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1.

a)

i) Explicit:

$$y = \pm\sqrt{r^2 - (x - x_o)^2} + y_o \rightarrow y = \pm\sqrt{9 - (x-1)^2} + 2$$

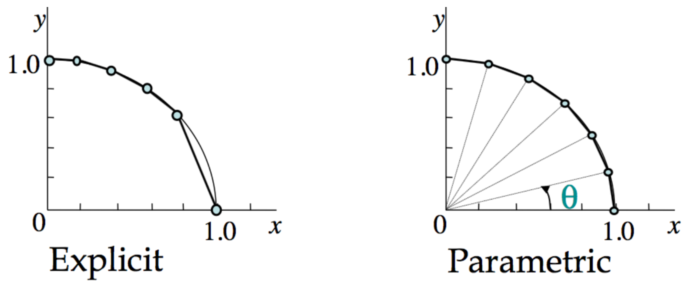
Implicit:

$$r^2 - (x - x_o)^2 - (y - y_o)^2 = 0 \rightarrow 9 - (x-1)^2 - (y-2)^2 = 0$$

Parametric:

$$\begin{cases} x = r\cos(2\pi u) + x_o \\ y = r\sin(2\pi u) + y_o \end{cases} \rightarrow \begin{cases} x = 3\cos(2\pi u) + 1 \\ y = 3\sin(2\pi u) + 2 \end{cases}, \quad u \in [0,1]$$

ii) Parametric representation. Circle drawn using parametric representation (particularly, the one formulated above*) will have its line segments uniform in length. This is due to the length of the circle line segment being proportional to the increment of the parameter (i.e. angle of circle arc). The picture below illustrates the difference between circle drawn using parametric representation and using explicit representation (taken from lecture slides):



* If the parametric representation used is a trivial one (i.e. one parameter assigned to x and substitute x in the explicit representation $y = f(x)$ to obtain y), the result would not be different than using explicit representation.

b)

i) $x = r\cos\alpha = (1 + \cos\alpha)\cos\alpha = \cos\alpha + \cos^2\alpha$

$y = r\sin\alpha = (1+\cos\alpha)\sin\alpha = \sin\alpha + \cos\alpha\sin\alpha$

Map α to u,

$\alpha \in [0,\pi], u \in [0,1] \rightarrow \alpha = 0 + u(\pi - 0) = \pi u$

$$\therefore \begin{cases} x = \cos(\pi u) + \cos^2(\pi u) \\ y = \sin(\pi u) + \cos(\pi u)\sin(\pi u) \end{cases}, \quad u \in [0,1]$$

ii) $r = \sqrt{x^2 + y^2}$

$= \sqrt{(\cos^2(2\pi u))^2 + (\cos(2\pi u)\sin(2\pi u))^2}$

$= \sqrt{\cos^4(2\pi u) + \cos^2(2\pi u)\sin^2(2\pi u)}$

$= \sqrt{\cos^2(2\pi u)\,(\cos^2(2\pi u) + \sin^2(2\pi u))}$

$= \sqrt{\cos^2(2\pi u)} \qquad (\because \cos^2\theta + \sin^2\theta = 1)$

$= \cos(2\pi u)$

Let $\alpha \in [0, 2\pi]$,

$\alpha \in [0, 2\pi], u \in [0, 1] \rightarrow \ \alpha = 0 + u(2\pi - 0) = 2\pi u$

$\therefore r = \cos(\alpha), \qquad \alpha \in [0, 2\pi]$

c) Let u be the parameter to draw the curve, and v be the parameter to rotate the curve.

Define the curve (pay attention to the domain x):

$x \in [-2, 2], u \in [0, 1] \rightarrow \ x = -2 + u\big(2 - (-2)\big) = 4u - 2$

$$\begin{cases} x = 4u - 2 \\ y = (4u-2)^2 - 1 \end{cases}, \qquad u \in [0, 1]$$

Rotate the curve around y-axis:

$$\begin{cases} x = (4u-2)\sin(2\pi v) \\ y = (4u-2)^2 - 1 \\ z = (4u-2)\cos(2\pi v) \end{cases}, \qquad u, v \in [0, 1]$$

2.

a) $\boldsymbol{N} \cdot (\boldsymbol{r} - \boldsymbol{r_o}) = 0$

$[3 \quad 4 \quad 5] \cdot [x-1 \quad y-2 \quad z-3] = 0$

$3(x-1) + 4(y-2) + 5(z-3) = 0$

$3x + 4y + 5z - 26 = 0$

b) Cylinder:

- Side: $0.2^2 - y^2 - z^2 \geq 0 \ \rightarrow \ 0.04 - y^2 - z^2 \geq 0$
- Left: $x \leq -0.9 \ \rightarrow \ -0.9 - x \geq 0$
- Right: $x \geq 0.7 \ \rightarrow \ x - 0.7 \geq 0$
- Final: (Side \ Left) \ Right

  $min(min(0.04 - y^2 - z^2, x + 0.9), 0.7 - x) \geq 0$

Hemisphere:

- Sphere: $0.7^2 - (x-0.7)^2 - y^2 - z^2 \geq 0 \ \rightarrow \ 0.49 - (x-0.7)^2 - y^2 - z^2 \geq 0$
- Top: $y \geq 0$
- Final: Sphere \ Top

  $min(0.49 - (x-0.7)^2 - y^2 - z^2, -y) \geq 0$

Final: Cylinder ∩ Hemisphere

$max(min(min(0.04 - y^2 - z^2, x + 0.9), 0.7 - x), min(0.49 - (x - 0.7)^2 - y^2 - z^2, -y)) \geq 0$

c) Let u be the parameter to draw the square side, v be the parameter to translate the side (forming a square), and w be the parameter to rotate the square.

Define the square side:

$x \in [1, 1.6], u \in [0, 1] \rightarrow x = 1 + u(1.6 - 1) = 0.6u + 1$

Translate the square side along y-axis:

$y \in [0, 0.6], v \in [0, 1] \rightarrow y = 0 + v(0.6 - 0) = 0.6v$

$$\begin{cases} x = 0.6u + 1 \\ y = 0.6v \end{cases}, \quad u, v \in [0, 1]$$

Rotate the square around y-axis (pay attention to the angle of rotation):

$$\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], w \in [0, 1] \rightarrow \theta = \frac{\pi}{2} + w\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) = \pi w + 0.5\pi$$

$$\begin{cases} x = (0.6u + 1)\sin(\pi w + 0.5\pi) \\ y = 0.6v \\ z = (0.6u + 1)\cos(\pi w + 0.5\pi) \end{cases}, \quad u, v, w \in [0, 1]$$

However, notice that the square side moves towards y-positive as it rotates. Initially ($w = 0$), the y-value is translated from 0 to 0.6, but at the end of the rotation ($w = 1$), the y-value is translated from 1 to 1.6. Therefore, the correct translation of the square side is:

$y \in [0 + w, 0.6 + w], v \in [0, 1] \rightarrow y = w + v(0.6 + w - w) = w + 0.6v$

$$\therefore \begin{cases} x = (0.6u + 1)\sin(\pi w + 0.5\pi) \\ y = w + 0.6v \\ z = (0.6u + 1)\cos(\pi w + 0.5\pi) \end{cases}, \quad u, v, w \in [0, 1]$$

3.

a)

i) $(0, 1, -2) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \div (-2) = \begin{bmatrix} 0 \\ -0.5 \\ 1 \end{bmatrix} \rightarrow (0, -0.5)$

$(2, 2, 1) \rightarrow (2, 2)$

$(2, 5, 0.5) = \begin{bmatrix} 2 \\ 5 \\ 0.5 \end{bmatrix} \div 0.5 = \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix} \rightarrow (4, 10)$

ii) $\begin{bmatrix} \cos 40° & \sin 40° & 0 \\ -\sin 40° & \cos 40° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-40°) & -\sin(-40°) & 0 \\ \sin(-40°) & \cos(-40°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta)$

The transformation matrix defines a rotation about the origin by $-40°$ counterclockwise (or 40° clockwise).

b) Shape B:

$$4(x+1)^2 + y^2 - 1 = 0 \ \rightarrow \ 1 - \frac{(x+1)^2}{0.25} - y^2 = 0$$

Shape B is an ellipse centered at $(-1,0)$ with $a = 0.5$ and $b = 1$.

Parametric representation:

$$\begin{cases} x = a\cos(2\pi u) + x_o \\ y = b\sin(2\pi u) + y_o \end{cases} \rightarrow \begin{cases} x = 0.5\cos(2\pi u) - 1 \\ y = \sin(2\pi u) \end{cases}, \qquad u \in [0,1]$$

Transformation:

$$\begin{cases} x = A_x(1-\tau) + B_x\tau \\ y = A_y(1-\tau) + B_y\tau \end{cases} \rightarrow \begin{cases} x = u\cos(2\pi u)\,(1-\tau) + (0.5\cos(2\pi u) - 1)\tau \\ y = (u+2)\sin(2\pi u)\,(1-\tau) + \sin(2\pi u)\,\tau \end{cases}, \qquad u,\tau \in [0,1]$$

Define τ:

$$\begin{aligned} \tau &= 1 - \cos\left(\frac{\pi}{2}\cdot\frac{k-1}{m-1}\right) \\ &= 1 - \cos\left(\frac{\pi}{2}\cdot\frac{k-1}{49-1}\right) \\ &= 1 - \cos\left(\frac{\pi(k-1)}{96}\right), \qquad 1 \le k \le 49, k \in \mathbb{N} \end{aligned}$$

c)

i) Steps:

1) Align vector $[-1 \quad 0 \quad \sqrt{3}]$ to z-axis

$a = -1, b = 0, c = \sqrt{3}$

$\lambda = \sqrt{b^2 + c^2} = \sqrt{3}$

$|V| = \sqrt{a^2 + b^2 + c^2} = 2$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{\lambda} & -\frac{b}{\lambda} & 0 \\ 0 & \frac{b}{\lambda} & \frac{c}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(The resulting matrix is an identity matrix because vector $[-1 \quad 0 \quad \sqrt{3}]$ is already in xz plane, hence can be disregarded)

$$R_y(-\varphi) = \begin{bmatrix} \frac{\lambda}{|V|} & 0 & -\frac{a}{|V|} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{|V|} & 0 & \frac{\lambda}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore R_y(-\varphi)R_x(\theta) = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Reflect about z-axis

$$\mathrm{Ref}_z = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3) Inverse transform step 1

$$R_y'(-\varphi) = \begin{bmatrix} \frac{\lambda}{|V|} & 0 & \frac{a}{|V|} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{a}{|V|} & 0 & \frac{\lambda}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_x'(\theta)$ can be disregarded

$$\therefore R_x'(\theta)R_y'(-\varphi) = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) Translate $[0 \quad 0 \quad 1]$ to origin

$$T(0 \quad 0 \quad -1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5) Uniform scale by a factor of 6

$$S(6 \quad 6 \quad 6) = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6) Inverse transform step 4

$$T(0 \quad 0 \quad 1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Final transformation matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii) Transformation matrix:

$$\begin{bmatrix} -3 & 0 & -3\sqrt{3} & 0 \\ 0 & -6 & 0 & 0 \\ -3\sqrt{3} & 0 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(1,0,0) \rightarrow \begin{bmatrix} -3 & 0 & -3\sqrt{3} & 0 \\ 0 & -6 & 0 & 0 \\ -3\sqrt{3} & 0 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -3\sqrt{3}-5 \\ 1 \end{bmatrix} \rightarrow (-3, 0, -10.196)$$

$$(0,1,0) \rightarrow \begin{bmatrix} -3 & 0 & -3\sqrt{3} & 0 \\ 0 & -6 & 0 & 0 \\ -3\sqrt{3} & 0 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -5 \\ 1 \end{bmatrix} \rightarrow (0, -6, -5)$$

$$(0,0,1) \rightarrow \begin{bmatrix} -3 & 0 & -3\sqrt{3} & 0 \\ 0 & -6 & 0 & 0 \\ -3\sqrt{3} & 0 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3\sqrt{3} \\ 0 \\ -2 \\ 1 \end{bmatrix} \rightarrow (-5.196, 0, -2)$$

(Rounded to 3 decimal places)

4.

a)

i) Providing the ambient light intensity is enough to define an ambient light source.

ii) The intensity of ambient reflection will not change if the viewer moves. Moving the viewer changes the viewing vector, but the viewing vector does not affect the intensity of ambient reflection.

b) Map the corners of the surface to the image (to help visualize the orientation of the texture):

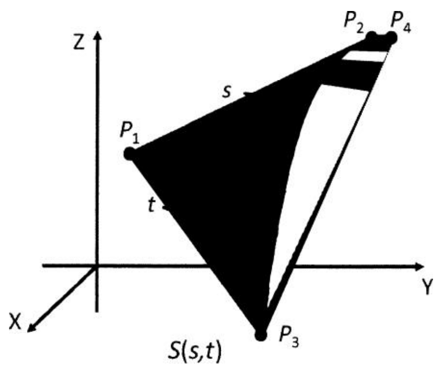
$P_1: s = 0, t = 0 \rightarrow u = 0, v = 0 \rightarrow (0, 0)$ on the image

$P_2: s = 1, t = 0 \rightarrow u = 1, v = 0 \rightarrow (1, 0)$ on the image

$P_3: s = 0, t = 1 \rightarrow u = 0, v = 1 \rightarrow (0, 1)$ on the image

$P_4: s = 1, t = 1 \rightarrow u = 1, v = 1 \rightarrow (1, 1)$ on the image

Result:



c) $I_s = 1$

$\boldsymbol{P} = [-30 \quad 41 \quad 0]$

$k_d = 0.8$

$k_s = 0.2$

$n = 2$

$\boldsymbol{S} = [0 \quad 1 \quad 0]$

$\boldsymbol{Q} = [12 \quad 10 \quad 0]$

$$\hat{\boldsymbol{L}} = \frac{\boldsymbol{P} - \boldsymbol{S}}{|\boldsymbol{P} - \boldsymbol{S}|} = \frac{[-30 \quad 40 \quad 0]}{\sqrt{(-30)^2 + 40^2 + 0^2}} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \end{bmatrix}$$

$$\widehat{\boldsymbol{V}} = \frac{\boldsymbol{Q} - \boldsymbol{S}}{|\boldsymbol{Q} - \boldsymbol{S}|} = \frac{[12 \quad 9 \quad 0]}{\sqrt{12^2 + 9^2 + 0^2}} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \end{bmatrix}$$

$\boldsymbol{N} = [0 \quad 1 \quad 1]$ (the light source is located at y-positive, hence $+[0 \quad 1 \quad 1]$ is used)

$$\widehat{\boldsymbol{N}} = \frac{\boldsymbol{N}}{|\boldsymbol{N}|} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\widehat{\boldsymbol{R}} = 2(\widehat{\boldsymbol{N}} \cdot \hat{\boldsymbol{L}})\widehat{\boldsymbol{N}} - \hat{\boldsymbol{L}}$$

$$= 2\left(0 + \frac{2\sqrt{2}}{5} + 0\right)\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} - \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{4}{5} & \frac{4}{5} \end{bmatrix} - \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

i) $I_d = k_d I_s(\widehat{\boldsymbol{N}} \cdot \hat{\boldsymbol{L}})$

$$= 0.8 \cdot 1 \cdot \left(0 + \frac{2\sqrt{2}}{5} + 0\right)$$

$= 0.453$ (Rounded to 3 decimal places)

ii) $I_s = k_s I_s(\widehat{\boldsymbol{V}} \cdot \widehat{\boldsymbol{R}})^n$

$$= 0.2 \cdot 1 \cdot \left(\frac{12}{25} + 0 + 0\right)^2$$

$= 0.046$ (Rounded to 3 decimal places)