19^{th} CSEC – Past Year Paper Solution 2018-2019 Sem 2
CE / CZ 1011 – Engineering Mathematics I

1) a) We note that

$$\cos\theta + j\sin\theta = e^{j\theta}$$

$$\sin\theta + j\cos\theta = j(\cos\theta - j\sin\theta) = je^{-j\theta}$$

Hence, we have

$$\frac{(\cos\theta + j\sin\theta)^4}{(\sin\theta + j\cos\theta)^5} = \frac{e^{j(4\theta)}}{j^5 e^{(-j)(5\theta)}} = \frac{e^{j(9\theta)}}{j} = \frac{-j^2}{j}e^{j(9\theta)}$$

$$= -je^{j(9\theta)}$$

$$= -j(\cos 9\theta + j\sin 9\theta)$$

$$= \sin 9\theta - j\cos 9\theta$$

b) Since z_1, z_2 are roots of $z^2 + az + b = 0$, we have

sum of roots $= z_1 + z_2 = -a$ and product of roots $= z_1 z_2 = b$

Given that O, z_1, z_2 forms an equilateral triangle, let $p_1 = 0, p_2 = z_1, p_3 = z_2$

$$p_1{}^2 + p_2{}^2 + p_3{}^2 = p_1 p_2 + p_2 p_3 + p_1 p_3$$

$$z_1{}^2 + z_2{}^2 = z_1 z_2$$

$$z_1{}^2 + z_2{}^2 + 2z_1 z_2 = 3z_1 z_2$$

$$(z_1 + z_2)^2 = 3z_1 z_2$$

$$(-a)^2 = 3b$$

$$a^2 = 3b \text{ (shown)}$$

c) Given that $proj_{\mathbf{u}}\mathbf{v} = proj_{\mathbf{u}}\mathbf{w}$,

$$\frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{\mathbf{w}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}$$

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$$

Given that **v** is orthogonal to **w**, we have $\mathbf{v}\cdot\mathbf{w} = 0$

Consider

$$\|\mathbf{u} - \mathbf{v} + \mathbf{w}\|^2 = (\mathbf{u} - \mathbf{v} + \mathbf{w})(\mathbf{u} - \mathbf{v} + \mathbf{w})$$

$$= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u} - \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w}$$

$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{w} - 2\mathbf{v}\cdot\mathbf{w}$$

$$= 1^2 + 2^2 + 3^2 - 2(0) \text{ since } \mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$$

$$= 14$$

$$\therefore \|\mathbf{u} - \mathbf{v} + \mathbf{w}\| = \sqrt{14}$$

d) Given **a** makes equal angles with **b** and **c**, let θ_1 and θ_2 be the angles between **a**, **b** and **a**, **c** respectively. We have

$$\theta_1 = \theta_2$$
$$\cos\theta_1 = \cos\theta_2$$
$$\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{\mathbf{a}\cdot\mathbf{c}}{\|\mathbf{a}\|\|\mathbf{c}\|}$$
$$\frac{\langle x,y,z\rangle\cdot\langle y,-2z,3x\rangle}{\|\mathbf{a}\|\sqrt{y^2+4z^2+9x^2}} = \frac{\langle x,y,z\rangle\cdot\langle 2z,3x,-y\rangle}{\|\mathbf{a}\|\sqrt{4z^2+9x^2+y^2}}$$
$$xy-2yz+3xz = 2xz+3xy-yz$$
$$2xy+yz-xz=0 \qquad \text{--- ①}$$

Given **a** is perpendicular to **d**, we have $\mathbf{a}\cdot\mathbf{d}=0 \Rightarrow x-y+2z=0$

We thus have $y = x + 2z$ --- ②

Given $\|\mathbf{a}\| = 2\sqrt{3} \Rightarrow x^2+y^2+z^2=12$ --- ③

Substitute ② into ①,

$$2xy+yz-xz=0$$
$$2x(x+2z)+(x+2z)z-xz=0$$
$$2x^2+4xz+2z^2=0$$
$$(x+z)^2=0 \Rightarrow x=-z \qquad \text{--- ④}$$

Substitute ②, ④ into ③,

$$x^2+y^2+z^2=12$$
$$x^2+(x+2(-x))^2+(-x)^2=12$$
$$3x^2=12 \Rightarrow x=\pm 2$$

It is clear from ②, ④ that **a** is $<-2,2,2>$ or $<2,-2,-2>$. However, since the angle between **a** and **j** is obtuse, we require $y < 0$. Thus, **a** = $<2,-2,-2>$.

2 a) Observe $T\begin{pmatrix}1\\0\end{pmatrix} = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\end{pmatrix}$. Thus, the projection matrix $A=\begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$

(Alternatively, use the standard matrix of the projection T of a vector **v** on a line l making an angle θ with the x-axis, $A=\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$.)

Now $T\begin{pmatrix}-1\\-1\end{pmatrix} = A\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$

b) i) $$\underbrace{\begin{pmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}}_{\mathbf{b}}$$

ii) $$\begin{pmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 2 \end{pmatrix} \xrightarrow[\mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_1]{\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

We are done and we have $U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

iii) *Editor's note:* The technique here is called LU decomposition. The submitted answer has been modified to follow LU decomposition algorithm.

From $L\mathbf{y} = \mathbf{b}$, we obtain

$$\mathbf{y} = L^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$

From $\mathbf{y} = U\mathbf{x}$, we obtain

$$\mathbf{x} = U^{-1}\mathbf{y} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

c)
$$\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right)$$
$$\xrightarrow{r_3 \to r_3 - \frac{5}{2} r_1} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{array}\right)$$
$$\xrightarrow{r_3 \to r_3 + \frac{1}{2} r_2} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{array}\right)$$

From the last row, we note that the system of equations is **inconsistent.**

d) **True**, as seen below.

$$A^2 - B^2 = (A+B)(A-B)$$
$$= A^2 - B^2 + BA - AB$$
$$AB = BA$$

3) a) i) We first arrange the data in ascending order.

5	24	25	28	29	30	30	30	30	32
32	35	35	35	39	40	42	44	48	50

1st quartile: rank $R = \frac{1}{4}(20+1) = 5.25 \Rightarrow I_R = 5,\ F_R = 0.25$

$$1^{\text{st}} \text{ quartile} = \text{data at rank } I_R + (\text{data at rank } I_{R+1} - I_R) \times F_R$$
$$= 29 + (30-29)(0.25) = 29.25$$

Median: rank $R = \frac{1}{2}(20+1) = 10.5 \Rightarrow I_R = 10,\ F_R = 0.5$

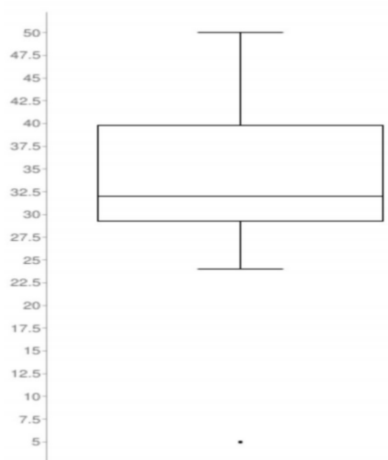
$$\text{median} = \text{data at rank } I_R + (\text{data at rank } I_{R+1} - I_R) \times F_R$$
$$= 32 + (32-32)(0.5) = 32$$

3rd quartile: rank $R = \frac{3}{4}(20+1) = 15.75 \Rightarrow I_R = 15,\ F_R = 0.75$

$$3^{\text{rd}} \text{ quartile} = \text{data at rank } I_R + (\text{data at rank } I_{R+1} - I_R) \times F_R$$
$$= 39 + (40-39)(0.75) = 39.75$$

Mean: $\bar{x} = \frac{\sum x}{n} = \frac{5+29+\cdots+50}{20} = 33.15$

Variance: $\sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = 91.0275$

ii) 

Editor's note: The answer here may be incomplete as the box plot lack details (e.g. important values).

b) i) $$\begin{aligned} P(Y>0 \mid X=+5) &= P(X+N>0 \mid X=5) \\ &= P(N>-5) \text{ where } N \sim N(0, 2.149^2) \\ &= P\left(Z > \frac{-5}{2.149}\right) \\ &= P(Z<2.33) \\ &= 0.990 \end{aligned}$$

ii) probability that device makes wrong decision

$$\begin{aligned} &= 0.7P(N \le -5) + 0.3P(N>5) \\ &= 0.7P(Z \le -2.33) + 0.3\text{P}(Z>2.33) \\ &= 0.7(1-0.99)+0.3(1-0.99) \\ &= 0.007+0.003 \\ &= 0.01 \end{aligned}$$

iii) Notice $K \sim B(n, p)$ and let $\lambda = np \Rightarrow p = \dfrac{n}{\lambda}$. We have

$$\begin{aligned} P(K=k) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!}\frac{n!}{(n-k)!n^k}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k} \end{aligned}$$

As $n \to \infty$, it is easy to see that

- $\dfrac{n!}{(n-k)!n^k} \to 1$ (notice that n! is polynomial of degree n, $(n-k)$! is a polynomial of degree $n - k$ and that n^k is a polynomial of degree k)
- $\left(1-\dfrac{\lambda}{n}\right)^n \to e^{-\lambda}$
- $\left(1-\dfrac{\lambda}{n}\right)^{-k} \to 1$

Hence, as $n \to \infty$, $P(K=k) = \dfrac{\lambda^k}{k!}\dfrac{n!}{(n-k)!n^k}\left(1-\dfrac{\lambda}{n}\right)^n\left(1-\dfrac{\lambda}{n}\right)^{-k} \to \dfrac{\lambda^k}{k!}e^{-\lambda}$

which is the probability distribution of $Po(\lambda)$.

4 a) i) After 6 hours, $F(6) = 1 - e^{-0.2(6)} = 1 - e^{-1.2} = 0.6988$

Let X be the number of packets out of 80 that turn sour after 6 hours

We have $X \sim B(80, 0.6988)$

It is then straightforward to compute the expectation and variance using known properties of the binomial distribution as follows

$$\mathrm{E}(X) = np = 80(0.6988) = 55.9$$

$$\mathrm{Var}(X) = np(1-p) = 80(0.6988)(1-0.6988) = 16.8$$

ii)

$$P(X > 50) = P\left(Z > \frac{50.5-56}{\sqrt{16.8}}\right) = P(Z > -1.34) = 0.9901$$

b) i) $\mathrm{H}_0 : \mu \geq 85$, $\mathrm{H}_\mathrm{A} : \mu < 85$

test statistic $z = \dfrac{\bar{x}-\mu}{s/\sqrt{n}} = \dfrac{82-85}{12/\sqrt{100}} = -2.5$

p-value $= P(Z \leq -2.5) = 0.0052$

As $\alpha = 0.01$, p-value < α, so there is sufficient evidence to reject H_0.

ii) The confidence interval is $\left[82 - 1.96\times\dfrac{12}{10}, 82 + 1.96\times\dfrac{12}{10}\right] = [79.648, 84.352]$

c) $\sum x = 1 + 2 + 3 + 4 = 10$

$\sum x^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

$\sum y = 1.1 + 1.4 + 2.1 + 2.5 = 7.1$

$\sum xy = 1(1.1) + 2(1.4) + 3(2.1) + 4(2.5) = 20.2$

Now

$$\begin{bmatrix} \sum x^2 & \sum x \\ \sum x & n \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} \sum xy \\ \sum y \end{bmatrix}$$

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 20.2 \\ 7.1 \end{bmatrix}$$

$$\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 20.2 \\ 7.1 \end{bmatrix}$$

$$\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 4/20 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}\begin{bmatrix} 20.2 \\ 7.1 \end{bmatrix} = \begin{bmatrix} 0.49 \\ 0.55 \end{bmatrix}$$

Hence the line $y = 0.49x + 0.55$

--End of Answers--

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