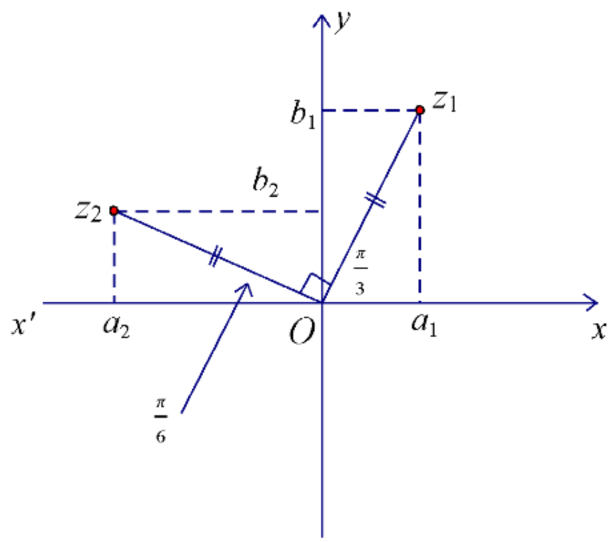
20th CSEC – Past Year Paper Solution *2017-2018 Sem 2*
CZ/CE 1011 – Engineering Mathematics I

1)

a)

i) $z_1 = a_1 + jb_1; \arg z_1 = \frac{\pi}{3} \Rightarrow z_1$ is in first quadrant, then $a_1 > 0, b_1 > 0$
$\Rightarrow a_1 = |z_1|\cos\frac{\pi}{3} = \frac{\sqrt{7}}{2}; b_1 = |z_1|\sin\frac{\pi}{3} = \sqrt{7}.\frac{\sqrt{3}}{2} = \frac{\sqrt{21}}{2}$. Thus, $z_1 = a_1 + jb_1 = \frac{\sqrt{7}}{2} + j\frac{\sqrt{21}}{2}$.

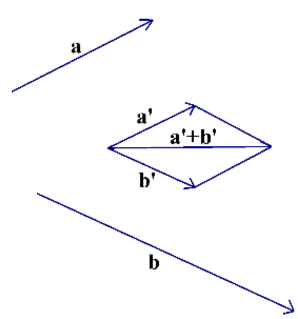
ii) Since the angle z_1 and z_2 (in second quadrant) is $\frac{\pi}{2}$ and $\arg(z_1) = \frac{\pi}{3}$, it follows that $\arg(z_2) = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$. We have $a_2 = |z_2|\cos\frac{5\pi}{6} = -\sqrt{7}.\frac{\sqrt{3}}{2} = -\frac{\sqrt{21}}{2}; b_2 = |z_2|\sin\frac{5\pi}{6} = \sqrt{7}\cdot\frac{1}{2} = \frac{\sqrt{7}}{2}$.
Thus, $z_2 = a_2 + jb_2 = -\frac{\sqrt{21}}{2} + j\frac{\sqrt{7}}{2}$.



b) $\frac{3(e^{jw}-e^{-jw})}{jw} = 3\cdot\frac{(\cos w + j\sin w)-(\cos(-w)+j\sin(-w))}{jw}$

$$= 3\cdot\frac{\cos w + j\sin w - \cos w + j\sin w}{jw} = \frac{3\sin w}{w}.$$

c) Normalizations the vectors $\boldsymbol{a} = (2,1,2)$ and $\boldsymbol{b} = (-3,0,4)$ we have $||\boldsymbol{a}|| = 3 \Rightarrow \widehat{\boldsymbol{a}} = (\frac{2}{3},\frac{1}{3},\frac{2}{3})$.
Similarly, $||\boldsymbol{b}|| = 5 \Rightarrow \widehat{\boldsymbol{b}} = \left(-\frac{3}{5},0,\frac{4}{5}\right)$.
Note that $|\widehat{\boldsymbol{a}}| = |\widehat{\boldsymbol{b}}| = 1$. That's why vector $\boldsymbol{c} = \widehat{\boldsymbol{a}} + \widehat{\boldsymbol{b}} = (\frac{1}{15},\frac{1}{3},\frac{22}{15})$ is direction vector of bisects of the angle between **a** and **b**. (diagonal of the rhombus)



d) $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k},\ \boldsymbol{b} = (1,5,-2) = 1\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}, \boldsymbol{c} = 3\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$

$\Rightarrow \boldsymbol{d} = \boldsymbol{a} - 2\boldsymbol{b} + 3\boldsymbol{c} = 9\boldsymbol{i} - 25\boldsymbol{j} + 14\boldsymbol{k} = (9,-25,14) \Rightarrow |\boldsymbol{a}| = \sqrt{9^2 + (-25)^2 + 14^2} = \sqrt{902}.$

2)

a) $p(x) = x^2 - 2x - 3$

$A = \begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix} \Rightarrow p(A) = -3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 2\begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix} + \begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix}^{-2}$

We have $\begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix}^2 = \begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix} = \begin{bmatrix}(-1)(-1) + 2.0 & (-1)(2) + 2.3\\ (0)(-1) + 3.0 & (0)(2) + 3.3\end{bmatrix} = \begin{bmatrix}1 & 4\\ 0 & 9\end{bmatrix}$

$p(A) = -3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - 2\begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix} + \begin{bmatrix}-1 & 2\\ 0 & 3\end{bmatrix}^2 = \begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix} - \begin{bmatrix}-2 & 4\\ 0 & 6\end{bmatrix} + \begin{bmatrix}1 & 4\\ 0 & 9\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

b) Matrix A has Row Echelon form $\begin{bmatrix}1 & 0 & \frac{1}{3}\\ 0 & 1 & -\frac{4}{3}\\ 0 & 0 & 0\end{bmatrix} \Rightarrow$ rank of matrix A is 2.

c) The linear system $\begin{cases}(5-k)x + y = 1\\ 6x + (6-k)y = k\end{cases}$ has $D = \begin{vmatrix}5-k & 1\\ 6 & 6-k\end{vmatrix} = (5-k)(6-k) - 6(1) = k^2 - 11k + 24$

The system has an unique solution: $D \neq 0 \Leftrightarrow k^2 - 11k + 24 \neq 0 \Leftrightarrow \begin{cases}k \neq 3\\ k \neq 8\end{cases}$

(Note that unique solution is $\begin{cases}x = \frac{D_x}{D}\\ y = \frac{D_x}{D}\end{cases}$ with $D_x = \begin{vmatrix}1 & 1\\ k & 6-k\end{vmatrix} = 6 - 2k; D_y = \begin{vmatrix}5-k & 1\\ 6 & k\end{vmatrix} = -k^2 + 5k - 6$).

The system is inconsistent $\Leftrightarrow \begin{cases}D = 0\\ D_x \neq 0\end{cases} or \begin{cases}D = 0\\ D_y \neq 0\end{cases} \Leftrightarrow$

$\begin{cases}k^2 - 11k + 24 = 0\\ 6 - 2k \neq 0\end{cases} or \begin{cases}k^2 - 11k + 24 = 0\\ -k^2 + 5k - 6 \neq 0\end{cases}$

$$\begin{cases}\left[\begin{array}{l}k = 3\\ k = 8\end{array}\right.\\ k \neq 3\end{cases} or \begin{cases}\left[\begin{array}{l}k = 3\\ k = 8\end{array}\right.\\ \begin{cases}k \neq 2\\ k \neq 3\end{cases}\end{cases} \Rightarrow k = 8$$

Editor's comment: Alternatively, do a row reduction.

$$\left(\begin{array}{cc|c}5-k & 1 & 1\\ 6 & 6-k & k\end{array}\right) \sim \left(\begin{array}{cc|c}6 & 6-k & k\\ 0 & -24 + 11k - k^2 & 6 - 5k + k^2\end{array}\right)$$

- For unique solution, it suffices to have 2 pivot points, thus we similarly need $k^2 - 11k + 24 \neq 0$
- For inconsistent solution, we similarly require $k^2 - 11k + 24 = 0$ and $k^2 - 5k + 6 \neq 0$

d) Denote matrix $A = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}, A_1 = \left[\begin{array}{ccc|c}1 & 0 & 1 & 1\\ 0 & 1 & -2 & 1\\ 0 & 0 & 0 & 0\end{array}\right]$

We have $A_1 \sim \left[\begin{array}{ccc|c}1 & 0 & 1 & 1\\ 0 & 1 & -2 & 1\end{array}\right]$

Clearly $r(A) = r(A_1) = 2$. Thus system is consistent. Choose x_2 and x_3 are basic variables, we have $\begin{cases}x_3 = 1 - x_1\\ x_2 - 2x_3 = 1\end{cases} \Rightarrow \begin{cases}x_3 = 1 - x_1\\ x_2 = 3 - 2x_1\end{cases}$

Let $x_1 = a \Rightarrow$ solutions of system are $x_1 = a, x_2 = 3 - 2a, x_3 = 1 - a$

e) $A^{-1}B^{-1} = (AB)^{-1} = (BA)^{-1} = B^{-1}A^{-1}$. Thus statement is true.

3)

a) Rearrange the table:

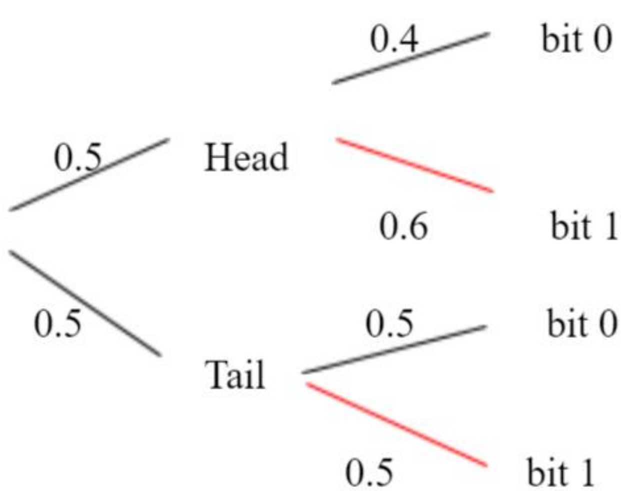
9	11	15	25	28	29	30	30	30	30
32	34	35	35	39	40	42	44	48	50

i) Mean = 31.8
Medium = (30+32)/2 = 31
Mode = 30
80th percentile:
Rank: $R = \frac{P(N+1)}{100} = 80 \times \frac{21}{100} = 16.8$
$IR = 16; FR = 0.8$
$Rank\ 16: 40;\ Rank\ 17: 42$
$\Rightarrow 0.8 \times (42 - 40) + 40 = 41.6$ is 80th percentile
Range: $50 - 9 = 41$
Sample Variance: $\frac{\sum(X-M)^2}{N-1} = 121.43, where\ X\ is\ every\ values, M\ is\ mean, N = 20$

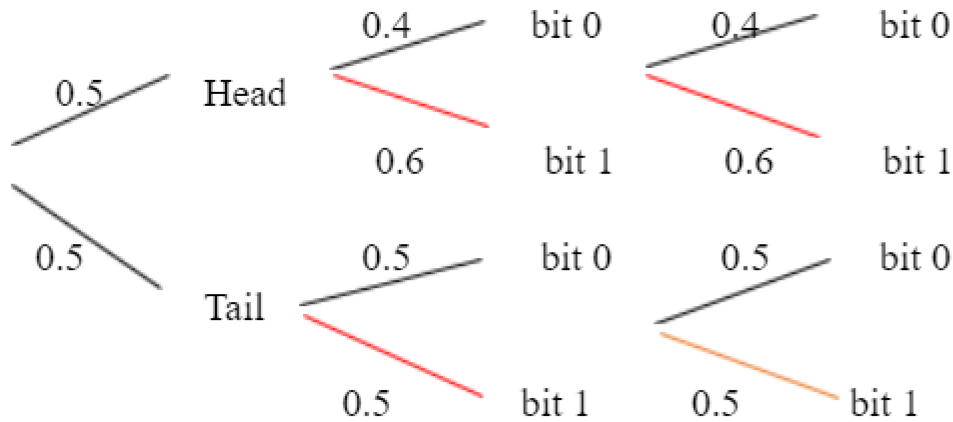
ii) 5 | 0
4 | 0 2 4 8
3 | 0 0 0 0 2 4 5 5 9
2 | 5 8 9
1 | 1 5
0 | 9

b)

i) Due to the tree diagram, the probability that first bit generated is 1 is:
$P = 0.5 \times 0.6 + 0.5 \times 0.5 = 0.55$



ii)



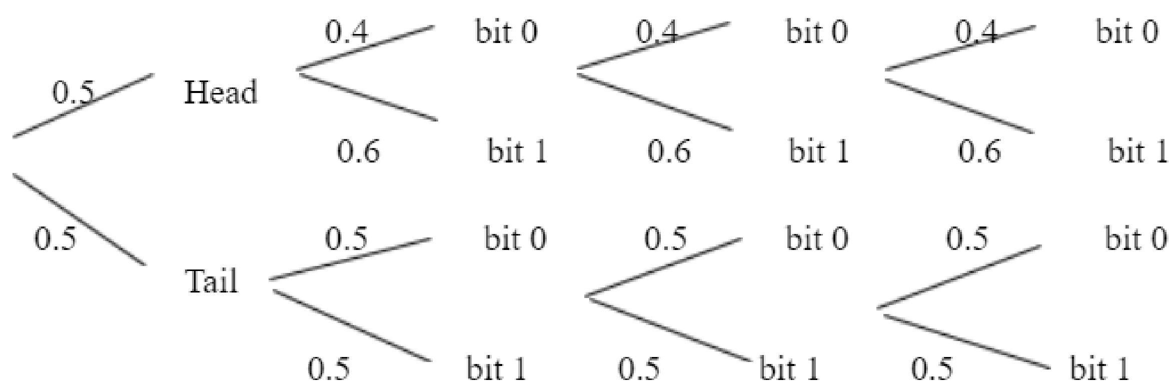
We have to find $P(Head|2bits = 1) = \frac{P(Head\ AND\ 2\ bits=1)}{P(2\ bits=1)}$

$P(Head\ AND\ 2\ bits = 1) = 0.5 \times 0.6 \times 0.6 = 0.18$

$P(2\ bits = 1) = 0.5 \times 0.6 \times 0.6 + 0.5 \times 0.5 \times 0.5 = 0.305$

$\Rightarrow P(Head\ |\ 2\ bits = 1) = \dfrac{0.18}{0.305} = 0.59$

iii)



Need to find $P(third\ bit = 0|first\ 2\ bits\ NOT\ BOTH\ 0) = P'$

$$P' = \frac{P1(third\ bit = 0\ AND\ first\ 2\ bits\ NOT\ BOTH\ 0)}{P2(first\ 2\ bits\ NOT\ BOTH\ 0)} = \frac{P1}{P2}$$

$$P1 = 0.4 \times 0.6 \times 0.6 \times 0.5 + 2 \times 0.4 \times 0.6 \times 0.4 \times 0.5 + 3 \times (0.5^4) = 0.3555$$

$$P2 = 0.5 \times 0.6 \times 0.6 + 2 \times 0.5 \times 0.4 \times 0.6 + 3 \times (0.5^3) = 0.795$$

$$P' = \frac{0.3555}{0.795} = 0.4472$$

c)

i) $Y = X + 5 \Rightarrow f(Y) = 0.01 \times (10 - |Y - 5|)$ when Y in [-5,15]

Y in [-5,5], $|Y - 5| = 5 - Y$

Y in [5,15], $|Y - 5| = Y - 5$

$\Rightarrow Expected\ value\ of\ Y\ is$:

$$E(Y) = \int Y \times f(Y) = \int_{-5}^{5} Y(0.01)\big(10 - (5 - Y)\big) + \int_{5}^{15} Y(0.01)\big(10 - (Y - 5)\big) = 5$$

Similiarly, we calculate expected value of Y^2

$$E(Y^2) = \int Y^2 \times f(Y)$$

$$= \int_{-5}^{5} Y^2(0.01)\big(10-(5-Y)\big) + \int_{5}^{15} Y^2(0.01)\big(10-(Y-5)\big) = \frac{125}{3}$$

$$\Rightarrow Variance = E(Y^2) - \big(E(Y)\big)^2 = \frac{50}{3} = 16.67$$

Editor's comment: It actually suffices to find E(X) and compute E(Y) = E(X) + 5
$E(X) = \int_{-10}^{10} 0.01(10-|x|)\,\mathrm{d}x = 0$ by symmetry, so E(Y) = 5
Similarly, Var(Y) = Var(X), so it also suffices to find Var(X) to obtain Var(Y).

ii) Since X>0 yields Y (=X+5) > 0 in all cases, then the probability is 1.

4)

a)

i) $n = 200, p = 0.3 \Rightarrow Average = 200 \times 0.3 = 60$

$$Variance = npq = 200 \times 0.3 \times 0.7 = 4.2$$

$$Standard\ deviation = \sqrt{4.2}$$

ii) Use continuity correction in normal distribution, we have to find:

$$P(69.5 < X < 70.5) = P\left(\frac{69.5-60}{6.48} < Z < \frac{70.5-60}{6.48}\right) = P(1.466 < Z < 1.62)$$

$$= P(Z < 1.62) - P(Z < 1.466) = 0.9474 - 0.9292 = 0.0182$$

b)

i) Let the new variable $\bar{x} = \overline{x_1} - \overline{x_2}$. Hence, the sample will be difference between 2 bottles ⇒ $n = 10$

n small, we use t-Distribution. ($r = n - 1 = 9$)

$Var(x1) = 3.51^2 = 12.3201; Var(x2) = 4.27^2 = 18.2329$

$\Rightarrow Var(x) = Var(X1) + Var(x2) = 30.553$

$Standard\ Deviation\ of\ x: \sigma_1 = \sqrt{30.553} = 5.53$

Mean of variable x is $299 - 294 = 5ml$

H_0 is that there are no difference in mean volume filled by 2 machines

H_A is that there are some difference in mean volume filled by 2 machines

⇒ This is a 2-sided test.

Hypothetical mean is 0.

p-value $= P(X < -5\ or\ X > 5) = P\left(Z < \frac{-5-0}{\frac{5.53}{\sqrt{10}}} or Z > \frac{5-0}{\frac{5.53}{\sqrt{10}}}\right) = P(Z < -2.86\ or\ Z > 2.86) = 2P(Z > 2.86)$

Due to t-Distribution table (r=9), $2.821 < t_9 < 3.250$

⇒ $2 \times 0.005 <$ p-value $< 2 \times 0.01 \Rightarrow 0.01 <$ p-value < 0.02, which is smaller than level of significance (5%) ⇒ Reject H_0

ii) We find the coefficient of 95% confidence interval:

$P(-a < X < a) = 0.95 \Rightarrow P(X < a) = 0.975 \Rightarrow a = 1.96$

Lower bound $= 299 - 1.96\left(\frac{4.27}{\sqrt{10}}\right) = 296.35\ ml$

Upper bound = $299 + 1.96\left(\frac{4.27}{\sqrt{10}}\right) = 301.65\ ml$

c) Bisection method and Newton method.

For Bisection method:

For a given function *f(x)*, the Bisection Method algorithm works as follows:

1. two values a and b are chosen for which *f(a) > 0* and *f(b) < 0* (or the other way around)

2. **interval halving**: a midpoint *c* is calculated as the arithmetic mean between *a* and *b*, *c= (a + b)/2*

3. the function *f* is evaluated for the value of *c*

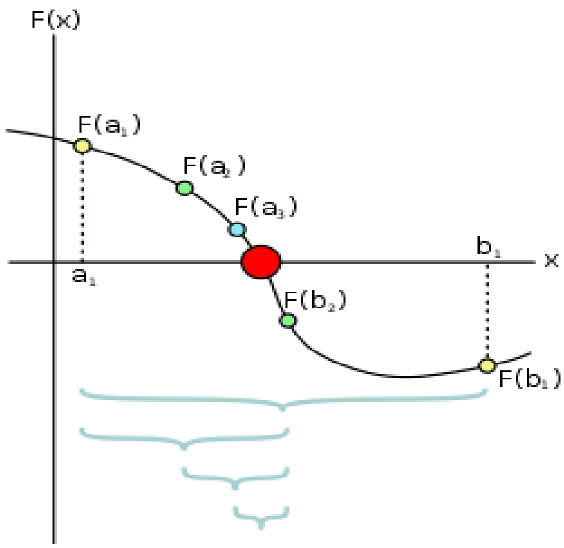
4. if *f(c) = 0* means that we found the root of the function, which is *c*

5. if *f(c) ≠ 0* we check the sign of *f(c)*:

if *f(c)* has the same sign as *f(a)* we replace *a* with c and we keep the same value for *b*

if *f(c)* has the same sign as *f(b)*, we replace *b* with *c* and we keep the same value for *a*

6. we go back to step 2. and recalculate *c* with the new value of *a* or *b*



Source: https://x-engineer.org/undergraduate-engineering/advanced-mathematics/numerical-methods/the-bisection-method-for-root-finding/

--End of Answers--

Solver: Tran Huu Hoang