Question 1

1)

a) Take any 2 points from line defined by $x = u - 1, y = u + 2, z = 2u - 1$. For simplicity, we take the points when $u = 0$ and $u = 1$. Let points $P_1 = (-1,2,-1), P_2 = (0,3,1)$, and $P_3 = (1,2,3)$. Thus, we can define the plane parametrically by:

$$P = P_1 + u(P_2 - P_1) + v(P_3 - P_1)$$
$$= (-1, 2, -1) + u(1, 1, 2) + v(2, 0, 4)$$
$$= (-1 + u + 2v, 2 + u, -1 + 2u + 4v)$$

where $u, v \in (-\infty, \infty)$. In other words,

$$x(u, v) = -1 + u + 2v$$
$$y(u, v) = 2 + u$$
$$z(u, v) = -1 + 2u + 4v$$

where $u, v \in (-\infty, \infty)$.

b) Use the formula :

$$x(u,v) = x_1 + u(x_2 - x_1) + v(x_3 - x_1 + u(x_4 - x_3 - x_2 + x_1))$$
$$y(u,v) = y_1 + u(y_2 - y_1) + v(y_3 - y_1 + u(y_4 - y_3 - y_2 + y_1))$$
$$z(u, v) = z_1 + u(z_2 - z_1) + v(z_3 - z_1 + u(z_4 - z_3 - z_2 + z_1))$$

Then, we get :

$$x(u,v) = -1.0 + u(1.5 - (-1.0)) + v(-1.2 - (-1.0) + u[1.0 - (-1.2) - 1.5 + (-1.0)])$$
$$= -1 + u(2.5) + v(-0.2 + u(-0.3))$$
$$= -1 + 2.5u - 0.2v - 0.3uv$$

$$y(u, v) = 1.0 + u(0.0 - 1.0) + v(0.5 - 1.0 + u(1.5 - 0.5 - 0.0 + 1.0))$$
$$= 1 + u(-1) + v(-0.5 + u(2))$$
$$= 1 - u - 0.5v + 2uv$$

$$z(u, v) = -1.0 + u(-1.0 - (-1.0)) + v(1.0 - (-1.0) + u[1.0 - 1.0 - (-1.0) + (-1.0)])$$
$$= -1 + 0 + v(2 + 0)$$
$$= -1 + 2v$$

where $u, v \in [0,1]$. Substitute $u = 0.3$ and $v = 0.3$:

$$x(0.3,0.3) = -0.337$$
$$y(0.3,0.3) = 0.73$$
$$z(0.3,0.3) = -0.4$$

Thus, the cartesian coordinates of point with parametric coordinate (0.3,0.3) is point (−0.337, 0.73, −0.4).

c) First, calculate the parametric equation of the sinusoidal curve in XY-plane:

$$x(u) = 1 + 0.2\, sin(3\pi u)$$
$$y(u) = -3 + 3u$$
$$z(u) = 0$$

where $u \in [0,1]$. Then, rotate the sinusoidal curve by 360° anti clockwise about Y-axis, the parametric representation for rotational sweeping is:

$$x(u, v) = (1 + 0.2\, sin(3\pi u))\, sin(2\pi v)$$

$y(u, v) = -3 + 3u$
$z(u, v) = (1 + 0.2\ sin(3\pi u))\ cos(2\pi v)$

where $u,v \in [0,1]$. Next, translate the sinusoidal curve by 3 units in positive Y-direction, the parametric representation of translational sweeping is:

$$x(v) = 0$$
$$y(v) = 3v$$
$$z(v) = 0$$

where $v \in [0,1]$. Thus, the parametric equations after doing both rotational sweeping and translational sweeping are :

$$x(u, v) = (1 + 0.2\ sin(3\pi u))\ sin(2\pi v)$$
$$y(u, v) = -3 + 3u + 3v$$
$$z(u, v) = (1 + 0.2\ sin(3\pi u))\ cos(2\pi v)$$

where $u,v \in [0,1]$.

Question 2

2)

a) To convert from Cartesian coordinates to polar coordinates, assume that

$$x = r cos(\alpha)$$
$$y = r sin(\alpha)$$

Then, the implicit equation becomes :

$$\begin{aligned} 0 &= 1 - (r\cos(\alpha) - 1)^2 - (r\sin(\alpha))^2 \\ &= 1 - r^2\cos^2(\alpha) + 2r\cos(\alpha) - 1 - r^2 - \sin^2(\alpha) \\ &= r^2(\cos^2(\alpha) + \sin^2(\alpha)) + 2r\cos(\alpha) \end{aligned}$$

Since $sin^2(\alpha) + cos^2(\alpha) = 1$:

$$-r^2 + 2r\ cos(\alpha) = 0$$

We get $r(\alpha) = 0$ or $r(\alpha) = 2\ cos(\alpha)$, where $\alpha \in [0,2\pi]$. Notice that $r(\alpha) = 0$ is impossible since $1-(x-1)^2-y^2 = 0$ is implicit equation of circle with radius 1 centered at (1,0), while $r(\alpha) = 0$ is polar coordinates of point (0,0). Thus, $r(\alpha) = 2\,cos(\alpha)$, $\alpha \in [0,2\pi]$.

b) The first side of triangle is line from (0,2) to (2,−1), the equation of this line is

$$3x + 2y = 4$$

To determine whether the shaded area contain in the region $3x + 2y \leq 4$ or $3x + 2y \geq 4$, pick any point in the shaded area, for example point (0,0). Substitute $x = 0, y = 0$ to get $3x + 2y = 3 \times 0 + 2 \times 0 = 0$. Since $0 \leq 4$, then

$$3x+2y \leq 4 \quad \Rightarrow \quad 4-3x-2y \geq 0$$

In the same way, we will get the functions of two other sides of triangle:

$$y + 1 \geq 0$$
$$3x - 2y + 4 \geq 0$$

Finally, the equation of the circle of radius 0.5 with center of (0, −0.5) is

$$x^2 + (y + 0.5)^2 = 0.5^2$$

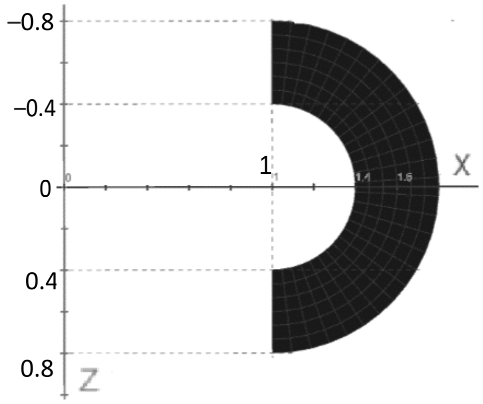
Since the shaded region are outside the circle, we have

$$x^2 + (y + 0.5)^2 \geq 0.5^2 \Rightarrow x^2 + (y + 0.5)^2 - 0.5^2 \geq 0$$

Then, we just need to use intersection of these implicit equations:

$$f(x, y) = \min(4 - 3x - 2y, \min(y + 1, \min(3x - 2y + 4, x^2 + (y + 0.5)^2 - 0.5^2))) \geq 0$$

c) First, we define parametrically the surface as shown in the figure.



The parametric equation of the surface is :

$x = 1 + R\ sin(\pi v)$

$y = 0$

$z = R\ cos(\pi v)$

where $R \in [0.4, 0.8]$, $v \in [0,1]$

We want R to be in interval [0,1], to do this notice that:

$$\frac{R - 0.4}{0.4} = u \in [0,1]$$

So,

$$R = 0.4u + 0.4$$

Thus, the parametric equation shown in the figure can be written as:

$$x = 1 + (0.4u + 0.4)sin(\pi v)$$
$$y = 0$$
$$z = (0.4u + 0.4)cos(\pi v)$$

where $u,v \in [0,1]$. Then, do rotational sweeping of 270° clockwise about Z-axis to get the solid object in the problem. Thus, we get:

$$x = [1 + (0.4u + 0.4)sin(\pi v)]\cos(-1.5\pi w)$$
$$y = [1 + (0.4u + 0.4)sin(\pi v)]\sin(-1.5\pi w)$$
$$z = (0.4u + 0.4)cos(\pi v)$$

where $u,v,w \in [0,1]$.

Question 3

3)

a) In 2D transformation, scaling, rotation, reflection, and shear transformations can be expressed as a 2x2 matrix multiplication, which allows to pre-multiply all the matrices together. However, translation cannot be expressed as a 2x2 matrix multiplication, which makes composition difficult. Same problem occurs in 3D transformation.

b) In homogeneous coordinates, R and T can be represented as 3x3 matrix :

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},\ T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Then we get:

$$RT = \begin{bmatrix} \cos\theta & -\sin\theta & t_x\cos\theta - t_y\sin\theta \\ \sin\theta & \cos\theta & t_x\sin\theta + t_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix},\ TR = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

We need to check whether RT = TR, thus:

$$t_x \cos\theta - t_y \sin\theta = t_x \Rightarrow -t_y \sin\theta = t_x(1 - \cos\theta) \text{ ... (1)}$$
$$t_x \sin\theta + t_y \cos\theta = t_y \Rightarrow t_x \sin\theta = t_y(1 - \cos\theta) \text{ ... (2)}$$

From this we consider 2 cases

Case 1 : when $\cos\theta = 1$, this means that $\theta = k2\pi$, where k is integer. So,

$$\sin\theta = \sin(k2\pi) = 0$$

Then substitute $\cos\theta = 1$ and $\sin\theta = 0$ to equation (1) and (2). From this, notice that $\cos\theta = 1$ and $\sin\theta = 0$ satisfies both equations. So, RT and TR will define same transformation when $\theta = k2\pi$, where k is integer.

Case 2 : when $\cos\theta \neq 1$, then from equation (1) and (2):

$$t_x = \frac{-t_y \sin\theta}{(1-\cos\theta)} \text{ ... } (*),\ t_y = \frac{t_x \sin\theta}{(1-\cos\theta)} \text{ ... } (**)$$

By substituting value of t_y in (**) to equation (*), we get :

$$t_x = \frac{-t_x \sin^2\theta}{(1-\cos\theta)^2} \Rightarrow (1 - \cos\theta)^2 t_x = -t_x \sin^2\theta$$
$$\Rightarrow t_x - 2t_x\cos\theta + t_x\cos^2\theta = -t_x\sin^2\theta$$

Since $\sin^2\theta + \cos^2\theta = 1$:

$$2t_x - 2t_x\cos\theta = 0$$
$$\Rightarrow 2t_x(1 - \cos\theta) = 0$$

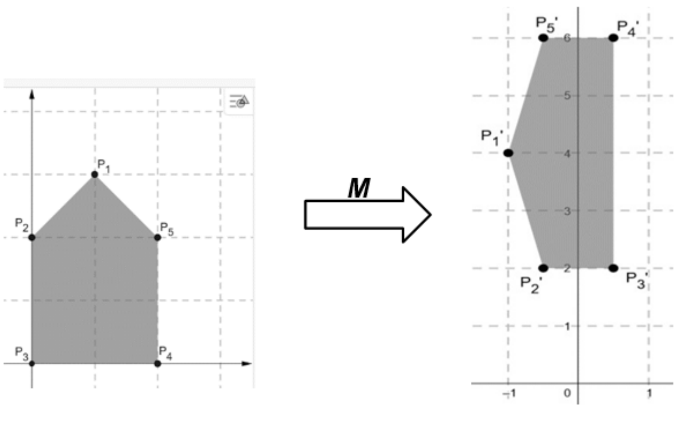
From assumption that $\cos\theta \neq 1$, we get $t_x = 0$. Substituting $t_x = 0$ to equation (2), we get $t_y = 0$. Notice that $t_x = 0$ and $t_y = 0$ satisfies equation (1) and (2). Thus, from this case RT and TR will define same transformation when $t_x = 0$ and $t_y = 0$.

In general, RT and TR will define same transformations when $\theta = k2\pi$ (k is integer) or when $t_x = 0$ and $t_y = 0$. Otherwise, $RT \neq TR$.

c)

i) By using homogeneous coordinates, let $P_1 = \begin{bmatrix}1\\3\\1\end{bmatrix}$, $P_2 = \begin{bmatrix}0\\2\\1\end{bmatrix}$, $P_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $P_4 = \begin{bmatrix}2\\0\\1\end{bmatrix}$, $P_5 = \begin{bmatrix}2\\2\\1\end{bmatrix}$.

After applying M = $\begin{bmatrix}0 & -0.5 & 0.5\\2 & 0 & 2\\0 & 0 & 1\end{bmatrix}$, we get $P_1' = \begin{bmatrix}-1\\4\\1\end{bmatrix}$, $P_2' = \begin{bmatrix}-0.5\\2\\1\end{bmatrix}$, $P_3' = \begin{bmatrix}0.5\\2\\1\end{bmatrix}$, $P_4' = \begin{bmatrix}0.5\\6\\1\end{bmatrix}$, $P_5' = \begin{bmatrix}-0.5\\6\\1\end{bmatrix}$.



As we can see in the figure above, the scaling factor are $s_x = 2$, and $s_y = 0.5$. Thus,the scaling matrix is $S = \begin{bmatrix}2 & 0 & 0\\0 & 0.5 & 0\\0 & 0 & 1\end{bmatrix}$. From the figure, we can see the rotation about 90° anti clockwise. Thus the rotation matrix is $R = \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Next, we also have translation by t_x = 0.5, and t_y = 2. Thus, the translation matrix is

$T = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. The order of these transformations is scaling first, then rotation, and finally translation. We can check that M = TRS.

d) First, do the transformation to the plane such that the normal of the plane are aligned to Z-axis (in other words, the plane coincides with XY-plane). To do this, translate the plane such that it passes through origin. We can do this by using translation matrix :

$$T = \begin{bmatrix} 1 & 0 & 0-1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then do the rotation by 45° clockwise through X-axis and continued with rotation by θ° (where $\cos\theta$ = $\sqrt{6}/3$ and $\sin\theta = \sqrt{3}/3$) clockwise through Y-axis. This can be written by matrices multiplication:

Next, do the reflection about XY-plane by using matrix:

$$Ref_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After perform rotation and translation back, this can be done with:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, combining the matrix multiplication we get:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 4

4)

a)

(i) Texture mapping: A mapping technique where texture or pattern is added to a surface. It will result in changes of color pattern of the surface.

(ii) Bump mapping: A mapping technique where a surface is roughened and it appears to have bumps.

(iii) Displacement mapping: A mapping technique that displaces the geometry of the surface.

b) The normal vector of the plane is [2 2 2], then the unit vector N = [$\sqrt{3}/3$ $\sqrt{3}/3$ $\sqrt{3}/3$]. From the problem statement, we got L= [0 0 1]. Thus, we have

$$cos\theta = N \bullet L = =\sqrt{3}/3$$

$$R = 2(N \bullet L)N - L = 2\sqrt{3}/3 \text{ (N – L)} = [2/3 \quad 2/3 \quad -1/3]$$

Since the peak of specular highlight occurs, then $R = V$. Let (x, y, z) be the position on the plane which cause the peak of specular highlight. We get:

$$[0 \quad 0 \quad 0] - [x \quad y \quad z] = kV = k[2/3 \quad 2/3 \quad -1/3]$$

Where k is a constant. Thus, [x y z] = [-2k/3 -2k/3 k/3]. Since (x, y, z) is point on the plane, we have:

$$2x + 2y + 2z = 1 \quad => \quad -\frac{4}{3}k - \frac{4}{3}k + \frac{2}{3}k = 1$$

So $k = -\frac{1}{2}$, then the point (x,y,z) = $\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{6}\right)$.

c) The parametric equation of a solid unit sphere centered at the origin is:

$$x(r, u, v) = r\ cos(2\pi u)$$
$$y(r,u,v)=r sin(2\pi u) cos(\pi v)$$
$$z(r,u,v) = r\ sin(2\pi u)\ sin(\pi v)$$

where $r, u, v \in [0,1]$. The parametric equation of a solid cylinder in the problem is:

$$x(r,u,v)=2r\cdot cos(2\pi u)+1$$
$$y(r,u,v)=2r\cdot sin(2\pi u)+1$$
$$z(r, u, v) = 6v$$

where $r,u,v \in [0,1]$. Since the morphing sequence has 200 frames and involves deceleration, define

$$\tau = sin\left(\frac{\pi}{2}\ \frac{k-1}{199}\right), \text{ where } k = 1,2, \ldots, 200$$

and thus, the mathematical model for the morphing is:

$$x(r, u, v) = (1 - \tau)(r \cdot cos(2\pi u)) + \tau(2r \cdot cos(2\pi u) + 1)$$
$$y(r, u, v) = (1 - \tau)(r \cdot sin(2\pi u) \cdot cos(\pi v)) + \tau(2r \cdot sin(2\pi u) + 1)$$
$$z(r, u, v) = (1 - \tau)(r \cdot sin(2\pi u) \cdot sin(\pi v)) + \tau(6v)$$

where

$$\tau = sin\left(\frac{\pi}{2}\ \frac{k-1}{199}\right), \text{ where } k = 1,2, \ldots, 200.$$

=== End of Answers ===

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