Curvature in the Eighties

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The notion of curvature is one of the central concepts of differential geometry; one could argue that it is *the* central one, distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic, or topological. In the words of Marcel Berger [**1**, p. 9], curvature is "the N° 1 Riemannian invariant and the most natural. Gauss and then Riemann saw it instantly."

Curvature also plays a key role in physics. The magnitude of a force required to move an object at constant speed along a curved path is, according to Newton's laws, a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein, by the curvature of space-time. All sorts of shapes, from soap bubbles to red blood cells, seem to be determined by various curvatures.

Given the vastness of the subject and the limited space available, I have restricted myself to some modest goals. First, to present a few topics that are easily accessible to a reader unfamiliar with the field. Second, to concentrate on results from the past ten years, to demonstrate the current vitality of the subject. And third, to give some feeling for the various forms of curvatures, both intrinsic and extrinsic.

The results we discuss are mainly of two types: inequalities involving integrals of curvatures, and implications of one or another type of curvature being constant. A huge subject that we shall barely touch on is that of consequences of curvature having a given sign or being constrained in other ways.

On the premise that anything published before 1980 can now be considered classic, we shall give explicit references chiefly to recent books and articles, where many further references may be found.

This article is divided into five parts.

1. Some background definitions and terminology.
2. Curvature inequalities for curves and surfaces.
3. Surfaces of constant mean curvature.
4. Higher dimensions.
5. Related results.

The first three sections concentrate on curves and surfaces in euclidean three-space, with a particular focus on recent work of Antonio Ros. A famous theorem of Alexandrov states that "all soap bubbles are spheres". In other words, soap bubbles, suitably idealized, must have constant mean curvature, and Alexandrov showed that the only compact surfaces of constant mean curvature, embedded in $\mathbb{R}^3$, are spheres. A new approach to the problem was initiated by Ros [**1**], and a subsequent paper of Montiel and Ros [**1**] contains a beautiful and elementary proof of Alexandrov's Theorem. It also contains an equally neat proof of a strong generalization to higher dimensions: the constancy of any of a whole series of different curvatures implies, for a hypersurface in $\mathbb{R}^n$, that it must be a hypersphere. A first case, for which partial results had been obtained by Cheng and Yau [**1**], was settled by Ros in his first paper. That was followed by different proofs of the general theorem by Ros [**2**] and Korevaar [**1**]. Montiel and Ros' proof is given in section 4.

Section 5 includes some further results on hypersurfaces of euclidean space and of spheres, as well as some purely Riemannian matters. We describe a recent breakthrough of Szabó [**1**] on a nearly fifty-year-old conjecture of Lichnerowicz, but do not go into the details of the proof.

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1. Background

For a plane curve one has the familiar definition of curvature:

$$\kappa = \frac{d\alpha}{ds} \tag{1}$$

where α is the angle made by the tangent vector to the curve (in the direction of travel) with a fixed direction, say the positive x-axis, and s measures arc length along the curve.

Note that curvature is assigned not only a magnitude, measuring the rate of deviation from "straight-aheadness," but also a sign, indicating the *direction* of deviation: positive if we veer off to the left, negative to the right.

For a curve in space, we have no way of attributing a sign to curvature. Again using the parameter s of arc length along the curve, and denoting by x the position vector on the curve, we have the unit tangent vector

$$T = \frac{dx}{ds}$$

and the curvature k defined by

$$k = \left|\frac{dT}{ds}\right| = \left|\frac{d^2x}{ds^2}\right|. \tag{2}$$

If the curve happens to lie in a plane, then the two definitions of curvature agree, up to sign:

$$k = |\kappa|. \tag{3}$$

Associated to a surface S in $\mathbb{R}^3$ are a number of curvatures:

$$\text{the } \textit{principal curvatures } k_1, k_2, \tag{4}$$

$$\textit{mean curvature } H = \frac{k_1 + k_2}{2}, \tag{5}$$

$$\textit{Gauss curvature } K = k_1 k_2. \tag{6}$$

As in the case of plane curves, the *signs* of these curvatures play an important role. For all except K, the sign depends on the orientation of the surface S, which in turn is determined locally by a choice of unit normal N. The intersection of S with a normal plane is a plane curve whose curvature, if not zero, is positive when it lies locally on the same side of the tangent line as N, and negative in the opposite case. The principal curvatures k_1, k_2 are the maximum and minimum of the curvatures obtained in this way over all normal planes. Note the following basic facts concerning the curvatures at a point p of S:

$$K < 0 \Leftrightarrow k_1 \text{ and } k_2 \text{ have opposite signs}$$
$$\Rightarrow S \text{ crosses its tangent plane in any neighborhood of } p, \text{ (a } \textit{saddle} \text{ point)}; \tag{7}$$

$$K = 0 \Leftrightarrow \text{ either (or both) } k_1, k_2 \text{ is zero} \tag{8}$$

$$K > 0 \Rightarrow S \text{ lies on one side of its tangent plane at } p;$$
$$\text{the same side as } N \text{ if } k_1, k_2 > 0, \text{ the opposite side if } k_1, k_2 < 0. \tag{9}$$

An important characterization of k_1, k_2, K, comes from the "Gauss map": the map

$$g: S \to \Sigma$$

where Σ is the unit sphere, defined by

$$g: p \mapsto N(p). \tag{10}$$

Then

$$|K| = |\det(dg_p)| = \lim_{U \downarrow \{p\}} \frac{\text{Area } g(U)}{\text{Area } U}, \tag{11}$$

where U is a neighborhood of p and the limit is taken as the neighborhood contracts down to the point p. If $K \neq 0$, then the sign of K is positive if orientation is preserved, and negative if orientation is reversed.

This approach to curvature is the one taken by Gauss in his fundamental paper on surfaces. (See Dombrowski [**1**].) The names "Gauss map" and "Gauss curvature" are not really justified, however, since Rodrigues [**1**] not only introduced and studied both of them considerably before Gauss, but made one very important observation that escaped Gauss' notice: the differential of the "Gauss map" is diagonalized by the principal curvature directions. Specifically, if $x(t)$ defines a curve C on the surface and $N(t)$ is the unit normal along C, then $N'(t) = \lambda x'(t)$ for some real λ if and only if the plane spanned by $x'(t)$ and $N(t)$ intersects S in a curve whose curvature is k_1 or k_2. The corresponding tangent direction $x'(t)$ is then called a *principal curvature direction*. Said differently, if u_1, u_2 are local parameters near a point p of S such that at p, they are in the principal curvature directions, then

$$\begin{aligned} \frac{\partial N}{\partial u_1} &= -k_1 \frac{\partial x}{\partial u_1} \\ \frac{\partial N}{\partial u_2} &= -k_2 \frac{\partial x}{\partial u_2}. \end{aligned} \tag{12}$$

Note that an immediate consequence is that

$$\frac{\partial N}{\partial u_1} \times \frac{\partial N}{\partial u_2} = k_1 k_2 \frac{\partial x}{\partial u_1} \times \frac{\partial x}{\partial u_2}. \tag{13}$$

Since the area of the portion of S corresponding to a domain D in the u_1, u_2-plane is given by

$$A = \int_D \left| \frac{\partial x}{\partial u_1} \times \frac{\partial x}{\partial u_2} \right| du_1\, du_2 \tag{14}$$

and the corresponding area of the image on the unit sphere is

$$\hat{A} = \int_D \left| \frac{\partial N}{\partial u_1} \times \frac{\partial N}{\partial u_2} \right| du_1\, du_2, \tag{15}$$

it follows from (11) and (13) that $|K| = |k_1 k_2|$. It also follows from (13) that the orientation is preserved or reversed according as $k_1 k_2$ is positive or negative. Thus the expression (6) for K as $k_1 k_2$ is equivalent to the description following (11) in terms of the Gauss map.

A final note regarding orientation. Given any local parameters u_1, u_2 on a surface in S, one has an induced orientation on the surface from the standard orientation in the $u_1 u_2$-plane. That induced orientation on S corresponds to the choice of unit normal:

$$N = \frac{\dfrac{\partial x}{\partial u_1} \times \dfrac{\partial x}{\partial u_2}}{\left| \dfrac{\partial x}{\partial u_1} \times \dfrac{\partial x}{\partial u_2} \right|}. \tag{16}$$

These facts and formulas are all that is needed in order to follow the arguments in sections 2 and 3 below. Section 4 involves some generalizations to higher dimensions.

First, if S is a hypersurface in $\mathbb{R}^n$, p a point of S, N a unit normal at p, then for any curve $x(s)$ on S with $x(0) = 0$, if s is the arclength parameter, then the curvature vector $x''(0)$ has a component in the direction N that depends only on the unit tangent vector $T = x'(0)$. In particular, if the curve C is the intersection of S with the plane through p spanned by N and a given unit tangent vector T, then the quantity obtained in the above fashion is just the ordinary curvature of the plane curve C. As the curve $x(s)$ varies, we obtain a set of values

$$x''(s) \cdot N = k(T) \tag{17}$$

called the *normal curvatures* of S at p. The right-hand side of (17) is the restriction to the unit sphere of a quadratic form Q on the tangent space. It follows that there exists an orthonormal basis $e_1, \dots e_{n-1}$ of the tangent space diagonalizing Q. The directions $e_1, \dots, e_{n-1}$ are called the *principal curvature directions* at p, and the corresponding quantities

$$k_1 = k(e_1), \dots, k_{n-1} = k(e_{n-1}) \tag{18}$$

are the *principal curvatures* of S at p.

Just as in $\mathbb{R}^3$ one can consider the Gauss map

$$g: p \mapsto N(p), \tag{19}$$

whose differential

$$dg_p: x'(t) \mapsto N'(t), \qquad (x(t) = p) \tag{20}$$

again satisfies Rodrigues' equations

$$dg_p(e_i) = -k_i e_i, \qquad i = 1, \dots, n-1. \tag{21}$$

We have the *mean curvature*

$$H = \frac{1}{n-1}(k_1 + \cdots + k_{n-1}) = -\frac{1}{n-1} tr(dg_p) \tag{22}$$

and the *Gauss-Kronecker curvature*

$$k_1 \cdots k_{n-1} = (-1)^{n-1} \det(dg_p), \tag{23}$$

but we also have the other symmetric functions of the principal curvatures, called the *higher-order mean curvatures*. We denote by H_j the jth-*order mean curvature*, normalized so that

$$\prod_{j=1}^{n-1} (1 + tk_j) = \sum_{j=0}^{n-1} \binom{n-1}{j} H_j t^j. \tag{24}$$

Thus, $H_1 = H$, the mean curvature, and H_{n-1} is the Gauss-Kronecker curvature. For $n = 3$, that is all there is, but in higher dimensions, there are intermediate ones. Among those, H_2 plays a special role. Except for a numerical constant, H_2 equals the *scalar curvature*, an important Riemannian invariant. Before describing it, we note that the remainder of the background material in this section is only needed for certain parts of the discussion in section 5 below.

We now move from the special case of a hypersurface to the general case of an m-dimensional submanifold M of $\mathbb{R}^n$. The important distinction made by Gauss in the case of surfaces in $\mathbb{R}^3$ also holds in all generality. *Intrinsic quantities* of M are those that are determined by measurements along M itself, while *extrinsic* ones depend on the way M lies in $\mathbb{R}^n$. Riemann introduced the notion of a *Riemannian manifold* in which lengths are defined by a formula like the one that holds for a submanifold of $\mathbb{R}^n$. About one hundred years later, John Nash showed that although the idea was new (and extremely fruitful) the objects themselves were not, in that no new manifolds arise that are not already among those that occur as submanifolds of $\mathbb{R}^n$.

For surfaces in $\mathbb{R}^3$ it is clear that the principal curvatures and mean curvature are not intrinsic, since rolling a sheet of paper into a cylinder alters them while leaving distances along the surface unchanged. The big surprise, Gauss' *Theorema Egregium*, was that the *product* $k_1 k_2$, the Gauss curvature, *is* intrinsic.

For an m-dimensional Riemannian manifold M, which we may if we like, by Nash's embedding theorem, consider as lying in some $\mathbb{R}^n$, Riemann defined at each point p the *sectional curvatures* as follows. Pick any pair of independent tangent vectors to M at p, say v and w. For every unit vector $u = \lambda v + \mu w$, there is a unique geodesic in M starting at p, with tangent vector u. The set of all such geodesics, as u describes the unit circle in the plane spanned by v and w, sweep out a surface whose Gauss curvature at p is called the *sectional curvature*

$$K_{\Pi} = K[v, w] \tag{25}$$

of the plane Π spanned by v and w.

Let $e_1, \dots, e_m$ be an orthonormal basis of the tangent space to M at p. Then the quantity

$$\tau = 2 \sum_{1 \le i < j \le m} K[e_i, e_j] \tag{26}$$

is independent of the choice of basis, and is called the *scalar curvature* of M at p.

Returning to the special case of a hypersurface in $\mathbb{R}^n$, we may choose the e_i to be principal curvature directions at p. Then a computation very similar to the one that Gauss used to prove that K is intrinsic leads to the equation

$$K[e_i, e_j] = k_i k_j, \tag{27}$$

so that the scalar curvature τ may be expressed in terms of the principal curvatures k_i by

$$\tau = 2 \sum_{1 \le i < j \le n-1} k_i k_j, \tag{28}$$

or by (24):

$$\tau = (n-1)(n-2) H_2. \tag{29}$$

With that as background, we can proceed to the results.

2. Curvature inequalities; curves and surfaces

We start with a classical theorem of Fenchel for a curve in $\mathbb{R}^3$ with curvature k.

THEOREM 2.1. *For a simple closed curve C in $\mathbb{R}^3$,*

$$\int_C k\, ds \ge 2\pi, \tag{30}$$

with equality if and only if C is a convex plane curve.

We give a slightly modified version of the proof in the book by do Carmo [**1**]. It has the unusual feature of "reducing" a theorem on curves to one on surfaces.

Let

$$K^+ = \max\{K, 0\}.$$

In other words, K^+ is the function on a surface whose value equals the Gauss curvature K wherever K is positive and is zero wherever K is negative or zero.

LEMMA 2.2. *For a compact surface S in $\mathbb{R}^3$,*

$$\int_S K^+\, dA \ge 4\pi. \tag{31}$$

Proof. The left-hand side of (31) represents the area of the image under the Gauss map (counting multiplicities) of the part of S where $K \ge 0$. Hence, it is sufficient to show that the image covers the whole unit sphere. For that, it suffices to take your surface, as Heinz Hopf used to say, and "set it down on the floor." The first point of S where it touches the floor has a normal vector pointing vertically downward (assuming that we are using the orientation of the surface corresponding to the outer normal) and has Gauss curvature $K \geq 0$, because at a point where $K < 0$, the surface is saddle-shaped and crosses its tangent plane, by (7). But the "floor" could be in any direction, and thus we get a point with $K \geq 0$ corresponding to every normal direction. Since the area of the unit sphere is 4π, the lemma is proved.

To prove Fenchel's theorem, we apply the lemma to a thin tube around the given curve. In fact the proof is simply a computation showing that for such a tube, the left-hand side of (31) is equal to twice the left-hand side of (30) over the original curve.

Let the curve C be given by

$$x(s), \qquad 0 \leq s \leq L,$$

where s is the parameter of arc length. At each point $x(s)$ of C, we set

$$e_1(s) = x'(s),$$

the unit tangent vector. Let $e_2(s)$ be a smooth unit vector field such that

$$e_2(s) \cdot e_1(s) \equiv 0$$

and set

$$e_3(s) = e_1(s) \times e_2(s).$$

Thus, e_2, e_3 span the normal space to the curve, and the tube of radius r around C is the surface

$$y(s,t) = x(s) + rN(s,t),$$

where

$$N(s,t) = \cos te_2(s) + \sin te_3(s).$$

Setting

$$e_i'(s) = \sum_{j=1}^{3} a_{ij}(s)e_j(s), \qquad i = 1,2,3,$$

and using $a_{ij} = -a_{ji}$ (by differentiating the equation $e_i(s) \cdot e_j(s) \equiv 0$), we find that

$$\frac{\partial y}{\partial t} \times \frac{\partial y}{\partial s} = r[1 + r(a_{21}\cos t + a_{31}\sin t)]N.$$

Thus, for small positive r, the surface $y(s,t)$ is a regular surface, and its unit normal is precisely the vector $N(s,t)$. Furthermore,

$$\frac{\partial N}{\partial t} \times \frac{\partial N}{\partial s} = -(a_{21}\cos t + a_{31}\sin t)N.$$

Since the definition of K amounts to the equation

$$\frac{\partial N}{\partial t} \times \frac{\partial N}{\partial s} = K\frac{\partial y}{\partial t} \times \frac{\partial y}{\partial s},$$

it follows that

$$K = -\frac{a_{21}\cos t + a_{31}\sin t}{r[1 + r(a_{21}\cos t + a_{31}\sin t)]}.$$

But the curvature k of C is

$$k = |x''(s)| = |e_1'(s)| = |a_{12}e_2 + a_{13}e_3| = |a_{21}e_2 + a_{31}e_3| = \sqrt{a_{21}^2 + a_{31}^2}\,.$$

Thus $k(s) = 0$ implies $K(s,t) \equiv 0$ for $0 \le t \le 2\pi$. At each point where $k(s) > 0$, we may set $a_{21} = k\cos\theta$, $a_{31} = k\sin\theta$, so that

$$K = -\frac{k\cos(t-\theta)}{r[1 + rk\cos(t-\theta)]}\,.$$

Thus $K > 0$ for $\pi/2 < t - \theta < 3\pi/2$. Since

$$\left|\frac{\partial y}{\partial t} \times \frac{\partial y}{\partial s}\right| = r[1 + rk\cos(t-\theta)],$$

for any interval $a < s < b$ where $k(s) > 0$, we have

$$\int K^+\,dA = \int_a^b \left[\int_{\theta+\pi/2}^{\theta+3\pi/2} [-k(s)\cos(t-\theta)]\,dt\right] ds = 2\int_a^b k(s)\,ds,$$

and for the entire tube, we get

$$\int K^+\,dA = 2\int_0^L k(s)\,ds.$$

Fenchel's inequality (30) now follows immediately from (31). A closer analysis shows that equality holds only in the case indicated. (For details, see do Carmo [**1**], p. 400.)

We next ask how Fenchel's inequality extends to surfaces. The answer depends on which curvature we adopt in place of k.

THEOREM 2.3. *Let S be a compact surface embedded in $\mathbb{R}^3$. Then*

$$\int_S |K|\,dA \ge 4\pi, \tag{32}$$

with equality if and only if S bounds a convex domain in $\mathbb{R}^3$.

Proof. Using Lemma 2.2,

$$\int_S |K|\,dA \ge \int_S K^+\,dA \ge 4\pi.$$

If $K < 0$ at any point, then $K < 0$ in a neighborhood, and the first inequality is strict. Hence equality implies $K \ge 0$ everywhere and by a theorem proved first by Hadamard when $K > 0$, and later by Chern and Lashof [**1**] in general, S must bound a convex domain.

If, instead of K, we consider the mean curvature H, we find several possible formulations.

THEOREM 2.4 (Willmore). *Let S be a compact surface in $\mathbb{R}^3$. Then*

$$\int_S H^2\,dA \ge 4\pi \tag{33}$$

with equality if and only if S is a standard sphere.

Proof.

$$H^2 = \left(\frac{k_1 + k_2}{2}\right)^2 = k_1 k_2 + \left(\frac{k_1 - k_2}{2}\right)^2 \geq K, \tag{34}$$

so that

$$\int_S H^2 \, dA \geq \int_S K^+ \, dA \geq 4\pi$$

by Lemma 3.2. Furthermore, inequality is strict unless $H^2 \equiv K$, in which case $k_1 \equiv k_2$ so that every point of S is umbilic. It is then a classical result (for example, do Carmo [**1**], p. 147) that S is a standard sphere.

A question that has received a great deal of attention in the past decade is the *Willmore Conjecture*. If S is topologically a torus, then

$$\int_S H^2 \, dA \geq 2\pi^2. \tag{35}$$

This conjecture has been verified for many cases, including when S is a tube about some curve, or of certain conformal type. There are also higher-dimensional versions, chiefly due to B.-Y. Chen. For these and other results on the conjecture, see Bryant [**1**], and the books by Willmore [**1**] and Chen [**1**]. More recently, Langer and Singer [**1**] proved the conjecture for tori of revolution. Leon Simon [**1**] obtained the important result that there exists an immersed torus minimizing the left side of (35), and a result of Li and Yau [**1**] guarantees that such a surface must be embedded. Ferus, Pedit, Pinkall, and Sterling [**1**] provide, in some sense, a complete classification of Willmore tori; that is, stationary surfaces for the integral in (35).

An inequality of a somewhat different character is one obtained by Minkowski for a surface S bounding a convex domain D of volume V. We may write the inequality as

$$\int_S H \, dA \leq \frac{A^2}{3V}, \tag{36}$$

where A is the area of S. Equality holds (for smooth surfaces) only when S is a standard sphere.

We come finally to the recent results of Ros. He proved a new inequality, in a way dual to (36).

THEOREM 2.5 (Ros [**1**]). *Let S be a compact embedded surface in $\mathbb{R}^3$ bounding a domain D of volume V. If the mean curvature H of S is positive everywhere, then*

$$\int_S \frac{1}{H} \, dA \geq 3V. \tag{37}$$

Equality holds if and only if S is a standard sphere.

Ros' original proof uses results of Reilly [**1**]. A subsequent paper of Montiel and Ros [**1**], based on ideas of Heintze and Karcher [**1**], uses a more geometric approach.

Proof. The idea is to use a well-known formula (whose derivation we include at the end of the proof) for the volume of a one-sided shell on the inside of S, where we allow for a shell of variable thickness (somewhat like an oyster shell). In other words, we compute the volume of the domain obtained by starting at each point p of S and going a distance $h(p)$ along the inner normal to S. The volume of this shell is given by

$$V = \int_S F\,dA \tag{38}$$

where

$$F = \int_0^{h(p)} |(1 - k_1 t)(1 - k_2 t)|\,dt = \int_0^{h(p)} |1 - 2Ht + Kt^2|\,dt, \tag{39}$$

provided there is not overlap; that is, provided no portion of the shell is counted twice along normals from different parts of S. To prevent overlap we define $h(p)$ as follows:

$$h(p) = \sup\{r : \text{the point } p \text{ is the unique nearest point on } S \text{ to the point } q \text{ at distance } r \text{ from } p \text{ along the normal to } S \text{ at } p\}. \tag{40}$$

From the definition of $h(p)$ it follows that the interior of the shell of variable width $h(p)$ has no overlap; every point of the interior lies on a unique normal to S. On the other hand, every point of D lies in the closed shell. Namely, given q in D, let d be the distance from q to S. Then the open ball $B_q(d)$ of radius d centered at q contains no point of S, but there is at least one point p of S on its boundary. For any point q' on the radius from q to p, if r is the distance from q' to p then the closed ball $\overline{B_{q'}(r)}$ lies in the interior of $B_q(d)$ except for the point p. It follows that p is the unique point of S realizing the distance r from q' to S. By the definition (40) of $h(p)$, we have $d \le h(p)$. Hence q lies in the closed shell, as claimed.

We conclude that the volume V of D is precisely equal to the volume of the shell given by (38), since all points of D are covered and the only points covered twice are in the image of the boundary, which is lower dimensional and contributes nothing to the volume.

The key to the argument is now the elementary but crucial observation:

$$\frac{1}{h(p)} \ge \max\{k_1(p), k_2(p)\} \tag{41}$$

for every point p on S. Namely, let q be the point at distance $h(p)$ from p along the normal. Then the open ball $B_q(h(p))$ cannot contain any points of S, since if there were such a point p', then its distance to q would be less than $h(p)$; for all points q' on the radius from q to p, sufficiently near q, the distance from q' to p' would then be less than the distance from q' to p, contradicting the definition of $h(p)$. But since the surface S lies outside the sphere $S_q(h(p))$, the surface and the sphere are tangent at p, and the normal curvatures of S at p are all bounded above by the normal curvatures of the sphere, which equal $1/h(p)$. This proves (41).

It follows from (41) that each of factors

$$(1 - k_1 t), \quad (1 - k_2 t)$$

in (39) is non-negative for $0 \leq t \leq h(p)$, so that

$$\int_0^{h(p)} |(1 - k_1 t)(1 - k_2 t)|\, dt = \int_0^{h(p)} (1 - k_1 t)(1 - k_2 t)\, dt. \tag{42}$$

By the inequality of the geometric and arithmetic mean,

$$(1 - k_1 t)(1 - k_2 t) \leq (1 - Ht)^2 \tag{43}$$

and by (41)

$$\frac{1}{h(p)} \geq H(p)$$

so that from (39)

$$F \leq \int_0^{1/H} (1 - Ht)^2\, dt = \frac{1}{3H}$$

and by (38)

$$V \leq \int_S \frac{1}{3H}\, dA,$$

proving (37). For equality to hold, it must hold in (43), which means that $k_1 = k_2$. But then by the classical result referred to earlier in the proof of Theorem 2.4, S must be a sphere. This proves the theorem.

For completeness, we show how one derives equations (38) and (39). It is sufficient to carry out the computation locally on S, since one can then add up to obtain the integral over the whole surface. So suppose we have S given locally in parametric form:

$$S\colon x(u_1, u_2),\ (u_1, u_2) \in D \subset \mathbb{R}^2.$$

At each point we have the unit normal $N(u_1, u_2)$, and we parametrize our "one-sided shell" by

$$y(u_1, u_2, u_3) = x(u_1, u_2) + u_3 N(u_1, u_2), \qquad (u_1, u_2) \in D,\ 0 < u_3 < h(u_1, u_2).$$

The volume of this shell is then given by

$$\iint_D \left[\int_0^{h(u_1, u_2)} \left| \frac{\partial(y_1, y_2, y_3)}{\partial(u_1, u_2, u_3)} \right| du_3 \right] du_1\, du_2,$$

provided the correspondence $(u_1, u_2, u_3) \mapsto (y_1, y_2, y_3)$ is one-to-one (which the definition (40) of $h(p)$ is designed to guarantee). To evaluate the integrand, we note that if u_1, u_2 are principal curvature directions, then Rodrigues' equations (12) imply

$$\frac{\partial y}{\partial u_1} = \frac{\partial x}{\partial u_1} + u_3 \frac{\partial N}{\partial u_1} = (1 - k_1 u_3) \frac{\partial x}{\partial u_1},$$

$$\frac{\partial y}{\partial u_2} = (1 - k_2 u_3) \frac{\partial x}{\partial u_2}, \qquad \frac{\partial y}{\partial u_3} = N;$$

$$\begin{aligned}\frac{\partial y}{\partial u_1} \times \frac{\partial y}{\partial u_2} &= (1 - k_1 u_3)(1 - k_2 u_3) \frac{\partial x}{\partial u_1} \times \frac{\partial x}{\partial u_2} \\ &= \left(1 - 2H u_3 + K u_3^2\right) \frac{\partial x}{\partial u_1} \times \frac{\partial x}{\partial u_2},\end{aligned}$$

where $-H$ and K are the trace and determinant of the differential of the Gauss map g given by (10); that is, if

$$dg\colon \frac{\partial x}{\partial u_i} \mapsto \frac{\partial N}{\partial u_i} = \sum c_{ij} \frac{\partial x}{\partial u_j},$$

then the trace and determinant of (c_{ij}) is independent of choice of parameters. The equation above for $(\partial y/\partial u_1) \times (\partial y/\partial u_2)$ is, therefore, valid for any parameters u_1, u_2. Then

$$\left|\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}\right| = \left|\left(\frac{\partial y}{\partial u_1} \times \frac{\partial y}{\partial u_2}\right) \cdot \frac{\partial y}{\partial u_3}\right| = |1 - 2Hu_3 + Ku_3^2| \left|\frac{\partial x}{\partial u_1} \times \frac{\partial x}{\partial u_2}\right|,$$

so that from (14)

$$\begin{aligned} V &= \iint_D \left[\int_0^{h(u_1, u_2)} |1 - 2Hu_3 + Ku_3^2| \, du_3\right] \left|\frac{\partial x}{\partial u_1} \times \frac{\partial x}{\partial u_2}\right| du_1 \, du_2 \\ &= \iint_S \left[\int_0^{h(u_1, u_2)} |1 - 2Hu_3 + Ku_3^2| \, du_3\right] dA, \end{aligned}$$

which gives equations (38), (39).

We have confined ourselves in this section to the case of compact surfaces. However there are many interesting results involving integrals of various curvatures over complete, non-compact surfaces. We note in particular a recent paper by Brian White **[1]** and earlier ones by Osserman **[1]**, **[2]**.

3. Surfaces of constant mean curvature

One consequence of Ros' inequality (37) is a remarkably elementary proof of a famous theorem of Alexandrov: *the only compact embedded surfaces of constant mean curvature in* $\mathbb{R}^3$ *are the standard spheres.*

To see that, Ros makes use of a general equation, valid for any oriented compact immersed surface S in $\mathbb{R}^3$:

$$A = -\int_S HN \cdot x \, dA, \tag{44}$$

where A is the area of S, H its mean curvature, N the unit normal and x the position vector. Equation (44) can be proved in various ways, the simplest, perhaps, being a special case of the first variation formula; if v is any smooth vector field along S, let S_r be the surface defined by

$$y = x + rv \tag{45}$$

and let $A(r)$ be the area of S_r. Then an elementary computation gives

$$A'(0) = -2\int_S HN \cdot v \, dA. \tag{46}$$

In this case, we choose $v = x$, so that (45) reduces to

$$y = (1 + r)x,$$

a uniform stretching of S by the factor $(1 + r)$, so that

$$A(r) = (1 + r)^2 A$$

and

$$A'(0) = 2A. \tag{47}$$

Combining (46) with (47) gives equation (44).

Now restrict to the case where S is an embedded surface of constant mean curvature H bounding a domain D. If we choose N to be the interior normal at each point, then $H > 0$, since S lies inside some large sphere, and contracting the sphere until it makes a first point of contact p with S, we find that both principal curvatures at p with respect to the inner normal must be positive. Applying the divergence theorem we deduce from (44):

$$A = -H\int_S N \cdot x\,dA = H\int_D \operatorname{div} x\,dV = 3HV \tag{48}$$

since

$$\operatorname{div} x \equiv 3.$$

Hence,

$$H = \frac{A}{3V}, \tag{49}$$

which means that equality holds in (37), and by Ros' theorem, S must be a standard sphere.

For a very pretty elementary proof of Alexandrov's theorem, of a more analytic nature, see Reilly **[2]**. Incidentally, Harold Rosenberg also noticed that equation (49) holds for constant mean curvature surfaces, and pointed out that as a result, equality holds in Minkowski's inequality (36). Since it is known that at least for smooth convex surfaces, equality can hold in (36) only for the sphere, this gives another proof of the original theorem of Liebmann, the precursor of Alexandrov's, that a *convex* surface of constant mean curvature must be a sphere, and could conceivably yield the full Alexandrov theorem if a suitable extension of Minkowski was found.

A generalization of Liebmann's theorem in a different direction than Alexandrov's was given by Heinz Hopf. He showed that any immersion of a surface, topologically a sphere, with constant mean curvature in $\mathbb{R}^3$ must be a standard sphere. In conjunction with Alexandrov's theorem, that led to one of the most stubborn questions in the field during a period of several decades: must every *immersed* compact surface of constant mean curvature in $\mathbb{R}^3$ be a standard sphere? That question, generally known as "Hopf's conjecture," was finally settled in 1986 by Wente **[1]** who demonstrated the existence of an immersed torus of constant mean curvature. Wente's work inspired a string of further results on compact tori of constant mean curvature (Abresch **[2]**, Spruck **[1]**, Pinkall-Sterling **[1]**). Using radically different methods, Kapouleas **[2]** has just shown that there also exist compact immersed surfaces with constant mean curvature of every genus $g \geq 3$. The existence of such surfaces with genus 2 remains an open question.

We may mention just a few further results of interest on constant mean curvature surfaces proved in the last few years.

In the case of compact surfaces, the first-variation formula (46) applied to a normal variation

$$v = fN$$

for an arbitrary function f, gives

$$A'(0) = -2\int\int_S fH\,dA.$$

If S bounds a domain of volume V, and the varied surfaces S_t bound domains of volume $V(t)$, then

$$V'(0) = \int\int_S f\,dA.$$

In particular, if the volume is held fixed, so that

$$V(t) \equiv V,$$

then $V'(0) = 0$, and the problem of minimizing surface area while fixing the enclosed volume leads to the analytic formulation of finding a surface S such that

$$\int\int_S fH\,dA = 0 \quad \text{whenever} \int\int_S f\,dA = 0. \tag{50}$$

But a standard argument shows that (50) holds if and only if H is constant. A surface S satisfying (50) is called *stationary* with respect to the given class of variations. It is called *stable* if in addition

$$A''(0) \geq 0.$$

Note that condition (50) makes sense for any immersed surface, as does stability.

THEOREM (Barbosa and do Carmo [**1**]). *A compact stable surface of constant mean curvature is a standard sphere.*

Thus, Wente's surfaces and their successors must all be unstable.

Do Carmo's theorem was extended by Palmer [**1**], Silveira [**1**] and Lopez and Ros [**1**] to complete surfaces, where the stability assumption applies to every compact subdomain and the surface is assumed to have non-zero constant mean curvature. This result, along with many others on complete surfaces of non-zero constant mean curvature, tends to restrict the possibilities for such surfaces. Other such theorems are those of Klotz-Osserman and Hoffman-Osserman-Schoen [**1**] stating that a complete surface of non-zero constant mean curvature must be a circular cylinder if, in the first case, its Gauss curvature K satisfies $K \leq 0$, and in the second, its image under the Gauss map lies in a closed hemisphere. The latter result was extended by Lopez and Ros [**1**] who showed that it is sufficient to assume that for some compact set on the surface, the image of its complement under the Gauss map lies in a closed hemisphere. These results are sharp, in that, as Seaman showed [**1**], there exist examples whose Gauss map covers an arbitrarily narrow strip surrounding the equator: namely certain of the surfaces of revolution, called *Delaunay surfaces*.

A beautiful very recent result by Korevaar, Kusner and Solomon [**1**] characterizes the Delaunay surfaces as the only doubly-connected surfaces (besides the circular cylinder) properly embedded in $\mathbb{R}^3$ with non-zero constant mean curvature.

In view of the paucity of examples, it represented a major breakthrough when Kapouleas [**1**] proved the existence of a whole new species of complete surfaces of non-zero constant mean curvature, including examples of infinite genus, and infinitely many topologically distinct ones of every finite genus. Each annular end converges to a Delaunay surface. That turns out not to be an accident, as was demonstrated shortly after in the paper of Korevaar, Kusner and Solomon [**1**].

They prove:

THEOREM. *Let S be a properly embedded surface of finite topological type with constant non-zero mean curvature. Then each end is asymptotic to a Delaunay surface.*

We have restricted our attention in this section to surfaces of *non-zero* constant mean curvature. Surfaces with $H \equiv 0$, *minimal surfaces*, constitute a whole world of their own, about which one can learn in a number of recent books: Barbosa and Colares **[1]**, Bourguignon, Lawson, and Margerin **[1]**, Nitsche **[1]**, Osserman **[3]**, **[4]**, and Yang **[1]** in particular.

4. Higher dimensions

Some of the results described in sections 2 and 3, such as Hopf's and Kapouleas' Theorems, use the two-dimensionality of the surfaces studied in an essential way. In fact, Hopf's theorem turns out to be false in general, as was proved by Wu-Yi Hsiang **[1]**, who constructed immersed spheres of constant mean curvature in $\mathbb{R}^n$, $n \geq 4$, different from the standard sphere. However, other results extend to hypersurfaces in higher dimensions, in many cases in a straightforward manner. In particular, Ros' fundamental inequality, along with its proof, extends to compact hypersurfaces S in $\mathbb{R}^n$ with $H > 0$ in the form

$$\int_S \frac{1}{H}\, dA \geq nV, \tag{51}$$

where S is assumed to be the boundary of a domain D of volume V, and again, equality holds only for the standard sphere. But also the first variation formula (46) and its consequences, (44) and (48), extend easily to give equality in (51) when H is constant, and hence proves Alexandrov's theorem in $\mathbb{R}^n$.

Ros' main goal, however, was to prove the conjecture that constant *scalar* curvature implies that an embedded compact hypersurface must be a sphere. He did that **[1]** using results of Reilly **[1]**, together with the fact that, as follows from (29), constant scalar curvature is equivalent to constant second-order mean curvature. That paper led Ros **[2]** and Korevaar **[1]** independently to a proof of the general theorem that the constancy of *any* of the H_j is possible only for a sphere. Korevaar showed how Alexandrov's original argument for H_1 could be carried over, whereas Ros adapted the reasoning in his earlier paper. Finally, Montiel and Ros **[1]** gave the new and elementary proof of Ros' inequality (51) that we described above.

To complete the proof for the higher order mean curvatures H_j, defined as we noted earlier by the equation

$$\prod_{j=1}^{n-1} (1 + tk_j) = \sum_{j=0}^{n-1} \binom{n-1}{j} H_j t^j,$$

Ros makes use of the Minkowski equations

$$\int_S (H_j + H_{j+1} N \cdot x)\, dA = 0. \tag{52}$$

Note that equation (44) is just the special case $j = 0$ of (52). But the general equations follow from the special case by computing the mean curvature $H(r)$ of a parallel hypersurface S_r to S, which may be expressed explicitly in terms of the quantities H_j on the original surface, and substituting in (44); setting the coefficient of each power of r equal to zero gives Minkowski's equations (see Montiel-Ros

[1]. Incidentally, these "Minkowski equations" were apparently first stated and proved by Kubota for convex hypersurfaces and then obtained in the general case by Hsiung [1]. What Minkowski did was the case $n = 3$, $j = 1$.)

To conclude the proof, Ros shows that if a particular H_j is positive on all of S, then the same holds for $H_1, \dots, H_{j-1}$, and the following inequalities hold:

$$H_i^{i-1} \le H_{i-1}^i, \qquad H_i \le H^i, \qquad i = 1, \dots, j. \tag{53}$$

Suppose now that H_j is constant on S. Since there exists some point where all the principal curvatures are positive (using the same argument of contact with an enveloping sphere as in $\mathbb{R}^3$), the constant value of H_j is also positive, and inequalities (53) hold. By (52),

$$\int_S H_{j-1}\, dA = -\int_S H_j N \cdot x\, dA = -H_j \int_S N \cdot x\, dA = H_j n V,$$

where we have used the n-dimensional version of (48) in the last step. Invoking the inequalities (53) then yields

$$nV = \frac{1}{H_j} \int H_{j-1}\, dA \ge \frac{1}{H_j} \int H_j^{(j-1)/j}\, dA \ge \frac{A}{H_j^{1/j}} = \int \frac{1}{H_j^{1/j}}\, dA \ge \int \frac{1}{H}\, dA.$$

But this is just the reverse of Ros' inequality (51). Hence equality holds in (51) and by Ros' Theorem, S is a standard sphere.

5. Related results

We now review briefly a few further topics, again with emphasis on some of those where exciting progress has been made in the eighties.

First, another innocuous-sounding question that flowered into a beautiful theory: what are the surfaces for which not only the mean curvature, but each of the individual principal curvatures is constant? That turns out to be such a strong condition, that even locally, any piece of such a surface must lie on a sphere, a plane, or a right circular cylinder. An interesting, and perhaps unexpected feature, is that it is impossible for the principal curvatures of a surface to be, even locally, distinct non-zero constants. Shortly after these results were obtained by Levi-Civita [1], they were extended to hypersurfaces of arbitrary dimension. Segre [1] proved that if all the principal curvatures are constant, then there could be at most two distinct values, and at most one of them different from zero. As a result, the hypersurface must again lie either on a sphere, a hyperplane, or a generalized cylinder $S^k \times \mathbb{R}^{n-k}$, $k = 1, \dots, n-1$. The whole subject, which was apparently finished shortly after it began, took on an entirely new life with a paper of Cartan [1] who posed the same question for hypersurfaces in a sphere. Before describing the results, let us recall the definitions.

Let C be a curve lying on a sphere S^n in $\mathbb{R}^{n+1}$. If C is given parametrically as $x(s)$ where s is the parameter of arclength and $x(s_0) = p$, then the *curvature vector* of C at p with respect to S^n is the projection of the Euclidean curvature vector $x''(s_0)$ onto the tangent plane of S^n at p. Note that a curve C has zero curvature vector with respect to S if and only if $x''(s)$ is normal to S^n. If that holds on an interval, then one can show that the corresponding arc of C lies on a great circle.

Now let M be an $(n-1)$-dimensional manifold lying on S^n, and p a point of M. By considering the curvature vectors at p with respect to S^n of curves through

p lying on M, we arrive at a set of *principal curvatures of M at p with respect to S^n* in the same manner that we defined in (18) the principal curvatures of a hypersurface in $\mathbb{R}^n$ by using the standard curvatures of curves on the hypersurface at a given point. It is those principal curvatures that Cartan requires to be constant.

In his original paper and a number of subsequent ones, Cartan shows that this class of surfaces, called *isoparametric hypersurfaces*, is far more interesting in S^n than in $\mathbb{R}^n$. The intersection of the cylinder

$$x_1^2 + \cdots + x_m^2 = r_1^2,$$

with the sphere S^n

$$x_1^2 + \cdots + x_{n+1}^2 = r^2 > r_1^2$$

is an isoparametric hypersurface. Cartan showed that every hypersurface of S^n with two distinct (constant) principal curvatures is one of those. They include a famous surface, the *Clifford torus* in S^3. Cartan also showed that there were examples with three distinct curvatures, and that they could all be represented as tubes over a special set of embeddings in certain S^n, including S^4. He then constructed an example with four distinct curvatures in S^5, and was naturally led to ask whether there exist examples with any prescribed number of distinct curvatures. That question remained unanswered for almost forty years, until the appearance in 1980 and 1981 of two remarkable papers by Münzner [**1**]. Among the results he obtains are

1. the only possibilities for the number of distinct curvatures are 1, 2, 3, 4 and 6;
2. every isoparametric hypersurface of S^n lies on an algebraic surface; if there are g distinct curvatures, then it lies on the intersection of S^n with a variety defined by a homogeneous polynomial of degree g.

An illustration of the second result is the theorem of Cartan mentioned earlier that all examples with two curvatures are given by the intersection of the sphere with a cylinder, defined by a quadratic polynomial.

Following the work of Münzner there has been a great burst of activity in the field. For an excellent discussion and references up to 1985, see Chapter 3 of the book by Cecil and Ryan [**1**]. A recent paper of interest with further references is that of Solomon [**1**].

Cartan also considered isoparametric hypersurfaces in hyperbolic space—that is, a space with constant negative sectional curvature. What Cartan showed [**1**] was that isoparametric surfaces in hyperbolic space were limited in the same manner as in Euclidean space, in that they could have at most two distinct principal curvatures.

We should say a word about how principal curvatures are defined for hypersurfaces in a Riemannian manifold, where the metric is only defined intrinsically. First of all, by Nash's embedding theorem, every such manifold M can be thought of as embedded in some Euclidean space $\mathbb{R}^N$, in which case any curve C in M has at each point its ordinary curvature vector $x''(s)$ in $\mathbb{R}^N$. The *geodesic curvature vector* of C at a point is simply the projection of $x''(s)$ onto the tangent space of M at the point. The critical fact is that the geodesic curvature vector can be expressed intrinsically in terms of the Riemannian metric on M. Consequently, one does not need to consider an embedding, but can define curvatures of curves, principal curvatures of hypersurfaces, and all the corresponding quantities in any Riemannian manifold.

As an example, once one has the principal curvatures of a hypersurface in hyperbolic space, one can define the mean curvature as their average, and all the higher-order mean curvatures as before. One can also define "spheres" in a Riemannian manifold, or "geodesic spheres," as they are usually called, centered at a point p, with radius r, as the set of points reached by going a distance r along an arbitrary geodesic starting at p. For a compact hypersurface M embedded in hyperbolic space, Korevaar [**1**] and Montiel-Ros [**1**] prove that if any of the mean curvatures is constant, then M must be a geodesic sphere. The same is true if M lies in a *hemisphere* of S^n. However, for hypersurfaces in the full sphere, the result is *not* true, since one has all the isoparametric examples, which clearly have all mean curvatures constant, since all principal curvatures are constant. One can narrow down the possibilities if one restricts, for example, the sectional curvature of M. (See Walter [**1**] and further references given there.)

We conclude our discussion of this subject with two further remarks. First, for an isoparametric hypersurface M in a constant curvature space, the parallel hypersurfaces to M are also isoparametric. Second, the family of parallel isoparametric hypersurfaces may be written as the level sets of a function F with the property that both the Laplacian of F and the magnitude of the gradient of F are functions of F itself. In fact this property characterizes isoparametric hypersurfaces and is the origin of their name.

We now enlarge our viewpoint a bit and ask the following question: *for which Riemannian manifolds do all geodesic spheres have constant mean curvature*? It turns out that manifolds with this property arose in the context of "harmonic spaces." These spaces, or manifolds, may be characterized in a number of ways—for example by the property that the mean value of a harmonic function over a geodesic sphere equals its value at the center, just as in Euclidean space (a theorem of Willmore. See also Ruse, Walker and Willmore [**1**] and Chapter 6 of Besse [**1**, in particular, 6.19].)

Among manifolds that satisfy the given condition is the famous class of "two-point homogeneous manifolds." They are defined by the property that for every two pairs of points p_1, q_1 and p_2, q_2, such that the distances $d(p_1, q_1), d(p_2, q_2)$ are equal, there is an isometry of the manifold taking p_1 to p_2 and q_1 to q_2. In particular, if $p_1 = p_2$, then for any two points q_1, q_2 of a geodesic sphere centered at p_1, there is an isometry of the manifold fixing the point p_1, hence necessarily taking the sphere to itself, and mapping q_1 to q_2. Thus, the mean curvature of the geodesic sphere is necessarily constant. In 1944, Lichnerowitz conjectured that the two-point homogeneous spaces were the *only* harmonic spaces. Very little progress was made for 45 years. Then in 1989, the conjecture was proved by Szabó [**1**] for all simply connected compact harmonic spaces. The proof draws on much work that had been done on both harmonic spaces and two-point homogeneous spaces separately, as well as some results linking them.

We mention here a few facts about two-point homogeneous spaces in which curvature plays a major role. We start with the celebrated "pinching theorem" of Toponogov, Berger, and Klingenberg:

THEOREM. *Let M be a compact simply-connected manifold whose sectional curvature K satisfies*

$$0 < \delta \leq K \leq 1.$$

Let d be the diameter of M. Then

i) $d \geq \pi$;

ii) *M is homeomorphic to a sphere if any of the following holds*:

a) $\delta > \frac{1}{4}$,

b) $\delta = \frac{1}{4}$ *and* dim M *is odd*,

c) $\delta = \frac{1}{4}$ *and* $d > \pi$;

iii) *if* dim M *is even*, $\delta = \frac{1}{4}$, *and* $d = \pi$, *then M is a two-point homogeneous space*.

For proofs of this and other pinching theorems see the books of Chavel [**1**] and Cheeger-Ebin [**1**], or the excellent recent article by Karcher [**1**].

Regarding part (iii) of the theorem, it turns out that one can list explicitly all the compact two-point homogeneous spaces; they are the standard spheres and real projective spaces, the complex and quaternionic projective spaces, and the projectivized Cayley plane, all with their standard Riemannian metrics.

This same list arose in Cartan's investigations of a certain class of spaces defined by a condition on curvatures. To explain that condition, let us start again with a manifold M embedded in Euclidean space. Let C be a curve in M, and let v be a vector field along C; that is, at each point $x(t)$ of C, $v(t)$ is a tangent vector to M at $x(t)$. Then v is said to be *parallel* along C if $v'(t)$ is normal to M at $x(t)$ for all t. This notion is the parallel translation of Levi-Civita, and like the closely related notion of geodesic curvature, it has the remarkable property that it can be defined intrinsically, independent of whether or how M is embedded in Euclidean space. Furthermore, given C and a tangent vector v_0 at a point $x(t_0)$ of C, there is a unique parallel field $v(t)$ along C with $v(t_0) = v_0$. Finally, if $v(t), w(t)$ are both parallel along C, then $v(t) \cdot w(t)$ is constant. It follows that if Π_0 is a two-dimensional tangent plane at a point $x(t_0)$ of C, then we can define parallel two-planes along C by taking a pair of orthonormal tangent vectors v_0, w_0 spanning Π_0, letting $v(t), w(t)$ be the parallel fields along C extending v_0, w_0 and defining $\Pi(t)$ to be the plane spanned by $v(t)$ $w(t)$, which are necessarily orthonormal for all t. By the properties we have mentioned, this one-parameter family of planes along C exists, is unique, and is independent of the choice of the original pair v_0, w_0.

We come now to the notion introduced by Cartan.

DEFINITION. A Riemannian manifold is *locally symmetric* if the sectional curvature is constant along every family of parallel planes $\Pi(t)$.

A locally symmetric space has *rank one* if none of its sectional curvatures is zero.

THEOREM. *The two-point homogeneous spaces are precisely the Euclidean spaces $\mathbb{R}^n$, the real projective spaces $\mathbb{R}P^n$, and the complete simply-connected locally symmetric spaces of rank one.*

For more on this subject, see the books of Chavel [**1**], Helgason [**1**] and Besse [**1**]. In this last reference, a Compact Rank One Symmetric Space is referred to acronymically as a CROSS. Thus, the theorems of Szabó and Berger referred to above may be stated: a compact simply-connected manifold which is either harmonic, or is even dimensional with diameter π and sectional curvatures pinched between $\frac{1}{4}$ and 1, must be a CROSS.

We conclude this section with a quick review of questions involving manifolds for which one or more of the intrinsically defined curvatures is constant.

First, complete Riemannian manifolds with constant sectional curvature—the "Clifford-Klein space forms"—were studied in detail by Hopf, and their classification was completed by Wolf [**1**]. The case of three-dimensional manifolds with constant negative curvature has been intensively studied in recent years, with the major impetus coming from Thurston [**1**], [**2**].

The requirement of constant sectional curvature imposes severe restrictions on the possible forms of the manifold. One can ask what restrictions arise from requiring that the *scalar* curvature be constant. The fundamental paper here is one of Yamabe from 1960. The question was whether there are *any* restrictions needed to have a metric of constant scalar curvature. Yamabe presented a proof that for compact manifolds there always exists such a metric conformally equivalent to any given metric on the manifold. However, his purported proof turned out to be incomplete, and after partial results by Trudinger, Aubin [**1**] proved the theorem for all manifolds of dimension $n \geq 6$ that were not conformally flat. To settle the problem completely, a new global type of argument was needed, and that was provided in 1984 by Schoen [**1**]. A general survey of the problem may be found in Lee and Parker [**1**]. Subsequent papers by Schoen [**2**], [**3**], [**4**] contain a number of further results, including an analysis of *all* metrics of constant scalar curvature in a given conformal class.

Thus, in contrast to the case of embedded hypersurfaces where, as we saw, constant scalar curvature was possible only for a sphere, for general compact manifolds there are no topological implications whatever. On the other hand, the *sign* of the scalar curvature does have significance. For a wealth of information on that topic we refer to the book of Lawson and Michelson [**1**].

The sectional curvature and the scalar curvature are the two most immediate curvature quantities that arise in the study of Riemannian manifolds. However, experience has shown that a third—intermediate—curvature is in many respects the "right one" to consider. To define it, we look more closely at the totality of sectional curvatures at a point p of a Riemannian manifold M.

Let v be a unit tangent vector to M at p. Let $H_v = v^{\perp}$ be the hyperplane in the tangent space orthogonal to v. For $w \in H_v$, set

$$Q_v(0) = 0, \qquad Q_v(w) = |w|^2 K(v, w) \text{ for } w \neq 0 \tag{54}$$

where

$$K(v, w) = \text{sectional curvature of the plane spanned by } v \text{ and } w. \tag{55}$$

Then Q_v is a quadratic form on H_v whose eigenvalues we denote

$$\lambda_1(v) \geq \lambda_2(v) \geq \cdots \geq \lambda_{n-1}(v) \tag{56}$$

where $\dim M = n$. The corresponding eigenvectors

$$e_1, \ldots, e_{n-1}$$

form an orthonormal basis of H_v, and

$$\lambda_i(v) = Q_v(e_i) = K(v, e_i)$$

is a sectional curvature of a plane through v, with λ_1 and λ_{n-1} representing the maximum and minimum of such curvatures.

DEFINITION. The *Ricci curvature* $R(v)$ of M at p in the direction v is

$$R(v) = \sum_{i=1}^{n-1} \lambda_i(v). \tag{57}$$

Thus, the Ricci curvature is the trace of the quadratic form Q_v, and since the trace is independent of basis, we can also write

$$R(v) = \sum_{i=1}^{n-1} K(v, e_i) \tag{58}$$

where $e_1, \dots, e_{n-1}$ is *any* orthonormal basis of H_v.

Geometrically, it would generally make more sense to consider the *average* rather than the *sum* of the sectional curvatures, and in some places that convention is adopted. However, (58) is the more common usage.

DEFINITION. An *Einstein manifold* is a Riemannian manifold with constant Ricci curvature.

The literature on Einstein manifolds is immense—we refer to the excellent book of Besse [**2**]—and there are many recent results of interest involving the Ricci curvature. Let us mention just one: the celebrated theorem of Hamilton [**1**].

THEOREM. *Let M be a compact manifold,* dim $M = 3$. *If there exists a metric on M with positive Ricci curvature, then there also exists one of constant positive sectional curvature.*

Since, as we mentioned, manifolds with constant sectional curvature are completely classified, Hamilton's Theorem implies that in three dimensions, positive Ricci curvature has powerful implications.

In view of the rich theory that grew out of replacing submanifolds of constant mean curvature by the more restrictive class of manifolds with each principal curvature constant—the isoparametric manifolds—it seems natural to consider a kind of intrinsic analog. We ask about manifolds satisfying the following condition:

$(*)$ *the set of eigenvalues* $\lambda_1(v), \dots, \lambda_{n-1}(v)$ *in* (56) *is the same for all p and v.*

Clearly, by (57), such manifolds are Einstein manifolds.

Although the analogy with isoparametric surfaces would be an obvious motivation for introducing condition $(*)$, that was not the original source of the notion. It arose from some results in ergodic theory giving a lower bound for entropy of geodesic flow in terms of curvature quantities (Osserman-Sarnak [**1**]) and from a later conjecture about a possible upper bound for the entropy (see Burns and Katok [**1**], Conjecture 5.4). A discussion of these matters with Christopher Croke led to condition $(*)$.

For rank one locally symmetric spaces the values of the $\lambda_i(v)$ in (56) are known explicitly: there are just two distinct values, with ratio $1/4$, and they are independent of p and v. Thus all such spaces satisfy condition $(*)$. In view of some of the results from ergodic theory, we are led to the following:

CONJECTURE. *The only nonflat manifolds satisfying condition* $(*)$ *are the rank one locally symmetric spaces.*

More details on the background to this conjecture are given in the paper of Chi [**1**], where it is shown that the conjecture does indeed hold in many cases: in particular for dimension 4, $2k + 1$, or $2(2k + 1)$. A later paper of Chi [**2**] settles several further cases. However, the general conjecture appears quite difficult. Of course it is possible that a counterexample may turn up, in which case the class of manifolds satisfying $(*)$ may turn out to be of separate interest.

For an interesting discussion of related questions and many more references, see the papers of Vanhecke [**1**], [**2**].

Conclusion

Of the many topics that could have been included in this discussion but were not, let me name just a few: there are the "almost flat manifolds" of Gromov, the classification of manifolds of non-positive curvature by Ballmann, Brin, Eberlein, Burns and Spatzier, and a whole series of results relating curvature bounds to topology. A few references for these subjects are included in the bibliography. I hope that at least the topics touched on here will convey some notion of the role of curvature in geometry, and a sense of its continuing fascination in current research.

Notes added in proof. Both Yu D. Burago and Joel Spruck have pointed out that for the special case of convex hypersurfaces, Ros' inequality (37) is a consequence of Minkowski's (36):

$$3V\int_S H\,dA \le A^2 = \left(\int_S \sqrt{H}\,\frac{1}{\sqrt{H}}\,dA\right)^2 \le \int_S H\,dA \int_S \frac{1}{H}\,dA.$$

The fact that equality holds in both Minkowski's and Ros' inequalities when H is constant turns out for star-shaped domains to be a special case of the equation $3V\overline{H} = A$ that holds for all such domains, where $\overline{H}$ is the volume average of the function obtained by extending the mean curvature H of the surface to be constant along radii. (See Theorem 9.1, p. 234 of Finn [**1**].)

An interesting additional reference on surfaces of constant mean curvature is the recent paper of Bobenko [**1**].

Finally, there are strong parallels between embedded surfaces of constant mean curvature in $\mathbb{R}^3$ and metrics of constant scalar curvature on conformally flat manifolds. Alexandrov's Theorem that the sphere is the only embedded compact surface of constant mean curvature has a counterpart in a uniqueness theorem of Obata, and the initial results of Kapouleas [**1**] were an offshoot of some of Schoen's work [**2**] related to the Yamabe problem.

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