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1.

a) $\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x}$ (L'Hôpital's rule)

$$= \frac{1}{e^{\infty}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

b) $\lim_{x\to 0} xe^{-x} = 0 \cdot e^{-0}$

$$= 0$$

c) Using the limit definition of a derivative,

$$\lim_{h\to 0}\frac{\sqrt{\sin(x+h)-\cos(x+h)+5}-\sqrt{\sin(x)-\cos(x)+5}}{h}$$

$$= \frac{d}{dx}(\sqrt{\sin(x)-\cos(x)+5})$$

$$= \frac{1}{2}(\sin(x)-\cos(x)+5)^{-\frac{1}{2}} \cdot \frac{d}{dx}(\sin(x)-\cos(x)+5) \quad \text{(Chain rule)}$$

$$= \frac{1}{2}(\sin(x)-\cos(x)+5)^{-\frac{1}{2}} \cdot (\cos(x)+\sin(x))$$

$$= \frac{\cos(x)+\sin(x)}{2\sqrt{\sin(x)-\cos(x)+5}}$$

d) $$\frac{dy}{dx} = \underbrace{-\sin(x)\sin(x)+\cos(x)\cos(x)}_{\text{(Product Rule)}} + \underbrace{\cos\left(\sqrt{x^2+1}\right)\cdot\frac{d}{dx}\left(\sqrt{x^2+1}\right)}_{\text{(Chain Rule)}}$$

$$= -\sin^2(x)+\cos^2(x)+\cos\left(\sqrt{x^2+1}\right)\cdot\underbrace{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot\frac{d}{dx}(x^2+1)}_{\text{(Chain Rule)}}$$

$$= \cos(2x)+\cos\left(\sqrt{x^2+1}\right)\cdot\frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}\cdot\cancel{2}x$$

$$= \cos(2x)+\frac{x\cos\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$$

2.

a) Using Implicit Differentiation,

$$\underbrace{y^2 + x \cdot 2y\frac{dy}{dx}}_{\text{(Product Rule)}} + 6y\frac{dy}{dx} = \underbrace{e^{xy} \cdot \frac{d}{dx}(xy)}_{\text{(Chain Rule)}}$$

$$y^2 + 2xy\frac{dy}{dx} + 6y\frac{dy}{dx} = e^{xy} \cdot (y + x\frac{dy}{dx})$$

$$2xy\frac{dy}{dx} + 6y\frac{dy}{dx} - xe^{xy}\frac{dy}{dx} = ye^{xy} - y^2$$

$$(2xy + 6y - xe^{xy})\frac{dy}{dx} = ye^{xy} - y^2$$

$$\frac{dy}{dx} = \frac{ye^{xy} - y^2}{2xy + 6y - xe^{xy}}$$

b) $\int(x + \sin(x) - \cos(x) + e^x - \frac{1}{x})\,dx = \frac{1}{2}x^2 - \cos(x) - \sin(x) + e^x - \ln|x| + C$

c) $\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$

$$= \int_0^3 \lfloor x \rfloor\,dx + \int_3^5 x\ln(x^2)\,dx$$

$$= (2-1)\cdot 1 + (3-2)\cdot 2 + \left[\frac{x^2\,(\ln(x^2) - 1)}{2}\right]_3^5$$

$$= 1 + 2 + \tfrac{5^2}{2}\cdot(\ln(5^2) - 1) - \tfrac{3^2}{2}\cdot(\ln(3^2) - 1)$$

$$= 3 + \frac{25}{2}\ln 25 - \frac{25}{2} - \frac{9}{2}\ln 9 + \frac{9}{2}$$

$$= -5 + \frac{25}{2}\ln 25 - \frac{9}{2}\ln 9$$

≈ 25.348 (Rounded to 3 decimal places)

Notes:

- To integrate $x\ln(x^2)$, use Integration by Parts.

  Let $u = \ln(x^2)$ and $dv = x\,dx$,

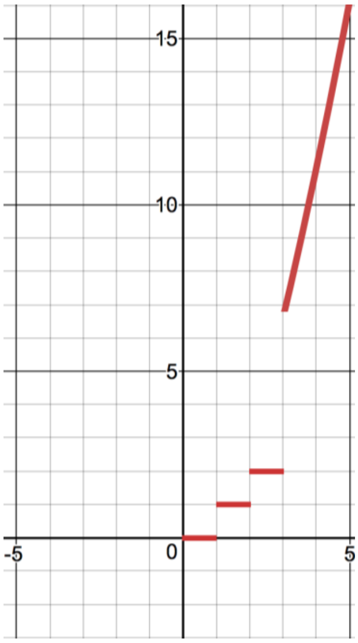
  Then, $du = \frac{2}{x}dx$ and $v = \frac{x^2}{2}$ (Hint: Use Chain Rule to differentiate $\ln(x^2)$)

$$\int x\ln(x^2)\,dx = \ln(x^2)\cdot\frac{x^2}{2} - \int \frac{x^{\cancel{2}}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{x}}dx$$

$$= \frac{x^2}{2}\ln(x^2) - \frac{x^2}{2}$$

$$= \frac{x^2\,(\ln(x^2) - 1)}{2}$$

- Graph of $f(x)$:



d) $y'' + 2y' + 1 = 0 \rightarrow$ Non-homogenous 2nd order linear ODE

i. Related homogenous equation: $y'' + 2y' = 0$

Characteristic equation: $r^2 + 2r = 0$

$r(r+2) = 0$

$r_1 = 0, r_2 = -2$

Solution of homogenous equation: $y(x) = C_1 + C_2e^{-2x}$

ii. $y'' + 2y' = -1 \rightarrow$ RHS is a constant

Look for particular solution in the form $y_p(x) = ax$

$(ax)'' + 2(ax)' = -1$

$2a = -1$

$a = -\dfrac{1}{2}$

Complete solution: $y(x) = C_1 + C_2e^{-2x} - \frac{1}{2}x$

iii. Plug in initial value conditions:

$y(0) = 0 \rightarrow C_1 + C_2e^0 - \frac{1}{2} \cdot 0 = 0 \rightarrow C_1 + C_2 = 0$

$y'(0) = 1 \rightarrow -2C_2e^0 - \frac{1}{2} = 1 \rightarrow -2C_2 - \frac{1}{2} = 1$

$C_2 = -\dfrac{3}{4}, C_1 = \dfrac{3}{4}$

$\therefore y(x) = \dfrac{3}{4} - \dfrac{3}{4}e^{-2x} - \dfrac{1}{2}x$

3.

a) $a_n = (-1)^n(\frac{(-1)^n \cdot n + 2^n}{n})$

$$= (-1)^{2n} + \frac{(-2)^n}{n}$$

$= 1 + \frac{(-2)^n}{n}$ ($\because$ $(-1)^{2n} = 1$ for n any positive integer)

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2\ln(n)}{\ln(3n)}$

$= \lim_{n\to\infty} \frac{\frac{2}{n}}{\frac{3}{3n}}$ (L'Hôpital's rule)

$$= \lim_{n\to\infty} \frac{2}{\cancel{n}} \cdot \frac{\cancel{3n}}{\cancel{3}}$$

$$= 2$$

Therefore, the sequence converges and $\lim_{n\to\infty} a_n = 2$

c) $-1 \leq \cos(n) \leq 1$ (By definition)

$$-\frac{1}{1+\sqrt{2n}} \leq \frac{\cos(n)}{1+\sqrt{2n}} \leq \frac{1}{1+\sqrt{2n}}$$

Since $\lim_{n\to\infty} -\frac{1}{1+\sqrt{2n}} = -\frac{1}{1+\sqrt{\infty}} = 0$ and $\lim_{n\to\infty} \frac{1}{1+\sqrt{2n}} = \frac{1}{1+\sqrt{\infty}} = 0$,

By Sandwich Theorem, $\lim_{n\to\infty} \frac{\cos(n)}{1+\sqrt{2n}} = 0$

Therefore, the sequence converges and $\lim_{n\to\infty} a_n = 0$

d) Rewrite the series in the form of geometric series.

$$\sum_{n=1}^{\infty}(2x-1)^n = \sum_{n=1}^{\infty}(2x-1)\cdot(2x-1)^{n-1}$$

which is a geometric series with $a = r = 2x - 1$.

For a geometric series to converge,

$|r| < 1$

$|2x - 1| < 1$

$-1 < 2x - 1 < 1$

$0 < x < 1$

For these values of x, the sum of the series is:

$$\frac{a}{1-r} = \frac{2x-1}{1-2x+1} = \frac{2x-1}{2-2x}$$

e) Using Comparison Test,

$$\frac{1}{\sqrt{5n^3+5}} \leq \frac{1}{\sqrt{5n^3}}$$

which is a constant $\left(\frac{1}{\sqrt{5}}\right)$ times p-series, with $p = \frac{3}{2} > 1$

Therefore, by Comparison Test, the series converges.

f)

i) $a = \frac{54{,}007 - 38{,}576}{2012 - 2009} = \frac{15{,}431}{3}$

$38{,}576 = \frac{15{,}431}{3} \cdot 2009 + b$

$\left(or\ 54{,}007 = \frac{15{,}431}{3} \cdot 2012 + b\right)$

$b = -\frac{30{,}885{,}151}{3}$

$\therefore y = \frac{15{,}431}{3} x - \frac{30{,}885{,}151}{3}$

$(= 5{,}143.667x - 10{,}295{,}050.333)$ (Rounded to 3 decimal places)

ii) $e(a, b) = (38{,}576 - 2009a - b)^2 + (46{,}569 - 2010a - b)^2 + (52{,}870 - 2011a - b)^2 + (54{,}007 - 2012a - b)^2$

$$\frac{\partial e(a,b)}{\partial a} = 2 \cdot (38{,}576 - 2009a - b) \cdot (-2009) + 2 \cdot (46{,}569 - 2010a - b) \cdot (-2010) + 2 \cdot (52{,}870 - 2011a - b) \cdot (-2011) + 2 \cdot (54{,}007 - 2012a - b) \cdot (-2012)$$

$$= -154{,}998{,}368 + 8{,}072{,}162a + 4{,}018b - 187{,}207{,}380 + 8{,}080{,}200a + 4{,}020b - 212{,}643{,}140 + 8{,}088{,}242a + 4{,}022b - 217{,}324{,}168 + 8{,}096{,}288a + 4{,}024b$$

$$= -772{,}173{,}056 + 32{,}336{,}892a + 16{,}084b \qquad \cdots (i)$$

(Hint: Use Chain Rule, treat b as a constant)

$$\frac{\partial e(a,b)}{\partial b} = 2 \cdot (38{,}576 - 2009a - b) \cdot (-1) + 2 \cdot (46{,}569 - 2010a - b) \cdot (-1) + 2 \cdot (52{,}870 - 2011a - b) \cdot (-1) + 2 \cdot (54{,}007 - 2012a - b) \cdot (-1)$$

$$= -77{,}152 + 4{,}018a + 2b - 93{,}138 + 4{,}020a + 2b - 105{,}740 + 4{,}022a + 2b - 108{,}014 + 4{,}024a + 2b$$

$$= -384{,}044 + 16{,}084a + 8b \qquad \cdots (ii)$$

(Hint: Use Chain Rule, treat a as a constant)

From (i) and (ii),

$a = 5{,}259.4$, $b = -10{,}526{,}018.2$

$\therefore y = 5{,}259.4x - -10{,}526{,}018.2$

4.

a) $f(x) = 5^{2x}$ $\qquad f(0) = 1$

$f'(x) = 5^{2x} \cdot \ln 5 \cdot 2$ $\qquad f'(0) = 2\ln 5$

$f''(x) = 5^{2x} \cdot (\ln 5 \cdot 2)^2$ $\qquad f''(0) = (2\ln 5)^2$

$\vdots$

$f^{(n)}(x) = 5^{2x} \cdot (\ln 5 \cdot 2)^n$ $\qquad f^{(n)}(0) = (2\ln 5)^n$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(2\ln 5)^n}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(2x\ln 5)^n}{n!}$$

Using Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x\ln 5)^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{(2x\ln 5)^n}}\right|$$

$$= \left|\frac{2x\ln 5}{n+1}\right| \to 0 \text{ as } n \to \infty$$

The series converges for all x, hence $R = \infty$

b) $\Delta x = \frac{b-a}{n} = \frac{2-0}{5} = 0.4$

$x_i = a + i \cdot \Delta x = 0.4i$

$$T_5 = \frac{\Delta x}{2} \cdot [f(x_0) + 2 \cdot (f(x_1) + f(x_2) + f(x_3) + f(x_4)) + f(x_5)]$$

$$= \frac{0.4}{2} \cdot [f(0) + 2 \cdot (f(0.4) + f(0.8) + f(1.2) + f(1.6)) + f(2)]$$

$= 0.704$ (Rounded to 3 decimal places)

$$\int_0^2 \sin(2x)\cos(x)\,dx = \int_0^2 2\sin(x)\cos(x) \cdot \cos(x)\,dx$$

$$= \int_0^2 2\sin(x)\cos^2(x)\,dx$$

Let $u = \cos(x)$, then $du = -\sin(x)\,dx$

$$\int_0^2 2\sin(x)\cos^2(x)\,dx = \int_{\cos(0)}^{\cos(2)} -2u^2\,du$$

$$= \left[-\frac{2}{3}u^3\right]_{\cos(0)}^{\cos(2)}$$

$$= -\frac{2}{3}\cos^3(2) + \frac{2}{3}\cos^3(0)$$

$= 0.715$ (Rounded to 3 decimal places)

The Trapezoidal Rule underestimates the actual integration.

c) The frequency of the components of the signal are:

$$\omega_{01} = \frac{5\pi}{3} \qquad \omega_{02} = \frac{10\pi}{4} \qquad \omega_{03} = \frac{3\pi}{2}$$

The fundamental frequency of the signal is:

$$\omega_0 = GCD\left(\frac{5\pi}{3}, \frac{10\pi}{4}, \frac{3\pi}{2}\right) = \frac{\pi}{6}$$

d)

$$f(t) = 2\cos(t\pi)\,[u(t+2) - u(t-2)] = \begin{cases} 2\cos(t\pi)\ , & -2 \le t \le 2 \\ 0\ , & |t| > 2 \end{cases}$$

$$F(\omega) = \int_{-2}^{2} 2\cos(t\pi)\, e^{-j\omega t}\, dt$$

$$= \left[\frac{2e^{-j\omega t}(j\omega\cos(\pi t) - \pi\sin(\pi t))}{\omega^2 - \pi^2}\right]_{-2}^{2}$$

$$= \frac{2e^{-j2\omega}}{\omega^2 - \pi^2}(j\omega\cos(2\pi) - \cancel{\pi\sin(2\pi)}) - \frac{2e^{j2\omega}}{\omega^2 - \pi^2}(j\omega\cos(-2\pi) - \cancel{\pi\sin(-2\pi)})$$

$$= \frac{j2\omega e^{-j2\omega}}{\omega^2 - \pi^2} - \frac{j2\omega e^{j2\omega}}{\omega^2 - \pi^2}$$

$$= \frac{j2\omega}{\omega^2 - \pi^2}\left(e^{-j2\omega} - e^{j2\omega}\right)$$

$$= \frac{j2\omega \cdot (-1) \cdot j2}{\omega^2 - \pi^2}\left(\frac{e^{j2\omega} - e^{-j2\omega}}{j2}\right)$$

$$= \frac{-j^2 4\omega}{\omega^2 - \pi^2} \cdot \sin(2\omega)$$

$$= \frac{4\omega\sin(2\omega)}{\omega^2 - \pi^2}$$

Steps to integrate $2\cos(t\pi)\, e^{-j\omega t}$:

Using Integration by Parts, let $u = \cos(\pi t)$ and $dv = 2e^{-j\omega t}\, dt$,

Then, $du = -\pi\sin(\pi t)\, dt$ and $v = \frac{2}{-j\omega}e^{-j\omega t} = \frac{j2}{\omega}e^{-j\omega t}$

$$\int 2\cos(t\pi)\, e^{-j\omega t}\, dt = \cos(\pi t) \cdot \frac{j2}{\omega}e^{-j\omega t} - \int \frac{j2}{\omega}e^{-j\omega t} \cdot -\pi\sin(\pi t)\, dt$$

$$= \frac{j2}{\omega}\cos(\pi t)\, e^{-j\omega t} + \int \frac{j2\pi}{\omega}\sin(\pi t)\, e^{-j\omega t}\, dt$$

Again, using Integration by Parts, let $u = \sin(\pi t)$ and $dv = \frac{j2\pi}{\omega} e^{-j\omega t}\, dt$,

Then, $du = \pi\cos(\pi t)\, dt$ and $v = \frac{j2\pi}{\omega\cdot(-j\omega)} e^{-j\omega t} = -\frac{2\pi}{\omega^2} e^{-j\omega t}$

$$\int 2\cos(t\pi)\, e^{-j\omega t}\, dt = \frac{j2}{\omega}\cos(\pi t)\, e^{-j\omega t} + \sin(\pi t)\cdot\left(-\frac{2\pi}{\omega^2} e^{-j\omega t}\right) - \int -\frac{2\pi}{\omega^2} e^{-j\omega t}\cdot \pi\cos(\pi t)\, dt$$

$$= \frac{j2}{\omega}\cos(\pi t)\, e^{-j\omega t} - \frac{2\pi}{\omega^2}\sin(\pi t)\, e^{-j\omega t} + \int \frac{2\pi^2}{\omega^2}\cos(\pi t)\, e^{-j\omega t}\, dt$$

Move $\int \frac{2\pi^2}{\omega^2}\cos(\pi t)\, e^{-j\omega t}\, dt$ to LHS

$$\int 2\cos(t\pi)\, e^{-j\omega t}\, dt - \int \frac{2\pi^2}{\omega^2}\cos(\pi t)\, e^{-j\omega t}\, dt = \frac{j2}{\omega}\cos(\pi t)\, e^{-j\omega t} - \frac{2\pi}{\omega^2}\sin(\pi t)\, e^{-j\omega t}$$

$$\int 2\left(1 - \frac{\pi^2}{\omega^2}\right)\cos(t\pi)\, e^{-j\omega t}\, dt = \frac{j2}{\omega}\cos(\pi t)\, e^{-j\omega t} - \frac{2\pi}{\omega^2}\sin(\pi t)\, e^{-j\omega t}$$

$$\int 2\left(\frac{\omega^2 - \pi^2}{\omega^2}\right)\cos(t\pi)\, e^{-j\omega t}\, dt = \frac{j2}{\omega}\cos(\pi t)\, e^{-j\omega t} - \frac{2\pi}{\omega^2}\sin(\pi t)\, e^{-j\omega t}$$

$$\int 2\cos(t\pi)\, e^{-j\omega t}\, dt = \frac{\omega^2}{\omega^2 - \pi^2}\left(\frac{j2}{\omega}\cos(\pi t)\, e^{-j\omega t} - \frac{2\pi}{\omega^2}\sin(\pi t)\, e^{-j\omega t}\right)$$

$$= \frac{2e^{-j\omega t}(j\omega\cos(\pi t) - \pi\sin(\pi t))}{\omega^2 - \pi^2}$$

Note:

The answers, whenever possible, have been checked for correctness using Wolfram|Alpha and Matlab. Still, do apologize if any errors occur within the solution. Should you have any questions, please do not hesitate to contact me via email at AKUSUMA002@e.ntu.edu.sg.

Wishing you all the best for your exams!