

V. EXERCISES (SOLUTIONS)

1. summation notation: $d \approx \sum_{i=1}^n R \Delta \phi \sec \phi_i$

Riemman sum: $d = \lim_{n \rightarrow \infty} \sum_{i=1}^n R \Delta \phi \sec \phi_i = \int_a^b R \sec \phi d\phi$

2. Using $\phi = \frac{\pi}{180} \theta$, where ϕ in radian and θ in degree

Latitude Line (in degrees north of the equator)	(in radians)
15°	$\pi / 12$
30°	$\pi / 6$
45°	$\pi / 4$
60°	$\pi / 3$
75°	$5\pi / 12$
90°	$\pi / 2$

3. For example, with $R = 6$, at latitude line $15^\circ = \frac{\pi}{12}$ radian,

Using Scilab to compute $d = \int_0^{\frac{\pi}{12}} 6 \sec \phi d\phi$

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              scilab-5.3.3

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Startup execution:
  loading initial environment

-->function y=f(x), y=6*sec(x), endfunction

-->intg(0,%pi/12,f)
ans =

    1.5890535

-->

```

Latitude Line (in degrees north of the equator)	Distance from Equator
15°	1.589
30°	3.296
45°	5.288
60°	7.902
75°	12.166
90°	

4. Convergence problem error ...

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-->intg(0,%pi/2,f)
      !--error 24
Convergence problem...

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latitude line 90° north of equator represents the north pole

5.
$$\int \sec \phi d\phi = \int \sec \phi \left(\frac{\sec \phi + \tan \phi}{\sec \phi + \tan \phi} \right) d\phi$$

$$= \int \frac{\sec^2 \phi + \sec \phi \tan \phi}{\sec \phi + \tan \phi} d\phi$$

Let $u = \sec \phi + \tan \phi$,

$$du = \sec \phi \tan \phi + \sec^2 \phi d\phi$$

So,

$$\begin{aligned} \int \sec \phi d\phi &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec \phi + \tan \phi| + C \end{aligned}$$

Another approach

$$\begin{aligned} &\frac{d}{d\phi} \ln|\sec \phi + \tan \phi| + C \\ &= \frac{1}{\sec \phi + \tan \phi} (\sec \phi \tan \phi + \sec^2 \phi) \\ &= \frac{1}{\sec \phi + \tan \phi} \sec \phi (\tan \phi + \sec \phi) \\ &= \sec \phi \end{aligned}$$

6. From Exercise 1 and Exercise 5, latitude line l measured in radians

$$d = \int_0^l 6 \sec \phi d\phi = 6 \ln|\sec \phi + \tan \phi| \Big|_0^l$$

Latitude Line (in degrees north of the equator)	Distance from Equator
15°	1.589
30°	3.296
45°	5.288
60°	7.902
75°	12.166
90°	∞

7. (a)

$$\begin{aligned}\int \sec \phi \, d\phi &= \int \frac{1}{\cos \phi} \, d\phi \\ &= \int \frac{1}{\sin\left(\frac{\pi}{2} - \phi\right)} \, d\phi\end{aligned}$$

$$\text{Let } u = \frac{1}{2}\left(\frac{\pi}{2} - \phi\right),$$

$$\text{then } du = -\frac{1}{2} \, d\phi$$

So,

$$\begin{aligned}\int \sec \phi \, d\phi &= \int \frac{-2}{\sin 2u} \, du \\ &= -\int \frac{1}{\sin u \cos u} \, du \\ &= -\int \frac{1}{\frac{\sin u}{\cos u} \cos^2 u} \, du \\ &= -\int \frac{\sec^2 u}{\tan u} \, du \\ &= -\ln|\tan u| + C \\ &= -\ln\left|\tan\left[\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)\right]\right| + C\end{aligned}$$

(b)

$$\begin{aligned}\int \sec \phi \, d\phi &= \int \frac{1}{\cos \phi} \, d\phi \\ &= \int \frac{1}{\sin\left(\frac{\pi}{2} + \phi\right)} \, d\phi\end{aligned}$$

$$\text{Let } u = \frac{1}{2}\left(\frac{\pi}{2} + \phi\right),$$

$$\text{then } du = \frac{1}{2} \, d\phi$$

So,

$$\begin{aligned}\int \sec \phi \, d\phi &= \int \frac{2}{\sin 2u} \, du \\ &= \int \frac{1}{\sin u \cos u} \, du \\ &= \int \frac{1}{\frac{\sin u}{\cos u} \cos^2 u} \, du \\ &= \int \frac{\sec^2 u}{\tan u} \, du \\ &= \ln|\tan u| + C \\ &= \ln\left|\tan\left[\frac{1}{2}\left(\frac{\pi}{2} + \phi\right)\right]\right| + C\end{aligned}$$

Another approach

(a)

$$\frac{d}{d\phi} - \ln\left|\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)\right)\right| + C = \frac{-1}{\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)\right)} \sec^2 \frac{1}{2}\left(\frac{\pi}{2} - \phi\right) \times \frac{-1}{2}$$

$$= \frac{1}{2 \sin\left(\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)\right) \cos\left(\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)\right)} = \frac{1}{\sin\left(\frac{\pi}{2} - \phi\right)} = \sec \phi$$

$$\begin{aligned}
\text{(b)} \quad \frac{d}{d\phi} \ln \left| \tan \left(\frac{1}{2} \left(\frac{\pi}{2} + \phi \right) \right) \right| + C &= \frac{1}{\tan \left(\frac{1}{2} \left(\frac{\pi}{2} + \phi \right) \right)} \sec^2 \frac{1}{2} \left(\frac{\pi}{2} + \phi \right) \times \frac{1}{2} \\
&= \frac{1}{2 \sin \left(\frac{1}{2} \left(\frac{\pi}{2} + \phi \right) \right) \cos \left(\frac{1}{2} \left(\frac{\pi}{2} + \phi \right) \right)} = \frac{1}{\sin \left(\frac{\pi}{2} + \phi \right)} = \sec \phi
\end{aligned}$$

8. No, because a Mercator map distorts the actual earth-distance. The further away from the equator, the more a Mercator map stretches the earth-distance. Similarly, a Mercator map cannot be used to compare areas since regions far from the equator will appear to be much larger than they actually are.