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1.

a) In order to make the sampled sequence x[n] periodic: $16 \times \pi \times \frac{N}{f} = k \times 2\pi$ where k is a positive integer and N is x[n]'s period.

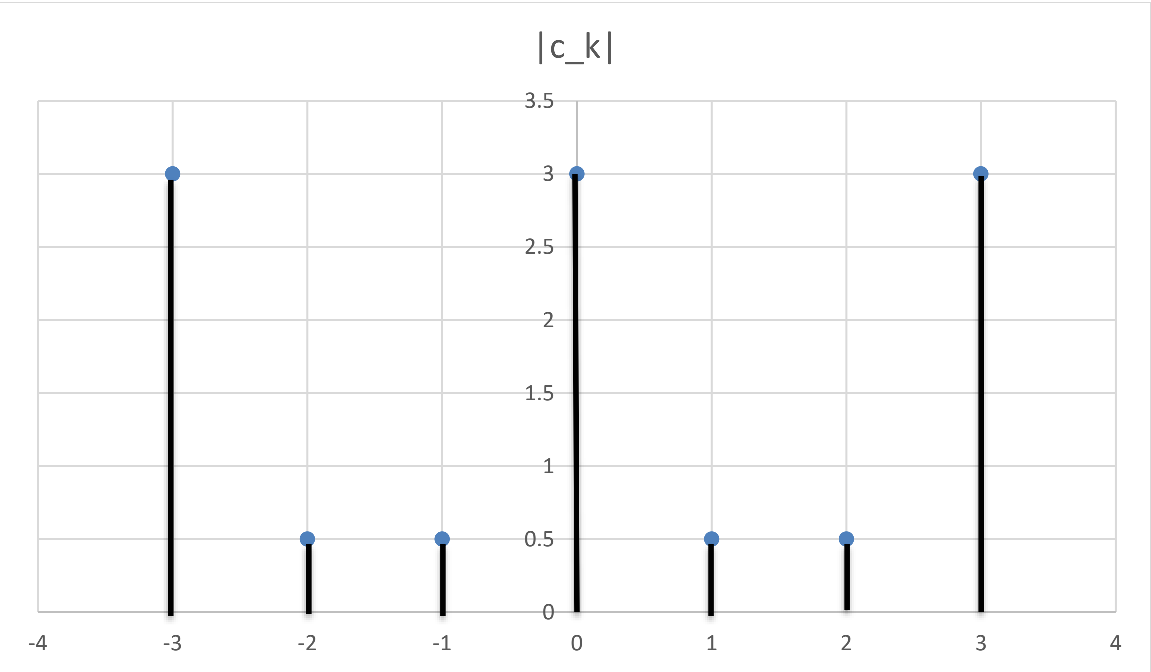
We can see that any rational value between 25Hz and 39Hz can do the trick, but we are only interested in value when k = 1. This results in only 1 possible value of sampling rate f = 32Hz

With f = 32Hz, the period if x[n] is N = 4

b) i) Choose N = 3 (also period of x[n])

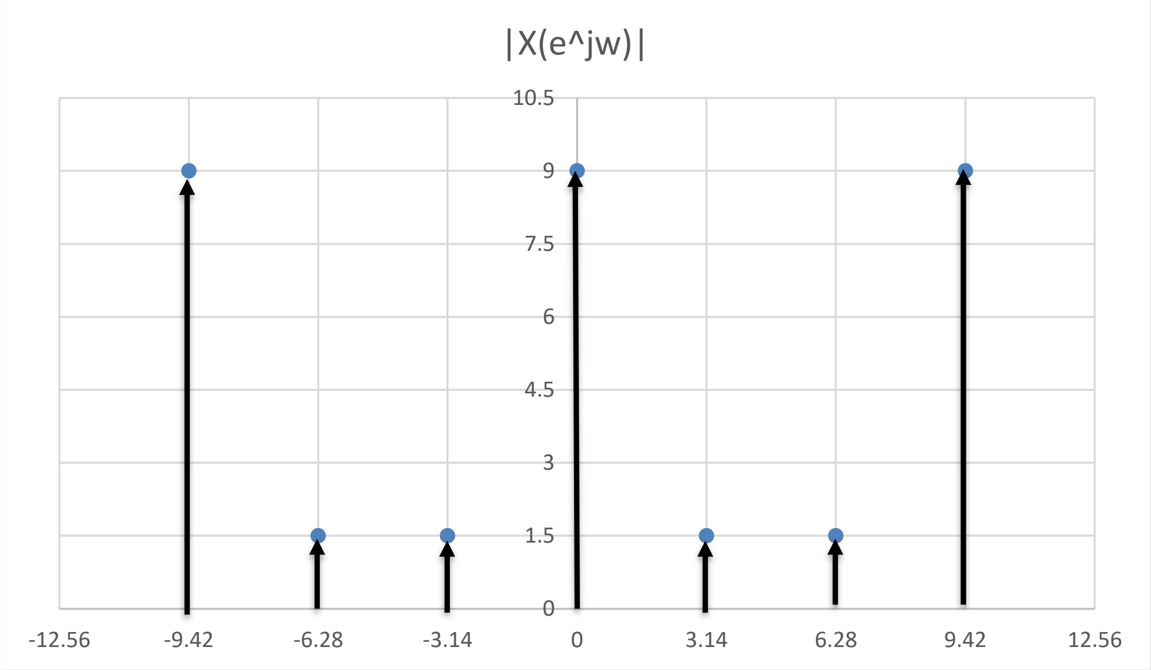
Apply DTFS equation: $c_k = \frac{1}{3} \times \sum_{n=0}^{2}(-3 + \sin\left(\frac{2\pi}{3}n\right))e^{\frac{-j2\pi kn}{3}}$

A few value: $|c_0| = -3, |c_1| = \frac{1}{2}, |c_2| = \frac{1}{2}$. Note that we only care about the magnitude



ii) The relationship between DTFT and DTFS: $X(e^{j\omega}) = Nc_k\delta(\omega - \frac{2\pi}{N}k)$ where N = 3.

From this relation, we can easily derive the DTFT. Again, we only care about magnitude (notice the arrow change in the sketch)



iii) The signal's power in time domain:

$$P = \frac{1}{3} \times \left((-3)^2 + \left(-3 + \sin\left(\frac{2\pi}{3}\right)\right)^2 + \left(-3 + \sin\left(\frac{4\pi}{3}\right)\right)^2\right) = 9.5$$

The signal's power in frequency domain:

$$P = 3^2 + 0.5^2 + 0.5^2 = 9.5$$

We can see that the signal's power is the same in both time and frequency domain. This obeys the Parseval's theorem

2.

a) Since the system is LTI, we have:

When $x[n] = \delta[n]$, $y[n] = h[n]$

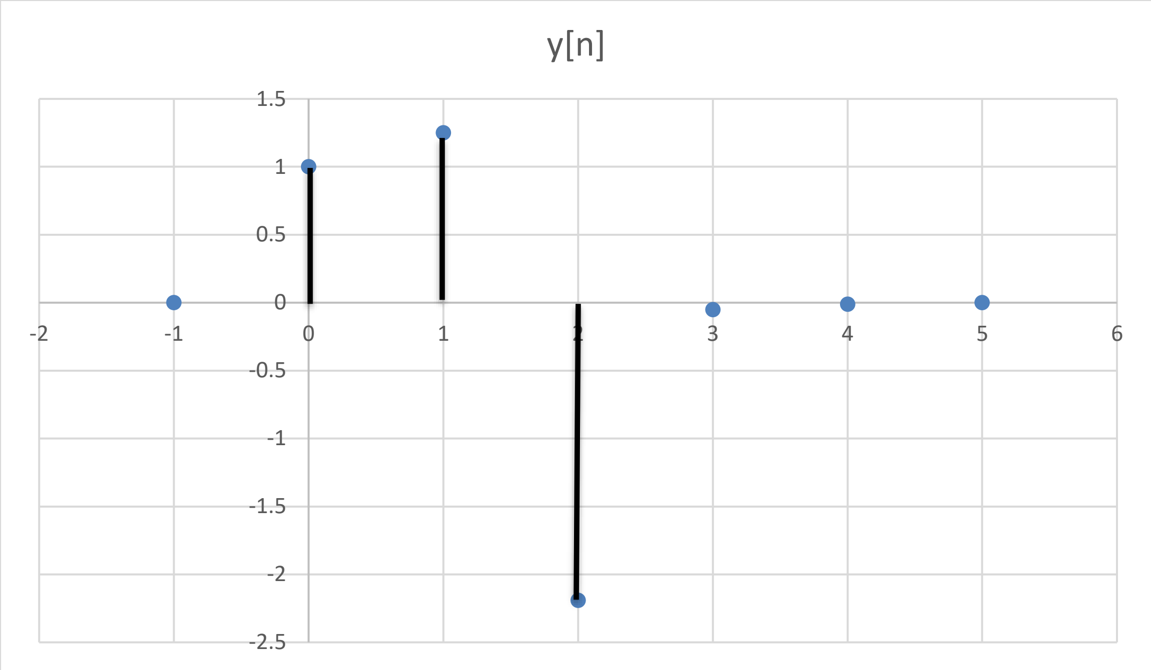
When $x[n] = \delta[n-1]$, $y[n] = h[n-1]$

When $x[n] = \delta[n] - \delta[n-1]$, $y[n] = h[n] - h[n-1]$

Calculate y[n] for a few values:

n	-1	0	1	2	3	4	5
Y[n]	0	1	1.25	-2.19	-0.05	-0.01	-0.00

Sketch y[n]:



ii) To find the system's transfer function, we find the Z transform of h[n]

Since $h[n] = \delta[n] + 2\delta[n-1] + \left(\frac{1}{4}\right)^n u[n-1]$ is causal, then

$$h[n] = \delta[n] + 2\delta[n-1] + \frac{1}{4} \times \left(\frac{1}{4}\right)^{n-1} u[n-1] \therefore\ H(z) = 1 + 2z^{-1} + \frac{\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}$$

(be careful for the last term in h[n])

b) i) To find the impulse response, we can find the transfer function first:

$$y[n] = 0.2x[n] + 0.9y[n-1] \therefore Y(z) = 0.2X(z) + 0.9Y(z)z^{-1}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{0.2}{1-0.9z^{-1}}$$

$$\therefore h[n] = 0.2 \times 0.9^n u[n]$$

(since the given filter is causal)

ii) $H(z) = \frac{0.2}{1-0.9z^{-1}} = \frac{0.2z}{z-0.9}$

So, it has pole at 0.9 and zero at 0

All poles is inside the unit circle, so the filter is stable.

iii) There's various ways to determine y[n], but the easiest one is to exploit the fact that x[n] is composed of 3 Eigen functions of the LTI system:

$$x[n] = 1 + 2\cos\left(\frac{\pi}{8}n\right) = e^{j0n} + e^{j\frac{\pi}{8}n} + e^{j\frac{-\pi}{8}n}$$

When each of them go through the LTI system, only magnitude and phase change

$$y[n] = \frac{0.2}{1-0.9} + e^{j\frac{\pi}{8}n} \times \frac{0.2}{1-0.9e^{\frac{-j\pi}{8}}} + e^{j\frac{-\pi}{8}n} \times \frac{0.2}{1-0.9e^{\frac{j\pi}{8}}}$$

$$= 2 + e^{\frac{j\pi}{8}n} \times (0.229 - j0.469) + e^{j\frac{-\pi}{8}n} \times (0.229 + j0.469)$$

$$= 2 + 0.458\cos\left(\frac{\pi}{8}n\right) + 0.938\sin\left(\frac{\pi}{8}n\right)$$

3.

a) i) According to Nyquist theorem, smallest possible sampling frequency is: $F_s = 2 \times \frac{\Omega_2}{2\pi} = \frac{\Omega_2}{\pi}$

ii) Frequency will be shifted by $(\Omega_1 + \Omega_2)/2$, so the new Nyquist rate is $2 \times \frac{\Omega_2 - \Omega_1}{2\times 2\pi} = \frac{\Omega_2 - \Omega_1}{2\pi}$

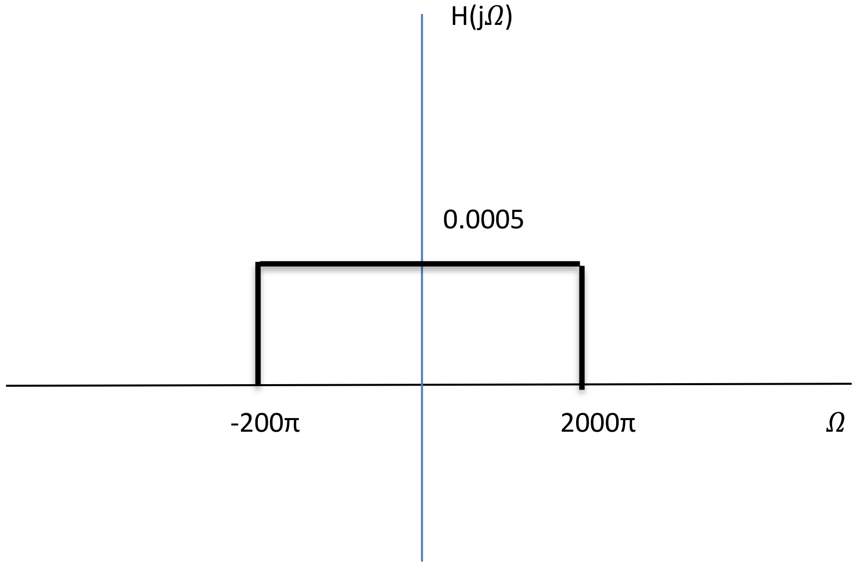
b) Fs = 2kHz, so Ts = 0.0005s

$x[n] = x_a(nT_s) = \sin\left(\frac{\pi}{2}n\right)$

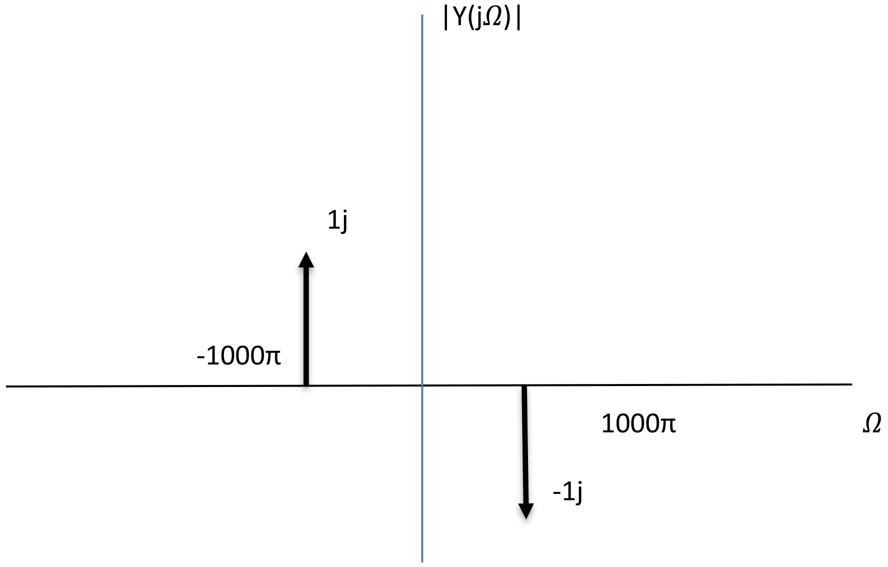
H = 2 means that every frequency is amplified by 2, or the entire signal is amplified by 2

So, $y[n] = 2\sin\left(\frac{\pi}{2}n\right)$

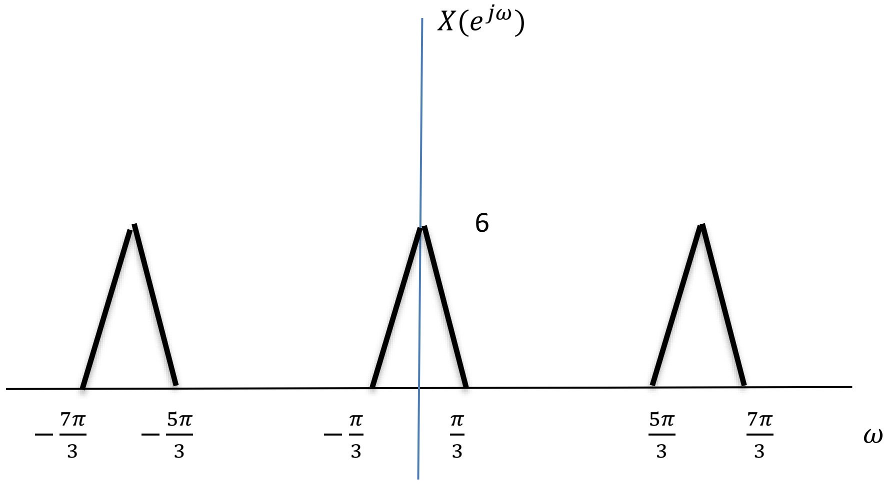
The frequency response of the ideal reconstruction filter in D/C:

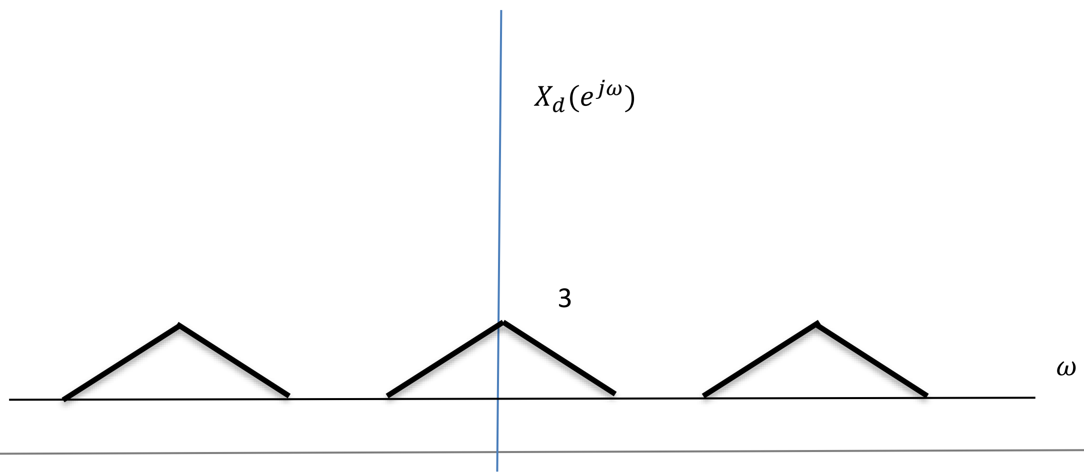


Fourier transform of $y_a(t)$:



c)





$-\frac{8\pi}{3}$ $-\frac{4\pi}{3}$ $-\frac{2\pi}{3}$ $\frac{2\pi}{3}$ $\frac{4\pi}{3}$ $\frac{8\pi}{3}$

4.

a) i) Substitute $s = j\Omega$ and $z = e^{j\omega}$ into the transformation function, we have:

$$j\Omega = \frac{1 + e^{-j\omega}}{1 - e^{-j\omega}}$$

$$\therefore j\Omega = \frac{e^{\frac{j\omega}{2}} + e^{\frac{-j\omega}{2}}}{e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}}$$

$$\therefore j\Omega = \frac{2\cos\left(\frac{\omega}{2}\right)}{2jsin\left(\frac{\omega}{2}\right)}$$

$$\therefore \Omega = \frac{-1}{\tan\left(\frac{\omega}{2}\right)}$$

We can see that Ω and ω change inversely, i.e. when Ω increases, ω decreases and vice versa (in terms of absolute value)

ii) Using the above transformation, we can easily determine the specification of the continuous-time low pass filter:

$$|H(\Omega)| \le 0.01 \qquad for\ |\Omega| \ge 1.73$$

$$0.95 \le |H(\Omega)| \le 1.05 \qquad for\ |\Omega| \le 1$$

b) i) 2 main characteristics of linear phase FIR filter:

- Phase response of the filter change linearly according to the frequency
- The impulse response is either symmetric or anti-symmetric

ii) 2 main advantages of FIR filters:

- FIR filter is always stable
- FIR filter does not require feedback as in IIR. This results in simpler implementation and less compounded rounding error