Solver: Cao Hannan

Email Address: HCAO001@e.ntu.edu.sg

1. (a): $\lim_{x\to-\infty}\frac{x^2-1}{x^3-1}$ = $\lim_{x\to-\infty}\frac{\frac{1}{x}-\frac{1}{x^3}}{1-\frac{1}{x^3}}$ = 0/1 = 0

(b): for any x ϵ R, we have:

-1 $\leq$ $\sin x$ $\leq$ 1, so

1 $\leq$ $|\sin x+2|$ $\leq$ 3

therefore, we have $\frac{1}{x^2+1} \leq \frac{|\sin x+2|}{x^2+1} \leq \frac{3}{x^2+1}$

So, by the Sandwich Theory, as $\lim_{x\to\infty}\frac{1}{x^2+1} \leq \lim_{x\to\infty}\frac{|\sin x+2|}{x^2+1} \leq \lim_{x\to\infty}\frac{3}{x^2+1}$,

and $\lim_{x\to\infty}\frac{1}{x^2+1}$ = 0 = $\lim_{x\to\infty}\frac{3}{x^2+1}$

So, $\lim_{x\to\infty}\frac{|\sin x+2|}{x^2+1}$ = 0

(c): As y = x*ln(x^2)

$\frac{dy}{dx} = \ln(x^2) + x * \frac{d(\ln(x^2))}{dx}$

$\frac{dy}{dx} = \ln(x^2) + 2x^2 * \frac{1}{x^2}$

$\frac{dy}{dx} = \ln(x^2) + 2$

(d): As y = $\frac{e^{2x}\cos 2x}{x}$

$$\frac{dy}{dx} = \frac{\frac{d(e^{2x}\cos 2x)}{dx}x - \frac{dx}{dx}e^{2x}\cos 2x}{x^2}$$

$$\frac{dy}{dx} = \frac{x(\frac{d(e^{2x})}{dx}\cos 2x + \frac{d(\cos 2x)}{dx}e^{2x}) - e^{2x}\cos 2x}{x^2}$$

$$\frac{dy}{dx} = \frac{x(2e^{2x}\cos 2x - 2\sin 2x e^{2x}) - e^{2x}\cos 2x}{x^2}$$

2. (a): f(x) = $e^x e^2(x+1)$

$\frac{d(f(x))}{dx} = e^2(\frac{d(e^x)}{dx}$(x + 1) + $\frac{d(x+1)}{dx}e^x)$

$\frac{d(f(x))}{dx} = e^2(e^x$(x + 1) + $e^x)$

$\frac{d(f(x))}{dx} = e^2(e^x$(x + 2))

substitute x = 1 into differentiated equation we can get $\frac{d(f(x))}{dx} = 2e^2$ at x = 0

as f(0) = e^2 we can get the tagent line at x = 0 for f(x) = $e^x e^2(x+1)$ which is

y - e^2 = $2e^2(x-0)$ => y = $2e^2 x + e^2$

(b): $\int(x^3 + \tan x + e^x + 1/x\text{^}\sqrt{2}\,)dx$

= $\frac{1}{4}x^4 + (\sec x)^2 + e^x + \frac{1}{-\sqrt{2}+1}x^{-\sqrt{2}+1}$ + C

(c): $\int(lnx + xln(x^2))\,dx$

= $\int 1 * lnx\,dx$ + $\int xln(x^2)\,dx$

by integration by parts $\int 1 * lnx\,dx$ = xln(x) + $\int x\frac{1}{x}dx$

= xln(x) + x + K

for $\int xln(x^2)\,dx$:

let u = x^2

so $\int xln(x^2)\,dx = (\int \ln(u)\,du)\frac{1}{2} = uln(u) + u + K = x^2\ln(x^2) + x^2 + K$

By combining these two equations together, we can get the equation for this problem, which is $xln(x) + x + x^2\ln(x^2) + x^2 + C$

(d) : $y = e^x$

So the differential equation is as following: $x^2 + 4x - 5 = 0$

by solving the equation, we can get x1 = 5, x2 = -1.

Therefore, we can get y =C1e$^{5x} + C2\,e^{-x}$

As y = 0 when x = 0, and $\frac{dy}{dx}$ = 2, when x = 0

so we can get $\begin{cases} c1 + c2 = 0 \\ 5c1 - c2 = 2 \end{cases}$

by solving the equation, c1 = $\frac{1}{3}$, c2 = -$\frac{1}{3}$

so y = $\frac{1}{3}$e$^{5x} - \frac{1}{3}e^{-x}$

3. (a): $\left\{\frac{(n+1)(n+2)}{n^2}\right\}_{n=1}^{n}$

(b): $-\frac{\pi}{2} \leq \tan^{-1} n \leq \frac{\pi}{2}$

$-\frac{\pi}{n^2} \leq \frac{\tan^{-1} n}{n^2} \leq \frac{\pi}{n^2}$

$\lim_{n\to\infty} -\frac{\pi}{n^2} = 0 = \lim_{n\to\infty} \frac{\pi}{n^2}$

By Sandwich Theory, $\lim_{n\to\infty} \frac{\tan^{-1} n}{n^2} = 0$.

(c): By looking at the structure of the expression, we can easily know that

$a_n = 4^{1/3^n} a_{n-1}$

so, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 4^{\overline{\sum_{n=1}^{n} 3^n}^{\,1}} = 4^{\lim_{n\to\infty} \sum_{n=1}^{n} 1/3^n}$

$\lim_{n\to\infty} \sum_{n=1}^{n} 1/3^n = \lim_{n\to\infty} \frac{a_{1(1-q^n)}}{1-q} = \frac{1}{2}$

therefore, $\lim_{n\to\infty} a_n = \frac{1}{2}$

(d): $x_0 = a - 2h, \quad x_1 = a - h, \quad x_2 = a + h, \; x_3 = a + 2h$

$p_3(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +$
$f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)(x - x_3)$

the derivative is

$p_3(x) = f[x_0, x_1] + f[x_0, x_1, x_2](2x - x_0 - x_1) + f[x_0, x_1, x_2, x_3](x - x_1)(x -$
$x_2) + (x - x_0)(x - x_2) + (x - x_0)(x - x_1)$

$f[x_0, x_1] = \frac{f(a-h) - f(a-2h)}{h}$

$f[x_0, x_1, x_2] = \frac{1}{6h^2}(f(a+h) - 3f(a-h) + 2f(a-2h))$

$f[x_0, x_1, x_2, x_3] = \frac{1}{12h^3}(f(a+2h) - 2f(a+h) - f(a-2h))$

By putting the above equations together, we can get the expression for f(a)

$$f(a)' = \frac{f(a-2h) - 8f(a-h) + 8f(a-h) - f(a-2h)}{12h}$$

4. (a): f(x) = $\begin{cases} \frac{x-\cos x}{x^2}, & x \neq 0 \\ \frac{1}{5}, & x = 0 \end{cases}$

to find Maclaurin series of f(x), we need to first find the differentiation of f(x) at

at x = 0. To find the differentiation of f(x) at x = 0, f(x) need to be continue at x = 0 first. However, $\lim_{x\to 0} \frac{x-\cos x}{x^2} \neq f(0)$. Therefore, f(x) is not continuous at x = 0. Therefore, the Maclaurin series of f(x) does not exist.

(b):

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

whereby n = 4, a = 0, b = 2

$\int_0^2 \frac{x^3+x}{1+x^3} = \frac{1}{4} * (f(0) + 2f(0.5) + 2f(1.0) + 2f(1.5) + f(2))$ = 1.584920635

(c):

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

. $\int_0^2 \frac{x^3+x}{1+x^2} = \int_0^2 x = \frac{1}{6} * (f(0) + 4f(0.5) + 2f(1.0) + 4f(1.5) + f(2))$ = 2

(d): $f(t) = \begin{cases} 2, 1-T \leq t \leq 1+T \\ 0, \quad otherwise \end{cases}$

Fourier transform is : $\int_{1-T}^{1+T} f(t)e^{-i\omega t}\,dt$

$= \int_{1-T}^{1+T} 2e^{-i\omega t}\,dt$

$= \frac{-2e}{i\omega}(e^T - e^{-T})$

$= \frac{-2e^{-i\omega}}{i\omega}(e^{-i\omega T} - e^{i\omega T})$

$= \frac{4e^{-i\omega}}{\omega}\sin \omega t$

Feel free to contact me if there is any problem with the solution.

Good luck for your exam, ☺