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1. (a)

A process is wide sense stationary iff:

i. Mean is constant
ii. Autocorrelation is a function of only time difference

Proof of i:

$$E[y(t)] = 3E[x(t)] + 2$$
$$\because E[x(t)] = 0$$
$$\therefore E[y(t)] = 2$$

The mean of the random process *y(t)* is a constant

Proof of ii:

$$\begin{aligned} R_y &= E[y(t)y(t+\tau)] \\ &= E[(3x(t)+2)(3x(t+\tau)+2)] \\ &= 9E[x(t)x(t+\tau)] + 6E[x(t)] + 6E[x(t+\tau)] + 4 \\ &= 9R_x(\tau) + 4 \\ &= 45e^{-3|\tau|} + 4 \end{aligned}$$

The autocorrelation of *y(t)* is a function of only the time difference

Thus it can be concluded that the process *y* is WSS.

(b)

(i)

By definition, a function's autocorrelation and power spectral density forms a FT pair. As such, the power spectral density can be obtained by applying Fourier transform on its autocorrelation function.

PSD of x(t):

$$\begin{aligned} S_m(f) &= F[80sinc(80\tau)] \\ &= rect(\frac{f}{80}) \end{aligned}$$
$$S_n(f) = \frac{N_0}{2}$$
$$\begin{aligned} \therefore S_x(f) &= S_m(f) + S_n(f) \\ &= rect(\frac{f}{80}) + \frac{N_0}{2} \end{aligned}$$

The autocorrelation function of x(t) can then be obtained by applying Inverse Fourier transform to its PSD:

$$\begin{aligned} R_x(\tau) &= F^{-1}[rect(\frac{f}{80}) + \frac{N_0}{2}] \\ &= 80sinc(80\tau) + \frac{N_0}{2}\delta(\tau) \end{aligned}$$

(ii)

The PSD of the filtered output y(t) is:

$$\begin{aligned} S_y(f) &= S_x(f) \cdot |H(f)|^2 \\ &= [rect(\frac{f}{80}) + \frac{N_0}{2}] \cdot rect(\frac{f}{100}) \\ &= rect(\frac{f}{80}) + \frac{N_0}{2} rect(\frac{f}{100}) \end{aligned}$$

Thus, the signal power is:

$$\begin{aligned} P_{out} &= \int_{-50}^{50} S_y(f) df \\ &= \int_{-50}^{50} rect(\frac{f}{80}) + \int_{-50}^{50} \frac{N_0}{2} rect(\frac{f}{100}) \\ &= 80 + 50N_0 \end{aligned}$$

The first term of the signal power corresponds to the average signal power and the second term corresponds to the average noise power. Thus the SNR in dB at filter output is:

$$10 \log_{10} \frac{80}{50N_0} = 2.04 - 10 \log_{10} N_0$$

2. (a)

(i)

When the signal is corrupted by an additive Gaussian white noise of zero mean and PSD of $N_0/2$, the probability density function of received signal will be a normal distribution with variance of $N_0/2$ and mean of bit energy without noise.

Energy of bit "1":

$$\begin{aligned} E_1 &= \int_0^T \sqrt{\frac{2E}{T}} \cos(2\pi f_c \tau) \cdot \sqrt{\frac{2}{T}} cos(2\pi f_c \tau) d\tau \\ &= \frac{2\sqrt{E}}{T} \int_0^T \cos^2(2\pi f_c \tau) d\tau \\ &= \frac{2\sqrt{E}}{T} \int_0^T \frac{\cos(4\pi f_c \tau) + 1}{2} d\tau \\ &= \frac{2\sqrt{E}}{T} \frac{1}{2} \tau \Big|_0^T \\ &= \sqrt{E} \end{aligned}$$

Energy of bit "0":

$$\begin{aligned} E_0 &= \int_0^T -\sqrt{\frac{E}{3T}} \cos(2\pi f_c \tau) \cdot \sqrt{\frac{2}{T}} cos(2\pi f_c \tau) d\tau \\ &= -\frac{1}{T} \sqrt{\frac{2E}{3}} \int_0^T \cos^2(2\pi f_c \tau) d\tau \\ &= -\frac{1}{T} \sqrt{\frac{2E}{3}} \int_0^T \frac{\cos(4\pi f_c \tau) + 1}{2} d\tau \\ &= -\frac{1}{T} \sqrt{\frac{2E}{3}} \frac{1}{2} \tau \Big|_0^T \\ &= -\frac{\sqrt{6E}}{6} \end{aligned}$$

Plugging in the numbers to the PDF formula, we can get:

$$f_1(n) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(n-\sqrt{E})^2}{N_0}}$$
$$f_0(n) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(n+\frac{\sqrt{6E}}{6})^2}{N_0}}$$

(ii)

The optimal detection threshold should be the average signal energy.

Since the transmitted bits are equally likely, the average signal energy is:

$[\sqrt{E} + -(\frac{\sqrt{6E}}{6})]/2 = \frac{6-\sqrt{6}}{12}\sqrt{E}$, hence, this is the optimal detection threshold.

(b)

Bipolar RZ code

(c)

16 level pulse = 4 bits per symbol

Symbol rate R_s = bit rate / bits per symbol = 6400bps / 4 = 1600 symbols per second

Nyquist transmission bandwidth = $R_s/2$ = 800 Hz

3. (a)

Based on the input–output relations, n = 8 and k = 4. Write out the original generator matrix described by the input-output relations:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To convert this generator matrix into a systematic generator matrix, only the 3 types of elementary row operations are allowed. The 4 rows are referred to as R1 to R4.

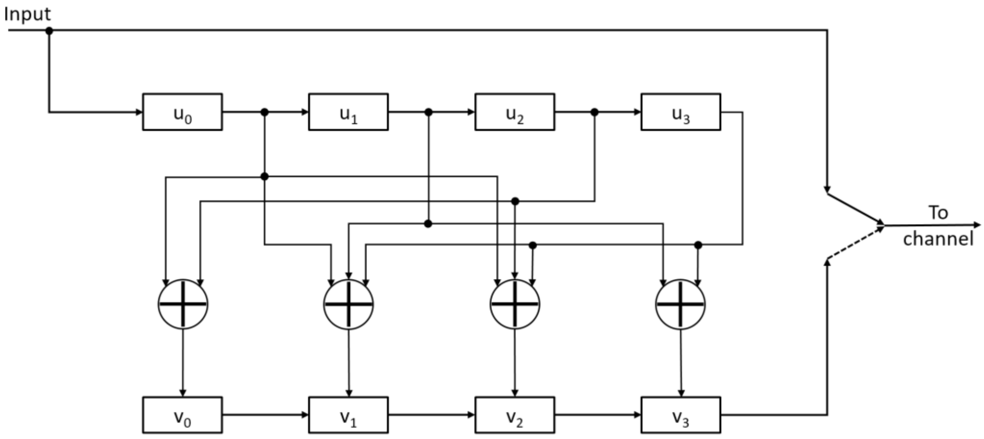
$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R3 \leftarrow R3+R4]{R1 \leftarrow R1+R2} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This systematic generator matrix is in the form of $[P \mid I_k]$. Thus, its corresponding parity check matrix is in the form of $[I_{n-k} \mid P^T]$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(b)



(c)

Message	Block code
0000	00000000
0001	01110001
0010	10100010
0011	11010011
0100	01010100
0101	00100101
0110	11110110
0111	10000111
1000	11101000
1001	10011001
1010	01001010
1011	00111011
1100	10111100
1101	11001101
1110	00011110
1111	01101111

The minimum distance of the code is the minimum weight of all block codes, excluding the all 0 code. Thus, the minimum distance of the code is 3.

(d)

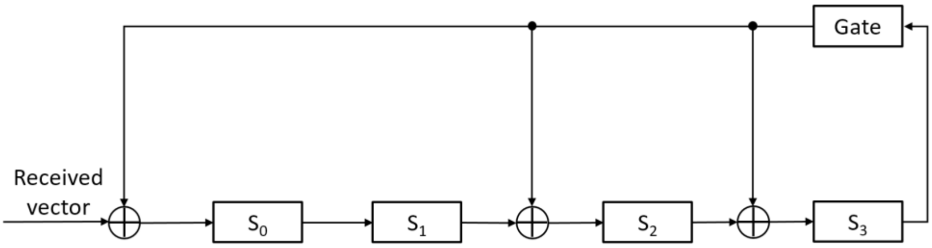
By definition, syndrome s = r*H^T, and all single-bit error patterns are just individual columns in parity check matrix.

Correctable error pattern	Syndrome
00000000	0000
10000000	1000
01000000	0100
00100000	0010
00010000	0001
00001000	1110
00000100	0101
00000010	1010
00000001	0111

4. (a)

(i)

Syndrome length = 7 – 3 = 4 bits

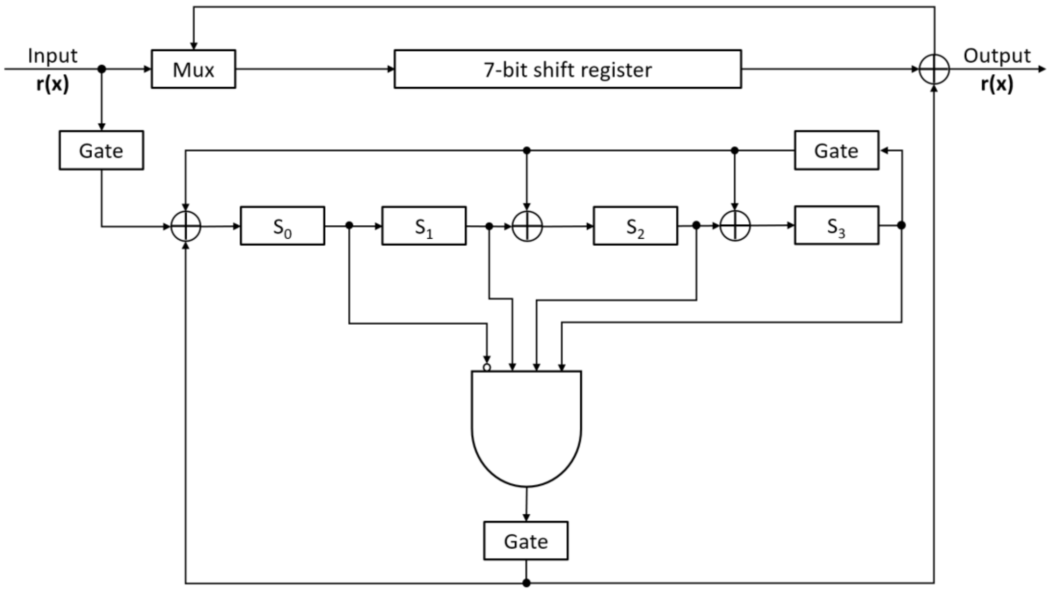


(ii)

To construct a Meggitt decoder, we first need to find the syndrome corresponding to the error in the highest power bit position. In this case, the highest bit position is x^6. The syndrome will then be the remainder of x^6 dividing the generator polynomial:

$$
\begin{array}{r|llllll}
 & x^2 & +x & & & & \\
\hline
x^4+x^3+x^2+1 & x^6 & & & & & \\
 & x^6 & +x^5 & +x^4 & & +x^2 & \\
\hline
 & & x^5 & +x^4 & & +x^2 & \\
 & & x^5 & +x^4 & +x^3 & & +x \\
\hline
 & & & & x^3 & +x^2 & +x
\end{array}
$$

Rearrange the remainder from lower power to higher power to convert to binary number (syndrome): 0111. A Meggitt decoder can then be constructed based on this syndrome:



(b)

No, this code cannot be considered as Huffman code as Huffman code requires the 2 longest code word to be of the same length and differs only by the last bit. Obviously, this code doesn't satisfy the requirement.

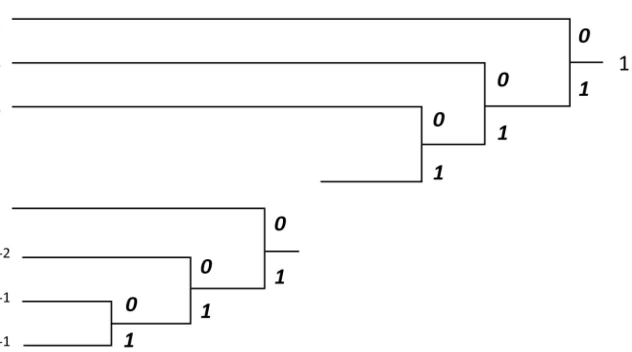
(c)

Source entropy:

$$
\begin{aligned}
H(X) &= -\sum p(x)\log_2 p(x) \\
&= -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \cdots - \frac{1}{2^{n-2}}\log_2\frac{1}{2^{n-2}} - 2\times\frac{1}{2^{n-1}}\log_2\frac{1}{2^{n-1}} \\
&= \sum_{i=1}^{n-1}\frac{1}{2^i}\log_2 2^i + \frac{1}{2^{n-1}}\log_2 2^{n-1} \\
&= \sum_{i=1}^{n-1}\frac{i}{2^i} + \frac{n-1}{2^{n-1}}
\end{aligned}
$$

Huffman coding scheme:

Code word length	Code word	Probability of symbols
1	0	1/2
2	10	1/4
3	110	1/8
.	.	.
.	.	.
.	.	.
n-2	11…10	$1/2^{n-2}$
n-1	111…10	$1/2^{n-1}$
n-1	111…11	$1/2^{n-1}$



Thus, average code word length:

$$
\begin{aligned}
L_{avg} &= \sum p(i)l(i) \\
&= \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \cdots + \frac{1}{2^n-2}\times(n-2) + 2\times\frac{1}{2^n-1}\times(n-1) \\
&= \sum_{i=1}^{n-1}\frac{i}{2^i} + \frac{n-1}{2^{n-1}}
\end{aligned}
$$

It is clear that the code word length of the binary Huffman code equals its source entropy.

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Thank you and all the best for your exams! ☺