Solver: Ricardo Jack Liwongan

Email Address: rica0003@e.ntu.edu.sg

1. (a)

(i) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$
$= 1 \times e^{-j\omega} + 2 \times e^{-2j\omega} + 3 \times e^{-3j\omega}$

(ii) In time domain:

$$x[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]x[n-k]$$
$$= x[0]x[n] + x[1]x[n-1] + x[2]x[n-2] + x[3]x[n-3] + \cdots$$
$$= 0 \times x[n] + 1 \times x[n-1] + 2 \times x[n-2] + 3 \times x[n-3] + 0 + 0 + \cdots$$

$x[n] = \delta[n-1] + 2\delta[n-2] + 3\delta[n-3]$
$1 \times x[n-1] = \delta[n-2] + 2\delta[n-3] + 3\delta[n-4]$
$2 \times x[n-2] = 2\delta[n-3] + 4\delta[n-4] + 6\delta[n-5]$
$3 \times x[n-2] = 3\delta[n-4] + 6\delta[n-5] + 9\delta[n-6]$
$x[n] * x[n] = 1 \times x[n-1] + 2 \times x[n-2] + 3 \times x[n-3]$
$= \delta[n-2] + 4\delta[n-3] + 10\delta[n-4] + 12\delta[n-5] + 9\delta[n-6]$

In frequency domain:

$$X(e^{j\omega}) \cdot X(e^{j\omega}) = (e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}) \cdot (e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega})$$
$$= e^{-2j\omega} + 2e^{-3j\omega} + 3e^{-4j\omega} + 2e^{-3j\omega} + 4e^{-4j\omega} + 6e^{-5j\omega} + 3e^{-4j\omega} + 6e^{-5j\omega} + 9e^{-6j\omega}$$
$$= e^{-2j\omega} + 4e^{-3j\omega} + 10e^{-4j\omega} + 12e^{-5j\omega} + 9e^{-6j\omega} + 6e^{-5j\omega}$$

$$X(e^{j\omega}) \cdot X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n] * x[n])e^{-j\omega n}$$

$x[2] * x[2] = 1$
$x[3] * x[3] = 4$
$x[4] * x[4] = 10$
$x[5] * x[5] = 12$
$x[6] * x[6] = 9$
$x[n] * x[n] = \delta[n-2] + 4\delta[n-3] + 10\delta[n-4] + 12\delta[n-5] + 9\delta[n-6]$

The result is the same when calculated in time and frequency domain

(b)

(i) $c_k = \frac{1}{N} \cdot \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$

$c_0 = \frac{1}{8} \cdot \sum_{n=0}^{7} x[n]e^{-j2\pi \cdot 0 \cdot \frac{n}{8}} = \frac{1}{8} \cdot (1 + 2 + 3) = \frac{3}{4}$

$$c_1 = \frac{1}{8} \cdot \sum_{n=0}^{7} x[n]e^{-j2\pi \cdot 1 \cdot \frac{n}{8}} = \frac{1}{8} \cdot \left(e^{-j2\pi \cdot 1 \cdot \frac{1}{8}} + 2e^{-j2\pi \cdot 1 \cdot \frac{2}{8}} + 3e^{-j2\pi \cdot 1 \cdot \frac{3}{8}}\right)$$

$$= \frac{1}{8} \cdot \left(e^{-\frac{j\pi}{4}} + 2e^{-\frac{j\pi}{2}} + 3e^{-\frac{j3\pi}{4}}\right) = \frac{1}{8} \cdot \left(e^{-\frac{j\pi}{4}} + 2e^{-\frac{j\pi}{2}} + 3e^{-\frac{j3\pi}{4}}\right) = 0.6289e^{-j1.8557}$$

$$c_2 = \frac{1}{8}\cdot\sum_{n=0}^{7} x[n]e^{-j2\pi\cdot 2\cdot\frac{n}{8}} = \frac{1}{8}\cdot\left(e^{-j2\pi\cdot 2\cdot\frac{1}{8}} + 2e^{-j2\pi\cdot 2\cdot\frac{2}{8}} + 3e^{-j2\pi\cdot 2\cdot\frac{3}{8}}\right) = \frac{\sqrt{2}}{4}e^{j\frac{3}{4}\pi}$$

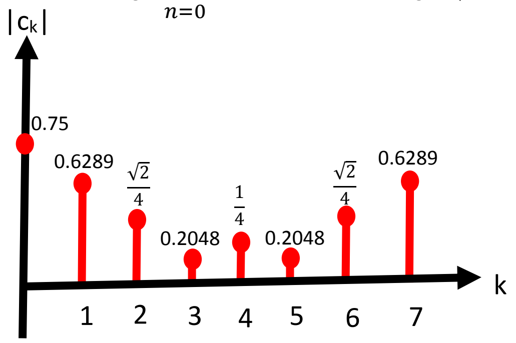
$$c_3 = \frac{1}{8}\cdot\sum_{n=0}^{7} x[n]e^{-j2\pi\cdot 3\cdot\frac{n}{8}} = \frac{1}{8}\cdot\left(e^{-j2\pi\cdot 3\cdot\frac{1}{8}} + 2e^{-j2\pi\cdot 3\cdot\frac{2}{8}} + 3e^{-j2\pi\cdot 3\cdot\frac{3}{8}}\right) = 0.20487e^{-j0.5299}$$

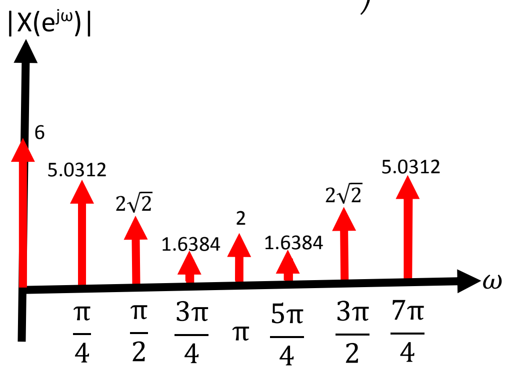
$$c_4 = \frac{1}{8}\cdot\sum_{n=0}^{7} x[n]e^{-j2\pi\cdot 3\cdot\frac{n}{8}} = \frac{1}{8}\cdot\left(e^{-j2\pi\cdot 4\cdot\frac{1}{8}} + 2e^{-j2\pi\cdot 4\cdot\frac{2}{8}} + 3e^{-j2\pi\cdot 4\cdot\frac{3}{8}}\right) = \frac{1}{4}e^{-\pi}$$

$$c_5 = \frac{1}{8}\cdot\sum_{n=0}^{7} x[n]e^{-j2\pi\cdot 3\cdot\frac{n}{8}} = \frac{1}{8}\cdot\left(e^{-j2\pi\cdot 5\cdot\frac{1}{8}} + 2e^{-j2\pi\cdot 5\cdot\frac{2}{8}} + 3e^{-j2\pi\cdot 5\cdot\frac{3}{8}}\right) = 0.20487e^{j0.5299}$$

$$c_6 = \frac{1}{8}\cdot\sum_{n=0}^{7} x[n]e^{-j2\pi\cdot 3\cdot\frac{n}{8}} = \frac{1}{8}\cdot\left(e^{-j2\pi\cdot 6\cdot\frac{1}{8}} + 2e^{-j2\pi\cdot 6\cdot\frac{2}{8}} + 3e^{-j2\pi\cdot 6\cdot\frac{3}{8}}\right) = \frac{\sqrt{2}}{4}e^{-j\frac{3}{4}}$$

$$c_7 = \frac{1}{8}\cdot\sum_{n=0}^{7} x[n]e^{-j2\pi\cdot 3\cdot\frac{n}{8}} = \frac{1}{8}\cdot\left(e^{-j2\pi\cdot 7\cdot\frac{1}{8}} + 2e^{-j2\pi\cdot 7\cdot\frac{2}{8}} + 3e^{-j2\pi\cdot 7\cdot\frac{3}{8}}\right) = 0.6289e^{j1.8557}$$





$$X(e^{j\omega}) = Nc_k\delta(\omega - \frac{2\pi}{N}k)$$

- To convert DTFS to DTFT along the x-axis, the k values are converted to frequencies $\frac{2\pi}{N}k$.
- In addition, the values of DTFS existing at k are represented as weighted impulses for DTFT
- The c_k coeffients are scaled by N for the DTFT values $X(e^{j\omega})$ at $\omega = \frac{2\pi}{N}k$

(ii) Using Parseval's theorem:

$$P = \frac{1}{N}\sum_{n=0}^{N}|x[n]|^2 = \sum_{k=0}^{N}|c_k|^2$$

$$LHS: \frac{1}{N}\sum_{n=0}^{N}|x[n]|^2 = \frac{1}{8}(1^2 + 2^2 + 3^2) = 1.75$$

$$RHS: \sum_{k=0}^{N}|c_k|^2 = ({c_1}^2 + {c_2}^2 + \cdots + {c_7}^2) = 1.7499165 \approx 1.75$$

2. (a)

$$H(z) = H_1(z) \times H_2(z)$$

$$= \frac{z}{z-0.8} \cdot (1+2z^{-1}) = \frac{z+2}{z-0.8} \rightarrow \frac{H(z)}{z} = \frac{z+2}{z(z-0.8)} = -\frac{2.5}{z} + \frac{3.5}{z-0.8}$$

$$\rightarrow H(z) = -2.5 + \frac{3.5z}{z-0.8} = -2.5 + \frac{3.5}{1-0.8z^{-1}}$$

$$H(z) \underset{IZT}{\longrightarrow} h[n] = -2.5\delta[n] + 3.5 \cdot 0.8^n u[n]$$

(b)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z+2}{z-0.8} = \frac{1+2z^{-1}}{1-0.8z^{-1}} \rightarrow Y(z)(1-0.8z^{-1}) = X(z)(1+2z^{-1})$$

$$y[n] - 0.8y[n-1] = x[n] + 2x[n-1]$$

(c)

$$X(e^{j\omega}) = \frac{e^{j\omega}+2}{e^{j\omega}-0.8}$$

$$\omega = 0 \rightarrow \frac{e^{j0}+2}{e^{j0}-0.8} = \frac{1+2}{1-0.8} = 15$$

$$\omega = \frac{\pi}{4} \rightarrow \frac{e^{j\frac{\pi}{4}}+2}{e^{j\frac{\pi}{4}}-0.8} = 3.923e^{-j1.4459}$$

$$\omega = \frac{\pi}{2} \rightarrow \frac{e^{j\frac{\pi}{2}}+2}{e^{j\frac{\pi}{2}}-0.8} = 1.746e^{-j1.782}$$

$$\omega = \frac{3\pi}{4} \rightarrow \frac{e^{j\frac{3\pi}{4}}+2}{e^{j\frac{3\pi}{4}}-0.8} = 0.88526e^{-j2.2024}$$

$$\omega = \pi \rightarrow \frac{e^{j\pi}+2}{e^{j\pi}-0.8} = 0.5556e^{-j\pi}$$

Filter type: Low pass, the magnitude response decreases as the frequecy increases

(d)

$$x[n] = x(nT) = A\sin(10\pi nT) = A\sin\left(\frac{10}{20}\pi n\right) = A\sin(0.5\pi n)$$

$$X(z) = \frac{\sin(0.5\pi)\,z^{-1}}{1-2\cdot\cos(0.5\pi)z^{-1}+z^{-2}} = \frac{z^{-1}}{1+z^{-2}} \rightarrow Y(z) = \frac{z^{-1}}{1+z^{-2}} \cdot \frac{1+2z^{-1}}{1-0.8z^{-1}}$$

$$Y(z) = \frac{z}{z^2+1} \cdot \frac{z+2}{z-0.8} \rightarrow Y(z) = \frac{z^2+2z}{(z^2+1)(z-0.8)}$$

3. (a)

Low pass filter with cut-off frequency of $\frac{1}{2} \times 30kHz = 15kHz$

(b)

(i) $H(s) = \frac{200\pi/s}{1+200\pi/s} = \frac{200\pi}{s+200\pi}$

Filter type: Lowpass filter

At the -3dB cut-off frequency, $|H(j\Omega_c)|^2 = \frac{1}{2}$

$, where\ \Omega_c\ is\ the\ analog\ angular\ cut-off\ frequency$

$$|H(j\Omega_c)|^2 = \frac{1}{2} \rightarrow \left|\frac{200\pi/j\Omega_c}{1+200\pi/j\Omega_c}\right|^2 = \frac{1}{2}$$

$$\rightarrow \left|\frac{1}{j\Omega_c/200\pi+1}\right|^2 = \frac{1}{2}$$

$$\rightarrow \left(\sqrt{\frac{1}{j\Omega_c/200\pi+1}\cdot\frac{1}{-j\Omega_c/200\pi+1}}\right)^2 = \frac{1}{2}$$

$$\rightarrow \frac{1}{(\frac{\Omega_c}{200\pi})^2+1^2} = \frac{1}{2} \rightarrow (\frac{\Omega_c}{200\pi})^2+1 = 2 \rightarrow \Omega_c = 200\pi$$

(ii) $f_s = 300Hz, \Omega_c = 200\pi$

$\omega_c = \frac{\Omega_c}{f_s} = \frac{200\pi}{300} = \frac{2\pi}{3}, \Omega = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) = \frac{2}{T}\tan\left(\frac{2\pi}{3\cdot 2}\right) = \frac{2}{T}\sqrt{3}$

$$H(s) = \frac{\Omega}{s+\Omega} = \frac{\frac{2}{T}\sqrt{3}}{s+\frac{2}{T}\sqrt{3}}, s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$H(z) = \frac{\Omega}{s+\Omega} = \frac{\frac{2}{T}\sqrt{3}}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)+\frac{2}{T}\sqrt{3}} = \frac{\sqrt{3}}{\left(\frac{z-1}{z+1}\right)+\sqrt{3}} = \frac{\sqrt{3}(z+1)}{z-1+\sqrt{3}(z+1)}$$

(iii)

Zero= -1

Pole= $\frac{1-\sqrt{3}}{1+\sqrt{3}}$

To verify that the resultant filter meets the -3dB cut-off frequency requirement, the square magnitude response at the cut-off frequency should be 0.5:

$$let\ z = e^{j\omega} \rightarrow |H(e^{j\omega})|^2 = \left|\frac{\sqrt{3}(e^{j\omega}+1)}{e^{j\omega}-1+\sqrt{3}(z+1)}\right|^2$$

$$with\ \omega = \omega_c = \frac{2\pi}{3} \rightarrow \left|\frac{\sqrt{3}\left(e^{j\frac{2\pi}{3}}+1\right)}{e^{j\frac{2\pi}{3}}-1+\sqrt{3}(e^{j\frac{2\pi}{3}}+1)}\right|^2$$

$$\rightarrow \left|\frac{\sqrt{3}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}j+1\right)}{-\frac{1}{2}+\frac{\sqrt{3}}{2}j-1+\sqrt{3}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}j+1\right)}\right|^2 = \left|\frac{\frac{1}{2}\sqrt{3}+\frac{3}{2}j}{-\frac{1}{2}+\frac{\sqrt{3}}{2}j-1+\frac{1}{2}\sqrt{3}+\frac{3}{2}j}\right|^2 = \frac{1}{2}$$

(iv) $H(z) = \frac{Y(z)}{X(z)} = \frac{\sqrt{3}(z+1)}{z-1+\sqrt{3}(z+1)} = \frac{\sqrt{3}(1+z^{-1})}{1-z^{-1}+\sqrt{3}(1+z^{-1})}$

$$\rightarrow Y(z)\left(1-z^{-1}+\sqrt{3}(1+z^{-1})\right) = X(z)(\sqrt{3}(1+z^{-1}))$$

$$\rightarrow y[n]-y[n-1]+\sqrt{3}y[n]+\sqrt{3}y[n-1] = \sqrt{3}x[n]+\sqrt{3}x[n-1]$$

$$\rightarrow h[n]-h[n-1]+\sqrt{3}h[n]+\sqrt{3}h[n-1] = \sqrt{3}\delta[n]+\sqrt{3}\delta[n-1]$$

$$h[n] = 0, n < 0$$

$$n = 0 \rightarrow h[0]-0+\sqrt{3}h[0]+0 = \sqrt{3}\delta[0]+0 \rightarrow h[0] = \frac{\sqrt{3}}{1+\sqrt{3}} = 0.63397$$

$$n = 1 \rightarrow h[1]-0.63397+\sqrt{3}h[1]+\sqrt{3}\cdot 0.63397 = \sqrt{3}\delta[1]+\sqrt{3}\delta[0]$$

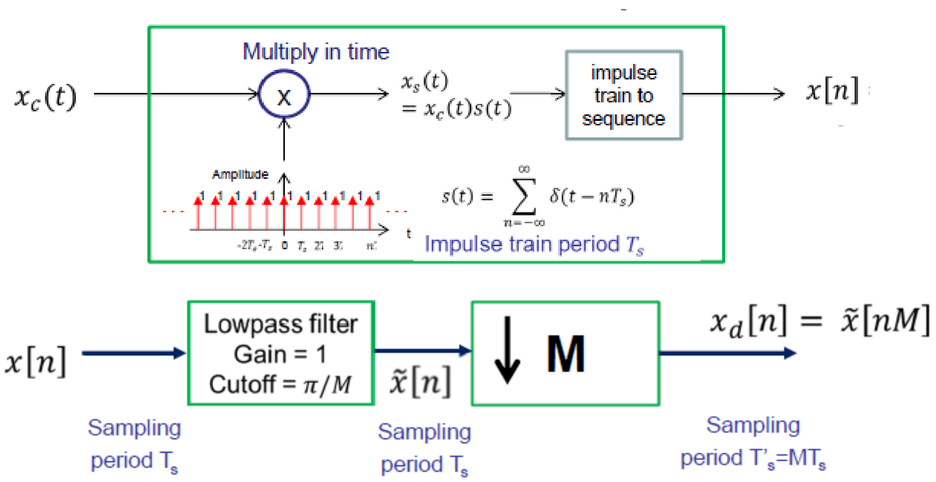
$$\rightarrow h[1]+\sqrt{3}h[1] = 0+\sqrt{3}+\left(1-\sqrt{3}\right)\cdot 0.63397 \rightarrow h[1] = 0.4641$$

$$n = 2 \rightarrow h[2]-0.4641+\sqrt{3}h[2]+\sqrt{3}\cdot 0.4641 = \sqrt{3}\delta[2]+\sqrt{3}\delta[1]$$

$$\rightarrow h[2]+\sqrt{3}h[2] = 0+0+\left(1-\sqrt{3}\right)\cdot 0.4641 \rightarrow h[2] = -0.124355$$

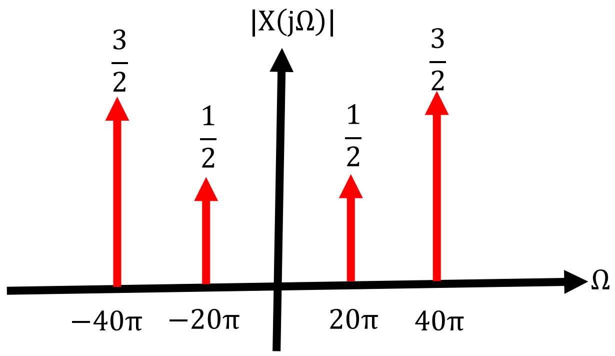
4. (a)

(i)

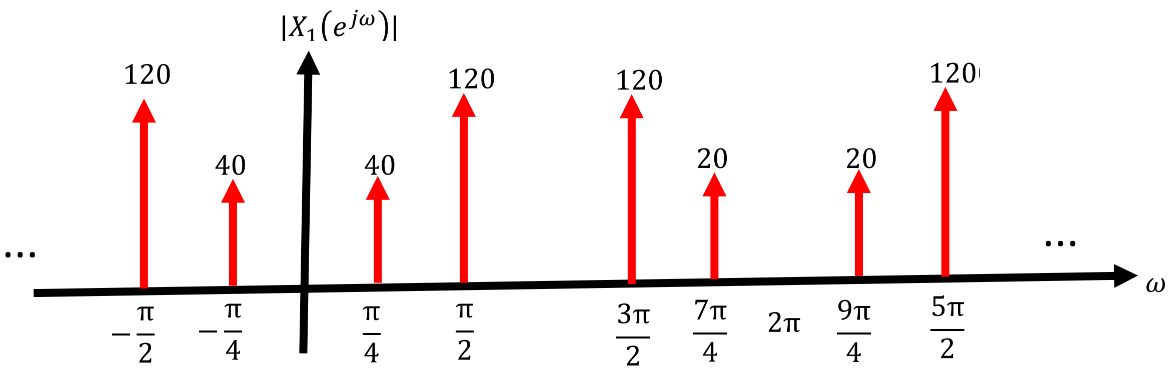


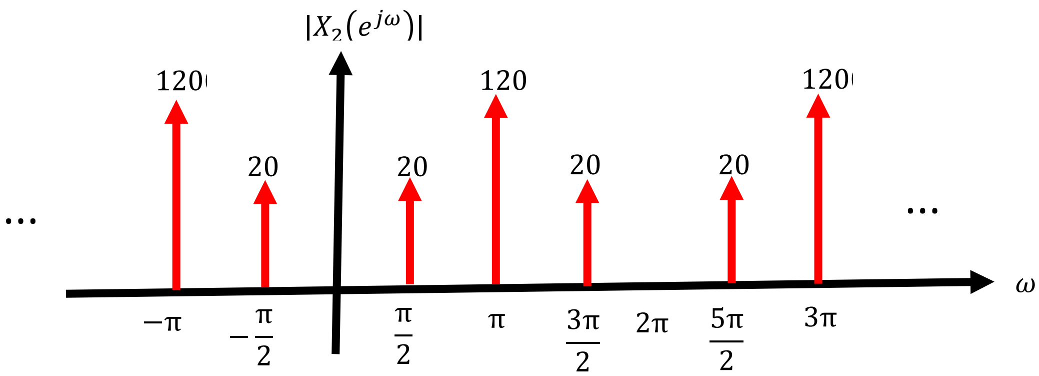
$T_s = 80Hz$, M= 2

(ii)



(iii)

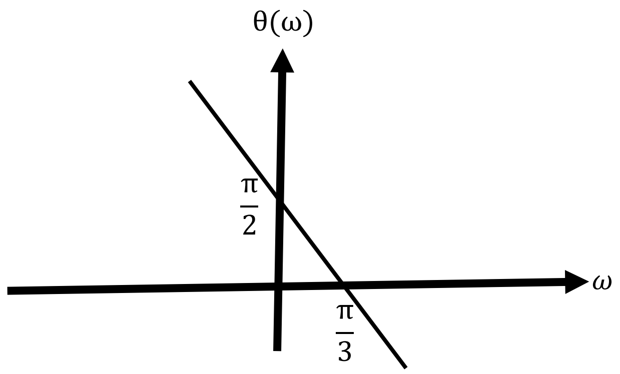




(b)

The impulse response is type IV(anti-symmetrical, odd N)

$$Phase\ Shift = e^{-j(\frac{3}{2}\omega-\frac{\pi}{2})} \rightarrow Phase\ response = \theta(\omega) = -(\frac{3}{2}\omega - \frac{\pi}{2})$$



(i)

- Yes, phase is linear
- No-phase distortion(for the frequencies to be passed)

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Thank you and all the best for your exams!