20th CSEC – Past Year Paper Solution *2017-2018 Sem 1*
CZ/CE 1011 – Engineering Mathematics I

1)

a) <u>Idea</u>: Transform z = a + jb to exponential form z = r*exp(jθ) to easily calculate z^3

<u>Calculation</u>:

$z = r * \exp(j\theta) \Rightarrow z^3 = r^3 * \exp(j * 3\theta)$

z^3 is real then 3θ = 2π *k for k is an integer

$\theta = 0\ or \frac{2\pi}{3}\ or - \frac{2\pi}{3}$

We have $b = a * \tan\theta \Rightarrow b = 0\ or - a\sqrt{3}\ or\ a\sqrt{3}$

b) <u>Calculation</u>:

$z_1 = 1 - j\sqrt{3} = 2\exp - \frac{\pi}{3}$

$z_1^3 = 8\exp(-\pi) = -8$

$z_2 = -\sqrt{3} + j = 2\exp\left(\frac{5\pi}{6}\right)$

$z_2^4 = 16\exp\left(\frac{10\pi}{3}\right) = 16\exp\left(\frac{\pi}{3}\right)$

$\frac{z_1^3}{z_2^4} = -\frac{1}{2}\exp\left(-\frac{\pi}{3}\right)$

c)

i) <u>Idea</u>: Vector equation of a line *l* passes through points A and B is:

$\boldsymbol{r} = \boldsymbol{OA} + \boldsymbol{AB} * \boldsymbol{t}$

<u>Calculation</u>:

$\boldsymbol{OA} = 8\boldsymbol{i} + 13\boldsymbol{j} - 2\boldsymbol{k}$

$\boldsymbol{OB} = 10\boldsymbol{i} + 14\boldsymbol{j} - 4\boldsymbol{k}$

$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

$\boldsymbol{r} = (8\boldsymbol{i} + 13\boldsymbol{j} - 2\boldsymbol{k}) + (2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}) * \boldsymbol{t}$

ii) <u>Idea</u>:

θ = angle between **CB** and line *l* = angle between **CB** and **AB**

Dot product of 2 vectors:

$\boldsymbol{CB} \cdot \boldsymbol{AB} = CB \cdot AB \cdot \cos\theta$

Length of a vector $\boldsymbol{MN} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is: $MN = \sqrt{x^2 + y^2 + z^2}$

<u>Calculation</u>:

$\boldsymbol{OC} = 9\boldsymbol{i} + 9\boldsymbol{j} + 6\boldsymbol{k}$

$\boldsymbol{CB} = \boldsymbol{OB} - \boldsymbol{OC} = \boldsymbol{i} + 5\boldsymbol{j} - 10\boldsymbol{k}$

$\boldsymbol{CB} \cdot \boldsymbol{AB} = 1 * 2 + 5 * 1 + (-10) * (-2) = 27$

$CB = 3\sqrt{14}$

$AB = 3$

$\Rightarrow \cos\theta = 0.802$

θ is an acute angle $\Rightarrow \theta = 36.7°$

d) <u>Idea</u>:

We have to prove 2 statements:

1. If **u** and **v** are parallel then $||\boldsymbol{u} + \boldsymbol{v}|| = ||\boldsymbol{u}|| + ||\boldsymbol{v}||$
2. If $||\boldsymbol{u} + \boldsymbol{v}|| = ||\boldsymbol{u}|| + ||\boldsymbol{v}||$ then **u** and **v** are parallel

Proof:

1.

$||\boldsymbol{u}+\boldsymbol{v}|| = ||\boldsymbol{u}|| + ||\boldsymbol{v}||$

$\Rightarrow ||\boldsymbol{u}+\boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2||\boldsymbol{u}|| * ||\boldsymbol{v}||$ (a)

We have:

$(\boldsymbol{u}+\boldsymbol{v})^2 = \boldsymbol{u}^2 + \boldsymbol{v}^2 + 2\boldsymbol{u}\cdot\boldsymbol{v}$

$||\boldsymbol{u}+\boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2||\boldsymbol{u}|| * ||\boldsymbol{v}|| * \cos(\boldsymbol{u},\boldsymbol{v})$ (b)

From (a) and (b) → $cos(\boldsymbol{u},\boldsymbol{v}) = 1$

The angle between **u** and **v** is 0 or π. Therefore, **u** and **v** are parallel.

2.

We have:

$(\boldsymbol{u}+\boldsymbol{v})^2 = \boldsymbol{u}^2 + \boldsymbol{v}^2 + 2\boldsymbol{u}\cdot\boldsymbol{v}$

$||\boldsymbol{u}+\boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2||\boldsymbol{u}|| * ||\boldsymbol{v}|| * \cos(\boldsymbol{u},\boldsymbol{v})$

u and **v** are parallel

$\cos(\boldsymbol{u},\boldsymbol{v}) = 1$

$(||\boldsymbol{u}+\boldsymbol{v}||)^2 = (||\boldsymbol{u}|| + ||\boldsymbol{v}||)^2$

$||\boldsymbol{u}+\boldsymbol{v}|| = ||\boldsymbol{u}|| + ||\boldsymbol{v}||$

2)

a) Idea:

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are variables.

Calculate BA^2 then let $BA^2 = A$ → a,b,c,d

Calculation:

$A^2 = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ 5 & 19 \end{bmatrix}$

$BA^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 4 & 15 \\ 5 & 19 \end{bmatrix} = \begin{bmatrix} 4a+5b & 15a+19b \\ 4c+5d & 15c+19d \end{bmatrix}$

$BA^2 = A$

$1 = 4a + 5b$

$3 = 15a + 19b$

$1 = 4c + 5d$

$4 = 15c + 19d$

$B = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$

b) Idea:

AX = B is a linear equation system.

We use Gauss-Jordon Elimination to [A|B]

Calculation:

$\begin{bmatrix} 1 & 3 & 2\,7 \\ 3 & 0 & 1/7 \\ 1 & 1 & 2\,5 \end{bmatrix} \gg \begin{bmatrix} 1 & 0 & 0\,2 \\ 0 & 1 & 0/1 \\ 0 & 0 & 1\,1 \end{bmatrix}$

$X = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

c) <u>Idea</u>:

For a linear equation system, Ax=b

M = [A|b]

Equations are consistent when rank(A) = rank(M)

<u>Calculation</u>:

$$M = \begin{bmatrix} 1 & 1 & 2\ 8 \\ 2 & 1 & -1/3 \\ 1 & -1 & -8\ k \end{bmatrix} \gg \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 5/ & 13 \\ 0 & 1 & 5 & (8-k)/2 \end{bmatrix}$$

$rank(A) = 2$

$if \frac{8-k}{2} = 13 \Rightarrow rank(M) = rank(A) = 2$

$\frac{if (8-k)}{2} \neq 13 \Rightarrow rank(M) = 3 \neq rank(A)$

$So, k = 2 \Rightarrow$ Equations are consistent

When k = -18, x = 3z -5, y = -5z + 13

d) The statement is true

3)

a)

i) N = 20 (measurements)

25th percentile = $\frac{3.5+3.6}{2} = 3.55$

50th percentile = $\frac{4.4+4.7}{2} = 4.55$

85th percentile = $\frac{6.3+6.3}{2} = 6.3$

Editor's note: Please use the method taught in lectures. For example, for 25th percentile, we need to take the $\frac{25}{100}(20+1) = 5.25^{th}$ value. That means $3.5 + 0.25(3.6 - 3.5) = 3.525$.

ii)

Interval	2.0 – 2.9	3.0 – 3.9	4.0 – 4.9	5.0 – 5.9	6.0 – 6.9
Frequency	2	4	6	4	4

b)

<u>Idea</u>:

Let D is the event a person suffers from disease

T is the event that person get a positive test result

$P(D) = 0.03$

$P(T|D) = 0.95$

$P(T|D') = 0.08$

i) $P(T) =?$

<u>Idea</u>:

$P(T|D) * P(D) = P(T\ and\ D)$

$P(T|D') * P(D') = P(T\ and\ D')$

$P(D') = 1 - P(D)$

$P(T\ and\ D) + P(T\ and\ D') = P(T)$

Calculation:

$P(T) = 0.1061$

ii) $P(D'|T) = ?$

Idea:

$P(D'|T) * P(T) = P(T\ and\ D')$

Calculation:

$P(D'|T) = 0.7314$

iii) $P(D|T') = ?$

Idea:

$P(D|T') * P(T') = P(D\ and\ T')$

$P(T') = 1 - P(T)$

$P(D\ and\ T') = P(D) - P(D\ and\ T)$

Calculation:

$P(D|T') = 1.678 * 10^{-3}$

c)

i) Idea:

For a probability density function: $f_x(x)$

Integral of $f_x(x)dx$ for x from $-\infty$ to $+\infty$ is equal to 1

$\frac{\int dx}{1+x^2} = \arctan x + C$

Calculation:

k = 2

ii) Idea:

pd function of Y is $f_y(y)$

cd function of Y is Integral of $f_y(y)dy$ for y from $-\infty$ to y

Integral of $f_y(y)dy$ for y from $-\infty$ to y is probability that y in the interval $-\infty$ to y

Probability that y in interval $-\infty$ to y is probability that x in the interval $x = \frac{1}{y}$ to $+\infty$

→cdf of Y = Integral of $f_x(x)dx$ for x from $x = \frac{1}{y}$ to $+\infty$

Calculation:

cdf of Y = $\frac{\pi}{2} - \arctan\left(\frac{1}{y}\right)$

4)

a) Idea:

n large, central limit theorem, normal distribution

Calculation:

$n = 43$

$E(X) = 1 * \frac{1}{6} + 2 * \frac{2}{6} + 3 * \frac{3}{6} = \frac{7}{3}$

$var(X) = \left(1 - E(X)\right)^2 * \frac{1}{6} + \left(2 - E(X)\right)^2 * \frac{2}{6} + \left(3 - E(X)\right)^2 * \frac{3}{6} = \frac{5}{9}$

$P(x < 2.1) = 0.0202$

$P(x < 2.4) = 0.7224$

$P(2.1 < x < 2.4) = 0.7022$

b)

i) Idea:

n large, central limit theorem, normal distribution

Calculation:

$n = 36$
$E(X) = mean = 438$
$var(X) = var(sample\ X) = 3600$
$H_0: mean > 450$
p-value= $0.1151 > 0.0500$
This claim is not acceptable

ii) Confidence level = $P(430 < x < +\infty) = 0.7881 = 78.81\%$

c) Idea:

Apply linear regression
$y = 1.767 + 0.03x$

--End of Answers--

Solver: Nguyen Ngoc Khanh