1)

a) $Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$

$$Z^{-1} = \frac{1}{Z}$$

$$Z^{-1} = \frac{1}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$Z^{-1} = \frac{R - j\left(\omega L - \frac{1}{\omega C}\right)}{\left[R - j\left(\omega L - \frac{1}{\omega C}\right)\right]\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]}$$

$$Z^{-1} = \frac{R - j\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$Z^{-1} = \frac{R - j\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)}$$

$$Re(Z^{-1}) = \frac{R}{R^2 + \left(\omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)}$$

Real part of $Z^{-1}$ is $\frac{R}{R^2 + \left(\omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}\right)}$.

b) $z^4 + 4z^2 + 8 = 0$

Let w be $z^2$

$$w^2 + 4w + 8 = 0$$

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a=1, b=4 and c=8 (coefficients of $x^2$, $x$ and constant term)

$$w = \frac{-4 \pm \sqrt{16 - 32}}{2}$$

$$w = \frac{-4 \pm j4}{2}$$

$$w = -2 \pm j2$$

Therefore,

$$z = \sqrt{-2 + j2} \; or \; z = \sqrt{-2 - j2}$$

$$z = \pm\sqrt{-2 + j2} \; or \; z = \pm\sqrt{-2 - j2}$$

$$z = \pm\sqrt{-2 + j2} \; or \; z = \pm\sqrt{-2 - j2}$$

$$|z| = 2\sqrt{2} \; or \; |z| = 2\sqrt{2} \Rightarrow |z| = 2\sqrt{2}$$

$$z = \pm 2\sqrt{2} e^{\frac{3\pi}{4}} \; or \; z = \pm 2\sqrt{2} e^{\frac{5\pi}{4}}$$

c)

i) $x_1 = (i + 2j - k) + \lambda(5i - 3j + 2k)$
$x_2 = (10i - 5j + 3k) + \mu(7i + 8j - 3k)$
For two vectors to be parallel, their direction cosines must be proportional. That is, their direction ratios must be equal.
Clearly, $5i - 3j + 2k \neq 7i + 8j - 3k$,
Therefore, the two vectors are NOT parallel.
To follow above alignment, (Home -> Multilevel List -> Change List Level)

ii) Let us **ASSUME** that the student is right, and that there exists a point of intersection (x,y,z) for the two lines.

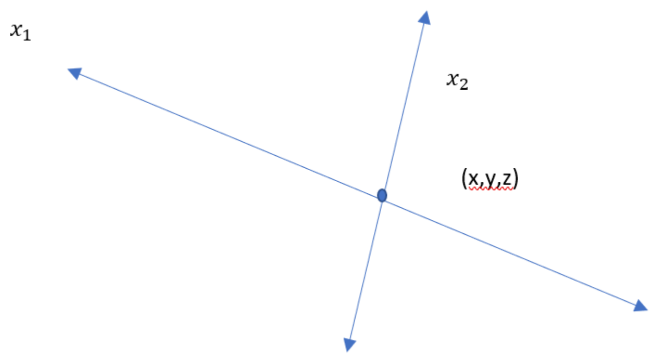



Figure 1: Lines intersecting

The Cartesian equation of x1:

$$\frac{x-1}{5} = \frac{y-2}{-3} = \frac{z+1}{2} = \lambda\ (say)$$

Point (x,y,z) lies on x1 and x2 by assumption. Therefore, (x,y,z) would be such that they satisfy the above given expression. Therefore,

$x = 5\lambda + 1$
$y = -3\lambda + 2$
$z = 2\lambda - 1$

Since we have assumed that this point also lies on x2, it must also satisfy the equation,

$$\frac{x-10}{7} = \frac{y+5}{8} = \frac{z-3}{-3}$$

Solving,

$$\frac{x-10}{7} = \frac{y+5}{8}$$

$$\frac{5\lambda + 1 - 10}{7} = \frac{-3\lambda + 2 + 5}{8}$$

$$=> \frac{5\lambda - 9}{7} = \frac{-3\lambda + 7}{8}$$

$=> 40\lambda - 72 = -21\lambda + 49$

$=> 61\lambda = 121$

$$=> \lambda = \frac{121}{61}$$

Now let us check this with the rest of the equation.

$$\frac{y+5}{8}=\frac{z-3}{-3}$$
$$\frac{-3\lambda+2+5}{8}=\frac{2\lambda-1-3}{-3}$$
$$=>\ \frac{-3\lambda+7}{8}=\frac{2\lambda-4}{-3}$$
$$=>\ 9\lambda-21=16\lambda-32$$
$$=>\ 11=7\lambda$$
$$=>\ \lambda=\frac{11}{7}$$

Clearly, the solutions do not match. We cannot find a $\lambda$ consistent with the equations. Therefore, our assumption that a point of intersection (x,y,z) exists is wrong.

d) $x=(\lambda i+j+4k)$

$y=(2i+6j+3k)$

$|y|=\sqrt{2^2+6^2+3^2}$

$|y|=\sqrt{49}$

$|y|=7$

$\boldsymbol{x}.\boldsymbol{y}=2\lambda+6+12$

$proj_a u=\frac{u.a}{||a||^2}a$

$4=\frac{2\lambda+6+12}{7}$

$28=2\lambda+18$

$10=2\lambda$

$\lambda=5$

The required value is 5.

2)

a) $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$A^2=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$A^2=\begin{bmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{bmatrix}$

Given, $A^2$=I,

Therefore,

$A^2=\begin{bmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$=>a^2+bc=1=>bc=1-a^2\ldots(1)$

$=>ab+bd=0=>b(a+d)=0\ldots(2)$

$=>ac+cd=0=>c(a+d)=0\ldots(3)$

$=>bc+d^2=1=>bc=1-d^2\ldots(4)$

Given that trace of A is equal to 0,

$=>a+d=0\ldots(5)$

From equations (1) and (2),

$1 - a^2 = 1 - d^2 => a^2 = d^2 => a = \pm d \ldots (6)$
But clearly, from equation (5), $a = -d$.
$|A| = ad - bc = -a^2 - (1 - a^2) = -a^2 - 1 + a^2 = -1$, since $a = -d$ and $bc = 1 - a^2$.
Hence, proved.

b) $A = \begin{bmatrix} \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \\ \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \end{bmatrix}$

$A = \begin{bmatrix} \frac{1}{2}(e^x - \frac{1}{e^x}) & \frac{1}{2}(e^x + \frac{1}{e^x}) \\ \frac{1}{2}(e^x + \frac{1}{e^x}) & \frac{1}{2}(e^x - \frac{1}{e^x}) \end{bmatrix}$

$A = \begin{bmatrix} \frac{1}{2e^x}(e^{2x} - 1) & \frac{1}{2e^x}(e^{2x} + 1) \\ \frac{1}{2e^x}(e^{2x} + 1) & \frac{1}{2e^x}(e^{2x} - 1) \end{bmatrix}$

$A = \frac{1}{2e^x}\begin{bmatrix} e^{2x} - 1 & e^{2x} + 1 \\ e^{2x} + 1 & e^{2x} - 1 \end{bmatrix}$

$cof(A) = \frac{1}{2e^x}\begin{bmatrix} e^{2x} - 1 & -(e^{2x} + 1) \\ -(e^{2x} + 1) & e^{2x} - 1 \end{bmatrix}$

$cof(A) = \frac{1}{2e^x}\begin{bmatrix} e^{2x} - 1 & -1 - e^{2x}) \\ -1 - e^{2x} & e^{2x} - 1 \end{bmatrix}$

$adj(A) = \frac{1}{2e^x}\begin{bmatrix} e^{2x} - 1 & -1 - e^{2x}) \\ -1 - e^{2x} & e^{2x} - 1 \end{bmatrix}$

$|A| = \frac{(e^{2x} - 1)^2 - (e^{2x} + 1)^2}{2e^x}$

$|A| = \frac{e^{4x} + 1 - 2e^{2x} - e^{4x} - 1 - 2e^{2x}}{2e^x}$

$|A| = \frac{-4e^{2x}}{2e^x}$

$|A| = -2$

Therefore,

$A^{-1} = \frac{adj(A)}{|A|}$

$A^{-1} = \frac{-1}{4e^x}\begin{bmatrix} e^{2x} - 1 & -1 - e^{2x}) \\ -1 - e^{2x} & e^{2x} - 1 \end{bmatrix}$

$A^{-1} = \frac{1}{4e^x}\begin{bmatrix} 1 - e^{2x} & 1 + e^{2x} \\ 1 + e^{2x} & 1 - e^{2x} \end{bmatrix}$

c) $x + y + z = -1$
$4x + 2y + z = 3$
$9x + 3y = \mu$
Let us represent the system of linear equations in the form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ \mu \end{bmatrix}$$

Let $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 0 \end{bmatrix} = A$, $\begin{bmatrix} -1 \\ 3 \\ \mu \end{bmatrix} = B$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X$.

For the equations to have infinitely many solutions, $adj(A)B = 0$.

$$cof(A) = \begin{bmatrix} -3 & 9 & -6 \\ 3 & -9 & 6 \\ -1 & 3 & -2 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} -3 & 3 & -1 \\ 9 & -9 & 3 \\ -6 & 6 & -2 \end{bmatrix}$$

$$adj(A)B = \begin{bmatrix} -3 & 3 & -1 \\ 9 & -9 & 3 \\ -6 & 6 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ \mu \end{bmatrix}$$

$$=> adj(A)B = \begin{bmatrix} 3+9-\mu \\ -9-27+3\mu \\ 6+18-2\mu \end{bmatrix}$$

$$\begin{bmatrix} 3+9-\mu \\ -9-27+3\mu \\ 6+18-2\mu \end{bmatrix} = 0$$

$$=> \ \mu = 12$$

d) False. (***Think about a complex number with no imaginary part. Then, |Re(z)|=|z|***)

3)

a) Let us first arrange the data in ascending order.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 1.4 | 1.6 | 2.0 | 2.1 | 2.5 | 3.1 | 3.1 | 3.8 | 4.5 |
| 4.6 | 4.7 | 5.0 | 5.4 | 5.7 | 5.8 | 6.2 | 6.2 | 6.4 | 6.9 |

i) To find 85th percentile:

Rank (R) = $\frac{85}{100}(20+1) = 17.85$

85th Percentile = (Value at $I_R$)+(Value at $I_{R+1}$-Value at $I_R$)*$F_R$

85th Percentile = 6.2+(6.2-6.2)*0.85

85th Percentile = 6.2

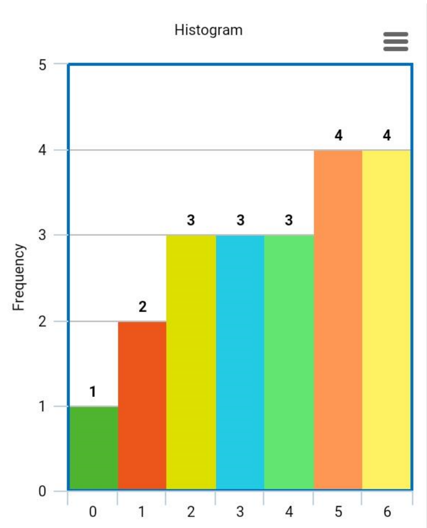
Mean = $\frac{0.5+1.4+1.6+2.0+2.1+2.5+3.1+3.1+3.8+4.5+4.6+4.7+5.0+5.4+5.7+5.8+6.2+6.2+6.4+6.9}{20} = \frac{81.5}{20} = 4.075$

$$\sum x^2 = 0.5^2 + 1.4^2 + 1.6^2 + 2.0^2 + 2.1^2 + 2.5^2 + 3.1^2 + 3.1^2 + 3.8^2 + 4.5^2 + 4.6^2 + 4.7^2 + 5.0^2 + 5.4^2 + 5.7^2 + 5.8^2 + 6.2^2 + 6.2^2 + 6.4^2 + 6.9^2 = 0.25 + 1.96 + 2.56 + 4 + 4.41 + 6.25 + 9.61 + 9.61 + 14.44 + 20.25 + 21.16 + 22.09 + 25 + 29.16 + 38.44 + 38.44 + 40.96 + 47.61 = 336.2$$

Variance = $\frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n} = \frac{336.2-332.1125}{20} = 0.204375$

ii)

| INTERVAL | FREQUENCY |
|---|---|
| **0-1** | 1 |
| **1-2** | 2 |
| **2-3** | 3 |
| **3-4** | 3 |
| **4-5** | 3 |
| **5-6** | 4 |
| **6-7** | 4 |

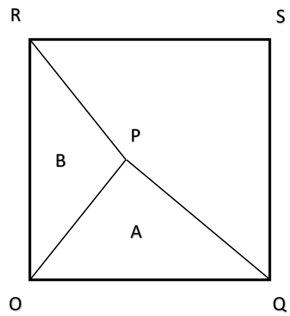



iii) Stem-and-leaf diagram

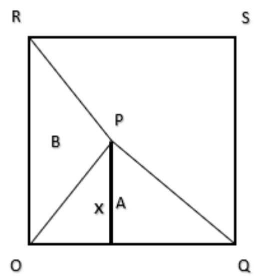
0 | 5
1 | 4 6
2 | 0 1 5
3 | 1 1 8
4 | 5 6 7
5 | 0 4 7 8
6 | 2 2 4 9

iv) A stem-and-leaf diagram provides a clear picture of data as well as frequency of data within a range, both together. Whereas, a histogram does not represent actual data. It only describes characteristics of a range.

b)




i)




Let x be the height of triangle OPQ.

Area of $\triangle\, OPQ = \frac{1}{2}(x)(1) = \frac{x}{2}$

Variance of A = $\int x^2 * \frac{x}{2} dx = \int \frac{x^3}{2} dx = \frac{x^4}{6}$

ii) Let x be the height of triangle OPQ.
Let y be the height of triangle OPR.
Probability density function = $\frac{x}{2} + \frac{y}{2}$

4)

a) Given that the probability that the milk turns bad in x days is given by

$$F(x) = 1 - e^{-0.2x}$$

i) For the packet to turn bad ON the second day, it must not have turned bad on the first day AND it must have turned bad on the second day.
Required probability = $(F(1)') * (F(2))$
Required probability = $(1 - (1 - e^{-0.2})) * (1 - e^{-0.4})$
Required probability = $(-1 + e^{-0.2}) * (1 - e^{-0.4})$
Required probability = $-1 + e^{-0.4} + e^{-0.2} - e^{0.08}$
Required probability = $e^{-0.4} + e^{-0.2} - e^{0.08} - 1$

ii) For the packet to turn bad ON the third day, it must not have turned bad on the first two days AND it must have turned bad on the third day.
Required probability = $(F(2)') * (F(3))$
Required probability = $(1 - (1 - e^{-0.4})) * (1 - e^{-0.6})$
Required probability = $(-1 + e^{-0.4}) * (1 - e^{-0.6})$
Required probability = $-1 + e^{-0.6} + e^{-0.4} - e^{0.24}$
Required probability = $e^{-0.6} + e^{-0.4} - e^{0.24} - 1$

b)

i) n=100

X=79

p=85

q=15

$$\frac{X}{n} = 0.79$$

$$var(X) = npq = 100 * 0.85 * 0.15 = 12.75$$

$$var\left(\frac{X}{n}\right) = \frac{pq}{n} = \frac{12.75}{100} = 0.1275$$

$$SD\left(\frac{X}{n}\right) = \sqrt{var\left(\frac{X}{n}\right)} = \sqrt{0.1275} = 0.3570$$

For 95% confidence interval,

$$L = \frac{X}{n} - \frac{SD\left(\frac{X}{n}\right)}{\sqrt{n}} = 0.79 - \frac{0.3570}{10} = 0.79 - 0.03570 = 0.7543$$

$$U = \frac{X}{n} + \frac{SD\left(\frac{X}{n}\right)}{\sqrt{n}} = 0.79 + \frac{0.3570}{10} = 0.79 + 0.03570 = 0.8257$$

The 95% confidence interval is (0.7543, 0.8257)

ii) Let the null hypothesis, $H_0$: $\mu = 85$

Let the alternative hypothesis, $H_0$: $\mu \neq 85$

$\alpha$=0.01

$$Z = \frac{\frac{X}{n} - p}{SD\left(\frac{X}{n}\right)}$$

$$Z = \frac{0.79 - 0.85}{0.3570}$$

$$Z = \frac{-0.6}{0.3570} = -1.68$$

$$p - value \ = \ 2 * (1 - 0.9535) = 2 * (0.0465) = 0.093$$

Since, p-value > $\alpha$,

The null hypothesis cannot be rejected.

Conclusion: The company’s claim that 85% percent of its people are happy with the service cannot be rejected.

--End of Answers--

Solver: Asuri Simhakutty Kamakshi