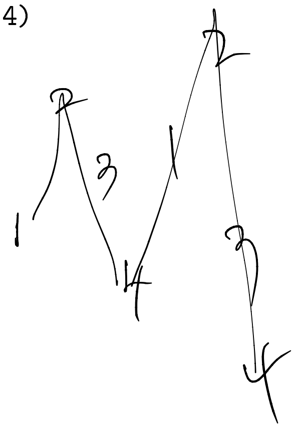
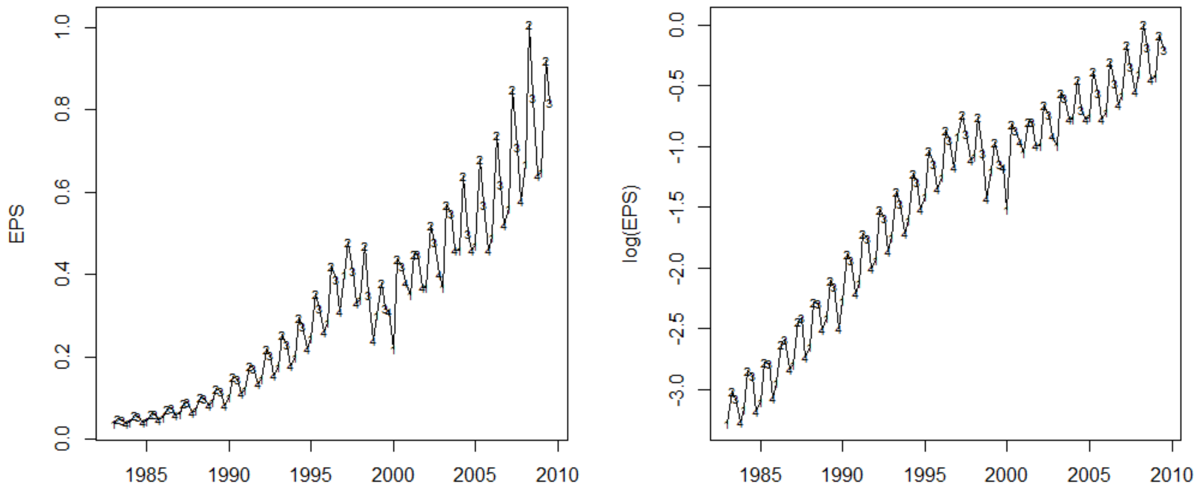
Lecture 7. Seasonal Models

1. Seasonal Differencing

Example 1.1. Consider quarterly earnings per share (EPS) of the Coca-Cola Company from 1Q 1983 to 3Q 2009.

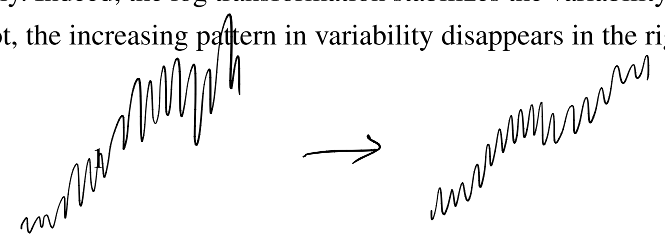
```
> mydat <- read.table("data1.txt", header = T)
> head(mydat)
                     EPS.
     pends   anntime  value
1 19830331 19830426 0.0375
...
6 19840630 19840720 0.0583
> EPS <- ts(mydat$value, start = c(1983, 1), freq = 4)
> eps <- log(EPS)
> par(mfrow = c(1, 2))
> plot(EPS, xlab = "", ylab = "EPS")
> c1 <- c("1", "2", "3", "4")
> points(EPS, pch = c1, cex = 0.6)
> plot(eps, xlab = "", ylab = "log(EPS)")
> points(eps, pch = c1, cex = 0.6)
```





length of cycle

- Two observations emerge. First, the quarterly EPS shows a strong seasonality; i.e., the seasonal pattern repeats itself every year. The periodicity of the series is 4. Second, the EPS grows exponentially, while the log EPS grows linearly. Indeed, the log transformation stabilizes the variability of the series; compared with the left plot, the increasing pattern in variability disappears in the right plot.



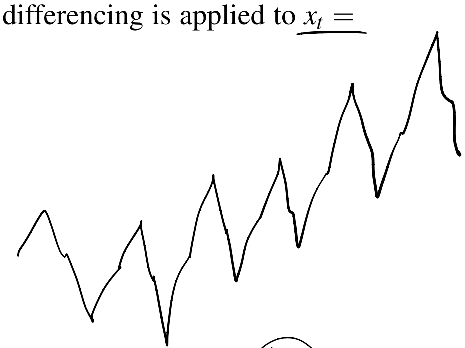
Definition 1.2. For a seasonal time series x_t with periodicity s, the operation $\Delta_s = 1 - L^s$ is called a *seasonal differencing*. The conventional difference $\Delta = 1 - L$ is referred to as a *regular differencing*.

- For instance, it shows

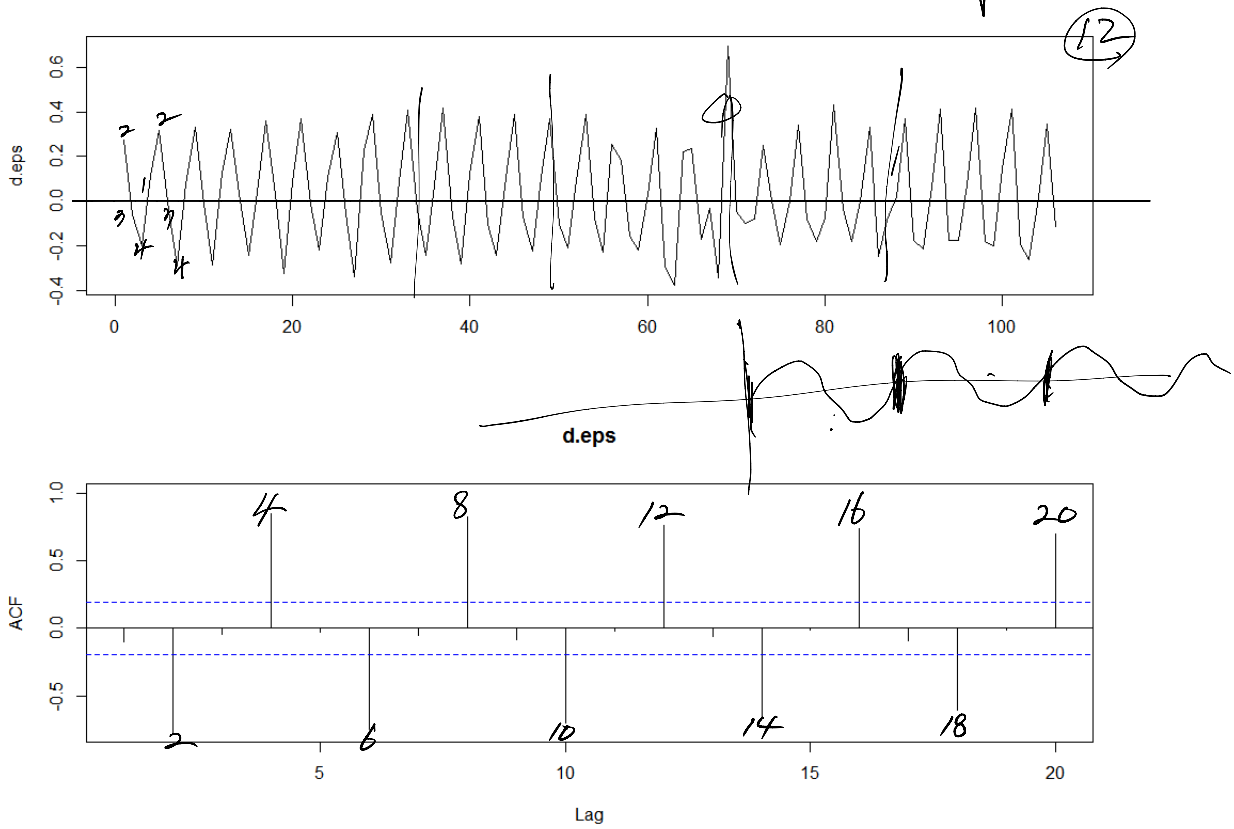
$$\Delta_4(\Delta x_t) = (1 - L^4)\Delta x_t = \Delta x_t - \Delta x_{t-4} = x_t - x_{t-1} - x_{t-4} + x_{t-5}.$$

Example 1.3. Consider quarterly earnings per share (EPS) of the Coca-Cola Company from 1Q 1983 to 3Q 2009. Since the log EPS series exhibits positive trend, the regular differencing is applied to $x_t = \log(EPS_t)$.

$\Delta x_t = \text{diff}(\log EPS_t)$

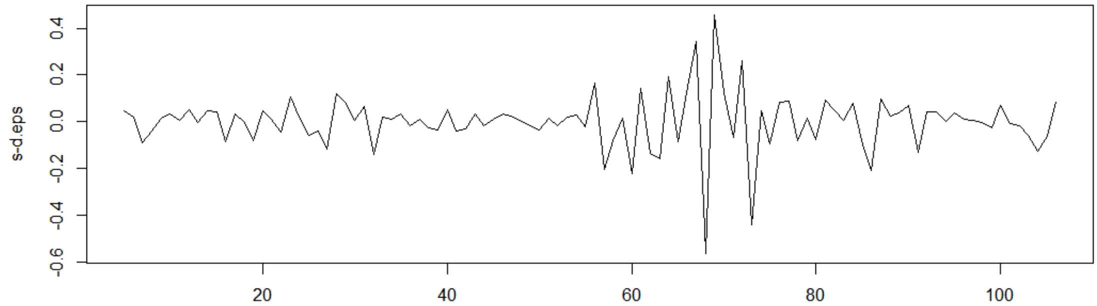


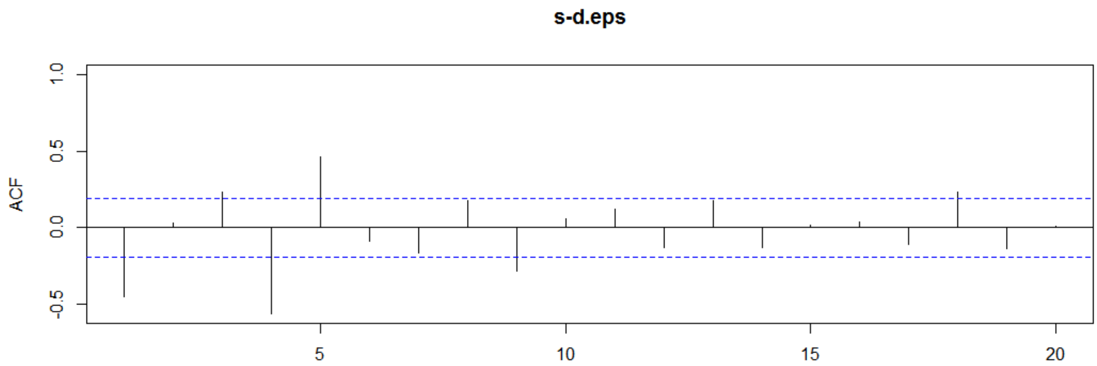
```
> eps.d <- diff(eps)
> par(mfrow = c(2, 1))
> plot(eps.d, xlab = "", ylab = "d.eps")
> acf(eps.d, lag = 20, main = "d.eps", xlim = c(1, 20))
```



- In the plot of Δx_t, the positive trend is removed, but a strong seasonality is present. In the ACF of Δx_t, the autocorrelations are large at lags, which are multiples of the periodicity 4.

```
> eps.sd <- diff(eps.d, 4)
> par(mfrow = c(2, 1))
> plot(eps.sd, xlab = "", ylab = "s-d.eps")
> acf(eps.sd, lag = 20, main = "s-d.eps", xlim = c(1, 20))
```





x_t	x_2	x_6	x_{10}	$\cdots \Rightarrow$ seasonal pattern
$\Delta_4(\Delta x_t)$	$x_2 - x_1$	$x_6 - x_5$	$x_{10} - x_9$	$\cdots \Rightarrow$ no seasonal pattern

- There is no seasonal pattern in the time series plot and the ACF of $\Delta_4(\Delta x_t)$.

$= (1-L^4)\Delta x_t = \Delta x_t - \Delta x_{t-4}$

$= x_t - x_{t-1} - x_{t-4} + x_{t-5}$

2. Multiplicative Seasonal Models

Definition 2.1. For a non-trending seasonal time series x_t, a *simple multiplicative seasonal model* has the form

$$(1-\phi_1 L - \cdots - \phi_p L^p)(1-L^s)x_t = (1+\theta_1 L + \cdots + \theta_q L^q)(1+\Theta L^s)\varepsilon_t,$$

$(x_t - x_{t-s})$ $(\varepsilon_t + \Theta\varepsilon_{t-s}$

where s is the periodicity of the series, $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$, and $|\Theta| < 1$.

- The terms $1-L^s$ and $1+\Theta L^s$ are referred to as *seasonal part*. So, the simple multiplicative seasonal model can be regarded as the ARIMA(p, 0, q) model augmented with the seasonal component.

Remark 2.2. If x_t is a trending seasonal time series, the seasonal model can be applied after taking a regular differencing:

$$(1-\phi_1 L - \cdots - \phi_p L^p)(1-L^s)(1-L)x_t = (1+\theta_1 L + \cdots + \theta_q L^q)(1+\Theta L^s)\varepsilon_t$$

which is regarded as the ARIMA(p, 1, q) model augmented with the seasonal component.

Remark 2.3. When $s \leq 4$, the ARIMA(0, 0, 1) or ARIMA(0, 1, 1) model augmented with the seasonal component is widely applicable in modeling seasonal time series. When $s > 4$, ARIMA(0, 0, 2) or ARIMA(0, 1, 2) model augmented with the seasonal component is often appropriate.

<Review>

AR(1)

$$x_t = \phi_0 + \phi_1 x_{t-1} + \varepsilon_t$$

$$\phi_0 + \phi_1 \mu = \mu$$
$$\mu(1-\phi_1) = \phi_0$$
$$\mu = \frac{\phi_0}{1-\phi_1}$$

MA(1).

$$x_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

ARMA(1,1).

$$x_t = \phi_0 + \phi_1 x_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

ARMA(p,q).

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \cdots - \phi_p x_{t-p} = \phi_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$
$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) x_t = \phi_0 + (1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)\varepsilon_t$$
$$\phi(L) x_t = \phi_0 + \theta(L)\varepsilon_t$$

If all solutions of $\phi(L)$ is greater than 1 in modulus, then ARMA model is stationary.

A sufficient condition for invertibility is that all solutions of the characteristic equation $\theta(z)$ are grater than 1 in modulus.

$$(x_t - \mu) = \sum_{i=1}^{p} \phi_i (x_{t-i} - \mu) + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$
$$\phi(L)(x_t - \mu) = \theta(L)\varepsilon_t$$

Wold Theorem.

: Any stationary process $\{x_t\}$ can be represented as a 'linear' time series in the form

$$x_t = \mu + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$, $\mu = E[x_t]$, $\psi_0 = 1$, $\sum_{i=0}^{\infty} \psi_i^2 < \infty$

If all solutions of $\theta(L)$ are greater than 1 in modulus, MA(q) process is invertible

If MA(q) invertible, there exists a corresponding AR process.

$$AR \underset{\text{invertible}}{\overset{\text{stationary}}{\rightleftarrows}} MA$$

wold

ARMA(p,q)

$$(1-\phi_1 L-\phi_2 L^2-\cdots-\phi_p L^p)x_t = \phi_0 + (1+\theta_1 L+\cdots+\theta_q L^q)\varepsilon_t$$

For a non-trending seasonal time series x_t

Simple mutiplicative seasonal model

$$(1-\phi_1 L-\cdots-\phi_p L^p)(1-L^s)x_t = (1+\theta_1 L+\cdots+\theta_q L^q)(1+\Theta L^s)\varepsilon_t$$

For a trending seasonal time series x_t

$$(1-\phi_1 L-\phi_p L^p)(1-L^s)(1-L)x_t = (1+\theta L+\cdots+\theta_q L^q)(1+\Theta L^s)\varepsilon_t$$

$$(1-\phi_1 L-\phi_p L^p)x_t - (1-\phi_1 L-\cdots-\phi_p L^p)x_{t-s}$$
$$= (1+\theta L+\cdots+\theta_q L^q)\varepsilon_t + \Theta(1+\theta L+\cdots+\theta_q L^q)\varepsilon_{t-s}$$

$$\phi(L)x_t = \theta(L)\varepsilon_t$$
$$\phi(L)x_{t-s} = \Theta\,\theta(L)\varepsilon_{t-s}$$

$$x_t = \alpha + \delta t + u_t$$
$$x_{t-s} = \alpha + \delta(t-s)\, u_{t-s}$$
$$x_t - x_{t-s} = \delta s + u_t - u_{t-s}$$
$$(1-L^s)x_t = \delta s + (1-L^s)u_t$$

Example 2.4. Consider quarterly earnings per share (EPS) of the Coca-Cola Company from 1Q 1983 to 3Q 2009.

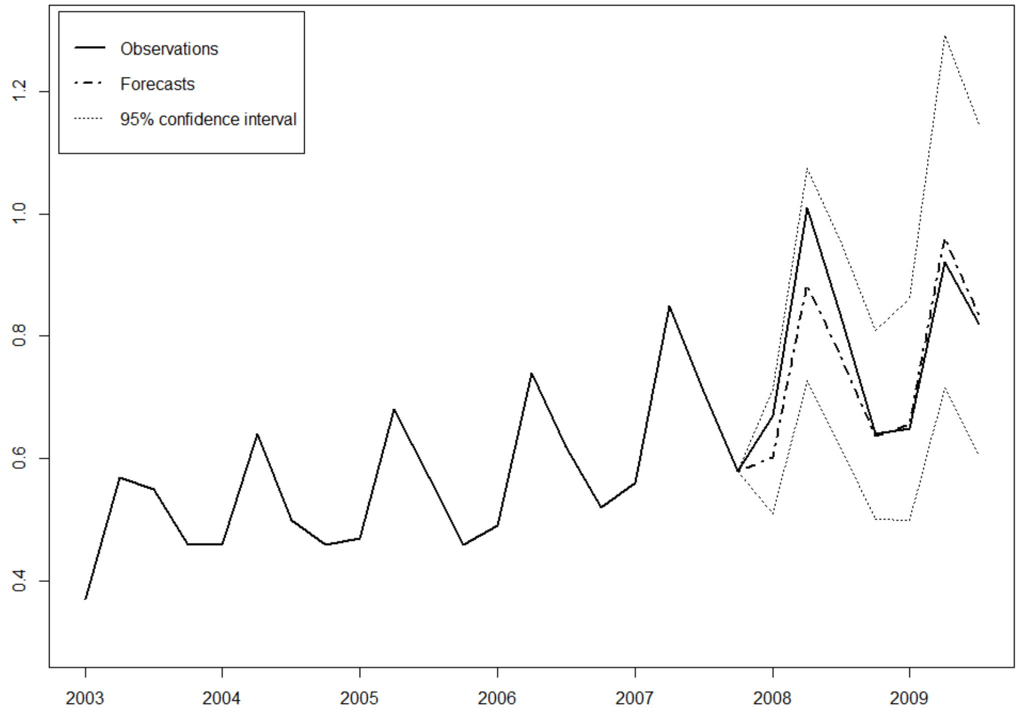
```
> (reg <- arima(eps, order = c(0, 1, 1), seasonal = list(order = c(0, 1, 1),
     period = 4), method = "ML"))
Coefficients:
          ma1     sma1
      -0.4096  -0.8203
s.e.   0.0866   0.0743
sigma^2 estimated as 0.00724: log likelihood = 104.25, aic = -202.5
```

ARIMA (0, 1, 1) model augmented with seasonal component

- The multiplicative seasonal model of the log EPS is estimated as

$$(1-L^4)(1-L)x_t = (1-0.4096L)(1-0.8203L^4)\varepsilon_t.$$

```
> eps1 <- window(eps, end = c(2007, 4))
> reg1 <- arima(eps1, order = c(0, 1, 1), seasonal = list(order = c(0, 1, 1),
     period = 4), method = "ML")
> x.ahead <- predict(reg1, 7)
> pred <- x.ahead$pred
> upper <- x.ahead$pred + 1.96*x.ahead$se
> lower <- x.ahead$pred - 1.96*x.ahead$se
> EPS1 <- window(EPS, start = c(2003, 1))
> plot(EPS1, lwd = 2, xlab = "", ylab = "", ylim = c(0.3, 1.3))
> Pred <- exp(pred)
> Upper <- exp(upper)
> Lower <- exp(lower)
> lines(ts(c(EPS1[20], Pred), start = c(2007, 4), freq = 4), lty = 4, lwd = 2)
> lines(ts(c(EPS1[20], Upper), start = c(2007, 4), freq = 4), lty = 3)
> lines(ts(c(EPS1[20], Lower), start = c(2007, 4), freq = 4), lty = 3)
> legend("topleft", c("Observations", "Forecasts", "95% confidence interval"),
     lty = c(1, 4, 3), lwd = c(2, 2, 1), inset = 0.01)
```


Observations
Forecasts
95% confidence interval
1.2
1.0
0.8
0.6
0.4
2003
2004
2005
2006
2007
2008
2009

EPS