Lecture 7. The Capital Asset Pricing Model

1. Market Equilibrium

risk-averse
non-satiation

μ_i 동일.
정보비대칭 ×, information free.

Assume that (a) everyone is a mean-variance optimizer, (b) all investors have the same estimates of means, variances, and covariances of asset returns, (c) there is a unique risk-free rate of borrowing and lending, and (d) there are no transaction costs.

value, price 개념을 많이 할수록
price = value (efficient).

- In the assumption, all investors have the same efficient frontier and, in turn, select the same risky portfolio, denoted by M, as an optimal risky portfolio. In equilibrium, it shows that the portfolio M must equal a *market portfolio* which contains shares of every stock in proportion to that stock's representation in the entire market.

In equilibrium where the market portfolio M is an optimal risky portfolio, it shows

$S_A \cdot \sigma_P$

$E[r_c] = r_f + S_P \cdot \sigma_c$

$$E[r_p] = r_f + \left(\frac{E[r_M] - r_f}{\sigma_M}\right)\sigma_P.$$

$E[r_p] = r_f + S_M \cdot \sigma_P$

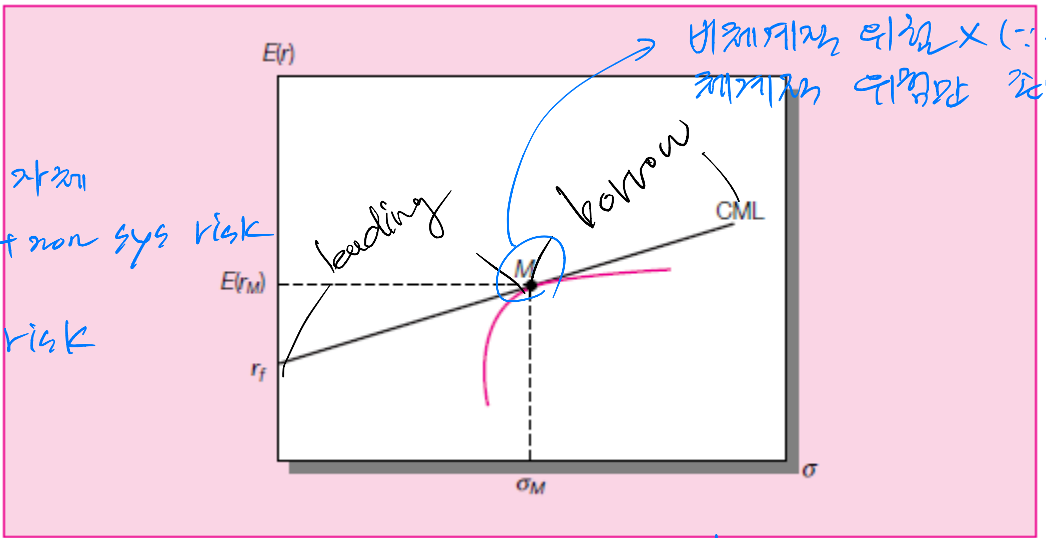
So, the efficient set consists of a single straight line, emanating from the risk-free rate and passing through the market portfolio. This line is called the *capital market line* (*CML*).

- The CML illustrates that the expected return of an efficient portfolio is linearly, positively related to its standard deviation; i.e., as the standard deviation increases, the expected return increases.

σ_i : risk 자체
sys + non sys risk

σ_p : sys risk

비체계적 위험 × (∵ 분산효과)
체계적 위험만 존재.



linear
portfolio $\sigma\uparrow \rightarrow E[r_p]\uparrow$

개별자산은 보장이 없음. 비체계, 체계 risk 등존

2. The Capital Asset Pricing Model

The *capital asset pricing model* (*CAPM*) relates the expected return of an *individual asset* to its *systematic risk*.

Theorem 1. *If the market portfolio M is efficient, the expected return $E[r_i]$ of an asset i satisfies*

asset i 의
risk-premium

$$E[r_i] - r_f = \beta_i(E[r_M] - r_f),$$

$\Downarrow$

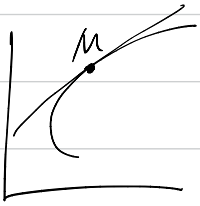
error

$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i$

with $E(\varepsilon_i) = 0$.
$cov(r_M, \varepsilon_i) = 0$.

In eq

(CAL
CML)



$$Var(r_i) = Var(\beta_i r_M + \varepsilon_i)$$
$$= \underbrace{\beta_i^2 Var(r_M)}_{sys} + \underbrace{Var(\varepsilon_i)}_{nonsys}$$

이 (market 연관없음)x

$$E[r_i] - r_f = \beta_i (E[r_M] - r_f)$$

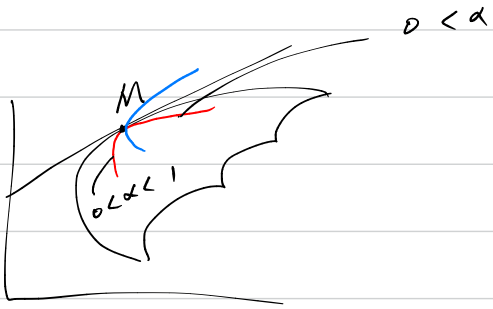
$$\Rightarrow r_i = r_f + \beta_i (r_M - r_f) + \varepsilon_i$$

with $E[\varepsilon_i] = 0$, $Cov(r_M, \varepsilon_i) = 0$

$$\Rightarrow Var[r_i] = \underbrace{\beta_i^2 Var[r_M]}_{systematic} + \underbrace{Var[\varepsilon_i]}_{non-systematic}$$

$$E[RP_i] = \beta_i E[RP_M]$$

$$RP_i = \alpha + \beta RP_M + \varepsilon_i$$



where

$$\beta_i = \frac{Cov[r_i, r_M]}{Var[r_M]}. \quad = \frac{\sigma_{iM}}{\sigma_M^2}$$

This relation is referred to as the CAPM.

Proof. Consider a portfolio consisting of a portion α invested in asset i and a portion $1-\alpha$ invested in the market portfolio. Then, the portfolio return is given by

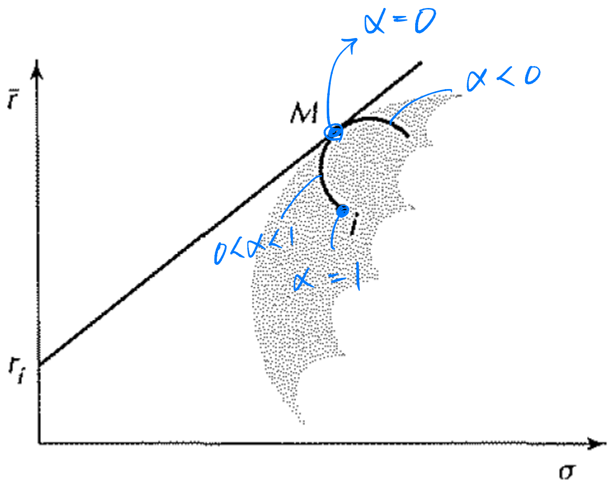
$$r_\alpha = \alpha r_i + (1-\alpha) r_M,$$

$E[r_\alpha] = E[\alpha r_i + (1-\alpha) r_M]$
$= \alpha \mu_i + (1-\alpha)\mu_M$
$Var[r_\alpha] = \alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2$

implying

$$\mu_\alpha = \alpha \mu_i + (1-\alpha)\mu_M$$
$$\sigma_\alpha = \left(\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2\right)^{1/2}.$$

As α varies, one traces out μ_α and σ_α in the mean-standard deviation diagram as follows:



When $\alpha = 0$, the curve must be tangent to the capital market line at M. This means that the slope of the curve is equal to the slope of the capital market line at the point M:

$$\left.\frac{d\mu_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{E[r_M] - r_f}{\sigma_M}.$$

Since first-derivatives of μ_α and σ_α with respect to α are given by

$$\frac{d\mu_\alpha}{d\alpha} = \mu_i - \mu_M$$
$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} - (1-\alpha)\sigma_M^2}{\left(\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2\right)^{1/2}},$$

$$\frac{\mu_i - \mu_M}{\frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}} = \frac{\mu_M - r_f}{\sigma_M}$$

$$\frac{\mu_i - \mu_M}{\sigma_{iM} - \sigma_M^2} = \frac{\mu_M - r_f}{\sigma_M^2}$$

$$(\mu_M - r_f)(\sigma_{iM} - \sigma_M^2) = \sigma_M^2(\mu_i - \mu_M).$$

$$\mu_M \sigma_{iM} - \cancel{\mu_M \sigma_M^2} - r_f \sigma_{iM} + r_f \sigma_M^2 = \sigma_M^2 \mu_i - \cancel{\sigma_M^2 \mu_M}$$

$$\sigma_{iM}(\mu_M - r_f) = \sigma_M^2(\mu_i - r_f)$$

$$E[r_i] - r_f = \frac{\sigma_{iM}}{\sigma_M^2}(E[r_M] - r_f)$$

$$\frac{\mu_i - \mu_M}{\frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}} = \frac{\mu_M - r_f}{\sigma_M^2}$$

$$\mu_i - \mu_M = \frac{(\mu_M - r_f)(\sigma}{\sigma_M^2}$$

one obtains

$$\left.\frac{d\mu_\alpha}{d\alpha}\right|_{\alpha=0} = \mu_i - \mu_M$$

$$\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}.$$

Then, it shows

$$\left.\frac{d\mu_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \left.\frac{d\mu_\alpha/d\alpha}{d\sigma_\alpha/d\alpha}\right|_{\alpha=0} = \frac{\mu_i - \mu_M}{(\sigma_{iM} - \sigma_M^2)/\sigma_M}. \quad (1)$$

Ex ante prediction.
사전적 예측이 가능하다
vs.
사후적 설명만 가능하다
→ Science.

Letting (1) equal to the slope of the CML at the point M leads to

$$\frac{\mu_i - \mu_M}{(\sigma_{iM} - \sigma_M^2)/\sigma_M} = \frac{\mu_M - r_f}{\sigma_M}$$

or

$$E[r_i] - r_f = \underbrace{\frac{\sigma_{iM}}{\sigma_M^2}}_{\beta_i}(E[r_M] - r_f).$$

□

- The value $E[r_i] - r_f$ represents the expected excess rate of return of asset i. The value $E[r_M] - r_f$ represents the expected excess rate of return of the market portfolio. In the CAPM, the expected excess return of asset i is proportional to the expected excess return of the market portfolio, and the proportionality factor is β_i.

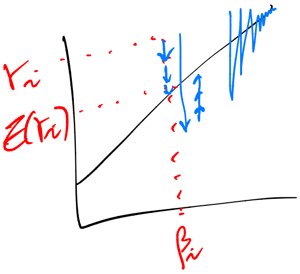
Remark 2. In the CAPM, β_i is a measure of the *systematic risk* of asset i, and the expected excess return of an asset is directly proportional to its covariance with the market. For instance, suppose $\beta_i = 0$, or equivalently, $\sigma_{iM} = 0$. Then, $E[r_i] = r_f$, no matter how big σ_i is; i.e., there is no necessary compensation for the risk measured with the standard deviation of the asset. This is because the risk associated with an asset that is uncorrelated with the market can be fully diversified away. In sum, the CAPM changes our concept of the risk of an asset from that of σ_i to that of β_i.

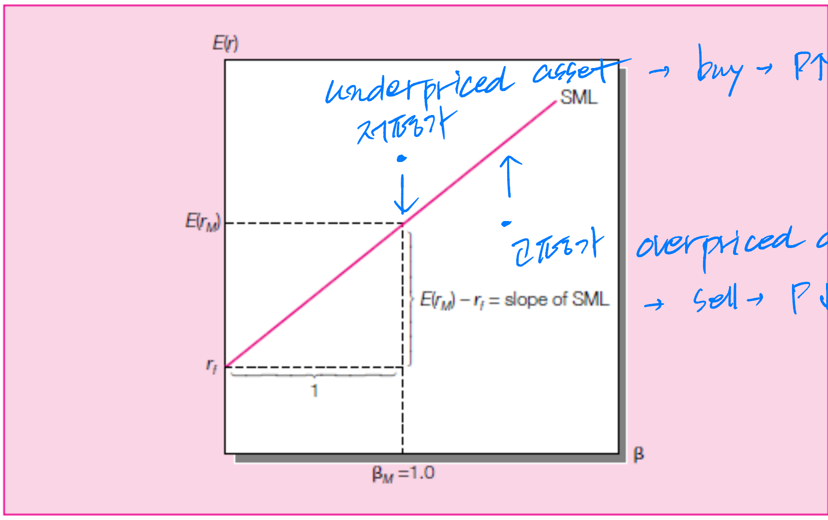
Market 과 관계없는 risk 는
non-sys → diversifiable.
시장은 sys risk에 대해서만 보상함.

3. The Security Market Line

The *security market line* (*SML*) is a graphical illustration of the CAPM. The SML represents all expected return-beta combinations of the individual stock i.

$$E(r_i) = r_f + \beta_i(E(r_M) - r_f)$$





- The difference between the actually observed and expected returns is called the asset's *alpha*, denoted by α:

실제 예측

$$\alpha_i = r_i - E[r_i].$$

underpriced : α > 0
overpriced : α < 0.

Any *fairly-priced* assets will plot exactly on the SML (i.e., $\alpha_i = 0$). In disequilibrium, *underpriced* assets will plot *above* the SML (i.e., $\alpha_i > 0$), while *overpriced* assets will plot *below* the SML (i.e., $\alpha_i < 0$).

Example 3. Suppose that $r_f = 6\%$, $\beta_i = 1.2$, and $E[r_M] = 14\%$. According to the CAPM, then, $E[r_i]$ should be equal to 15.6%, since

$$E[r_i] = 6 + 1.2(14 - 6) = 15.6\%.$$

If one observes that the actual return r_i is 17%, then the implied alpha is 1.4% and this asset is a good buy because it is underpriced.

4. Systematic Risk

According to the CAPM, the return of asset i is written as

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i$$

where ε_i is a random error with $E[\varepsilon_i] = 0$, $Var[\varepsilon_i] \neq 0$, and $Cov[r_M, \varepsilon_i] = 0$. So, the variance of r_i is given by

$$\begin{aligned} Var[r_i] &= Var[r_f + \beta_i(r_M - r_f) + \varepsilon_i] \\ &= \beta_i^2 Var[r_M] + Var[\varepsilon_i]. \end{aligned}$$

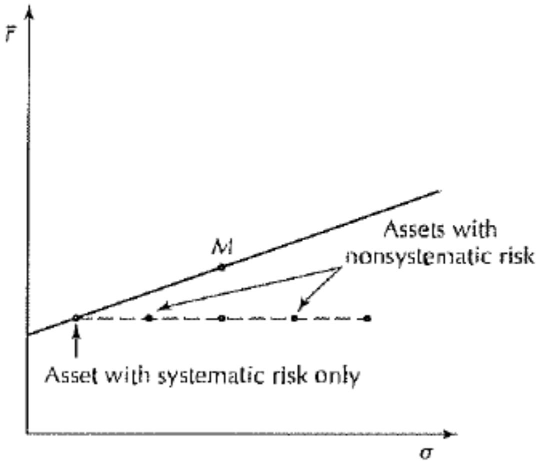
체계적위험과 return ↑ x.

↳ 이 부분에 대한 보상이 이뤄짐.

- $\beta_i^2 Var[r_M]$ is the systematic risk that is associated with the market and thus cannot be reduced by diversification. $Var[\varepsilon_i]$ is nonsystematic (or idiosyncratic) risk that is uncorrelated with the market

체계적 위험에 대한 보상만 이루어짐.

and thus can be diversified away. That is, the variance or standard deviation of the asset measures the *total risk* that consists of the systematic risk and the nonsystematic risk.



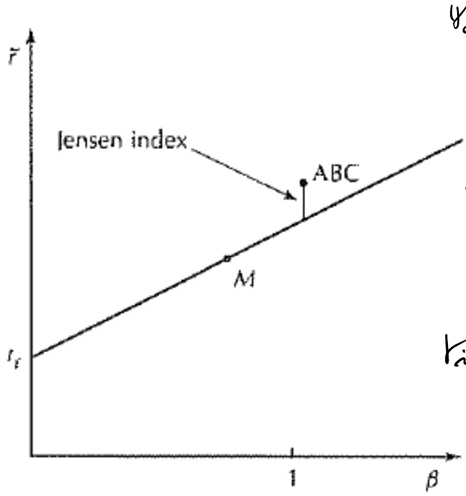
5. Performance Evaluation

The CAPM is often used to evaluate the performance of an asset (or an investment portfolio). Let α_i be the alpha of the asset i: $\alpha_i = r_i - E[r_i]$. In a portfolio context, α_i is called *Jensen's index.*

- When α_i is positive (or negative), it is said that the asset i outperformed (or underperformed) relative to the expected return predicted by the CAPM given the asset's beta. In other words, Jensen's index measures the vertical distance between the portfolio return and the point on the SML given the beta value.

벤치마크: CAPM.
(가정 CAPM이 통계적이 맞다고 가정).
올바른 performance measure.

벤치마크 설정 이유
: risk 고려.



$r_i - r_f = \alpha_i + \beta_i(r_M - r_f) + \varepsilon_i$

$\alpha_i = r_i - E[r_i]$

$= r_i - r_f - (\underline{E(r_i) - r_f})$

$\beta_i(r_M - r_f) + \varepsilon_i$

$\alpha_i = 0$ SML 선상

$\alpha_i > 0$ SML 위

〃 < 0 〃 아래

For the asset i, the CAPM implies

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i(r_{M,t} - r_{f,t}) + \varepsilon_{i,t}$$

or more compactly

$$Z_{i,t} = \alpha_i + \beta_i Z_{M,t} + \varepsilon_{i,t}$$

for $t = 1, \dots, T$, where $Z_{i,t} = r_{i,t} - r_{f,t}$, $Z_{M,t} = r_{M,t} - r_{f,t}$, $r_{i,t}$ is the return on asset i in time t, $r_{M,t}$ is the return on a market index portfolio in time t, $r_{f,t}$ is the return on a risk-free asset in time t, and $\varepsilon_{i,t} \sim N(0, \sigma_\varepsilon^2)$.

- In practice, the market index portfolio is some well diversified portfolio like the S&P 500 index. $r_{f,t}$ is taken as the T-bill rate to match the investment horizon associated with $r_{i,t}$.
- The parameters α_i and β_i are estimated using the *ordinary least squares* (*OLS*) estimation which minimizes the sum of squared residuals:

$$
\begin{aligned}
SSR &= \sum_{t=1}^{T} \varepsilon_{i,t}^2. \\
&= \sum_{t=1}^{T} (Z_{i,t} - \alpha_i - \beta_i Z_{M,t})^2.
\end{aligned}
$$

$y = x\beta + \varepsilon$ $\varepsilon \sim /$

$b = (x'x)^{-1}x'y$

모수 parameter. $b \sim N(\beta, \sigma^2(x'x)^{-1})$

The OLS estimates a_i and b_i that minimize SSR are given by

$$
b_i = \frac{\sum_{t=1}^{T}(Z_{i,t} - \bar{Z}_i)(Z_{M,t} - \bar{Z}_M)}{\sum_{t=1}^{T}(Z_{M,t} - \bar{Z}_M)^2}
$$

and

추정치.

$$
a_i = \bar{Z}_i - b_i \bar{Z}_M
$$

where $\bar{Z}_i = (1/T)\sum_{t=1}^{T} Z_{i,t}$ and $\bar{Z}_M = (1/T)\sum_{t=1}^{T} Z_{M,t}$.

Theorem 4. *With* ***some assumptions****, an appropriate test statistic for testing the null hypothesis* $H_0 : \alpha_i = 0$ *versus the alternative hypothesis* $H_1 : \alpha_i \neq 0$ *is given by*

$$
t_\alpha = \frac{a_i}{s.e.(a_i)}
$$

where

$$
\begin{aligned}
s.e.(a_i) &= s\sqrt{\frac{1}{T} + \frac{\bar{r}_M^2}{\sum_{t=1}^{T}(r_{M,t} - \bar{r}_M)^2}} \\
s &= \sqrt{\frac{\sum_{t=1}^{T} e_{i,t}^2}{T-2}} \\
e_{i,t} &= Z_{i,t} - a_i - b_i Z_{M,t}.
\end{aligned}
$$

Under H_0*, it shows*

$$
t_\alpha \sim t_{T-2}
$$

and in turn H_0 *is rejected at the 5% significance level if*

$$
|t| > t_{T-2,0.025}.
$$

Alternatively, H_0 *is rejected at the 5% level if the p-value is less than 5%.*

Example 5. Consider the CAPM regression for Microsoft using monthly return over the periods 4/1986 to 6/2012:

$$Z_{MSFT,t} = \alpha_{MSFT} + \beta_{MSFT} Z_{M,t} + \varepsilon_{MSFT,t}$$

```
> mydat <- read.csv("MSFT.csv", header = T)
> mydat[1:5, ]
      Date       MSFT  MKT_RF   RF   SMB   HML
1 4/1/1986   28.57143  -1.31 0.52  2.82 -2.91
2 5/1/1986    0.00000   4.59 0.49 -1.30 -0.11
3 6/2/1986  -11.11111   0.90 0.52 -0.89  1.38
4 7/1/1986    0.00000  -6.49 0.52 -3.43  4.71
5 8/1/1986    0.00000   6.16 0.46 -4.17  3.53
> attach(mydat)
> fit <- lm(I(MSFT-RF) ~ MKT_RF)
> summary(fit)
Coefficients:
            Estimate Std.Error  t value Pr(>|t|)
(Intercept)   1.5144    0.5041   3.004  0.00288 **
MKT_RF        1.2351    0.1075  11.493  < 2e-16 ***
---
Signif.  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1 ' ' 1
Residual standard error:  8.89 on 313 degrees of freedom
Multiple R-squared:  0.2968, Adjusted R-squared:  0.2945
F-statistic:  132.1 on 1 and 313 DF, p-value:  < 2.2e-16
```

Jense.

$\alpha_i > 0$

outperformance.

$\alpha > 0$ buy

$\alpha < 0$ sell

CAPM → α계산.

The estimated value for β_{MSFT} is 1.2351, so that Microsoft is judged to be riskier than the market index. The p-value for $H_0 : \alpha_{MSFT} = 0$ is less than 0.05, meaning that α_i is statistically significantly different from zero at the 5% level. If the CAMP is true, the MSFT stock is underpriced.

가정.

Remark 6. Jensen's index does not tell if the portfolio p is efficient, or equivalently, on the CML. In order to measure the efficiency, *Sharpe index* of the portfolio p, denoted by SR_p, is defined by

평균치.

$$SR_p = \frac{\bar{r}_p - \bar{r}_f}{s_p}$$

SR_M : market sharpe index.

for $t = 1, \ldots, T$, where $\bar{r}_p = (1/T)\sum_{t=1}^{T} r_{p,t}$, $\bar{r}_f = (1/T)\sum_{t=1}^{T} r_{f,t}$, $s_p = \sqrt{(T-1)^{-1}\sum_{t=1}^{T}(r_{p,t} - \bar{r}_p)^2}$, and $r_{p,t}$ is the portfolio return in time t. If the value of SR_p is equal to the value of SR_M (i.e, Sharpe index for the market), the portfolio p lies on the CML. If $SR_p < SR_M$, it concludes that the portfolio p is inefficient.

market portfolio

efficient

기울기 값 비교

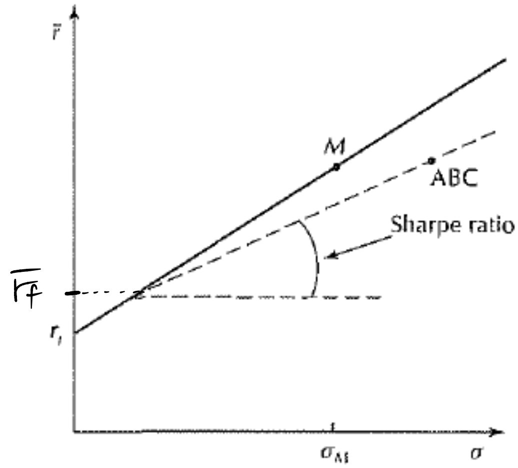
sharperatio

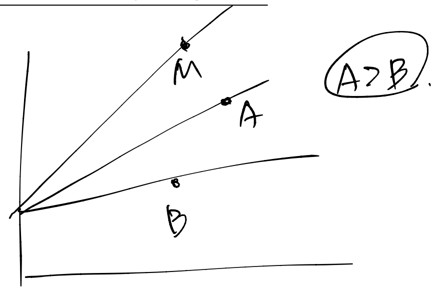
market portfolio 대비

벗어나도

efficient

inefficient





Example 7. Suppose that the portfolio p has the 10-year record of returns as follows:

t	$r_{p,t}$	$r_{M,t}$	$r_{f,t}$	$Z_{p,t}$	$Z_{M,t}$	$(Z_{p,t}-\bar{Z}_p)(Z_{M,t}-\bar{Z}_M)$	$(Z_{M,t}-\bar{Z}_M)^2$
1	14	12	7	7	5	0.96	0.36
2	10	7	7.5	2.5	-0.5	14.21	24.01
3	19	20	7.7	11.3	12.3	46.61	62.41
4	-8	-2	7.5	-15.5	-9.5	290.51	193.21
5	23	12	8.5	14.5	3.5	-8.19	0.81
6	28	23	8	20	15	154.76	112.36
7	20	17	7.3	12.7	9.7	38.69	28.09
8	14	20	7	7	13	13.76	73.96
9	-9	-5	7.5	-16.5	-12.5	370.11	285.61
10	19	16	8	11	8	20.16	12.96
Average	13	12	7.6	5.4	4.4		
Standard Deviation	12.4	9.4					
Sum						941.58	793.78

The value of β_p is estimated as

$$b_p = \frac{\sum_{t=1}^{T}(Z_{p,t}-\bar{Z}_p)(Z_{M,t}-\bar{Z}_M)}{\sum_{t=1}^{T}(Z_{M,t}-\bar{Z}_M)^2} = \frac{941.58}{793.78} = 1.19.$$

The value of Jensen's index for the portfolio p is given by

outperform
underprice.

$$\alpha_p = \bar{Z}_p - b_p\bar{Z}_M = 5.4 - 1.19 \times 4.4 = 0.164 > 0,$$

thereby concluding that the portfolio p outperformed relative to the expected return implied by the CAPM. However, this portfolio is not efficient, since

$$SR_p = \frac{\bar{r}_p - \bar{r}_f}{s_p} = \frac{13-7.6}{12.4} = 0.44$$

$$SR_M = \frac{\bar{r}_M - \bar{r}_f}{s_M} = \frac{12-7.6}{9.4} = 0.47.$$

CAPM이 맞다 가정 → Jensen's
: performance estimate.

6. Maximum Likelihood Estimation

MLE

Definition 8. The probability density function (pdf) of the *normal distribution* with mean μ and variance σ^2, denoted by $N(\mu, \sigma^2)$, is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

for $-\infty < x < \infty$.

Definition 9. If $X_1, \ldots, X_n$ are an *iid* sample from a population with pdf $f(x|\theta)$, the *likelihood function* is defined by

$$L(\theta|\tilde{x}) = L(\theta|x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i|\theta).$$

marginal pdf.

Example 10. Let $X_1, \ldots, X_n$ be an *iid* sample from $N(\mu, 1)$. Then, the likelihood function is

$$\begin{aligned} L(\theta|\tilde{x}) &= \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{(x_i-\mu)^2}{2}\right) \\ &= \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right). \end{aligned}$$

Definition 11. For each sample point $\tilde{x}$, let $\hat{\theta}(\tilde{x})$ be a parameter value at which $L(\theta|\tilde{x})$ attains its maximum as a function of θ, with $\tilde{x}$ held fixed. A *maximum likelihood estimator* (MLE) of the parameter θ based on a sample $\tilde{X} = (X_1, \ldots, X_n)$ is $\hat{\theta}$.

- If the likelihood function is differentiable in θ, the MLE is the value that solves

$$\frac{\partial L(\theta|\tilde{x})}{\partial \theta} = 0$$

and satisfies

$$\frac{\partial^2 L(\theta|\tilde{x})}{\partial \theta^2} < 0.$$

monotonous increase.
ordering 유지.

Remark 12. In most cases, it is easier to work with the natural logarithm of $L(\theta|\tilde{x})$, which is known as the *log likelihood*, than it it is to work with $L(\theta|\tilde{x})$ directly; that is, the MLE is the value that solves

$$\frac{\partial \ln L(\theta|\tilde{x})}{\partial \theta} = 0$$

e^x

and satisfies

$$\frac{\partial^2 \ln L(\theta|\tilde{x})}{\partial \theta^2} < 0.$$

This is possible because the log function is strictly increasing on $(0, \infty)$, which implies that the extrema of $L(\theta|\tilde{x})$ and $\ln L(\theta|\tilde{x})$ coincide.

Example 13. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$. Then one obtains that

$$L(\mu, \sigma^2|\tilde{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}\right)$$

and

$$\ln L(\mu,\sigma^2|\tilde{x}) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}.$$

The partial derivatives, with respect to μ and σ^2, are

$$\begin{aligned}
\frac{\partial \ln L(\mu,\sigma^2|\tilde{x})}{\partial\mu} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) \\
\frac{\partial \ln L(\mu,\sigma^2|\tilde{x})}{\partial\sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2.
\end{aligned}$$

OLS
MLE

Letting the derivatives equal to zero, one solves for $\hat{\mu}$ and $\hat{\sigma}^2$ as

$$\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n}x_i \\
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2.
\end{aligned}$$

$E(\hat{\sigma}^2) \neq \sigma^2$

not $(n-1)$ $\Rightarrow$: baised.

7. Gibbons, Ross, and Shanken (1989)

7.1. The CAPM Test

Define Z_t as an $N \times 1$ vector of excess returns for N assets: i.e.,

$$Z_t = \begin{bmatrix} r_{1,t} - r_{f,t} \\ \vdots \\ r_{N,t} - r_{f,t} \end{bmatrix}.$$

For these N assets, the CAPM describes Z_t as

$$Z_t = \alpha + \beta Z_{M,t} + \varepsilon_t,$$

$Z_{i,t} = \alpha_{i,t} + \beta_{i,t}(r_{i,t} - r_{f,t}) + \varepsilon_{i,t}$

for $i = 1,\ldots,N$, where $Z_{M,t} = r_{M,t} - r_{f,t}$,

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix}, \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \vdots \\ \varepsilon_{N,t} \end{bmatrix}.$$

- The CAPM implies that "all" of the elements of the vector α are zero. Equivalently speaking, the CAPM holds if

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$$

is not rejected. Gibbons, Ross, and Shanken (1989) rely on MLE to test for H_0.

F-test.

reject 하지 못하여야 CAPM이 맞는거.

7.2. MLE Estimation

One assumes the followings:

$$
\begin{aligned}
E[Z_{M,t}] &= \mu_M, \\
Var[Z_{M,t}] &= \sigma_M^2 \\
E[\varepsilon_t] &= 0_{(N\times 1)} \\
Var[\varepsilon_t] &= \Sigma_{(N\times N)} \\
Cov[Z_{M,t}, \varepsilon_t] &= 0_{(N\times 1).}
\end{aligned}
$$

Assuming the joint normality of Z_t, one obtains the probability density function of Z_t conditional on $Z_{M,t}$ as

$$
\begin{aligned}
f(Z_t|Z_{M,t}) &= (2\pi)^{-N/2}|\Sigma|^{-1/2} \\
&\quad \times \exp\left[-\frac{1}{2}(Z_t-\alpha-\beta Z_{M,t})'\Sigma^{-1}(Z_t-\alpha-\beta Z_{M,t})\right].
\end{aligned}
$$

Assume that each element of Z_t is *i.i.d.* Then, the likelihood function is given by

$$
\begin{aligned}
L(\alpha,\beta,\Sigma|\tilde{Z}_t) &= \prod_{t=1}^{T} f(Z_t|Z_{M,t}) \\
&= \prod_{t=1}^{T}(2\pi)^{-N/2}|\Sigma|^{-1/2} \\
&\quad \times \exp\left[-\frac{1}{2}(Z_t-\alpha-\beta Z_{M,t})'\Sigma^{-1}(Z_t-\alpha-\beta Z_{M,t})\right],
\end{aligned}
$$

so that the log-likelihood function is

$$
\begin{aligned}
\ln L(\alpha,\beta,\Sigma|\tilde{Z}_t) &= -\frac{NT}{2}\log(2\pi) - \frac{T}{2}\log|\Sigma| \\
&\quad -\frac{1}{2}\sum_{t=1}^{T}\frac{1}{2}(Z_t-\alpha-\beta Z_{M,t})'\Sigma^{-1}(Z_t-\alpha-\beta Z_{M,t}).
\end{aligned}
$$

- The MLE estimates are the values of the parameters which maximize $\ln L$. The first-order conditions (FOCs) produce

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} &= \Sigma^{-1}\left[\sum_{t=1}^{T}(Z_t-\alpha-\beta Z_{M,t})\right] = 0 \\
\frac{\partial \ln L}{\partial \beta} &= \Sigma^{-1}\left[\sum_{t=1}^{T}(Z_t-\alpha-\beta Z_{M,t})Z_{M,t}\right] = 0 \\
\frac{\partial \ln L}{\partial \Sigma} &= -\frac{T}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\left[\sum_{t=1}^{T}(Z_t-\alpha-\beta Z_{M,t})(Z_t-\alpha-\beta Z_{M,t})'\right]\Sigma^{-1} = 0.
\end{aligned}
$$

Solving the FOCs for α, β, and Σ, the MLE estimates are given by

$$\begin{aligned}
\hat{\alpha} &= \bar{Z} - \hat{\beta}\bar{Z}_M \\
\hat{\beta} &= \frac{\sum_{t=1}^{T}(Z_t - \bar{Z})(Z_{M,t} - \bar{Z}_M)}{\sum_{t=1}^{T}(Z_{M,t} - \bar{Z}_M)^2} \\
\hat{\Sigma} &= \frac{1}{T}\sum_{t=1}^{T}(Z_t - \hat{\alpha} - \hat{\beta}Z_{M,t})(Z_t - \hat{\alpha} - \hat{\beta}Z_{M,t})'
\end{aligned}$$

where

$$\bar{Z} = \frac{1}{T}\sum_{t=1}^{T} Z_t \text{ and } \bar{Z}_M = \frac{1}{T}\sum_{t=1}^{T} Z_{M,t}.$$

Remark 14. Under the *some* conditions, the conditional distributions of the MLE estimates are given by

$$\begin{aligned}
\hat{\alpha} &\sim N\left[\alpha, \frac{1}{T}\left(1 + \frac{\bar{Z}_M^2}{\hat{\sigma}_M^2}\right)\Sigma\right] \\
\hat{\beta} &\sim N\left[\beta, \frac{1}{T\hat{\sigma}_M^2}\Sigma\right] \\
T\hat{\Sigma} &\sim W_N\left[T-2, \Sigma\right]
\end{aligned}$$

χ^2

where

$$\hat{\sigma}_M^2 = \frac{1}{T}\sum_{t=1}^{T}(Z_{M,t} - \hat{\mu}_M)^2$$

and $W_N[T-2,\Sigma]$ represents a Wishart distribution with $(T-2)$ degrees of freedom and variance matrix Σ.

Theorem 15. *Assume $\varepsilon_{i,t} \sim i.i.d.N(0, \sigma_\varepsilon^2)$. Gibbons, Ross, and Shanken (1989) derive a finite-sample F distribution for the null hypothesis $H_0 : \alpha_1 = \cdots = \alpha_N = 0$ as*

$$\frac{T-N-1}{N}\left(1 + \frac{\bar{Z}_M^2}{\hat{\sigma}_M^2}\right)^{-1}\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim F_{N,T-N-1}.$$

//.

⇒ CAPM이 맞는 모형이냐 아니냐 판단.

This is called the GRS test statistic.

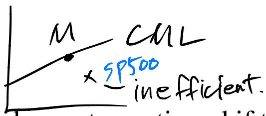
7.3. Empirical Results

The GRS test is conducted using monthly returns on ten portfolios from January 1965 through December 1994. Stocks listed on the NYSE and on the AMEX are allocated to the portfolios based on the market value of equity and are value-weighted within the portfolios. The CRSP value-weighted index is used as a proxy for the market portfolio, and the one-month U.S. T-bill return is used for the risk-free return. The results are quoted from Campbell, Lo, and MacKinlay (1997).

Time	GRS statistic	p-value	
Panel A. Five-year subperiods			
1/65-12/69	**2.038**	**0.049**	
1/70-12/74	**2.136**	**0.039**	
1/75-12/79	1.914	0.066	
1/80-12/84	1.224	0.300	
1/85-12/89	1.732	0.100	
1/90-12/94	1.153	0.344	
Overall	**77.224**	**0.004**	→ reject
Panel B. Ten-year subperiods			
1/65-12/74	2.400	0.013	
1/75-12/84	**2.248**	**0.020**	
1/85-12/94	1.900	0.053	
Overall	**57.690**	**0.001**	→ reject
Panel C. Thirty-year period			
1/65-12/94	**2.159**	**0.020**	→ reject. ∴ CAPM (+).

- The CAPM is dead! For instance, the p-value for the overall thirty-year period is 0.020, indicating that $H_0 : \alpha = 0$ is rejected at the 5% significance level. The five- and ten-year subperiod results also suggest the empirical evidence against the CAPM.

8. Roll's Critique



CML위에 market portfolio 가 존재한다고 가정하고 시작.

In testing the CAPM, one has not questioned if the CRSP value-weighted index is a "true" proxy for the market portfolio that is actually *unobservable*. For this reason, Roll (1977) emphasizes that tests of the CAPM using the proxy of the true market portfolio only reject the mean-variance efficiency of the proxy and that the CAPM could not be rejected if the true market portfolio is used instead.

- Stambaugh (1982) shows that the CAPM is rejected if the market portfolio is proxied by a stock-based proxy, a stock- and bond-based proxy, or a stock-, bond-, and real-estate-based proxy. Thus, he concludes that the rejection of the CAPM is not sensitive to the different measures of the market portfolio.

  Roll : SP500 가 market portfolio 가 아니라 (CAPM이 틀린 게 아니다).

- Shanken (1987) estimates an upper bound on the correlation between the proxy and the true market portfolio which is necessary to overrun the rejection of the CAPM. The finding is that if the correlation exceeds 0.7, then the rejection of the CAPM with the market proxy would imply the rejection of the CAPM with the true market portfolio. Therefore, if one believes that the CRSP value-weighted index reasonably approximates the true market portfolio, the empirical evidence against the CAPM remains robust and is free from the Roll's critique.

9. Fama-French Three-factor Model

9.1. Empirical Regularities Against the CAPM

The *CAPM anomalies* are empirical regularities that seem to be inconsistent with the CAPM. Since anomalies are defined relative to a model of "normal" return behaviors, the CAPM anomalies indicate that the CAPM would be *inadequate*.

The Size Effect Banz (1981) finds that small-capitalization firms on the NYSE earned higher average returns than is predicted by the CAPM from 1936 - 1975. Put differently, he finds the positive abnormal alpha from

$$(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i(r_{M,t} - r_{f,t}) + \varepsilon_{i,t},$$

where $r_{i,t}$ is the return on a portfolio of small companies.

$P = \frac{EPS}{r} + \underline{PVGO}$

The Value Effect Basu (1977) notes that firms with high earnings-to-price (E/P) ratios earn positive abnormal returns relative to the CAPM. Others find that positive abnormal returns seem to accrue to portfolios of stocks with high dividend yield (D/P) or to stocks with high book-to-market (B/M) values.

The Momentum Effect Jegadeesh and Titman (1993) find a momentum effect that recent past winners out-perform recent past losers. That is to say, a zero-cost portfolio taking a long position on winners and taking a short position on losers generates the positive abnormal returns relative to the CAPM.

9.2. Fama and French (1992)

Fama and French (1992) perform the cross-sectional test to see if *size*, proxied by ln(ME), and *book-to-market*, proxied by ln(B/M), have explanatory power beyond the CAPM β.

β	size ln(ME)	value ln(BE/ME)	ln(A/ME)	ln(A/BE)	E/P Dummy	E(+)/P
0.15 (0.46)						
	−0.15 (−2.58)					
−0.37 (−1.21)	−0.17 (−3.41)					
		0.50 (5.71)				
			0.50 (5.69)	−0.57 (−5.34)		
					0.57 (2.28)	4.72 (4.57)
	−0.11 (−1.99)	0.35 (4.44)				
	−0.11 (−2.06)		0.35 (4.32)	−0.50 (−4.56)		
	−0.16 (−3.06)				0.06 (0.38)	2.99 (3.04)
	−0.13 (−2.47)	0.33 (4.46)			−0.14 (−0.90)	0.87 (1.23)
	−0.13 (−2.47)		0.32 (4.28)	−0.46 (−4.45)	−0.08 (−0.56)	1.15 (1.57)

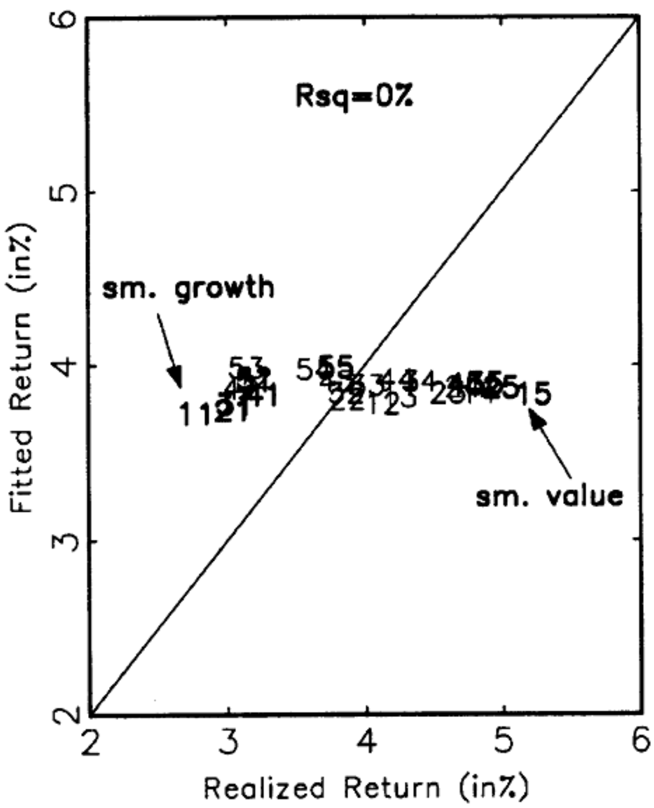
- *Size explains the cross-section of average stock returns.* The slope from the monthly regressions of returns on size alone is -0.15% (with a t-statistic of -2.58). This reliable negative relation persists no matter which other explanatory variables are in the regression.

- *Book-to-market explains the cross-section of average stock returns.* The slope from the monthly regressions of returns on ln(BE/ME) alone is 0.50% (with a t-statistic of 5.71). Book-to-market does not replace size in explaining average returns; i.e., when both ln(ME) and ln(BE/ME) are included in the regressions, the average size slope is -0.11% (with a t-statistic of -1.99) and the book-to-market slope is 0.35% (with a t-statistic of 4.44).

- *In contrast to the consistent explanatory power of size and book-to-market, the CAPM β does not explain average stock returns.* The slope from the regressions of returns on β alone is 0.15% per month with a t-statistic of 0.46.

It plots the CAPM fitted expected returns for 25 size and book-to-market sorted portfolios against their realized average returns. If the CAPM might fit perfectly, all the points in this figure would lie along the 45-degree line. The figure shows clearly that few do.

25개 portfolio는
CAPM으로 설명
불가.



Remark 16. Fama and French (1992) conclude that the size and value effects are empirical regularities of cross-sectional asset pricing data. By contrast, they find that the CAPM demonstrates virtually no power to explain the cross section of average returns on assets sorted by size and book-to-market.

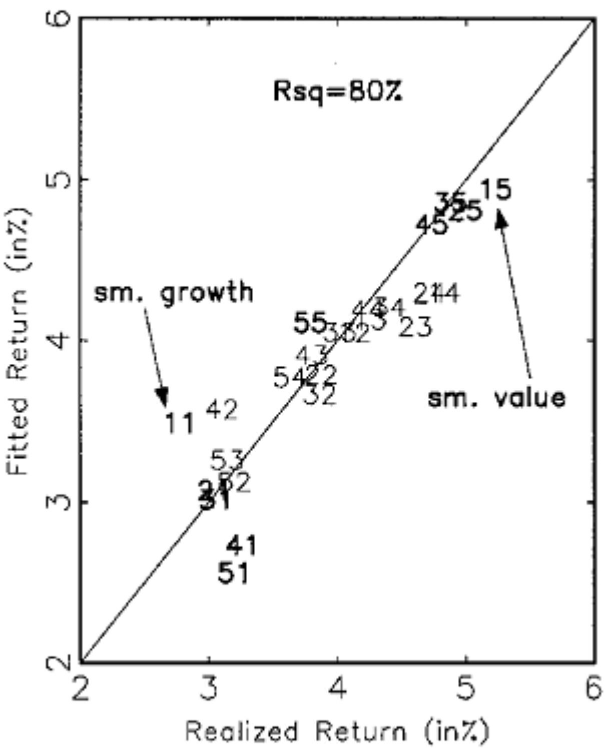
9.3. Fama and French (1993)

Fama and French (1993) develop a three-factor model with factors related to *firm size*, *book-to-market*, and the *aggregate stock market*. The cross-sectional variation in expected returns in excess of the risk-free rate of return can be captured by

$$E[r_i] - r_f = \beta_{i,1}E[MKT] + \beta_{i,2}E[SMB] + \beta_{i,3}E[HML],$$

where MKT is the return on the market portfolio in excess of the risk-free rate of return, SMB is a zero-cost portfolio that is long in small firm stocks and short in large firm stocks, and HML is a zero-cost portfolio that is long in high book-to-market stocks and short in low book-to-market stocks.

- Fama and French (1993) demonstrate, in sharp contrast to the CAPM, that the three-factor model successfully explains the cross section of average returns on the portfolios sorted by size and book-to-market.



Remark 17. Fama and French (1996) show that the Fama-French three-factor model successfully explains many of the market anomalies, excluding momentum effects.

- momentem 은 여기에서 설명할 수 X.