V. EXERCISES (SOLUTIONS)

1. summation notation: $d \approx \sum_{i=1}^{n} R\Delta\phi \sec\phi_i$

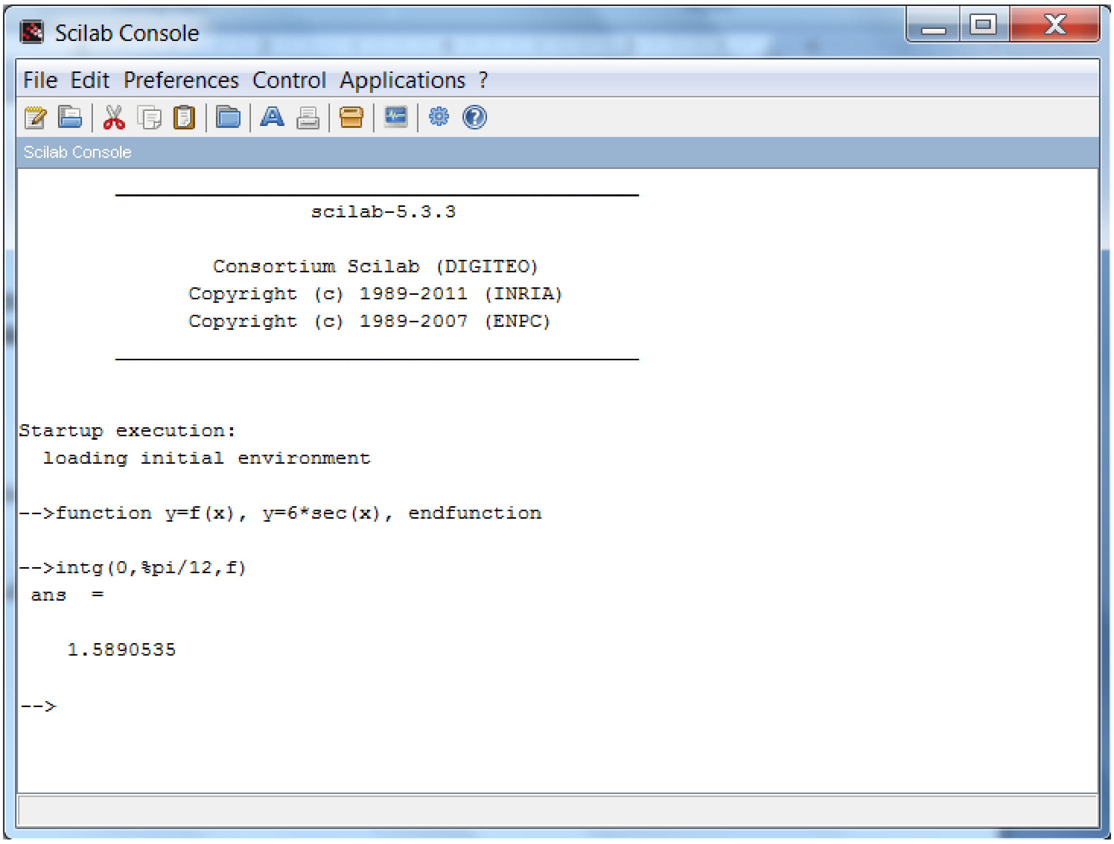
   Riemman sum: $d = \lim_{n\to\infty} \sum_{i=1}^{n} R\Delta\phi \sec\phi_i = \int_a^b R\sec\phi \, d\phi$

2. Using $\phi = \dfrac{\pi}{180}\theta$, where ϕ in radian and θ in degree

Latitude Line (in degrees north of the equator)	(in radians)
$15°$	$\pi/12$
$30°$	$\pi/6$
$45°$	$\pi/4$
$60°$	$\pi/3$
$75°$	$5\pi/12$
$90°$	$\pi/2$

3. For example, with $R = 6$, at latitude line $15° = \dfrac{\pi}{12}$ radian,

Using Scilab to compute $d = \int_0^{\frac{\pi}{12}} 6 \sec \phi \, d\phi$



Latitude Line (in degrees north of the equator)	Distance from Equator
15°	1.589
30°	3.296
45°	5.288
60°	7.902
75°	12.166
90°	

4. Convergence problem error ...

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-->intg(0,%pi/2,f)
                 !--error 24
Convergence problem...
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latitude line $90°$ north of equator represents the north pole

5. $\int \sec \phi \, d\phi = \int \sec \phi \left(\frac{\sec \phi + \tan \phi}{\sec \phi + \tan \phi} \right) d\phi$

$$= \int \frac{\sec^2\phi + \sec\phi\tan\phi}{\sec\phi + \tan\phi}\, d\phi$$

Let $u = \sec\phi + \tan\phi$,

$du = \sec\phi\tan\phi + \sec^2\phi\, d\phi$

So,

$$\int \sec\phi\, d\phi = \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\sec\phi + \tan\phi| + C$$

Another approach

$$\frac{d}{d\phi}\ln|\sec\phi + \tan\phi| + C$$
$$= \frac{1}{\sec\phi + \tan\phi}(\sec\phi\tan\phi + \sec^2\phi)$$
$$= \frac{1}{\sec\phi + \tan\phi}\sec\phi(\tan\phi + \sec\phi)$$
$$= \sec\phi$$

6. From Exercise 1 and Exercise 5, latitude line l measured in radians

$$d = \int_0^l 6\sec\phi\, d\phi = 6\ln|\sec\phi + \tan\phi|\,\Big]_0^l$$

Latitude Line (in degrees north of the equator)	**Distance from Equator**
15°	1.589
30°	3.296
45°	5.288
60°	7.902
75°	12.166
90°	∞

7. (a)

$$\int \sec\phi\, d\phi = \int \frac{1}{\cos\phi}\, d\phi$$
$$= \int \frac{1}{\sin\left(\frac{\pi}{2}-\phi\right)}\, d\phi$$

Let $u = \frac{1}{2}\left(\frac{\pi}{2}-\phi\right)$,

then $du = -\frac{1}{2}\, d\phi$

So,

$$\int \sec\phi\, d\phi = \int \frac{-2}{\sin 2u}\, du$$
$$= -\int \frac{1}{\sin u \cos u}\, du$$
$$= -\int \frac{1}{\frac{\sin u}{\cos u}\cos^2 u}\, du$$
$$= -\int \frac{\sec^2 u}{\tan u}\, du$$
$$= -\ln\left|\tan u\right| + C$$
$$= -\ln\left|\tan\left[\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)\right]\right| + C$$

(b)

$$\int \sec\phi\, d\phi = \int \frac{1}{\cos\phi}\, d\phi$$
$$= \int \frac{1}{\sin\left(\frac{\pi}{2}+\phi\right)}\, d\phi$$

Let $u = \frac{1}{2}\left(\frac{\pi}{2}+\phi\right)$,

then $du = \frac{1}{2}\, d\phi$

So,

$$\int \sec\phi\, d\phi = \int \frac{2}{\sin 2u}\, du$$
$$= \int \frac{1}{\sin u \cos u}\, du$$
$$= \int \frac{1}{\frac{\sin u}{\cos u}\cos^2 u}\, du$$
$$= \int \frac{\sec^2 u}{\tan u}\, du$$
$$= \ln\left|\tan u\right| + C$$
$$= \ln\left|\tan\left[\frac{1}{2}\left(\frac{\pi}{2}+\phi\right)\right]\right| + C$$

Another approach

(a)

$$\frac{d}{d\phi} - \ln\left|\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)\right)\right| + C = \frac{-1}{\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)\right)}\sec^2\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)\times\frac{-1}{2}$$

$$= \frac{1}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)\right)} = \frac{1}{\sin\left(\frac{\pi}{2}-\phi\right)} = \sec\phi$$

(b) $\frac{d}{d\phi}\ln\left|\tan\left(\frac{1}{2}\left(\frac{\pi}{2}+\phi\right)\right)\right|+C=\frac{1}{\tan\left(\frac{1}{2}\left(\frac{\pi}{2}+\phi\right)\right)}\sec^2\frac{1}{2}\left(\frac{\pi}{2}+\phi\right)\times\frac{1}{2}$

$$=\frac{1}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}+\phi\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}+\phi\right)\right)}=\frac{1}{\sin\left(\frac{\pi}{2}+\phi\right)}=\sec\phi$$

8. No, because a Mercator map distorts the actual earth-distance. The further away from the equator, the more a Mercator map stretches the earth-distance. Similarly, a Mercator map cannot be used to compare areas since regions far from the equator will appear to be much larger than they actually are.