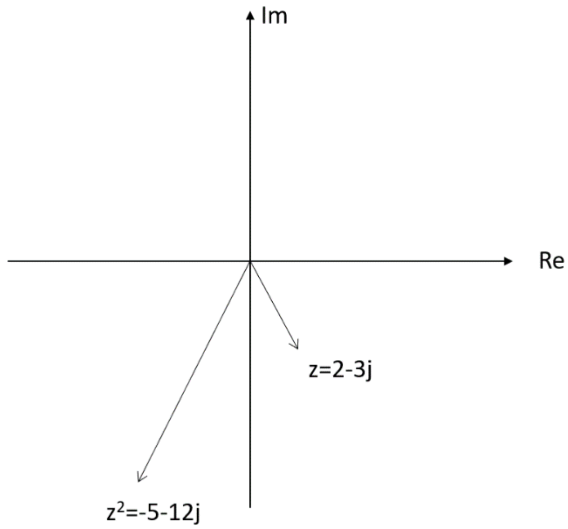
1)

a) $z = 2 - j3 \Rightarrow z^2 = (2 - j3)^2 = 4 - 2(2)(j3) + j^2 9 = -5 - j12$

Argand diagram:



Notice that z is the 4th Quadrant and z^2 is the 3rd Quadrant.

$\arg z = \tan^{-1}\left(-\frac{3}{2}\right) \approx -0.983$, $\arg(z^2) = 2\arg z = -1.966$

Therefore the angle between them is $-0.983 - (-1.966) = 0.983\ rad$

b) Notice that z is in the 2nd Quadrant

$$z = -1 + j \Rightarrow \begin{cases} |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \\ \arg z = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4} \end{cases}$$

Let $z_1^3 = z$, we have:

$$\begin{cases} |z_1^3| = |z_1|^3 = 2^{\frac{1}{2}} \\ \arg(z_1^3) = 3\arg z_1 = -\frac{\pi}{4} + 2k\pi (k \in Z) \end{cases} \Rightarrow \begin{cases} |z_1| = 2^{\frac{1}{6}} \\ \arg z_1 = \frac{1}{3}\left(-\frac{\pi}{4} + 2k\pi\right) (k \in Z) \end{cases}$$

We know that $-\pi < \arg z_1 \leq \pi$

$$\Rightarrow -\pi < \frac{1}{3}\left(-\frac{\pi}{4} + 2k\pi\right) \leq \pi$$

$$\Rightarrow \frac{-11}{8} < k \leq \frac{13}{8} \Rightarrow k = -1,0,1$$

$$\Rightarrow \arg z_1 = \frac{1}{3}\left(-\frac{\pi}{4} - 2\pi\right), \frac{1}{3}\left(-\frac{\pi}{4}\right), \frac{1}{3}\left(-\frac{\pi}{4} + 2\pi\right) = -\frac{3\pi}{4}, -\frac{\pi}{12}, \frac{7\pi}{12}$$

$$\Rightarrow z_1 = 2^{\frac{1}{6}} e^{-\frac{j3\pi}{4}}, 2^{\frac{1}{6}} e^{-\frac{j\pi}{12}}, 2^{\frac{1}{6}} e^{\frac{j7\pi}{12}}$$

c) $\boldsymbol{u} \cdot \overline{\boldsymbol{v}} = (2 + j3, 4 - j, 2j) \cdot (3 + j2, 5, 4 + j6)$

$= (2 + j3)(3 + j2) + (4 - j)(5) + (2j)(4 + j6)$

$= 6 + j13 + j^2 6 + 20 - j5 + j8 + j^2 12$

$= 8 + j16$

$\Rightarrow |\boldsymbol{u} \cdot \overline{\boldsymbol{v}}| = \sqrt{8^2 + 16^2} = 8\sqrt{5}$

d) $(6\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-2\boldsymbol{b}) = 6\boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{a}-12\boldsymbol{a}\cdot\boldsymbol{b}-2\boldsymbol{b}\cdot\boldsymbol{b}$
$= 6|\boldsymbol{a}|^2 - 11|\boldsymbol{a}||\boldsymbol{b}|\cos\frac{\pi}{3} - 2|\boldsymbol{b}|^2$
$= 6(1) - 11(1)(1)\left(\frac{1}{2}\right) - 2(1) = -\frac{3}{2}$

e) Let $\boldsymbol{v} = (v_1, v_2, \dots, v_n) \Rightarrow k\boldsymbol{v} = (kv_1, kv_2, \dots, kv_n)$
$\Rightarrow ||k\boldsymbol{v}|| = \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2}$
$= |k|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
$= |k|||\boldsymbol{v}||$

2)

a) Notice that $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I$
Assume that there exist a matrix S such that $A = S^{-1}BS$
$LHS = A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$
$RHS = S^{-1}BS = S^{-1}(2I)S = 2S^{-1}IS = 2S^{-1}S = 2I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
$LHS \neq RHS$
⇒ The assumption is wrong, i.e., there is no such S.

b) $$\begin{bmatrix} 0 & 7 & 9 & 0 \\ 2 & 1 & -1 & 0 \\ 5 & 6 & 2 & 0 \end{bmatrix} \xRightarrow[r_2 \leftarrow \frac{1}{2}r_2]{r_1 \leftarrow \frac{1}{7}r_1} \begin{bmatrix} 0 & 1 & \frac{9}{7} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 5 & 6 & 2 & 0 \end{bmatrix} \xRightarrow{r_3 \leftarrow r_3 - 5r_2} \begin{bmatrix} 0 & 1 & \frac{9}{7} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{7}{2} & \frac{9}{2} & 0 \end{bmatrix} \xRightarrow{r_3 \leftarrow r_3 - \frac{7}{2}r_1} \begin{bmatrix} 0 & 1 & \frac{9}{7} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y + \frac{9}{7}z = 0 \\ x + \frac{1}{2}y - \frac{1}{2}z = 0 \\ 0 = 0 \end{cases}$$

Let $z = 7t$, we have $y + 9t = 0 \Rightarrow y = 9t$
Substitute back to r_2: $x - \frac{9}{2}t - \frac{7}{2}t = 0 \Rightarrow x = 8t$

$$\Rightarrow \begin{cases} x = 8t \\ y = -9t, t \in R \\ z = 7t \end{cases}$$

c) From the statement of the question, we know that: $b_{ij} = ka_{ij} \forall j \in \{1,2,\dots,n\}$
where a_{ij} is an i^th row element of A and b_{ij} is an i^th row element of B
To calculate $|A|$, we perform cofactor expansion along the i^th row:

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} \frac{1}{k} b_{ij}C_{ij} = \frac{1}{k}\sum_{j=1}^{n} b_{ij}C_{ij} = \frac{1}{k}|B|$$

since expect those in the i^th row, all elements in A is equal to the elements in B.

d) False. The cofactor of element a_{ij} of a matrix A is the **determinant** of the matrix A_{ij} obtained by deleting from A its i^th row and j^th column.

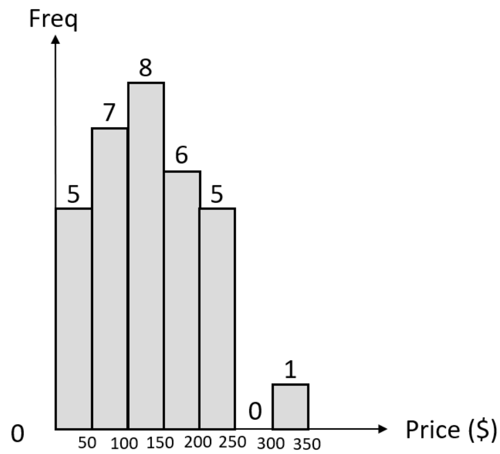
3)

a)

i) From the Cumulative Frequency table we can obtain the Frequency table

Price ($)	[0,50)	[50,100)	[100,150)	[150,200)	[200,250)	[250,300)	[300,350)
Frequency	5	7	8	6	5	0	1

The histogram is as follows:



ii) $Mean\ \overline{x} = \frac{1}{32}\sum_{i=1}^{7} f_i x_i$

$= \frac{1}{32}(5 \times 25 + 7 \times 75 + 8 \times 125 + 6 \times 175 + 5 \times 225 + 0 \times 275 + 1 \times 300)$

$= 128.90625$

$Mode = 125$

$n = 32 \Rightarrow$ Median is the average of 16th and 17th element, i.e. 16.5th, which falls in the interval [100,150)

$$Median = 100 + \frac{16.5 - 5 - 7}{8} \times (150 - 100) = 128.125$$

iii) The mean is greatly affected by the extreme values. The mode and median might not be affected greatly by the extreme values.

b) Let T denote "the traffic is congested", R denote "it is a rainy day" and L denote "one will be late for school".

Then $P(T|R) = 0.6, P(T|\overline{R}) = 0.2$

$P(L|T|R) = 0.8, P(L|\overline{T}|R) = 0.3$

$P(\overline{L}|T|\overline{R}) = 0.65, P(\overline{L}|\overline{T}|\overline{R}) = 0.8$

$P(R) = 0.25$

i) $P(\overline{T}|R) = 1 - P(T|R) = 1 - 0.6 = 0.4$

$P(L|R) = P(L \wedge (T \vee \overline{T})|R) = P(L \wedge T|R) + P(L \wedge \overline{T}|R)$

$= P(L|T|R)P(T|R) + P(L|\overline{T}|R)P(\overline{T}|R)$

$= 0.8 \times 0.6 + 0.3 \times 0.4 = 0.6$

(Justification: The probability of being late is the sum of the probability of being late and the traffic is congested and the probability of being late and the traffic is NOT congested, given it is a rainy day.)

ii) $P(\overline{T}|\overline{R}) = 1 - P(T|\overline{R}) = 1 - 0.2 = 0.8$

$P(\overline{L}|\overline{R}) = P(\overline{L} \wedge (T \vee \overline{T})|\overline{R}) = P(\overline{L} \wedge T|\overline{R}) + P(\overline{L} \wedge \overline{T}|\overline{R})$

$= P(\overline{L}|T|\overline{R})P(T|\overline{R}) + P(\overline{L}|\overline{T}|\overline{R})P(\overline{T}|\overline{R})$

$= 0.65 \times 0.2 + 0.8 \times 0.8 = 0.77$

(Justification: The probability of NOT being late is the sum of the probability of NOT being late and the traffic is congested and the probability of NOT being late and the traffic is NOT congested, given it is NOT a rainy day.)

$P(\overline{L}|R) = 1 - P(L|R) = 1 - 0.6 = 0.4$

$P(\overline{R}) = 1 - P(R) = 1 - 0.25 = 0.75$

$P(\overline{L}) = P\left(\overline{L} \wedge (R \vee \overline{R})\right) = P(\overline{L} \wedge R) + P(\overline{L} \wedge \overline{R})$

$= P(\overline{L}|R)P(R) + P(\overline{L}|\overline{R})P(\overline{R})$

$= 0.4 \times 0.25 + 0.77 \times 0.75 = 0.6775$

(Justification: The probability of NOT being late is the sum of the probability of NOT being late and that it is a rainy day and the probability of NOT being late and it is NOT a rainy day.)

iii) $P(L) = 1 - P(\overline{L}) = 0.3225$

$P(L|\overline{R}) = 1 - P(\overline{L}|\overline{R}) = 1 - 0.77 = 0.23$

$P(\overline{R}|L) = \dfrac{P(L|\overline{R})P(\overline{R})}{P(L)} = \dfrac{0.23 \times 0.75}{0.3225} \approx 0.535$

c) Construct a table as follows:

X	$P(Sent = X)$	$P(Received = 01 \| Sent = X)$	$P(Received = 01 \wedge Sent = X)$
00	$0.4 \times 0.4 = 0.16$	$0.8 \times 0.2 = 0.16$	$0.16 \times 0.16 = 0.0256$
01	$0.4 \times 0.6 = 0.24$	$0.8 \times 0.8 = 0.64$	$0.24 \times 0.64 = 0.1536$
10	$0.6 \times 0.4 = 0.24$	$0.2 \times 0.2 = 0.04$	$0.24 \times 0.04 = 0.0096$
11	$0.6 \times 0.6 = 0.36$	$0.2 \times 0.8 = 0.16$	$0.36 \times 0.16 = 0.0576$

$P(Received = 01) = 0.0256 + 0.1536 + 0.0096 + 0.0576 = 0.2464$

The distribution table is as follows (given 01 is received):

X	00	01	10	11
$P(Sent = X)$	$\frac{0.0256}{0.2464} \approx 0.104$	$\frac{0.1536}{0.2464} \approx 0.623$	$\frac{0.0096}{0.2464} \approx 0.039$	$\frac{0.0576}{0.2464} \approx 0.234$

4)

a)

i) $Sample\ Space\ S = \begin{Bmatrix} \{10,10,10\}, \{10,10,20\}, \{10,10,30\}, \{10,20,20\}, \{10,20,30\}, \\ \{10,30,30\}, \{20,20,20\}, \{20,20,30\}, \{20,30,30\}, \{30,30,30\} \end{Bmatrix}$

ii) Construct a table as follows:

Sample	Mean	Median	Probability
{10,10,10}	10	10	$\left(\frac{1}{6}\right)^3 = \frac{1}{216}$
{10,10,20}	$\frac{40}{3}$	10	$\binom{3}{1}\left(\frac{3}{6}\right)\left(\frac{1}{6}\right)^2 = \frac{1}{24}$
{10,10,30}	$\frac{50}{3}$	10	$\binom{3}{1}\left(\frac{2}{6}\right)\left(\frac{1}{6}\right)^2 = \frac{1}{36}$
{10,20,20}	$\frac{50}{3}$	20	$\binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)^2 = \frac{1}{8}$
{10,20,30}	20	20	$\binom{3}{1}\left(\frac{1}{6}\right)\binom{2}{1}\left(\frac{3}{6}\right)\left(\frac{2}{6}\right) = \frac{1}{6}$
{10,30,30}	$\frac{70}{3}$	30	$\binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)^2 = \frac{1}{18}$
{20,20,20}	20	20	$\left(\frac{3}{6}\right)^3 = \frac{1}{8}$
{20,20,30}	$\frac{70}{3}$	20	$\binom{3}{1}\left(\frac{2}{6}\right)\left(\frac{3}{6}\right)^2 = \frac{1}{4}$
{20,30,30}	$\frac{80}{3}$	30	$\binom{3}{1}\left(\frac{3}{6}\right)\left(\frac{2}{6}\right)^2 = \frac{1}{6}$
{30,30,30}	90	30	$\left(\frac{2}{6}\right)^3 = \frac{1}{27}$

(Justification: The probability of non-repeated combination is larger since they have several permutations, e.g. $\{10,10,20\} = (10,10,20) \vee (10,20,10) \vee (20,10,10)$. Therefore we can think of it as choose a slot (out of 3) to place 20, and then we place the 10s to the other 2 slots, and the formula will be $\binom{3}{1}\left(\frac{3}{6}\right)\left(\frac{1}{6}\right)^2$.)

From this table we can obtain the sampling distribution:

X	10	$\frac{40}{3}$	$\frac{50}{3}$	20	$\frac{70}{3}$	$\frac{80}{3}$	90
$P(Mean)$	$\frac{1}{216}$	$\frac{1}{24}$	$\frac{1}{36}+\frac{1}{8}=\frac{11}{72}$	$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$	$\frac{1}{18}+\frac{1}{4}=\frac{11}{36}$	$\frac{1}{6}$	$\frac{1}{27}$

Y	10	20	30
$P(Median = Y)$	$\frac{1}{216}+\frac{1}{24}+\frac{1}{36}=\frac{2}{27}$	$\frac{1}{8}+\frac{1}{6}+\frac{1}{8}+\frac{1}{4}=\frac{2}{3}$	$\frac{1}{18}+\frac{1}{6}+\frac{1}{27}=\frac{7}{27}$

b)

i) $n = 35, \overline{X} = 2.8, s^2 = 0.7, \alpha = 0.01$

$H_0: \mu \geq 3, H_A: \mu < 3$ (One-sided test)

$$Test\ statistic\ z = \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{2.8 - 3}{\sqrt{\frac{0.7}{35}}} = -\sqrt{2} \approx -1.41$$

From the table we can obtain $P(Z < -1.41) = 1 - (Z \geq 1.41) = 1 - 0.9207 = 0.0793 > 0.01$

Therefore we cannot reject the null hypothesis, i.e., there is sufficient evidence to support the claim.

ii) From the table we can find $P(Z \leq 1.96) = 0.9750 \Rightarrow P(-1.96 \leq Z \leq 1.96) = 2 - 2 \times 0.9750 = 0.05$

Therefore the confidence limits are $L = \overline{X} - z\left(\frac{s}{\sqrt{n}}\right) = 2.8 - 1.96 \times \sqrt{\frac{0.7}{35}} = 2.523$ and $U = \overline{X} + z\left(\frac{s}{\sqrt{n}}\right) = 2.8 + 1.96 \times \sqrt{\frac{0.7}{35}} = 3.078$

The 95% confidence interval is [2.523, 3.078]

c) Newton's Method

--End of Answers--

Solver: Zhao Jingyi