20th CSEC – Past Year Paper Solution *2019-2020 Sem 1*
MH 1812 – Discrete Mathematics

1)

$$\mathrm{T} \rightarrow (E \vee M) \ldots (1)$$
$$\mathrm{S} \rightarrow \neg \mathrm{E} \ldots (2)$$
$$\mathrm{T} \wedge \mathrm{S} \ldots (3)$$

Observe that from equation (3) we can conclude by Conjunctive Simplification that

$$\therefore \ \mathrm{T} \ldots (4)$$
$$\therefore \ \mathrm{S} \ldots (5)$$

Then by using Modus Ponens for equation (4) and (1) we can conclude that

$$\therefore E \ \vee M \ldots (6)$$

Similarly, using Modus Ponens for equation (5) and (2) we have

$$\therefore \ \neg \mathrm{E} \ldots (7)$$

Hence, by Disjunctive Syllogism for equation (6) and (7)

$$\therefore \ \mathrm{M} \ldots (6)$$

So, the argument is **VALID**

2)

a) Firstly, observe that the equation is equivalent to

$$a_n + 1 = 2a_{n-1} + 2$$
$$a_n + 1 = 2[(a)_{n-1} + 1]$$

So, let's define new sequence $b_n = a_n + 1$, and then we have recurrence equation for b_n is:

$$b_n = 2b_{n-1}$$

With initial condition $b_1 = a_1 + 1 = 2$
Using backtracking method we have that:

$$b_n = 2b_{n-1} = 2^2 b_{n-2} = 2^3 b_{n-3} = \cdots = 2^{n-1} b_1$$

Hence, $b_n = 2^{n-1}.2 = 2^n$ so that

$$a_n = b_n - 1 = 2^n - 1$$

b) To prove by mathematical induction, let us prove the base case, which is when $n = 1$
That is:

$$\frac{1}{2} = 2 - \frac{3}{2} = 2 - \frac{1+2}{2^1}$$

which is true for $\mathrm{n} = 1$. Hence, the base case is proven.
Then for the inductive step, let us assume for $\mathrm{n} = \mathrm{k}$ the equation is true.
That is

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$$

We will prove that the equation remains true when $\mathrm{n} = \mathrm{k} + 1$
Observe that

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = \left(\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{k}{2^k}\right) + \frac{k+1}{2^{k+1}}$$
$$= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$$
$$= 2 - \frac{(2k+4) - (k+1)}{2^{k+1}}$$
$$= 2 - \frac{k+3}{2^{k+1}}$$
$$= 2 - \frac{(k+1)+2}{2^{k+1}}$$

and the equation remains true when $\mathrm{n} = \mathrm{k} + 1$
Hence it is proven that, for all n,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

3)

a) Since there is no restriction in the selection, it's just same as we have to choose any 5 persons from 11 persons , which the number of ways is $\binom{11}{5} = \frac{11!}{5!6\,!} = \frac{11.10.9.8.7.6!}{5.4.3.2.1.6!} = 462$ ways

b) Because the committee must include exactly 2 teachers, then number of students is (5-2) = 3 students on committee. Hence committee must consist of 2 teachers and 3 students.
So, the number of ways is $\binom{4}{2}.\binom{7}{3} = 6 \times 35 = 210$ ways

c) Since the committee must include at least 3 teachers, the committee must be in form of (3 teachers, 2 students) or (4 teachers , 1 students)
So, for this case number of ways is $\binom{4}{3}.\binom{7}{2} + \binom{4}{4}.\binom{7}{1} = 4 \times 21 + 1 \times 7 = 84 + 7 = 91$ ways

d) We will look for the complement, which is when a particular teacher is on same committee with a particular student. Then, 2 members of committee is fixed by those persons while remaining 3 is free from remaining 9 persons. Hence number of ways is $\binom{9}{3} = 84$ ways
So, if they are not in same committee, number of ways is 462 -84 = 378 ways

4) We will prove by showing each other's subset. Firstly, consider $x \in A$, there is 2 case:
1. $x \in C$ then implies $x \in (A \cap C)$ and so $x \in (B \cap C)$. Hence $x \in B$
2. $x \notin C$ then we have since $x \in A$ then $x \in (A \cup C)$ and so $x \in (B \cup C)$.
But $x \notin C$ and so we conclude that $x \in B$

In both case we have that for every $x \in A$ then $x \in B$. So, we have $A \subseteq B$
Hence, in a similar way of above by reversing B with A, we can also have for every $x \in B$ then $x \in A$. So, $B \subseteq A$ and we conclude that $A = B$

Alternatively, we can also prove by membership table:

A	B	C	$A \cap C$	$B \cap C$	$A \cup C$	$B \cup C$
1	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	**0**	**0**	**0**	**1**	**1**
1	0	1	1	0	1	1
1	0	0	0	0	1	0
0	1	1	0	1	1	1
0	1	0	0	0	0	1
0	**0**	**1**	**0**	**0**	**1**	**1**
0	**0**	**0**	**0**	**0**	**0**	**0**

At the only possible row, we have that no matter what every element at A should be at B (and vice versa) and also if that element not in A then it also not in B (and vice versa)

Hence, we conclude that $A = B$

5) R is REFLEXIVE because 3 | $x^2 - x^2$ and so $x \; R \; x$
R is SYMMETRY because if $x \; R \; y$ which is 3 | $x^2 - y^2$ then we also have 3 | $y^2 - x^2$ and hence $y \; R \; x$
R is TRANSITIVE because if $x \; R \; y$ and $y \; R \; z$ which is 3 | $x^2 - y^2$ and 3 | $y^2 - z^2$ then we also have 3 | $(x^2 - y^2) + (y^2 - z^2) \leftrightarrow$ 3 | $x^2 - z^2$ and hence $x \; R \; z$
∴ So, R is a EQUIVALENCE RELATION

Then for equivalence classes, observe that when $x \equiv 0 \; (mod\ 3)$ then $x^2 \equiv 0 \; (mod\ 3)$ and $x^2 \equiv 1 (mod\ 3)$ otherwise.
∴ Hence equivalence classes are $[0] = \{\dots, -3,0,3,6, \dots\}$ and $[1] = \{\dots, -2, -1,1,2,4, \dots\}$

6)

a) Definition of function is mapping from A to B such that every element at A is matched with one of element of B. For each element of A, there is 3 choice of match and hence number of functions are $3^4 = 81$ functions

b) First, let us define S be set of images from A. ($S = \{f(1), f(2), f(3), f(4)\}$)
Since the function must be onto then $\{a, b, c\} \subseteq S$ and since S must consist of 4 elements, last element can be chosen free from {a,b,c}
WLOG, the last element is a. Hence S = {a,a,b,c}
To count how many functions which have range like S, firstly we chose two elements of A which the image is a → There are $\binom{4}{2}$ ways
Then we chose one element from the remaining 2 which the image is b → There are $\binom{2}{1}$ ways
Hence, total functions are $3.\binom{4}{2}.\binom{2}{1} = 3 \; x \; 6 \; x2 = 36$ functions

c) We claim that there is no function which is one to one. Let's prove by contradiction
Assume the function exists. Hence $f(1), f(2), f(3), f(4)$ must all have different values.

However, there is only 3 elements at range of f and by pigeonhole principle, there are two functions which have same value. This is a contradiction and hence no functions one-to one exist

7) We claim that Sam must grab minimum of 13 socks in order to guarantee he has at least 4 socks of the same color

First, we prove that 13 is the minimum. That is same as prove if Sam take 12 socks, there is a case when he does not get 4 socks of same color. It is possible as at worse condition, Sam could grab 3 red socks, 3 blue socks, 3 green socks and 3 yellow socks make it no 4 socks of same color.

Then, we prove that no matter what 13 socks Sam grab, there must be 4 socks of same color. This is same as pigeonhole principle, as there are 13 socks and 4 color so there exists color which consisted of $\left\lceil \frac{13}{4} \right\rceil = 4$ socks. So, he is sure that he will get 4 socks of same color if he takes 13 socks.

--End of Answers--

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