IV. THEORY

From Chapter 7.7: Single Variable Calculus (Book)

Approximate integral can be based on Riemann sum. Given an integral range of [a,b], we can divide it into n subintervals of equal length Δx = (b – a)/n, then we have

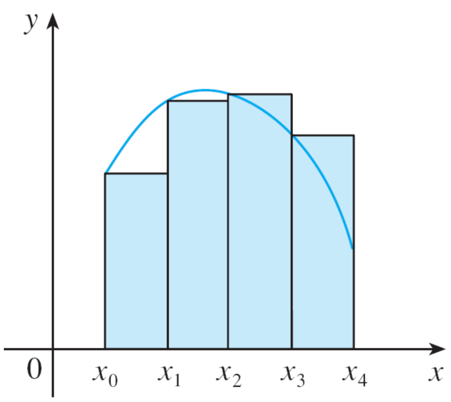
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is any point in the i^{th} subinterval $[x_{i-1},x_i]$. If x_i^* is chosen to be the left endpoint of the interval, then we have

$$\boxed{1} \qquad \int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

which is called "left endpoint approximation"

If f(x) ≥0, then Eq. (1) represents an approximation of the area by the rectangles as shown in the following figure.

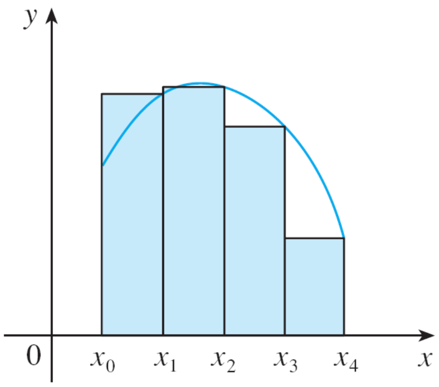


Left endpoint approximation

If we choose x_i^* to be the right endpoint, then $x_i^* = x_i$ and we have

$$\boxed{2} \qquad \int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

which is called "right endpoint approximation". The following figure shows this approximation.



Right endpoint approximation

We can also have the case where x_i* is chosen to be the midpoint of the subinterval $[x_{i-1}, x_i]$. The following figure shows the midpoint approximation, which appears to be better than either left or right endpoint approximations. The midpoint approximation is formally defined as follows:

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $$\Delta x = \frac{b-a}{n}$$

and $$\bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

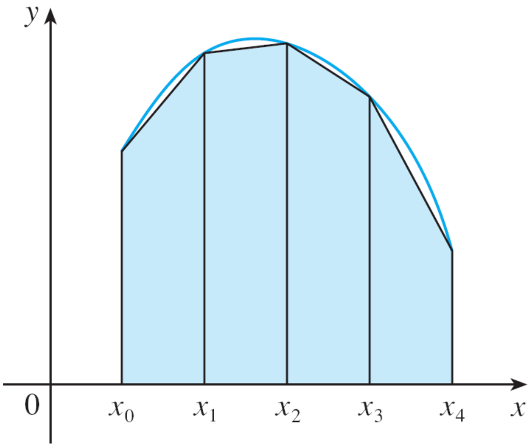
Another approximation, called the Trapezoidal Rule, results from averaging the approximations in Equations (1) and (2). The definition is given as follows:

Trapezoidal Rule

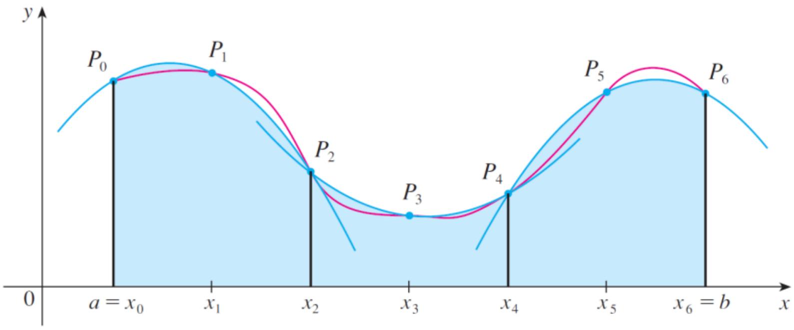
$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\,\Delta x$.

The following figure shows the Trapezoidal Rule.



Another alternative for approximate integration is by using parabolas instead of straight line segments to approximate a curve. We divide $[a,b]$ into n subintervals of equal length $h = \Delta x = (b-a)/n$, where n is an even number. Then on each consecutive pair of intervals we approximate the curve by a parabola as shown in the following figure.



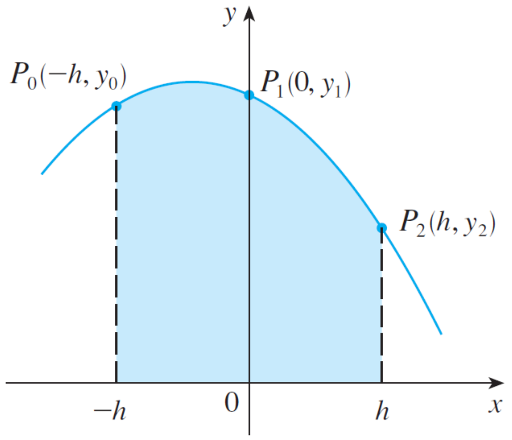
If $y_i = f(x_i)$, then $P_i(x_i,y_i)$ is the point on the curve lying above x_i. A typical parabola passes through three consecutive points P_i, $P_{i+1}$and P_{i+2}. Thomas Simpson (1710–1761), the English mathematician, proposed a reasonable approximation for any continuous function and is called Simpson's Rule.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n is even and $\Delta x = (b - a)/n$.

The main concept of the Simpson's rule is as follows. We first consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ as shown in the figure below.



The parabola through P_0, P_1, and P_2 can be expressed as $y = Ax^2 + Bx + C$, and the area under the parabola from $x = -h$ to $x = h$ is

$$\int_{-h}^{h} (Ax^2 + Bx + C)\, dx = 2\int_{0}^{h} (Ax^2 + C)\, dx$$

$$= 2\left[A\frac{x^3}{3} + Cx\right]_0^h$$

$$= 2\left(A\frac{h^3}{3} + Ch\right) = \frac{h}{3}(2Ah^2 + 6C)$$

The parabola passes through $P_0(-h,y_0)$, $P_1(0,y_1)$, and $P_2(h,y_2)$, we have

$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$

$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

and, as a result, we have $y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$.

The area under the parabola is expressed as follows:

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Let us consider the next points, P_2, P_3, and P_4 from $x = x_2$ to $x = x_4$.
Following the same step, the areas under all the parabolas can be obtained from

$$\boxed{\begin{aligned}\int_a^b f(x)\, dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) \\ &\quad + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)\end{aligned}}$$

The similar result is then applicable to any continuous function.

V. EXERCISES

1. ***Midpoint rule:*** Write a program using the midpoint rule to determine

$$\int_0^2 \frac{x}{1 + x^2}\, dx,$$

for n as given in the table. Compare it with an analytical solution (i.e., with an explicit integral).

Code:

```
p = [5:5:50]

Lower = 0
Upper = 2
ans = 0
irange = upper - lower

for i = 1:10
```

```
sum = 0
subinterval = irange/p(i)

x = [lower:subinterval:upper]

for j = 2:length(x)
x_mid=(x(j-1)+x(j)) / 2
y = (x_mid/(1+x_mid^2)) * subinterval

sum = sum+y
end

ans = [ans, sum]
end

vpa(ans,7)
```

n	Midpoint rule
5	0.8123942
10	0.8065977
15	0.8055509
20	0.8051864
25	0.8050179
30	0.8049265
35	0.8048714
40	0.8048357
45	0.8048112
50	0.8047936

2. ***Trapezoidal rule***: Repeat Exercise 1 using trapezoidal rule.

Code:

```
p = [5:5:50]

lower = 0
upper = 2
ans2 = 0
irange = upper-lower

for i = 1:10
subinterval = irange/p(i)

x = [lower:subinterval:upper]

y = x./(1+x.^2)

ans1 = trapz(x,y)
```

```
ans2 = [ans2, ans1]
end

vpa(ans2,6)
```

n	Trapezoidal rule
5	0.789550
10	0.800972
15	0.803057
20	0.803785
25	0.804121
30	0.804304
35	0.804414
40	0.804486
45	0.804535
50	0.804570

3. ***Simpson's rule***: Repeat Exercise 1 using Simpson's rule.

Code:

```
N = [10:10:50]
A = 0
B = 2
lrange = 0
ans2 = 0

for j = 1:5
h=(b-a)/n(j)

sum=0
for i = 1:n(j)-1
x = lrange+i*h

if mod(i,2) == 0
```

```
sum=sum+2*(x/(1+x^2))
else
sum=sum+4*(x/(1+x^2))
end

end

ans1 = (h/3)*((a/(1+a^2))+(b/(1+b^2))+sum)
ans2 = [ans2, ans1]
end

vpa(ans2,7)
```

n	Simpson’s rule
10	0.80467
20	0.80465
30	0.80466
40	0.80463
50	0.80469