MA2001 Important Theorems

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Matrix

1. $A(BC) = (AB)C$
2. $A(B_1 + B_2) = AB_1 + AB_2$
3. $(C_1 + C_2)A = C_1A + C_2A$
4. $c(AB) = (cA)B = A(cB)$
5. $A0 = 0$
6. $AI = A = IA$
7. $A^mA^n = A^{m+n}$
8. $(AB)^2 = (AB)(AB)$
9. $(A^T)^T = A$
10. $(A + B)^T = A^T + B^T$
11. $(cA)^T = cA^T$
12. then $(AB)^T = B^TA^T$

Invertible Matrix

1. A is invertible $\Leftrightarrow AB = I$ and $BA = I$
2. if A is invertible and $AB_1 = AB_2$, then $B_1 = B_2$, B could be non-square matrix
3. The inverse of a matrix is unique
4. $(cA)^{-1} = \frac{1}{c}A^{-1}$
5. $(A^T)^{-1} = (A^{-1})^T$
6. $(A^{-1})^{-1} = A$

7. $(AB)^{-1} = B^{-1}A^{-1}$
8. $(A^n)^{-1} = (A^{-1})^n$
9. If A is singular, then both AB and BA are singular

Determinant

1. If A is an triangular matrix, then $det(A) =$ product of diagonal entries
2. $det(A^T) = det(A)$
3. $det(AA^{-1}) = det(A) \times det(A^{-1}) = 1$
4. a square matrix with two identical rows or columns has 0 determinant
5. If $A \xrightarrow{kR_i} B$, then $det(B) = k \times det(A)$
6. If $A \leftrightarrow B$, then $det(B) = -det(A)$
7. If $A \xrightarrow{R_j + kR_i} B$, then $det(B) = det(A)$
8. If A is a $n \times n$ matrix, then $det(cA) = c^n det(A)$
9. $det(AB) = det(A) \times det(B)$
10. $det(A^{-1}) = \frac{1}{det(A)}$
11. $A^{-1} = \frac{1}{det(A)} \times adj(A)$

Linear Span

1. Linear span = the set of all linear combinations of $\{c_1u_1 + c_2u_2 + ... + c_ku_k \mid c_1, c_2, ..., c_k \in \mathbb{R}\}$
2. $0 \in span(S)$
3. For any $v_1, v_2, ..., v_r \in span(S)$ and $c_1, c_2, ..., c_r \in \mathbb{R}$, $c_1v_1 + c_2v_2 + ... + c_rv_r \in span(S)$
4. $span(u_1, u_2, ..., u_k) \subset span(v_1, v_2, ..., v_k) \Leftrightarrow u_i$ is a linear combination of $v_1, v_2, ..., v_k$

Subspace

1. If it is a span of something, it is a subspace
2. $span 0$ is the smallest subspace, a.k.a trivial subspace
3. If it satisfies:
   (a) $0 \in span(S)$
   (b) For any $v_1, v_2, ..., v_r \in span(S)$ and $c_1, c_2, ..., c_r \in \mathbb{R}$, $c_1v_1 + c_2v_2 + ... + c_rv_r \in span(S)$

   then it is a subspace
4. The solution set of a homogenous system of linear equations in n variables is a subspace of $\mathbb{R}^n$, and it is called the solution space

Linear Independence

1. If a set of vectors is linearly dependent, then there exists at least one redundant vector in the set
2. If a set of vectors is linearly independent, then tehre is no redundant vector in the set
3. In $\mathbb{R}^n$, if the set has more vector than n, then the set must be linearly dependent

Bases

1. let V be a vector space and $S = \{u_1, u_2, ..., u_k\}$ a subset of V, then S is called a basis of V if
   (a) S is linearly independent
   (b) S spans V
2. For any $u, v \in V, u = v \Leftrightarrow (u)_S = (v)_S$
3. For any $v_1, v_2, ..., v_r \in V$ and $c_1, c_2, ..., c_r \in \mathbb{R}$,
   $(c_1v_1 + c_2v_2 + ... + c_rv_r)_S = c_1(v_1)_S + c_2(v_2)_S + ... + c_r(v_r)_S$
4. $v_1, v_2, ..., v_r$ are linearly independent $\Leftrightarrow (v_1)_S, (v_2)_S, ..., (v_r)_S$ are linearly independent vectors in $\mathbb{R}^k$
5. $span\{v_1, v_2, ..., v_r\} = V \Leftrightarrow span\{(v_1)_S, (v_2)_S, ..., (v_r)_S\} = \mathbb{R}^k$

Dimensions

1. Let V be a vector space which has a basis with k vectors,

   (a) Any subset of V with more than k vectors is always linearly dependent

   (b) Any subset of V with less than k vectors cannot spans V

   this means that every basis for V have the same size k

2. The dimension of a vector space V, denoted by $dim(V)$, is defined to be the number of vectors in a basis for V

Transition Matrix

1. a transition matrix is invertible

2. P^{-1} is the transition matrix with an opposite direction

3. A transition matrix from S to T, write it in S is a linear combination of T, then form the matrix with

$$\begin{pmatrix} [s_1]_T & [s_2]_T & [s_3]_T \end{pmatrix}$$

   .

Row Space and Column Space

1. The nonzero rows of REF of A form a basis for the row space of REF of A and A

2. Let $W = span\{u_1, u_2, u_3\}$, methods to find the basis of W:

   (a) i. Place the vectors as row vectors to form a matrix
       ii. Get the row space from REF
       iii. The basis of row space is the basis of W

   (b) i. Place the vectors as column vectors to form a matrix
       ii. The basis of the column space is the basis of W

3. A linear system $Ax = b$ is consistent $\Leftrightarrow$ b lies in the column space of A

Ranks

1. The row space and the column space of a matrix has the same dimension

2. The rank of a matrix is the dimension of its row space and its column space

3. $rank(0) = 0$ and $rank(I_n) = n$
4. $rank(A) \leq min\{m, n\}$
5. If $rank(A) = min\{m, n\}$, then A is said to have full rank
6. A square matrix A is of full rank $\Leftrightarrow det(A) \neq 0$
7. $rank(A) = rank(A^T)$
8. A linear system $Ax = b$ is consistent $\Leftrightarrow A$ and $(A|b)$ have the same rank
9. $rank(AB) \leq min\{rank(A), rank(B)\}$

Nullspaces and Nullities

1. The solution space of $Ax = 0$ is the nullspace of A
2. The dimension of the nullspace of A is the nullity of A
3. $rank(A) + nullity(A) =$ the number of columns of A

The Dot Product

1. The norm(length) of $u = ||u|| = \sqrt{u_1^2 + u_2^2 + ... + u_k^2}$
2. $d(u, v) = ||u - v|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + ... + (u_k - v_k)^2}$
3. $u \cdot v = u_1v_1 + u_2v_2 + ... + u_kv_k$
4. $\theta = cos^{-1}(\frac{u \cdot v}{||u||||v||})$
5. $u \cdot v = v \cdot u$
6. $(u + v) \cdot w = u \cdot w + v \cdot w$ and $w \cdot (u + v) = w \cdot u + w \cdot v$
7. $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$
8. $||cu|| = |c|||u||$
9. $u \cdot u \geq 0; u \cdot u = 0 \Leftrightarrow u = 0$

Orthogonal and Orthonormal Bases

1. u and v are orthogonal if $u \cdot v = 0$
2. $\frac{1}{||u||}U$ is a unit vector
3. An orthogonal set is linearly independent

4. To show S is an orthogonal basis for V, we need to check:
   (a) S is orthogonal
   (b) $|S| = dim(V)$ or $span(S) = V$
5. Let $S = \{u_1, u_2, ..., u_k$ be an orthogonal basis for V, then for any $w \in V$:
   $w = \frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2} u_2 + ... + \frac{w \cdot u_k}{u_k \cdot u_k} u_k$
6. From above, w is the projection onto V
7. Gram-Schmidt Process:
   Let $\{u_1, u_2, ..., u_k\}$ be a basis for V
   $v_1 = u_1$
   $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$
   $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$

Best Approximation

1. Let $Ax = b$ be a linear system,
   then u is a least square solution to the system $Ax = b \Leftrightarrow u$ is a solution to $A^T Ax = A^T b$

Orthogonal Matrices

1. A square matrix A is orthogonal if $A^{-1} = A^T$
2. A is orthogonal
   $\Leftrightarrow$ The rows of A form an orthonormal basis for $\mathbb{R}^n$
   $\Leftrightarrow$ The columns of A form an orthonormal basis for $\mathbb{R}^n$

Eigenvalues and Eigenvectors

1. λ is an eigenvalue of A
   $\Leftrightarrow Au = \lambda u$
   $\Leftrightarrow \lambda u - Au = 0$
   $\Leftrightarrow (\lambda I - A)u = 0$
   $\Leftrightarrow$ the linear system $(\lambda I - A)u = 0$ has non-trivial solutions $\Leftrightarrow det(\lambda I - A) = 0$
2. Eigenspaces is the solution space of the linear system $(\lambda I - A) = 0$

Diagonalization

1. A square matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. P is said to diagonalize A
2. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors
3. Steps to diagonalize a matrix of n order:
   (a) Find all distinct eigenvalues $\lambda_1, ..., \lambda_k$
   (b) For each eigenvalue find a basis $S_{\lambda i}$ for the eigenspace $E_{\lambda i}$
   (c) Let $S = S_{\lambda 1} \cup ... \cup S_lambda_k$
   If $|S| < n$, then A is not diagonalizable
   If $|S| = n$, then A is diagonalizable
   (d) If an n order matrix A has n distinct eigenvalues, then A is diagonalizable

section*Orthogonal Diagonalization

1. A square matrix A is called orthogonally diagonalizable if there exists an orthogonal matrix such that P^TAP is a diagonal matrix, the matrix P is said to orthogonally diagonalize A
2. A square matrix A is orthogonally diagonalizable $\Leftrightarrow$ A is symmetric, i.e. $A^T = A$
3. Steps to diagonalize a matrix of n order:
   (a) Find all distinct eigenvalues $\lambda_1, ..., \lambda_k$
   (b) For each eigenvalue
      i. find a basis $S_{\lambda i}$ for the eigenspace $E_{\lambda i}$
      ii. use te Gram-Schmidt Process to transform $S_{\lambda i}$ to an orthonormal basis $T_{\lambda i}$
   (c) Let $T = T_{\lambda 1} \cup ... \cup T_lambda_k$
4. If A is symmetric, eigenvectors from different eigenspaces of A are always orthogonal to each other

Linear Transformation

1. $T : V \to W$ is called a linear transformation $\Leftrightarrow$ $T(cu+dv) = cT(u)+dT(v)$
2. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation,
   (a) $T(0) = 0$
   (b) If $u_1, ..., u_k \in \mathbb{R}^n$ and $c_1, ..., c_k \in \mathbb{R}$,
   then $T(c_1u_1 + ... + c_ku_k) = c_1T(u_1) + ... + c_kT(u_k)$

Ranges and Kernels

1. The range $R(T)$ is the column space of a standard matrix
2. The rank of T is $rank(T) = dim(R(T)) = rank(A)$
3. The kernel $Ker(T)$ is the nullspace of a standard matrix
4. The nullity of T is $nullity(T) = dim(Ker(T)) = nullity(A)$
5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$,
   $rank(T) + nullity(T) = n$

Invertible Matrix

The following statements are equivalent:

1. A is invertible
2. The linear system $Ax = 0$ has only the trivial solution
3. The $RREF$ of A is I
4. A can be expressed as a product of elementary matrices
5. $det(A) \neq 0$
6. The rows of A form a basis for $\mathbb{R}^n$
7. The columns of A form a basis for $\mathbb{R}^n$
8. $rank(A) = n$
9. 0 is not an eigenvalue of A