1 尝试证明

我们电动力学书上是直接给出梯度散度旋度拉普拉斯算符的，我对此一直很困惑

定义 1.1 (梯度).

$$\boldsymbol{\nabla} = \sum_i \frac{1}{h_i}\frac{\partial}{\partial x_i}\hat{\mathbf{x}}_i \tag{1.1}$$

定义 1.2 (散度).

$$\boldsymbol{\nabla}\cdot\mathbf{y} = \frac{1}{\prod_i h_i}\sum_i\left[\frac{\partial}{\partial x_i}(h_i' y_i)\right], \quad h_i' = \prod_{j\neq i} h_j \tag{1.2}$$

我发现：
只要令 Eq. (1.2) 中 $\mathbf{y}$ 等于 $\boldsymbol{\nabla}$
即：令 $y_i = \frac{1}{h_i}\frac{\partial}{\partial x_i}$ 塞进 Eq. (1.2)

定义 1.3 (拉普拉斯算子).

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} \equiv \nabla^2 = \frac{1}{\prod_i h_i}\sum_i\left[\frac{\partial}{\partial x_i}\left(\frac{h_i'}{h_i}\frac{\partial}{\partial x_i}\right)\right], \quad H_i = \frac{\prod_{j\neq i} h_j}{h_i} = \frac{h_i'}{h_i} \tag{1.3}$$

定义 1.4 (旋度).

$$\boldsymbol{\nabla}\times\mathbf{y} = \frac{1}{\prod_i h_i}\begin{vmatrix} \dots & h_i\hat{\mathbf{x}}_i & \dots \\ \dots & \frac{\partial}{\partial x_i} & \dots \\ \dots & h_i y_i & \dots \end{vmatrix} \tag{1.4}$$

先证明梯度，其他看起来大同小异。直接列式子要写一大堆，用线性代数矩阵乘法来做要简洁些 (满足结合律)。

2 开始证明

2.1 铺垫

2.1.1 旋转矩阵

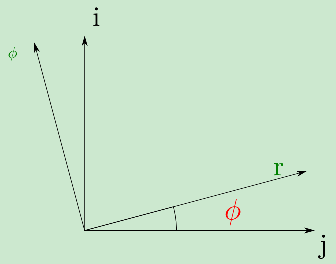

图 1. 直角坐标基矢变换为极坐标基矢

$$
\begin{aligned}
\hat{\mathbf{r}} &= \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \\
\hat{\boldsymbol{\phi}} &= \cos\Big(\phi + \frac{\pi}{2}\Big)\hat{\mathbf{i}} + \sin\Big(\phi + \frac{\pi}{2}\Big)\hat{\mathbf{j}}
\end{aligned} \tag{2.1}
$$

写成矩阵形式

$$
\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \end{bmatrix} \tag{2.2}
$$

定义 2.1 (旋转矩阵 $\hat{R}$).

$$
\hat{R} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \tag{2.3}
$$

评注. 极坐标的旋转矩阵好写，就是看图转下，ϕ 多加 90 度；柱坐标也好写，在极坐标基础上多加一维；球坐标的旋转矩阵，就是在柱坐标基础上再转下，在柱坐标旋转矩阵基础上再乘一个矩阵

通过观察法凑出基矢变换矩阵 (对于球坐标也就是转两次的“旋转矩阵”) 可以这样写：

$$
\begin{aligned}
\hat{R} &= \begin{bmatrix} \frac{1}{h_1}\frac{\partial x}{\partial r} & \frac{1}{h_1}\frac{\partial y}{\partial r} & \frac{1}{h_1}\frac{\partial z}{\partial r} \\ \frac{1}{h_2}\frac{\partial x}{\partial \theta} & \frac{1}{h_2}\frac{\partial y}{\partial \theta} & \frac{1}{h_2}\frac{\partial z}{\partial \theta} \\ \frac{1}{h_3}\frac{\partial x}{\partial \phi} & \frac{1}{h_3}\frac{\partial y}{\partial \phi} & \frac{1}{h_3}\frac{\partial z}{\partial \phi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{h_1} & 0 & 0 \\ 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & \frac{1}{h_3} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{h_1} & 0 & 0 \\ 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & \frac{1}{h_3} \end{bmatrix} \hat{J}^{\mathrm{T}}
\end{aligned} \tag{2.4}
$$

2.1.2 雅可比矩阵

定义 2.2 (雅可比矩阵).

$$
\begin{aligned}
\hat{J} &= \left(\left(\frac{\partial}{\partial \mathbf{x}}\right)(\mathbf{y})^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix}\right)^{\mathrm{T}} \\
&= \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{r}}{\partial r} & \frac{\partial \mathbf{r}}{\partial \theta} & \frac{\partial \mathbf{r}}{\partial \phi} \end{bmatrix}
\end{aligned} \tag{2.5}
$$

2.1.3 拉梅系数

$$
h_i = \left\| \frac{\partial \mathbf{y}}{\partial x_i} \right\| \quad \text{或写成} \quad h_i = \left\| \frac{\partial \mathbf{r}}{\partial q_i} \right\| \tag{2.6}
$$

如表1所示。

看起来对雅可比矩阵的一列取模就是拉梅系数的值了？

表 1. 拉梅系数.

(q_1,q_2,q_3)	直角 $(x,y,z)^{\mathrm{T}}$	柱 $(\rho,\phi,z)^{\mathrm{T}}$	球 $(r,\theta,\phi)^{\mathrm{T}}$
h_1	1	1	1
h_2	1	ρ	r
h_3	1	1	$r\sin\theta$

2.2 证明的关键：旋转矩阵乘雅可比行列式

利用链式法则和式 (2.5) 可得

$$\begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \phi}\\ \frac{\partial u}{\partial \theta}\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial z}\end{bmatrix} = \hat{J}^{\mathrm{T}} \begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial z}\end{bmatrix} \tag{2.7}$$

$$\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial z}\end{bmatrix} = \begin{bmatrix}\frac{\partial r}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial \theta}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial \phi}{\partial y} & \frac{\partial \theta}{\partial y}\\ \frac{\partial r}{\partial z} & \frac{\partial \phi}{\partial z} & \frac{\partial \theta}{\partial z}\end{bmatrix} \begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \phi}\\ \frac{\partial u}{\partial \theta}\end{bmatrix} = (\hat{J}^{\mathrm{T}})^{-1} \begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \phi}\\ \frac{\partial u}{\partial \theta}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial z}\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \phi}\\ \frac{\partial u}{\partial \theta}\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\frac{\partial r}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial \theta}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial \phi}{\partial y} & \frac{\partial \theta}{\partial y}\\ \frac{\partial r}{\partial z} & \frac{\partial \phi}{\partial z} & \frac{\partial \theta}{\partial z}\end{bmatrix}^{\mathrm{T}} = ((\hat{J}^{\mathrm{T}})^{-1})^{\mathrm{T}} \begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \phi}\\ \frac{\partial u}{\partial \theta}\end{bmatrix} = \hat{J}^{-1} \begin{bmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \phi}\\ \frac{\partial u}{\partial \theta}\end{bmatrix}$$

直角坐标的梯度可以这样写：

$$\nabla u = \begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y}\end{bmatrix} \begin{bmatrix}\hat{\mathbf{i}}\\ \hat{\mathbf{j}}\\ \hat{\mathbf{k}}\end{bmatrix} \tag{2.8}$$

把式 (2.3)和式 (2.7) 带入式 (2.8)，并把式 (1.1)写成矩阵形式：

$$\begin{aligned}\nabla u &= \begin{bmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \theta}\end{bmatrix} \hat{J}^{-1}\hat{R}^{\mathrm{T}} \begin{bmatrix}\hat{\mathbf{r}}\\ \hat{\phi}\\ \hat{\theta}\end{bmatrix}\\ &= \begin{bmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial \theta}\end{bmatrix} \begin{bmatrix}\frac{1}{h_1} & 0 & 0\\ 0 & \frac{1}{h_2} & 0\\ 0 & 0 & \frac{1}{h_3}\end{bmatrix} \begin{bmatrix}\hat{\mathbf{r}}\\ \hat{\phi}\\ \hat{\theta}\end{bmatrix}\end{aligned} \tag{2.9}$$

我猜想红色的式子是普遍相等的

$$\hat{J}^{-1}\hat{R}^{\mathrm{T}} = \begin{bmatrix}\frac{1}{h_1} & 0 & 0\\ 0 & \frac{1}{h_2} & 0\\ 0 & 0 & \frac{1}{h_3}\end{bmatrix}$$

$$(\hat{J}^{-1}\hat{R}^{\mathrm{T}})^{-1} = \hat{R}\hat{J} = \begin{bmatrix}h_1 & 0 & 0\\ 0 & h_2 & 0\\ 0 & 0 & h_3\end{bmatrix}$$

将式 (2.4) 带入：

$$
\begin{aligned}
\hat{R}\hat{J} &= \begin{bmatrix} \frac{1}{h_1} & 0 & 0 \\ 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & \frac{1}{h_3} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{h_1} & 0 & 0 \\ 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & \frac{1}{h_3} \end{bmatrix} \hat{J}^{\mathrm{T}}\hat{J}
\end{aligned}
$$

也就是证明：

$$
\hat{J}^{\mathrm{T}}\hat{J} = \begin{bmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{bmatrix} \tag{2.10}
$$

由拉梅系数定义式 (2.6) 知，只需要证明雅可比矩阵正交

```
(J = D[#, {{r, \[Theta], \[Phi]}}] & [{r Sin[\[Theta]] Cos[\[Phi]],
     r Sin[\[Theta]] Sin[\[Phi]], r Cos[\[Theta]]}]) // MatrixForm
Transpose[J] . J // Simplify // MatrixForm
```

用 Mathematica 验证发现对于极坐标,柱坐标,球坐标,雅可比矩阵都满足式 (2.10)

Out[12]//MatrixForm=

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}
$$

。

2.3 没证明出来的地方

- 如果给出任意变换，如何写出基矢变换矩阵 (2.4)
- 雅可比矩阵正交