CX1012 Engineering Maths II 2018/19 - S1 : Deng Jinyang
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1.(a)

$$y = \frac{1}{2+\sqrt{X}}$$

$$y' = \lim_{h\to 0} \frac{\frac{1}{2+\sqrt{X+h}} - \frac{1}{2+\sqrt{X}}}{h} \quad \text{(by definition)}$$

$$= \lim_{h\to 0} \frac{\frac{(2+\sqrt{X})-(2+\sqrt{X+h})}{(2+\sqrt{X+h})(2+\sqrt{X})}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{X}-\sqrt{X+h}}{(2+\sqrt{X+h})(2+\sqrt{X})\,h} \quad \times \frac{X+\sqrt{X+h}}{X+\sqrt{X+h}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{X}-\sqrt{X+h})(X+\sqrt{X+h})}{(2+\sqrt{X+h})(2+\sqrt{X})\,h\,(X+\sqrt{X+h})}$$

$$= \lim_{h\to 0} \frac{X-(X+h)}{(2+\sqrt{X+h})(2+\sqrt{X})\,h\,(X+\sqrt{X+h})}$$

$$= \lim_{h\to 0} \frac{-h}{(2+\sqrt{X+h})(2+\sqrt{X})\,h\,(X+\sqrt{X+h})}$$

cancelling h on top & below, sub in $h \approx 0$

$$\approx \frac{-1}{(2+\sqrt{X})(2+\sqrt{X})(X+\sqrt{X})}$$

When $X=1$, $y'(1) = \frac{-1}{(2+\sqrt{1})(2+\sqrt{1})(1+\sqrt{1})} = -\frac{1}{18}$ //

(b)

$$\lim_{x\to 0} (\cos x)^{\frac{1}{x}}$$

$$= \lim_{x\to 0} \left(e^{\frac{1}{x}\ln(\cos x)}\right)$$

breaks down to

$$f(x) = \lim_{x\to 0} \frac{1}{x}\ln(\cos x)$$

L'Hopital

$$= \lim_{x\to 0} \frac{-\tan(x)}{1}$$

$$= 0$$

Since $e^{f(x)}$ is continuous at $x=0$, question can be rewritten as

$$\lim_{f(x)\to 0} \left(e^{f(x)}\right) = e^0 = 1$$ //

(c) $\frac{d}{dx}\left(\frac{e^2 \ln x^2}{x}\right) \xrightarrow{\text{take out constant}} = e^2 \frac{d}{dx}\left(\frac{\ln x^2}{x}\right)$

$$= e^2 \frac{\left(\frac{2}{x}\right)(x) - (1)(\ln x^2)}{x^2} \quad \text{quotient rule}$$

$$= e^2 \frac{2 - \ln x^2}{x^2} \quad //$$

(d) $\frac{d}{dx}\left(\sqrt{1-x}\,\sin^{-1}(e^{2x})\right) = \left(\frac{-1}{2\sqrt{1-x}}\right)\left(\sin^{-1}(e^{2x})\right) + \left(\sqrt{1-x}\right)\left(\frac{2e^{2x}}{\sqrt{1-(e^{2x})^2}}\right)$

$$= \frac{-\sin^{-1}(e^{2x})}{2\sqrt{1-x}} + \frac{2\sqrt{1-x}\,e^{2x}}{\sqrt{1-e^{4x}}} \quad //$$

Main point is in realising that $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \frac{d}{dx}(x)$,

so when x is subbed with e^{2x},

$$\frac{d}{dx}(\sin^{-1}(e^{2x})) = \frac{1}{\sqrt{1-(e^{2x})^2}}(2e^{2x})$$

(e)

"best time" = incur lowest costs

wait for cheap house price

cannot wait for too long, as rental needs to be paid.

Let t be number of months between best month and Dec 2018, x be costs.

$$x = 100(10000 + 5(t-20)^2) + 2000t$$

$$= 100000 + 500(t^2 - 40t + 400) + 2000t$$

$$= 120000 + 500t^2 - 18000t$$

$$\frac{dx}{dt} = 1000t - 18000$$

$$\frac{dx}{dt} = 0 \text{ when } t = 18$$

18 months from Dec 2018 is June 2020. //

2.(a) $\int (x^4 + x^{-1} + \sin(x) - 4)\,dx$

$= \frac{x^5}{5} + \ln(x) - \cos(x) - 4x + C$

(b) let $u = t^2$

$du = 2t\,dt$

Thus, $\int t^3 \sin(t^2)\,dt$ can be rewritten as $\frac{1}{2}\int u\sin(u)\,du$

Integration by parts:

$f = u \longrightarrow df = 1\,du$

$dg = \sin(u)du \longrightarrow g = -\cos(u)$

$\frac{1}{2}\left(u(-\cos(u)) - \int(-\cos(u))\,du\right)$

$= -\frac{1}{2}u\cos(u) + \frac{1}{2}\int \cos(u)\,du$

$= -\frac{1}{2}u\cos(u) + \frac{1}{2}\sin(u) + C$) sub in $u = t^2$

$= -\frac{1}{2}t^2\cos(t^2) + \frac{\sin(t^2)}{2} + C$

(c) Area of triangle $= \frac{1}{2}(\underset{\text{height}}{4})(\underset{\text{base}}{2}) = 4$

Line $x = a$ cuts triangle into 2 areas of 2 each.

$\int_0^a (4-2x)\,dx = 2$

$[4x - x^2]_0^a = 2$

$4a - a^2 = 2$

$a^2 - 4a + 2 = 0$

$a = 0.5857864376$ or 3.414213562 (rejected as it is >2)

(d) $x^2 - 3x + 2 = 0$

$x = 2$ or 1

$y = ae^x + be^{2x} \longrightarrow 0 = a + b \longrightarrow b = 3$

$y' = ae^x + 2be^{2x} \longrightarrow 3 = a + 2b \longrightarrow a = -3$

$\therefore y = -3e^x + 3e^{2x}$

3.(a)

No solution available from solver & friends.

(b)

$-1 \leqslant \sin n \leqslant 1$

$\sin^2 n \leqslant 1$

$0 \leqslant \sin^2 n + \sin n + 1 \leqslant 3$ (sandwich theorem)

$$\lim_{n\to\infty} \frac{0}{n^{10}+n^3+n+17} = 0$$

$$\lim_{n\to\infty} \frac{3}{n^{10}+n^3+n+17} = 0$$

$$\therefore \lim_{n\to\infty} \frac{\sin^2 n+\sin n+1}{n^{10}+n^3+n+17} = 0$$

(c) ratio test:

common ratio : (x^2-9)

$-1 \leqslant (x^2-9) \leqslant 1$

$8 \leqslant x^2 \leqslant 10$

$\sqrt{8} \leqslant x \leqslant \sqrt{10}$ or $-\sqrt{10} \leqslant x \leqslant -\sqrt{8}$

(d) Firstly: since $\frac{\pi}{4} < 1$, $\lim_{n\to\infty} (-1)^{n+1}\left(\frac{\pi}{4}\right)^{2n+1} = 0$

so it can be simplified to

$$\lim_{n\to\infty} \frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}.$$

notice:

$$\left(\frac{\pi}{3}\right)^{2n+1} = \overbrace{\frac{\pi}{3} \times \frac{\pi}{3} \times \frac{\pi}{3} \times \ldots \times \frac{\pi}{3}}^{(2n+1)\text{ terms}}$$

$$(2n+1)! = 1 \times 2 \times 3 \times \ldots \times (2n+1)$$

From 2 onwards, the difference between 2, 3, 4... and $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$... only increases, all the way until $(2\times\infty+1)$ and $\frac{\pi}{3}$ as $n\to\infty$

$\therefore a_n$ converges, Limit $= 0$ as $n \to \infty$

(e) $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}(n+6)}{4^{n+3}(2n+3)!}\times\frac{4^{n+2}(2n+1)!}{(-1)^n x^{2n+1}(n+5)}\right|$

$=\lim_{n\to\infty}\left|\frac{(-1)x^2(n+6)}{4(2n+2)(2n+3)}\times\frac{1}{n+5}\right|$

$=\lim_{n\to\infty}\frac{x^2(n+6)}{4(2n+2)(2n+3)(2n+5)} < 1$

$\therefore$ The series converge for $x \in (-\infty, \infty)$

4.(a) $f(x) = x^2-4x+4+\sin(2-x)$

$= (x-2)^2 + \sin(2-x)$

$= (1+(x-3))^2 + \sin(2-x)$

$= \sum_{n=0}^{\infty}\binom{2}{n}(x-3)^n + \sum_{n=0}^{\infty}(-1)^n\frac{(2-x)^{2n+1}}{(2n+1)!}$

$= \sum_{n=0}^{\infty}\binom{2}{n}(x-3)^n + (-1)^n\frac{(2-x)^{2n+1}}{(2n+1)!}$ (general term)

Expanding

$=\left[1+2(x-3)+\frac{2(1)}{2!}(x-3)^2\right]+\left[(2-x)-\frac{(2-x)^3}{3!}+\frac{(2-x)^5}{5!}\right]$

$= 1+2(x-3)+(x-3)^2+(2-x)+\frac{(x-2)^3}{6}-\frac{(x-2)^5}{120}$

$= 1+2x-6+x^2-6x+9+(2-x)+\frac{(x-2)^3}{6}-\frac{(x-2)^5}{120}$

$= x^2-5x+6 + \frac{(x-2)^3}{6}-\frac{(x-2)^5}{120}$

$= -(x-2)+(x-2)^2+\frac{(x-2)^3}{6}-\frac{(x-2)^5}{120}$ (expanded to 4 terms, centered at $x=2$)

(b)

$f(x) = xe^{-5x}$ $\quad f(0)=0$

$f'(x) = e^{-5x}(1-5x)$ $\quad f'(0)=1$

$f''(x) = 5e^{-5x}(5x-2)$ $\quad f''(0)=-10$

$f'''(x) = -25e^{-5x}(5x-3)$ $\quad f'''(0)=75$

$f^{4}(x) = 125e^{-5x}(5x-4)$ $\quad f^{4}(0)=-500$

$f(x) = 0 + x + \frac{-10}{2!}x^2+\frac{75}{3!}x^3+\frac{-500}{4!}x^4+\dots$ (expanded to 4 terms)

$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(5)^{n-1}}{n!}x^n$ (general term)

Series is convergent for all x, thus radius of convergence is ∞.

(c)

$n = 4, \quad \text{interval} = \frac{\pi}{4}$

Let $f(x) = x\sin^2 x$

$f(0) = 0$

$f(\frac{\pi}{4}) = \frac{\pi}{8}$

$f(\frac{\pi}{2}) = \frac{\pi}{2}$

$f(\frac{3\pi}{4}) = \frac{3\pi}{8}$

$f(\pi) = 0$

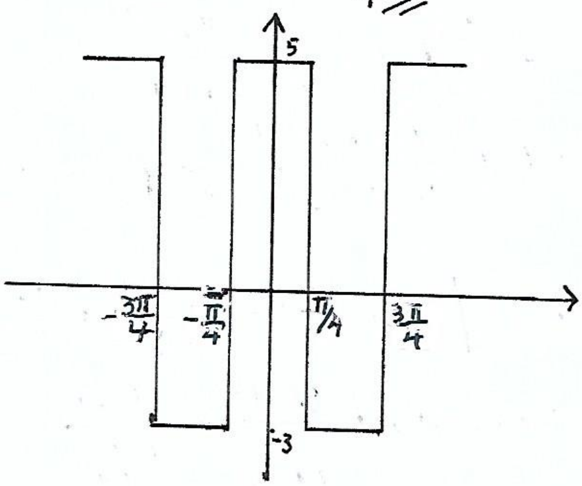
Simpson rule:

$$\frac{\frac{\pi}{4}}{3}\left(0 + 4(\tfrac{\pi}{8}) + 2(\tfrac{\pi}{2}) + 4(\tfrac{3\pi}{8}) + 0\right)$$
$$= \frac{\pi^2}{4}$$

Trapezoidal rule:

$$\frac{\frac{\pi}{4}}{2}\left(0 + 2(\tfrac{\pi}{8}) + 2(\tfrac{\pi}{2}) + 2(\tfrac{3\pi}{8}) + 0\right)$$
$$= \frac{\pi^2}{4}$$

(d)



(UNSURE)*

Coefficients: $W_0 = 2$

$$f(x) = \begin{cases} 5 & -\frac{\pi}{4} < x < \frac{\pi}{4} \\ -3 & \frac{\pi}{4} < x < \frac{3\pi}{4} \end{cases}$$

$$\frac{1}{\pi}\left[5\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-jk(2)x}\,dx - 3\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{-jk(2)x}\,dx\right]$$

$$= \frac{1}{\pi}\frac{1}{-2jk}\left[5 \times e^{-2jkx}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - 3 \times e^{-2jkx}\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\right]$$

$$= \frac{1}{\pi}\frac{1}{-2jk}\left(5e^{-\frac{\pi}{2}jk} - 5e^{\frac{\pi}{2}jk} - 3e^{-\frac{3\pi}{2}jk} - 3e^{-\frac{\pi}{2}jk}\right)$$

$$= \frac{1}{-2\pi jk}\left(2e^{-\frac{\pi}{2}jk} - 5e^{\frac{\pi}{2}jk} - 3e^{-\frac{3\pi}{2}jk}\right)$$