homoscedastic : all its r.v have same finite variance

Lecture 9. Conditional Heteroskedastic Models

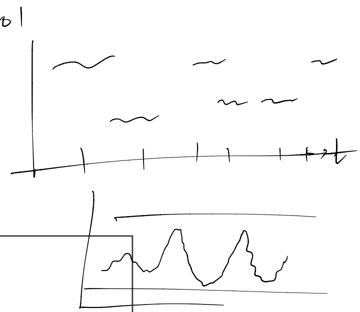
absence of homo

different dispersion : different variance

1. Empirical Regularities of Volatility

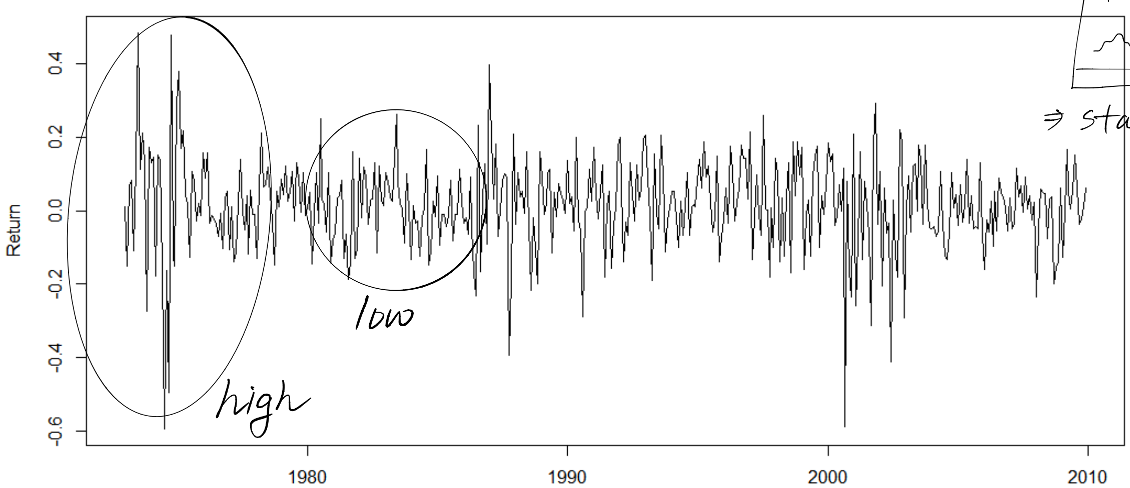
There are some empirical regularities for stock return volatility. First, there exits volatility clustering in that volatility is high for certain time periods and low for other periods. Second, volatility evolves over time in a continuous manner, and volatility jumps are rare. Third, volatility varies within some fixed range, meaning that volatility is often stationary. Fourth, volatility reacts differently to a big price increase and a big price drop with the latter having a greater impact, referred to as the "leverage effect."

Example 1.1. Consider the monthly log stock returns for Intel Corporation from January 1973 to December 2009.




```
> mydat <- read.table("data1.txt", header = T)
> rtn <- ts(log(mydat$intc + 1), frequency = 12, start = c(1973,1))
> plot(rtn, xlab = "", ylab = "Return")
```

⇒ stationary.



- High volatile periods are followed by high volatile periods, and low volatile periods are by low volatile periods; i.e., a return series has the "time-varying" volatility *conditional on changes in past volatility*.

$\sigma_t <$ σ_t^H if σ_{t-1}^H ; σ_t^L if σ_{t-1}^L

$\sigma_t = f(\sigma_{t-1})$

$\sigma_t = \alpha + \beta \cdot \sigma_{t-1}$ $\beta > 1$

2. The ARCH Model

2.1. Properties

Let r_t be the log return of an asset at time t. The conditional mean and variance of r_t given I_{t-1} are denoted by

$$\mu_t = E[r_t|I_{t-1}]$$
$$\sigma_t^2 = Var[r_t|I_{t-1}] = E[(r_t - \mu_t)^2|I_{t-1}],$$

$= E[(r_t - \underbrace{E[r_t|I_{t-1}]}_{\mu_t})^2 | I_{t-1}]$

where I_{t-1} denotes the information set available at time $t-1$.

cumulate up to $t-1$

- The conditional mean μ_t is referred to as the *mean equation* for r_t. The conditional variance σ_t^2 is referred to as the *volatility equation* for r_t. Conditional heteroskedastic models are concerned with the evolution of σ_t^2.

The *Autoregressive Conditional Heteroskedasticity* (ARCH) model of Engle (1982) specifies that r_t follows an ARMA(p, q)-ARCH(m) process in the form

$r_t = \mu_t + \varepsilon_t$

$\varepsilon_t = z_t \sigma_t$

$z_t \sim WN(0,1)$

$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$

$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + \cdots + a_m \varepsilon_{t-m}^2$

$$\begin{aligned} r_t &= \mu_t + \varepsilon_t \\ \varepsilon_t &= z_t \sigma_t \\ z_t &\sim WN(0,1) \\ \mu_t &= \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} \\ \sigma_t^2 &= a_0 + a_1 \varepsilon_{t-1}^2 + \cdots + a_m \varepsilon_{t-m}^2, \end{aligned} \tag{2.1}$$

$\mu_t = E[r_t | I_{t-1}]$

$\sigma_t^2 = Var[r_t | I_{t-1}]$
$= E[(r_t - \mu_t)^2 | I_{t-1}]$

$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$

$r_t - \varepsilon_t = ''$
$\parallel$
$\mu_t = ''$

where $a_0 > 0$ and $a_i \geq 0$ for $i > 0$.

- In (2.1), large past squared shocks $\varepsilon_{t-1}^2, \ldots, \varepsilon_{t-m}^2$ imply a large conditional variance σ_t^2, thereby meaning that large shocks tend to be followed by another large shock. This conforms to the notion of the volatility clustering.

Theorem 2.1. *For random variables X and Y, the law of iterated expectations states*

$$E[X] = E[E[X|Y]].$$

Proposition 2.2. *Suppose that* $\varepsilon_t = z_t \sigma_t$, $z_t \sim WN(0,1)$, *and* σ_t *is a function of variables in* I_{t-1}. *Then, it shows that (a)* $E[\varepsilon_t] = 0$ *for all t and (b)* $Var[\varepsilon_t] = E[\sigma_t^2]$.

Proof. Notice that

$$\begin{aligned} E[\varepsilon_t] &= E[E[\varepsilon_t | I_{t-1}]] \\ &= E[E[z_t \sigma_t | I_{t-1}]] \\ &= E[\sigma_t E[z_t | I_{t-1}]] \\ &= E[\sigma_t E[z_t]] \\ &= 0 \end{aligned}$$

for all t and

$$\begin{aligned} Var[\varepsilon_t] &= E[\varepsilon_t^2] \\ &= E[E[\varepsilon_t^2 | I_{t-1}]] \\ &= E[E[z_t^2 \sigma_t^2 | I_{t-1}]] \\ &= E[\sigma_t^2 E[z_t^2 | I_{t-1}]] \\ &= E[\sigma_t^2 E[z_t^2]] \\ &= E[\sigma_t^2]. \end{aligned}$$

ARMA(p,q) − ARCH(m)

$$\mu_t = E[r_t \mid I_{t-1}]$$
$$\sigma_t^2 = Var[r_t \mid I_{t-1}] = E[(r_t - \mu_t)^2 \mid I_{t-1}]$$

$$r_t = \mu_t + \varepsilon_t$$
$$\varepsilon_t = z_t \sigma_t$$
$$z_t \sim WN(0,1).$$
$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$
$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + \cdots + a_m \varepsilon_{t-m}^2.$$

$$r_t \sim ARMA(p,q).$$
$$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$
$$r_t - \varepsilon_t = \mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

$$
\begin{aligned}
E[\varepsilon_t] &= E[E[\varepsilon_t \mid I_{t-1}]] \\
&= E[E[z_t\sigma_t \mid I_{t-1}]] \\
&= E[\sigma_t E[z_t \mid I_{t-1}]] \\
&= E[\sigma_t E[z_t]] = 0
\end{aligned}
$$

$$
\begin{aligned}
Var[\varepsilon_t] &= E[\varepsilon_t^2] \quad (\because E[\varepsilon_t] = 0) \\
&= E[E[\varepsilon_t^2 \mid I_{t-1}]] \\
&= E[E[z_t^2\sigma_t^2 \mid I_{t-1}]] \quad = E[E[z_t^2 \mid I_{t-1}]\, E[\sigma_t^2 \mid I_{t-1}]] \\
&= E[\sigma_t^2 E[z_t^2 \mid I_{t-1}]] \quad = E[E[\sigma_t^2 \mid I_{t-1}]] \\
&= E[\sigma_t^2 E[z_t^2]] \quad = E[\sigma_t^2] \\
&= E[\sigma_t^2] \quad (\because E[z] = 0,\ Var[z_t] = 1,\ E[z^2] = 1).
\end{aligned}
$$

□

2.2. *Order Determination*

Define $\eta_t = \varepsilon_t^2 - \sigma_t^2$ and assume it quite small. Then, the volatility equation (2.1) becomes

$\sigma_t^2 = \varepsilon_t^2 - \eta_t$

$$\varepsilon_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \cdots + a_m\varepsilon_{t-m}^2 + \eta_t,$$

$r_t = \mu_t + \varepsilon_t$
$= \hat{\mu}_t + \hat{\varepsilon}_t$

which is an AR(m) model for ε_t^2, except that η_t is "not" a white noise error.

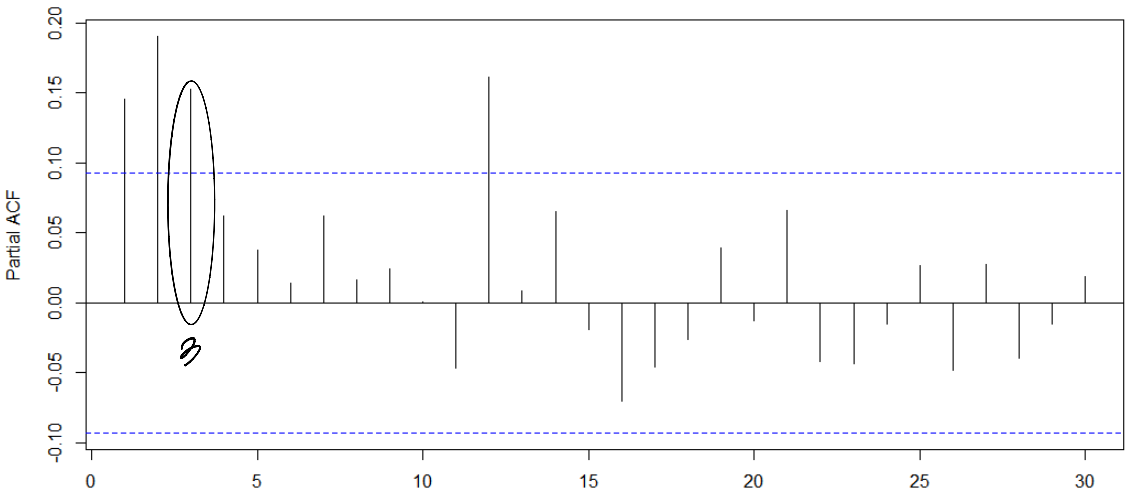
- In practice, one can use the PACF of $\hat{\varepsilon}_t^2 = (r_t - \hat{\mu}_t)^2$ to determine the ARCH order m.

Remark 2.3. Suppose that the mean equation is $\mu_t = \phi_0$. Then, $r_t = \phi_0 + \varepsilon_t$, which implies $E[r_t] = \phi_0$. Since the sample mean of r_t (denoted by $\bar{r}$) is an unbiased estimate of $E[r_t]$ or ϕ_0, one obtains

$$\hat{\varepsilon}_t = r_t - \hat{\mu}_t = r_t - \hat{\phi}_0 = r_t - \bar{r}.$$

Example 2.4. Consider the monthly log stock returns for Intel Corporation. Suppose that the mean equation is $\mu_t = \phi_0$. Then, one can use $\hat{\varepsilon}_t^2 = (r_t - \bar{r})^2$, where $\bar{r}$ is the sample mean of r_t.

```
> adj.rtn <- rtn - mean(rtn)
> pacf(c(adj.rtn^2), lag = 30, main = "")
```



- The PACF plot selects $m = 3$, and consequently the ARCH(3) model is specified as follows:

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + a_3\varepsilon_{t-3}^2.$$

2.3. *Estimation*

Consider an ARMA(p, q)-ARCH(m) model in the form

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$
$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \cdots + a_m\varepsilon_{t-m}^2.$$

The unknown parameters $\phi_0, \ldots, \phi_p$, $\theta_1, \ldots, \theta_q$, and $a_0, \ldots, a_m$ are estimated using conditional maximum likelihood estimation (MLE). See Bollerslev, Engle, and Nelson (1994) for details.

- With the parameter estimates, one constructs estimates of σ_t^2 as

$$\hat{\sigma}_t^2 = \hat{a}_0 + \sum_{i=1}^{m} \hat{a}_i \hat{\varepsilon}_{t-i}^2,$$

where $\hat{\varepsilon}_{t-i} = r_{t-i} - \hat{\mu}_{t-i}$.

Remark 2.5. In practice, z_t is assumed to follow a standard normal, a standardized Student t distribution, or a generalized error distribution (GED).

Example 2.6. Consider the monthly log stock returns for Intel Corporation. Consider an ARMA(0, 0)-ARCH(3) model of the form

from PACF

$$\mu_t = \phi_0$$
$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + a_3\varepsilon_{t-3}^2.$$

Assume that z_t is iid standard normal.

```
> library(rugarch)
> spec1 <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(garchOrder = c(3, 0)))
> (fit <- ugarchfit(spec = spec1, data = rtn))
       Estimate Std. Error t value Pr(>|t|)
mu     0.012564   0.005512  2.2792 0.022653
omega  0.010426   0.001244  8.3829 0.000000
alpha1 0.236038   0.112973  2.0893 0.036678
alpha2 0.075176   0.047596  1.5795 0.114229
alpha3 0.052143   0.045422  1.1480 0.250975
```

- The fitted model is

$$\mu_t = 0.012564$$
$$\sigma_t^2 = 0.010426 + 0.236038\varepsilon_{t-1}^2 + 0.075176\varepsilon_{t-2}^2 + 0.052143\varepsilon_{t-3}^2.$$

The estimates of a_2 and a_3 are statistically insignificant. One may drop them for refinement.

```
> spec2 <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(garchOrder = c(1, 0)))
> (fit <- ugarchfit(spec = spec2, data = rtn))
       Estimate Std. Error t value Pr(>|t|)
mu     0.013133   0.005315  2.4711 0.013470
omega  0.011048   0.001199  9.2126 0.000000
alpha1 0.378502   0.113552  3.3333 0.000858
```

ARMA(0,0) - ARCH(1)

- The fitted model is

$$\mu_t = 0.013133$$
$$\sigma_t^2 = 0.011048 + 0.378502\varepsilon_{t-1}^2.$$

2.4. Model Checking

Since $z_t = \varepsilon_t/\sigma_t$ are iid random variables by construction, for a properly specified ARCH model, the standardized residuals

$$\hat{z} = \tilde{\varepsilon}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$$

표기 중요X

ratio가 중요한 것

$$\varepsilon_t = z_t \sigma_t$$
$$z_t = \frac{\varepsilon_t}{\sigma_t} \overset{iid}{\sim} N(0,1)$$
$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t} \overset{iid}{\sim} N(0,1)$$

should not be autocorrelated.

- One can check the adequacy of a fitted ARCH model by performing the Ljung-Box test for $\{\tilde{\varepsilon}_t\}$.

Example 2.7. Consider the ARMA(0, 0)-ARCH(1) model for the monthly log stock returns for Intel Corporation.

$$\hat{\varepsilon}_t = r_t - \hat{\mu}_t$$
$$\hat{\varepsilon}_t^2 = a_0 + a_1 \hat{\varepsilon}_{t-1}^2 + \cdots + a_m \hat{\varepsilon}_{t-m}^2 + \eta_t$$
$$\eta_t = \varepsilon_t^2 - \sigma_t^2$$

```
> std.e <- residuals(fit, standardize = T)
> Box.test(std.e, lag = 20, type = "Ljung")
Box-Ljung test
data: std.e
X-squared = 24.3062, df = 20, p-value = 0.2293
```

- The null hypothesis of no autocorrelation cannot be rejected. So, the ARMA(0, 0)-ARCH(1) model is adequate.

Ljung – Box Test

: H_0 : no residual autocorrelation

2.5. Volatility Forecasting

Proposition 2.8. *For an ARCH(m) model, it shows*

$$E[\sigma_{t+k}^2 z_{t+k}^2 | I_t] = E[\sigma_{t+k}^2 | I_t].$$

$$= \sigma_{t+k}^2 E[z_{t+k}^2 | I_t] = \sigma_{t+k}^2$$

Proof. From (2.1), one sees

$$\begin{aligned} \sigma_{t+k}^2 &= a_0 + a_1\varepsilon_{t+k-1}^2 + a_2\varepsilon_{t+k-2}^2 + \cdots + a_m\varepsilon_{t+k-m}^2 \\ &= a_0 + a_1 z_{t+k-1}^2 \sigma_{t+k-1}^2 + a_2 z_{t+k-2}^2 \sigma_{t+k-2}^2 + \cdots + a_m z_{t+k-m}^2 \sigma_{t+k-m}^2, \end{aligned}$$

which means that σ_{t+k}^2 is a function of $\{z_{t+k-1}^2, z_{t+k-2}^2, \ldots\}$ and $\{\sigma_{t+k-1}^2, \sigma_{t+k-2}^2, \ldots\}$. Similarly, one sees that σ_{t+k-1}^2 is a function of $\{z_{t+k-2}^2, z_{t+k-3}^2, \ldots\}$ and $\{\sigma_{t+k-2}^2, \sigma_{t+k-3}^2, \ldots\}$ and so on. So, σ_{t+k}^2 is a function of $\{z_{t+k-1}^2, z_{t+k-2}^2, \ldots\}$; consequently, σ_{t+k}^2 and z_{t+k}^2 are independent. Therefore, it shows

$$\begin{aligned} E[\sigma_{t+k}^2 z_{t+k}^2 | I_t] &= E[\sigma_{t+k}^2 | I_t] E[z_{t+k}^2 | I_t] \\ &= E[\sigma_{t+k}^2 | I_t] E[z_{t+k}^2] \\ &= E[\sigma_{t+k}^2 | I_t]. \end{aligned}$$

□

$$E[\varepsilon_{t+k}^2 \mid I_t] = E[\sigma_{t+k}^2 Z_{t+k}^2 \mid I_t] = E[\sigma_{t+k}^2 \mid I_t]$$

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + a_2 \varepsilon_{t-2}^2 + \cdots + a_m \varepsilon_{t-m}^2$$

$$\sigma_{t+k}^2 = a_0 + a_1 \varepsilon_{t+k-1}^2 + a_2 \varepsilon_{t+k-2}^2 + \cdots + a_m \varepsilon_{t+k-m}^2$$

$$= a_0 + a_1 Z_{t+k-1}^2 \underline{\sigma_{t+k-1}^2} + \cdots + a_m Z_{t+k-m}^2 \underline{\sigma_{t+k-m}^2}$$

σ_{t+k}^2 is function of $\{Z_{t+k-1}^2, Z_{t+k-2}^2, \cdots\}$

$\{\underline{\sigma_{t+k-1}^2}, \sigma_{t+k-2}^2, \cdots\}$

σ_{t+k-1}^2 is function of $\{Z_{t+k-2}^2, Z_{t+k-3}^2, \cdots\}$

$\{\sigma_{t+k-2}^2, \sigma_{t+k-3}^2, \cdots\}$

$\therefore$ σ_{t+k}^2 is function of $\{Z_{t+k-1}^2, Z_{t+k-2}^2, \cdots\}$

$\therefore$ σ_{t+k}^2, Z_{t+k}^2 are independent.

forecast.

$$\hat{\sigma}_t^2[1] = E[\sigma_{t+1}^2 \mid I_t]$$

$$= E[a_0 + a_1 \varepsilon_t^2 + a_2 \varepsilon_{t-1}^2 + \cdots + a_m \varepsilon_{t+1-m}^2 \mid I_t]$$

$$= a_0 + a_1 \varepsilon_t^2 + a_2 \varepsilon_{t-1}^2 + \cdots + a_m \varepsilon_{t+1-m}^2$$

$$\hat{\sigma}_t^2[2] = E[\sigma_{t+2}^2 \mid I_t]$$

$$= E[a_0 + a_1 \varepsilon_{t+1}^2 + a_2 \varepsilon_t^2 + \cdots + a_m \varepsilon_{t+2-m}^2 \mid I_t]$$

$$= a_0 + a_1 E[\varepsilon_{t+1}^2 \mid I_t] + a_2 \varepsilon_t^2 + \cdots + a_m \varepsilon_{t+2-m}^2$$

$$= a_0 + a_1 E[\sigma_{t+1}^2 Z_{t+1}^2 \mid I_t] + a_2 \varepsilon_t^2 + \cdots + a_m \varepsilon_{t+2-m}^2$$

$$= a_0 + a_1 E[\sigma_{t+1}^2 \mid I_t] E[Z_{t+1}^2 \mid I_t] + \quad ''$$

$$= a_0 + a_1 \hat{\sigma}_t^2[1] + a_2 \varepsilon_t^2 + \cdots + a_m \varepsilon_{t+2-m}^2$$

Remark 2.9. This proposition holds for a general case in which $\varepsilon_t = z_t\sigma_t$, $z_t \sim WN(0,1)$, and σ_t is a function of variables in I_{t-1}.

- For an ARCH(m) model, the s-step ahead volatility forecast is computed in a recursive manner. The 1-step ahead volatility forecast is given by

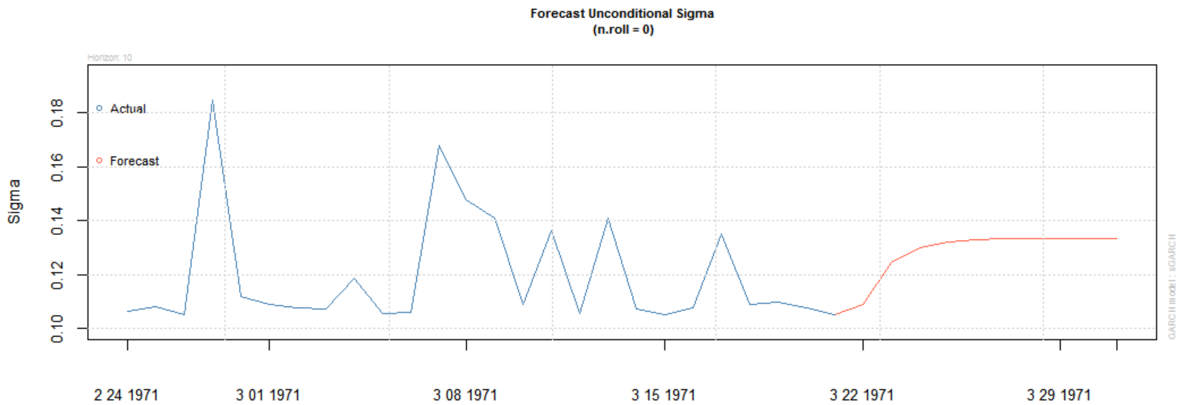
$$\begin{aligned}\hat{\sigma}_t^2[1] &= E[\sigma_{t+1}^2|I_t]\\ &= E[a_0 + a_1\varepsilon_t^2 + a_2\varepsilon_{t-1}^2 + \cdots + a_m\varepsilon_{t+1-m}^2|I_t]\\ &= a_0 + a_1\varepsilon_t^2 + a_2\varepsilon_{t-1}^2 + \cdots + a_m\varepsilon_{t+1-m}^2.\end{aligned}$$

The 2-step ahead volatility forecast is given by

$$\begin{aligned}\hat{\sigma}_t^2[2] &= E[\sigma_{t+2}^2|I_t]\\ &= E[a_0 + a_1\varepsilon_{t+1}^2 + a_2\varepsilon_t^2 + \cdots + a_m\varepsilon_{t+2-m}^2|I_t]\\ &= a_0 + a_1E[\varepsilon_{t+1}^2|I_t] + a_2\varepsilon_t^2 + \cdots + a_m\varepsilon_{t+2-m}^2\\ &= a_0 + a_1E[z_{t+1}^2\sigma_{t+1}^2|I_t] + a_2\varepsilon_t^2 + \cdots + a_m\varepsilon_{t+2-m}^2\\ &= a_0 + a_1E[\sigma_{t+1}^2|I_t] + a_2\varepsilon_t^2 + \cdots + a_m\varepsilon_{t+2-m}^2\\ &= a_0 + a_1\hat{\sigma}_t^2[1] + a_2\varepsilon_t^2 + \cdots + a_m\varepsilon_{t+2-m}^2.\end{aligned}$$

Example 2.10. Consider the ARMA(0, 0)-ARCH(1) model for the monthly log stock returns for Intel Corporation.

```
> pred <- ugarchforecast(fit, n.ahead = 10)
> plot(pred, which = 3)
```



3. The GARCH Model

The ARCH model requires a long lag length m, so that a large number of parameters should be estimated. An alternative and more flexible lag structure is provided by the *Generalized Autoregressive Conditional Heteroskedasticity* (GARCH) model of Bollerslev (1986) who replaces (2.1) with

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i\varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j\sigma_{t-j}^2,$$

ARCH (m).

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2$$

GARCH (m, n)

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j \sigma_{t-j}^2$$

most case
$m = 1$, $n = 1$
sufficient.

r_t 아님 ??

where $a_0 > 0$, $a_i \geq 0$, $b_j \geq 0$, and $\sum_{i=1}^{\max(m,n)}(a_i + b_i) < 1$. This model is referred to as a GARCH(m, n) model.

$\sigma_t^2 \approx Var(\varepsilon_t | I_t)$?

- The conditional variance of ε_t depends on (a) the squared residuals ε_{t-i}^2 in the previous m periods and (b) the variance σ_{t-j}^2 in the previous n periods. In many applications, a GARCH(1, 1) model is enough to obtain a good fit for financial time series.

$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2$

3.1. *Properties*

Consider the simplest GARCH(1, 1) model with

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + \overset{\approx 0.9}{b_1} \sigma_{t-1}^2, \tag{3.1}$$

$\Rightarrow$ σ_t^2 can be explained with σ_{t-1}^2

where $a_0 > 0$, $a_1 \geq 0$, $b_1 \geq 0$, and $a_1 + b_1 < 1$.

- Let $\eta_t = \varepsilon_t^2 - \sigma_t^2$ and assume it quite small. Then, one writes (3.1) as

$\varepsilon_t = z_t \sigma_t$
$z_t \sim WN(0,1)$

$$\begin{aligned} \varepsilon_t^2 &= a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2 + \eta_t \\ &= a_0 + a_1 \varepsilon_{t-1}^2 + b_1(\varepsilon_{t-1}^2 - \eta_{t-1}) + \eta_t \\ &= a_0 + (a_1 + b_1)\varepsilon_{t-1}^2 + \eta_t - b_1 \eta_{t-1}. \end{aligned} \tag{3.2}$$

The squared residual terms can be expressed as an ARMA(1, 1) model, although η_t is not a white noise process. So, a GARCH model can be regarded as an application of the ARMA idea to the squared residual ε_t^2.

Remark 3.1. The ARMA representation of ε_t^2 states that the GARCH(1, 1) model is stationary, provided $\phi_1 = a_1 + b_1 < 1$. In many cases of weekly or daily financial time series, the GARCH coefficient b_1 is found to be around 0.9.

but ?? $\eta_t \neq WN$

3.2. *Estimation*

An ARMA(p, q)-GARCH(m, n) model is in the form

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j \sigma_{t-j}^2.$$

- The unknown parameters $\phi_0, \ldots, \phi_p$, $\theta_1, \ldots, \theta_q$, $a_0, \ldots, a_m$, and $b_1, \ldots, b_n$ are estimated simultaneously using a conditional MLE method. With the parameter estimates, one constructs estimates of the volatility σ_t^2 as

$$\hat{\sigma}_t^2 = \hat{a}_0 + \sum_{i=1}^{m} \hat{a}_i \hat{\varepsilon}_{t-i}^2 + \sum_{j=1}^{n} \hat{b}_j \hat{\sigma}_{t-j}^2,$$

where $\hat{\varepsilon}_t = r_t - \hat{\mu}_t$.

GARCH(2,2) model. $\eta_t = \varepsilon_t^2 - \sigma_t^2$

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + b_1\sigma_{t-1}^2 + b_2\sigma_{t-2}^2$$

$$\varepsilon_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + a_2\varepsilon_{t-2}^2 + b_1(\varepsilon_{t-1}^2 - \eta_{t-1}) + b_2(\varepsilon_{t-2}^2 - \eta_{t-2}) + \eta_t$$

$$= a_0 + (a_1+b_1)\varepsilon_{t-1}^2 + (a_2+b_2)\varepsilon_{t-2}^2 + \eta_t - b_1\eta_{t-1} - b_2\eta_{t-2}$$

Although η_t is not a white noise process squared residual term can be expressed as an ARMA(2,2) model

---

forecast.

GARCH model is stationary.

$a_1 + b_1 < 1$, $a_2 + b_2 < 1$.

$$\hat{\sigma}_t^2[1] = E[\sigma_{t+1}^2 \mid I_t]$$
$$= E[a_0 + a_1\varepsilon_t^2 + b_1\sigma_t^2 \mid I_t]$$
$$= a_0 + a_1\varepsilon_t^2 + b_1\sigma_t^2$$

$$\hat{\sigma}_t^2[2] = E[\sigma_{t+2}^2 \mid I_t]$$
$$= E[a_0 + a_1\varepsilon_{t+1}^2 + b_1\sigma_{t+1}^2 \mid I_t]$$
$$= a_0 + a_1 E[\varepsilon_{t+1}^2 \mid I_t] + b_1 E[\sigma_{t+1}^2 \mid I_t]$$
$$= a_0 + a_1 E[z_{t+1}^2\sigma_{t+1}^2 \mid I_t] + b_1\hat{\sigma}_t^2[1]$$
$$= a_0 + a_1\{E[z_{t+1}^2 \mid I_t]E[\sigma_{t+1}^2 \mid I_t]\} + b_1\hat{\sigma}_t^2[1]$$
$$= a_0 + a_1 E[\sigma_{t+1}^2 \mid I_t] + b_1\hat{\sigma}_t^2[1]$$
$$= a_0 + (a_1+b_1)\hat{\sigma}_t^2[1]$$

$$\hat{\sigma}_t^2[s] = E[\sigma_{t+s}^2 \mid I_t]$$
$$= E[a_0 + a_1\varepsilon_{t+s-1}^2 + b_1\sigma_{t+s-1}^2 \mid I_t]$$
$$= a_0 + a_1 E[z_{t+s-1}^2 \mid I_t] + b_1 E[\sigma_{t+s-1}^2 \mid I_t]$$
$$= a_0 + (a_1+b_1)\hat{\sigma}_t^2[s-1]$$
$$= a_0 + (a_1+b_1)(a_0 + (a_1+b_1)\hat{\sigma}_t^2[s-2])$$

$= a_0 + (a_1+b_1)a_0 + (a_1+b_1)^2 \hat{\sigma}_t^2[s-2]$

$= a_0 + (a_1+b_1)a_0 + (a_1+b_1)^2 a_0 + (a_1+b_1)^3 \hat{\sigma}_t^2[s-3]$

$= a_0 + (a_1+b_1)a_0 + \cdots + (a_1+b_1)^{s-2} a_0 + (a_1+b_1)^{s-1} \hat{\sigma}_t^2[s-(s-1)]$

$= a_0 + (a_1+b_1)a_0 + \cdots + (a_1+b_1)^{s-2} a_0 + (a_1+b_1)^{s-1} \hat{\sigma}_t^2[1]$

Since $a_1 + b_1 < 1$.

$= \dfrac{a_0(1-(a_1+b_1)^{s-1})}{1-a_1-b_1} + (a_1+b_1)^{s-1} \hat{\sigma}_t^2[1]$

GARCH (2, 2) model forecast.

$\hat{\sigma}_t^2[1] = E[\sigma_{t+1}^2 \mid I_{t}]$

$= E[a_0 + a_1\varepsilon_t^2 + a_2\varepsilon_{t-1}^2 + b_1\sigma_t^2 + b_2\sigma_{t-1}^2 \mid I_t]$

$= a_0 + a_1\varepsilon_t^2 + a_2\varepsilon_{t-1}^2 + b_1\sigma_t^2 + b_2\sigma_{t-1}^2$

$\hat{\sigma}_t^2[2] = E[\sigma_{t+2}^2 \mid I_t]$

$= E[a_0 + a_1\varepsilon_{t+1}^2 + a_2\varepsilon_t^2 + b_1\sigma_{t+1}^2 + b_2\sigma_t^2 \mid I_t]$

$= a_0 + (a_1+b_1)\hat{\sigma}_t^2[1] + a_2\varepsilon_t^2 + b_2\sigma_t^2$

$\hat{\sigma}_t^2[s] = E[\sigma_{t+s}^2 \mid I_t]$

$= E[a_0 + a_1\varepsilon_{t+s-1}^2 + a_2\varepsilon_{t+s-2}^2 + b_1\sigma_{t+s-1}^2 + b_2\sigma_{t+s-2}^2 \mid I_t]$

$= a_0 + (a_1+b_1)\hat{\sigma}_t^2[s-1] + (a_2+b_2)\hat{\sigma}_t^2[s-2]$

$= a_0 + (a_1+b_1)(a_0 + (a_1+b_1)\hat{\sigma}_t^2[s-2] + (a_2+b_2)\hat{\sigma}_t^2[s-3])$
$+ (a_2+b_2)\hat{\sigma}_t^2[s-2]$

$= a_0 + (a_1+b_1)a_0 + (a_1+b_1)^2\hat{\sigma}_t^2[s-2] + (a_1+b_1)(a_2+b_2)\hat{\sigma}_t^2[s-3]$
$+ (a_2+b_2)\hat{\sigma}_t^2[s-2]$

...

3.3. Forecasting

Consider the simplest GARCH(1, 1) model in (3.1). The 1-step ahead forecast is

$$\begin{aligned}\hat{\sigma}_t^2[1] &= E[\sigma_{t+1}^2|I_t] \\ &= a_0 + a_1\varepsilon_t^2 + b_1\sigma_t^2.\end{aligned}$$

The 2-step ahead forecast is

$$\begin{aligned}\hat{\sigma}_t^2[2] &= E[\sigma_{t+2}^2|I_t] \\ &= E[a_0 + a_1\varepsilon_{t+1}^2 + b_1\sigma_{t+1}^2|I_t] \\ &= E[a_0 + a_1\sigma_{t+1}^2 z_{t+1}^2 + b_1\sigma_{t+1}^2|I_t] \\ &= a_0 + a_1E[\sigma_{t+1}^2 z_{t+1}^2|I_t] + b_1E[\sigma_{t+1}^2|I_t] \\ &= a_0 + a_1E[\sigma_{t+1}^2|I_t]E[z_{t+1}^2|I_t] + b_1E[\sigma_{t+1}^2|I_t] \\ &= a_0 + a_1E[\sigma_{t+1}^2|I_t]E[z_{t+1}^2] + b_1E[\sigma_{t+1}^2|I_t] \\ &= a_0 + (a_1+b_1)E[\sigma_{t+1}^2|I_t] \\ &= a_0 + (a_1+b_1)\hat{\sigma}_t^2[1].\end{aligned}$$

In general, one obtains

$$\hat{\sigma}_t^2[s] = a_0 + (a_1+b_1)\hat{\sigma}_t^2[s-1].$$

- In the GARCH(1, 1) model, the s-step ahead forecast is given by

$$\begin{aligned}\hat{\sigma}_t^2[s] &= a_0 + (a_1+b_1)\hat{\sigma}_t^2[s-1] \\ &= a_0 + (a_1+b_1)\left[a_0 + (a_1+b_1)\hat{\sigma}_t^2[s-2]\right] \\ &= a_0 + (a_1+b_1)a_0 + (a_1+b_1)^2\hat{\sigma}_t^2[s-2] \\ &\;\vdots \\ &= \frac{a_0[1-(a_1+b_1)^{s-1}]}{1-a_1-b_1} + (a_1+b_1)^{s-1}\hat{\sigma}_t^2[1]\end{aligned}$$

for $s > 1$. Similarly, one can construct the s-step ahead forecast of the general GARCH(p, q).

$\to \mu_t = \phi_0$

Example 3.2. Consider an ARMA(0, 0)-GARCH(1, 1) model for the monthly log stock returns for Intel Corporation. Assume that $z_t \overset{iid}{\sim} N(0,1)$.

```
> spec1 <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(garchOrder = c(1, 1)))
> (fit1 <- ugarchfit(spec = spec1, data = rtn))
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.011270    0.005393   2.0898 0.036639
omega   0.000918    0.000389   2.3624 0.018158
alpha1  0.085916    0.026449   3.2484 0.001160
beta1   0.852779    0.039490  21.5949 0.000000
```

distribution.model = "normal"
default

m
w
d
mst g

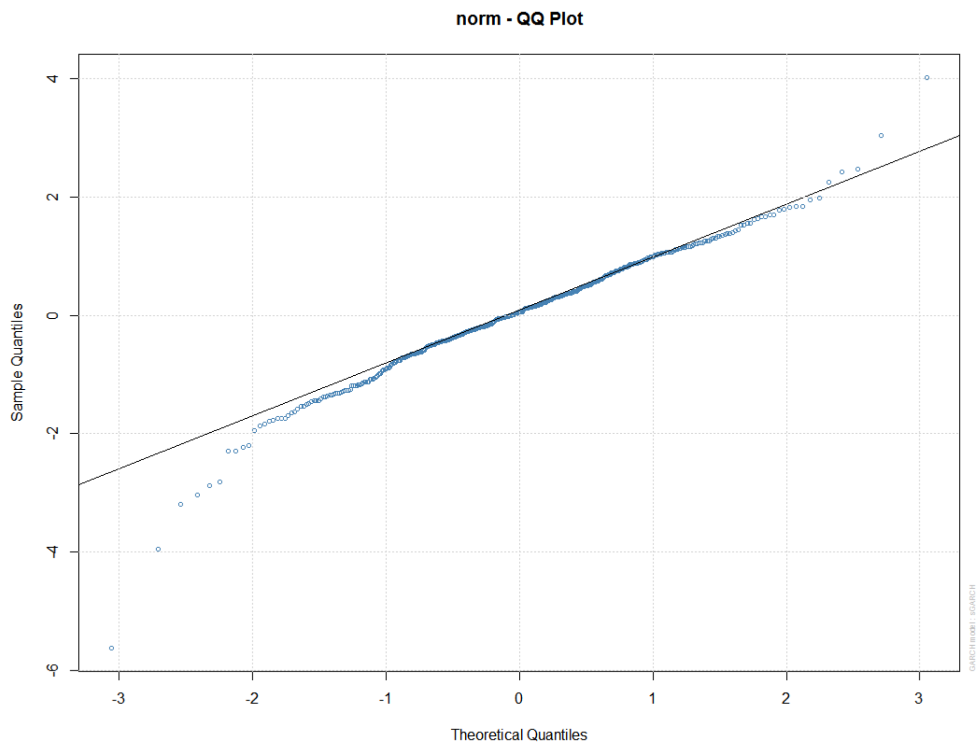
- The fitted model is

$$\mu_t = 0.011270$$
$$\sigma_t^2 = 0.000918 + 0.085916\varepsilon_{t-1}^2 + 0.852779\sigma_{t-1}^2.$$

```
> std.e <- residuals(fit1, standardize = T)
> Box.test(std.e, lag = 20, type = "Ljung")
Box-Ljung test
data: std.e
X-squared = 16.431, df = 20, p-value = 0.6896
```

- The null hypothesis of no autocorrelation for $\tilde{\varepsilon}_t$ cannot be rejected.

```
> plot(fit1, which = 9)
```



- There is significant deviation in both tails from the normal qq-line, so that the normality assumption of z_t may not be appropriate.

```
> spec2 <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(garchOrder = c(1, 1)),
                      distribution.model = "std")
> (fit2 <- ugarchfit(spec = spec2, data = rtn))
        Estimate Std. Error t value Pr(>|t|)
mu      0.016508   0.005103  3.2348 0.001217
omega   0.001156   0.000578  2.0002 0.045476
```

t dist^n

```
alpha1 0.105542   0.037176  2.8390 0.004526
beta1  0.817265   0.058094 14.0679 0.000000
shape  6.787868   1.861988  3.6455 0.000267
```

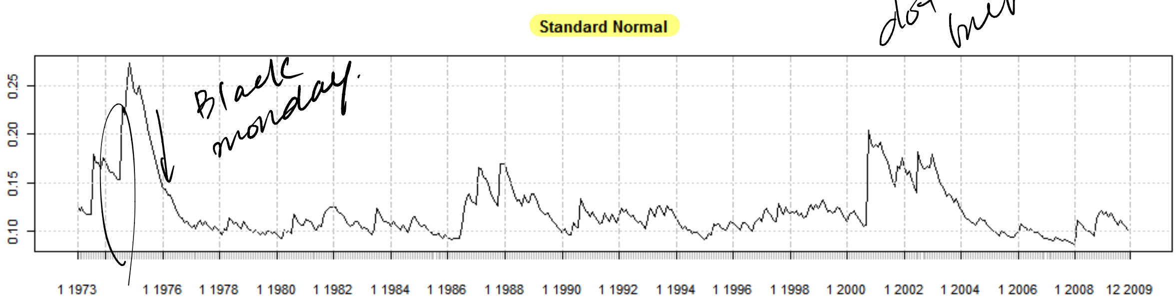
- Assuming that z_t follows an iid standard Student t distribution, one obtains the fitted model as

$$\mu_t = 0.016508$$
$$\sigma_t^2 = 0.001156 + 0.1055426\varepsilon_{t-1}^2 + 0.817265\sigma_{t-1}^2$$
$$z_t \sim t_{6.787868}.$$

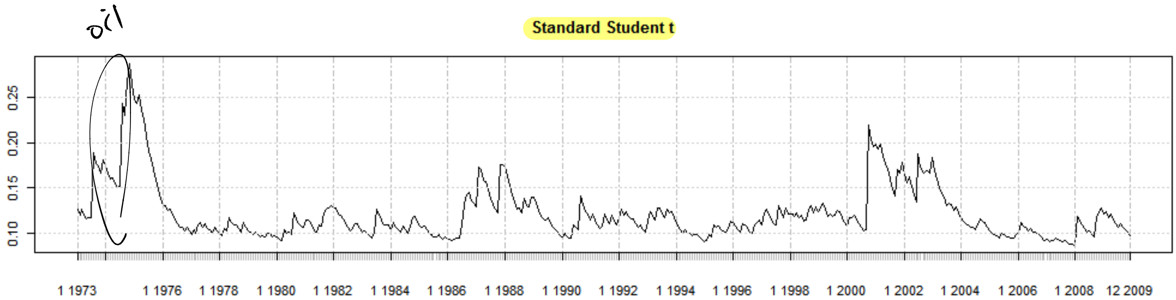
```
> vol1 <- sigma(fit1)
> vol2 <- sigma(fit2)
> par(mfrow = c(2, 1))
> plot(vol1, main = "Standard Normal")
> plot(vol2, main = "Standard Student t")
```

dot-com bubble

Black monday



oil



- The volatility estimates are essentially the same. So, the difference between two innovations is tiny.

$\eta_t = \epsilon_t^2 - \sigma_t^2$

$\sigma_t^2 = a_0 + a_1\epsilon_{t-1}^2 + b_1\sigma_{t-1}^2$

$\epsilon_t^2 = a_0 + a_1\epsilon_{t-1}^2 + b_1\sigma_{t-1}^2 + \eta_t$

$= a_0 + a_1\epsilon_{t-1}^2 + b_1(\epsilon_{t-1}^2 - \eta_{t-1}) + \eta_t$

$= a_0 + (a_1+b_1)\epsilon_{t-1}^2 + \eta_t - b_1\eta_{t-1}$

4. The Integrated GARCH (IGARCH) Model

If the AR part of the GARCH(1, 1) representation in (3.2) has a unit root (i.e., $a_1 + b_1 = 1$), an IGARCH(1, 1) model emerges; that is, the IGARCH(1, 1) model is the unit-root GARCH(1, 1) model.

If $a_1+b_1 = 1$

$\epsilon_t^2 = a_0 + \epsilon_{t-1}^2 + \eta_t - b_1\eta_{t-1}$

IGARCH(1,1).

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2$$

① $\eta_t = \varepsilon_t^2 - \sigma_t^2$

$$\begin{aligned}
\varepsilon_t^2 &= a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2 + \eta_t \\
&= a_0 + a_1 \varepsilon_{t-1}^2 + b_1 (\varepsilon_{t-1}^2 - \eta_{t-1}) + \eta_t \\
&= a_0 + (a_1 + b_1) \varepsilon_{t-1}^2 + \eta_t - b_1 \eta_{t-1}
\end{aligned}$$

If $a_1 + b_1 = 1, \quad a_0 = 0$

$$\varepsilon_t^2 = \varepsilon_{t-1}^2 + \eta_t - b_1 \eta_{t-1}$$

②
$$\begin{aligned}
\sigma_t^2 &= a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2 \\
&= (1-b_1) \varepsilon_{-1}^2 + b_1 ((1-b_1) \varepsilon_{t-2}^2 + b_1 \sigma_{t-2}^2) \\
&= (1-b_1) \varepsilon_{t-1}^2 + b_1 (1-b_1) \varepsilon_{t-2}^2 + b_1^2 \sigma_{t-2}^2 \\
&= (1-b_1)(\varepsilon_{t-1}^2 + b_1 \varepsilon_{t-2}^2) + b_1^2 ((1-b_1) \varepsilon_{t-3}^2 + b_1 \sigma_{t-3}^2) \\
&= (1-b_1)(\varepsilon_{t-1}^2 + b_1 \varepsilon_{t-2}^2 + b_1^2 \varepsilon_{t-3}^2) + b_1^3 \sigma_{t-3}^2 \\
&= (1-b_1)(\varepsilon_{t-1}^2 + b_1 \varepsilon_{t-2}^2 + b_1^2 \varepsilon_{t-3}^2 + b_1^3 \varepsilon_{t-4}^2 + \cdots)
\end{aligned}$$

$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2$

- From (3.1), one sees that the IGARCH(1, 1) model if $a_0 = 0$ has the form

$$\begin{aligned}
\sigma_t^2 &= (1-b_1)\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2 \\
&= (1-b_1)\varepsilon_{t-1}^2 + b_1((1-b_1)\varepsilon_{t-2}^2 + b_1\sigma_{t-2}^2) \\
&= (1-b_1)\varepsilon_{t-1}^2 + b_1(1-b_1)\varepsilon_{t-2}^2 + b_1^2\sigma_{t-2}^2 \\
&= (1-b_1)(\varepsilon_{t-1}^2 + b_1\varepsilon_{t-2}^2) + b_1^2\sigma_{t-2}^2.
\end{aligned}$$

By repeated substitution, one obtains

$$\sigma_t^2 = (1-b_1)\left[\varepsilon_{t-1}^2 + b_1\varepsilon_{t-2}^2 + b_1^2\varepsilon_{t-3}^2 + \cdots\right], \tag{4.1}$$

which is the exponential smoothing formation with b_1 being the discounting factor. Specifically, the closer b_1 is to one, the more weight is put on previous return shocks (i.e., $\varepsilon_{t-2}^2, \varepsilon_{t-3}^2, \ldots$).

Remark 4.1. J.P. Morgan's RiskMetrics® methodology is based on estimates of volatility in (4.1), which is an approach for computing value at risk (VaR).

Example 4.2. Consider an ARMA(0, 0)-IGARCH(1, 1) model with Gaussian innovations for the monthly log stock returns for Intel Corporation.

$Z_t \overset{iid}{\sim} N(0,1)$

```
> spec <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                     variance.model = list(model = "iGARCH",
                                           garchOrder = c(1, 1)))
> (fit1 <- ugarchfit(spec = spec, data = rtn))
       Estimate  Std. Error  t value Pr(>|t|)
mu      0.009665   0.005340   1.8100 0.070298
omega   0.000345   0.000182   1.8990 0.057561
alpha1  0.127533   0.033626   3.7927 0.000149
beta1   0.872467         NA       NA       NA
```

$b_1 = 1 - a_1$

automatically computed from $1-a_1$

- The fitted model is

$$\begin{aligned}
\mu_t &= 0.009665 \\
\sigma_t^2 &= 0.000345 + 0.127533\varepsilon_{t-1}^2 + 0.872467\sigma_{t-1}^2.
\end{aligned}$$

5. The GARCH-M Model

Engle, Lilien, and Robins (1987) extend a basic GARCH model so that the volatility can generate a risk premium:

$g(\sigma_t)$: σ_t : archpow = 1; σ_t^2 : archpow = 2

$$\mu_t = c + \alpha g(\sigma_t)$$
$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j \sigma_{t-j}^2.$$

This extended model is referred to as *GARCH-in-the-mean* (GARCH-M) model.

GARCH(1,1).

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2$$

$$a_1 + b_1 < 1.$$

IGARCH(1,1) assume $a_1 + b_1 = 1$

$a_0 = 0$

$$\begin{aligned}\sigma_t^2 &= a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2 \\ &= (1-b_1)\varepsilon_{t-1}^2 + b_1 [(1-b_1)\varepsilon_{t-2}^2 + b_1 \sigma_{t-2}^2] \\ &= (1-b_1)\varepsilon_{t-1}^2 + (1-b_1) b_1 \varepsilon_{t-2}^2 + b_1^2 \sigma_{t-2}^2 \\ &= (1-b_1)[\varepsilon_{t-1}^2 + b_1 \varepsilon_{t-2}^2 + b_1^2 \varepsilon_{t-3} + \cdots]\end{aligned}$$

$\mu_t = c + \alpha \cdot \sigma_t^2$

$\downarrow$

$E[r_t | I_{t-1}]$ $\alpha > 0$

$r_t = \mu_t + \varepsilon_t$

β = sys

σ_t^2 = T.R = sys + unsys

$c + \alpha \sigma_t^2$

$\mu_t = \phi_0.$

$\mu_t = c + \alpha g(\sigma_t)$

$\mu_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \theta_1 \varepsilon_{t-1} + \cdots$

$\mu_t = \phi_0.$

$r_t = \mu_t + \varepsilon_t$

not necessary true positive relation. α.

- The GARCH-M model suggests that there are some interactions between the expected returns and risk as measured by volatility; for instance, a positive α indicates that the expected return is positively related to volatility. The function $g(\sigma_t)$ can be standard deviation or variance.

Example 5.1. Consider an GARCH(1, 1)-M model with Gaussian innovations for the monthly log stock returns for Intel Corporation. ARMA(0,0) - GARCH(1,1)-M

```
> spec1 <- ugarchspec(mean.model = list(armaOrder = c(0, 0), archm = T,
                                        archpow = 2),   ⇒ g(σt) = σt²
                      variance.model = list(garchOrder = c(1, 1)))
> (fit1 <- ugarchfit(spec = spec1, data = rtn))
       Estimate Std. Error   t value Pr(>|t|)
mu     -0.000182   0.013692 -0.013278 0.989406
archm   0.855492   0.949116  0.901357 0.367398   → insignificant
omega   0.000944   0.000392  2.408995 0.015997     μt & σt²
alpha1  0.087242   0.026658  3.272664 0.001065   ⇒ no relationship.
beta1   0.849716   0.039235 21.657337 0.000000
```

- The fitted model is

$$\mu_t = -0.000182 + 0.855492\sigma_t^2$$
$$\sigma_t^2 = 0.000944 + 0.087242\varepsilon_{t-1}^2 + 0.849716\sigma_{t-1}^2.$$

The estimated coefficient $\hat{\alpha}$ is positive but statistically insignificant at the 5% level.

```
> spec2 <- ugarchspec(mean.model = list(armaOrder = c(0, 0), archm = T,
                                        archpow = 1),   ⇒ g(σt) = σt
                      variance.model = list(garchOrder = c(1, 1)))
> (fit2 <- ugarchfit(spec = spec2, data = rtn))
       Estimate Std. Error  t value Pr(>|t|)
mu     -0.022993   0.038020 -0.60475 0.545345
archm   0.300049   0.330614  0.90755 0.364115
omega   0.000959   0.000391  2.45110 0.014242
alpha1  0.085507   0.026920  3.17633 0.001491
beta1   0.849888   0.038924 21.83459 0.000000
```

- The fitted model is

$$\mu_t = -0.022993 + 0.300049\sigma_t$$
$$\sigma_t^2 = 0.000959 + 0.085507\varepsilon_{t-1}^2 + 0.849888\sigma_{t-1}^2.$$

Still, the estimate of α is positive but insignificant.

6. The Exponential GARCH Model

Consider an ARMA(p, q)-GARCH(m, n) model in the form

$$
\begin{aligned}
r_t &= \mu_t + \varepsilon_t \\
\mu_t &= \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} \quad \text{mean eq.} \\
\sigma_t^2 &= a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j \sigma_{t-j}^2, \quad \text{variance eq.}
\end{aligned}
$$

where $\varepsilon_t = z_t \sigma_t$ and $z_t \sim WN(0,1)$.

- In the mean equation μ_t, the innovation z_t amounts to news to asset returns. Specifically, positive z_t implies good news (i.e., $\varepsilon_t > 0$), while negative z_t implies bad news (i.e., $\varepsilon_t < 0$). In the variance equation σ_t^2, squared residuals ε_{t-i}^2 are entered, so that the signs of z_t have "no" effects on conditional volatility.

Remark 6.1. A stylized fact of financial volatility is that bad news tend to have a larger impact on volatility than good news. According to Black (1976), for instance, bad news tends to drive down the stock price and thus increase the leverage (i.e., the debt-equity ratio) of the stock, thereby causing the stock to be more volatile (i.e., more risky). This asymmetric news impact is often referred to as the leverage effect.

Nelson (1991) proposes the *exponential GARCH* (EGARCH) model of the form

$$h_t = a_0 + \sum_{i=1}^{m} (a_i z_{t-i} + \gamma_i(|z_{t-i}| - E[|z_{t-i}|])) + \sum_{j=1}^{n} b_j h_{t-j},$$

where $h_t = \log \sigma_t^2$. The coefficient a_i captures the sign effect and $\gamma_i (> 0)$ does the size effect.

- Consider the simplest EGARCH(1, 1) model of the form

$$h_t = a_0 + a_1 z_{t-1} + \gamma_1 (|z_{t-1}| - E[|z_{t-1}|]) + b_1 h_{t-1}.$$

Using $h_t - b_1 h_{t-1} = \ln(\sigma_t^2/\sigma_{t-1}^{2b_1})$ and $E[|z_t|] = \sqrt{2/\pi}$ for the standard normal random variable z_t, one obtains

$$\ln(\sigma_t^2/\sigma_{t-1}^{2b_1}) = a_0 + a_1 z_{t-1} + \gamma_1 \left(|z_{t-1}| - \sqrt{\frac{2}{\pi}} \right)$$

or

$$
\begin{aligned}
\sigma_t^2 &= \sigma_{t-1}^{2b_1} \exp(a_0^*) \exp(a_1 z_{t-1} + \gamma_1 |z_{t-1}|) \\
&= \sigma_{t-1}^{2b_1} \exp(a_0^*) \begin{cases} \exp((\gamma_1 + a_1) z_{t-1}) & \text{for } z_{t-1} > 0 \\ \exp((\gamma_1 - a_1)(-z_{t-1})) & \text{for } z_{t-1} < 0, \end{cases}
\end{aligned}
$$

if $a_1 < 0$

$\gamma_1 + a_1 < \gamma_1 - a_1 \Rightarrow$ bad news have large impact

where $a_0^* = a_0 - \gamma_1 \sqrt{2/\pi}$. So, if bad news have a larger impact, one expects a_1 to be "negative."

$z_{t-1} = -2$ ~~13~~ $\rightarrow$
$z_{t-1} = 2 \rightarrow$
$$\frac{e^{(\gamma_1 - a_1)(-z_{t-1})}}{e^{(\gamma_1 + a_1) z_{t-1}}} > 1 \Rightarrow a_1 \text{ negative effect.}$$

$$r_t = \mu_t + \varepsilon_t$$

ARMA(p,q): $\mu_t = \phi_0 + \sum \phi_i r_{t-i} + \sum \theta_j \varepsilon_{t-j}$

GARCH(1,1): $\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2$

$a_1 \geq 0$

$\varepsilon_t = z_t \sigma_t \qquad z_t \sim WN(0,1).$

Suppose $z_t > 0 \quad \varepsilon_t > 0 \rightarrow r_t \uparrow$

$z_t < 0 \quad \varepsilon_t < 0 \rightarrow r_t \downarrow$

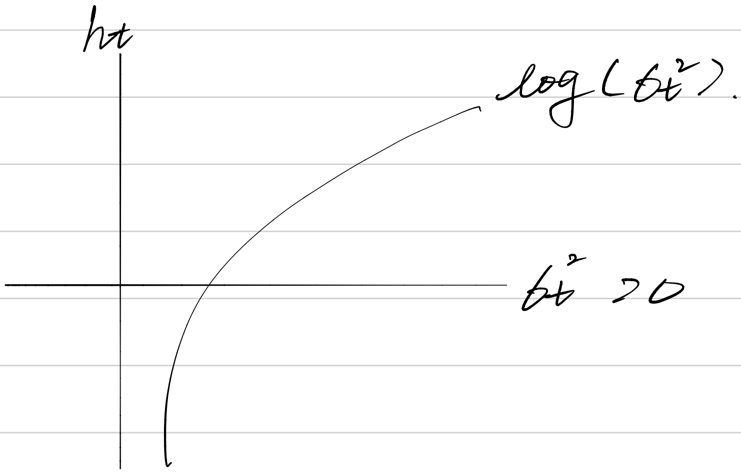
$z_{t-1} > 0 \quad \varepsilon_{t-1} > 0 \quad \sigma_t^2 \uparrow$ by a_1

$z_{t-1} < 0 \quad \varepsilon_{t-1} < 0 \quad \sigma_t^2 \uparrow$ by a_1

$\Rightarrow$ Shock effect symmetric volatility

M.E $\downarrow$ = price $\downarrow \times$ #.

D/E $\uparrow^{3}$ = $\dfrac{\text{M. Debt}}{\text{M. Equity}} \downarrow_2$ Prob. default $\uparrow$.



$$h_t = a_0 + a_1 z_{t-1} + r_1(|z_{t-1}| - E[|z_{t-1}|]) + b_1 h_{t-1}$$

$$\Rightarrow \log\left(\frac{\sigma_t^2}{\sigma_{t-1}^{2b_1}}\right) = a_0 + a_1 z_{t-1} + r_1\left(|z_{t-1}| - \sqrt{\tfrac{2}{\pi}}\right)$$

$X \sim N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$E[|X|] = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_{-\infty}^{0} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$h_t = a_0 + \sum_{i=1}^{m} (a_i z_{t-i} + r_i(|z_{t-i}| - E[|z_{t-i}|])) + \sum_{j=1}^{n} b_j h_{t-j}$$

$h_t = \log \sigma_t^2$ $\quad a_i$ capture sign effect $\quad r_i$ does size effect

EGARCH (1,1).

$$h_t = a_0 + a_1 z_{t-1} + r_1(|z_{t-1}| - E[|z_{t-1}|]) + b_1 h_{t-1}$$

$$h_t - b_1 h_{t-1} = \log \frac{\sigma_t^2}{\sigma_{t-1}^{2b_1}} = a_0 + a_1 z_{t-1} + r_1(|z_{t-1}| - E[|z_{t-1}|])$$

$E[|z_t|] = \sqrt{\frac{2}{\pi}}$ where $z_t \sim N(0,1)$.

$$\log\left(\frac{\sigma_t^2}{\sigma_{t-1}^{2b_1}}\right) = a_0 + a_1 z_{t-1} + r_1\left(|z_{t-1}| - \sqrt{\frac{2}{\pi}}\right)$$

$$\frac{\sigma_t^2}{\sigma_{t-1}^{2b_1}} = e^{a_0} \cdot e^{a_1 z_{t-1}} e^{r_1(|z_{t-1}| - \sqrt{\frac{2}{\pi}})}$$

$$\therefore \sigma_t^2 = \sigma_{t-1}^{2b_1} e^{a_0} \cdot e^{a_1 z_{t-1}} \cdot e^{r_1(|z_{t-1}| - \sqrt{\frac{2}{\pi}})}.$$

$$= \sigma_{t-1}^{2b_1} e^{\underbrace{a_0 - r_1\sqrt{\frac{2}{\pi}}}_{= a_0^*}} \cdot e^{a_1 z_{t-1} + r_1 |z_{t-1}|}$$

$$= \begin{cases} \sigma_{t-1}^{2b_1} e^{a_0 - r_1\sqrt{\frac{2}{\pi}}} \cdot e^{(r_1 + a_1) z_{t-1}} & z_{t-1} > 0 \\ \sigma_{t-1}^{2b_1} e^{a_0 - r_1\sqrt{\frac{2}{\pi}}} \cdot e^{(r_1 - a_1)(-z_{t-1})} & \end{cases}$$

Example 6.2. Consider an ARMA(0, 0)-EGARCH(1, 1) model with Gaussian innovations for the monthly log returns for IBM stock from January 1967 to December 2009. $z_t \sim N(0,1)$ iid

```
> mydat <- read.table("data2.txt", header = T)
> head(mydat)
      date       ibm       sp
1 19670131 0.075370 0.078178
...
6 19670630 0.067024 0.017512
> library(timeSeries)
> date <- as.character(mydat$date)
> ibm <- log(mydat$ibm + 1)
> rtn <- timeSeries(data = ibm, charvec = date)
> spec <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                     variance.model = list(model = "eGARCH",
                                           garchOrder = c(1, 1)))
> (fit <- ugarchfit(spec = spec, data = rtn))
        Estimate Std. Error t value Pr(>|t|)
mu      0.006649   0.002963  2.2442 0.024820
omega  -0.423206   0.223672 -1.8921 0.058480
alpha1 -0.094813   0.039373 -2.4081 0.016037
beta1   0.920485   0.041729 22.0587 0.000000
gamma1  0.218710   0.060802  3.5971 0.000322
```

- The fitted model is

$$\mu_t = 0.006649$$
$$h_t = -0.423206 - 0.094813 z_{t-i} + 0.218710(|z_{t-i}| - E[|z_{t-i}|]) + 0.920485 h_{t-1}.$$

Since the estimate of a_1 is negative and statistically significant, there exists the leverage effect.

```
> pos.shock <- exp((coef(fit)[5] + coef(fit)[3])*2)
> neg.shock <- exp((coef(fit)[5] - coef(fit)[3])*2)
> as.numeric(neg.shock/pos.shock)
[1] 1.461189
```

- For a standard normal shock with two standard deviations, one obtains

$$\frac{\sigma_t^2(\cancel{\epsilon_{t-1}}\, z_{t-1} = -2)}{\sigma_t^2(\cancel{\epsilon_t}\, z_{t-1} = 2)} = \frac{\exp((\hat{\gamma}_1 - \hat{a}_1)2)}{\exp((\hat{\gamma}_1 + \hat{a}_1)2)} = 1.461189.$$

So, the impact of a negative shock is about 46% higher than that of a positive shock of the same size.

7. The Threshold GARCH Model

TGARCH

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} (a_i \varepsilon_{t-i}^2 + \eta_t D_{t-i} \varepsilon_{t-i}^2) + \sum_{j=1}^{n} b_j \sigma_{t-j}^2$$

The *threshold GARCH* (TGARCH) assumes the form

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} (a_i \varepsilon_{t-i}^2 + \eta_i D_{t-i} \varepsilon_{t-i}^2) + \sum_{j=1}^{n} b_j \sigma_{t-j}^2,$$

where D_{t-i} is an indicator taking on value of 1 for $\varepsilon_{t-i} < 0$ and 0 otherwise. Depending on whether ε_{t-i} is above or below the threshold value of “zero,” ε_{t-i}^2 has different effects on the conditional variance σ_t^2.

- When good news occurs, the total effect of ε_{t-i} on σ_t^2 is $a_i \varepsilon_{t-i}^2$. When bad news occurs, the total effect of ε_{t-i} is $(a_i + \eta_i)\varepsilon_{t-i}^2$. If bad news have a larger impact, thus, the value of η_i is expected to be “positive.”

Remark 7.1. The *GJRGARCH* model of Glosten, Jagannathan, and Runkle (1993) has the very similar form

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} (a_i \varepsilon_{t-i}^2 + \gamma_i D_{t-i} \varepsilon_{t-i}^2) + \sum_{j=1}^{n} b_j \sigma_{t-j}^2,$$

where

$$D_{t-i} = \begin{cases} 1 & \text{if } \varepsilon_{t-i} < \mu \\ 0 & \text{if } \varepsilon_{t-i} \geq \mu \end{cases}$$

and μ is unknown. That is, the threshold value is μ instead of zero.

Example 7.2. Consider an ARMA(0, 0)-TGARCH(1, 1) model and an ARMA(0, 0)-GJRGARCH(1, 1) model with Gaussian innovations for the daily log returns, in percentage, of the exchange rate between US Dollar and Euro from January 4, 1990, to August 20, 2010.

```
> mydat <- read.table("data3.txt", header = T)
> head(mydat)
  year mon day rate
1 1999   1 4 1.1812
...
6 1999   1 11 1.1534
> fx <- log(mydat$rate)
> eu <- diff(fx)*100
> spec1 <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.mode = list(model = "fGARCH",
                                           garchOrder = c(1, 1),
                                           submodel = "TGARCH"))
> (fit1 <- ugarchfit(spec = spec1, data = eu))
       Estimate Std. Error  t value Pr(>|t|)
mu     0.012338   0.011273   1.0944 0.273778
omega  0.002570   0.000936   2.7469 0.006015
alpha1 0.033411   0.002502  13.3526 0.000000
beta1  0.970304   0.001484 653.6720 0.000000
eta11  0.174481   0.086079   2.0270 0.042663
```

$\eta_t \neq 0$.

leverage effect exists.

- The fitted ARMA(0, 0)-TGARCH(1, 1) model is

$$\mu_t = 0.012338$$
$$\sigma_t^2 = 0.002570 + (0.033411 + 0.174481 D_{t-1})\varepsilon_{t-1}^2 + 0.970304\sigma_{t-1}^2.$$

The leverage effect is present and significant at the 5% level.

```
> spec2 <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                      variance.model = list(model = "gjrGARCH",
                                            garchOrder = c(1, 1)))
> (fit2 <- ugarchfit(spec = spec2, data = eu))
       Estimate Std. Error  t value Pr(>|t|)
mu     0.012265   0.010726   1.1435 0.252826
omega  0.001274   0.000554   2.3014 0.021370
alpha1 0.022400   0.004277   5.2371 0.000000
beta1  0.968711   0.001779 544.4527 0.000000
gamma1 0.012437   0.007027   1.7699 0.076742
```

- The fitted ARMA(0, 0)-GJRGARCH(1, 1) model is

$$\mu_t = 0.012265$$
$$\sigma_t^2 = 0.001274 + (0.022400 + 0.012437 I_{t-1})\varepsilon_{t-1}^2 + 0.968711\sigma_{t-1}^2.$$

The leverage effect is present but insignificant at the 5% level.

8. Exogenous Variables

The mean and variance equations can include exogenous variables:

$$\mu_t = \phi_0 + \sum_{i=1}^{p}\phi_i r_{t-i} + \sum_{j=1}^{q}\theta_j\varepsilon_{t-j} + \beta'\text{mxreg}_t$$
$$\sigma_t^2 = a_0 + \sum_{i=1}^{p}a_i\varepsilon_{t-i}^2 + \sum_{j=1}^{q}b_j\sigma_{t-j}^2 + \delta'\text{vxreg}_t$$

where mxreg_t and vxreg_t are exogenous variables.

Example 8.1. Consider the following dynamic version of market regression

ARMA(0,0) with exogenous variable $r_{M,t}$ - GARCH(1,1)

$$r_t = \mu_t + \varepsilon_t$$
$$\mu_t = \phi_0 + \beta_1 r_{M,t}$$
$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2,$$

market return

⇒ ARMA(0,0) with exogenous variable $r_{M,t}$

⇒ GARCH(1,1)

where $\varepsilon_t = \sigma_t z_t$ and z_t is a standard normal random variable.

```
> mydat <- read.csv("data4.csv", header = T)
> head(mydat)
```

ARCH(m)

$$r_t = \mu_t + \varepsilon_t$$

$$\varepsilon_t = z_t \sigma_t$$

$$z_t \sim WN(0,1)$$

$$\mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 \quad \text{where} \quad a_0 > 0, \quad a_i \geq 0 \quad i = 1, \cdots, m$$

GARCH(m,n)

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j \sigma_{t-j}^2 \qquad \underline{a_0 > 0, \quad a_i \geq 0, \quad b_j \geq 0}$$

$$\eta_t = \varepsilon_t^2 - \sigma_t^2, \qquad GARCH(m,m).$$

$$\varepsilon_t^2 = a_0 + \sum_{i=1}^{m} (a_i + b_i) \varepsilon_{t-i}^2 + \eta_t - \sum_{j=1}^{m} b_j \eta_{t-j}$$

IGARCH(m,n)

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2, \quad \sum_{i=0}^{\max(m,n)} (a_i + b_i) < 1$$

$IGARCH(1,1)$, $a_0 = 1$.

$$\sigma_t^2 = (1-b_1)\varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2$$

$$= (1-b_1)\varepsilon_{t-1}^2 + b_1((1-b_1)\varepsilon_{t-2}^2 + b_1 \sigma_{t-2}^2)$$

$$= (1-b_1)(\varepsilon_{t-1}^2 + b_1 \varepsilon_{t-2}^2 + b_1^2 \varepsilon_{t-3}^2 + \cdots)$$

$b_1 \approx 1$, many period take.

GARCH(m,n) − M

$$\mu_t = c + \alpha g(\sigma_t). \qquad g(\sigma_t) = \sigma_t \text{ or } \sigma_t^2$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j \sigma_{t-j}^2$$

EGARCH(m,n).

$$h_t = a_0 + \sum_{i=1}^{m} (a_i z_{t-i} + r_i (|z_{t-i}| - E[|z_{t-i}|])) + \sum_{j=1}^{n} b_j h_{t-j}$$

where $h_t = \log \sigma_t^2$ $\quad$ a_i captures sign effect

r_i $\quad$ " $\quad$ size effect

EGARCH(1,1).

$$h_t = a_0 + a_1 z_{t-1} + r_1(|z_{t-1}| - E[|z_{t-1}|]) + b_1 h_{t-1}$$

$$h_t - b_1 h_{t-1} = \log \frac{\sigma_t^2}{\sigma_{t-1}^{2b_1}} = a_0 + a_1 z_{t-1} + r_1(|z_{t-1}| - \sqrt{\tfrac{2}{\pi}}).$$

$$\frac{\sigma_t^2}{\sigma_{t-1}^{2b_1}} = e^{a_0} \cdot e^{a_1 z_{t-1}} \cdot e^{r_1 |z_{t-1}|} \cdot e^{-r_1\sqrt{\frac{2}{\pi}}}$$

$$\sigma_t^2 = \sigma_{t-1}^{2b_1} \cdot e^{a_0 - r_1\sqrt{\frac{2}{\pi}}} \cdot e^{a_1 z_{t-1} + r_1 |z_{t-1}|}$$

$$= \sigma_{t-1}^{2b_1} \cdot e^{a_0 - r_1\sqrt{\frac{2}{\pi}}} \cdot \begin{cases} e^{(r_1+a_1) z_{t-1}} & z_{t-1} > 0 \\ e^{(r_1-a_1)(-z_{t-1})} & z_{t-1} < 0 \end{cases}$$

If leverage effect exist a_1 should be negative.

TGARCH(m,n)

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} (a_i \varepsilon_{t-i}^2 + \eta_i D_{t-i} \varepsilon_{t-i}^2) + \sum_{j=1}^{n} b_j \sigma_{t-j}^2$$

where $D_{t-i} = \begin{cases} 1 & \varepsilon_{t-i} < 0 \\ 0 & \text{o.w} \end{cases}$

total effect of ε_{t-i} on σ_t^2 $\begin{cases} a_i \varepsilon_{t-i}^2 & \varepsilon_{t-i} > 0 \\ (a_i + \eta_i) \varepsilon_{t-i}^2 & \varepsilon_{t-i} < 0 \end{cases}$

leverage impact exist η_i should be positive.

GJRGARCH (m,n).

$$\sigma_t^2 = a_0 + \sum_{i=1}^{m} (a_i \varepsilon_{t-i}^2 + \gamma_i D_{t-i} \varepsilon_{t-i}^2) + \sum_{j=1}^{n} b_j \sigma_{t-j}^2$$

where $D_{t-i} = \begin{cases} 1 & \varepsilon_{t-i} < u \\ 0 & \varepsilon_{t-i} \geq u \end{cases}$ u unknown.

```
      Date         MKT        FORD
1 2/2/1984 0.003348537 0.025236590
...
6 2/9/1984 -0.004493365 -0.016556290
> mkt <- log(1 + mydat$MKT)
> ford <- log(1 + mydat$FORD)
> spec <- ugarchspec(mean.model = list(armaOrder = c(0, 0),
                                  external.regressors = as.matrix(mkt)),
                     variance.model = list(garchOrder = c(1, 1)))
> (fit <- ugarchfit(spec = spec, data = ford))
       Estimate Std. Error   t value Pr(>|t|)
mu    -0.000054   0.000284  -0.19149  0.84814
mxreg1 1.211067   0.031944  37.91205  0.00000
omega  0.000001   0.000001   1.30969  0.19030
alpha1 0.032400   0.004033   8.03402  0.00000
beta1  0.962475   0.004115 233.87713  0.00000
```

r_t : ford

$mxreg_t$: mkt

fitted ARMA(0,0) with exogenous variable $r_{m,t}$ – GARCH(1,1)

positive relation

$\mu_t = -0.000054 + 1.211067\, mx_t$

$\sigma_t^2 = 0.000001 + 0.032400\, \varepsilon_{t-1}^2 + 0.962475\, \sigma_{t-1}^2$

≈ 1

- The coefficient β_1 is identified by `mxreg1` in the output and significantly positive.

Example 8.2. In Tauchen and Pitts (1983), the impact of trading volume on the volatility is explored with a GJRGARCH model in the form

$$\mu_t = \phi_0$$
$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \gamma_1 I_{t-1}\varepsilon_{t-1}^2 + b_1\sigma_{t-1}^2 + \delta_1 \Delta vol_t,$$

where Δvol_t represents the changes in trading volume.

```
> mydat <- read.csv("data5.csv", header = T)
> head(mydat)
       Date       Ret       dVol
1 8/25/1993 -2.0548668 -0.5540951
...
6 9/1/1993 -1.3152316 0.2600586
> rtn <- log(1 + mydat$Ret/100)
> dvol <- as.matrix(mydat$dVol)
> spec <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                     variance.model = list(model = "gjrGARCH",
                                           garchOrder = c(1, 1),
                                           external.regressors = dvol))
> (fit <- ugarchfit(spec = spec, data = rtn))
       Estimate Std. Error   t value Pr(>|t|)
mu     0.000548   0.000769  0.713667 0.475433
omega  0.000217   0.000003 67.951630 0.000000
```

orders

(orders initi by sellers / buyers

I Ask — market buy order

I BID

Selling / buying trading volume.

→ Total trading volume.

$Vol_t - Vol_{t-1}$

```
alpha1 0.000311    0.014506    0.021451 0.982886
beta1  0.604040    0.000882  684.603138 0.000000
gamma1 0.154624    0.030851    5.011906 0.000001
vxreg1 0.000802    0.000009   87.890299 0.000000
```

significant

positive

- The coefficient δ_1 is identified by vxreg1. The significantly positive δ_1 implies that positive changes in trading volume lead to volatility increases.

9. Alternative Approach: Use of High Frequency Data

non-parmetric

Let r_t^m be the monthly log return at month t and $r_{t,d}$ be the daily log return in the month. Assume that there are n trading days in the month. Then, one obtains

$$r_t^m = \sum_{d=1}^{n} r_{t,d}$$

and

$$Var[r_t^m | I_{t-1}] = \sum_{d=1}^{n} Var[r_{t,d}|I_{t-1}] + 2\sum_{d_1<d_2} Cov[r_{t,d_1}, r_{t,d_2}|I_{t-1}]. \tag{9.1}$$

random walk

- If $r_{t,d}$ is assumed to be a white noise, then (9.1) simplifies to

Weak form of market efficiency

$$Var[r_t^m|I_{t-1}] = nVar[r_{t,d}].$$

Thus, the estimated monthly volatility at time t is computed as

$$\hat{\sigma}_{m,t} = \sqrt{n\hat{\sigma}_{t,d}^2},$$

where

$$\hat{\sigma}_{t,d}^2 = \frac{\sum_{d=1}^{n}(r_{t,d} - \bar{r}_t)^2}{n-1}$$

and $\bar{r}_t$ is the sample mean of $\{r_{t,d}\}_{d=1}^{n}$. This volatility measure is referred to as the *realized volatility* of monthly returns.

Example 9.1. Consider the volatility of monthly log returns of the S&P 500 index from January 1980 to August 2010. In the first approach, one uses daily returns to compute the realized volatility of monthly returns. The second approach applies an ARMA(0, 0)-GARCH(1, 1) model with Gaussian innovations.

```
> mydat <- read.table("data6.txt", header = T)
> head(mydat)
  Mon Day Year   Open   High    Low  Close   Volume Adjclose
1   1   3 1980 105.76 106.08 103.26 105.22 50480000   105.22
...
6   1  10 1980 109.05 110.86 108.47 109.89 55980000   109.89
> rtn <- c(NA, diff(log(mydat$Adjclose)))
> datmat <- cbind(mydat, rtn)
> rel.vol <- NA
```

```
> for (i in 1980:2010){
      tempdat <- datmat[datmat$Year == i, ]
      for (j in 1:12){
           pos <- 12*(i - 1980) + j
           tempdat1 <- tempdat[tempdat$Mon == j, ]
           n <- dim(tempdat1)[1]
           rel.vol[pos] <- sqrt(var(tempdat1$rtn, na.rm = T)*n)
           }
      }
> rel.vol <- ts(rel.vol[-c(369:372)], start = c(1980, 1), freq = 12)
> mydat <- read.table("data7.txt", header = T)
> head(mydat)
  Mon Day Year  Open  High   Low Close   Volume Adjclose
1   1   3 1967 80.33 88.17 79.43 86.61 10426100    86.61
...
6   6   1 1967 89.08 93.28 87.19 90.64 10020000    90.64
> prcs <- ts(mydat$Adjclose, start = c(1967, 1), freq = 12)
> rtn <- diff(log(prcs))
> rtn1 <- window(rtn, start = c(1980, 1), end = c(2010, 8))
> spec <- ugarchspec(mean.model = list(armaOrder = c(0, 0)),
                     variance.model = list(garchOrder = c(1, 1)))
> fit <- ugarchfit(spec = spec, data = rtn1)
> vol <- ts(sigma(fit), start = c(1980, 1), freq = 12)
> plot(rel.vol, main = "", ylab = "", xlab = "")
> lines(vol, lty = 3)
> legend("topleft", c("Realized Volatility", "GARCH(1, 1)"), lty = c(1, 3),
         inset = 0.01)
```