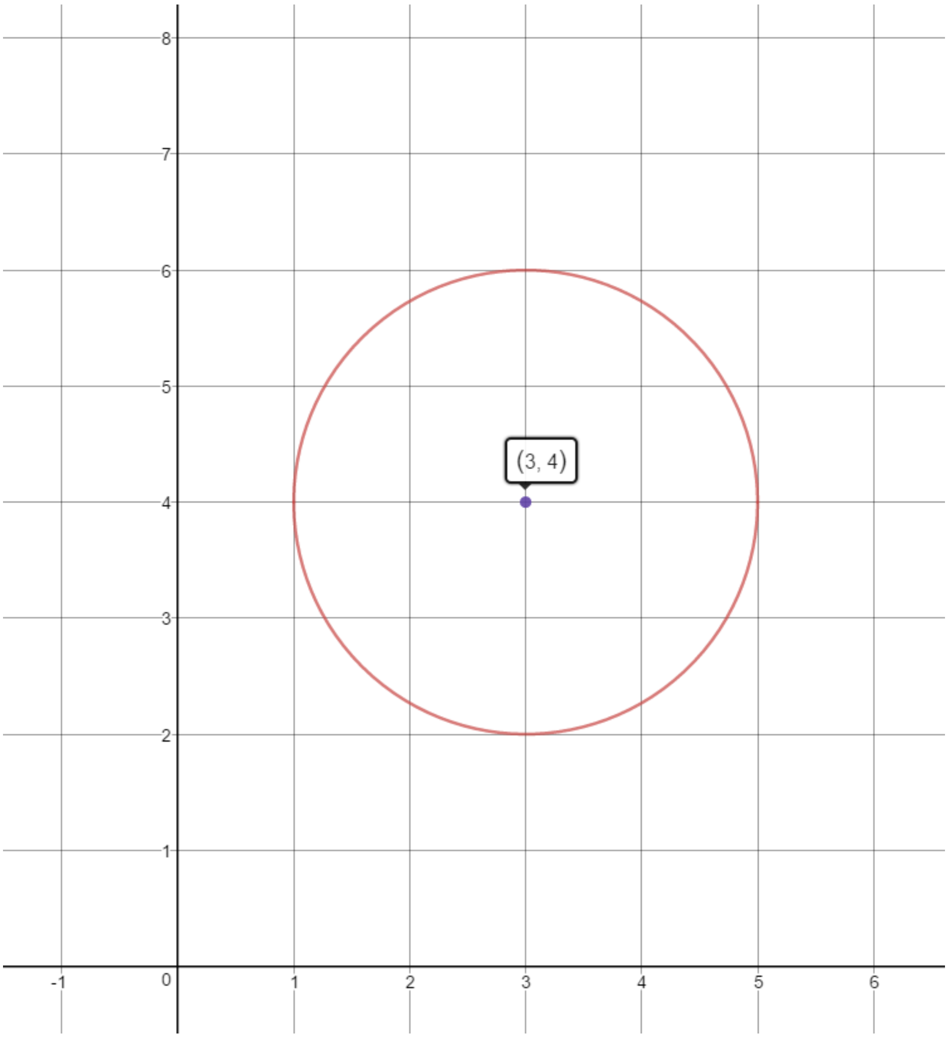
Solver: Marcellino

Email Address: marc0036@e.ntu.edu.sg

1. (a)

Information given:

- A circle
- $r = 2$
- Centre: $x = 3, y = 4$
- Right-handed Cartesian coordinate system XY (the positive x and y axes point right and up. Positive rotation is **counterclockwise** about the axis of rotation.)



A picture for your reference, drawn using desmos.com

Solution:

The general **explicit** equation for circle is

$$\boldsymbol{y = b \pm \sqrt{r^2 - (x - a)^2}}$$

Where $(\boldsymbol{a}, \boldsymbol{b})$ is the centre of the circle and $\boldsymbol{r}$ is the radius of the circle.

So, substituting in, the explicit definition of the circle is:

$$y = 4 \pm \sqrt{2^2 - (x - 3)^2}$$
$$y = 4 \pm \sqrt{4 - (x - 3)^2}$$

The general **implicit** equation of a circle is:

$$\boldsymbol{(x-a)^2+(y-b)^2=r^2}$$

Where $(\boldsymbol{a}, \boldsymbol{b})$ is the centre of the circle and $\boldsymbol{r}$ is the radius of the circle.
So, substituting in, the implicit definition of the circle is:

$$(x-3)^2+(y-4)^2=2^2$$
$$(x-3)^2+(y-4)^2=4$$

Side note: if you memorized the implicit equation of a circle, you can derive the explicit equation without memorizing it:

$$\boldsymbol{(x-a)^2+(y-b)^2=r^2}$$
$$(y-b)^2=r^2-(x-a)^2$$
$$y-b=\pm\sqrt{r^2-(x-a)^2}$$
$$\boldsymbol{y=b\pm\sqrt{r^2-(x-a)^2}}$$

which is the explicit equation for a circle.

And lastly, the general parametric equation of a circle is:

$$\boldsymbol{x=r\cos(\theta)+a}$$
$$\boldsymbol{y=r\sin(\theta)+b}$$
$$\boldsymbol{\theta\in[0,2\pi]}$$

Where $(\boldsymbol{a}, \boldsymbol{b})$ is the centre of the circle and $\boldsymbol{r}$ is the radius of the circle.
So, substituting in, the parametric definition of the circle is:

$$\boldsymbol{x=2\cos(\theta)+3}$$
$$\boldsymbol{y=2\sin(\theta)+4}$$
$$\boldsymbol{\theta\in[0,2\pi]}$$

(b)

$\boldsymbol{x(u)=0.6u\cos(u\times 2\pi)-0.6}$
$\boldsymbol{y(u)=-0.6u\sin(u\times 2\pi)}$
$\boldsymbol{u\in[0,1]}$

The $\boldsymbol{0.6}$ **factor** in front of the cosine and sine function is the spiral **radius**.
The $\boldsymbol{-0.6}$ in the $x(u)$ function is to shift the centre of the spiral to left by -0.6 units.
And **the negative factor** in $y(u)$ is to flip the spiral with respect to $x-axis$ so it starts to curve from bottom right clockwise instead of upper right counterclockwise.

(c) $\boldsymbol{x(u)=0.5\cos(5u\times 2\pi)}$
$\boldsymbol{y(u)=1.5u-0.5}$
$\boldsymbol{z(u)=0.5\sin(5u\times 2\pi)}$
$\boldsymbol{u\in[0,1]}$

The 0.5 in x and z is the radius. The 5 factor inside the cosine and the sine function is to make it rotate 5 times. And the y is obtained from the Linear Interpolation model (Module 8 slides 21):

$$V(\tau)=(1-\tau)A+\tau B, 0\le\tau\le 1$$

So in our case,

$y(u) = (1-u) \times (-0.5) + u \times 1$

$\boldsymbol{y(u) = 1.5u - 0.5}$

2. (a)

From Module 3 Page 42:
For any point $\boldsymbol{r_o = (x_o, y_o, z_o)}$: $\boldsymbol{N \cdot (r - r_o) = 0}$
Here our $\boldsymbol{r_o}$ is $(\boldsymbol{4, 5, 6})$ and vector $\boldsymbol{N = [1\ 2\ 3]}$.
Substituting in,

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \left[r - \begin{pmatrix}4\\5\\6\end{pmatrix}\right] = 0$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \left[\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}4\\5\\6\end{pmatrix}\right] = 0$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \left[\begin{pmatrix}x-4\\y-5\\z-6\end{pmatrix}\right] = 0$$

$1 \times (x-4) + 2 \times (y-5) + 3 \times (z-6) = 0$
$x - 4 + 2y - 10 + 3z - 18 = 0$
$\boldsymbol{x + 2y + 3z - 32 = 0}$

Note: One way to verify if your answer is correct is to substitute in $\boldsymbol{r_o(4, 5, 6)}$ to your final equation and see if the equation stands true, i.e.:
$x + 2y + 3z - 32 = 0$
$4 + 2(5) + 3(6) - 32 = 0$
$4 + 10 + 18 - 32 = 0$
$32 - 32 = 0$
$0 = 0$ (true)
Don't forget to verify the normal vector as well by taking the coefficient each from x, y, and z, which is 1, 2, and 3 respectively.

(b)
The cylinder:

$$x^2 + y^2 \leq 0.5^2$$
$$x^2 + y^2 \leq 0.25$$
$$\boldsymbol{0.25 - x^2 - y^2 \geq 0}$$

The plane that we do not want:

$$z \geq 1$$
$$\boldsymbol{z - 1 \geq 0}$$

The second plane that we do not want:

$$z \leq 0$$
$$\boldsymbol{-z \geq 0}$$

Subtracting the cylinder with the planes that we do not want:

$$\min\left(0.25 - x^2 - y^2, -(z-1), -(-z)\right)$$
$$\boldsymbol{\min\left(0.25 - x^2 - y^2, 1 - z, z\right) \geq 0}$$

The square hole:

$$y \leq 0.2 \text{ and } y \geq -0.2 \text{ and } x \leq 0.2 \text{ and } x \geq -0.2$$
$$0.2 - y \geq 0 \text{ and } y + 0.2 \geq 0 \text{ and } 0.2 - x \geq 0 \text{ and } x + 0.2 \geq 0$$

So, to intersect all of them to make the square hole:

$$\boldsymbol{\min\left(0.2 - y, y + 0.2, 0.2 - x, x + 0.2\right) \geq 0}$$

Combining (intersecting) the cylinder with the hole:

$$\boldsymbol{f(x,y,z) = \min\left(\min\left(0.25 - x^2 - y^2, 1 - z, z\right), \min(0.2 - y, y + 0.2, 0.2 - x, x + 0.2\right) \geq 0}$$

(c)

Reference: Module 3 Slides 118

Firstly, we have to recalibrate the x function as the parameter is fixed to $u, v \in [0,1]$.

$$\frac{u - u_o}{u_1 - u_o} = \frac{x - x_o}{x_1 - x_o}$$
$$\frac{u - 0}{1 - 0} = \frac{x - (-0.8)}{0.8 - (-0.8)}$$
$$\frac{u}{1} = \frac{x + 0.8}{1.6}$$
$$\boldsymbol{x = 1.6u - 0.8}$$

Secondly, substitute x to y:

$$y = x^2 + 0.2$$
$$\boldsymbol{y = (1.6u - 0.8)^2 + 0.2}$$

And finally, rotate the curve:

$\boldsymbol{x = 1.6u - 0.8}$
$\boldsymbol{y = \left((1.6u - 0.8)^2 + 0.2\right) \times \cos(v \times 2\pi)}$
$\boldsymbol{z = \left((1.6u - 0.8)^2 + 0.2\right) \times \sin(v \times 2\pi)}$
$\boldsymbol{u, v \in [0, 1]}$

3. (a)

(i) **Reference: Module 7 Slide 17**

Rotation about the z-axis:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos 60° & -\sin 60° & 0 & 0 \\ \sin 60° & \cos 60° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Step I: Translate $(-2,0,-1)$
Step II: Reflect
Step III: Translate back $(2,0,1)$

The matrix:
$(Step\ III) \cdot (Step\ II) \cdot (Step\ I)$
$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

(b)

(i) **Reference: Module 8 Slide 12**
$$\tau = f(k) = \sin\left(\frac{\pi}{2}\frac{k-1}{m-1}\right)$$
Where k = the frame index, $1 \leq k \leq m$
And m = total number of frames
So,
$$\tau = f(k) = \sin\left(\frac{\pi}{2}\frac{k-1}{m-1}\right)$$
$$\tau = f(k) = \sin\left(\frac{\pi}{2}\frac{k-1}{100-1}\right)$$
$$\tau = f(k) = \sin\left(\frac{\pi}{2}\frac{k-1}{99}\right)$$
$$\boldsymbol{\tau = f(k) = \sin\left(\frac{k-1}{99}\frac{\pi}{2}\right)}$$

(c)
Extracting all the important information from the question:
Plane $y - 1 = 0$
$y = 1$
Normal $N = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, taken from the coefficient of x, y, and z in the above equation, which is 0, 1, and 0 respectively.
$I_{source} = 0.8$ located at $(-30, 41, 0)$
Diffuse reflection coefficient $k_d = 0.7$
Specular reflection coefficient $k_s = 0.2$
Specular exponent $n = 3$
Ambient reflection coefficient $k_a = 0.3$
Ambient intensity $I_a = 0.1$
Observer located at $(6, 9, 0)$
Calculate the reflection intensity I at point $(0, 1, 0)$

The formula for reflection intensity I:

$$I = k_a I_a + k_d I_{source} \cos\theta + k_s I_{source} \cos^n \emptyset$$

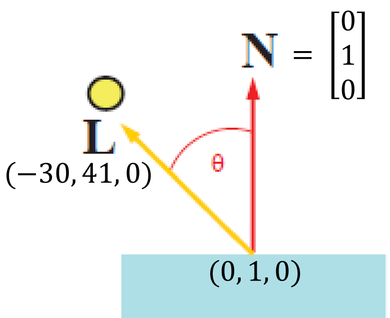
The only unknowns in the formula are $\cos\theta$ and $\cos\emptyset$ so we have to find those values.

From Module 4 Slide 40 we should know that:

$$\cos\theta = N \cdot L$$

$$\cos\emptyset = V \cdot R$$

Where $|N| = |L| = |V| = |R| = 1$ (unit vectors)



$$L = \begin{bmatrix} -30 \\ 41 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$L = \begin{bmatrix} -30 \\ 40 \\ 0 \end{bmatrix}$$

Calculate the unit vector $\hat{L}$:

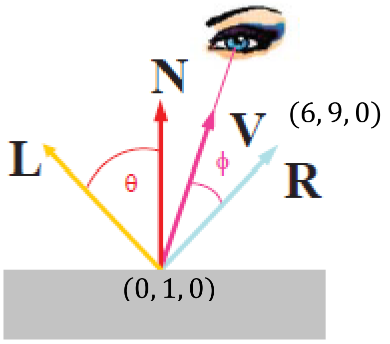
$$\hat{L} = \frac{\begin{bmatrix} -30 \\ 40 \\ 0 \end{bmatrix}}{\sqrt{(-30)^2 + (40)^2 + 0^2}}$$

$$\hat{L} = \frac{\begin{bmatrix} -30 \\ 40 \\ 0 \end{bmatrix}}{50}$$

$$\hat{L} = \begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}$$

$$N \cdot L = |N||L| \cos\theta$$
$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -30 \\ 40 \\ 0 \end{bmatrix} = \sqrt{0^2 + 1^2 + 0^2} \times \sqrt{(-30)^2 + (40)^2 + 0^2} \times \cos\theta$$

$$40 = 1 \times 50 \times \cos\theta$$
$$\cos\theta = \frac{40}{50}$$
$$\cos\theta = \frac{4}{5}$$



$$R = 2(N \cdot L)N - L$$

Where L and N are both unit vectors. Refer to Module 4 Page 41 for derivation and more details.

$$R = 2\left(\frac{4}{5}\right)\begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}-\frac{3}{5}\\ \frac{4}{5}\\ 0\end{bmatrix}$$

$$R = \left[0, \frac{8}{5}, 0\right] - \left[-\frac{3}{5}, \frac{4}{5}, 0\right]$$

$$R = \left[\frac{3}{5}, \frac{4}{5}, 0\right]$$

Calculate V:

$$V = \begin{bmatrix}6\\9\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$V = \begin{bmatrix}6\\8\\0\end{bmatrix}$$

Calculate the unit vector $\hat{V}$:

$$\hat{V} = \frac{\begin{bmatrix}6\\8\\0\end{bmatrix}}{\sqrt{6^2 + 8^2 + 0^2}}$$

$$\hat{V} = \frac{\begin{bmatrix}6\\8\\0\end{bmatrix}}{\sqrt{36+64+0}}$$

$$\hat{V} = \frac{\begin{bmatrix}6\\8\\0\end{bmatrix}}{\sqrt{100}}$$

$$\hat{V} = \frac{\begin{bmatrix}6\\8\\0\end{bmatrix}}{10}$$

$$\hat{V} = \begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\\0\end{bmatrix}$$

$$V \cdot R = \begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\\0\end{bmatrix} \cdot \begin{bmatrix}\frac{3}{5}\\\frac{4}{5}\\0\end{bmatrix}$$

$$V \cdot R = \frac{3}{5} \times \frac{3}{5} + \frac{4}{5} \times \frac{4}{5} + 0 \times 0$$

$$V \cdot R = \frac{9}{25} + \frac{16}{25} + 0$$

$$V \cdot R = 1$$

$$\cos \emptyset = V \cdot R = 1$$

So, $\cos\theta = \frac{4}{5}$ and $\cos\emptyset = 1$. Substitute all the variables into the formula:

$$\boldsymbol{I = k_a I_a + k_d I_{source} \cos\theta + k_s I_{source} \cos^n \emptyset}$$

$$I = 0.3 \times 0.1 + 0.7 \times 0.8 \times \frac{4}{5} + 0.2 \times 0.8 \times 1^3$$

$$I = 0.03 + 0.448 + 0.16$$

$$I = 0.638$$

4. (a)
From Module 6 Slide 28:
Homogenous → Cartesian

$$\begin{bmatrix} x_h \\ y_h \\ h \end{bmatrix} = \begin{bmatrix} \frac{x_h}{h} \\ \frac{y_h}{h} \\ 1 \end{bmatrix} \xrightarrow{\textbf{inhomogenizing}} \begin{bmatrix} \frac{x_h}{h} \\ \frac{y_h}{h} \\ h \end{bmatrix}$$

$(1, 0, 0, -2) \rightarrow (-\frac{1}{2}, 0, 0, 1) \rightarrow$ Cartesian coordinates $\left(-\frac{1}{2}, 0, 0\right)$
$(2, 0, 2, 2) \rightarrow (1, 0, 1, 1) \rightarrow$ Cartesian coordinates $(1, 0, 1)$
$(-2, 0, 1, 1) \rightarrow (-2, 0, 1, 1) \rightarrow$ Cartesian coordinates $(-2, 0, 1)$
$(3, 9, 6, 3) \rightarrow (1, 3, 2, 1) \rightarrow$ Cartesian coordinates $(1, 3, 2)$
$(2, 1, 5, 0.5) \rightarrow (4, 2, 10, 1) \rightarrow$ Cartesian coordinates $(4, 2, 10)$

(b)
Reference: Module 6 Slide 53
Affine transformations can always be represented by

$$x' = ax + by + m$$
$$y' = cx + dy + n$$

Where

- a, b, c, d, m, n are constants
- (x, y) are the coordinates of the point to be transformed
- (x', y') are the coordinates of the transformed point

The general matrix form of affine transformations is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & m \\ c & d & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

So we need to find a, b, c, d, m, and n by substituting the known points.

$A_1 = (1, 0) \rightarrow B_1(0, 3)$

$A_2 = (2, 0) \rightarrow B_2(1, 5)$

$A_3 = (2, 2) \rightarrow B_3(-1, 7)$

$A_4 = (0, 1) \rightarrow B_4(-2, 2)$

Using A_1 and B_1:

$x' = ax + by + m$

$0 = a(1) + b(0) + m$

$0 = a + m$ (1)

$y' = cx + dy + n$

$3 = c(1) + d(0) + n$

$3 = c + n$ (2)

Using A_2 and B_2:

$x' = ax + by + m$

$1 = a(2) + b(0) + m$

$1 = 2a + m$ ③

$y' = cx + dy + n$

$5 = c(2) + d(0) + n$

$5 = 2c + n$ ④

Solving ① and ③ gives $a = 1$ and $m = -1$
Solving ② and ④ gives $c = 2$ and $n = 1$

Using A_3 and B_3:

$x' = ax + by + m$

$-1 = 1(2) + b(2) - 1$

$b = -1$

$y' = cx + dy + n$

$7 = 2(2) + d(2) + 1$

$d = 1$

$a = 1, b = -1, c = 2, d = 1, m = -1, n = 1$

So the matrix for the affine transformation is:

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Final answer is verified using matrix multiplication calculator at
http://onlinemschool.com/math/assistance/matrix/multiply/

(c)

Reference: Module 5 Slide 9 onwards

For the image, $u \in [0, 199]$, $v \in [0, 99]$

For the sphere, $s \in [0, 2\pi]$, $v \in [0, \pi]$

Mapping:

$$\frac{u-u_0}{u_1-u_0} = \frac{s-s_0}{s_1-s_0}$$

$$\frac{u-0}{199-0} = \frac{s-0}{2\pi-0}$$

$$\frac{u}{199} = \frac{s}{2\pi}$$

$$u = \frac{199}{2\pi} s$$

Do the same for v and t:

$$\frac{v - v_0}{v_1 - v_0} = \frac{t - t_0}{t_1 - t_0}$$

$$\frac{v - 0}{99 - 0} = \frac{t - 0}{\pi - 0}$$

$$\frac{v}{99} = \frac{t}{\pi}$$

$$v = \frac{99}{\pi} t$$

Parametric equation of a sphere:

$x = r \cos s \sin t$

$y = r \sin s \sin t$

$z = r \cos t$

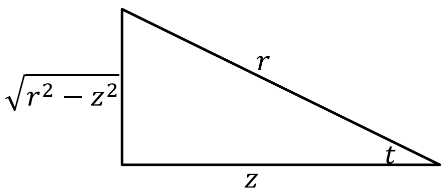
Solve for s and t. Using the z equation,

$z = r \cos t$

$\cos t = \frac{z}{r}$

$t = \cos^{-1} \frac{z}{r}$

Find $\sin t$ from $\cos t$:



$\sin t = \frac{\sqrt{r^2 - z^2}}{r}$

Using the x equation,

$x = r \cos s \sin t$

$x = r \cos s \times \frac{\sqrt{r^2 - z^2}}{r}$

$x = \cos s \times \sqrt{r^2 - z^2}$

$\cos s = \frac{x}{\sqrt{r^2 - z^2}}$

$s = \cos^{-1}\frac{x}{\sqrt{r^2-z^2}}$

Substitute s to the u equation:

$$u = \frac{199}{2\pi}s$$

$$\boldsymbol{u = \frac{199}{2\pi}\cos^{-1}\frac{x}{\sqrt{r^2 - z^2}}}$$

Substitute t to the v equation:

$$v = \frac{99}{\pi}t$$

$$\boldsymbol{v = \frac{99}{\pi}\cos^{-1}\frac{z}{r}}$$

THE END

Go to https://www.dropbox.com/sh/28qzcg76eb73w3d/AADvY78eKIHUmyayr7QWXipva?dl=0

or bit.ly/2bOBLF6 for VRML files for question 1b, 1c, and 2c.

For reporting of errors and errata, please visit pypdiscuss.appspot.com
Thank you and all the best for your exams! ☺