Lecture 8. General Principles

1. Utility Functions

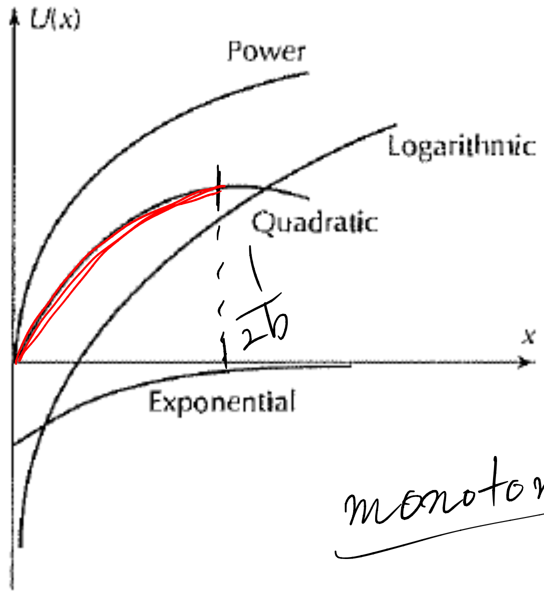
money

Definition 1. A *utility function* is a function $u(\cdot)$ defined on the nonnegative real numbers and giving a real value:

$$u : \mathscr{R}^{+} \cup \{0\} \rightarrow \mathscr{R}.$$

Example 2. An *exponential utility function* is given by $u(x) = -e^{-ax}$ for $a > 0$. A *logarithm utility function* is given by $u(x) = \ln(x)$ for $x > 0$. A *power utility function* is given by $u(x) = bx^b$ for $0 < b \leq 1$. A *quadratic utility function* is given by $u(x) = x - bx^2$ for $b > 0$ and for $x < 1/(2b)$.

$-b(x-\frac{1}{2b})^2+\frac{1}{4b}$ 정의역



monotonous

"order"

Remark 3. With the defined utility function, all *random* outcomes are ranked by evaluating their *expected utility function*. One prefers x to y *if and only if* $E[u(x)]$ is greater than $E[u(y)]$:

"선호"

$$x \succeq y \;\Leftrightarrow\; E[u(x)] \geq E[u(y)].$$

Example 4. Suppose that you have two investment opportunities, A and B. The option A is to buy T-bills, which will give you \$6 million for sure in time 1. The option B is to buy stocks, which will give you \$10 million with a probability of 0.2, \$5 million with a probability of 0.4, and \$1 million with a probability of 0.4, respectively. You decide to rank the investment opportunities using the power utility function $u(x) = x^{1/2}$. The expected utilities are

$$E[u(A)] = \sqrt{6} = 2.45$$

$E[u(A)] = \sqrt{6}$.

$E[u(B)] = 0.2 \cdot \sqrt{10} + 0.4 \cdot \sqrt{5} + 0.4 \cdot 1$

and

$$
\begin{aligned}
E[u(B)] &= \sum_{i=1}^{n} Pr(X = x_i)u(x_i) \\
&= 0.2 \times u(10) + 0.4 \times u(5) + 0.4 \times u(1) \\
&= 0.2 \times \sqrt{10} + 0.4 \times \sqrt{5} + 0.4 \times \sqrt{1} \\
&= 1.93.
\end{aligned}
$$

Since $E[u(A)] > E[u(B)]$, the investment option A is preferred to the investment option B.

Theorem 5. *Given a utility function $u(x)$, any function of the form*

$$v(x) = au(x) + b$$

*with $a > 0$ is a utility function **equivalent** to $u(x)$. That is, the equivalent utility function $v(x)$ gives exactly the same ranking as the original function $u(x)$.*

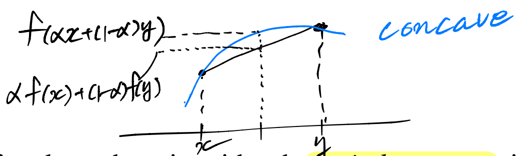
u(x), v(x) 값 자체는 의미X.
rank 결과만 일치하면 됨.
order 만 유지.

Proof. Suppose $x \succeq y$. So, $E[u(x)] \geq E[u(y)]$. It shows

$$
\begin{aligned}
E[v(x)] - E[v(y)] &= (aE[u(x)] + b) - (aE[u(y)] + b) \\
&= a(E[u(x)] - E[u(y)]) \geq 0.
\end{aligned}
$$

Likewise, one can easily show that if $E[v(x)] \geq E[v(y)]$, then $x \succeq y$. □

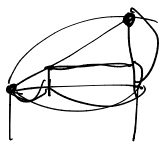
- A utility function provides a ranking among alternatives, but its actual numerical value has no real meaning. This means that any utility function can be modified freely, so long as the rankings that it provides do not change.



2. Risk Aversion

Definition 6. A function f defined on an interval $[a,b]$ of real numbers is said to be *strictly concave* if for any α with $0 < \alpha < 1$ and any x and y in $[a,b]$ there holds

$$f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y).$$



- The condition for strict concavity is that the straight line drawn between any two points on the function must lie below the function itself.

Definition 7. A utility function u is said to be *risk averse on* $[a,b]$ if it is strictly concave on $[a,b]$. If u is strictly concave everywhere, it is said to be *risk averse*.

- $u(x)$ is increasing if $u' > 0$. $u(x)$ is strictly concave if $u'' < 0$. For instance, the exponential utility function $u(x) = -e^{-ax}$ is increasing and strictly concave, so that it can be used to represent risk aversion.

Suppose that you have two investment opportunities, A and B. The risky option A is that you obtain either x or y, each with probability of 0.5. In the riskless option B, you obtain $0.5x + 0.5y$ with certainty.

$A \prec B$

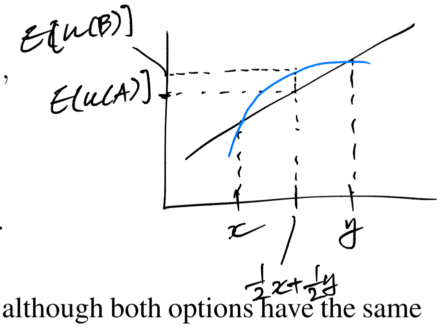
$\Leftrightarrow E[u(A)] < E[u(B)]$

- Both options have the same expected payoff; i.e., $x^* = 0.5x + 0.5y$. The expected utility of the option A is given by

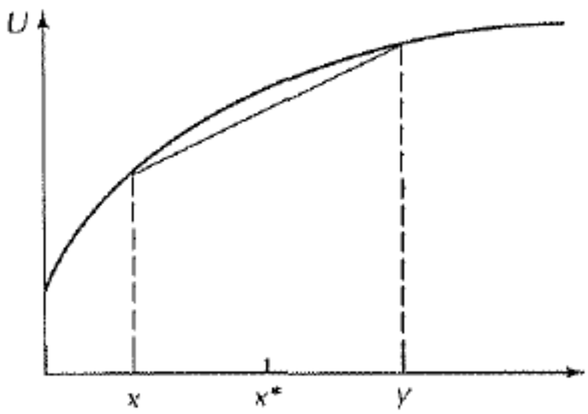
$$E[u(A)] = \frac{1}{2}u(x) + \frac{1}{2}u(y),$$

while the expected utility of the option B is given by

$$E[u(B)] = u\left(\frac{1}{2}x + \frac{1}{2}y\right).$$



When u is strictly concave, $E[u(B)] > E[u(A)]$, so that $B \succ A$, although both options have the same expected payoff x^*.



∴ concave ⇒ risk averse 를 표현함.

Remark 8. The degree of risk aversion is increasing with the magnitude of of the bend in the utility function; the stronger the bend, the greater the risk aversion. This relationship is formally defined by the *Arrow-Pratt absolute risk aversion coefficient*, denoted by $a(x)$, which is given by

$$a(x) = -\frac{u''(x)}{u'(x)}.$$

$x \uparrow \rightarrow$ 덜 위험 회피.

Example 9. Suppose that $u(x) = \ln x$. Then $a(x) = 1/x$, meaning that as x increases, risk aversion decreases. Treating x as wealth, it is equivalent to say that you are willing to take more risk when you are financially secure.

Definition 10. The *certainty equivalent* (*CE*) of a random payoff x is defined by a *certain* amount that has a utility level equal to the expected utility of x. In other words, the CE of x is the value c satisfying

$$u(c) = E[u(x)].$$

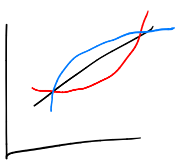
Example 11. Consider an investment option that gives you \$10 or \$0, each with probability of 0.5. Your utility function is given by $u(x) = x - 0.4x^2$. The CE of the option is computed as follows:

$$c - 0.4c^2 = \frac{1}{2}(10 - 0.4(10^2)) + \frac{1}{2}(0) = \cancel{3},$$

−15

E[X] = 5 이지만 C=3.49.

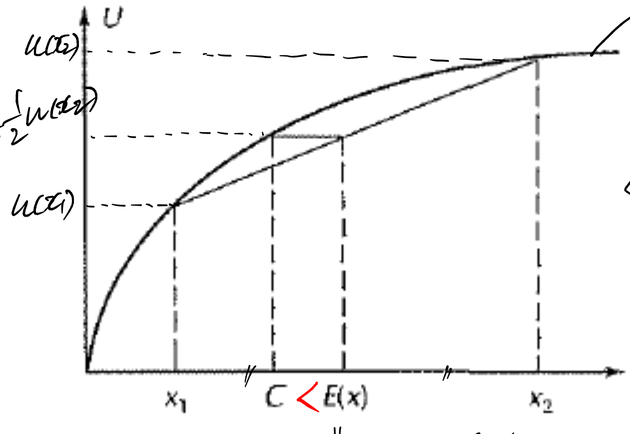
so that $c = \$3.49$. Thus you are indifferent between getting \$3.49 for sure and having a 50-50 chance of getting \$10 or \$0.



risk-averse
risk-lover
risk

risk-averse 때문.
risk-asset 의 E(x) = 5.
riskless-asset 의 C : 3.49.

Remark 12. For a strictly concave utility function it always holds $c < E[x]$.



$E[u(x)] = \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2)$

→ u : risk-averse

x_1 : w/p 0.5
x_2 : w/p 0.5

$E(x) = \frac{1}{2}x_1 + \frac{1}{2}x_2$

$E[u(x)] = u[E(x)]$ #

3. Kuhn-Tucker Necessary Conditions

Suppose that we have a function $f = f(x_1, \ldots, x_n)$, which we wish to maximize, together with inequality constraints, $g_j(x_1, \ldots, x_n) \le c_j$ for $j = 1, \ldots, m$, which must be satisfied: i.e.,

$$\max f(x_1, \ldots, x_n)$$
$$\text{subject to: } g_i(x_1, \ldots, x_n) \le c_i \text{ for } j = 1, \ldots, m$$
$$x_i \ge 0 \text{ for } i = 1, \ldots, n.$$

Like the previous optimization problem with equality constraints, we form the Lagrangian as

$$\mathscr{L} = f(x_1, \ldots, x_n) + \lambda_1(c_1 - g_1(x_1, \ldots, x_n)) + \cdots + \lambda_m(c_m - g_m(x_1, \ldots, x_n)).$$

Remark 13. This is a maximum only problem. To do a minimization, you need to maximize the function $-f(x_1, \ldots, x_n)$.

Theorem 14. *The Kuhn-Tucker conditions, which are **necessary** (but not sufficient) for a point to be a maximum are as follows:*

$\frac{\partial L}{\partial x} \le 0$ $\lambda \ge 0$
$x\frac{\partial L}{\partial x} = 0$ $g_j(1, \ldots, n) \le c_j$
$x \ge 0$
$\lambda_j(c_j - g_j(x_1, \ldots, x_n)) = 0$

$$\frac{\partial \mathscr{L}}{\partial x_i} \le 0$$
$$x_i \ge 0$$
$$x_i \frac{\partial \mathscr{L}}{\partial x_i} = 0$$
$$g_j(x_1, \ldots, x_n) \le c_j$$
$$\lambda_j \ge 0$$
$$\lambda_j(c_j - g_j(x_1, \ldots, x_n)) = 0$$

for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$.

- Each point that is a solution to this equation system is a possible candidate for the maximum. Once we have established all the points, we need to check them individually to see which is the real maximum.
- These conditions are called the *complementary slackness* conditions. This is because for each set of three conditions, either the first or the second condition can be slack (i.e. not equal to zero), but the third condition ensures that they cannot both be non-zero.

Example 15. Maximize $f(x_1,x_2) = 4x_1 + 3x_2$ subject to $g(x_1,x_2) = 2x_1 + x_2 \leq 10$ and $x_1,x_2 \geq 0$. One first forms the Lagrangian:

$$\mathscr{L} = 4x_1 + 3x_2 + \lambda(10 - 2x_1 - x_2).$$

The necessary conditions are:

$$\begin{aligned}
\mathscr{L}_1 &= 4 - 2\lambda \leq 0 && (1)\\
x_1 &\geq 0 \\
x_1(4-2\lambda) &= 0 && (2)\\
\mathscr{L}_2 &= 3 - \lambda \leq 0 && (3)\\
x_2 &\geq 0 \\
x_2(3-\lambda) &= 0 && (4)\\
2x_1 + x_2 - 10 &\leq 0 \\
\lambda &\geq 0 \\
\lambda(2x_1 + x_2 - 10) &= 0. && (5)
\end{aligned}$$

One solves this set of inequalities and equations to find points which may be maxima. (2) implies $x_1 = 0$ or $\lambda = 2$. Suppose $\lambda = 2$. Then (3) does not hold. Hence, one must have $x_1 = 0$. (4) implies $x_2 = 0$ or $\lambda = 3$. If $x_2 = 0$ (together with $x_1 = 0$), then (5) implies $\lambda = 0$, which contradicts (1). So, one must have $x_1 = 0$ and $\lambda = 3$. Hence, (5) implies $x_2 = 10$. Finally, one finds the only solution, which is $x_1 = 0$, $x_2 = 10$, and $\lambda = 3$, and calculates the maximum as $4 \times 0 + 3 \times 10 = 30$.

Remark 16. What you need to do is just plug away at every possible combination and eliminate those that do not fit the conditions. When you have eliminated all impossible points, you will be left with a few candidate points which you must then check by substitution into $f(x)$ to find whether they are the maximum.

4. Portfolio Choice

4.1. Law of One Price

Suppose that asset i generating random payoff x_i in the future is priced at $P_i = p(x_i)$ today. The *law of one price* (*LOOP*) implies that if $x_1 = x_2$, then $P_1 = P_2$; i.e., assets with the same payoffs should be equally priced.

Theorem 17. *(Linear pricing) The law of one price implies that the price of $\alpha x_1 + \beta x_2$ must be $\alpha P_1 + \beta P_2$:*

$$p(\alpha x_1 + \beta x_2) = \alpha p(x_1) + \beta p(x_2).$$

Proof. Consider two assets, 1 and 2, paying payoffs x_1 and x_2, respectively. Since the price of x_1 is P_1, the price of αx_1 should equal to αP_1. Likewise, $p(\beta x_2) = \beta P_2$. According to the LOOP, an asset paying $\alpha x_1 + \beta x_2$ should equal to $\alpha P_1 + \beta P_2$, since its payoff is as same as the sum of two payoffs, αx_1 and βx_2. □

Remark 18. Suppose that there are n assets and each asset has payoff x_i for $i = 1, \ldots, n$. Let θ be an $n \times 1$ vector of $\theta_1, \ldots, \theta_n$, where θ_i represents the amount of asset i in the portfolio. Then the payoff of the portfolio is

$$x = \sum_{i=1}^{n} \theta_i x_i,$$

and the LOOP implies that the price of the portfolio is

$$p(x) = p\left(\sum_{i=1}^{n} \theta_i x_i\right) = \sum_{i=1}^{n} \theta_i p(x_i).$$

4.2. Optimal Portfolio Choice Problem

Suppose that an investor has a strictly increasing utility function u and an initial wealth W. Given n assets, the investor wants to form a portfolio to maximize the expected utility of the portfolio payoff $x = \sum_{i=1}^{n} \theta_i x_i$. random payoff

weight

- The investor's problem is given by

$$\max_{\theta_1,\ldots,\theta_n} E\left[u\left(\sum_{i=1}^{n} \theta_i x_i\right)\right]$$

weight 보다는 수량을 나타냄

$$\text{subject to } \sum_{i=1}^{n} \theta_i P_i \le W$$

예산제약식

$$x_i \ge 0 \text{ for } i = 1, \ldots, n.$$

Theorem 19. *If $x^* = \sum_{i=1}^{n} \theta_i^* x_i$ is a solution to the optimal portfolio problem, then*

$$E\left[u'(x^*) x_i\right] = \lambda P_i$$

for $i = 1, \ldots, n$, where $\lambda > 0$

Proof. Write the Lagrangian function as

$$\mathscr{L} = E\left[u\left(\sum_{i=1}^{n} \theta_i x_i\right)\right] + \lambda\left(W - \sum_{i=1}^{n} \theta_i P_i\right).$$

The Kuhn-Tucker necessary conditions are

$$
\begin{aligned}
\mathscr{L}_i &= E\left[u'(x)x_i\right] - \lambda P_i \leq 0 \\
\theta_i \quad x_i &\geq 0 \\
\mathscr{L}_i x_i &= 0 \\
\mathscr{L}_\lambda &= W - \sum_{i=1}^{n} \theta_i P_i \leq 0 \\
\lambda &\geq 0 \\
\mathscr{L}_\lambda \lambda &= 0
\end{aligned}
$$

for $i = 1, \ldots, n$. Since the utility function is strictly increasing, the budget constraint should be binding at the solution; i.e., $\mathscr{L}_\lambda = 0$. Thus the condition $\mathscr{L}_\lambda \lambda = 0$ implies $\lambda > 0$. The condition $\mathscr{L}_i x_i = 0$ requires either $\mathscr{L}_i = 0$ or $x_i = 0$. It makes no sense to have $x_i = 0$, so that $\mathscr{L}_i = 0$ or

$$E\left[u'(x)x_i\right] = \lambda P_i.$$

In sum, the Kuhn-Tucker condition becomes

$$
\left(
\begin{aligned}
\mathscr{L}_i &= E\left[u'(x)x_i\right] - \lambda P_i = 0 \\
\mathscr{L}_\lambda &= W - \sum_{i=1}^{n} \theta_i P_i = 0
\end{aligned}
\right).
$$

for $i = 1, \ldots, n$. Since there are $n+1$ equations for $n+1$ unknowns $\theta_1, \ldots, \theta_n$ and λ, the optimal solution θ^* can be obtained. □

4.3. Example

An investor is considering the possibility of investing in a venture to produce a firm. There are three possible outcomes: (a) with probability 0.3 her investment will be multiplied by a factor of 3, (b) with probability 0.4 the factor will be 1, and (c) with probability 0.3 she will lose the entire investment. One of these outcomes will occur a year later. She also has the opportunity to earn 20% risk free over this period. Assume that her utility function is $u(x) = \ln x$.

- Let θ_1 and θ_2 be the amounts of money invested in the venture and the risk-free opportunity, respectively. Let W be her initial wealth. The expected utility is

$$E\left[u\left(\theta_1 x_1 + \theta_2 x_2\right)\right] = 0.3\ln(3\theta_1 + 1.2\theta_2) + 0.4\ln(\theta_1 + 1.2\theta_2) + 0.3\ln(1.2\theta_2),$$

$$E[u(x)] = u(3\theta_1 + 1.2\theta_2) \times 0.3 + u(\theta_1 + 1.2\theta_2) \times 0.4 + u(1.2\theta_2) \times 0.3$$

so the maximization problem is

$$\max_{\theta_1,\theta_2} 0.3\ln(3\theta_1+1.2\theta_2)+0.4\ln(\theta_1+1.2\theta_2)+0.3\ln(1.2\theta_2)$$
$$\text{subject to } \theta_1+\theta_2 \leq W$$
$$\theta_1 \geq 0$$
$$\theta_2 \geq 0.$$

- The Lagrange function is

$$\mathscr{L} = 0.3\ln(3\theta_1+1.2\theta_2)+0.4\ln(\theta_1+1.2\theta_2)+0.3\ln(1.2\theta_2)+\lambda(W-\theta_1-\theta_2)$$

and the Kuhn-Tucker conditions are

$$\begin{aligned}
\mathscr{L}_1 &= \frac{0.9}{3\theta_1+1.2\theta_2}+\frac{0.4}{\theta_1+1.2\theta_2}-\lambda = 0 \\
\mathscr{L}_2 &= \frac{0.36}{3\theta_1+1.2\theta_2}+\frac{0.48}{\theta_1+1.2\theta_2}+\frac{0.36}{1.2\theta_2}-\lambda = 0 \\
\mathscr{L}_\lambda &= W-\theta_1-\theta_2 = 0.
\end{aligned}$$

Solving for θ_1, θ_2, and λ, one obtains $\theta_1^* = 0.089W$, $\theta_2^* = 0.911W$, and $\lambda^* = 1/W$. Thus she commits 8.9% of her wealth to the venture and the rest to the risk-free opportunity.