Solver: Lek Jie Ling

1) a) By L'hopital rule, $\lim_{x\to -1} \frac{x^2-1}{x^2-4x-5} = \lim_{x\to -1} \frac{2x}{2x-4} = \frac{1}{3}$

b) $\lim_{x\to -1^+}(x+1)$ is a very small positive number and $\lim_{x\to -1^+}\frac{1}{x+1}$ is a large positive number. A small number less than 1 to the power of a large number will tend to zero.

c) $\frac{dy}{dx} = \cos x - x\sin x + 2xe^{x^2} + \frac{1}{x}$

d) $xy^2 = x^2 + 2xy + y^3$
$y^2 + 2xy\frac{dy}{dx} = 2x + 2y + 2\frac{dy}{dx}x + 3y^2\frac{dy}{dx}$
$\frac{dy}{dx}(2xy - 2x - 3y^2) = 2x + 2y - y^2$
$\frac{dy}{dx} = \frac{2x+2y-y^2}{2xy-2x-3y^2}$

2) a) $f(x) = xe^x$ on [0,1)
$f'(x) = e^x + xe^x$
$f'(x) = 0, x = -1$
Since f(x) is consistently increasing, min point is at x=0 and max point is x=1.
Absolute min point is (0,0) and there is no absolute max point because 1 is not in the domain.

b) $\int x + e^2 + e^x + \sin x + xe^x dx = \frac{x^2}{2} + e^2x + e^x - \cos x + xe^x - e^x + C,\ C \in R$

c) Differential Equation is $x^2 + 2x - 8 = 0$
$x = 4\ or - 2$
$\frac{dy}{dx} = 4C_1e^{4x} - 2C_2e^{-2x}$
$At\ y = 0, x = 0 \to 0 = C_1 + C_2$
$At\ x = 0, \frac{dy}{dx} = 2 \to 2 = 4C_1 - 2C_2$
By solving simultaneous equations, $C_1 = \frac{1}{3}, C_2 = -\frac{1}{3}$
$y = \frac{1}{3}e^{4x} - \frac{1}{3}e^{-2x}$

3) a) If n is even, $a_n = 0,$
If n is odd, $a_n = (-1)(-1)^{\frac{n+1}{2}}(5 + (n-1)\times 6)$

b) $$a_n = \frac{(-1)^{2n+1}n}{n + n^{\frac{1}{3}}} = -\frac{n}{n + n^{\frac{1}{3}}} = -\frac{1}{1 + \frac{1}{n^{\frac{2}{3}}}}$$
As n tends to infinity, $\frac{1}{n^{\frac{2}{3}}}$ tends to 0. Hence a_n tends to -1.

c) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$
$(a_1 - b_1) + \cdots + (a_n - b_n) = (a_1 + \cdots + a_n) - (b_1 + \cdots + b_n)$ [Proven]
Condition: Series must have absolute convergence

More details:
https://math.stackexchange.com/questions/112413/whether-sum-a-sum-b-sum-ab-is-correct-or-not

d) Approx. $f'(0.25\pi) =$

$$\frac{\cos(0.25\pi - 2\times 10^y) - 8\cos(0.25\pi - 10^y) + 8\cos(0.25\pi + 10^y) - \cos(0.25\pi + 2\times 10^y)}{12\times 10^y}$$

Actual value: $\frac{dy}{dx}_{x=0.25\pi} = -\sin(0.25\pi) = -0.0137074$

Using GC, to find y where approx. $f'(0.25\pi) - actual\ f'(0.25\pi) < 10^{-10}, y = -2$

e) $$\int_1^2 \ln x\,dx = \frac{1/6}{3}\left(f(1) + 4f\left(\frac{7}{6}\right) + 2f\left(\frac{8}{6}\right) + 4f\left(\frac{9}{6}\right) + 2f\left(\frac{10}{6}\right) + 4f\left(\frac{11}{6}\right) + f(2)\right)$$
$$= 0.38629$$

Actual value according to GC = 0.38629

4) a) $f(x) = (27-x)^{-\frac{1}{3}} \to f(0) = \frac{1}{3}$

$f'(x) = \frac{1}{3}(27-x)^{-\frac{4}{3}} \to f'(0) = \frac{1}{243}$

$f''(x) = \frac{4}{9}(27-x)^{-\frac{7}{3}} \to f''(0) = \frac{4}{9}\times\frac{1}{2187}$

$f'''(0) = \frac{4}{9}\times -\frac{7}{3}(27-x)^{-\frac{10}{3}} = \frac{28}{27}\times\frac{1}{3^{10}} = \frac{28}{3^{13}}$

$f(x) = \frac{1}{3} + \frac{1}{243}x + \frac{1}{19683}\frac{x^2}{2} + \frac{14}{3^{14}}x^3$

General Term = $\sum\frac{(1/3)^n}{n!}$

It will converge to 0 as n tends to infinity.

b) $f'''(x) = \frac{4}{9}\times -\frac{7}{3}(27-x)^{-\frac{10}{3}} = \frac{28}{27}$

$(-x)e^{-x} - e^{-x} + C = (-x)\left(1-(-x)+\frac{(-x)^2}{2!}-\frac{(-x)^3}{3!}\right) - \left(1-(-x)+\frac{(-x)^2}{2!}-\frac{(-x)^3}{3!}\right) + \left(1-(-x)+\frac{(-x)^2}{2!}-\frac{(-x)^3}{3!}\right)$

c) $C_k = \frac{1}{T_0}\int_{-2}^{2} x(t)e^{-jkw_0t}\,dt$

$C_k = \frac{1}{4}\left\{\int_{-2}^{-1}(-2)e^{-jkw_0t}\,dt + \int_{-1}^{0}(-1)e^{-jkw_0t}\,dt + \int_0^1 2e^{-jkw_0t}\,dt + \int_1^2 e^{-jkw_0t}\,dt\right\}$

$= \frac{1}{-4jkw}\left\{-e^{jkw} + 2e^{2jkw_0} - 3 + e^{-jkw_0} + e^{-2jkw_0}\right\}$

$= \frac{1}{-2\pi jk}\left\{-2j\sin(kw) + 2e^{2jkw_0} - 3 + e^{-2jkw_0}\right\}$

--End of Answers--