Homework #4

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Problem 1.
Work out (in LaTeX) the equations for Absolute/Relative Risk Premia for CARA/CRRA respectively

Solution.
(1) Since risk premia are closely related to one's utility of money, calculate the risk premia, we can perform taylor-expansion on utility $U(x)$ like follows:

$$U(x) \approx U(\bar{x}) + U'(\bar{x}) \cdot (x - \bar{x}) + \frac{1}{2} U''(\bar{x}) \cdot (x - \bar{x})^2 \tag{1}$$
$$U(x_{CE}) \approx U(\bar{x}) + U'(\bar{x}) \cdot (x_{CE} - \bar{x}) \tag{2}$$

Since we have $\mathbb{E}[U(x)] = U(x_{CE})$, the left hand side is the expectation of $U(x)$:

$$\mathbb{E}[U(x)] \approx U(\bar{x}) + \frac{1}{2} \cdot U''(\bar{x}) \cdot \sigma_x^2 = U(x_{CE}) \approx U(\bar{x}) + U'(\bar{x}) \cdot (x_{CE} - \bar{x}) \tag{3}$$
$$U'(\bar{x}) \cdot (x_{CE} - \bar{x}) \approx \frac{1}{2} \cdot U''(\bar{x}) \cdot \sigma_x^2 \tag{4}$$

Moving $U'(\bar{x})$ to the right-hand side (RHS) and U'' to the left-hand side (LHS) of the equation, we have:

$$x_{CE} - \bar{x} \approx \frac{1}{2} \cdot \frac{U''(\bar{x})}{U'(\bar{x})} \cdot \sigma_x^2 \tag{5}$$
$$\pi_A = \bar{x} - x_{CE} \approx -\frac{1}{2} \cdot A(\bar{x}) \cdot \sigma_x^2, \tag{6}$$

where $A(\bar{x}) = \frac{U''(\bar{x})}{U'(\bar{x})}$.

For Constant Absolute Risk-Aversion (CARA), consider the utility function $U(x) = \frac{-e^{-ax}}{a}$:

$$U'(\bar{x}) = e^{-ax} \tag{7}$$
$$U''(\bar{x}) = -a \cdot e^{-ax} \tag{8}$$
$$\frac{U''(\bar{x})}{U'(\bar{x})} = -a \tag{9}$$

Plug it into the π_A equation before, we have:

$$\pi_A = \mu - x_{CE} = -\frac{1}{2} \cdot A(\bar{x}) \cdot \sigma_x^2 = \frac{a\sigma^2}{2} \tag{10}$$

where μ is the expected value of the uncertain payment and x_{CE} is the value of certainty equivalent.

Certainty equivalent refers to the amount that a person is willing to pay; in a risk-avert scenario, this value is typically lower than μ.

(2) For Constant Relative Risk-Aversion (CRRA), consider $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$:

$$R(x) = \frac{-U'' \cdot x}{U(x)} = \gamma \tag{11}$$

When $\gamma \neq 1$:

$$\mathbb{E}[U(x)] = \frac{e^{\mu(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2}}{1-\gamma} \tag{12}$$

$$= \frac{e^{(1-\gamma)\cdot(\mu+\frac{\sigma^2}{2}(1-\gamma))}}{1-\gamma} \tag{13}$$

Therefore:

$$x_{CE} = e^{\mu+\frac{\sigma^2}{2}(1-\gamma)} \tag{14}$$

$$\pi_R = 1 - \frac{x_{CE}}{\bar{x}} = 1 - \frac{x_{CE}}{\frac{e^{\mu+\frac{\sigma^2}{2}}}{1-e^{-\frac{\sigma^2\gamma}{2}}}} \tag{15}$$

$$= 1 - e^{-\frac{\sigma^2\gamma}{2}} \tag{16}$$

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