Economics of Information and Uncertainty

Notes taken by Junwoo Yang

Based on lecture by In-Koo Cho in spring 2021

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Chapter 1

Introduction

1.1 Expected utility theory

Lecture 1. introduction Wed, Mar 3

Why needed? A decision is often made under uncertainty.

- Objective uncertainty: The value of a decision may depend upon the contingency, which is not observable at the time of decision but the probability of the event is known.
- Subjective uncertainty: The value of a decision may depend upon the decisions of the other players, which are not observable at the time of decision. The probability of the event is endogeneous.

We need a formal theory to evaluate a choice whose value is not a deterministic value, but a probability distribution over values.

History John von Neumann and Oskar Morgenstern developed the theory of games. They immediately recognized that a decision maker faces uncertainty. A decision maker does not know the actual value from his decision, but realizes the value only after he made the decision. The value of a decision is more like a probability distribution than a number. In order to model his decision problem, we need a formal way to evaluate a probability distribution.

They developed the expected utility theory, as a way to investigate the interactive decision problem. The expected utility theory appears in the appendix rather than in the main text of their classic book.[1]

In this edition, Harold Kuhn and Ariel Rubinstein contributed short essays in the front and the back of the book, illustrating the status of economic theory 60 years after the first edition of the classic book is first published. Students are strong encouraged to read the short essay by Ariel Rubinstein in the postscript, as well as the original foreward by von Neumann and Morgenstern who foresaw the future of the game theory.

Reference This lecture is drawn from David M. Kreps [1988][2] which is probably the best reference for the choice under uncertainty.

[1] John von Neumann and Oskar Morgenstern [2007]: Theory of Games and Economic Behavior: 60th Anniversary Commemorative Edition (Princeton Classic Editions).

[2] David M. Kreps. (1988). *Notes on the Theory of Choice*. Westview press.

1.2 Description

Let Z be a finite set of outcomes, or attributes, from which a decision maker generates utility. Let $p\colon Z \to [0,1]$ be the probability distribution over Z. That is, $p(z) \geq 0\ \forall z$, and $\sum_{z\in Z} p(z) = 1$.

We use lotteries to describe risky alternatives. Suppose first that the number of possible outcomes is finite. Fix a set of outcomes $C = \{c_1, \cdots, c_N\}$. Let p_n be the probability that outcome $c_n \in C$ occurs and suppose these probabilities are objectively known.

Definition 1 (Lottery). A (simple) lottery $L = (p_1, \cdots, p_n)$ is an assignment of probabilities to each outcome c_n, where $p_n \geq 0$ for all n and $\sum_n p_n = 1$.

The collection of such lotteries can be written as

$$\mathcal{L} = \Big\{(p_1, \ldots, p_N) \Big| \sum_{n=1}^{N} p_n = 1,\ p_n \geq 0 \text{ for } n = 1, \ldots, N\Big\}.$$

We can also think of a **compound lottery** $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$, where $\alpha_k \geq 0$, $\sum_k \alpha_k = 1$, which allows the outcomes of a lottery to be lotteries. It is immediate to see that any compound lottery can be reduced to a simple lottery defined as above.

Example. $C = \{c_1, c_2\}$, $L_1 = (p, 1-p)$, $L_2 = (q, 1-q)$. Then,

$$(L_1, L_2; \alpha, 1-\alpha) = (\alpha p + (1-\alpha)q, \alpha(1-p) + (1-\alpha)(1-q)).$$

Hence, we can only focus on simple lotteries. One special and important class of lotteries is money lotteries, whose outcomes are real numbers, i.e., $C = \mathbb{R}$. A money lottery can be characterized by a cumulative distribution function F, where $F : \mathbb{R} \to [0,1]$ is nondecreasing. $F(x)$ is the probability of receiving a prize less than or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

If an individual has reasonable preferences about consumption in different circumstances, we will be able to use a utility function to describe these preferences just as we do in other contexts. However, the fact that we are considering choice under uncertainty adds some special structures to the choice problem, which we will see below. Historically, the study of individual behavior under uncertainty is originated from attempts to understand (and hopefully to win) games of chance. One may think that the key determinant of behavior under uncertainty is the expected return of the gamble. However, people are generally reluctant to play fair games.

Example (St. Petersburg Paradox)**.** Consider the following gamble: you toss a coin repeatedly until the head comes up. If this happens in the nth toss, the gamble gives a monetary payoff of 2^n. What is the expected return of this game? How much would you pay to play this gamble?

Lottery We call p a lottery, which is an object of the choice by the decision maker. Let P be the collection of all lotteries. Mathematically, P is the unit simplex in $\mathbb{R}^L$ where L is the number of elements in Z. For this reason, we sometimes write $\Delta(Z)$ in place of P to emphasize the relationship between Z and the probability distribution over Z.

Our goal is to explain how a decision maker chooses a particular lottery from the set of feasible lotteries. The main difference from the consumer theory is that the decision maker does not observe $z \in Z$ before he chooses a lottery. In order to examine the decision making process under uncertainty, we need to formulate how a decision maker choose a lottery, or formalize the preference ordering over P.

We have many theories for the decision making under uncertainty. The most prominet theory is the expected utility theory by John von Neumann and Oskar Morgenstern, which was developed as a part of developing the theory of games.

Expected utility Let $\succeq$ be an ordering over P, which represents the decision maker's preference over lotteries. If $p \succeq q$, then we say that p is preferred to q. The only difference from the conventional consumer theory is that p and q are probabilities, rather than attributes (or goods) which the decision maker draw utility.

Definition 2. $\succeq$ is complete if $\forall p, q \in P$, $p \succeq q$ or $q \succeq p$. $\succeq$ is transitive if $\forall p, q, r \in P$, $p \succeq q$ and $q \succeq r$ imply $p \succeq r$. We say that $\succeq$ is a preference ordering if $\succeq$ is complete and transitive.

Axiom 1. $\succeq$ is a preference ordering over P.

This axiom is hardly controversial, although experimental evidence shows that the ordering of a human being is often not complete or not transitive. Throughout this class, we maintain the assumption that $\succeq$ is complete and transitive.

Definition 3. $\forall p, q \in P$, $\forall a \in [0, 1]$, a composite lottery is $ap + (1 - a)q$.

If one interpret $a \in [0, 1]$ as a probability, one can interpret a composite lottery as a lottery over lotteries. One can interpret a as the amount of lottery a in the portfolio. A stock is a lottery, because the value of a stock depends upon the profitability and the market condition, but the decision maker does not observe the true state when he purchases a stock. A mutual fund is a composite lottery.

An important observation is that P is a convex set. Therefore, a composite lottery is an element of P.

The second axiom is called the substitution axiom, the independence axiom or the linearity axiom.

Axiom 2 (Substitution, Independence, Linearity)**.** $\forall p, q, r \in P$, $\forall a \in (0, 1]$, if $p \succeq q$, then $ap + (1 - a)r \succeq aq + (1 - a)r$.

The preference between two composite lotteries is determined by the preference between p and q, independently of r. In that sense, this axiom is called the

independence axiom.
Note. If $p \succeq q$, then the preference between the two composite lotteries is independent of the size of a. This is the crucial feature of linear preferences, which this axiom implies.

As important as this assumption is for the expected utility theory, the linearity of the preference has been challenged by many experiments. In reponse, many alternative axioms were proposed. Still, the linearity of the expected utility allows us to use the mathematical expectation to formulate the optimization problem. For this reason, this axiom endures the challenges.

Axiom 3 (Continuity, Archimedian)**.** $\forall p, q, r \in P$, if $p \succeq q \succeq r$, then $\exists a, b \in (0, 1)$ such that $ap + (1-a)r \succeq q \succeq bp + (1-b)r$.

This axiom is called the continuity axiom or Archimedian axiom. A key implication is that the utility must be finite. Suppose that the utility from p is infinite. Then, it would be impossible to find $b \in (0, 1)$ to construct a composite lottery so that

$$q \succeq pb + (1-b)r.$$

Similarly, if you assign $-\infty$ utility to lottery p (such as death with probability 1), then it would be impossible to construct a composite lottery in which the proportion of p is $a \in (0, 1)$ such that

$$ap + (1-a)r \succeq q.$$

The fundamental theorem by von Neumann and Morgenstern is that we can represent any preference satisfying three axioms by the expected value of a utility.

Theorem 1 (Expected utility theorem)**.** $\succeq$ satisfies three axiom if and only if there exists a utility function $u\colon Z \to \mathbb{R}$ such that $\forall p, q \in P$, $p \succeq q$ if and only if

$$U(p) = \sum_{z \in Z} u(z)p(z) > \sum_{z \in Z} u(z)q(z) = U(q).$$

Moreover, if u represents $\succeq$, then v represents $\succeq$ if and only if $\exists c > 0$, $\exists d \in \mathbb{R}$ such that $v(z) = cu(z) + d$.

The function U is often called a von Neumann-Morgenstern (vNM) expected utility function.

Discussion

- The first part is the existing of utility function u, which measures how much utils the decision maker obtains by consuming $z \in Z$. This utility function is called von Neumann Morgenstern utility function. The probability distribution is used to calculate the (mathematical) expected value of u, which the name expected utility came from.
- We call $f(z) = cz + d$ where $c > 0$ and $z, d \in \mathbb{R}$ an affine function, which is a linear function with a constant term. The second part is the uniqueness

result up to the affine transformation. That is, if u represents $\succeq$, its affine transformation represents the same preference.

- In the neoclassical consumer theory, we assumed the ordinal preference, which is represented by a utility function. An important result is that the utility function is unique up to monotonic transformation. If f is a strictly increasing function, and u represents the preference, then $v = f(u)$ represents the same preference. Since f preserves the order of the preference, but not the cardinal value of the utility, we regard this result as the mathematical formulation of the ordinal utility function.
- An affine function is a strictly increasing function, but not vice versa. An affine function does not preserve the cardinal value of the utility, but the uniqueness does not extend to all strictly increasing functions. In this sense, a von Neumann Morgenstern utility function is a cardinal utility.

1.3 Challenges

Experimental challenges Expected utility theory allows us to use the statistical method to formulate and solve the optimization problem to examine the decision making process under uncertainty. The probability enters the decision problem linearly, which simplifies the problem tremendously.

However, expected utility theory has been challenged by experimental data for a long time. Let us discuss three best known examples.

Allais's paradox

Many different versions of the same experiment have been conducted over time. We examine the one by Kahneman and Tversky. Consider the following experiments which consist of two parts.

In the first part, a group of subjects is asked to choose between two lotteries: A and B where

$$A = \begin{cases} 2500 & \text{with probability } 0.33 \\ 2400 & \text{with probability } 0.66 \\ 0 & \text{with probability } 0.01 \end{cases}$$

and

$$B = 2400 \text{ with probability } 1.$$

In the second part, the same group of subjects is asked to choose between two lotteries: C and D where

$$C = \begin{cases} 2500 & \text{with probability } 0.33 \\ 0 & \text{with probability } 0.67 \end{cases}$$

and

$$D = \begin{cases} 2400 & \text{with probability } 0.34 \\ 0 & \text{with probability } 0.66 \end{cases}.$$

At the end of the second round, the experimenter compares the choice by the subjects in the first and the second rounds. The focus is the consistency of the choice across different pairs of options.

82% choose B over A, while 83% chooses C over D, which means at least 65% ($\simeq$ 82% $\times$ 83%) chooses B over A and C over D.

A plausible heuristic explanation is that between A and B, \$2400 for sure would be better than a little bit of uncertainty, while between C and D, the difference between the size of the prize outweights the difference of the probability.

Whatever the reason might be, the behavior of a substantial portion of subjects is inconsistent with the prediction of the expected utility theory. The inconsistency is due to the violation of the independence axiom.

If the preference of a subject satisfies three axioms, we have

$$u \colon \{2500, 2400, 0\} \to \mathbb{R}.$$

Suppose that $u(2500)$, $u(2400)$, $u(0)$ are the von Neumann Morgenstern utilities of 2500, 2400 and 0 prizes.

If the subject chooses B over A, then

$$u(2400) > 0.33u(2500) + 0.66u(2400) + 0.01u(0)$$

which is equivalent to

$$0.33u(2500) - 0.34u(2400) + 0.1u(0) < 0. \tag{1.1}$$

If the same subject chooses C over D, then

$$0.33u(2500) - 0.34u(2400) + 0.1u(0) > 0. \tag{1.2}$$

But, (1.1) and (1.2) are inconsistent.

The outcome is an evidence of the violation of the independence axiom. Suppose that a subject has a well defined ordering between two lotteries:

$$X = \begin{cases} 2500 & \text{with probability } \frac{33}{34} \\ 0 & \text{with probability } \frac{1}{34} \end{cases}$$

and

$$Y = 2400 \text{with probability } 1.$$

Independence axiom says that $Y \succeq X$ if and only if

$$0.34Y + 0.66 \cdot 2400 = B \succeq A = 0.34X + 0.66 \cdot 2400.$$

Similarly, $Y \succeq X$ if and only if

$$0.34Y + 0.66 \cdot 0 = D \succeq C = 0.34X + 0.66 \cdot 0.$$

If a subject chooses B over A but C over D, his preference must violate the independence axiom.

Framing effect (Tversky and Kahneman (1981))

Imagine that the U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume that the exact scientific estimate of the consequences of the program are as follows.

If program A is adopted, 200 people will be save. If program B is adopted, there is 2/3 probability that no one will be saved, and 1/3 probability that 600 people will be saved.

If program C is adopted, 400 people will die with certainty. If program D is adopted, there is 2/3 probability that 600 people will die, and 1/3 probability that no one will die.

- 72% of subjects say $B \preceq A$.
- 78% of subjects say $C \preceq D$.
- Thus, $.72 \times .78 \simeq 50\%$ of subjects say $B \preceq A$ and $C \preceq D$.
- But, $A = C$ and $B = D$.

Ellsberg paradox (1961)

I have an urn with 300 balls in it. Some of the balls are red, some blue and some yellow. All the balls are the same size and weight, and they are not distinguished in any way except in color. I am willing to tell you that precisely 100 of the balls are red. I am unwilling to say how many are blue and how many are yellow, except that, of course, the total number of blue and yellow is 200. I want to know your preferences between gambles based on the outcome of this random event. In all these gambles, you will either win \$1000 or you will win nothing.

Gambles 1

A: Get \$1000 if the ball drawn out is red, and \$0 if it is blue or yellow.
B: Get \$1000 if the ball drawn out is blue, and \$0 if it is red or yellow.

Gambles 2

C: Get \$1000 if the ball drawn out is blue or yellow, and \$0 if it is red.
D: Get \$1000 if the ball drawn out is red or yellow, and \$0 if it is blue.

A typical response is $A \succeq B$ and $C \succeq D$. You know the odd in A, but not in B. You know the odd in C but not in D. What is wrong with this observation? $A \succeq B$ if you think the number of blue balls is less than 100. If so, $D \succeq C$. We typically do not like ambiguous problems.

1.4 Attitude toward risk

Lecture 2. attitude Mon, Mar 8

Why useful? Despite experimental evidence against the axioms, the expected utility theory is widely used. We can describe and analyze the decision problem using the same mathematical tool to compute expectations. The vNM utility provides a convenient way of formulating the attitude toward risk.

Attitude toward risk In many economic environments, individuals display aversion to risk. We formalize the notion of risk aversion and study some of its properties.

Utility on money We focus on money lotteries, i.e., risky alternatives whose outcomes are amounts of money. It is convenient to treat money as a continuous variable. We have so far assumed a finite number of outcomes to derive the expected utility representation. How to extend this?

It is convenient to assume that $X = [0, \infty)$ is money, and consider a lottery over X. Let u be the vNM utility over X. Any probability distribution over X can be represented by cumulative distribution functions (or cdf) $F\colon \mathbb{R} \to [0,1]$ where $F(x) = P(x' \leq x)$.

We assume that F is differentiable and $f(x) = F'(x)$, which is known as the density funtion. If X is discrete, $f(x)$ corresponds to the probability of event $x \in X$.

Expected utility framework on monetary outcomes We describe a monetary lottery by means of a cumulative distribution functions $F : \mathbb{R} \to [0,1]$. $F(x)$ is the probability that the realized payoff is less that or equal to x. That is, if t is distributed according to F, then $F(x) = P(t \leq x)$.

Expected utility Consider a preference relation $\succsim$ on $\mathcal{L}$. It has an expected utility representation if $F \succsim F' \Leftrightarrow U(F) \geq U(F')$, where

$$U(F) = \int_{-\infty}^{\infty} u(x)\, dF(x)$$

or

$$U(F) = \int_{-\infty}^{\infty} u(x) f(x)\, dx$$

if F is differentiable and $f = dF/dx$.

Note. U is defined on lotteries whereas u is defined on money.

To differentiate the two objects, we often call U the (von Neumann Morgenstern) expected utility function and $u(\cdot)$ the Bernoulli utility function or von Neumann Morgenstern utility of money.

We assume that u is (strictly) increasing, implying that the marginal utility of money is strictly positive, and twice continuously differentiable, for analytic convenience.

Attitude toward risk

Definition 4 (Attitude toward risk)**.** Let u be a utility function defined on money outcomes that represents $\succsim$. We say that $\succsim$ exhibits

$$\begin{pmatrix} \text{risk aversion} \\ \text{risk neutrality} \\ \text{risk loving} \end{pmatrix} \iff \int u(x)\, dF(x) \begin{pmatrix} < \\ = \\ > \end{pmatrix} u\left(\int x\, dF(x) \right)$$

for all lotteries F.

Equivalently, $\succsim$ exhibits risk aversion if $\mathbb{E}[u(X)] < u(\mathbb{E}[X])$. Notice that if $\succsim$ is risk averse (neutral, loving), then u is concave (linear, convex).

Risk averse decision maker Consider $X = \{x_g, x_b\}$ where $x_g > x_b$. Recall that u shows risk aversion if

$$u(\pi x_b + (1-\pi)x_g) > \pi u(x_b) + (1-\pi)u(x_g).$$

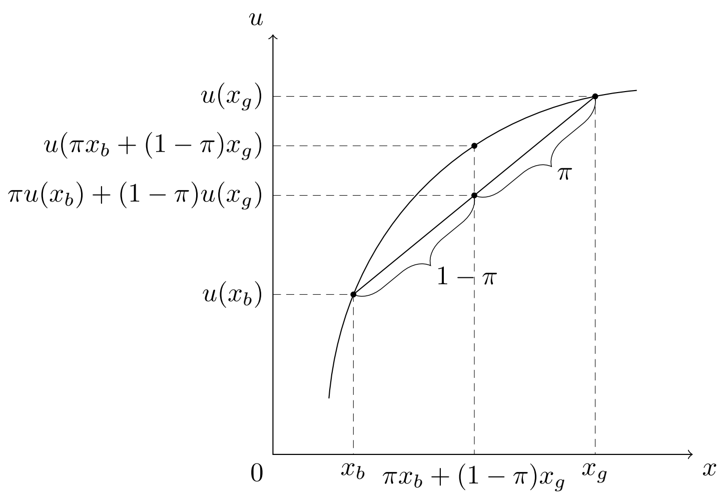


Figure 1.1: Risk aversion

Sure thing and fair gamble If u is concave, Jensen's inequality says

$$\int u(z)\, dF(z) = \mathbb{E}[u(z)] \leq u\left(\int z\, dF(z)\right) = u(\mathbb{E}[z]).$$

The left hand side is the expected utility from the bet whose return z is distributed according to F. The right hand side is the utility from money whose amount is equal to the expected value of the random variable.

Definition 5 (Sure thing)**.** By a sure thing, we mean a deterministic outcome z. A bet is a random variable. A fair bet is a random variable whose expected return is equal to the sure thing.

Risk averse and fair bet Let ε be a random variable whose expected value is 0. Given z^e, a fair bet to z^e is $z^e + \varepsilon$. Let $z^e = \mathbb{E}[z]$, and $\varepsilon = z - z^e$ whose distribution function is G. Then,

$$\int u(z^e + \varepsilon)\, dG(\varepsilon) = \mathbb{E}[u(z)] \leq u\left(\int z\, dF(z)\right) = u(\mathbb{E}[z]) = u(z^e).$$

We often say that u shows risk averse attitude if and only if the decision maker prefers a sure thing over a fair bet.

Risk neutral decision maker If a decision maker is risk neutral, then

$$u(\pi x_b + (1-\pi)x_g) = \pi u(x_b) + (1-\pi)u(x_g).$$

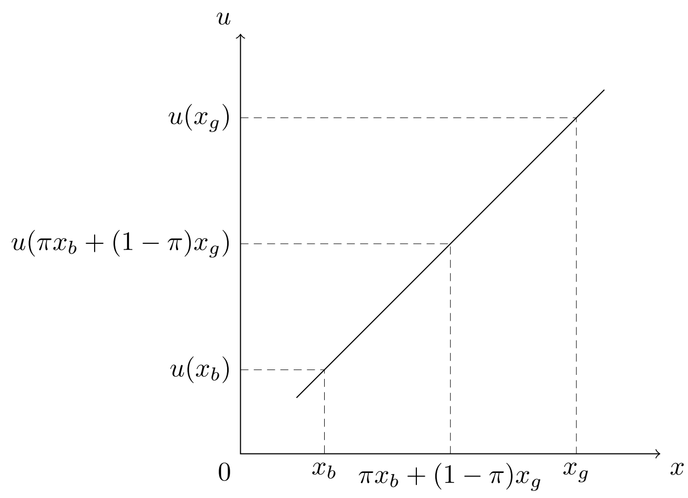


Figure 1.2: Risk neutrality

Risk loving A decision maker is risk loving if

$$u(\pi x_b + (1-\pi)x_g) < \pi u(x_b) + (1-\pi)u(x_g).$$

You may think that only a professional gambler might be risk loving. A policy with a good intention can turn a risk neutral decision maker into a risk loving decision maker.

Credit guarantee Suppose that a firm is a risk neutral decision maker whose Bernouille utility function (or vNM utility) is $u(z) = z$. The firm has a fixed cost D, but the return is a random variable R distributed over $[0, \infty)$. The profit of the firm is $u(R - D) = R - D$ which is a random variable. Note that if $R - D < 0$, then the firm loses money, which may lead to default.

It is not unusual that a government sometimes offers a rescue plan, by covering the loss in the bad state. Suppose that the government covers the loss whenever the firm incurs loss. The subsidy S is therefore, $S = -\min(R - D, 0)$. The firm's utility is now $R - D + S = \max(R - D, 0)$ which is a convex function. The intervention of the government changes the behavior of the firm from a risk neutral decision maker to a risk loving decision maker.

1.5 Certainty equivalent

Alternative ways The expected utility theory provides alternative ways to represent the attitude toward risk other than the shape of the vNM utility is one way. Let us discuss a couple of widely used methods.

Certainty equivalence A risk averse individual prefers a sure thing to a fair gamble. Is there a smaller amount of certain wealth that would be viewed as equivalent to the gamble?

Definition 6 (Certainty equivalent)**.** The certainty equivalent (CE) of F is the amount of money for which the individual is indifferent between the gamble F and the certain amount CE; that is, $u(CE) = \int u(x)\, dF(x)$.

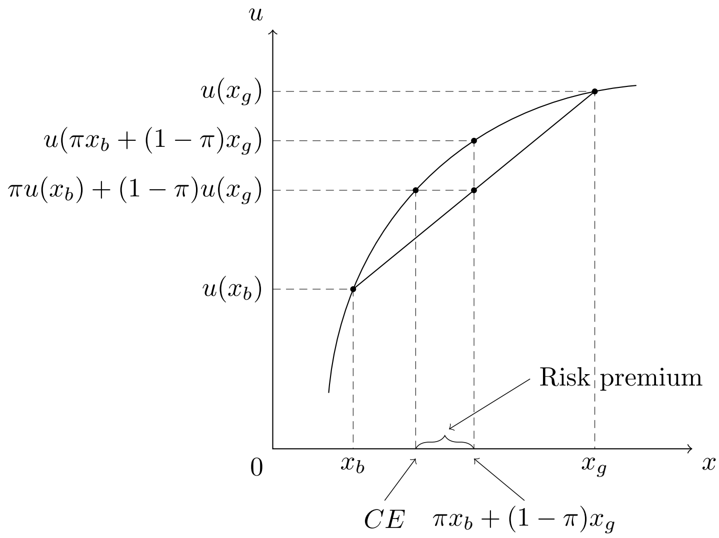


Figure 1.3: Certainty equivalent and risk premium

Note. u is concave if and only if $CE < \pi x_b + (1-\pi)x_g$.

If a risk averse decision maker is offered two options: CE and $\pi x_b + (1-\pi)x_g$, then he will accept the expected return.

This behavior provides an alternative way to represents the attitude toward risk.

Definition 7 (Risk premium)**.** The risk premium (RP) associated with F is the maximum amount of money an individual is prepared to pay to avoid the game; that is, $\mathbb{E}[u(X)] = u(\mathbb{E}[X] - RP)$. Clearly, $RP = \mathbb{E}[X] - CE$.

Proposition 1. TFAE:

(1) u exhibits risk aversion.

(2) u is concave.

(3) $CE \leq \mathbb{E}[X]$ (i.e., $RP \geq 0$).

Proof. (2) $\Rightarrow$ (1) If u is concave, then Jensen's inequality immediately implies

$$\int u(x)\, dF(x) \leq u\left(\int x\, dF(x)\right). \tag{1.3}$$

(1) ⇒ (3) By (1.3), we have

$$u(CE) = \int u(x)\, dF(x) \leq u\left(\int x\, dF(x)\right) = u(\mathbb{E}[X]),$$

so, $CE \leq \mathbb{E}[X]$.

(3) ⇒ (2) Suppose that (2) does not hold. Then, there must exist x, y, and $\lambda \in (0,1)$ such that $u(\lambda x + (1-\lambda)y) < \lambda u(x) + (1-\lambda)u(y)$. Now, consider a binary distribution $F(\cdot)$ according to which x is drawn with probability λ while y is drawn with $(1-\lambda)$. Then,

$$u(\mathbb{E}[X]) = u(\lambda x + (1-\lambda)y) < \lambda u(x) + (1-\lambda)u(y) = u(CE).$$

Thus, $\mathbb{E}[X] < CE$. ↯

□

1.6 Measurement of Risk Aversion

We sometimes have to rank two decision makers according to their attitude toward risk by saying that a decision maker is more risk averse than the other. Intuitively, the more concave the utility function, the more risk averse the consumer. Thus, the second derivative of u is a natural candidate for the measure risk aversion.

Recall that vNM utility is invariant with respect to affine transformation. Thus, if we change u by $\alpha u + \beta$ for some $\alpha > 0$, the attitude toward risk does not change. The problem of u'' as the measure of the risk aversion is that it is not invariant with respect to the affine transformation.

As an example, consider a decision maker with $v(\cdot) = 2u(\cdot)$, who has the same preference over the bet as the decision maker with u. But, $v''(\cdot) = 2u''(\cdot) \neq u''(\cdot)$.

Definition 8 (Arrow-Pratt measure of absolute risk aversion).

$$r_A(x,u) := -\frac{u''(x)}{u'(x)}$$

The idea of constructing r_A is intuitive. We normalize the degree of concavity by u' so that the measure is invariant with respect to affine transformation. More precisely,

$$-\frac{u''}{u'} = -\frac{du'/dx}{u'} = -\frac{du'/u'}{dx} = -\frac{\text{\% change in MU}}{\text{absolute change in } x}.$$

$r_A(x)$ is positive, negative, or zero as the agent is risk averse, risk loving, or risk neutral.

Another interpretation Let us consider two outcomes: bad outcome $x_b = w + r_b z$ and good outcome $x_g = w + r_g z$. Draw indifferent cure:

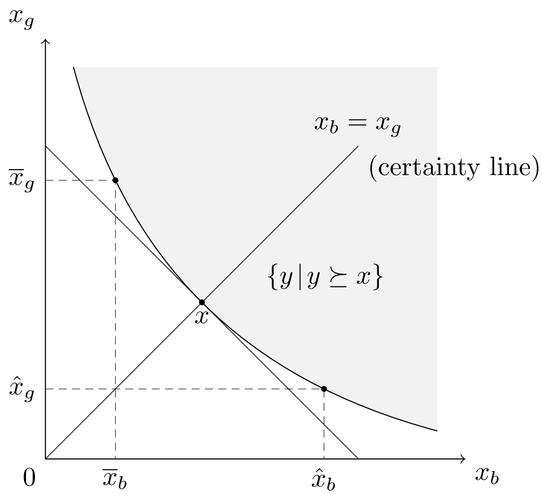
$$\pi u(x_b) + (1-\pi)u(x_g) \equiv \overline{u}.$$

By totally differentiating both sides, we implicitly derive the marginal rate of substitution

$$\pi u'(x_b) + (1-\pi)u'(x_g)\frac{dx_g}{dx_b} = 0. \tag{1.4}$$

Hence, the marginal rate of substitution (MRS) is

$$\frac{dx_g}{dx_b} = -\frac{\pi}{1-\pi}\frac{u'(x_b)}{u'(x_g)}. \tag{1.5}$$



$$\left|\frac{dx_g}{dx_b}\right| \begin{pmatrix} = \\ < \\ > \end{pmatrix} \frac{\pi}{1-\pi} \text{ when } x_b \begin{pmatrix} = \\ > \\ < \end{pmatrix} x_g, \text{ showing that } u(\cdot) \text{ is concave.}$$

Define the consumer's preferred set at x to be the set of all outcome the consumer will prefer to x, i.e., $\{y \,|\, y \succeq x\}$.

Suppose now we have two consumers, i and j. It is natural to say that consumer i is (locally) more risk averse than consumer j if consumer i 's preferred set at x is contained in j's one. Consumer i's indifferent curve is more curved than consumer j's one at x.

Differentiate (1.4) one more with respect to x_b,

$$\pi u''(x_b) + (1-\pi)u''(x_g)\left(\frac{dx_g}{dx_b}\right)\left(\frac{dx_g}{dx_b}\right) + (1-\pi)u'(x_g)\left(\frac{d^2x_g}{dx_b^2}\right) = 0.$$

Using (1.5), we have

$$\frac{d^2x_g}{dx_b^2} = \frac{\pi}{(1-\pi)^2}\left[-\frac{u''(x)}{u'(x)}\right] \text{ when } x_b = x_g = x.$$

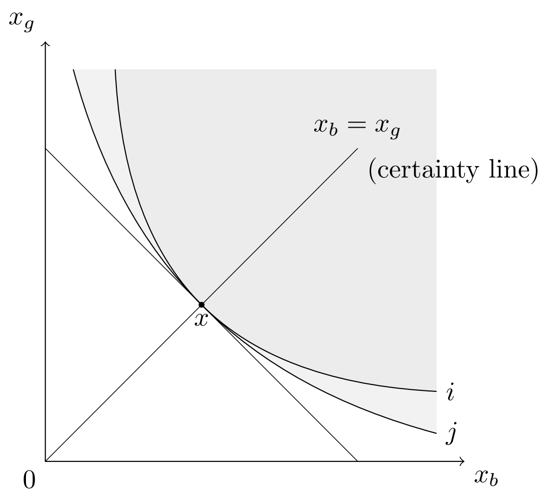


Figure 1.4: Arrow-Pratt measure of absolute risk aversion

Comparison of attitude toward risk In the cases of Proposition 2 below, we say that $u_i(\cdot)$ is more risk averse than $u_j(\cdot)$.

Proposition 2. Given two utility functions $u_i(\cdot)$ and $u_j(\cdot)$, TFAE:

(1) $r_A(x, u_i) \geq r_A(x, u_j)$ for all x. That is, consumer i has a higher degree of risk aversion than consumer j everywhere.

(2) There exists an increasing concave function $\psi(\cdot)$ such that $u_i(x) = \psi(u_j(x))$ for all x. In other words, $u_i(\cdot)$ is more concave than $u_j(\cdot)$.

(3) $CE_i \leq CE_j$ (or $RP_i \geq RP_j$, i.e., i would be willing to pay more to avoid a given risk than j would.)

Proof. (1) $\Leftrightarrow$ (2) Note that we always have $u_2(x) = \psi(u_1(x))$ for some increasing function ψ (this is true because u_1 and u_2 are ordinary identical). Differentiating, we get $u_2'(x) = \psi'(u_1(x))u_1'(x)$ and so $\log u_2'(x) = \log \psi'(u_1(x)) + \log u_1'(x)$. Differentiating again, we have

$$\frac{u_2''(x)}{u_2'(x)} = \frac{\psi''(u_1(x))}{\psi'(u_1(x))}u_1'(x) + \frac{u_1''(x)}{u_1'(x)} \Leftrightarrow r_A(x, u_2) = r_A(x, u_1) - \frac{\psi''(u_1(x))}{\psi'(u_1(x))}u_1'(x).$$

Thus, $r_A(x, u_2) \geq r_A(x, u_1)$ for all x if and only if $\psi''(u_1) \leq 0$ for all u_1.

(2) $\Rightarrow$ (3) By Jensen's inequality,

$$\begin{aligned} u_i(CE_i) &= \mathbb{E}[u_i(X)] = \mathbb{E}[\psi(u_j(X))] \\ &\leq \psi(\mathbb{E}[u_j(X)]) = \psi(u_j(CE_j)) = u_i(CE_j). \end{aligned}$$

Thus, $CE_i < CE_j$.

□

Chapter 2

Hidden Information

2.1 Economy with uncertainty

Lecture 3. information Wed, Mar 10

State contingent claim We have learned the equilibrium model under certainty, where a decision maker knows all characteristics of the goods, and the state at the time when he makes a decision. As we move from a model with certainty to uncertainty, we had to develop a new way of evaluating an object, a lottery, over the set of commodities. A fundamental question is whether the presence of uncertainty changes the equilibrium allocation of the competitive market.

Arrow and Debreu showed the condition under which the presence of uncertainty does not matter. We can apply exactly the same exercise as we learned from the model with certainty. More importantly, the first and the second welfare theorems continue to hold.

Complete market This condition is called the complete market hypothesis: each commodity has a market where it can be traded. In order to make the notion of complete market precise, Arrow invented the notion of contingent commodity.

We first state the model of competitive market satisfying the complete market hypothesis. We do so, because it provides an important benchmark against which an economy with incomplete market is examined, providing a fundamental insight into the role of uncertainty to the equilibrium outcome of the market.

Uncertainty By a state, we mean any factor that affects the decision of an economic agent. The quality of a product is a good example, which may or may not be known to the decision maker at the time of his decision. Let S be the set of states, and $s \in S$ be a generic element. Let us assume that S is finite.

> **Definition 9.** The economy is subject to uncertainty, if a state is not revealed to a decision maker at the time of his decision making.

A lottery is one of the examples. The value of the lottery is a state, which is not revealed at the time when a decision maker purchases a lottery at a certain price.

Let us consider a finite exchange economy, which is populated by I consumers endowed with neoclassical utility function, L commodities and S states, each of which has $\#I$, $\#L$ and $\#S$ elements.

We start with the description of the initial endowment, which is a complete specification of endowment for all possible contingencies. Let $\omega = (\omega_\ell)_{\ell \in L} \in \mathbb{R}^{\#L}$ be the profile of commodities. Because his endowment is affected by state $s \in S$, we need to spell out the profile of endowments for all states in S. Thus, the endowment of agent i is

$$\omega_i = (\omega_{s,i}) \in \mathbb{R}^{\#L \times \#S}.$$

Commodity We differentiate a commodity by state. This is a fundamental innovation of Arrow and Debreu. That is, a commodity is differentiated by an attribute which is relevant to the decision of an economic agent, including a state. For example, an umbrella when it rains is a different commodity from an umbrella when it shines. A stock when the firm is generating a large profit is a different commodity form a stock when the same firm is bankrupted.

Contract Another important interpretation of a commodity by Arrow and Debreu is that a commodity is a contract which promises to deliver the same physical goods if such state arises.

Depending upon a model, it is more convenient to regard an umbrella in rain as a contract which promises to deliver one unit of physical umbrella when it rains. Certainly, this contract carries a price. Similarly, a contract that promises to deliver one stock when the firm is generating a large profit is different from a contract that promises to deliver one stock when the firm is bankrupted. The two commodities would carry different prices.

We can measure the quantity of a commodity according to a basic unit, known as contigent commodity.

Definition 10. A contingent commodity (or Arrow security) is a contract to deliver one unit of a good if a particular state is realized.

As we expand the economy by incorporating uncertainty, we are expanding the notion of commodity from a profile of physical characteristics to profile of physical characteristics and states. For each of L commodities, we have to consider as many as $\#S$ contingent commodities. As a result, we have $\#L \times \#S$ commodities.

Let x_{ls} be the contract to deliver x_l units of l-th good if state s occurs. The collection of commodities is now $\mathbb{R}^{\#L \times \#S}$. Let $\succeq_i$ be the preference over $\mathbb{R}^{\#L \times \#S}$.

Example. Suppose that π_s is the probability that state s is realized, and $u_i(x, s)$ is the utility of $x \in \mathbb{R}^{\#L}$ under state s. Let $x = (x_s)_{s \in S} \in \mathbb{R}^{\#L \times \#S}$ and $x' = (x'_s)_{s \in S} \in \mathbb{R}^{\#L \times \#S}$ be a pair of state contingent commodity bundles. Then, $x \succ x'$ if

$$\sum_{x \in S} u(x_s, s)\pi_s > \sum_{x \in S} u(x'_s, s)\pi_s.$$

It is important to know that the expected utility is one of many different ways to evaluate the state contingent commodity bundles.

Sequence of moves

(1) Before a state $s \in S$ is realized, there is a market to trade commodities.

(2) After trading commodities (or contracts), a state is realized.

(3) The good is delivered according to the contract.

(4) Goods are consumed, and utility is generated.

The first and third steps warrant a careful examination.

Forward market A commodity is trade in a market, before a state is realized. Thus, is is more convenient to interpret a commodity as a contingent contract, which promises to deliver the specific amount of goods if a state arises.

In the sense that the contingent contract is traded in a forward market, the contract is often called a forward contract.

Enforcement When the contract is traded, a buyer of the commodity pays money to the seller, and receives a piece paper with a promise on it. The good is not delivered, until a state is realized.

An important assumption is that the contract is enforced without any exception, or the seller is committed to carry out the contract. If the enforcement is not complete, or if the seller has a limited commitment, then the contract may not be traded, or will fetch a lower price than under the full commitment.

For example, a debt contract is a promise that a borrower will pay back the principal and the interest back to the lender by a specific time. In Arrow Debreu economy, a debt contract will be enforced without any exception.

Symmetric information At the time when the good is traded, no agent in the economy observes the state. Every decision maker faces uncertainty. In this sense, uncertainty is symmetric.

Complete market hypothesis A fundamental assumption of Arrow Debreu economy is that every contingent commodity has a market where it can be traded. This assumption is called the complete market hypothesis. Because a forward contract is traded, we sometimes say that Arrow Debreu economy assumes a complete set of forward markets.

With a complete set of forward markets, and with full commitment, we can follow exactly the same analysis as for the economy with certainty to establish the first and the second welfare theorems. Any failure of the fundamental welfare theorems can be traced back to the missing market.

2.2 Informational efficiency

Symmetric vs. asymmetric information

- An economy with uncertainty is subject to a state which is not revealed to an agent at the time of decision.
- If no agent observes a state, the economy is subject to uncertainty, but the uncertainty is symmetric.
- If an agent observe a state, but another agent does not observe the same state, asymmetric information exists.

Rational expectations Presence of asymmetric information does not necessarily lead to inefficient allocation, as the competitive market can aggregate dispersed information into the market clearing price. Fredrick von Hayek called this property informational efficiency of competitive market.[1]

Information aggregation Let us consider an exchange economy with uncertainty with two consumers with identical utility function:

$$u_i(x_{1,i}, x_{2,i}) = \beta \ln x_{1,i} + x_{2,i}$$

where $\beta \in \{1, 2\}$ with probability $P(\beta = 1) = 0.5$. Assuming that the second good is a numeraire, let p be the price of the first commodity. Each consumer has 1 unit of the first good as an initial endowment, which makes the aggregate supply of the first good 2 units.

Symmetric uncertainty Suppose that neither agent observes the actual realized value of $\beta \in \{1, 2\}$ at the time of deciding the demand. That is, uncertainty is symmetric.

The demand for the first good should equate the expected marginal utility of the first good to the market clearing price p:

$$\frac{1}{2}\left[\frac{1}{x_{1,i}} + \frac{2}{x_{1,i}}\right] = p$$

and the uniformed demand of consumer i would be

$$x_{1,i}(p) = \frac{3}{2p} \quad \forall i.$$

The market clearing price would be $x_{1,1}(p) + x_{1,2}(p) = 2$ implying $p = \frac{3}{2}$.

Full information If consumer i observes the true state $\beta \in \{1, 2\}$, then his demand is conditioned on β and p. A simple calculation shows consumer i's demand for good 1 at price p in state β is

$$x_{1,i}(p; \beta) = \frac{\beta}{p}.$$

The aggregate demand in state β is therefore

$$x_{1,1}(p; \beta) + x_{1,2}(p; \beta) = \frac{2\beta}{p}$$

and the market clearing price in state β is $p = \beta$.

[1] Friedrich Hayek. (September 1945). The use of knowledge in society. *The American economic review*, 35(4), 519-530.

Note. The market price is one-to-one correspondence to the underlying state. If a rational agent outside of the market observes the market price, he can infer the underlying state.

Asymmetric information Suppose that consumer 1 observes the actual state, while consumer 2 does not. Because consumer 1 observes the true state, his demand is conditioned on β:

$$x_{1,1}(p;\beta) = \frac{\beta}{p} \quad \forall \beta \in \{1,2\}$$

while consumer 2's demand is independent of β:

$$x_{1,2}(p) = \frac{3}{2p}.$$

The market clearing price solves

$$\frac{\beta}{p} + \frac{3}{2p} = 2$$

implying

$$p = \frac{1}{2}\left[\beta + \frac{3}{2}\right].$$

Note. The market clearing price is one-to-one correspondence to the state.

Rational expectations If consumer 2 is rational, he can infer the true state from the market clearing price. Consumer 2's decision is not optimal, because his belief does not incorporate all information available.

If he is rational, he should be able to infer the underlying state, and should behave as if he observes the underlying state. As a result, the equilibrium price must be fully revealing. The market clearing price must be $p = \beta$.

Private information of consumer 1 is aggregated into the market clearing price so that all consumers in the economy can behave optimally for each state. This property is known as informational efficiency of competitive market. We can trace back the idea to Wealth of Nation by Adam Smith, but Frederick von Hayek is generally credited for the idea of informational efficiency.

Arrow Debreu Symmetric uncertainty. It sounds strong assumption in decentralized economy. extend to asymmetric uncertainty.

Hayek Positive. Informational efficiency of competitive market. Even if asymmetric information exists, competitive market should remain efficient.

Akerlof His example opens up possibility that asymmetric information can lead to inefficiency, more dramatically complete collapse of market. That prompted us to investigate how we can recover efficiency of market in a decentralized manner.

2.3 Lemon's problem

Lecture 4. lemon Mon, Mar 15

Symmetric vs. asymmetric information If an agent observes the underlying state, while another agent does not, we say that asymmetric information exists. The economic impact of asymmetric information was first demonstrated by an example of used cars in Akerlof [1970].

At the time, the profession believes in the informational efficiency of the competitive market. He constructed two simple examples of markets with asymmetric information, which completely changed the way how we understand the role of asymmetric information.

Lemon's market By a lemon, we mean a used car which has a low quality but cannot be differentiated from a good quality car. A buyer can see a used car, but cannot tell whether is a good or bad quality used car, before paying for the car, if he chooses to buy one. The seller observes the true quality of the used car, before the car is put on the market. Asymmetric information exists in the market.

The state is $S = \{H, L\}$. If the state is H, the quality is ϕ_H and the outside option for the high quality used car is c_H. Similarly, if the state is L, the quality is ϕ_L and the outside option for the low quality used car is c_L which we normalized to be 0.

We assume that $\phi_H > \phi_L$ which is the utility from consuming the used car. A high quality car generates more utility than a lower quality car. Similarly, we assume $c_H > c_L = 0$. The outside option of a high quality used car is higher than that of a low quality used car. One can also interpret c_H and c_L as the cost for the good and bad used cars. In order to have a good used car, the seller should spend more money to keep the car in good condition.

Key assumptions A model with lemon's problem satisfies the following three conditions. Let $\pi_H = P(s = H)$ be the probability that the true state is H. This π_H is not the proportion of high quality car in the used car market for sale but one among all used car being used by current owner.

(1) Gains from trading: $\phi_H > c_H > \phi_L > c_L$.

(2) Single crossing property: $\phi_H - c_H > \phi_L - c_L$.

(3) Severe lemon's problem: $\pi_H \phi_H + (1 - \pi_H)\phi_L < c_H$.

Gains from trading If $\phi_H > c_H > \phi_L > c_L$, then the gain from trading is positive in each state. If the true state is H, the gain from trading is $\phi_H - c_H > 0$, and if the true state is L, the gain from trading is $\phi_L - c_L > 0$. Every agent in the economy knows that the gain from trading is always positive. Gains from trading is common knowledge. By moving used car from seller to buyer, society can realize that positive gain from trading.

Single crossing property The gain from trading increases in a high state: $\phi_H - c_H > \phi_L - c_L$. In order to achieve an efficient allocation, it is necessary that trading occurs in the high state with a positive probability. In efficiency allocation, high quality used car must be traded. If not, allocation can not be efficient.

Severe lemon's problem In an economy with a complete set of markets, the used car should fetch a price equal to its utility ϕ_s. Before the state is revealed, the market clearing price, if one exists, must be equal to the average quality $\pi_H \phi_H + (1-\pi_H)\phi_L$. If the market price is lower than c_H, then an owner of a high quality used car would not put the car on sale, because he can fetch a higher price from the outside source, or he cannot recover the cost c_H.

Lemon's market

Theorem 2. The market clearing price is ϕ_L, and only the low quality product is traded.

Proof. We show that ϕ_L is the only possible market clearing price. Suppose that p is the

$p > \phi_H$ is not possible, because no consumer will buy a used car whose quality cannot exceed ϕ_H. ↯

$c_H \leq p \leq \phi_H$. Since $p \geq c_H > c_L$, all low quality sellers will put their low quality cars in the market. As a result, the average quality of a used car in the market cannot be more than $\pi_H \phi_H + (1-\pi_H)\phi_L$ which is strictly less than c_H by the last assumption. Thus, no high quality used car will be on the market, which implies that the quality of the used car is exactly $\phi_L < c_H$ by the first assumption. Since $c_H \leq p$, no buyer will pay p to buy a used car with quality $\phi_L < c_H \leq p$. Hence, p cannot be an equilibrium price. ↯

$\phi_L < p < c_H$. Since $p < c_H$, only the low quality car will be in the market. No buyer is willing to pay a price more than ϕ_L. Thus, p cannot be an equilibrium price. ↯

$p < \phi_L$. Because buyers compete for a used car whose utility is ϕ_L, the market experiences excess demand. ↯

If $p = \phi_L$, only the low quality used car will be on the market and a buyer is willing to pay for his utility for the car. □

Discussion

- The equilibrium allocation is evidently inefficient, because no high quality used car will be traded.
- A surprising part is that the high quality good is driven out of the market, even though the gain from trading is larger and every agent in the economy knows the existence of the positive gain from trading. The logic behind Gresham's law is exactly the same as the lemon's market.
- The nature of uncertainty should be noted. The quality of the used car determines the cost of the seller and the utility of the buyer. In this sense, the quality of the car is the common value of the two players. One of the key components of the lemon's problem is that the seller has private information about the common value.

- The lemon's problem arises in many different cases of asymmetric information over the common value components. Because the lemon's problem leads to an inefficient allocation, it has become a fundamental challenge for economist to find a way to alleviate the implications of the lemon's problem.

Continuous distribution The lemon's problem persists even if we admit more than two types. For example, suppose that the quality is distributed according to continuous density function f over interval $[\phi_l, \phi_h]$. Let $c(\phi)$ be the cost (or the outside option) of the used car with quality ϕ. Assume that $c(\phi)$ is a strictly increasing continuously differentiable function, and $c(\phi) < \phi$ so that there is gain from trading regardless of the quality of a used car. We assume that

$$c(\phi_h) > \mathbb{E}[\phi] = \int_{\phi_l}^{\phi_h} \phi f(\phi)\, d\phi$$

which implies that if the market clearing price is slightly higher than the average quality, the highest quality used car owner would not put his car on the market.

Second example of Akerlof

Theorem 3. If the market clearing price is determined according to the average quality of the products in the market, then the lemon's problem arises and the only equilibrium price is ϕ_l.

Proof. Let p be an equilibrium price. Since the average utility of the products determines the market clearing price,

$$p \leq \mathbb{E}[\phi] = \int_{\phi_l}^{\phi_h} \phi f(\phi)\, d\phi.$$

$c(\phi_h) > \mathbb{E}[\phi]$ and c is a continuous function. $\exists \varepsilon_1$ such that $\forall \phi \in (\phi_h - \varepsilon_1, \phi_h]$ will not put the product in the market since $c(\phi) > \mathbb{E}[\phi]$, where

$$c(\phi_h - \varepsilon_1) = \mathbb{E}[\phi].$$

Then, the average expected price cannot be higher than

$$p \leq \mathbb{E}[\phi \,|\, \phi \leq \phi_h - \varepsilon_1] = \int_{\phi_l}^{\phi_h - \varepsilon_1} \phi f(\phi \,|\, \phi \leq \phi_h - \varepsilon_1)\, d\phi.$$

If we iterate the same process for n rounds, we have ε_n so that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) = \mathbb{E}\left[\phi \,\middle|\, \phi \leq \phi_h - \sum_{k=1}^{n-1} \varepsilon_k\right].$$

By applying the same logic, we conclude that

$$c\left(\phi_h - \sum_{k=1}^{n} \varepsilon_k\right) < \mathbb{E}\left[\phi \,\middle|\, \phi \leq \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

Since $c(\cdot)$ is continuous, $\exists \varepsilon_{n+1} > 0$ so that

$$c\left(\phi_h - \sum_{k=1}^{n+1} \varepsilon_k\right) < \mathbb{E}\left[\phi \,|\, \phi \leq \phi_h - \sum_{k=1}^{n} \varepsilon_k\right].$$

This process continues as long as

$$\phi_h - \sum_{k=1}^{n+1} \varepsilon_k > \phi_l.$$

Thus, ϕ_l is the only equilibrium price. □

Discussion

- In case of a continuous distribution, the conclusion is even more pessimistic than what the discrete example says. In the first example of the discrete model, we expect that a low quality car will be traded at ϕ_l, whose mass is as much as $1 - \pi_h > 0$.
- In case of a continuous distribution, the mass of ϕ_l quality car is infinitesimal. With probability 1, no trading occurs. The market collapses.
- Most students probably have heard Gresham's law, saying that a low quality gold coin drives out a high quality gold coin. The underlying logic is identical with the lemon's problem.

2.4 Extension

Endogenous trigger From the first two examples, one might conclude that the lemon's problem arises because of the parameters of the models are assumed in a specific way. The next example is to show that the lemon's problem can be triggered endogenously through the optimization behavior of an agent.

Merger A corporate raider is trying to buy a firm. Let π be the profit of the firm under the present management, which is distributed uniformed over $[0, 1]$. Under the new management, the profit will be 1.5π $\forall \pi \in [0, 1]$.

In the first period, the raider makes a tender offer p. By the end of the first round, the profit π is realized, and observed by the present manager, but not by the raider. Conditioned on π, the manager decides to weather to accept or reject the tender offer p.

If the management accepts the offer, the management receives p, and the firm will be under the new management appointed by the raider. The profit will be the new profit minus the cost of taking over the firm: $1.5\pi - p$. The management receives the tender offer p, by giving away the firm. If the management rejects the offer, the management receives π and the corporate raider receives 0.

Calculation of an equilibrium We can solve the problem backward, from the optimization problem of the management, who has to decide whether to accept or reject p, conditioned on the realized profit of π. The management accept p only if $p \geq \pi$.

We next calculate the expected return to the raider from purchasing the firm at price p. Since the raider does not observe π, he has to infer the value of π from the response of the management to his tender offer p. To purchase the firm, the management must accept the offer which happens only if $p \geq \pi$. Thus, the expected profit from purchasing the firm at p should be

$$\mathbb{E}\left[\frac{3}{2}\pi \,|\, p \geq \pi\right].$$

The net is then

$$\mathbb{E}\left[\frac{3}{2}\pi \,|\, p \geq \pi\right] - p = \frac{3}{2}\frac{1}{2}p - p = -\frac{p}{4} \leq 0.$$

Therefore, the best response of the raider is to offer 0, at which no trading will occur with a positive probability.

Discussion

- In an efficient allocation, the firm must be under the new management who can increase the profit by 50 percent.
- The challenge of the raider is to infer the profit of the firm, which the raider can obtain only if the management agrees to accept the offer.
- In this case, a lemon is a firm with a low profit. Only the firm with a lower profit than p is willing to accept the offer. The raider is served lemons endogenously by the decision of the present management.
- Knowing the response of the management, the corporate raider would not make any meaningful offer. As a result, the social gain under the new management is not realized.

Asymmetric information on common value component The heart of the lemon's problem is the presence of asymmetric information on common value component.

In the original model of Akerlof, the seller observes the quality of the car, but the buyer dose not. The quality affects the payoff of both the seller and the buyer. Similarly, in the model of merger, the management observes the profit of the firm, but not the corporate raider. The profit affects the payoff of both parties.

The outcome is often a complete collapse of the transaction, hence inefficient allocation. In order to alleviate the lemon's problem, we need to understand the mechanism how asymmetric information on common value component undermines otherwise mutually beneficial transaction.

Two fundamental mechanisms are the absence of screening (lemons) and the inability of credibly signaling (the quality of good used cars).

Buyers cannot screen out bad cars At any price $p > c_l$, an owner of a lemon is willing to put his car in the market. Since $c_h > c_l$, it is inevitable that the low quality cars enter the market, if $p \geq c_h$ so that the high quality car owner has incentive to put his car in the market.

Without screening out low quality cars, a buyer cannot keep low quality cars from entering the market. As a result, the proportion of bad used cars in the market becomes significant, pushing down the quality and leading to the lemon's problem.

Sellers cannot signal good cars In the used car market of Akerlof, a owner of a good used car does not have any mechanism to convince a buyer of the quality of his car. Even if an owner can talk to a buyer, a seller does not have an instrument to credibly signal the quality of the car, because an owner of a bad quality car has incentive to imitate the owner of a good car.

Unless a good car can separate itself away from a bad quality car, a good car cannot fetch a higher price than a bad quality car. If good and bad quality cars fetch the same price, an owner of a good quality car will receive a price below what is worth of his car. If the market price is below his cost of maintaining high quality car, he will pull out his car out of the market so that the lemon's problem arises.

Information economics George A. Akerlof [1970][2] is important, as he pointed out the presence of asymmetric information can hinder a competitive market from achieving an efficient allocation (failure of the first welfare theorem). In response, economists have pursued two possible ways to alleviate the lemon's problem.

Screening model A buyer who does not observe the true quality of a car wants to use an instrument to screen out bad quality cars from good quality cars. Insurance company uses the past information of clients to sort out clients according to the risk, offering different types of contracts. Michael Rothschild and Joseph E. Stiglitz [1976][3] is a seminal paper on this topic.

Signaling model A seller with a good quality car wants to signal the good quality of his car so that his car can be separated away from the rest of cars. Because the low quality car owners have the incentive to imitate, the seller with a high quality car must rely on an instrument, which can prevent an owner of a low quality car from imitating. A good example would be a generous warranty for the car, which incurs prohibitively large cost if the car is bad, but only a modest cost, if the car is good. A. Michael Spence [1973][4] started this line of research.

[2] George A. Akerlof [1970]: The Market for Lemons: Quality Uncertainty and the Market Mechanism. Quarterly Journal of Economics, Vol. 84, No. 3, pp. 488-500

[3] Michael Rothschild and Joseph E. Stiglitz [1976] Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information, Quarterly Journal of Economics, Vol. 90, No. 4, pp. 629-649

[4] A. Michael Spence [1973]: Job Market Signaling, Quarterly Journal of Economics, Vol. 87, No. 3, pp. 355-374

Information economics

- Akerlof [1970], Rothschild and Stiglitz [1976] and Spence [1973] opened up a new area of economics which examines the economic implications of information and uncertainty.
- Their pioneering work accompanied by development in game theory in early 1980's, which allows us to investigated the problem rigorously by equilibrium in games with incomplete information.
- This development ushered the game theory into the main stream of economics, which is often called the games with incomplete information revolution.

Chapter 3

Primer of Information Economics

3.1 Review

Lecture 5. signaling Wed, Mar 17

Complete market

- Arrow-Debreu economy presumes a compete set of markets so that each commodity can be trade at a market clearing price.
- Without market, externality prevails and the first welfare theorem fails.
- All market failure can be traced back to the absence of a market.
- Inefficiency in the lemon's market can be explained by the absence of a market for information.

Market for information Creating a market for information is extremely difficult.

- The value of information is not concave, because information is often not divisible. For example, unless you have a complete set of service manual, you do not have information about how to maintain an airplane.
- Observation and verification by a third party is difficult, or even illegal. Observation of the medical record of an agent requires a complex legal process. Verifying an information by a third party is impossible in an international relationship, because the international organization often lacks the power of enforcement.

The goal of information economics is to examine and understand various implications of information.

Two sources of asymmetric information Asymmetric information arises from two sources.

- Hidden information. An agent has information about the state, while others do not observe the same state.

- Hidden action. The action of an agent is not observe by the other agent, whose payoff is affected by the action of the first agent.

A natural question would be whether there are other sources which we need to consider. The seminal work by Roger Myerson [1983] demonstrated that we can focus on these two sources without loss of generality. That is, within the context of Bayesian decision making, any decision problem involving asymmetric information can be reduced to the problem with hidden information or hidden action or the combination of both.

3.2 Definitions

Hidden information

Definition 11 (Adverse selection). An incentive problem arising from hidden information is called adverse selection problem.

The term originates from insurance industry. If an insurance company offers a generous automobile insurance coverage, it was discovered that the same coverage attracts risky drivers, rendering the pool of clients adversely selected.

In a certain sense, the lemon's market of Akerlof is subject to the adverse selection problem. In order to induce the high quality car to be on the market, the market price must be at least $c_H > c_L$, which provides incentive for lower quality cars to enter the market, driving down the average quality of the car.

Hidden action

Definition 12 (Moral hazard). An incentive problem arising from hidden action is called moral hazard problem.

This term also originates from insurance industry. When an insurance company offers a generous coverage against automobile accidents, the driver tends to drive more recklessly, paying less attention than otherwise.

An insurance company cannot monitor how carefully a driver drives, and paying attention generates disutility to the driver. Because the driver knows the inability of the insurance company to monitor the driver, he has incentive to pay less attention, if the same coverage is offered against accident caused by careless driving or by chance despite careful driving.

Moral? Moral hazard problem is an incentive problem, and has little to do moral. Economists began to regard the moral hazard problem as an incentive problem rather than as a moral problem, following a series of exchanges between Kenneth Arrow[1][2] and Mark Pauly[3].

[1] Kenneth Arrow [1963]: Uncertainty and the Welfare Economics of Medical Care. American Economic Review, Vol. 53, No. 5, pp 941-973

[2] Kenneth Arrow [1968]: The Ecconomics of Moral Hazard: Further Comment. American Economic Review, Vol. 58, No. 3, Part 1, pp. 537-539

[3] Mark Pauly [1968]: Moral Hazard: Comment, American Economic Review, Vol. 58, No. 3, Part 1, pp. 531-537

Why important? If one regards adverse selection and moral hazard as a consequence of bad moral behavior, economists have little to offer. On the other hand, if we regard them as a consequence of an incentive problem of a rational agent, economists can design the environment of the decision maker in such a way to guide his behavior in a socially beneficial way.

Arrow's observation changes what was considered simply a bad behavior to what can be fixed or alleviated by a suitable design of the institution. His insight lays the foundation of the incentive problem.

3.3 Baseline model

Labor market Let us use the labor market model with asymmetric information as the laboratory to examine the consequence of adverse selection and its remedy.

Asymmetric information Let $\theta \in \{\theta_l, \theta_h\}$ be the productivity of the worker with $\theta_h > \theta_l$. The probability that $\theta = \theta_h$ is π. There are firms with want to hire the worker. Because of competitive pressure, a firm has to pay the expected productivity based upon the information of the firm, as wage to hire a worker.

If a firm observes the productivity of the worker, the market wage will be $\theta \in \{\theta_l, \theta_h\}$ depending upon the productivity, but no firm observes the productivity of a worker at the time when the firm hires the worker. The wage w is then

$$w = \mathbb{E}[\theta \,|\, w \text{ is accepted by the worker}].$$

Lemon's problem Let us assume that a worker will not accept an offer, unless the wage is at least as much as his productivity:

$$w \geq \theta.$$

Thus, if w is accepted, the expected productivity of workers is

$$\mathbb{E}[\theta \,|\, \theta \leq w] \leq w$$

and the equality holds if and only if $w = \theta_l$. High productivity workers are driven out of the market.

How to alleviate? We will consider two approaches to alleviate the lemon's problem, depending upon which side of the market makes a move.

- Signaling. The worker, who is informed of the underlying state, makes a move, before he is on the market: the informed party moves first. Because his move is conditioned on the state which he observed, the firm should be able to infer what the worker knows (i.e., his productivity) from the signal.
- Screening. The firm, who is not informed of the underlying state, makes a move, before the firm is on the market: the uninformed party moves first. The firm offers a menu of contracts, which can separate workers with high ability from those with low ability.

3.4 Signaling

Signaling model of Spence [1970] We need to spell out what the signaling is, and how the signaling affects the payoff of each agent. Before a worker looks for a job, he decides how much education he will take. Let e be the amount of education: $0 \leq e < \infty$. The worker's utility is determined by three elements:

- θ. His productivity, or the true state
- w. Wage
- e. Education

Let $u(\theta, w, e)$ be the utility function of a worker in state θ, wage w, and education e. We need to impose a structure to $u(\theta, w, e)$ by specifying how (θ, w, e) determines the utility.

Utility of a worker For simplicity, we assume a quasi linear function:

$$u(\theta, w, e) = w - c(e, \theta)$$

where c is the disutility of education, conditioned on state θ. $\forall e \geq 0$, $\forall \theta$,

- $c(0, \theta) = 0$. No education, no disutility.
- $\partial c(e, \theta)/\partial e > 0$. Marginal disutility of education is positive.
- $\partial^2 c(e, \theta)/\partial e^2 > 0$. Marginal disutility of education is increasing.
- $\partial^2 c(e, \theta)/\partial e \partial \theta < 0$. Marginal disutility of education is decreasing with respect to productivity.

Two key features The first three conditions are easily motivated.

- The only role of education is to generate disutility for the worker. It is an extreme assumption, because education is an important tool to enhance the skill and the human capital of a worker. While restrictive and unrealistic, it strengthens the main conclusion. It is easy to see that if good education improves the productivity, a worker would like to endure disutility of education in return for higher wage in the future. Spence demonstrated that even if education does not improve the productivity, a high ability worker may have incentive to take education only to separate from the lower ability workers.
- The last condition $\partial^2 c(e, \theta)/\partial e \partial \theta < 0$ is known as single crossing property or (Spence-)Mirrlees condition, which warrants additional discussion.

Single crossing property In general, the single crossing property is the monotonicity of the marginal utility with respect to the state.

Definition 13 (Single crossing property). $u(\theta, e, w)$ satisfies the single crossing property if $\partial u(\theta, e, w)/\partial e$ is strictly monotonic with respect to θ. If it is weakly monotonic, we say that u satisfies the weak single crossing property.

In applications, the marginal (dis)utility of signal (education) can be strictly increasing or decreasing, depending upon the sign of the marginal utility itself. In case of the labor market signaling model, it is strictly decreasing, because the sign of the marginal utility is negative.

- The level of education is the signal for the productivity, because the decision to take education is conditioned on the productivity, and the firm knows the link, if not the productivity.
- The signaling is costly, because the marginal utility of education is strictly negative. In some models, the signaling is costless like cheap talk.
- The single crossing property was first invented by James Mirrlees to characterize the optimal taxation scheme through the first order condition, hence his name in the condition. The single crossing property is a sufficient condition under which the first order condition implies the local maximum.
- In the labor market signaling model, Spence used the same condition to allow the high productivity worker to separate from the low productivity worker. The single crossing property is an essential component for the construction of a separating equilibrium (or signaling equilibrium) in which different types of workers choose different levels of educations so that a firm can infer the productivity of a worker from the education.

Signaling equilibrium Let e_θ be the level of education selected by worker with productivity $\theta \in \{\theta_l, \theta_h\}$.

Definition 14. (e_l, e_h) is a signaling equilibrium if $u(\theta_l, e_l, \theta_l) \geq u(\theta_l, e_h, \theta_h)$ and $u(\theta_h, e_h, \theta_h) \geq u(\theta_h, e_l, \theta_l)$.

Discussion

- The original approach by Spence is not game theoretic, and remains vague about what the strategy space is, and what the solution concept is. Let us stick to the original approach, to appreciate his insight.
- A signaling equilibrium is a pair of different education levels, where neither type of workers has incentive to imitate the education level of the other type.
- Because different types of workers choose different levels of education, $\mathbb{E}[\theta \,|\, e_\theta] = \theta$ so that each worker's wage is exactly the productivity of the worker.
- The inequality says that in the signaling equilibrium, type θ worker must have a right incentive to choose the equilibrium education level e_θ instead of choosing the education level of the other type of worker. For this reason, we call the inequality the incentive compatibility constraint.

Why important?

- If a signaling equilibrium exists, the high productivity worker can fetch the wage he deserves, escaping from the trap of the lemon's problem. Thus, the high quality worker may enter the market, and the gains from trading may occurs.
- Signaling model of Spence is the first example to show how we can escape from the lemon's problem, without the third party (i.e., government) intervention. If an outcome is not efficient, the intervention of the government is justified. The insight of Spence is significant, because the lemon's problem can be alleviated without the government's intervention.

Existence

Theorem 4. Let $e_l = 0$, and define e_h implicitly as an education level where the incentive compatibility constraint of θ_l worker is satisfies:

$$u(\theta_l, e_l, \theta_l) \geq u(\theta_l, e_h, \theta_h).$$

Then (e_l, e_h) is a signaling equilibrium.

Proof. We need to verify the incentive compatibility constraint of each type of workers. By the construction of (e_l, e_h), the incentive compatibility constraint of the low productivity worker is satisfied. By the single crossing property, if

$$u(\theta_l, e_l, \theta_l) \geq u(\theta_l, e_h, \theta_h),$$

then

$$u(\theta_h, e_h, \theta_h) \geq u(\theta_h, e_l, \theta_l),$$

and therefore, the incentive compatibility condition of the high productivity worker is satisfied. □

Define e_h^* implicitly $u(\theta_l, e_l, \theta_l) = u(\theta_l, e_h^*, \theta_h)$ as the education level that binds the incentive compatibility constraint. (e_l, e_h^*) is the most efficient signaling equilibrium among all signaling equilibrium, which is called the Riley outcome.

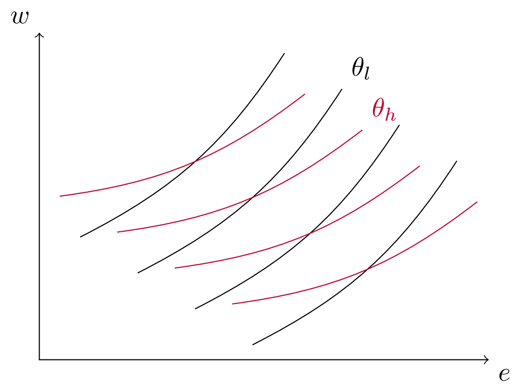


Figure 3.1: Indifference curves of θ_l and θ_h

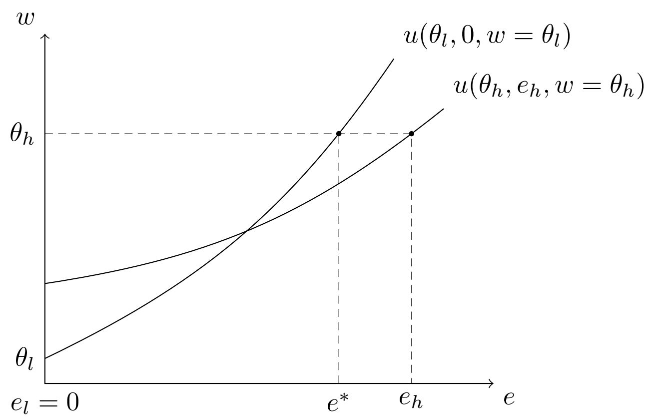


Figure 3.2: Signaling equilibrium in application

Efficient allocation If Spence shows how we can alleviate lemon's problem without the intervention of the government, is the allocation efficient?

- Yes, because both high and low productivity workers are hired by the firm, receiving the same wage as in the complete information about their productivity.
- No, because the incentive compatibility constraint must be satisfied. To do so, the positive amount of eduction must be taken.

First and second best solution If asymmetric information is present, the incentive compatibility constraint must be satisfied in order to reveal the private information about the state truthfully, incurring social cost.

Definition 15. An allocation is the first best solution, if it is a Pareto efficient allocation. An allocation is the second best solution, if it is a Pareto efficient allocation, subject to the incentive compatibility constraint.

The allocation of the Riley equilibrium is not the first best solution, but the second best solution. Typically, in models with asymmetric information, the first best solution is not feasible, unless the allocation satisfies the incentive compatibility constraint.

3.5 Screening

Lecture 6. screening Mon, Mar 22

Firm To escape from the lemon's problem, or to prevent the lower productive workers from entering the employee pool, a firm uses a mechanism design to screen lower quality workers.

The firm has to rely on the difference of the marginal rate of substitution between the wage and the education to screen out one from another group.

Assumptions We maintain the same assumptions on the utility function of the worker, and the firm. Let us summarize the assumptions.

$$u(\theta, w, e) = w - c(e, \theta),\ c(0, \theta) = 0,\ \frac{\partial c(e, \theta)}{\partial e} > 0,\ \frac{\partial^2 c(e, \theta)}{\partial e^2} > 0,$$

and

$$\frac{\partial c(e, \theta)}{\partial \theta} < 0,\ \frac{\partial^2 c(e, \theta)}{\partial e \partial \theta} < 0.$$

Interpretation We continue to assume that the education is only to generate disutility of the workers. We can regard education as (unpleasant) task which must be completed in return for the job (and wage).

First best solution It is easy to see that if productivity θ_i is known to the firm, the firm has to pay for the productivity, without any unpleasant task in an efficient allocation.

Proposition 3. Suppose that the worker's ability is public information. Then,

$$(w_i^*, e_i^*) = (\theta_i, 0) \quad \forall i \in \{h, l\}.$$

and the firms obtain 0 profit.

Proof. Since θ_i is known to the firm, it is easy to see that the wage must be equal to the productivity. Since the only function of the unpleasant task is to generate disutility on the part of the worker, no unpleasant task should be imposed in an efficient allocation (the first best solution).

The difficult part is to show that the firms cannot entertain positive profit. Let us assume that there are two firms competing each other as the Bertrand competitor. The case of the multiple firms can be analyzed in the same way.

Let Π_k be the profit of firm k. Define $\Pi = \Pi_1 + \Pi_2$. Suppose that $\Pi > 0$. Since all firms are identical, we can assume without loss of generality that

$$\Pi_1 \leq \frac{\Pi}{2}.$$

Suppose that firm 1 offers $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$ instead of (w_h^*, e_h^*) and (w_l^*, e_l^*). Since (w_l^*, e_l^*) is an equilibrium for low productive workers, the incentive compatibility condition of θ_l worker must satisfy:

$$w_h^* - c(\theta_l, e_h^*) \leq w_l^* - c(\theta_l, e_l^*).$$

By adding equal amount of ε on both sides, we know that $(w_l^* + \varepsilon, e_l^*)$ is also satisfying the incentive compatibility constraint:

$$(w_h^* + \varepsilon) - c(\theta_l, e_h^*) \leq (w_l^* + \varepsilon) - c(\theta_l, e_l^*).$$

By applying the same logic to θ_h worker, we also conclude that $(w_h^* + \varepsilon, e_h^*)$

is incentive compatible. Choose $\varepsilon > 0$ sufficiently small so that

$$\Pi - \varepsilon > \frac{\Pi}{2} > 0.$$

If firm 1 offers menu of contracts of $(w_h^* + \varepsilon, e_h^*)$ and $(w_l^* + \varepsilon, e_l^*)$, then all θ_h workers will take $(w_l^* + \varepsilon, e_l^*)$, thus generating profit of $\Pi - \varepsilon$ for firm 1. By assumption,

$$\Pi - \varepsilon > \frac{\Pi}{2} \geq \Pi_1$$

which implies that Π_1 is no longer an equilibrium payoff. This is a contradiction to the hypothesis that Π_1 is an equilibrium profit. $\lightning$ $\square$

Asymmetric information Suppose that the workers observe their productivity, but no firms observe the productivity of workers. Akerlof indicated that the market is exposed to the lemon's problem.

In contrast to the signaling model of Spence [1973] in which the workers make move, Rothschild and Stiglitz [1976] demonstrated that the uniformed firms can design a menu of contract which allows the firm to escape from the lemon's problem.

Key concepts A contract is (w, e), which specifies wage w and the level $e \geq 0$ of task associated with the job. A menu of contracts is the list of state contingent contracts:

$$M = ((w_h, e_h), (w_l, e_l))$$

where (w_i, e_i) is supposed to be accepted by θ_i workers.

Incentive compatibility constraint A menu of contracts is feasible if the menu satisfies the incentive compatibility constraint of θ_l worker

$$w_h - c(\theta_l, e_h) \leq w_l - c(\theta_l, e_l)$$

and the incentive compatibility constraint of θ_h worker

$$w_l - c(\theta_h, e_l) \leq w_h - c(\theta_h, e_h).$$

Thanks to the single crossing property, whenever the first inequality holds, the second inequality holds. By the incentive compatibility constraint, we usually mean the first constraint, if we assume the single crossing property.

Time line The economy is populated by a continuum of infinitesimal workers and a finite number of (say, two) of identical risk neutral firms. A state is realized so that the productivity of π portion of workers is θ_h and the remaining portion of worker has productivity θ_l where $\theta_h > \theta_l$.

(1) Workers observe their productivity, but no firm observe the productivity.

(2) Each firm offers a menu of contracts.

(3) Each workers chooses a contract from a menu of contracts from a particular firm. If a worker does not choose a contract, he receives 0.

(4) Payoff is realized.

Equilibrium concept Let $M^i = ((w_h^i, e_h^i), (w_l^i, e_l^i))$ be a menu of contracts offered by firm i. Given (M^1, M^2), let q_j^i be the proportion of θ_j workers who accept w_j^i. Since the proportion of θ_j worker is π_j, $q_j^i \pi_j$ mass of type θ_j workers receives wage w_j^i. Let

$$\mathcal{U}^i(M^1, M^2) = \sum_{j' \neq j \in \{h,l\}} [(q_j^i \pi_j \theta_j + (1 - q_{j'}^i)\pi_{j'}\theta_j) - (q_j^i \pi_j + (1 - q_{j'}^i)\pi_{j'})w_j^i]$$

be the expected payoff of firm $i \in \{1, 2\}$.
Note. The firm's payoff may be negative, if a high wage contract draws too many low productivity workers.

Definition 16 (Equilibrium). (M^1, M^2) is an equilibrium if $\forall i$, M^i is a menu of feasible contracts (satisfying incentive compatibility constraint), where

$$w_j^i - c^i(\theta_j, e_j^i) \geq 0 \quad \forall i \in \{1, 2\}, \forall j \in \{h, l\}$$

and M^i is a best response against $M^{i'}$ among all possible contracts of firm i $\forall i \neq i' \in \{1, 2\}$.

We will focus on symmetric equilibrium where the two firms offer identical menus: $M^1 = M^2$, thus dropping the superscript to simplify analysis.

Main conclusions Let us summarize the main findings of Rothschild and Stiglitz [1976].

Theorem 5. (1) In any equilibrium, firm's profit is 0.

(2) No pooling equilibrium exists.

(3) If a separating equilibrium exists, (w_l, e_l) and (w_h, e_h) satisfy

$$w_l = \theta_l,\ e_l = 0;\ w_h = \theta_h$$

and e_h is defined implicitly by the incentive compatibility constraint of θ_l worker:

$$\theta_h - c(\theta_l, e_h) = \theta_l - c(\theta_l, 0).$$

Proof. We prove the main conclusions in multiple steps, which reveals how the hidden information affects the incentive of the workers, and how the firm can exploit the worker's incentive to screen out different workers.

We follow the same logic as in the previous classes to show that the equilibrium profit of the firm must be 0.

Lemma 1. In any equilibrium, each firm receive 0 profit.

In Spence [1973], the single crossing property allows the high productivity worker to signal his productivity credibly to the firm, to fetch a wage equal to his true productivity. In Rothschild and Stiglitz [1976], the single crossing property of the worker's utility allows the firm to screen them out.

Lemma 2. No pooling equilibrium exists. That is, if $((w_h, e_h), (w_l, e_l))$ is an equilibrium menu, then $(w_h, e_h) \neq (w_l, e_l)$.

If $(w_h, e_h) \neq (w_l, e_l)$, we say it is a separating equilibrium. In a separating equilibrium, (w_j, e_j) is accepted only by θ_j worker $\forall j \in \{h, l\}$. Since the firm makes at least 0 profit, the wage must be equal to the productivity.

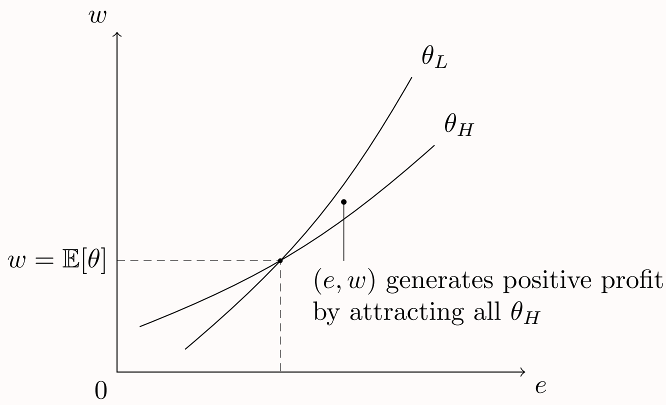


Figure 3.3: No pooling equilibrium exists

Lemma 3. In a separating equilibrium, $w_i = \theta_i \; \forall i \in \{h, l\}$.

In a separating equilibrium, the competitive pressure forces each firm to offer $e_l = 0$ so that the low productivity workers endure no unpleasant task. This result is similar to the property of the signaling equilibrium of Spence [1973] where the low productivity worker does not take any (unpleasant) education. The difference is that in Spence [1973], the decision by the worker is motivated completely by the negative payoff of taking education, while in Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer no task for low productivity workers.

Lemma 4. In a separating equilibrium, $e_l = 0$.

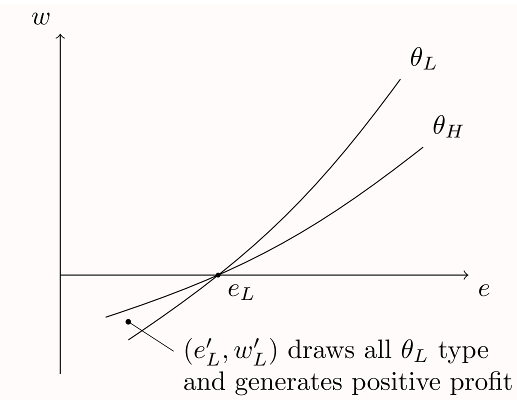


Figure 3.4: $e_L = 0$ in a separating equilibrium

To be a feasible menu, the incentive compatibility condition must hold in a separating equilibrium $((w_h, e_h), (w_l, e_l))$.

$$\theta_h - c(\theta_l, e_h) \leq \theta_l - c(\theta_l, e_l).$$

In Spence [1973], there are multiple signaling equilibria where the weak inequality holds strictly. Only in the Riley outcome, the weak inequality holds with inequality. In Rothschild and Stiglitz [1976], the competitive pressure forces each firm to offer a menu in which the incentive compatibility constraint is binding (i.e., the weak inequality holds with equality).

Lemma 5. In a separating equilibrium $((w_h, e_h), (w_l, e_l))$,

$$\theta_h - c(\theta_l, e_h) = \theta_l - c(\theta_l, e_l).$$

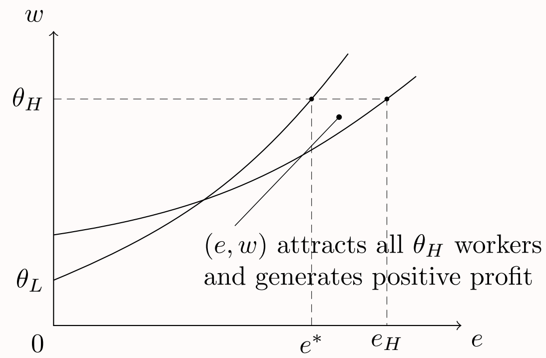


Figure 3.5: $e_H = e^*$ in a separating equilibrium

Existence A separating equilibrium may not exists in Rothschild and Stiglitz [1976]. Because no pooling equilibrium exists, no equilibrium exists if a separating equilibrium fails to exists in Rothschild and Stiglitz [1976].

The conditions under which no equilibrium exists in Rothschild and Stiglitz [1976] further reveals the close relationship between Spence [1973] and in Rothschild and Stiglitz [1976].

Let us consider the Riley outcome, which is the best possible signaling equilibrium for the workers. Because of the disutility of education, a pooling equilibrium can generate higher (ex ante) profit for workers.

As it turns out, Rothschild and Stiglitz [1976] fails to have an equilibrium, if and only if a pooling equilibrium generates higher (ex ante) profit than the Riley outcome in Spence [1973].

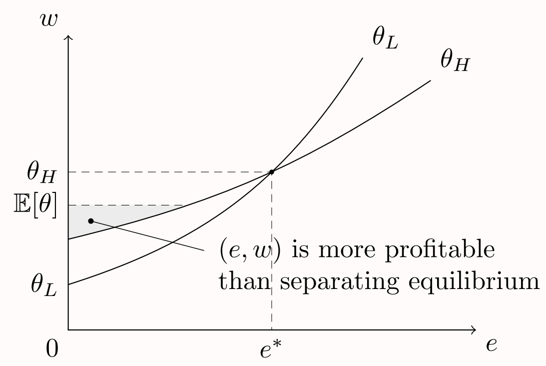


Figure 3.6: No separating equilibrium exists

□

Note. Rothschild and Stiglitz [1976] did not establish the existence of a separating equilibrium. Later studies modified the original model of Rothschild and Stiglitz [1976] to ensure the existence of an equilibrium for all parameter values.

3.6 Hidden action

Definition 17 (Moral hazard problem)**.** An incentive problem arising from hidden action is called the moral hazard problem.

Arrow [1963] observed the moral hazard problem in the health care industry, and first treated it as an incentive problem rather than a moral problem. His observation is ground breaking. If it is a moral problem, an economist has little to say about the remedy. If the problem arises as a consequence of incentive of a rational agent, an economist can propose an alternative rule of the game to guide the incentive to generate more efficient outcome.

Principal agent problem The moral hazard problem is a core concept in the contract literature, which is too extensive for us to cover in this class. Instead, we examine a textbook example of a principal agent problem, which is a classic case of contract subject to moral hazard problem.

Story Consider a risk neutral principal who has to write a contract to ask a risk averse agent to produce an object. A share cropping is an old example of the principal agent problem, where the landlord (the principal) rents out a plot of land so that a tenant farmer produce grain to generate profit.

The profit is then to shared between the two parties so that the principal can receive the rent. The contract stipulates the wage conditioned on the profit of the agent.

The profit is an increasing function of the effort of the agent plus noise. The profit is a noisy signal of the effort level. The principal can only imperfectly monitor the agent's effort level because of the noise. Because agent's action (i.e., effort) is hidden from the principal, the moral hazard problem arises.

Informational structure Let π and e be the profit and the effort. The distribution of π is given by the density unction $f(\pi|e)$. We make two assumptions.

- Imperfect monitoring: $\forall e$, $f(\pi|e) > 0$.
- Stochastic increasing: if $e > e'$, then $f(\pi|e)$ first order stochastically dominates $f(\pi|e')$: $\forall \pi$

$$\int_{-\infty}^{\pi} f(x|e)\,dx < \int_{-\infty}^{\pi} f(x|e')\,dx.$$

Because the moral hazard problem arises whenever the monitoring of the agent's action is imperfect, we often call the class of moral hazard problems the models with imperfect monitoring.

Principal A contract is the wage schedule a function of profit: $w = w(\pi)$. The principal is risk averse, whose expected payoff is $\int_{\pi}(\pi - w(\pi))f(\pi|e)\,d\pi$.

Agent Agent's utility function is $u(w, e)$ where w is wage and e is effort.

- Risk averse: $u_w(w, e) > 0$ and $u_{ww}(w, e) < 0$.
- Increasing marginal disutility of effort: $u_e(w, e) < 0$ and $u_{ee}(w, e) < 0$.

In many applications, we use additively separable utility

$$u(w, e) = v(w) - g(e)$$

with $v' > 0$, $v'' < 0$, $g' > 0$ and $g'' > 0$.

First best solution Suppose that the monitoring is perfect so that the principal is not subject to the moral hazard problem.

The principal is risk neutral while the agent is risk averse. In the first best solution, the principal should take all the risk of the agent so that the agent's income stream must be constant.

Because the principal can monitor the effort e, he can include the level of effort in the contract along with the wage schedule. Thus, his optimization problem is

$$\max_{e,(w(\pi))_\pi} \int (\pi - w(\pi))f(\pi|e)\,d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) \geq \underline{u}$$

for some utility level $\underline{u}$, which is typically the outside option of the agent. We usually normalize $\underline{u} = 0$.

Useful observation The optimization problem is complex, because the principal is maximizing over the set of wage schedules instead of wages. We make two observations.

Lemma 6. If $(e^*, w^*(\pi))$ is an optimal solution, then the individual rationality constraint must be binding:

$$\int v(w^*(\pi))f(\pi|e^*)\,d\pi - g(e^*) = \underline{u}.$$

Proof. Suppose that

$$\int v(w^*(\pi))f(\pi|e^*)\,d\pi - g(e^*) > \underline{u}.$$

Then, the principal can offer slightly less wage, still satisfying

$$\int v(w^*(\pi) - \varepsilon)f(\pi|e^*)\,d\pi - g(e^*) > \underline{u}.$$

Since the utility of the principal is strictly decreasing in wage, the principal has higher expected payoff from $(e^*, w^*(\pi) - \varepsilon)$. We treat the individual rationality constraint as an equality constraint. For each e, we solve

$$\max_{(w(\pi))_\pi} \int (\pi - w(\pi))f(\pi|e)\,d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) = \underline{u}.$$

Then, we choose e^* to maximize objective function. □

First order condition For fixed e, consider the Lagrangian

$$\mathcal{L} = \int (\pi - w(\pi))f(\pi|e)\,d\pi + \lambda\left[\int v(w(\pi))f(\pi|e)\,d\pi - g(e) - \underline{u}\right].$$

The first order condition with respect to $w(\pi)$ for each realized value of π is

$$f(\pi|e) - \lambda v'(w(\pi))f(\pi|e) = 0.$$

After canceling out $f(\pi|e) \neq 0$ from both sides, we have

$$v'(w(\pi)) = \frac{1}{\lambda} \quad \forall \pi.$$

Since $v'' > 0$, v' is invertible. In any optimal value of $w(\pi)$ given e,

$$w(\pi) = [v']^{-1}\left(\frac{1}{\lambda}\right).$$

Note. The right hand side is independent of π.

Full coverage insurance Thus, the wage conditioned on π must be constant with respect to π, for any given e. In the first best solution w^*,

$$w^* = [v']^{-1}\left(\frac{1}{\lambda}\right)$$

so that the agent receives constant stream of wage regardless of the output level. The risk neutral principal offers a full insurance to the agent, in return for his effort e^* so that $v(w^*) - g(e^*) = 0$.

Discussion

- The first best solution relies critically on the assumption that the principal can perfectly monitor the effort level e^*.
- If the agent knows that his effort level is not perfectly monitored, then he has effort level is not perfectly monitored, then he has incentive to reduce his effort because his income w^* is independent of his effort, which is the moral hazard problem.
- The first best solution is infeasible if the monitoring is imperfect.
- To prevent the moral hazard problem, the principal has to design the contract to link the output to the effort level, exposing the agent to the risk. Because the output is stochastically increasing with respect to effort, lowering effort level leads to lower profit and therefore, lower wage on average.
- Instead of offering a full insurance against risk, the principal has to offer a partial insurace, if the moral hazard problem is present.

Hidden action If the principal has no way to get information about e, then there is no hope to have a contract between the principal and the agent. On the other hand, if the principal can perfectly monitor e, no moral hazard problem arises and the first best solution is feasible.

By hidden action, we mean that the principal can imperfectly monitor e so that the first best solution is not feasible. The moral hazard problem becomes relevant, as we can design a contract to guide the incentive of the agent to a socially desirable outcome, if the first best solution is not feasible.

Incentive compatibility constraint The central idea is to design the wage schedule $w(\pi)$ and the effort e so that it is better for the agent to follow the (implicitly) assigned effort level than to disobey the instruction:

$$\int v(w(\pi))f(\pi|e)\,d\pi - g(e) \geq \int v(w(\pi))f(\pi|e')\,d\pi - g(e') \quad \forall e'.$$

The left hand side is the expected utility if the agent follows the contract $(w(\pi), e)$, while the right hand side is the expected payoff if the agent chooses e', given wage schedule $w(\pi)$.

This constraint is called the incentive compatibility constraint. Let us call $(w(\pi), e)$ a feasible contract if $(w(\pi), e)$ satisfies the incentive compatibility constraint.

Optimization problem of the principal The optimization problem under imperfect monitoring is similar to the optimization problem under perfect monitoring, but has the additional constraint of incentive compatibility:

$$\max_{e,(w(\pi))_\pi} \int (\pi - w(\pi)) f(\pi|e)\, d\pi$$

subject to the individual rationality condition of the agent

$$\int v(w(\pi)) f(\pi|e)\, d\pi - g(e) \geq \underline{u}$$

and the incentive compatibility

$$\int v(w(\pi)) f(\pi|e)\, d\pi - g(e) \geq \int v(w(\pi)) f(\pi|e')\, d\pi - g(e') \quad \forall e'.$$

Second best solution Since $e' = e$ must be the best response of the agent under $w(\pi)$, the first order condition of the utility maximization must hold if $e' = e$:

$$\int v(w(\pi)) \frac{\partial f(\pi|e)}{\partial e}\, d\pi - g'(e) = 0.$$

The optimal solution must satisfy the incentive compatibility constraint, thus generating lower profit for the principal. The optimal solution subject to the incentive compatibility constraint is called the second best solution.

Chapter 4

Basic Auctions

Lecture 7.
basicauction
Wed, Mar 24

Review: Games with incomplete information

- The idea of screening different types of workers is essentially to design a program so that the agent is willing to reveal his private information.
- This idea of designing a program to extract private information of an agent is known as the mechanism design problem, which is one of the most important topics as it has a broad range of important applications.
- Instead of introducing to the mechanism design literature directly, let us use a monopolistic market with incomplete information as the laboratory to explore the area of mechanism design, while analyzing yet another extremely important topic of auctions.

4.1 Introduction

Monopolistic market is the trading protocol in which a single seller is trading against many buyers. The number of commodities available to the buyer is usually assumed to be no more than the number of buyers so that the buyer have to compete among themselves.

The monopolistic seller tries to exploit the competitive pressure among buyers to extract larger than the competitive equilibrium surplus.

By an auction, we mean a monopolistic trading protocol. A classic example would be a textbook model of the monopolistic market in which a single seller is facing a continuum of infinitesimal buyer, and sells the infinitely divisible goods by posting a single price.

Why study auctions? Auction is an important institution through which a large amount of goods and services is traded. It is probably one of the oldest trading protocol, going back to Babylon according to Herodotus in allocating brides to grooms. As auctions are widely used for a broad range of economic environment, the auction has many institutional and informational variations.

We study an auction to understand how the price is formed, which is intentionally left vague in Arrow Debreu economy. As the game theoretic tools develop, we can rigorously analyze the equilibrium outcome of the monopolistic trading protocol, and examine how the parameters of the model such as preference, technology and budget affect the delivery price of the good.

4.1.1 Institutional variations

Institutional assumptions Let us assume that there is a single seller who has one unit of a good for sale. We can also consider multiple units for sale, which is an important topic to study. For this class, we focus on the case of the single object for sale.

The seller has no role in the auction, other than placing the good for sale. The seller is risk neutral. We normalize the valuation of the seller to be 0. If the good is sold with probability π and the delivery price is p, then his expected payoff is πp.

Standard auctions There are N buyers, whom we often call bidders. Let v_i be the valuation of buyer i. The buyer has a linear preference, thus risk neutral. If he pays p, his surplus is $v_i - p$. Throughout this class, we focus on the class of standard auctions.

> **Definition 18** (Standard auction)**.** An auction is a standard auction, if a buyer with the highest bid wins the object.

Four basic auctions We start with four basic auctions among standard auctions.

- First price auction
- Dutch auction
- English auction
- Second price auction

Let us describe verbally the rule of each of four basic auctions.

First price auction

- Each bidder place a bid simultaneously. Let b_i be the bid.
- The highest bidder wins the object.
- The winner pays his bid.

Because each bidder does not observe the actions of other bidders, this auction is often called closed auction. The class of auctions in which the winner pays his bid is called pay your bid auction.

Dutch auction The name comes from the fact that one of the main exports of Netherlands is tulip, which requires a quick transaction. In fact, the auction of fish often follows the same format, which also requires a quick transaction.

In front of bidders, a bulletin board displays a price. Starting from a price so high no one will ever buy the object, the displayed price drops continuously. When a bidder stops the clock, the auction stops. The bidder who stops the clock wins the object, paying the price displayed on the bulletin board.

Difference Because the price drops, this class of auctions is called the descending bid auction. An interesting feature of this auction compared to the first price (sealed bid) auction is that a losing bid is not observed, even after the auction is over. Even though a bid is sealed, an economist can access the record of losing bids, if a good is auctioned according to the first price auction. In the Dutch auction, the auction stops immediately when a bidder stops the clock. Because losing bidders have no chance to take an action, an economist cannot access the record of what corresponds to losing bids.

English auction Auction houses like Christie's or Sothby's sell paintings and other goods through an auction, known as English auction.

The seller places a good on the table, and the auctioneer calls the starting bid. If a bidder responds to the bid by a signal (or through a proxy), the auctioneer increases the bid. The bidding continues until no one responds to the call of the auctioneer. Then, the good is delivered to the last bidder.

Because the bidding is done openly, and the price is increasing as the auction progresses, this class of auctions is called the open ascending bid auction.

Button auction The English auction in Christie's or Sothby's is quite complex to analyze. Instead, we examine a simple version of the open ascending bid auction, called the button auction or the Japanese auction.

The auction room has a bulletin board, which displays a price. Each bidder is placed in a cubicle so that he cannot see what other bidders are doing. Each cubicle has a button. A bidder places the button, if he wants to remain in the auction, competing for the good. If two or more bidders are pressing the button, the price in the bulletin board is increasing over time. As the price increases, bidders may drop out. As soon as only one bidder is left pressing the button, the auction ends. The last bidder wins the object, paying the price displayed on the bulletin board.

Simplifying features We do not allow re-entry. We assume that when a bidder stops pressing the button, he leaves the auction once and for all. In reality, re-entry is common, but the analysis of a model with re-entry of bidders is difficult.

For now, we assume that each bidder does not observe whether others are pressing the button or not, because each bidder is placed in a cubicle. This assumption is mainly for simplifying the analysis, even though the same assumption eliminates the feature of open auction from the button auction.

After describing and analyzing the basic model where each bidder is placed in a cubicle, we investigate an open auction version of the button auction, where each bidder observes at which price other bidders quit.

Depending upon the informational structure of the auction, the openness of the button auction makes an important difference to the strategic behavior of bidders.

Second price auction Remember how the first price (sealed bid) auction is conducted. The second price auction is almost identical with the first price auction, except that the winner pays the second highest bid (thus, the highest losing bid), as the name suggests.

This auction format is rarely used, as it is. The main reason is that we consider the bid as the signal of bidder's willingness to pay. If the winner pays less than his bid, we tend to be suspicious of the rule of the auction.

Vickrey auction This auction was invented by William Vickrey. For his contribution, the second price auction is called Vickrey auction.

As the ensuing analysis reveals, this auction format has an important property, and its equilibrium outcome serves as the benchmark for the outcomes of other auctions. For that reason, the auction theory class usually starts with the second price auction.

Even though its original form is rarely used in reality, its equivalent form is widely used. We have yet to explain what we mean by being equivalent and to point out what the equivalent auction is.

Informational assumptions

Private value When a bidder attends the auction, the bidder knows his own valuation of the object, but not others. To generate utility, the bidder need to compete for the product, raising his bid. But, to obtain positive surplus, he would not pay more than what the good is worth to him.

Common value If the government auctions off the right to explore mineral or oil buried underground, the bidders do not know the value of the product, but observe only the samples which are noisy signal of the value of the product. The competitive pressure pushes the bid upward. Because the bidder does not know the true value, but knows only the estimated value based upon his information, it is possible that the winner may end up paying too much for the object. This phenomenon is known the winner's curse.

4.1.2 Informational variations

Inter-related utility In many auctions, the object has two component. A house is a good example. A house provides valuable service as a shelter. Its private value is the present discounted value of the service provided by the house. But, a house has an investment value, for which the bidder has only a noisy information.

We start with the private value model, spending most time analyzing the private value models. We later examine classic cases of (pure) common value models.

IPV The basic institutional and informational assumptions for the private value auctions are known independent private value (IPV) models.

A1 Private value: Bidder i observes his reservation value v_i drawn from $[\underline{v}_i, \overline{v}_i]$ according to cumulative differentiable distribution function $F_i(v_i)$. The seller's reservation value is normalized to 0.

A2 $\forall i \neq j$, $F_i \perp\!\!\!\perp F_j$: v_i and v_j are independent.

A3 Symmetry: $F_1 = \cdots = F_n = F$, $\underline{v}_i = 0$ and $\overline{v}_i = 1$.

A4 Risk neutral: The bidders and the seller are risk neutral. The expected payoff of a bidder can be represented as $p_i v_i - x_i$ where p_i is the probability of winning the object, and x_i is the expected payment.

A5 No reserve price: If the highest bid is higher than the seller's reservation value (which is normalized to 0), the good is sold.

A6 No entry fee: A losing bidder receives 0.

Discussion

- The private value assumption is a substantive assumption. Observing his reservation value at the start of the auction is the defining characteristics of the private value model.

- Independence and symmetry is imposed largely for the simplicity of exposition. If the private value is correlated, we have a number of interesting questions which we will not cover at this point. The symmetry is mainly for the simplicity, but also justify the selection of symmetric Nash equilibrium. We normalize $\underline{v} = 0$ and $\overline{v} = 1$.

Risk neutral players Risk neutral bidders and seller are substantive assumptions. Because the bidder's objective is summarized into the probability of winning and the expected payment, we greatly simplify the analysis. If the bidder is risk averse, the bidder is also concerned of the second moment of the event that he is winning, which changes the nature of bidding behavior substantially.

Reserve price and entry fee No reserve price and no entry fee assumptions are institutional assumptions. These are the instruments which the seller can use to control the competitive pressure and to raise additional revenue. By raising the reserve price, the seller increases the probability of no sales, but hopefully insure himself against unusually low winning price. Entry fee generates revenue. If a bidder has low reservation value, he would find it not worthwhile attending the auction paying the entry fee because his chance of winning the object is too low. As a result, fewer bidders will participate which lower the competitive pressure among bidders and the revenue of the seller. The optimal ways of setting reserve price and entry fee were studied and well understood.

4.1.3 Four basic auctions

Four basic auctions Let us formulate the four basic auctions as a game with incomplete information. A strategy bi of bidder i is a function from his reservation value to a bid. Formally, $\sigma_i \colon [0,1] \to \mathbb{R}$ where $\sigma_i(v_i) = b_i$ is called a bid of bidder i. Let Σ_i be the strategy space of bidder i. To complete the description of each one of four basic auctions, we have to describe the payoff function:

$$\mathcal{U}_i \colon \Sigma_1 \times \cdots \Sigma_n \to \mathbb{R}$$

It would be more convenient to describe the state contingent payoff $u_i(b_1, \cdots, b_n, v)$ conditioned on the underlying state and a profile $(b_1, c \ldots, b_n)$ of bids. We can define ex ante expected utility (which is the payoff function)

$$\mathcal{U}_i(\sigma_1, \cdots, \sigma_n) = \mathbb{E}_v[u_i(\sigma_1(v_1), \cdots, \sigma_n(v_n), v_1, \cdots, v_n)].$$

We define interim expected utility (which is more relevant in making a decision)

$$U_i(b_i, v_i) = \mathbb{E}_{v-i}[u_i(\sigma_{-i}(v_{-i}), b_i, v_1, \cdots, v_n)|v_i]$$

where

$$\sigma_{-i}(v_{-i}) = (\sigma_1(v_1), \cdots, \sigma_{i-1}(v_{i-1}), \sigma_{i+1}(v_{i+1}, \cdots, \sigma_n(v_n))$$

is the profile of strategies except for bidder i's strategy.
Note. A bidder makes a decision conditioned on v_i, but not on v_{-i}, because he only observes his valuation but not others.

Therefore, the interim expected utility function is very much useful in describing the decision by bidder i.

First price auction A bidding function specifies the number (i.e., bid) submitted by bidder i conditioned on v_i, which is formally $\sigma_i \colon [0,1] \to \mathbb{R}$. The delivery rule of the auction is that the highest bidder wins the object, paying his bid.

The state contingent payoff function of bidder i is then

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

where k is the number of the bidders with the highest bids.

We assume no entry fee so that a losing bidder's payoff is 0. We also assume that if there are multiple winners, the good is allocated randomly among multiple winners with an equal probability.

Dutch auction A bidding function specifies the the price at which bidder i conditioned on v_i raises his hand to stop the clock, which is formally $\sigma_i \colon [0,1] \to \mathbb{R}$. The delivery rule of the auction is that the bidder who first raises his hand (or stops the clock first) wins the object, paying his bid.

The state contigent payoff function of bidder i is then

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

where k is the number of the bidders with the highest bids.
Note. The mathematical specification of the strategy and the payoff of the first price and the Dutch auctions are identical. The only difference is the name of a strategy.

Strategic equivalence Because a rational player is free from framing effect, his decision should not be affected by the name of a strategy. His decision is completely determined by the strategy and the payoff function. If two games have the same strategy and payoff function, the outcomes played by rational players must be identical.

> **Definition 19.** If two games have the same normal form game, then the two games are strategically equivalent.

Because the first and the Dutch auctions are strategically equivalent, we only examine the first price auction.

Second price auction As the name suggests, the delivery rule of the second price auction is that the highest bidder wins the object, but pays the second highest bid. Thus, the state contingent utility of bidder i is

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - \max_{j \neq i} b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - \max_{j \neq i} b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Because the payoff function is different from the first price auction, the second price auction is a different game from the first price auction. Despite the formal different, the ensuing analysis will show that the equilibrium outcomes of these two auctions are quite similar.

English auction Let us consider the button auction version of English auction. The strategy of bidder i is to choose a price conditioned on vi at which he drops out. Given bi , the delivery rule of the button auction is identical with the second price auction. Thus, the state contingent utility of bidder i is

$$u_i(b_{-i}, b_i, v) = \begin{cases} v_i - \max_{j \neq i} b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{k}(v_i - \max_{j \neq i} b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

The second price auction and the English auction are strategically equivalent.

Closed and open auctions Among four basic auctions, the first and the Dutch auctions are strategically equivalent, and so are the second and the English auctions. We will examine only the first and the second auctions.

Because the first price auction does not show the bids of other bidders, it is called a closed auction. In (some versions) of English auction, a bidder can see what other bidders are doing. In this sense, the second price auction is often called the open auction.

We start with the analysis of the second price auction.

4.2 Open Auctions

Lecture 8. openauction Mon, Mar 29

4.2.1 Vickrey auction

Second price auction

Definition 20. $\sigma_i(v_i) = v_i$ is the truthful bidding strategy of bidder i.

Definition 21. σ_i is a dominant strategy if σ_i is a best response to any σ_{-i}:

$$\mathcal{U}_i(\sigma_i, \sigma_{-i}) \geq \mathcal{U}_i(\sigma_i', \sigma_{-i}) \quad \forall \sigma_i', \forall \sigma_{-i}.$$

Dominance solvable

Theorem 6. The truthful bidding strategy is a dominant strategy.

Proof. Since every player is ex ante identical, let us consider the optimization problem of bidder 1. Let

$$z = \max(\sigma_2(v_2), \cdots, \sigma_n(v_n)).$$

Let $A(b, z, v)$ be the utility of bidder 1, if his reservation value is v, bid b and the highest competing bid is z:

$$A(b, z, v) = \begin{cases} v - z & \text{if } b > z \\ \frac{1}{k}(v - z) & \text{if } b = z \\ 0 & \text{if } b < z \end{cases}$$

where k is the number of the highest bidders.

We show that for any z, $b = v$ is a best response. First, if $v \geq z$, winning is better than losing. By winning the object, the surplus is $v - z \geq 0$. If $v > z$, bidder 1 can win by bidding $b = v$. Second, if $v < z$, losing is better than winning. He can do so by bidding $b = v$. Thus,

$$A(v, z, v) \geq A(b, z, v) \quad \forall b, z.$$

Let $G(z)$ be the distribution of z. The (interim) expected payoff from bidding b is

$$U(b, v) = \int A(b, z, v)\, dG(z).$$

The (interim) expected payoff of truthful bidding strategy $\sigma(v) = v$ is

$$U(v, v) = \int A(v, z, v)\, dG(v) \geq \int A(b, z, v)\, dG(v) = U(b, v) \quad \forall b,$$

implying that the truthful bidding strategy is a best response against any profile of bidding strategies of other bidders. $\square$

This result has a profound implication. Each bidder has private information about the valuation of the object. If a social planner wants to allocate a single object efficiently, he has to deliver the good to the bidder with the highest reservation value.

If the reservation value is private information, eliciting the information from the bidder is a major challenge, because the bidder has incentive to report his valuation to his advantage, and need not report truthfully.

The fundamental contribution of Vickrey is that if the utility function of the bidder is quasi-linear (or linear with respect to money), then we can design a trading protocol under which each agent has incentive to tell the truth, because truthful bid is a best response. The second price auction, thus named Vickrey auction, is an example that an economist can design a market in which each bidder is willing to tell the truth voluntarily.

Discussion

- The proof reveals that the truthful bid remains an optimal response, even after z is observed by the bidder, which is called the ex post stage. The truthful bid is optimal in the interim stage, as well as ex post stage.
- If each bidder uses a dominant strategy, then he bids according to his reservation value. Thus, the good is delivered to the bidder with the highest reservation value, if $\max(v_1, \cdots, v_n) \geq 0$ so that the gain from trading exists. The allocation is efficient.
- The ability of the second price auction to induce each bidder to reveal his reservation value voluntarily, to achieve an efficient allocation makes the second price auction and its equivalent auction, the English auction, an important benchmarket for designing the market.

Nash equilibrium Suppose that $\sigma_i^*(v_i) = v_i$ is a truthful bidding strategy. Since σ_i^* is a best response against any profile of bidding strategies of other players, it is also a best response against a profile of truthful bidding strategy.

Theorem 7. A profile of truthful bidding strategies is a Nash equilibrium.

Other Nash equilibria The second price auction has a continuum of Nash equilibria. An example would be

$$\sigma_1(v_1) = 1,\ \sigma_2(v_2) = 0,\ \cdots,\ \sigma_n(v_n) = 0 \quad \forall (v_1, \cdots, v_n).$$

To show that the profile constitutes a Nash equilibrium, note that the delivery price is 0, since the second highest bid is 0 and bidder 1 gets the object at the price of 0. Since bidder 1 wins the object at the price of 0 with probability 1, his strategy is a best response.

We need show that $\sigma_j \neq 1$ is a best response. Observe that bidder j's equilibrium payoff is 0. To win the object, bidder j has to bid at least 1, because $\sigma_1(v_1) = 1$. If bidder j bids higher than bidder 1, the second highest bid is 1 (which is the bid by bidder 1). Thus, the payoff from winning the object is

$$v_j - 1 \leq 0.$$

Thus, any deviation from $\sigma_j(v_j) = 0$ cannot generate a positive payoff. Thus, σ_j is a best response.

All other Nash equilibria than the one with the truthful bidding strategy shares common properties.

- The equilibrium strategy of some player must be dominated by the truthful bidding strategy, because the truthful bidding strategy is a dominant strategy. If we apply the elimination of dominated strategies, all Nash equilibria except for the truthful bidding strategy equilibrium are eliminated.
- The equilibrium allocation is not efficient with a positive probability, in contrast to the truthful bidding strategy Nash equilibrium, which always induces an efficient allocation ex post. It is due to the fact that the Nash equilibrium strategy is not symmetric so that the highest bid may not come from someone with the highest reservation value.

From now on, whenever we refer to the equilibrium outcome of the second price auction, we mean the outcome induced by the truthful bidding strategy of each bidder, which is efficient ex post.

4.2.2 Revenue

Auction is a monopolistic trading protocol. If the monopolist has a choice over the protocol, the expected revenue from an auction rule is a key figure to guide the choice of the monopolist.

In the second price auction, the winner is the bidder with the highest reservation value, but pays the second highest bid which is equal to the second highest reservation value. We need to calculate the expected value of the second highest reservation value out of $\{v_1, \cdots, v_n\}$.

Order statistics Let $v_{(k)}$ be the k-th highest reservation value among $v_1, \cdots, v_n$ drawn from the same distribution F. We call it k-th order statistics. If we need to highlight the number of samples, we sometimes write $v_{(k:n)}$ in place of $v_{(k)}$. Since the bidder with the highest reservation value wins, and pays the second highest reservation value, $v_{(1)}$ and $v_{(2)}$ are the two random variables of interest. Let $F_{(k)}$ be the cumulative distribution function of k-th order statistics. Our goal is to calculate $F_{(k)}$ and the expected value of $v_{(k)}$ for $k \in \{1, 2\}$.

First order statistics

$$F_{(1)}(v) = P(v_{(1)} \leq v) = F^n(v)$$

Therefore, the density function is

$$f_{(1)}(v) = nF^{n-1}(v)f(v).$$

Note that

$$\lim_{n\to\infty} F^n(v) \begin{cases} 1 & \text{if } v \geq 1 \\ 0 & \text{if } v < 1. \end{cases}$$

Thus, $v_{(1)}$ converges weakly to 1.

Second order statistics

$$
\begin{aligned}
F_{(2)}(v) &= P(v_{(2)} \le v) \\
&= P(v_{(1)} \le v) + P(v_{(2)} \le v < v_{(1)}) \\
&= F^n(v) + nF^{n-1}(v)(1-F(v)) \\
&= nF^{n-1}(v) - (n-1)F^n(v) \\
f_{(2)}(v) &= n(n-1)F^{n-2}(v)(1-F(v))f(v)
\end{aligned}
$$

Example. Suppose that $F(v)$ is the uniform distribution over $[0,1]$. Then,

$$
\begin{aligned}
&f_{(1)}(v) = nv^{n-1} \\
&E(v_{(1)}) = \int_0^1 v f_{(1)}(v)\, dv = \frac{n}{n+1} \\
&f_{(2)}(v) = n(n-1)v^{n-2}(1-v) \\
&E(v_{(2)}) = \int_0^1 n(n-1)v^{n-1}(1-v)\, dv = n(n-1)\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{n-1}{n+1}.
\end{aligned}
$$

4.2.3 English auction

When we illustrate the English auction, we presented two different versions of the auction.

- Basic version. Each bidder is placed in a cuble so that he cannot observe the actions of other bidders. In particular, he cannot observe whether other bidders are still in the auction or have dropped out at a specific price.

- Open version. Each bidder can observe what others do.

The first version of the English auction is strategically equivalent to the second price auction. Let us examine the second version, which is indeed an open auction, in the sense that the behavior of every bidder is opened to others.

Open ascending bid auction In an open version of the English auction, the strategy space of each bidder is larger than the strategy space of the basic version of the English auction.

As the auction progresses, bidder i can observe the price at which another bidder drops out, and based upon the information, he can change the price at which he drops out. His strategy is now to choose a price to drop out, conditioned on who dropped out at what prices. Since every bidder is ex ante identical, the name of the bidder is not relevant. We can focus on the sequence of prices at which bidder have dropped out.

Let $h_k = \{p_{n-1}, \cdots, p_{k+1}\}$ with $h_{n-1} = \emptyset$ be a history of prices at which $n-k$ bidders have dropped out. Among $n-1$ competing bidders, one of the competitors has dropped out at price p_{n-1}, and p_{k+1} is the latest observation of the price at which one of $k+1$ competing bidders has dropped out. Conditioned on h_k, bidder i has k competing bidders.

Let H_k be the set of all possible h_k. Bidder i's strategy conditioned on h_k

$$\sigma_{i,k} \colon [0,1] \times H_k \to \mathbb{R}$$

specifies the price at which bidder i drops out. Bidder i's strategy is

$$\sigma_i = (\sigma_{i,n-1}, \cdots, \sigma_{i,1}).$$

Dominance solvable

Definition 22. A game is dominance solvable, if the game has a unique strategy profile which survives the repeated elimination of dominated strategies.

If a game has a dominant strategy like the second price auction, the game has a unique strategy profile after one round of elimination of dominated strategies. We now allow the elimination process can continue more than a single round. Let us consider the truthful bidding strategy in the open ascending bid auction:

$$b_{i,k}(v_i, h_k) = v_i \quad \forall v_i,\, \forall h_k,\, \forall k \in \{1, \cdots, n-1\}.$$

That is, bidder i drops out at his reservation value following any history.

Theorem 8. The open ascending bid auction is dominance solvable. The only strategy surviving the repeated elimination of dominated strategies is the truthful bidding strategy following every history.

Proof. Fix h_1. If bidder i has a single competing bidder in the open ascending bid auction, he has the same decision problem as in the second price auction. His dominant strategy is to drop out at his reservation value.

Suppose that we have shown that dropping at at his reservation value is a dominant strategy for $h_1, \cdots, h_{k-1}$. Fix h_k so that k competing bidders are left. If $j \leq k-1$ bidders quit, then bidder i will play again $k-j$ bidders, where the truthful bid is shown to be a dominant strategy. If k bidders quit simultaneously, then he will be the winner.

Following the same logic as in the proof to show that the truthful bid is a dominant strategy in the second price auction, we can show that conditioned on being a winner against k bidders, it is a dominant strategy to bid his reservation value. Thus, conditioned on h_k, it is a dominant strategy to drop out at his reservation value. $\square$

The crucial assumption is the private value model. Because each bidder observes the valuation of the object, the observation of other player's action does not change his valuation.

This is not the case in the common value model, in which a bidder does not know the true value, but has an estimate of the underlying value. By observing the price at which other bidders drop out, he can use the information to update his estimate of the value, and adjust his bid. The equilibrium outcome of the basic version of the English auction differs from the outcome of the open version of the English auction.

Discussion Because we have to apply the elimination process from the end of the game, we have to apply repeatedly the elimination of dominated strategies. In the end, we select a truthful bidding strategy, which leads to the efficient allocation.

The key assumption is that the valuation of the object does not change, conditioned on history. This assumption holds in the private value model, but not in the common value model which we will discuss later.

4.3 Closed Auctions

Lecture 9. closedauction Wed, Mar 31

4.3.1 Introduction

First price auction In the second price auction, the truthful bidding strategy is a dominant strategy. In an open ascending bid auction, it is the only strategy that survives the repeated elimination of dominated strategies.

In a sharp contrast, the same strategy does not survive the repeated elimination of dominated strategies.

Proposition 4. The truthful bidding strategy does not survive the repeated elimination of dominated strategies in the first price auction.

Proof. The payoff from the truthful bidding strategy is always 0. If he wins, the payment is equal to his reservation value so that the surplus is 0. If he loses, the surplus is 0.

If bidder i places a bid $v_i - \varepsilon$, then with a positive probability $v_i - \varepsilon$ is the highest bid and his expected payoff is bounded away from 0. A strategy which admits a bid above the reservation value is a dominated strategy. With probability $F^{n-1}(v_i - \varepsilon)$, all reservation value of other bidders is less than $v_i - \varepsilon$ and therefore, $v_i - \varepsilon$ will be a winning bid to generate a positive surplus for bidder i. Thus, the truthful bid is dominated by $v_i - \varepsilon$, after we eliminate all bidding strategies which admits a bid above the reservation value. □

The first price auction does not have a dominant strategy or is not dominance solvable. The calculation of a Nash equilibrium is more involved than for the second price auction.

4.3.2 Symmetric equilibrium

Definition 23 (Symmetric equilibrium)**.** A Nash equilibrium $(\sigma_1^*, \cdots, \sigma_n^*)$ is a symmetric Nash equilibrium, if $\sigma_1^* = \cdots = \sigma_n^*$.

Bidders are ex ante symmetric: Until a bidder observes his reservation value v_i (which differs across different bidders), each bidder has the same type of utility function and the distribution over this reservation value is identical.

For this reason, a symmetric Nash equilibrium has been the focal point of analysis in the literature.

Interim probability In games with incomplete information, like the auctions we are examining now, the information is revealed as the game progresses. In a typical model, a bidder makes a decision in a interim stage, when bidder i observes v_i , but does not observes the valuation of others, v_{-i}.

A related, and important, concept is ex ante and ex post. Before any bidder observes his own reservation value, we call the stage ex ante. Naturally, the ex ante expected utility must be the right way to calculate the return from his choice.

The ex post stage is essentially the time when the game is over, so that each bidder observes the reservation value of every player. It is too late to make a decision at the ex post stage. Ex post stage is still relevant for the efficiency of allocation, and possibility of re-trading.

Because the decision is made in the interim stage, it is convenient to write down the relevant probability conditioned on the information of bidder i, which include his reservation value and his bid.

Because we focus on the symmetric Nash equilibrium, we will drop subscript i from the variable. For example, instead of v_i and b_i , we write v and b.

Notation Let $\hat{Q}_i(b_i)$ be the probability that a bidder wins the object if he places bid b_i. Formally,

$$\hat{Q}_i(b_i) = P\left(b_i \geq \max_{j \neq i} \sigma_j(v_j)\right).$$

Note. $\hat{Q}$ does not depend on bidder i's reservation value.

Since bidder i does not observe v_{-i}, $\hat{Q}$ depends only upon b_i, given σ_{-i}. Then, the (interim) expected payoff of bidder i with reservation value v_i is

$$\Pi_i(v_i, b_i) = (v_i - b_i)\hat{Q}_i(b_i).$$

Since we consider a symmetric Nash equilibrium, we drop subscript i and write

$$\Pi(v, b) = (v - b)\hat{Q}(b).$$

Symmetric Nash equilibrium

Theorem 9. Define

$$\sigma^*(v) = v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{n-1} dx.$$

$(\sigma^*, \cdots, \sigma^*)$ is the unique symmetric Nash equilibrium.

Since $F(x) > 0$ for $x > 0$, $\sigma^*(v) < v$ for $v > 0$. In equilibrium, bidder places a bid less than his reservation value. The difference is called bid shading.

Note.

$$\frac{F(x)}{F(v)} = P(v_j \leq x \,|\, v_j \leq v).$$

Thus,

$$\begin{aligned}\left[\frac{F(x)}{F(v)}\right]^{n-1} &= P\left(\max_{j \neq i} v_j \leq x \,\middle|\, \max_{j \neq i} v_j < v\right) \\ &= P(v_{(1:n-1)} \,|\, v_{(1:n-1)} \leq v)\end{aligned}$$

The symmetric equilibrium strategy has two alternative representations, which reveal different properties of the equilibrium strategy. By integrating by part,

$$
\begin{aligned}
v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{n-1} dx &= v - x \left[\frac{F(x)}{F(v)}\right]^{n-1}\Bigg|_0^v + \int_0^v x\, d\left[\frac{F(x)}{F(v)}\right]^{n-1} \\
&= \int_0^v x\, d\left[\frac{F(x)}{F(v)}\right]^{n-1} \\
&= \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-1}(v)}\, dx
\end{aligned}
$$

Note.

$$
\int_0^v x\, d\left[\frac{F(x)}{F(v)}\right]^{n-1}
$$

is the expected value of a random variable over $[0, v]$ whose distribution is

$$
\left[\frac{F(x)}{F(v)}\right]^{n-1}
$$

which is the distribution function of the highest reservation value among $n-1$ competing bidders, conditioned on the event that the highest reservation value does not exceed v. Recall that we write the first order statistics among $n-1$ samples as $v_{(1:n-1)}$.

Thus, the integration can be written as

$$
E(v_{(1:n-1)})\,|\,v_{(1:n-1)} \leq v).
$$

For later reference, let us collect three different formla of the symmetric Nash equilibrium strategy.

$$
\begin{aligned}
\sigma^*(v) &= v - \int_0^v \left[\frac{F(x)}{F(v)}\right]^{n-1} dx \\
&= E(v_{(1:n-1)})\,|\,v_{(1:n-1)} \leq v) \\
&= \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-1}(v)}\, dx.
\end{aligned}
$$

The first formula is useful to show the existence and the size of the bid shading. The second expression provides an economic interpretation of the equilibrium bid.

The last expression is used to prove the theorem. In particular, the last formula is used to show that the bidding function is strictly increasing with respect to v.

First and second price auctions Despite different rules of auctions, the two auctions share important properties.

- Since the bidding function is strictly increasing with respect to v, and every bidder uses the same strategy, the bidder with the highest reservation value wins the object with probability 1.

- The expected payoff of the bidder whose reservation value is the lowest (i.e., $v = 0$) is 0.

These are two main conditions for the revenue equivalence theorem which is the fundamental result in the auction theory. Roughly speaking, in the frame work of IPV, any Nash equilibrium in any auction which satisfies these two conditions generates the expected revenue for the seller equal to $E(v_{(2:n)})$ (that is the expected revenue of the second price auction).

If you compare the first and the second price auctions, the first price auction appears to have an advantage in generating higher revenue for the seller, because the winner has to pay the highest bid, instead of the second highst bid. Since the bidders are rational, the bidders bid less aggressively in the first price auction than in the second price auction. In the second price auction, a bidder is willing to bid up to his reservation value, but in the first price auction, his bid is strictly less than his reservation value. The revenue equivalence theorem says that the impacts of these two factors are perfectly cancelled out in any Nash equilibrium satisfying the two properties.

Example. Suppose that v is uniformly distributed over [0, 1]. We know that

$$E(v_{(2:n)}) = \frac{n-1}{n+1}.$$

$$\sigma^*(v) = v - \int_0^v \left(\frac{x}{v}\right)^{n-1} dx = v - \frac{1}{v^{n-1}} \frac{v^n}{n} = v - \frac{1}{n} v.$$

The amount of bid shading is $\frac{v}{n}$. As the number of bidders increases, a bidder is compelled to place a bid closer to his reservation value.

Since the winner pays his bid, the seller's expected revenue from the first price auction is

$$\begin{aligned} E(\sigma^*(v_{(1:n)})) &= E\left(\frac{n-1}{n} v_{(1:n)}\right) \\ &= \frac{n-1}{n} E(v_{(1:n)}) = \frac{n-1}{n} \frac{n}{n+1} = \frac{n-1}{n+1} \end{aligned}$$

which is exactly the same as the expected revenue from the second price auction.

Now we prove Theorem 9.

Proof (Theorem 9)**.** First, we show that $(\sigma^*, \cdots, \sigma^*)$ is the symmetric Nash equilibrium.

Step 1. $\sigma^*(v)$ is strictly increasing in $v > 0$. We use the third formula of $\sigma^*(v)$, in combination with the first formula.

$$\sigma^*(v) = \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-1}(v)} dx.$$

By differentiating the right hand side with respect to v, we have

$$\frac{d\sigma^*(v)}{dv} = \frac{(n-1)(v-\sigma^*(v))f(v)}{F(v)} > 0$$

since

$$v - \sigma^*(v) = \int_0^v \left(\frac{x}{v}\right)^{n-1} dx > 0 \quad \forall v > 0.$$

Since $\sigma^*(v)$ is strictly increasing, the inverse function of $\sigma^*(v)$ exists. Let

$$\varphi^*(b) = (\sigma^*)^{-1}(b)$$

be the inverse function of σ^*. Recall that $\hat{Q}(b)$ is the interim probability that bidder i wins the object if he bids b.

Step 2. $\hat{Q}(b) = F^{n-1}(\varphi^*(b))$

$$P(b \geq \sigma^*(v)) = P(\varphi^*(b) \geq v) = F(\varphi^*(b)).$$

Since the strategy of other players is independent,

$$\begin{aligned}\hat{Q}(v) &= P(b \geq \max_{j\neq i} \sigma^*(v_j)) \prod_{j\neq i} P(b \geq \sigma^*(v)) \\ &= \prod_{j\neq i} F(\varphi^*(b)) = F^{n-1}(\varphi^*(b)).\end{aligned}$$

We know that the interim expected utility of a bidder with reservation value v when he bids b is

$$\Pi(v,b) = (v-b)F^{n-1}(\varphi^*(b)).$$

First order condition.

$$\begin{aligned}\frac{\partial \Pi(v,b)}{\partial b} = (n-1)(v-b)F^{n-2}(\varphi^*(b))f(\varphi^*(b))(\varphi^*)'(b) \\ - F(\varphi^*(b))^{n-1} = 0\end{aligned} \tag{4.1}$$

if $b = \sigma^*(v)$. Because

$$(\varphi^*)'(b) = \frac{1}{\sigma^{*\prime}(v)}, \quad \varphi^*(\sigma^*(v)) = v,$$

we have

$$\frac{\partial \Pi(v,b)}{\partial b} = -F^{n-1}(v) + (n-1)(v-\sigma^*(v))F^{n-2}(v)f(v)\frac{1}{\sigma^{*\prime}(v)} = 0.$$

By collecting the terms, we have

$$(\sigma^*)'(v) = \frac{(n-1)(v-\sigma^*(v))F^{n-2}(v)f(v)}{F^{n-1}(v)}$$

which is the first order differential equation of σ^*.

Second order condition. Before we proceed, we must check whether the first order condition identifies a local maximum. To do so, we prove that the second order condition of the maximization is satisfied. This task is trickier than usual, because the density function is continuous but may not be differentiable. We need to show

$$\Pi_b(v, b) = \frac{\partial \Pi(v, b)}{\partial b}$$

is a decreasing function in a neighborhood of $b = \sigma^*(v)$, even if f is not differentiable. Recall that (4.1). We need to show that in a small neighborhood of $b = \sigma^*(v)$, if $b' > \sigma^*(v)$, then

$$\frac{\partial \Pi(v, b')}{\partial b} < 0.$$

Note

$$\frac{\partial^2 \Pi(v, b)}{\partial b \partial v} = (n-1)F^{n-2}(\varphi^*(b))f(\varphi^*(b))(\varphi^*)'(b) > 0.$$

Since σ^* is differentiable and strictly increasing, $\forall b' > \sigma^*(v)$ in a small neighborhood of $b = \sigma^*(v)$, $\exists v' > v$ so that $b' = \sigma^*(v')$. In particular, $b' = \sigma^*(v') > \sigma^*(v) = b$. Note that

$$\frac{\partial \Pi(v, \sigma^*(v))}{\partial b} = \frac{\partial \Pi(v', \sigma^*(v'))}{\partial b} = 0.$$

Since $\frac{\partial \Pi(v,b)}{\partial b}$ is strictly increasing in v, and $v' > v$,

$$0 = \frac{\partial \Pi(v', \sigma^*(v'))}{\partial b} > \frac{\partial \Pi(v, \sigma^*(v'))}{\partial b} = \frac{\partial \Pi(v, b')}{\partial b}$$

so that

$$0 = \frac{\partial \Pi(v, \sigma^*(v))}{\partial b} > \frac{\partial \Pi(v, b')}{\partial b}$$

for $b' > \sigma^*(v)$ as desired. By following the same logic, we have

$$\frac{\partial \Pi(v, \sigma^*(v))}{\partial b} < \frac{\partial \Pi(v, b')}{\partial b}$$

for $b' < \sigma^*(v)$.

Solving differential equation. From the first order condition, we have a differential equation.

$$(\sigma^*)'(v) = \frac{(n-1)(v - \sigma^*(v))F^{n-2}(v)f(v)}{F^{n-1}(v)}.$$

Moving terms, we have

$$(\sigma^*)'(v)F^{n-1}(v) + (n-1)\sigma^*(v)F^{n-2}(v)f(v) = (n-1)vF^{n-2}(v)f(v).$$

Note that the right hand side is the derivation of $[\sigma^*(v)F^{n-2}(v)]$. Therefore,

$$[\sigma^*(v)F^{n-2}(v)]' = (n-1)vF^{n-2}(v)f(v).$$

By the fundamental law of calculus,

$$\sigma^*(v)F^{n-2}(v) - \sigma^*(0)F^{n-2}(0) = \int_0^v (n-1)xF^{n-2}(x)f(x)\,dx.$$

By assumption, $F(0) = 0$. Therefore,

$$\sigma^*(v) = \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-2}(v)}\,dx$$

which is the third formula of the symmetric Nash equilibrium strategy.

Uniqueness It takes more work to show that the symmetric Nash equilibrium is unique. Because the method we used in the proof is replicated in different models, we will go through the proof.

We show that in any symmetric Nash equilibrium $(\sigma, \cdots, \sigma)$ the following three properties must hold.

A. $\sigma(0) = 0$.

B. σ is differentiable over $(0, 1)$.

C. σ is strictly increasing.

We first show that the three properties imply that σ must be the third formula of the symmetric Nash equilibrium strategy. Then, we establish three properties in several steps.

Proposition 5. If σ satisfies A, B, and C, then $\sigma(v) = \sigma^*(v)$.

Proof. By A and C, $\sigma(0) = 0$ and is strictly increasing over $[0,1]$. Since σ is strictly increasing, it is invertible. Define $\varphi(b) = \sigma^{-1}(b)$. In a symmetric equilibrium in which the equilibrium strategy is strictly increasing, the probability of winning the object is

$$P\left(v \geq \max_{j \neq i} v_j\right) = F^{n-1}(v).$$

Thus, the interim expected equilibrium payoff is

$$\Pi(v, \sigma(v)) = (v - \sigma(v))F^{n-1}(v).$$

Since σ is an equilibrium strategy, any deviation should not be profitable. In particular, fix any $v, z \in [0,1]$ with $v > z$. In equilibrium, reservation value v bidder should not have incentive to imitate the bid of reservation value z, and vice versa. Thus,

$$\begin{aligned}(v - \sigma(v))F^{n-1}(v) &\geq (v - \sigma(z))F^{n-1}(z)\\ (z - \sigma(z))F^{n-1}(z) &\geq (z - \sigma(v))F^{n-1}(v).\end{aligned}$$

From the first constraint,

$$v[F^{n-1}(v) - F^{n-1}(z)] \geq \sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z)$$

and from the second constraint,

$$\sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z) \geq z[F^{n-1}(v) - F^{n-1}(z)].$$

Combining the two inequality, and using $v > z$, we have

$$\begin{aligned}v\frac{F^{n-1}(v) - F^{n-1}(z)}{v - z} \geq \frac{\sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z)}{v - z}\\ \geq z\frac{F^{n-1}(v) - F^{n-1}(z)}{v - z}.\end{aligned}$$

Because these two inequalities must hold any $v - z > 0$, we can let $v - z \to 0$ to have

$$\sigma(v)F^{n-1}(v)]' = v[F^{n-1}(v)]' = (n-1)vF^{n-2}(v)f(v).$$

By the fundamental law of calculus,

$$\sigma(v)F^{n-1}(v) - \sigma(0)F^{n-1}(0) = \int_0^v (n-1)xF^{n-2}(x)f(x)\,dx.$$

Since $\sigma(0) = F(0) = 0$,

$$\sigma(v) = \int_0^v \frac{(n-1)xF^{n-2}(x)f(x)}{F^{n-1}(v)}\,dx$$

which is exactly the third formula of $\sigma^*(v)$. □

We establish A, B, and C in several steps.

Step 1. $\exists v_0$ such that $\forall v \leq v_0$, $\sigma(v) = 0$ if $Q(\sigma(v)) > 0$. Note that

$$\Pi(v, b) = (v - b)\hat{Q}(b)$$

is a strictly increasing function of v if $\hat{Q}(b) > 0$, and $\Pi(0, b) \leq 0$, $\forall b$. Define

$$v_0 = \sup\{\hat{v} \mid \sup_b \Pi(\hat{v}, b) \leq 0\}.$$

By the definition of v_0, $\forall v < v_0$, $\Pi(v, b) < 0$ if $\hat{Q}(b) > 0$.

Step 2. $\sigma(v)$ is a weakly increasing function of $v \geq v_0$. Fix any $v, z \geq v_0$. Without loss of generality, assume $v > z \geq v_0$. By the definition of v_0, $v - \sigma(v) > 0$ and $z - \sigma(z) > 0$. By the incentive compatibility constraint,

$$\begin{aligned}(v - \sigma(v))\hat{Q}(\sigma(v)) &\geq (v - \sigma(z))\hat{Q}(\sigma(z)) \\ (z - \sigma(z))\hat{Q}(\sigma(z)) &\geq (z - \sigma(v))\hat{Q}(\sigma(v)).\end{aligned}$$

Adding up both sides, and simplifying the terms, we have

$$(v - z)(\hat{Q}(\sigma(v)) - \hat{Q}(\sigma(z))) \geq 0.$$

Since $v - z > 0$, we have

$$\hat{Q}(\sigma(v)) - \hat{Q}(\sigma(z)) \geq 0$$

which is short of proving that $\hat{Q}(b)$ is weakly increasing, because we have yet to show that $\sigma(v)$ is weakly increasing.

By the definition of v_0, bidder with reservation value v receives strictly positive payoff. Thus,

$$\hat{Q}(\sigma(v)) > 0$$

which implies

$$\frac{\hat{Q}(\sigma(z))}{\hat{Q}(\sigma(v))} \leq 1.$$

From the incentive compatibility constraint of bidder with reservation value z, we know

$$\frac{\hat{Q}(\sigma(z))}{\hat{Q}(\sigma(v))} \geq \frac{z - \sigma(v)}{z - \sigma(z)}.$$

Therefore, $\sigma(v) \geq \sigma(z)$ from which we conclude that $\hat{Q}(b)$ is weakly increasing. The following is a useful corollary.

> **Corollary.** $\hat{Q}(\sigma(v))$ is weakly increasing in v.

Step 3. $\sigma(v)$ is strictly increasing over $(v_0, 1]$. Suppose that $\sigma(v)$ is weakly increasing, but not strictly increasing around the neighborhood of some reservation value:

$$\exists z, w \in (v_0, 1] \text{ so that } \sigma(z) = \sigma(w).$$

Since σ is weakly increasing, $\forall z' \in (z, w)$,

$$\sigma(z) = \sigma(z')\sigma(w).$$

Recall that $\hat{Q}(b) = [P(\sigma(v) \leq b)]^{n-1}$. The probability inside of the bracket is the probability that b can beat bidder j whose is using strategy $\sigma(v)$. Since all bidders are ex ante identical, and there are $n-1$ competing bidders, we multiply the probability $n-1$ times. Since $\sigma(v)$ is flat over interval $[z, w]$,

$$\lim_{\varepsilon \to 0} \hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z) - \varepsilon) > 0$$

which should have been 0, if σ is strictly increasing.

In particular, if there are multiple winners, the object is distributed according to random allocation rule. Thus,

$$\lim_{\varepsilon \to 0} \hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z)) > 0.$$

At $v = z$, bidder i's equilibrium payoff is

$$(z - \sigma(z))\hat{Q}(\sigma(z)).$$

If he increases his bid by $\varepsilon > 0$, the expected payoff is

$$(z - \sigma(z) - \varepsilon)\hat{Q}(\sigma(z) + \varepsilon).$$

Note that

$$\begin{aligned}
&(z - \sigma(z) - \varepsilon)\hat{Q}(\sigma(z) + \varepsilon) - (z - \sigma(z))\hat{Q}(\sigma(z)) \\
&\qquad = (z - \sigma(z))[\hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z))] \\
&\qquad\qquad - \varepsilon[\hat{Q}(\sigma(z) + \varepsilon) - \hat{Q}(\sigma(z))].
\end{aligned}$$

Since $z > v_0$, $z - \sigma(z) > 0$. As $\varepsilon \to 0$, the first term is bounded away from 0, while the second term vanishes. Thus, for a sufficiently small $\varepsilon > 0$, a slight increase of the bid from $\sigma(z)$ increases the profit, which contradicts the hypothesis that $\sigma(z)$ is an equilibrium bid. The following is a useful corollary.

Corollary. $\sigma(v) > 0 \; \forall v > v_0$.

Step 4. $v_0 = 0$ and $\lim_{v \to 0} \sigma(v) = 0$. We now know

$$\sigma(v) > 0 \quad \text{and} \quad \hat{Q}(b(v)) > 0 \quad \forall v > v_0.$$

Note that if a bidder place a bid equal to $\varepsilon > 0$, his chance of winning the object is at least $F^{n-1}(v_0 + \varepsilon)$. $\forall v \leq v_0$, a bidder can generate at least

$$(v - \varepsilon)F^{n-1}(v_0 + \varepsilon) \quad \forall \varepsilon > 0.$$

By the definition of v_0, the expected payoff of bidder with reservation value $v \leq v_0$ is 0. Thus,

$$0 \leq \lim_{\varepsilon \to 0}(v - \varepsilon)F^{n-1}(v_0 + \varepsilon) = vF^{n-1}(v_0) \leq \Pi(v, \sigma(v)) = 0.$$

Thus, $v_0 = 0$. We know $\forall v > v_0$, $v - \sigma(v) > 0$. If

$$\lim_{v \to 0} \sigma(v) > 0,$$

then there is a small $v > 0$ where $v - \sigma(v) < 0$. Since $\forall v > v_0 = 0$, $\hat{Q}(v) > 0$, $v - \sigma(v) < 0$ implies that $\Pi(v, \sigma(v)) < 0$, which is a contradiction to the hypothesis that σ is an equilibrium strategy. $\lightning$

Step 5. σ is a differentiable and

$$\sigma'(v) = \frac{(n-1)(v - \sigma(v))f(v)}{F(v)}.$$

Since $\sigma(v)$ is strictly increasing over $[0, 1]$,

$$\hat{Q}(\sigma(v)) = F^{n-1}(v).$$

Since the incentive compatibility constraint must hold $\forall v, z \in (0, 1)$ with $v > z > 0$,

$$\begin{aligned} (v - \sigma(v))F^{n-1}(v) &\geq (v - \sigma(z))F^{n-1}(z) \\ (z - \sigma(z))F^{n-1}(z) &\geq (z - \sigma(v))F^{n-1}(v) \end{aligned}$$

which implies that

$$\begin{aligned} v(F^{n-1}(v) - F^{n-1}(z) &\geq \sigma(v)F^{n-1}(v) - \sigma(z))F^{n-1}(z) \\ \sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z) &\geq z(F^{n-1}(v) - F^{n-1}(z)). \end{aligned}$$

Hence,

$$\begin{aligned} v\frac{F^{n-1}(v) - F^{n-1}(z)}{v - z} &\geq \frac{\sigma(v)F^{n-1}(v) - \sigma(z)F^{n-1}(z)}{v - z} \\ &\geq z\frac{F^{n-1}(v) - F^{n-1}(z)}{v - z}. \end{aligned}$$

As $v - z \to 0$, the first and the last term converge to the same limit:

$$v[F^{n-1}(v)]'.$$

Then,
$$(\sigma(v)F^{n-1}(v))' = v(F^{n-1}(v))'$$
from which the conclusion follows.

□

Chapter 5

Mechanism Design

5.1 Revenue comparison

Lecture 10. mechanism Mon, Apr 5

Revenue equivalence theorem

- Auction is a monopolistic trading protocol. The expected revenue generated by an auction is an important factor for a monopolist to use the format.
- Many institutional and informational variations of the auction make it challenging to compare the equilibrium payoff of the monopolist.
- It is useful to have a benchmarket for the revenue comparison by identying a set of conditions under which a broad set of auctions generates the same expected revenue of the monopolist: revenue equivalence theorem.
- The importance of the revenue equivalence theorem is not in the fact that we prove that a broad class of auctions generates the same expected revenue, but in the conditions under which the revenue equivalence can be established. Based upon the revenue equivalence theorem, we can infer the source of the difference in the revenue of two different auctions.

Theorem 10 (Revenue equivalence theorem)**.** Consider IPV. Fix any Nash equilibrium of any auction, which satisfies the two properties.

(1) the bidder with the highest reservation value wins the object with probability 1.

(2) the expected payoff the lowest reservation value is 0.

The expected revenue of the seller from such a Nash equilibrium is $E(v_{(2:n)})$ which is the expected revenue from the second price auction.

The first condition implies that the outcome must be efficient, because the good is delivered to a bidder with the highest reservation value with probability 1. The second condition implies that the auction should not have a positive entry fee.

Discussion

- Strong. The theorem applies to a broad class of IPV models, imposing little restriction on the institutional details of the auction.
- Weak. The conditions of the theorem are the properties of an endogenous variable (equilibrium strategy) rather than the primitives of the model.

5.2 Mechanism design

Revelation principle The proof appears to be an impossible task.

We need to calculate a Nash equilibrium from each IPV model, checking whether the Nash equilibrium satisfies the two properties and then, check the seller's revenue from the Nash equilibrium is equal to the revenue from the second price auction.

The paper by Myerson [1981][1] shows otherwise, opening up a new and important field of mechanism design.

Hurwicz triangle An auction is a social institution to allocate a good to bidders. The bidders communicate with the auction through their strategies. Based upon the strategies, the auction decides the probability of winning the object and the expected payment of each bidder.

Let us consider an abstract general model of auction, beyond the framework of IPV. Let

$$\sigma_i \colon [0,1] \to A_i$$

be a strategy of bidder. A_i is the space of feasible actions, which can be the price (or bid) or any signal. We impose no restriction on what A_i should be. Depending upon the rule of an auction, A_i can be highly complex. The complexity of A_i poses the major challenge in proving the revenue equivalence theorem.

The auction (or the auctioneer) does not observe $(v_1, \cdots, v_n)$ but only observes the realized value of the strategy of each player

$$\sigma(v) = (\sigma_1(v_1), \cdots, \sigma_n(v_n)).$$

Based upon the data, the auction decides

$$(\hat{Q}_i(\sigma(v)), \hat{p}_i(\sigma(v)))$$

where $\hat{Q}_i(\sigma(v))$ is the probability that the good is delivered to bidder i and $\hat{p}_i(\sigma(v))$ is the expected payment by bidder i. Let us call

$$(\hat{Q}, \hat{p}) = \{(\hat{Q}_i(\sigma(v)), \hat{p}_i(\sigma(v)))\}_{i=1}^n$$

the allocation rule.

Since $\hat{Q}_i(\sigma(v))$ is a probability,

$$\sum_{i=1}^n \hat{Q}_i(\sigma(v)) \leq 1$$

[1] Roger B. Myerson [1981] "Optimal Auction Design" Mathematics of Operations Research, Vol 6, No 1, pp 58-73

but does not have to be equal to 1, because the good may not be sold with a positive probability.

Similarly, the auctioneer may impose entry fee, or some sort of fee so that bidder i may have to pay even if he does not receive the good. Thus, it is entirely possible that for some realization of $(v_1, \cdots, v_n)$, $\hat{p}_i(\sigma(v)) > 0$ but $\hat{Q}_i(\sigma(v)) = 0$.

Given allocation rule $(\hat{Q}, \hat{p})$, each bidder chooses $\hat{\sigma}_i$ conditioned on his valuation v_i, maximizing his interim expected utility against the profile $\hat{\sigma}_{-i}$ of other bidders' strategies: $\forall i$, $\forall v_i \in [0, 1]$,

$$E[\hat{Q}_i(\sigma(v))v_i - \hat{p}_i(\hat{\sigma}(v)) \,|\, v_i] \\ \geq E[\hat{Q}_i(\hat{\sigma}_{-i}(v_{-i}), b_i)v_i - \hat{p}_i(\hat{\sigma}_{-i}(v_{-i}), b_i) \,|\, v_i] \quad \forall b_i.$$

Note. The deviation b_i from the equilibrium strategy $\hat{\sigma}_i(v_i)$ can be conditioned on v_i.

As it is stated, the calculation of a Nash equilibrium is practically an impossible task, because the action space A_i can be extremely complicated. The fundamental contribution by Myerson [1981] is to show that we can construct another Bayesian game which has a Nash equilibrium with the same allocation as the Nash equilibrium of the original game, while the action space is significantly simpler.

Revelation game Given allocation rule $(\hat{Q}, \hat{p})$, and Nash equilibrium $\hat{\sigma}$ of an auction, let us consider a revelation game, in which the action space is the reservation values of bidder i:

$$\sigma_i \colon [0, 1] \to [0, 1].$$

In this game, each bidder reveals (not necessarily truthfully) his reservation value, thus the name of the game.

The allocation rule of the revelation game is defined as $Q(v) = \hat{Q}(\hat{\sigma}(v))$ and $p(v) = \hat{p}(\hat{\sigma}(v))$ $\forall v = (v_1, \cdots, v_n) \in [0, 1]^n$.

Theorem 11 (Revelation principle)**.** The constructed revelation game has a Nash equilibrium in which the truthful revelation

$$\sigma_i(v_i) = v_i \quad \forall i, \forall v_i$$

is a Nash equilibrium strategy. The allocation of the truthful Nash equilibrium is the same as the allocation of the Nash equilibrium of the original game $\forall v$.

Proof. Since $\hat{\sigma}$ is a Nash equilibrium of the original game with allocation rule $(\hat{Q}, \hat{p})$,

$$E[\hat{Q}_i(\sigma(v))v_i - \hat{p}_i(\hat{\sigma}(v)) \,|\, v_i] \\ \geq E[\hat{Q}_i(\hat{\sigma}_{-i}(v_{-i}), b_i)v_i - \hat{p}_i(\hat{\sigma}_{-i}(v_{-i}), b_i) \,|\, v_i] \quad \forall b_i.$$

By the definition of new allocation rule (Q, p), we can write the same equilibrium condition as

$$E[Q_i(v)v_i - p_i(v) \,|\, v_i] \geq E[Q_i(v_{-i}, b_i)v_i - p_i(v_{-i}, b_i) \,|\, v_i] \quad \forall b_i.$$

Interpreting b_i as a different reservation value than v_i, we conclude that the inequality is precisely the incentive compatibility constraint of the truth telling Nash equilibrium. By the construction of allocation rule (Q, p),

$$Q(v) = \hat{Q}(\hat{\sigma}(v)) \quad \text{and} \quad p(v) = \hat{p}(\hat{\sigma}(v)) \quad \forall v,$$

thus inducing the same allocation as the Nash equilibrium of the original game. □

Discussion

- This result is known as the revelation principle. For any Nash equilibrium of any auctions, we can construct a revelation game which has a truth telling Nash equilibrium with the same allocation. In that sense, we can consider a revelation game and a truthful Nash equilibrium without loss of generality, where allocation rule (Q, p) satisfies the incentive compatibility constraint:

$$E[Q_i(v)v_i - p_i(v) \,|\, v_i] \geq E[Q_i(v_{-i}, v_i')v_i - p_i(v_{-i}, v_i') \,|\, v_i].$$

- The revelation principle shows that the complexity of the action space of different auctions is irrelevant for the analysis of the Nash equilibrium. The revelation game is not a social institution, but an analytic tool we construct to investigate the social institution of interest. For this reason, we sometimes refers to the revelation game associated with an auction (which is a social institution) the auction mechanism.

- In the revelation game, the profile of reservation values directly determines the allocation, while in the original game, the allocation is determined by the profile of reservations values, indirectly through the auction rule. First, the bidders reports the actions to the auctioneer conditioned on the realized value of reservation value, who then determine the allocation based on the actions, not the reservation values. For this reason, we often call the revelation game the direct mechanism.

- We saw the same incentive compatibility constraint while we examine the adverse selection problem. The uninformed party has to design a contract to elicit the truthful information. In this sense, we transfer the auction into an adverse selection problem, which turns out to be much more manageable than the original problem.

- Allocation rule (Q, p) satisfying the incentive compatibility constraint will be the focus of analysis. Since the design of (Q, p) determines the economic properties of the mechanism, our exercise is often called mechanism design problem.

5.3 Revenue equivalence theorem

Lecture 11. optimalauction Wed, Apr 7

Interim expectation Let us define the interim values of (Q,p).

$$Q_i(v_i) = \int Q_i(v_{-i}, v_i)\, dF_{-i}(v_{-i})$$
$$p_i(v_i) = \int p_i(v_{-i}, v_i)\, dF_{-i}(v_{-i})$$

If bidder with reservation value v_i reports truthfully, his interim expected payoff conditioned on v_i is

$$\Pi_i(v_i) = Q_i(v_i)v_i - p_i(v_i).$$

Increasing probability of winning Since the incentive compatibility constraint must satisfied, $\forall v_i, v_i'$ with $v_i > v_i'$,

$$\Pi_i(v_i) = Q_i(v_i)v_i - p_i(v_i) \geq Q_i(v_i')v_i - p_i(v_i')$$
$$\Pi_i(v_i') = Q_i(v_i')v_i' - p_i(v_i) \geq Q_i(v_i)v_i' - p_i(v_i)$$

By arranging terms, we have

$$Q_i(v_i)[v_i - v_i'] \geq \Pi_i(v_i) - \Pi_i(v_i') \geq Q_i(v_i')[v_i - v_i'].$$

Since $v_i > v_i'$,

$$Q_i(v_i) \geq Q_i(v_i')$$

which is the first important implication of the incentive compatibility constraint.

For the seller, a buyer with a high reservation value is more valuable than the one with a low valuation. Since the reservation value is a private information, the seller needs to provide an incentive for the buyer with a high reservation value to report his reservation value truthfully. To do so, the probability of winning should be increasing with respect to the reported reservation value. In a certain sense, the buyer is extracting informational rent from the seller.

Increasing equilibrium payoff The second important implication of the incentive compatibility constraint is that

$$\Pi_i(v_i) \geq \Pi_i(v_i')$$

or the equilibrium payoff must be increasing with respect to the reservation value.

Since $v_i > v_i'$, we can write the incentive compatibility constraint as

$$Q_i(v_i) \geq \frac{\Pi_i(v_i) - \Pi_i(v_i')}{v_i - v_i'} \geq Q_i(v_i').$$

Since Π_i is increasing, it is almost everywhere differentiable. Whenever it is differentiable,

$$\Pi_i'(v_i) = Q_i(v_i).$$

Interim payoff and probability of winning By the fundamental law of calculus,

$$\Pi_i(v_i) = \Pi_i(0) + \int_0^{v_i} Q_i(x)\,dx.$$

This equation is the third, and probably the most important, implication of the incentive compatibility constraint.

Recall that

$$\Pi_i(v_i) = Q_i(v_i)v_i - p_i(v_i).$$

In principle, the seller has to manipulate two functions of Q_i and p_i to control the incentive of the bidder. If (Q, p) satisfies the incentive compatibility constraint, $\Pi_i(v_i)$ is the function of $\Pi_i(0)$ and Q_i. Instead of controlling two functions, the seller controls one number $\Pi_i(0)$ and one function Q_i subject to the incentive constraint (Q_i is increasing), which is significantly simpler problem.

The expected payment is then derived according to

$$p_i(v_i) = Q_i(v_i)v_i - \int_0^{v_i} Q_i(x)\,dx - \Pi_i(0).$$

Revenue equivalence While the revenue equivalence theorem is a statement about the expected revenue of the seller, we can infer the equilibrium payment of each bidder in a Nash equilibrium which satisfies the two conditions of the revenue equivalence theorem.

Since the bidder with the highest reservation value wins the object with probability 1, $Q_i(v_i)$ is exactly the probability that

$$v_i \geq \max_{j \neq i} v_j$$

and therefore,

$$Q_i(v_i) = F^{n-1}(v_i).$$

Since the bidder with reservation value 0 receives 0,

$$\Pi_i(0) = 0.$$

Hence, in any Nash equilibrium satisfying the two conditions of the revenue equivalence theorem, the interim expected payment of bidder with reservation value v_i is

$$\Pi_i(v_i) = F^{n-1}(v_i)v_i - \int_0^{v_i} F^{n-1}(x)\,dx.$$

For example, in the first price and the second price auction, the expected payment of a bidder is the same.

Discussion

- We should take this statement with a grain of salt. The same conclusion remains silent about the second moment of the actual payment (which is a random variable). Because the bidder is risk neutral, his objective function is the expected payoff. Thus, from the view point of a risk neutral bidder, the expected payment of the two auctions is the same.

- If a bidder is risk averse, then his expected payoff is affected by the mean but also by the variance of the distribution of the payment. The first and the second price auctions induce the equipment payment with the same mean but different variance. (More later)

Back to proof

Proof (Theorem 10). Recall that

$$p_i(v_i) = Q_i(v_i) - \int_0^{v_i} Q_i(x)\,dx - \Pi_i(0).$$

Since the seller does not observe the reservation value of bidder i, the seller's expected revenue from bidder i is

$$\int_0^1 p_i(v_i)\,dF_i(v_i)$$
$$= \int_0^1 Q_i(v_i)v_i f(v_i)\,dv_i - \int_0^1 \left[\int_0^{v_i} Q_i(x)\,dx\right] f(v_i)\,dv_i - \Pi_i(0).$$

By integrating by parts,

$$\begin{aligned}\int_0^1 \left[\int_0^{v_i} Q_i(x)\,dx\right] f(v_i)\,dv_i &= \int_0^{v_i} Q_i(x)\,dxF(v_i)\Big|_0^1 - \int_0^1 Q_i(v_i)F(v_i)\,dv_i \\ &= \int_0^1 Q_i(v_i)\,dx - \int_0^1 Q_i(v_i)F(v_i)\,dv_i \\ &= \int_0^1 Q_i(v_i)(1-F(v_i))\,dx.\end{aligned}$$

After substitution, the expected revenue from bidder i is

$$\int_0^1 Q_i(v_i)v_i f(v_i)\,dv_i - \int_0^1 (1-F(v_i))Q_i(v_i)\,dv_i - \Pi_i(0)$$
$$= \int_0^1 \left[v - \frac{1-F(v)}{f(v)}\right] f(v)Q_i(v)\,dv - \Pi_i(0).$$

If the good is sold, the seller's revenue is

$$\sum_{i=1}^n \left[\int_0^1 \left[v - \frac{1-F(v)}{f(v)}\right] f(v)Q_i(v)\,dv - \Pi_i(0)\right].$$

The good is not sold with probability

$$1 - \sum_{i=1}^n \int_0^1 Q_i(v)f(v)\,dv.$$

If the seller's value of the good is v_s, the seller's revenue is

$$\begin{aligned}\Pi_s &= v_s\left[1-\sum_{i=1}^{n}\int_0^1 Q_i(v)f(v)\,dv\right]\\&\quad+\sum_{i=1}^{n}\left[\int_0^1\left[v-\frac{1-F(v)}{f(v)}\right]f(v)Q_i(v)\,dv-\Pi_i(0)\right]\\&= v_s+\sum_{i=1}^{n}\left[\int_0^1\left[v-\frac{1-F(v)}{f(v)}-v_s\right]f(v)Q_i(v)\,dv\right]-\sum_{i=1}^{n}\Pi_i(0).\end{aligned}$$

By the first condition,

$$Q_i(v)=F^{n-1}(v)\quad\forall i$$

and by the second condition,

$$\Pi_i(0)=0\quad\forall i.$$

As we normalize $v_s=0$,

$$\begin{aligned}\Pi_s &= v_s+n\left[\int_0^1\left[v-\frac{1-F(v)}{f(v)}-v_s\right]f(v)F^{n-1}(v)\,dv\right]\\&=\int_0^1\left[v-\frac{1-F(v)}{f(v)}\right]nf(v)F^{n-1}(v)\,dv\\&=\int_0^1\left[v-\frac{1-F(v)}{f(v)}\right]dF^{n}(v)\\&=\int_0^1\left[v-\frac{1-F(v)}{f(v)}\right]dF_{(1:n)}(v).\end{aligned}$$

It remains to show that

$$\begin{aligned}\Pi_s &= E(v_{(2:n)})\\&=\int_0^1 v\,dF_{(2:n)}(v)\\&=\int_0^1 v\,d(F^{n}(v)+n(1-F(v))F^{n-1}(v))\\&=\int_0^1 v\,d(F_{(1:n)}(v)+n(1-F(v))F^{n-1}(v)).\end{aligned}$$

Note

$$
\begin{aligned}
& \int_0^1 \left[v - \frac{1-F(v)}{f(v)}\right] dF_{(1:n)}(v) \\
= & \int_0^1 v\, dF_{(1:n)}(v) - \int_0^1 \frac{1-F(v)}{f(v)}\, dF_{(1:n)}(v) \\
= & \int_0^1 v\, dF_{(1:n)}(v) - \int_0^1 \frac{1-F(v)}{f(v)}\, dF^n(v) \\
= & \int_0^1 v\, dF_{(1:n)}(v) - \int_0^1 \frac{1-F(v)}{f(v)} nF^{n-1}(v) f(v)\, dv \\
= & \; vF_{(1:n)}(v)\big|_0^1 - \int_0^1 F_{(1:n)}(v)\, dv - \int_0^1 \frac{1-F(v)}{f(v)} nF^{n-1}(v) f(v)\, dv \\
= & \int_0^1 1 - (F_{(1:n)}(v) + n(1-F(v))F^{n-1}(v))\, dv \\
= & \int_0^1 1 - F_{(2:n)}(v)\, dv \\
= & \; v[1 - F_{(2:n)}(v)]\big|_0^1 - \int_0^1 v\, d(1 - F_{(2:n)}(v)) \\
= & \int_0^1 v\, dF_{(2:n)}(v).
\end{aligned}
$$

□

Price dispersion We now know that the first and the second price auctions are very similar. Both are efficient, and the expected revenue of the seller is the same. A risk neutral seller would treat the two auction the same. What if the seller is risk averse? Which auction does a risk averse seller prefer? Because the two auctions generate the same expected revenue, we need to check the variance of the revenue of each auction.

The variance is a measure of how a random variable is spread out. An alternative, but closely related, way to compare the dispersion of a random variable is the mean preserving spread.

Mean preserving spread

Definition 24 (Mean preserving spread)**.** Y is a mean preserving spread of X if

$$E[Y|X = x] = x \quad \forall x.$$

If Y is a mean preserving spread of X, then Y can be represented as

$$Y = X + \varepsilon$$

where $E[\varepsilon|X] = 0$. While X and Y have the same mean, the variance of Y must be larger than the variance of X.

Dispersion comparison Let σ'' and σ' be the symmetric Nash equilibrium strategies of the second and the first price auctions. The expected revenue from the second price auction is $E[\sigma''(v_{(2:n)})]$ while the expected revenue from the first price auction is $E[\sigma'(v_{(1:n)})]$.

Proposition 6. The seller's revenue from the second price auction is the mean preserving spread of the seller's revenue from the first price auction:

$$E[\sigma''(v_{(2:n)})\,|\,\sigma'(v_{(1:n)}) = b] = b.$$

Proof.

$$\begin{aligned}
E[\sigma''(v_{(2:n)})\,|\,\sigma'(v_{(1:n)}) = b] &= E[\sigma''(v_{(2:n)})\,|\,v_{(1:n)} = (\sigma')^{-1}(b)] &(5.1)\\
&= E[v_{(2:n)}\,|\,v_{(1:n)} = (\sigma')^{-1}(b)] &(5.2)\\
&= E[v_{(1:n-1)}\,|\,v_{(1:n-1)} < (\sigma')^{-1}(b)] &(5.3)\\
&= \sigma'((\sigma')^{-1}(b)) &(5.4)\\
&= b.
\end{aligned}$$

(5.1) follows from the fact that σ' is strictly increasing and therefore, invertible. (5.2) is implied by the fact that the truthful bid is the equilibrium strategy in the second price auction.

(5.3) is the trickiest. Consider n samples of the reservation value, ranked from the highest to the lowest. The highest value of the reservation value is fixed to $(\sigma')^{-1}(b)$. The remaining $n-1$ samples remain random, which must be smaller than $(\sigma')^{-1}(b)$.

Because the highest value of the sample is fixed, the second highest value of n sample must be equal to the highest value of the remaining $n-1$ samples: $v_{(2:n)} = v_{(1:n-1)}$ if $v_{(1:n)}$ is fixed.

Similarly, the event that the highest value among n samples is fixed to $(\sigma')^{-1}$ is the same event that the second highest among n samples, or the highest among remaining $n-1$ samples, must be less than $(\sigma')^{-1}(b)$.

To understand (5.4), remember the second formula of the equilibrium strategy of the first price auction

$$\sigma'(v) = E[v_{(1:n-1)}\,|\,v_{(1:n-1)} \leq v].$$

If $v = (\sigma')^{-1}(b)$, we have (5.4). □

5.4 Optimal auction

Because an auction is a monopolistic trading protocol, the monopolist would like to implement a trading protocol which maximizes his expected profit. The exercise to find the most profitable monopolistic trading mechanism is known as the optimal auction design problem.

Similar This exercise is closely related but fundamentally different from the revenue equivalence problem. As in the proof of the revenue equivalence theorem, we confront the task of comparing the seller's revenue among all feasible monopolistic trading protocols. Thanks to the revelation principle, we focus on the set of the allocation rules (Q, p) satisfying the incentive compatibility constraint, and identify the most profitable allocation rule among incentive compatible allocation rules.

Different The objective function of the optimal auction is the expected profit of the monopolist, not the social welfare. Any Nash equilibrium satisfying the two conditions of the revenue equivalence theorem induces an efficient allocation. Because the objective of the monopolist generally differs from the objective of the society, an optimal auction is typically inefficient.

Textbook example In the intermediate microeconomics, we learn the monopolistic market in which a single seller is setting a product to a large number of buyers. A classic example is Cournot's mineral water example.

Monopoly market Consider a monopolistic seller of mineral water which is coming out of a spring. We normalize the marginal cost of production to be 0. A unit mass of a continuum of infinitesimal buyers is in the market, who values one unit of mineral water $v \in [0, 1]$. The valuation of buyers is distributed uniformly over $[0, 1]$.

Because the size of individual buyers is infinitesimal, the aggregate market demand function is

$$D(p) = 1 - p$$

over $\in [0, 1]$.

We can then proceed to compute the profit maximizing quantity of mineral water as 0.5, which gives market clearing price 0.5 and the profit of 0.25. Because the marginal production cost is 0, and the lowest reservation value of consumers is 0, the efficient allocation requires the firm to produce 1 unit, and charge 0 to mineral water. The market outcome is inefficient, because consumers with reservation value less than 0.5 cannot trade, even though there is a positive gain from trading.

What we do not teach undergraduates is to point out this is a particular form of a monopolistic market, and to wars students that it is too hasty to conclude that any monopolistic market leads to inefficient allocation. As we already find out, if the monopolist allocate the good according to the second or the first price auction, the allocation must be efficient.

Post price mechanism If one looks into the trading protocol, the rule of the game is not necessarily most realistic. The monopolist chooses a price, and post the price. We call the trading protocol the post price mechanism.

One may wonder whether the monopolist can implement more elaborate way to discriminate buyers to elicit information about the reservation value. For example, if a monopolist can observe the reservation value of a buyer, then he might be able to extract all gain from trading surplus by setting the price to be equal to the reservation price of a buyer. This practice is known the first order price discrimination, which leads to efficient allocation, while the seller receives the entire gain from trading.

Price discrimination When we teach students first order price discrimination, we tend to remain vague about whether such a practice is feasible, if the reservation value is private information of a buyer, or if the monopolist has only limited control over the secondary market to prevent the initial buyer from reselling the product to someone else.

In the end, we have to ask whether the monopolist can design a trading protocol in which he can generate higher profit than what he could have received from the post price mechanism. Myerson [1981] answered the question formally and elegantly.

Optimal auction We know that any incentive compatible allocation rule (Q,p) generates the expected surplus of the seller

$$\Pi_s = v_s + \left[\sum_{i=1}^{n}\int_0^1 \left[v_i - \frac{1-F(v_i)}{f(v_i)} - v_s\right] Q_i(v_i) f(v_i)\, dv_i - \Pi_i(0)\right].$$

Optimization problem The optimization problem of the seller is

$$\max_{(Q,p)} \Pi_s$$

subject to (Q,p) satisfying the incentive compatibility constraint and interim individual rationality of bidder i:

$$\Pi_i(v_i) \geq 0 \quad \forall v_i \in [0,1].$$

Since

$$p_i(v_i) = Q_i(v_i)v - \int_0^v Q_i(x)\, dx - \Pi_i(0),$$

the seller's problem is

$$\max_{(Q,p)} \Pi_s$$

subject to Q satisfying the incentive compatibility constraint, and $(\Pi_i(0))_{i=1}^n$ satisfying the interim individual rationality.

In order to maximize the profit, the seller should set $\Pi_i(0)$ as small as possible. To satisfy the interim individual rationality of $v_i = 0$,

$$\Pi_i(0) = 0 \quad \forall i.$$

Then, the seller's optimization problem is reduced to

$$\max_{(Q,p)} v_s + \sum_{i=1}^{n}\left[\int_0^1 \left[v_i - \frac{1-F(v_i)}{f(v_i)} - v_s\right]\sum_{i=1}^{n} Q_i(v_i)\right] f(v_i)\, dv$$

subject to $Q = (Q_1, \cdots, Q_n)$ satisfying the incentive compatibility constraint.

Relaxed problem One of the implications of the incentive compatibility constraint on Q_i is that Q_i must be increasing. Let us consider a relaxed problem

$$\max_{(Q,p)} v_s + \sum_{i=1}^{n}\left[\int_0^1 \left[v_i - \frac{1-F(v_i)}{f(v_i)} - v_s\right] Q_i(v_i) f(v_i)\, dv_i\right]$$

subject to Q_i is increasing $\forall i$.

We then show that the solution of the relaxed problem is indeed the solution of the maximization problem.

Challenge A natural candidate of optimal Q_i is

$$Q_i(v) = \begin{cases} 1 & \text{if } v - \frac{1-F(v)}{f(v)} - v_s \geq 0 \\ 0 & \text{if } v - \frac{1-F(v)}{f(v)} - v_s < 0 \end{cases}.$$

The problem is that the constructed Q_i is not increasing unless

$$v - \frac{1-F(v)}{f(v)} - v_s$$

is an increasing function of v.

Note that

$$\frac{1-F(v)}{f(v)}$$

is an inverse function of hazard rate. Thus, if the distribution function has the decreasing hazard rate property, then

$$v - \frac{1-F(v)}{f(v)} - v_s$$

is an increasing function of v, which Myerson [1981] called the regular case, as this condition is satisfied by many well know distributions such as uniform and Gaussian distributions.

Regular case Suppose that

$$v - \frac{1-F(v)}{f(v)} - v_s$$

is an increasing function. Then, we can find a unique $v^* \in [0,1]$ so that

$$v - \frac{1-F(v)}{f(v)} - v_s \geq 0 \quad \Longleftrightarrow \quad v \geq v^*.$$

Define

$$Q_i(v) = \begin{cases} 1 & \text{if } v \geq v^* \\ 0 & \text{if } v < v^* < 0 \end{cases}.$$

Clearly, Q_i maximizes the objective function. The remaining step is to show that Q_i is incentive compatible. To this end, we calculate the interim expected payment of bidder i.

Incentive compatible Recall that $\Pi_i(0) = 0$ and

$$p_i(v_i) = Q_i(v_i)v_i - \int_0^{v_i} Q_i(x)\,dx.$$

A simple calculation shows

$$p_i(v_i) = \begin{cases} v^* & \text{if } v_i \geq v^* \\ 0 & \text{if } v_i < v^* \end{cases}.$$

The interim payoff is

$$\Pi_i(v_i) = \begin{cases} v_i - v^* & \text{if } v_i \geq v^* \\ 0 & \text{if } v_i < v^* \end{cases}.$$

Now, we are ready to show that $Q_i(v_i)$ we have constructed satisfies the incentive compatibility constraint.

Suppose that $v_i < v^*$. By reporting truthfully, the bidder receives 0. If $v_i' \neq v_i$ but $v_i' < v^*$, then reporting v_i' instead of v_i does not change the interim payoff. If $v_i' > v^*$, then the interim payoff is

$$v_i - v^* < 0$$

because $v_i' > v^*$ triggers the good is delivered to the bidder at price of v^*. Thus, if $v_i < v^*$, it is optimal for bidder i to report truthfully.

Suppose that $v_i \geq v^*$. By reporting truthfully, the bidder receives

$$v_i - v^* \geq 0.$$

If $v_i' \neq v_i$ and $v_i' \geq v^*$, reporting v_i' instead of v_i does not change the interim payoff of the bidder. If $v_i' < v^*$, he receives 0 intead of $v_i - v^* \geq 0$. Thus, the truthful reporting is an optimal strategy.

Second price auction with reserve price Let us examine how the optimal auction operates in the context of the Counot's mineral water example. Each bidder comes to the shop, and reports his reservation value. If his reservation value is above the threshold v^*, then the buyer will receive one unit of the mineral water, paying v^*. Otherwise, he will not get any mineral water, and his expected payoff is 0.

The monopolist is essentially running the second price auction with reserve price v^* against the buyer. Since one buyer is served each time whenever the reported price is above v^*, the delivery price is always v^* whenever the good is sold. For this reason, the optimal auction is sometimes called the second price auction with reserve price.

To see how the optimal auction is implement, let us assume that F is the uniform distribution over $[0, 1]$ and $v_s = 0$. To calculate v^*,

$$v - \frac{1 - F(v)}{f(v)} - v_s = v - (1 - v) = 0$$

at $v = v^*$. Thus, $v^* = \frac{1}{2}$, which is exactly the same price as the post price mechanism. That is, the maximum profit the monopolist can generate in any incentive compatible trading protocol is exactly the monopolist profit from the post price mechanism.

Chapter 6

Public Goods

Lecture 12. public Mon, Apr 12

6.1 Public good

6.2 Grove Clarke scheme

Chapter 7

Efficient Mechanism

7.1 Efficient mechanism

Lecture 13.
efficient
Wed, Apr 14

7.2 Basic properties of VCG

Chapter 8

Search and Matching

8.1 Search

Lecture 14. search Mon, Apr 19

Friction Arrow Debreu economy is the foundation of modern microeconomics, where we establish the two fundamental welfare theorems.

Definition 25. By a friction, we mean any institutional or informational restriction imposed upon the optimization problem of an agent in the economy.

Asymmetric information is a friction in this sense, as the agent without private information cannot identify the true characteristics of the property, and there is no market for insurance against the unknown states. We often call asymmetric information an example of informational friction.

Search friction

Market with search friction

Theorem 12. $\forall \varepsilon > 0$, the unique equilibrium price is $p_s = b$ for every seller s.

Proof. Because every seller is identical, the equilibrium price $p = p_s \, \forall s$. We first show that

$$(\underbrace{0, \ldots, 0}_{s})$$

is not an equilibrium. Suppose seller s increases the price by $\varepsilon/2$. If the buyer who is assign to seller s accept the price, his surplus is

$$b - \frac{\varepsilon}{2}$$

and if he rejects, his surplus cannot exceed $b - \varepsilon$ because of the vacancy cost. Thus, it is optimal for a buyer to accept $\varepsilon/2$. By the same reasoning, we conclude that for any $p < b$,

$$\underbrace{(p, \dots, p)}_{s}$$

is an equilibrium. Thus, the only equilibrium price is $p_s = b \, \forall s$. $\square$

This is a sharp contrast to the economy without any friction. Even if the supply exceeds the demand, it takes only an arbitrarily small search cost to shift the equilibrium from the competitive price 0 to the monopolistic price b.

Discussion

Chapter 9

Repeated Games with Perfect Monitoring

Lecture 15.
repeatedgame
Wed, Apr 21

9.1 Introduction

9.2 Questions

9.3 Finitely repeated game

Chapter 10

Infinitely Repeated Game

10.1 Infinitely repeated game

Lecture 16. infinitelyrepeated Mon, Apr 26

We discuss three well known repeated game strategies.

Example (*D* forever)**.** Define

$$\sigma_i(h_t) = D \quad \forall h_t.$$

The pair $\sigma = (\sigma_1, \sigma_2)$ constitutes a subgame perfect equilibrium.

Proof. First, we show that σ is a Nash equilibrium of G^∞. In the equilibrium, each player receives average payoff of 1, since the outcome path

$$f(\sigma) = ((D, D), \dots, (D, D), \dots).$$

If player i deviates following any history along the equilibrium path, he cannot receives more than 1, because D is the strictly dominant strategy of the component game. Therefore, player i has no profitable deviation.

Second, we prove that σ induces a Nash equilibrium in every subgame $G|_{h_t}$. Note that following any history h_t, the continuation play of $G|_{h_t}$ is identical with the outcome of G. Following the same logic, we can show that σ induces a Nash equilibrium in $G|_{h_t}$. Thus, σ is a subgame perfect equilibrium. ◇

We can generalize this result. Fix any normal form game G and a Nash equilibrium $s = (s_1, s_2)$ of G. Define a repeated game strategy for G^∞ as

$$\sigma_i(g_t) = s_i \quad \forall g_t.$$

Such a strategy is called the repetition of one shot Nash equilibrium.

Proposition 7. The repetition of one shot Nash equilibrium is a subgame perfect equilibrium in G^∞.

The repetition of one shot Nash equilibrium is an important benchmark, aginst which the virtue of a long term relationship is compared.

Example (Grim trigger). Define $\sigma_i(h_1) = C$ and for $t \geq 2$,

$$\sigma_i(h_t) = \begin{cases} C & \text{if } h_t = ((C,C),\dots,(C,C)) \\ D & \text{otherwise} \end{cases}$$

as the grim trigger strategy.

The strategy triggers punishment D, if anyone plays D. Otherwise, the strategy dictates to play C. Once D starts, player i will play D forever. The punishment is grim.

The pair of grim trigger strategies induce outcome path

$$(C,C),\dots,(C,C),\dots$$

and the average payoff of each player is 3, which Pareto dominates the Nash equilibrium outcome of the component game (which is 1).

Proposition 8. The pair of grim trigger strategies is a subgame perfect equilibrium.

Definition 26. Let V be the set of feasible payoff vectors. $v \in V$ is individually rational, if $v_i \geq \underline{v}_i \, \forall i$.

The following theorem was discovered by a number of people, thus given the name Folk theorem.

Theorem 13 (Aumann and Shapley; Rubinstein). $\forall v \in V$ which is individually rational, there exists a subgame perfect equilibrium σ such that $v = u(\sigma)$.

Proposition 9. The pair of tit-for-tat strategies constitutes a Nash equilibrium, but not a subgame perfect equilibrium in G^{∞}.

Proof. Along the equilibrium path,

$$(C,C),(C,C),\dots,(C,C),\dots$$

and the average payoff of each player is 3. □

Chapter 11

Repeated Games with Imperfect Monitoring Examples and Illustration

11.1 Introduction

Lecture 17.
imperfectmonitor
Wed, Apr 28

11.2 Repeated game with imperfect monitoring

11.3 Example

Chapter 12

Repeated Moral Hazard Problem

12.1 Repeated principal agent problem

Lecture 18.
imperfectmonitor2
Mon, May 10

12.2 Results

Chapter 13

Nash Bargaining Problem

13.1 Plan

Competitive equilibrium In a competitive market, decisions are made in a decentralized manner. A consumer optimizes subject to the budget constraint and the price. A producer maximizes profit for a given price.

It is not obvious how the market clearing price is determined. We often use a fictitious auctioneer *Walrasian auctioneer* who construct the aggregate demand and supply from individual demand and supply curves. He then finds the intersection of the two curves and announces the market clearing price. Although it is a useful educational tool to explain the market clearing mechanism, the presence of the Walraisn auctioneer goes directly against the very spirit of the decentralized trading of the competitive market: the invisible hand.

Informational efficiency Hayek observed that the competitive market price aggregates dispersed information so that the individual agents in the economy can take and action in a decentralized manner, but can coordinate to achieve and efficient allocation.

The general equilibrium model of Arrow and Debreu formulates the first welfare theorem but remains vague about the price determination process and the information aggregation process.

Decentralized trading model We need to understand the mechanism that aggregates the private information of the individual agents into the market clearing price.

If the trading is done decentralized, it is not clear how there should a single market clearing price. While we analyze the information aggregation process, we can explain the law of single price.

Dynamic decentralized trading model We build a decentralized trading model from a smallest unit of trading: bargaining. To trade something, you need at least two players. Bargining is a trading protocol between two players. We introduce a matching process as we examined in the model of Peter Diamond so that the trading partner may change over time. We examine the prices at which trading occurs, and how the difference among different trading partners vanish, and converge to the competitive equilibrium price.

Plan

- Bargaining
  - Axiomatic model of Nash [1950]
  - Strategic model of Rubinstein [1982]
- Matching and bargaining

13.2 Bargaining

Trading institution between one seller and one buyer to determine the price and the delivery condition of good or service. Also called bilateral monopoly problem.

Finest unit of trading, which forms the foundation of the market. To trade goods, you need at least two people. The bargaining is a trading unit with two people: one seller vs. one buyer.

Old and widely used, and has many institutional variations. Difficult to formulate and anlyze.

13.3 (Nash) Divide a dollar

A game between two players.
$A_i = [0, 1]$

$$u_i(a_1, a_2) = \begin{cases} a_i & \text{if } a_1 + a_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Any division of 1 dollar can be sustained by a Nash equilibrium. We have a theory, which cannot make any useful prediction. This is often considered as a failure of the theory, revealing the difficulty of the problem, as the bargaining outcome is affected by the details of the rule of the game.

Why difficult? The bargaining problem is difficult, not because we need a fancy mathematical tool. The analysis must satisfy two conditions.

(1) The rule of the bargaining must be reasonable, in which the two parties have comparable bargaining power and can influence the outcome, A dictatorial game is a form of bargaining, but is not considered a reasonable model of bargaining.

(2) The model must make a sharp prediction to tell us what would the most likely outcome from the bargaining process. Divide a dollar game is a reasonable bargaining model. We cannot learn anything from Nash equilibrium, because every efficient division is a Nash equilibrium.

13.4 Nash bargaining problem

Approach by Nash [1950] In 1950, we did not have tools to tackle the difficulties of the bargaining problems. It is before John Nash invented (Nash) equilibrium concept. Instead, Nash chose to bypass the difficulties, but search for a way to say something useful regarding the bargaining process.

The major innovation of Nash's approach is to suppress the details of the procedure, treating it as a black box, and he focuses on the properties of the reasonable outcome.

A bargaining problem is regarded as a mapping from the data which consists of the preference of the players and the structure of the surplus to the division of the surplus.

He asked the following question: A reasonable bargaining outcome must satisfy a certain set of properties, what is the mathematical formula to calculate the outcome?

Bargaining solution Let S be the set of surplus and d be the disagreement point. S is the collection of all possible outcomes attainable, if agreement is reached. d is the pair of payoffs associated with disagreement.

Example. In case of the divide dollar,

$$\{(a_1, a_2) : a_i \geq 0, a_1 + a_2 \leq 1\}$$

and $d = (0, 0)$.

Definition 27. A bargaining problem is (S, d), where $S \subset \mathbb{R}^2$ is compact and convex and $\exists s \in S$ such that $s_i > d_i \; \forall i \in \{1, 2\}$.

We admit randomized contact which makes the set of all feasible utilities convex. $S \subset \mathbb{R}^2$ if and only if S is closed and bounded. If S is unbounded, then the bargaining can be meaningless, because each party can get what he wants without negotiation. The closeness of S is a technical condition to ensure the existence of a solution of the optimization problem.

The last condition ensures that the bargaining is not degenerate. If every feasible payoff vector is Pareto dominated by the disagreement outcome, there is no point of negotiation.

Let S be the set of all bargaining problems.

Definition 28. A bargaining solution $f(S, d) = (u_1, u_2)$ is the rule that specifies which outcome is determined: $f : \mathcal{S} \to \mathbb{R}^2$.

Note. A bargaining solution is not conditioned on a particular bargaining problem. Instead, the way how a bargaining outcome is determined should be spelled out before a particular bargaining problem is selected.

Example (Dictatorial). Let

$$\overline{u}_1 = \arg\max_{u_1'}\{u_1' \mid \exists u_2,\, (u_1', u_2) \in S,\, u_2 \geq d_2\}$$

be the best outcome of player 1 in S.

$$f(S, d) = \{(\overline{u}_1, u_2) \mid (\overline{u}_1, u_2) \in S\}.$$

chooses the best possible outcome.

Example (Always disagree). $f(S, d) = d$.

Discussion

- Dictatorial bargaining solution does not sound reasonable, because the lack of symmetry. By a bargaining situation, we refer to a situation where each party has some, if not equal, control over the outcome of the negotiation. Dictatorial solution does not allow any room for negotiation by player 2.
- If a negotiation always breaks downm the outcome is not efficient. Alluding to Coase theorem, such a bargaining rule should be replaced by another rule which generates more efficient outcome.

Axioms Let us spell out the properties which any reasonable bargaining solution must satisfy. John Nash call these properties axioms on the ground that they are evidently reasonable. Let us state four axioms, along with discussions.

A reasonable bargaining solution should be such that its outcome is affected by the units of the utils.

Axiom 4 (Invariance). Consider the two bargaining problems, (S, d) and (S', d') where (S', d') is obtained by applying a positive affine transformation to (S, d): $\forall i$, $\exists \alpha_i \geq 0$ and $\beta_i \in \mathbb{R}$ such that

$$s_i' = \alpha_i s_i + \beta_i$$

f satisfies the invariance axiom if

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i \quad \forall i.$$

A reasonable bargaining solution should not produce an outcome which is Pareto dominated by another feasible outcome.

Axiom 5 (Pareto). f satisfies the Pareto axiom if $\exists (t_1, t_2) \in S$ such that $t_i > s_i$ $\forall i$ implies $f(S, d) \neq (s_1, s_2)$.

Definition 29. A bargaining problem (S, d) is symmetric if $(s_1, s_2) \in S$ implies that $(s_2, s_1) \in S$.

If the bargaining problem is symmetric, then the name of a player should not matter, implying that the two parties have equal bargaining power.

Axiom 6 (Symmetry). f satisfies the symmetry axiom if for any symmetric bargaining problem (S, d), $f_1(S, d) = f_2(S, d)$.

The symmetry axiom does not require that the bargaining outcome must be equal for all players. The axiom applies only to a symmetric bargaining problem. The same axiom imposes no restriction on bargaining problems which are not symmetric.

The first three axioms (INV, PAR and SYM) are the restrictions on f over individual bargaining problems. The next axiom specifies how the solutions from two different bargaining problems should be related.

Axiom 7 (Independence of Irrelevant Alternatives). Consider two bargaining problems, (S, d) and (T, d) with $T \subset S$. f satisfies the axiom of independence of irrelevant alternatives, if $f(S, d) \in T$ implies $f(T, d) = f(S, d)$.

Discussion

- While INV, PAR and SYM appear to be considered evidently reasonable, IIA need motivation, as it imposes a restriction on the relationship between the solutions of two bargaining problems.
- IIA is essentially identical with the weak axiom of choice. If a decision maker choose an object from T shich contains S, but the selected commodity bundle is in S, then the consumer should choose the same bundle when he is constrained to choose from S. The alternatives in $T \setminus S$ are irrelevant. We know that the weak axiom in this sense implies that the consumer behavior can be described as a consequence of utility maximization.
- IIA is reasonable, if we accept the view that a reasonable bargaining solution should be a solution of a social welfare function. Otherwise, it is not. The existence of a social welfare function is not always guaranteed.
- For me, IIA appears to be reasonable, but has room to be improved. Quite a few people followed up Nash [1950] proposing alternative set of axioms. Usually, INV, PAR and SYM are not touched, but IIA is replaced by something else.

Consistency and uniqueness We choose INV, PAR, SYM and IIA because they are considered reasonable. We did not consider whether four axioms are consistent with each other. If they are not consistent, no bargaining solution satisfying four axioms exists.

If a bargaining solution satisfying four axioms exists, we need to ask how many solutions satisfy the axioms. If too many solutions satisfy the axioms, there is a room to impose additional axioms.

Nash [1950] answer these two question rigorously and elegantly.

Nash bargaining solution Nash [1950] proposes a bargaining solution.

Definition 30. f^N is the Nash bargaining solution if

$$f^N(S,d) = \arg \max_{(s_1,s_2)\in S, s_i \geq d_i} (s_1 - d_1)(s_2 - d_2).$$

$s_i - d_i$ represents the gain from reaching agreement over the disagreement payoff. We call $s_i - d_i$ the Nash gain, and $(s_1 - d_1)(s_2 - d_s)$ the Nash product.

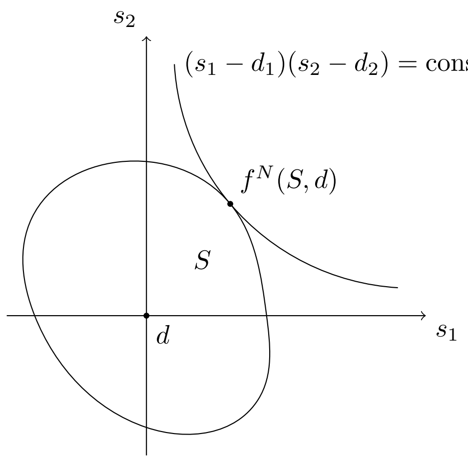


Figure 13.1: Nash bargaining solution (Osborne and Rubinstein [1990])

Note. The Nash product $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_s)$ is strictly quasi concave continuous function. Since S is convex and compact, the maximizer exists and is unique. Thus, the Nash bargaining solution is well defined.

Characterization The fundamental theorem of Nash [1950] is that the Nash bargaining solution is the only bargaining solution which satisfies INV, PAR, SYM and IIA.

Theorem 14. Bargaining solution f satisfies INV, PAR, SYM and IIA if and only if $f = f^N$.

Proof. We prove that the Nash bargaining solution satisfies INV, PAR, SYM and IIA.

INV Suppose that $s_i' = \alpha_i s_i + \beta_i$ and $d_i' = \alpha_i d_i + \beta_i$. Note

$$(s_1' - d_1')(s_2' - d_2') = \alpha_1\alpha_2(s_1 - d_1)(s_2 - d_2).$$

Thus, if (s_1^*, s_2^*) maximizes the right hand side, then $(\alpha_1 s_1^* + \beta_1, \alpha_2 s_2^* + \beta_2)$ maximizes the left hand side.

PAR Since $W(s_1, s_2) = (s_1 - d_1)(s_2 - d_2)$ is a strictly increasing function of (s_1, s_2), an optimal solution must be at the Pareto frontier of S.

SYM Fix a symmetric problem (S,d) where $d_1 = d_2 = d$. Note

$$W(s_1,s_2) = (s_1-d)(s_2-d) = (s_2-d)(s_1-d),$$

If (s_1^*,s_2^*) maximizes the left hand side, then $(s_2^*,s_1^*) \in S$ maximizes the right hand side. It remains to show $s_1^* = s_2^*$.

Suppose that $s_1^* \neq s_2^*$. Since $W(s_1,s_2)$ is strictly quasi concave and S is convex, $\forall \lambda \in (0,1)$,

$$\lambda(s_1^*,s_2^*) + (1-\lambda)(s_2^*,s_1^*) \in S$$

and

$$W(\lambda(s_1^*,s_2^*) + (1-\lambda)(s_2^*,s_1^*)) > W(s_1^*,s_2^*) = W(s_2^*,s_1^*)$$

which contradicts to the hypothesis that (s_1^*,s_2^*) maximizes W over S.

IIA Suppose that (s_1^*,s_2^*) maximizes $W(s_1,s_2)$ over T and

$$(s_1^*,s_2^*) \in S \subset T.$$

Since

$$W(s_1^*,s_2^*) \geq W(s_1,s_2) \quad \forall (s_1,s_2) \in T,$$

and $S \subset T$,

$$W(s_1^*,s_2^*) \geq W(s_1,s_2) \quad \forall (s_1,s_2) \in S.$$

Thus, $(s_1^*,s_2^*) \in S$ must maximize W over S.

The difficult part is to show that Nash bargaining solution is the only solution satisfying four axioms. □

Lemma 7. $\forall (s_1,s_2) \in S$, $s_1 + s_2 \leq 1$.

Proof. Suppose that $\exists (t_1,t_2) \in S$ such that $t_1 + t_2 > 1$. Recall that

$$\left(\frac{1}{2},\frac{1}{2}\right) \in S$$

and S is convex. Thus, $\forall \lambda \in (0,1)$,

$$(1-\lambda)\left(\frac{1}{2},\frac{1}{2}\right) + \lambda t \in S.$$

Note

$$
\begin{aligned}
&\left[(1-\lambda)\frac{1}{2}+\lambda t_1\right]\left[(1-\lambda)\frac{1}{2}+\lambda t_2\right]\\
=&\left[\frac{1}{2}+\lambda\left(t_1-\frac{1}{2}\right)\right]\left[\frac{1}{2}+\lambda\left(t_2-\frac{1}{2}\right)\right]\\
=&\frac{1}{4}+\frac{\lambda}{2}(t_1+t_2-1)+\lambda^2\left(t_1-\frac{1}{2}\right)\left(t_2-\frac{1}{2}\right)>\frac{1}{4}
\end{aligned}
$$

for a sufficiently small $\lambda > 0$, because $t_1 + t_2 - 1 > 0$. But, this contradicts the fact that $f^N(S, 0) = (1/2, 1/2)$. $\square$

Chapter 14

Alternating Offer Bargaining

14.1 Bargaining

14.2 Infinite horizon Rubinstein

14.3 Nash equilibrium

14.4 Subgame perfect equilibrium

Chapter 15

Dynamic Monopoly

15.1 Introduction

Monopoly market Monopolistic market is a trading protocol in which a single seller is facing many buyers. We teach the monopolistic market in the undergraduate class through a market with a linear demand, and show that the profit maximization by the monopolist leads to a market clearing price higher than the marginal production cost. The market clearing price in the monopolist market is above the marginal cost, and the equilibrium quantity is less than the competitive equilibrium quantity. The monopolist market leaves unrealized gains from trading, thus inefficient.

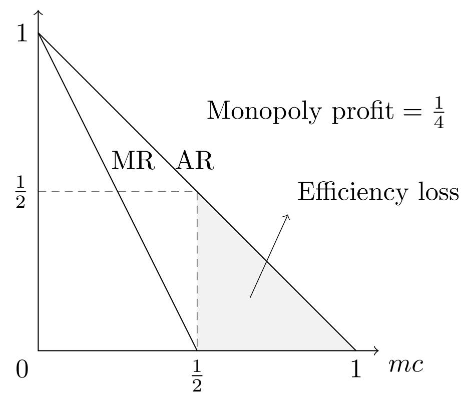


Figure 15.1: $MC = MR$

Monopolistic power The ability to charge a price higher than the marginal cost is the monopolistic power, measured by the difference between the market clearing price and the marginal production cost. A fundamental question is about the source of the monopolistic power.

Discussion The fact that a single seller controls the price of the good is not the source of the monopolistic power, as was first discovered by Ronald H. Coase. Understanding the foundation is important for the policy maker to develop a remedy to regulate the monopolistic market to recover the efficiency. Dynamic monopoly problem is one of a few economic problems, which are important for theory and policy.

15.2 Heuristics

Coase [1973] Ronald H. Coase examines a simple monopoly market in which the monopolist is selling out (commercial) land. The important characteristics of the commercial land is durability.

15.3 Rational expectations

Coasian dynamics

Lemma 8. If $\{p_t\}_{t=1}^{\infty}$ is an optimal pricing rule, then $q_t - q_{t-1} > 0$.

If $q_t = q_{t-1}$, then the monopolist did not make any sales in period t, wasting time. By skipping p_t and offering p_{t+1}, the monopolist can increase profit. Thus, $q_t - q_{t-1} > 0$.

Corollary (Coasian dynamics)**.** In any optimal pricing rule, $p_t > t_{t+1}$.

If $p_t \leq p_{t+1}$, then no consumer will purchase at p_{t+1}, since p_t is lower and offered earlier than p_{t+1}.

15.4 Subgame perfect equilibrium

Extensive form game We take advantage of the continuum of infinitesimal consumers, assuming that the action of a single buyer does not make any difference. A history of the game at the beginning of period t is $h_t = (p_1, \dots, p_{t-1})$.

A strategy of the seller is $\sigma(h)t) = p_t$ and the buyer's strategy is summarized by the optimal decision rule: accept p_t if

$$v - p_t > \delta(v - \sigma(h_t, p_t)).$$

Thanks to the successive skimming property, we can write the critical type v_t

15.5 Examples

Let us consider a market demand function which is not continuously downward sloping. One half of population has reservation value $v = 3$ and the remaining half of the consumers has reservation value $v = 1$.

To calculate the static monopolistic profit maximizing solution, we cannot use the differential calculus to equate the marginal cost to the marginal revenue. We need to rely on basic reasoning.

- The seller will never charge less than 1.
- If above 1, the price must be 3.

- The seller never charges above 3.
- 3 will generate profit of 1.5 while 1 will generate 1.

The monopolist profit maximizing price is 3. the profit is 1.5, serving only the high reservation value consumers. Thus, the allocation is inefficient.

Gap. vs. No gap The lowest reservation value buyer is 1, which is above the marginal cost 0, in this example. If the lowest reservation value of the buyer is above the marginal cost, then we call it gap case.

In the model of Stokey, the lowest reservation value of a buyer is 0, which is equal to the marginal production. It is called no gap case.

We use the lowest reservation value of the buyer as the competitive benchmark.

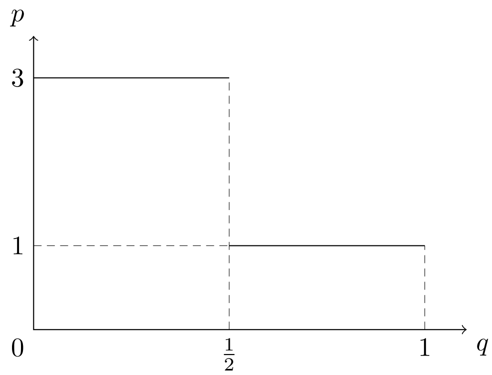


Figure 15.2: Market demand function

Dynamic monopoly problem Suppose that the monopolist opens the market until all buyers are served, and the monopolist has sufficient capacity to serve all buyers in the market. The time span of each period is Δ and discount rate is $r > 0$, and therefore, the discount factor is $\delta = e^{-\Delta r}$.

Preliminaries

Lemma 9. $\exists T < \infty$ such that all consumers are seved in a subgame perfect equilibrium.

Proof. Suppose that a positive mass, say $q > 0$, of consumers is never served. If so, the monopolist can offer ε so that

$$1 - \varepsilon > \delta(1 - 0).$$

Consumers with reservation value 1 will find it optimal to accept $\varepsilon > 0$ right away, because even if the monopolist offers 0 tomorrow, it is better to accept $\varepsilon > 0$ right away. Thus, the monopolist can recover $\varepsilon q > 0$ profit which wolud have been wasted. $\square$

This is the most important consequence of the gap case. Because the lowest reservation value of the buyer is strictly higher than the marginal production cost, the seller's profit is bounded away from 0.

Terminal round

Lemma 10. $p_T = 1$.

If T is the last round when every consumer is served, then $p_T \leq 1$. If $p_T < 1$, then the buyer will accept, because T is the last round. Thus, the equilibrium offer must be $p_T = 1$.

Penultimate round In T round, if any consumer is still active, the consumer must have reservation value 1. If some consumer has reservation value 3, then in period $T-1$, no reservation value 3 consumer has purchased, and therefore, the monopolist has wasted one round. We know in any equilibrium, the monopolist has to sell a positive amount to consumers. Thus, no consumer with reservation value 3 should be left in period T.

We conclude tat in period T, only the reservation value 1 consumers remain to be served at delivery price $p_T = 1$.

In the penultimate round $T-1$, the price will be higher than 1 so that only reservation value 3 consumers will be served. Thus, p_{T-1} must satisfy

$$3 - p_{T-1} \geq \delta(3 - p_T) = \delta(3-1) = 2\delta.$$

Thus, the highest price which reservation value 3 consumer is willing to accept is

$$p_{T-1} = 3 - 2\delta.$$

Since there are two types of consumers, $T = 2$.

Weak stationary equilibrium

Definition 31 (Weak stationary equilibrium)**.** A subgame perfect equilibrium is a stationary equilibrium, if $\forall h_t, \sigma(h_t)$ depends only upon the residual demand by the end of h_t. A subgame perfect equilibrium is a weakly stationary equilibrium, if $\forall h_t, \sigma(h_t)$ depends only upon the residual demand and the previous round's offer along h_t.

The economy is populated by a unit mass of consumers, each of whom has reservation value $v \in [\underline{v}, \overline{v}]$ and demand one unit of the good. Let

$$F(v) = P(v' \leq v)$$

and therefore,

$$q = 1 - F(p)$$

is the market demand if the seller makes a take-it-or-leave-it offer. As we change p, we can derive the demand, which we call the market demand.

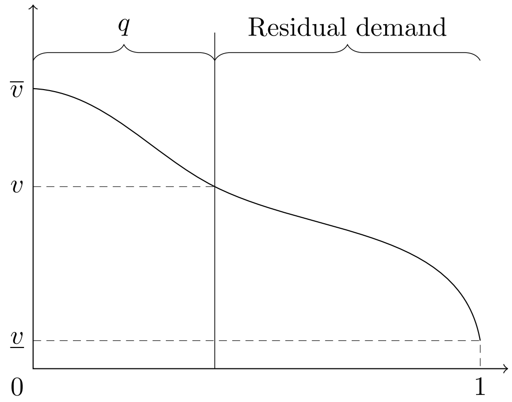


Figure 15.3: Residual demand curve

while we can admit more general demand curve, let us assume that $F(p)$ is strictly increasing continuous function. Thus, the market demand function is strictly decreasing. Thanks to the successive skimming property, a residual demand is parametrized by the mass of consumers who have been served. By residual demand q, we mean the portion of the aggregate demand without consumers with $v \geq 1 - q$.

Chapter 16

Bargaining under Uncertainty

16.1 Delay in Bargaining

16.2 Analysis

16.3 Delay and uncertainty

Chapter 17

Uncertainty and Delay

17.1 Uncertainty and delay

17.2 Stationarity

17.3 Gains from trading

17.4 Common value

Chapter 18

Search and Matching

18.1 Introduction

18.2 Overview

18.3 Modeling features

Chapter 19

Synthetic Market

19.1 Introduction

19.2 Model

19.3 Analysis

19.4 Variations