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<table>
<tr><td>1</td><td>(a)</td><td>The radius of the sphere is 2; from the diagram, we can infer that:
– the projection of each point on the sphere surface on the xz plane is represented as $2\cos\alpha$.
  o the projection of each point on the sphere surface on the xz plane is projected to the x-axis, making $x = 2\cos\alpha\sin\beta$.
  o when this same projection is again projected to the z-axis, $z = 2\cos\alpha\cos\beta$.
– the projection of each point on the sphere surface on the y-axis is $2\sin\alpha$. Hence, by that alone, $y = 2\sin\alpha$.

So,
$$x = 2\cos\alpha\sin\beta, \qquad y = 2\sin\alpha, \qquad z = 2\cos\alpha\cos\beta$$</td></tr>
<tr><td></td><td>(b)</td><td>Notice that you can represent combine those three parametric equations into one that tells all possible Cartesian coordinates on the line:
$$(x, y, z) = (1, 2, -2) + u(1, 1, 1), \qquad u \in [-\infty, \infty]$$
Immediately, notice that since this line lies on a plane, then the direction vector $(1, 1, 1)$ lies on the plane.

We are also given that the point $(1, 3, 3)$ lies on the plane. Based on the equation we formed above, we also know that the point $(1, 2, -2)$ also lies on the plane. So, we can deduce, since these two points lie on the plane, then the direction vector that connects them ($(1, 3, 3) - (1, 2, -2) = (0, 1, 5)$) also lies on the plane.

Hence, we know the vectors $(1, 1, 1)$ and $(0, 1, 5)$ lie on the plane. So just find their common perpendicular:
$$(1 \times 5 - 1 \times 1, 1 \times 0 - 1 \times 5, 1 \times 1 - 1 \times 0) = (4, -5, 1)$$
Hence the equation of the plane becomes:
$$4x - 5y + z = d$$
To find d, simply substitute any point on the plane to the equation above:
$$d = 4(1) - 5(3) + 3 = -8$$
So, the implicit equation (remember, implicit equation means it needs to be expressed in the form $f(x, y, z) = 0$) of the plane is:
$$4x - 5y + z + 8 = 0$$</td></tr>
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1 (c) (i)

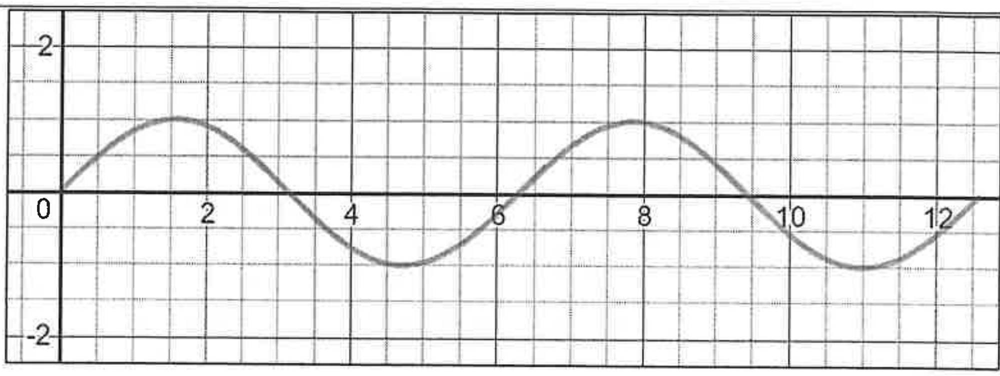


Figure above shows $y = \sin(x)$, $x \in [0, 4\pi]$.

We need to transform it somehow to the diagram in Figure Q1b.

Obviously, $x = 0.3 + 0.7u$.

For y, look at the diagram and notice that, the amplitude of the sine curve is 0.25 and it is displaced 0.25 units upward. Immediately, notice that:

$$y = 0.25 + 0.25\sin(u), \quad u \in [0, 4\pi].$$

Now change u's domain to $u \in [0, 1]$:

$$y = 0.25 + 0.25\sin(4\pi u), \quad u \in [0, 1]$$

So,

$$x = 0.3 + 0.7u$$
$$y = 0.25 + 0.25\sin(4\pi u)$$
$$u \in [0, 1]$$

(ii)

From the two equations I wrote, apply a rotational sweeping of $\frac{3\pi}{4}$ radians. Since the rotation is about the y -axis, so $x = r\cos(0.75\pi v)$ and $z = -r\sin(0.75\pi v)$. Note that z will start out negative in this rotation, so we need to adjust the sign for z.

$$x = (0.3 + 0.7u)\cos(0.75\pi v)$$
$$y = 0.25 + 0.25\sin(4\pi u)$$
$$z = -(0.3 + 0.7u)\sin(0.75\pi v)$$
$$u, v \in [0, 1]$$

2 (a)

We are given this polar equation:

$$r = 2 + 4\cos(\alpha), \quad \alpha \in [0, 2\pi]$$

Converting this into Cartesian coordinates is easy since we know that $x = r\cos\alpha$ and $y = r\sin\alpha$.

		So, $$x = (2 + 4\cos(\alpha))\cos(\alpha)$$ $$y = (2 + 4\cos(\alpha))\sin(\alpha)$$ $$\alpha \in [0, 2\pi]$$ Now let's convert α into the parameter u where $u \in [0,1]$. $$x = (2 + 4\cos(2\pi u))\cos(2\pi u)$$ $$y = (2 + 4\cos(2\pi u))\sin(2\pi u)$$ $$u \in [0,1]$$
(b)	(i)	**Cube:** We know the cube is centered at the origin and has dimensions $2 \times 2 \times 2$. All the points in the cube must satisfy these conditions: $$x \geq -1, \quad x \leq 1, \quad y \geq -1, \quad y \leq 1, \quad z \geq -1, \quad z \leq 1$$ $$\downarrow$$ $$x+1 \geq 0, \quad 1-x \geq 0, \quad y+1 \geq 0, \quad 1-y \geq 0, \quad z+1 \geq 0, \quad 1-z \geq 0$$ Here are the equations: $$f(x,y,z) = \min(x+1, 1-x, y+1, 1-y, z+1, 1-z) \geq 0$$ **Sphere:** We know the sphere has a radius of 1 and is centered at the point $(0,1,0)$. All points in the sphere hence satisfy this equation: $$f(x,y,z) = 1 - x^2 - (y-1)^2 - z^2 \geq 0$$ This is also the function $f(x,y,z) \geq 0$ that defines the sphere. (hint: $$f(x,y,z) = r^2 - (x-x_0)^2 - (y-y_0)^2 - (z-z_0)^2 \geq 0$$ is the equation of a sphere with radius r centered with Cartesian coordinates (x_0, y_0, z_0)). **Cylinder:** We know that the cylinder is parallel to the z-axis and since its axis passes through the point $(0,-1,0)$, we know that if the z-axis were removed, the cylinder would turn into a circle centered at Cartesian coordinates $(0,-1)$. We also know that the radius of the cylinder is 0.5. With that in mind, the next page shows the equation that represents the cylinder: $$f(x,y,z) = 0.25 - x^2 - (y+1)^2 \geq 0$$

(ii) Combine the sphere and the cube and remove the cylinder. Hence the equation becomes:

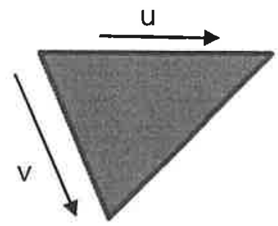
$$f(x,y,z) = \min(\max(\min(x+1, 1-x, y+1, 1-y, z+1, 1-z), 1-x^2 - (y-1)^2 - z^2), -0.25 + x^2 + (y+1)^2) \geq 0$$

(iii) Bounding box:

$$x \geq -1, \quad x \leq 1, \quad y \geq -1, \quad y \leq 2, \quad z \geq -1, \quad z \leq 1$$

Centre of this bounding box is the coordinate $(0, 0.5, 0)$. The size of this bounding box (which I assume is the volume) is $2 \times 3 \times 2 = 12$ volume units.

(c) Let's first break this to parts and define, by using the bilinear surface approach, the triangle with coordinates $(-0.25, 0.8)$, $(0.5, 0.8)$, and $(0.0, 0.2)$.



Derive these equations first:

$$P' = P_1 + u(P_2 - P_1)$$
$$P'' = P_3 + u(P_4 - P_3)$$
$$P = P' + v(P'' - P')$$
$$u, v \in [0,1]$$

$$P_1(-0.25, 0.8), P_2(0.5, 0.8), P_3(0.0, 0.2), P_4(0.0, 0.8)$$

Substitute these 4 points to form:

$$P' = \begin{pmatrix} -0.25 \\ 0.8 \end{pmatrix} + u \begin{pmatrix} 0.75 \\ 0 \end{pmatrix}$$
$$P'' = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}$$

$$P = \begin{pmatrix} -0.25 \\ 0.8 \end{pmatrix} + u \begin{pmatrix} 0.75 \\ 0 \end{pmatrix} + v\left(\begin{pmatrix} 0 \\ 0.2 \end{pmatrix} - \begin{pmatrix} -0.25 \\ 0.8 \end{pmatrix} - u \begin{pmatrix} 0.75 \\ 0 \end{pmatrix}\right)$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.25 + 0.75u + 0.25v - 0.75uv \\ 0.8 - 0.6v \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.25(1-3u)(1-v) \\ 0.8 - 0.6v \end{pmatrix}, \qquad u, v \in [0,1]$$

We've found parametric equations for x and y for the triangle. There is one more step, which is to apply rotational sweeping parallel to the z-axis from

$z = -0.5$ to $z = 0.75$. x and y are unaffected in this sweeping. Hence the equation becomes:

$$x = -0.25(1 - 3u)(1 - v)$$
$$y = 0.8 - 0.6v$$
$$z = -0.5 + 1.25w$$
$$u, v, w \in [0, 1]$$

3 (a)

$I(4, 6)$ will map to $s = \frac{4-1}{10} = 0.3$ and $t = \frac{6-1}{10} = 0.5$. Substitute this into the parametric equations and you can find the Cartesian coordinates:

$$x = 10(0.3) = 3$$
$$y = 10(0.3) + 10(0.5) = 8$$
$$z = 10 - 10(0.3) - 10(0.5) + 20(0.3)(0.5) = 5$$

Cartesian coordinates are $(3, 8, 5)$.

(b)

Affine transformation matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \\ 1 \end{bmatrix} = \begin{bmatrix} x_{out} \\ y_{out} \\ 1 \end{bmatrix}$$

Form the equations from A_1 to B_1, then for A_2 to B_2, then for A_3 to B_3.

$$-3a + b + c = 2$$
$$-3d + e + f = -1$$

$$-3a + 2b + c = 3$$
$$-3d + 2e + f = 1$$
$$-2a + 2b + c = 4$$
$$-2d + 2e + f = 2$$

Subtract the first equation from the third equation and you obtain $b = 1$.
Subtract the third equation from the fifth equation and you obtain $a = 1$.
Substitute $a = b = 1$ to the first equation to obtain $c = 4$.

Subtract the second equation from the fourth equation and you obtain $e = 2$.
Subtract the fourth equation from the sixth equation and you obtain $d = 1$.
Substitute $d = 1$ and $e = 2$ to the second equation to obtain $f = 0$.

Hence the affine transformation is:

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) (i)

This is how the VRML code should be interpreted:

Apply a translation of 1 unit in the direction of the y-axis, followed by a scaling factor of 1, 2, and 1 about the x-, y-, and z-axes, respectively, followed by a counterclockwise rotation of $\frac{\pi}{2}$ radians with respect to the line connecting the origin and the coordinates $(5, 3, 4)$.

Translation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation:

Rotate anticlockwise by $\tan^{-1}\frac{5}{3}$ about the z-axis.

$$\begin{bmatrix} \frac{3}{\sqrt{34}} & -\frac{5}{\sqrt{34}} & 0 & 0 \\ \frac{5}{\sqrt{34}} & \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then rotate anticlockwise by $\tan^{-1}\frac{\sqrt{34}}{4}$ about the x-axis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4}{\sqrt{50}} & -\sqrt{\frac{34}{50}} & 0 \\ 0 & \sqrt{\frac{34}{50}} & \frac{4}{\sqrt{50}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then rotate anticlockwise by $\frac{\pi}{2}$ radians about the z-axis.

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then rotate clockwise by $\tan^{-1}\frac{\sqrt{34}}{4}$ about the x-axis before rotating clockwise by $\tan^{-1}\frac{5}{3}$ about the z-axis. Then the total matrix becomes:

$$\begin{bmatrix} \frac{3}{\sqrt{34}} & \frac{5}{\sqrt{34}} & 0 & 0 \\ -\frac{5}{\sqrt{34}} & \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4}{\sqrt{50}} & \sqrt{\frac{34}{50}} & 0 \\ 0 & -\sqrt{\frac{34}{50}} & \frac{4}{\sqrt{50}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4}{\sqrt{50}} & -\sqrt{\frac{34}{50}} & 0 \\ 0 & \sqrt{\frac{34}{50}} & \frac{4}{\sqrt{50}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{34}} & -\frac{5}{\sqrt{34}} & 0 & 0 \\ \frac{5}{\sqrt{34}} & \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Well, it can't get any worse than this, right?

(ii) Darn.

It's like they expect you to be a calculator with this question.

Let's process this slowly. $(0,0,0)$ becomes $(0,1,0)$ after translation and becomes $(0,2,0)$ after scaling.

Then after crunching numbers like crazy on your calculator, you get ugly numbers.

Coordinates are (rounded to two decimal places):

$$(-0.53, 0.36, 1.89)$$

Fingers crossed this is correct.

4 (a) Similarities:

- both appear to have bumps
- both appear to be 3-dimensional
- appears to have rough surface

Differences:

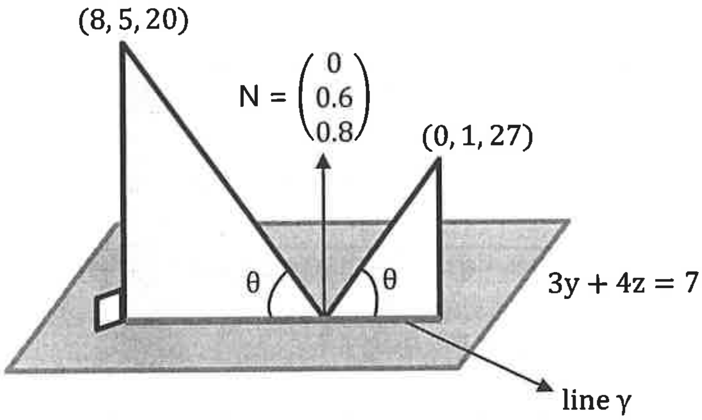
- bump mapping only perturbs the normal, displacement mapping displaces the surface (the bumps are real and 3-dimensional)
- self-occlusion in displacement mapping
- smooth silhouette in bump mapping, vs the rough one for displacement
- bump mapping does not change the geometry of the surface, while displacement mapping does

	(b)		First off, parametrize the two curves so that they have the same parameter u. Curve A: $$x = -4\cos(u), \quad y = 2\sin(u), \quad u \in [0, \pi]$$ Curve B: $$x = 2u, \quad y = \cos(2u), \quad u \in [0, \pi]$$ Now apply parameter τ to describe the morphing from Curve A to Curve B: $$x = 2u\tau - 4(1-\tau)\cos(u)$$ $$y = \tau\cos(2u) + (1-\tau)\sin(u)$$ $$u \in [0, \pi], \quad \tau \in [0, 1]$$ Where if the animation were to involve acceleration and 200 frames, $$\tau = 1 - \cos\left(\frac{\pi}{2} * \frac{k-1}{199}\right)$$ where k represents the frame number, $1 \le k \le 199$ and $k \in \mathbb{Z}$.
(c)	(i)		Here is what we know: $\text{intensity} = 0.8$ Point light intensity at $(0, 1, 27)$ Plane equation: $3y + 4z = 7$ $k_d = 0.5$ Observer at $(8, 5, 20)$ To compute intensity of diffuse reflection, we need to find the angle that the light ray makes with the normal perpendicular to the plane. $\mathrm{L} = (0,1,27) - (0,1,1) = (0,0,26)$ Normalizing L gives $\mathrm{L} = (0,0,1)$. $\mathrm{N} = (0,3,4)$, is perpendicular to the plane. Normalizing N gives $\mathrm{N} = (0, 0.6, 0.8)$ Diffuse reflection intensity = $k_d \times \text{intensity} \times (\mathrm{N} \cdot \mathrm{L}) = 0.5 \times 0.8 \times 0.8$ $= 0.32$.
	(ii)		Finding the ideal reflected vector: $$\mathrm{R} = 2(\mathrm{N}\cdot\mathrm{L})\mathrm{N} - \mathrm{L} = 1.6(0, 0.6, 0.8) - (0,0,1) = (0, 0.96, 0.28)$$ Finding viewing vector: $$\mathrm{V} = (8,5,20) - (0,1,1) = (8,4,19)$$ Normalize V: $$\mathrm{V} = \left(\frac{8}{21}, \frac{4}{21}, \frac{19}{21}\right)$$

(iii) These 4 marks aren't worth your time. But if you really do want an answer to this question, here you go.

We are looking for a point (x, y, z) in the plane where $V = R$.

Think of it like the last past year paper, but more messed up, because the line in the plane where this one point lies in isn't even parallel to any of the axes.



We know that this point lies on line γ, which lies on the plane. To find the equation of this line, we need to first find the two projected coordinates from the light source and the observer when the two points are each 'projected' perpendicularly to 'crash' the plane.

Here is the projection line perpendicular to the plane from the observer: (the reason why the direction vector in the line is $(0, -3, -4)^T$ instead of $(0, 3, 4)^T$ is because I'm trying to find a positive u which the line 'crashes' the plane. It still works if the direction vector is $(0, 3, 4)^T$, but eh.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 20 \end{pmatrix} + u \begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix}$$

Now find the coordinates where this line 'crashes' into the plane, by substituting the equation of the line above into the plane:

$$3(5 - 3u) + 4(20 - 4u) = 7$$
$$u = 3.52$$

Coordinates are $(8, -5.56, 5, 92)$

Now do the same thing with the light source:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 27 \end{pmatrix} + u \begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix}$$
$$3(1 - 3u) + 4(27 - 4u) = 7$$
$$u = 4.16$$

Coordinates are $(0, -11.48, 10.36)$

Now refer to the diagram. We're talking about similar triangles here. Find the closest distance between the observer and the plane and between the light source and the plane:

Between the observer and the plane:

$$\sqrt{(8-8)^2 + (5+5.56)^2 + (20-5.92)^2} = 17.6$$

Between the light source and the plane:

$$\sqrt{(0-0)^2 + (1+11.48)^2 + (27-10.36)^2} = 20.8$$

And one more figure, between the two coordinates on the plane we just found:

$$\sqrt{(8-0)^2 + (-5.56+11.48)^2 + (5.92-10.36)^2} \approx 10.9$$

Keep referring to the diagram.

Now, for the triangles to be similar, find the distance d such that:

$$\frac{17.6}{d} = \frac{20.8}{10.9-d}$$

$$d \approx 4.99 \approx 5.00$$

For the final coordinates, just project the two points on the plane:

$$(x, y, z) = (8, -5.56, 5.92) + \frac{4.99}{10.9}((0, -11.48, 10.36) - (8, -5.56, 5.92))$$

And the coordinates are:

$$\left(\frac{13}{3}, -\frac{1241}{150}, \frac{1591}{200}\right)$$

or, in decimal form, (rounded to two decimal places)

$$(4.33, -8.27, 7.96)$$