Fixed Income Review

junyan.xu*

January 12, 2019

Contents

*junyanxu5513@gmail.com

1 Forward Measure

1.1 Intuitive Definition

To start with, recall in the HJM framework we have the discount value of a bond price is a martingale in risk neutral measure. This is true because any tradable assets must be a martingale in risk neutral measure otherwise you have arbitrage opportunity

$$d(D(t)B(t,T)) = -\sigma^* D(t)B(t,T)d\widetilde{W}(t) \tag{1}$$

Then recall the definition of the fundamental theorem that there exists a measure between two tradable assets that those two can form a martingale.

If we further assume the measure is at time T, which is the same expiration time of the zero-coupon bonds, then we have:

$$\frac{V(t)}{B(0,T)} = E^T \frac{V(T)}{B(T,T)} = E^T V(T) \tag{2}$$

$$V(t) = B(0,T)E^T V(T) \tag{3}$$

The idea behind this is that, we cannot find a way using bond to hedge our payoff then get money at any condition. Recall the similar risk neutral pricing formula for any contract. But first, think of risk neutral measure as a measure that you can find a martingale such that it prevents you from being using bank account to make arbitrage

$$V(t) = E^B(V(T)D(T)) \tag{4}$$

Since T, B measures are both defined on same event space. The we can naturally get the transformation of the probability on same event to be

$$\widetilde{P}^T(A) = \frac{1}{B(0,T)} \int_A D(T) d\widetilde{P}^B(A) \tag{5}$$

1.2 Link To Risk Neutral Measure

1.2.1 Non-Model Based

We often hear someone say *forward price is martingale under T measure* (*1 over forward price is martingale under S measure*). Firstly, without proof, this statement can be think of as:

- First forward price is the number of bond you hold to hedge the price at time T. Aka the ratio between the expected payoff of contract vs the bond
- Under T measure, this value should equal to $\frac{V(t)}{B(t,T)}$ at time t

- In fact, the ratio value mentioned in first item should equal to $\frac{V(0)}{B(0,T)}$ at time 0
- According to the Tower Theorem, this second and third items are chained up to force the evolution of f from time 0 to time t, the f is a martingale

More intuitively, you can see using T measure just include/offset the factor of $B(0,T)$ and $D(T)$ in the risk neutral measure.

$$f(0) = \frac{E^B V(T)D(T)}{E^B D(T)} = B(0,T)E^B V(T)D(T) = E^T f(T) \tag{6}$$

1.2.2 Change of Numeraire

Thinking intuitively, T measure is just some drift transformation of normal distribution. Under risk-neutral measure, assume we have two process $S(t)$ and $N(t)$ (both are tradable assets).

$$D(t)S(t) = D(0)S(0)\exp(\int_0^t \sigma(t)d\widetilde{W}(u) - 0.5\int_0^t \left\|\sigma(t)^2\right\| du) \tag{7}$$

$$D(t)N(t) = D(0)N(0)\exp(\int_0^t \mu(t)d\widetilde{W}(u) - 0.5\int_0^t \left\|\mu(t)^2\right\| du) \tag{8}$$

So therefore under this risk neutral pricing measure, we can have

$$\begin{aligned}\frac{S(t)}{N(t)} &= \frac{S(0)}{N(0)}\exp(\int_0^t (\sigma(t)-\mu(t))d\widetilde{W}(u) - 0.5\int_0^t (\left\|\sigma(t)^2\right\| - \left\|\mu(t)^2\right\|)du) \\ &= \frac{S(0)}{N(0)}\exp(\int_0^t (\sigma(t)-\mu(t))d\widetilde{W}(u) - 0.5\int_0^t (\sigma(t)-\mu(t))^2 du) - \int_0^t \sum_1^d \mu_i(t)(\sigma_i(t)-\mu_i(t))du\end{aligned} \tag{9}$$

So we can find a measure

$$d\widetilde{W}^N(u) = \begin{bmatrix} d\widetilde{W}_1 - \mu_1(u) \\ d\widetilde{W}_2 - \mu_2(u) \\ \vdots \\ d\widetilde{W}_d - \mu_d(u) \end{bmatrix} \tag{10}$$

This equation is telling you that under measure N, the normal distribution in risk neutral have positive drift $\mu(u)$

Example 1.1. Under single stock measure, the log normal stock actually have σ^2 drift

$$d(\log(S)) = (r - \frac{1}{2}\sigma^2)dt + \sigma\cdot\sigma dt + \sigma d\widetilde{W}^S(t) \tag{11}$$

Example 1.2. Quanto Option, suppose XAU-EUR is $S_1(t)$ and USD-EUR $S_2(t)$. Then we have

$$\frac{d(S_1)}{S_1} = r_f dt + \sigma_1 d\widetilde{W_1}^f(t) \tag{12}$$

$$\frac{d(S_2)}{S_2} = (r_f - r_d)dt + \sigma_2 d\widetilde{W_2}^f(t) \tag{13}$$

The inverse of exchange rate process under EUR is

$$d(\frac{1}{S_2}) = \frac{1}{S_2}((r_d - r_f + \sigma_2^2)dt - \sigma_2 d\widetilde{W_2}^f(t)) \tag{14}$$

Under EUR risk neutral measure, the XAU by USD have (using $dXY = XdY + Xdx + dXdY$)

$$\frac{dS_3}{S_3} = (r_d + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 d\widetilde{W_1}^f(t) - \sigma_2 d\widetilde{W_2}^f(t) \tag{15}$$

when we change measure EUR to measure USD then we have

$$\begin{aligned} \frac{dS_3}{S_3} &= (r_d + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + (\sigma_1 - \rho\sigma_2)(d\widetilde{W_1}^d(t) + \rho\sigma_2 dt) \\ &- \sigma_2\sqrt{1-\rho^2}(d\widetilde{W_2}^d(t) + \sqrt{1-\rho^2}\sigma_2 dt \\ &= r_d dt + \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \cdot d\widetilde{W_3}^d(t) \end{aligned} \tag{16}$$

The last equation shows that after exchange rate consideration, the XAU in USD is still a martingale under USD measure.

1.3 Normal Based T Measure

Based on the analysis above. Since discount zero coupon bond is a martingale under risk neutral measure.

$$d(D(t)B(t,T)) = D(t)B(t,T) - \sigma^*(t)d\widetilde{W}(t) \tag{17}$$

Use equation 10 can effciently get

$$d\widetilde{W}^T(t) - \sigma^*(t) = d\widetilde{W}(t) \tag{18}$$

1.4 Use T Measure To Price Option

Even if interest rate is random, we can still get BS formula in a very general way.

$$\widetilde{E}(D(T)(S(T)-K))^+ = \widetilde{E}(D(T)S(T)\mathbb{1}[S(T)>K]) - K\widetilde{E}(D(T)\mathbb{1}[S(T)>K]) \tag{19}$$

The second part is easy to show. The σ here is the vol of forward price, which is comprised of both vol of spot and vol of zero bond(interest rate)

$$
\begin{aligned}
K\widetilde{E}(D(T)\mathbb{1}[S(T) > K]) &= KB(0,T)\widetilde{E}^T(F(T,T) > K) \\
&= KB(0,T)P^T(\exp(-\frac{1}{2}\sigma^2 t + \sigma\widetilde{W}^T(T)) > \frac{K}{F(0,T)}) \\
&= KB(0,T)N(d_2)
\end{aligned} \tag{20}
$$

The first part is quite similar

$$
\widetilde{E}(D(T)S(T)\mathbb{1}[S(T) > K]) = S(0)\widetilde{E}^S(F(T,T) > K) \tag{21}
$$

The tricky part is in fact, under the definition of S measure, the inverse of F is a martingale with vol $-\sigma$

$$
\frac{B(0,T)}{S(0)} = \frac{1}{F(0,T)} = E^S(\frac{B(t,T)}{S(t)}) = E^S(\frac{1}{F(t,T)}) \tag{22}
$$

The using the similar technique as for second part proof we can get the expression for first term.

$$
S(0)N(d_1) \tag{23}
$$

2 HJM and LMM

2.1 HJM Framework

2.1.1 No Arbitrage Condition

A key things to remember is that all CIR process, Vasicek process are Markov models, and lies in the HJM framework.

$$
f(t,T) - f(0,T) = \int_0^t (\alpha(u,T)du + \sigma(u,T))dW(u) \tag{24}
$$

At time t, since the zero bond $B(t,T)$ is a martingale.

$$
log(B(t,T))) = -\int_t^T f(t,u) \tag{25}
$$

$$
\begin{aligned}
dlog(B(t,T))) &= f(t,t)dt - (\int_t^T \alpha(t,u)du)dt + \int_t^T \sigma(t,u)dW(t) \\
&= (f(t,t) - \alpha^*(t))dt + \sigma^*(t)dW(t)
\end{aligned} \tag{26}
$$

Then we have

$$\begin{aligned} d(B(t,T))) &= f(t,t)dt - (\int_t^T \alpha(t,u)du)dt + \int_t^T \sigma(t,u)dW(t) \\ &= B(t,T)(f(t,t) - \alpha^*(t) + \frac{1}{2}\sigma^{*2}(t))dt - \sigma^*(t)dW(t) \end{aligned} \tag{27}$$

Then we have

$$\begin{aligned} d(D(t)B(t,T))) &= f(t,t)dt - (\int_t^T \alpha(t,u)du)dt + \int_t^T \sigma(t,u)dW(t) \\ &= D(t)B(t,T)(-\alpha^*(t) + \frac{1}{2}\sigma^{*2}(t))dt - \sigma^*(t)d\widetilde{W}(t) \end{aligned} \tag{28}$$

So we need to have

$$(-\alpha^*(t,T) + \frac{1}{2}\sigma^{*2}(t,T)) = \Theta(t)|forT \in [0,T] \tag{29}$$

Then take derivative with T, we get

$$\alpha(t,T) = \sigma^*(t)\sigma(t,T) \tag{30}$$

2.1.2 Some Important Feature

Once we get no-arbitrage condition, we can have several important conclusion.

First, the T measure under this is vs risk neutral measure.

$$d\widetilde{W}^T(t) - \sigma^*(t) = d\widetilde{W}(t) \tag{31}$$

Second, The forward rate's drift is. This means the forward rate can be solely determined by sigma process.

$$f(t,T) - f(0,T) = \int_0^t (\sigma^*(u)\sigma(u,T)du + \sigma(u,T)dW(u)) \tag{32}$$

2.2 LMM Framework

2.2.1 Forward Payment Replicate

The first things to remember is how to replicate a pay off for a FRA that pays $L(T,T)$ at time $T+\delta$. In fact we can have portfolio of $\frac{1}{\delta}B(t,T) - \frac{1}{\delta}B(t,T+\delta)$ to exactly replicate the payoff.(At time T we need to reinvest our earning of first leg to $B(T,T+\delta)$ and have payoff $\frac{1}{\delta B(T,T+\delta)}$)

In fact, the red part is $B(t,T+\delta)L(t,T)$ and we can think of it as a $T+\delta$ bond times forward rate

2.2.2 $B(t, T+\delta)$ Measure Pricing Example (T measure)

Since $B(t, T+\delta)L(t, T)$ is a price of contract (tradable), so under $B(t, T+\delta)$ measure $L(t, T)$ is a martingale. So for example the price of caplet paying $(L(T, T) - K)^+$ at time $T + \delta$ can be valued as

$$C(t, T) = B(t, T+\delta)E^{T+\delta}(L(T, T) - K)^+ \tag{33}$$

Since at $B(t, T+\delta)L(t, T)$ measure the $L(t, T)$ satisfy $d(L(t, T)) = \gamma(t, T)L(t, T)d\widetilde{W}^{T+\delta}(t)$. Same as HJM, we need to find a vol process for $L(t, T)$ to price the caplet using blacks formula

In fact, we can get the $\gamma(t, T)$ by using zero bond at risk neutral measure. Since $F(t, T) = (\frac{1}{\delta}B(t, T) - \frac{1}{\delta}B(t, T+\delta))/B(t, T+\delta)$

So we have

$$\begin{aligned}
F(t, T) + \frac{1}{\delta} &= \frac{B(t, T)}{B(t, T+\delta)} \\
&= \frac{B(0, t)}{B(0, T)} \exp(\int_0^t (\sigma^*(t, T) - \sigma^*(t, T+\delta)) \cdot d\widetilde{W}(u) \\
&- 0.5 \int_0^t \|\sigma^*(t, T) - \sigma^*(t, T+\delta)\|^2 \, du \\
&- \int_0^t \sum_1^d \sigma_i^*(t, T+\delta)(\sigma_i^*(t, T) - \sigma_i^*(t, T+\delta)du)
\end{aligned} \tag{34}$$

So

$$\begin{aligned}
d(F(t, T)) = &(F(t, T) + \frac{1}{\delta}) \\
&- (\sum_1^d \sigma_i^*(t, T+\delta)(\sigma_i^*(t, T) - \sigma_i^*(t, T+\delta)du + (\sigma^*(t, T) - \sigma^*(t, T+\delta)) \cdot d\widetilde{W}(u)) \\
&= (F(t, T) + \frac{1}{\delta})(\sigma^*(t, T) - \sigma^*(t, T+\delta)) \cdot d\widetilde{W}^{T+\delta}(u)
\end{aligned} \tag{35}$$

If we keep HJM notation choose negative σ

$$\gamma(t, T) = \frac{1 + \delta F(t, T)}{F(t, T)} \frac{1}{\delta}(\sigma^*(t, T+\delta) - \sigma^*(t, T)) \tag{36}$$

This is the vol process under $B(t, T+\delta)$ measure

So you can do PCA on historical vol to get component. Then use implied vol of zero coupon bonds or Caplets to get the factored implied vol. Finally you can use monte-carlo engine to price the exotics in forward rate process. The use that to do the pricing.of Caplets or Swaptions.

2.2.3 Price A Claim Using Terminal Bond Measure

Assume a payoff at time t_n depend on the forward rate at $t1, t2, t3$, and now we are at t. Then we can to use the terminal bond measure (You always have to use a measure to price as base. The example is pay something base on the average past forward rates, which are the use of series of rates for payments)

If we use $B(t, t_n)$ measure to price the forward of the forward contract that pays $F(t_{n-2}, t_{n-1})$ at time t_{n-1}. Then we have

$$E^{B(u,t_n)}\left(\frac{F(u, t_{n-2}, t_{n-1}) - F(t, t_{n-2}, t_{n-1})}{B(u, t_n)}(1 + \delta F(u, t_{n-1}, t_n))\right)|_{u=t_n} = 0 \tag{37}$$

The equation above can be rewrite as

$$\begin{aligned} F(t, t_{n-2}, t_{n-1}) &= \frac{E^{B(u,t_n)}_{u=t_n}(F(u, t_{n-2}, t_{n-1})(1 + \delta F(u, t_{n-1}, t_n))}{E^{B(u,t_n)}_{u=t_n}(1 + \delta F(u, t_{n-1}, t_n))} \\ &= \frac{COV^{B(u,t_n)}_{u=t_n}(F(u, t_{n-2}, t_{n-1})(1 + \delta F(u, t_{n-1}, t_n))}{E^{B(u,t_n)}_{u=t_n}(1 + \delta F(u, t_{n-1}, t_n))} + E^{B(u,t_n)}_{u=t_n} F(u, t_{n-2}, t_{n-1}) \\ &= \frac{\delta \rho_{t_{n-2}, t_{n-1}} \sigma_{t_{n-2}} \sigma_{t_{n-1}}}{E^{B(u,t_n)}_{u=t_n}(1 + \delta F(u, t_{n-1}, t_n))} + E^{B(u,t_n)}_{u=t_n} F(u, t_{n-2}, t_{n-1}) \\ &= \frac{\delta \rho_{t_{n-2}, t_{n-1}} \sigma_{t_{n-2}} \sigma_{t_{n-1}}}{1 + \delta F(t, t_{n-1}, t_n))} + E^{B(u,t_n)}_{u=t_n} F(u, t_{n-2}, t_{n-1}) \end{aligned} \tag{38}$$

So the drift is $-\frac{\rho_{t_{n-2}, t_{n-1}} \sigma_{t_{n-2}} \sigma_{t_{n-1}}}{1 + \delta F(t, t_{n-1}, t_n))}$

3 Real Data Calibration

3.1 LMM

To calibrate LMM, we need two kinds of data: volatility and correlation of forward rate.

- Caps are for calculating volatility (sum of individual vol)
- Swaptions are for calulating volatility and correlation (average of forward rate)
- Correlation can also be derived from historical analysis