly arrives. The probability that the trader at time t is a buyer is 0.5. A



buyer has an endowment of cash to buy one share, and a seller holds one share to sell. A trader may decide to do nothing and keep his existing positions. *consumption*

- A trader arriving at time t has a utility function $U(c_1, c_2) = \underline{c}_1 + \beta_t c_2$, where β_t is a time preference parameter and is randomly distributed over an interval $(\underline{\beta}, \bar{\beta})$ across traders. With a low (high) β_t , a seller is eager (disinclined) to sell and a buyer is disinclined (eager) to buy. β is $\underline{\beta}$: 



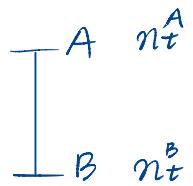
3.2. Trading Mechanism

The bid and ask prices are fixed, with $B < V < A$. The limit order book contains the number of shares on the bid and ask sides, n_t^B and n_t^A , immediately prior to the arrival of a trader at time t .

- If the order book is not empty, a market order can be executed against the book. A limit order is added to the order book, and limit orders in the book are executed in *first-in, first-out time priority* (i.e., no outbidding is allowed). 

The order book is updated upon the arrival of the trader t as

$$n_{t+1}^B = \begin{cases} n_t^B + 1 & \text{if the trader } t \text{ submits a limit buy order} \\ n_t^B - 1 & \text{if } n_t^B \geq 1 \text{ and the trader } t \text{ submits a market sell order} \end{cases}$$



and

$$n_{t+1}^A = \begin{cases} n_t^A + 1 & \text{if the trader } t \text{ submits a limit sell order} \\ n_t^A - 1 & \text{if } n_t^A \geq 1 \text{ and the trader } t \text{ submits a market buy order.} \end{cases}$$

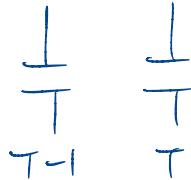
3.3. Optimal Strategies

Suppose that the trader T is a seller. He never submits a limit sell order, as there is no chance for a market buy order to arrive. Instead, he submits a market sell order if the utility of selling a share (i.e., B) is greater than the utility of doing nothing (i.e., $\beta_T V$). With the distribution of β_T , one computes the probability of submitting a market sell order: i.e.,

$$\Pr(\text{Seller } T \text{ submits a market sell order}) = \Pr\left(\beta_T < \frac{B}{V}\right). \quad B > \beta_T V$$

Suppose that the trader $T-1$ is a buyer. If $n_{T-1}^B \geq 1$, he never submit a limit buy order, since the order is not first in the execution queue and thus cannot be executed in the one remaining period. Instead, he submits a market buy order if the utility of buying a share (i.e., $\beta_{T-1}V$) is greater than the utility of doing nothing (i.e., A). With the distribution of β_{T-1} , the probability of submitting a market buy order is given by

$$\Pr(\text{Buyer } T-1 \text{ submits a market buy order}) = \Pr\left(\beta_{T-1} > \frac{A}{V}\right).$$



If $n_{T-1}^B = 0$, he can submit a limit buy order, since it can be executed if the trader T submits a market sell order with the probability of $\Pr(\beta_T < B/V)$. The execution probability of a limit buy order at time $T - 1$ is given by

$$\Pr(\text{A limit buy order at time } T-1 \text{ is executed}) = \begin{cases} \Pr\left(\beta_T < \frac{B}{V}\right) & \text{if } n_{T-1}^B = 0 \\ 0 & \text{if } n_{T-1}^B \geq 1. \end{cases}$$

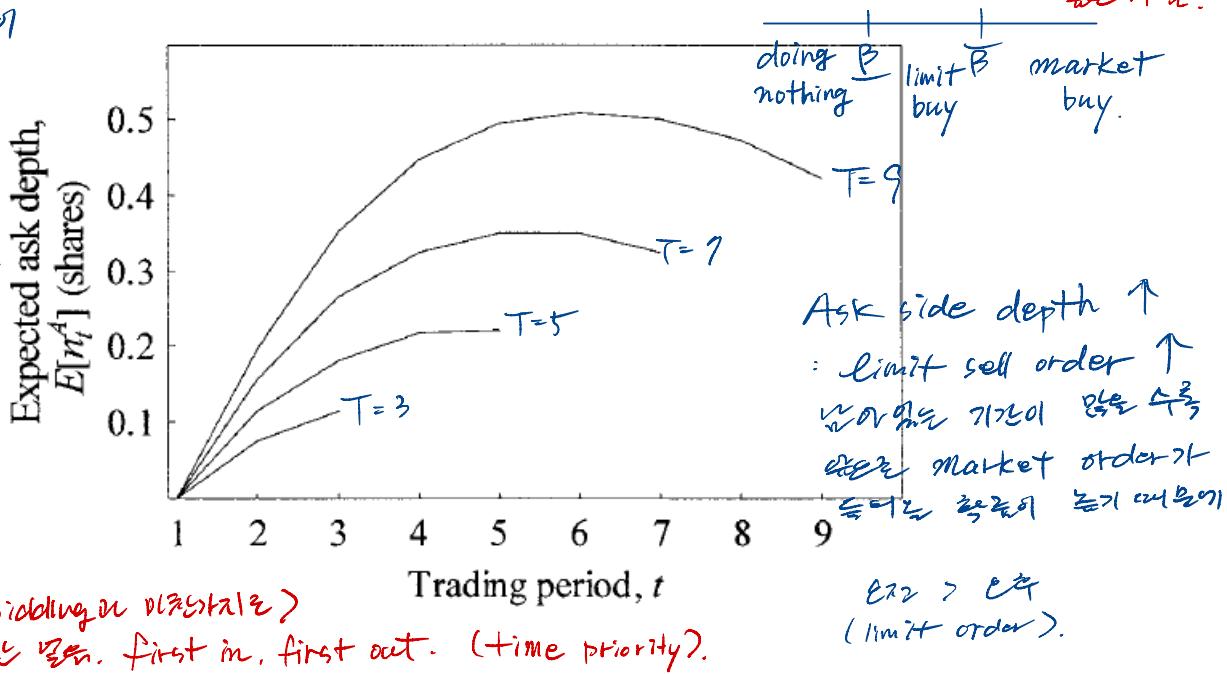
- 1. T-assel market
Seller of c₁₂.
- 2. T-1 assel
limit order book

Remark 3.1. Regarding the trader's optimal strategies, Parlour (1998) demonstrates that there are cutoffs $\beta_{\text{Limit}}^{\text{Buy}}$ and $\bar{\beta}_{\text{Limit}}^{\text{Buy}}$ (which are functions of time, the state of the book, and the execution probabilities) such that the buyer $T - 1$ does nothing if $\beta_{T-1} < \beta_{\text{Limit}}^{\text{Buy}}$, submits a limit buy order if $\beta_{\text{Limit}}^{\text{Buy}} < \beta_{T-1} < \bar{\beta}_{\text{Limit}}^{\text{Buy}}$, or submits a market buy order if $\bar{\beta}_{\text{Limit}}^{\text{Buy}} < \beta_{T-1}$.

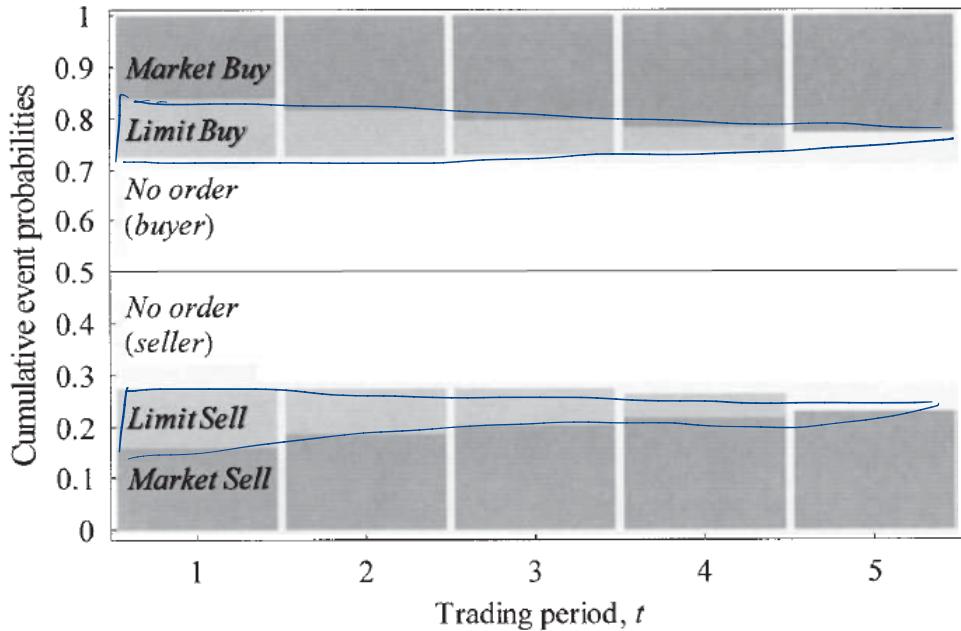
Remark 3.2. With the distribution of β_{T-1} , one computes the probabilities that each event occurs, e.g., $\Pr(\text{Buyer } T-1 \text{ does nothing}) = \Pr(\beta_{T-1} < \beta_{\text{Limit}}^{\text{Buy}})$. These probabilities define the execution probabilities for limit orders submitted at time $T-2$, which defines the optimal strategies of the trader (who is either a buyer or a seller) arriving at time $T-2$, and so on.

- 1. market buy
 - 2. limit buy B
 - 3. nothing T

Example 3.3. Assume that $V = 5.5$, $B = 5$, $A = 6$, and β_t are uniformly distributed over $(0, 2)$. The expected ask-side depth of the order book over time for trading days of $T = 3, 5, 7$, and 9 periods is plotted. The book fills more rapidly with larger T . This is because with longer trading days, a limit order is more attractive since there are more opportunities for it to be executed. Near the end of the trading day the depth drops, however.



Example 3.4. Consider the model with $T = 5$. The following figure depicts the probabilities of various events at each trading opportunity. Limit order usage declines over time, while market order usage increases.



3.4. Implications

ex ante prediction

The Parlour model provides empirical implications for the relationship between limit order usage and the state of the book. Overall, the model predicts that *both sides of the market will affect a trader's decision*. Specific predictions are as follows:

Prediction 1. There will be the order flow persistence, first observed by Biais et al. (1995), in that market sell orders are followed by market sell orders rather than market buy orders, i.e.,

$$\Pr(\text{Market Sell}_{t+1} | \text{Market Sell}_t) \geq \Pr(\text{Market Sell}_{t+1} | \text{Market Buy}_t).$$

A market buy order at time t reduces the queue at the ask and thus increases the execution probability of existing limit sell orders. So, a seller arriving at time $t + 1$ is more inclined to submit a limit sell order, or equivalently, less likely to submit a market sell order.

market buy 1
 \Rightarrow *limit sell* 2
 $n_t^A - 1$
 $i.e.,$
 \Rightarrow *priority* 3
(limit sell order)

Prediction 2. A “crowding-out mechanism” will occur in that a limit sell order at time t , by lengthening the queue at the ask, can make a limit sell order unattractive to a subsequent trader, i.e.,

$$\Pr(\text{Limit Sell}_{t+1} | \text{Other than Limit Sell}_t) \geq \Pr(\text{Limit Sell}_{t+1} | \text{Limit Sell}_t). \quad (3.1)$$

market buy 1
 \downarrow
 \uparrow
 prob
 T
 \uparrow
 $\text{market buy } T$
 $i.e.,$

Prediction 3. A market buy order will affect a limit sell order, since the market buy order at time t consumes liquidity at the ask and thus makes more attractive for the next trader to submit a limit sell order: i.e.,

market sell yet
limit sell or attractive

$$\Pr(\text{Limit Sell}_{t+1} | \text{Market Buy}_t) \geq \Pr(\text{Limit Sell}_{t+1} | \text{Market Sell}_t).$$

market buy T
 \Rightarrow *market sell T+1 ...* \downarrow
limit sell T+1, ... \uparrow

Prediction 4. The longer queue at the bid will induce more limit sell orders, i.e.,

$$\Pr(\text{Limit Sell}_{t+1} | \text{Limit Buy}_t) \geq \Pr(\text{Limit Sell}_{t+1} | \text{Limit Sell}_t).$$

Suppose that a limit buy order is placed at time t . Due to the crowding out of incoming limit buy orders, as predicted in (3.1), a seller at time $t+1$ anticipates more incoming market buy orders, which makes limit sells more attractive.

4. The Foucault (1999) Model

4.1. Model Description

good news

bad news

palauke
adverse risk.

if $V < BID$.

A trader arrives at each time $t = 1, \dots, T$, where T is a random stopping time. If trading terminates, the security pays $V_T = v_0 + \sum_{t=1}^T \varepsilon_t$, where ε_t is an independently distributed value innovation and $\Pr(\varepsilon_t = \sigma) = \Pr(\varepsilon_t = -\sigma) = 1/2$. The trader t has the reservation price, $R_t = v_t + y_t$, where $\Pr(y_t = L) = \Pr(y_t = -L) = 1/2$. Notice that y_t is independent of v_t at all leads and lags; i.e., y_t is not informative about the value but arises from "liquidation demand."

- If a trade is executed at a price P , a buyer (arriving when $y_t = L$) has utility $U(y_t) = (V_t + y_t) - P$ and a seller (arriving when $y_t = -L$) has utility $U(y_t) = P - (V_t + y_t)$. The order book at time t is characterized with $s_t = \{A_t, B_t\}$, where A_t and B_t are the best ask and bid prices.

The trader t can submit either a market order or a limit order. The market order ensures immediate execution at the unfavorable price. The limit order can be executed at the favorable price but is subject to adverse-selection risk as well as nonexecution risk. This is because limit prices are fixed over time (i.e., limit orders cannot be revised) and thus limit orders can become mispriced when new public information arrives. *cancel* *cancel*.

4.2. Predictions

Overall, the Foucault model argues that the asset volatility (i.e., the volatility of an underlying value which is not directly observable) is a determinant of the mix between market and limit orders. Specific predictions are as follows:

uninformed
limit buy order. $\frac{1}{T}$

$\Rightarrow V \uparrow \Rightarrow$ *more* *longer* *queue* *at* *bid* *order*
 $V \downarrow \Rightarrow$ " " " "

8
but monitoring *etc.*
Market *etc.* *etc.* *etc.*
not observe Foucault model
etc. *etc.*

* \leftarrow HSK *etc.*
limit order \leftarrow *etc.*
non-execution
adverse-selection
palauke *etc.*

- ↗ 1. non-execution risk limit buy ↑ priority ↓
 ↗ 2. adverse selection (winner's curse). good news
 bad news. → limit buy \rightarrow (market sell orders).
 but bad price.

Faucault - update $\frac{1}{2}$ pt.

market order ↴ negatively relate to the fill rate (i.e., the ratio of executed limit orders to a total number of submitted limit orders). When the asset volatility is higher, limit order traders are more exposed to adverse selection risk and tend to submit less aggressive orders (to ask for a larger compensation); consequently, the cost of market order trading increases. More traders find it optimal to carry their trades using limit orders, which in turn lowers execution probabilities since market order trading is less frequent.

Vol ↑ → good or bad news \rightarrow selection + aggressive x.
or \downarrow \rightarrow adverse selection + passive x.

Prediction 2. The proportion of limit orders will be positively related to the average size of the spread. Traders shade more limit orders and use market orders less frequently when the asset volatility increases. This creates a positive correlation between the size of the spread and the proportion of limit orders.

BID-ASK
spread ↑

order merit

Prediction 3. The increase in trading costs at the end of the trading day will be negatively related to the level of competition between limit order traders. In this paper, limit order traders react to increasing nonexecution risk at the time closer to closing by posting larger spreads, so that spreads enlarge at the end of the trading day.

Prediction 4. The average trading cost for market buy (sell) orders increases (decreases) with the ratio of buy to sell orders. If potential sellers decrease, the execution probability of a limit buy order decreases; as a consequence, potential buyers find it optimal to submit market buy orders. As the maximum ask prices at which buyers are willing to accept increase, limit sell orders are more attractive for potential sellers and thus bid prices must increase to attract market sell orders.

5. Empirical Findings

BIDT limit buy
seller market sell submit

The evolution of a market is described by a sequence of well-defined events such as order submissions (buy or sell, market or limit) and cancellations at different prices. The event occurrence is empirically modeled via a *multinomial logit model* or a *ordered choice model*.

- Many studies find that if the spread is wide, order choice tilts in favor of a limit order. Ellul et al. (2002), Hasbrouck and Saar (2003), and Renaldo (2004) test the Parlour model, and find that (a) the prediction that the same-side depth favors a market order is supported but (b) the prediction that the opposite-side depth favors a limit order is less clear. Smith (2000) and Hasbrouck and Saar (2003) provide mixed evidence for the Foucault model prediction regarding the effects of volatility.

Remark 5.1. Parlour (1998) and Foucault (1999) discover that traders' optimal strategies depend on conjectures of other traders' strategies. To simplify the analysis, both papers assume that trades solves "static" problems in which they are allowed to take only one action (i.e., submitting a market or a limit order without the ability to return to the market and update their strategies), which is far from a nature of dynamics and reality.

static - mz. (주소불가)

dynamic - 영역적. 유동적

dynamic -
(e.g. Equilibrium in a
Dynamic Limit order market.).

static P

三

: non execution -

: (non-execution
adverse

Prediction 4.

Seller \downarrow ($\frac{b}{s} \uparrow$).

limit buy \downarrow execut \downarrow

mar buy

Ask \uparrow

limit sell \uparrow \rightarrow bid \uparrow

mark sell \uparrow

(market sell ∞
execute \rightarrow mkt sell \uparrow).

\therefore mkt sell trader cost \downarrow .

✓ 거래 수수료 차이가 문제인 X.

- lecture 1 ~ 8 대.

제한 주문이 많음.

매수 주문이 많음 X.

제한 주문이 많음 때문인 문제를 가짐.

마켓 주문이 많음.
제한 주문이 많음인 이유는?

(9월).



증권 문제 초기 X.

provide economic intuition.

explain why

특정 시장에서 거래량이 많은 경우 제한 주문이 많음.

✓ 거래 수수료 X.

(22 / 20).

제한 주문

✓ 수수료 차이가 문제인 X.

제한 주문이 많음 X.

✓ 다른 시장 주문이 많음.