# Differential Manifolds

此模板修改自 *jsk-lecnotes*

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# 1. Analogy and Introduction

Let's look at the theory of flat ground. Humans cannot distinguish flatness: From a local perspective, the human eye cannot differentiate between a plane and a slightly curved surface. Such a curved surface is locally similar to a plane, and with this intuition, we now have a general understanding of what local similarity means.

Following this concept upwards, can the Earth be considered a manifold? Of course it can. Although the Earth's surface is curved, the local area around each point can be considered similar to a plane. Therefore, the Earth is locally similar to a Euclidean plane everywhere, and thus the Earth's surface can be viewed as a manifold. The local coordinate system that this person just established is called a local coordinate chart.

If we express this in strict mathematical language, manifolds and coordinate charts should be defined as follows:

**定义 1.1（Manifold）**

For a topological space $M$, if $\forall p \in M$, there always exists an open neighborhood $U$ of $p$, an open set $V$ in $\mathbb{R}^n$, and a homeomorphism $\varphi : U \to V \subseteq \mathbb{R}^n$, then $M$ is called locally Euclidean. If $M$ is also a second-countable Hausdorff space, then $M$ is called a (topological) manifold. The number $n$ is called the dimension of the manifold. ♣

**定义 1.2（相容性与转移映射）**

设 $M$ 和 $N$ 是两个拓扑空间，$M$ 上的一个坐标卡是一个同胚映射 $\varphi : U \to V$， 其中 $U$ 是 $M$ 的开子集，$V$ 是欧几里得空间 $\mathbb{R}^n$ 的开子集。如果 $U$ 覆盖了 $M$，那么所有这样的坐标卡构成了 $M$ 上的一个**图册**。

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \tag{1.1}$$

是微分同胚，则称这两个坐标卡是相容的。♣

**命题 1.1（光滑映射的复合）**

如果 $f \in C^\infty(M, N)$ 且 $g \in C^\infty(N, P)$，那么 $g \circ f \in C^\infty(M, P)$. Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aeque doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere. ♠

来阐明余切矢量作为**微分形式**的性质, 以及它们如何与**微分形式**相联系。

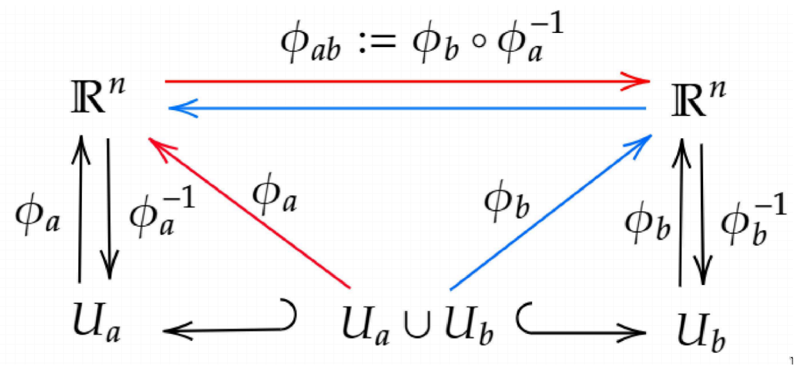


图 1.1 用来描述上述映射之间关系的交换图

**例题 1.1 (证明)**:

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \tag{1.2}$$

$$\begin{aligned}
&[X,[Y,Z]](f)\\
&=X([Y,Z])(f)-[Y,Z](X(f))\\
&=X(Y(Z(f)))-Y(Z(X(f)))-Z(X(Y(f)))+Z(Y(X(f)))
\end{aligned} \tag{1.3}$$

Inspired by

## 1.1. 散度、旋度

$$\begin{aligned}
\nabla\cdot\boldsymbol{a} &= \star\, d\, \star\left(h_1a_1du^1+h_2a_2du^2+h_3a_3du^3\right)\\
&= \star\, d\left(a_1h_2h_3du^2\wedge du^3+a_2h_3h_1du^3\wedge du^1+a_3h_1h_2du^1\wedge du^2\right)\\
&= \star\left(\frac{\partial a_1h_2h_3}{\partial u^1}du^1\wedge du^2\wedge du^3+\frac{\partial a_2h_3h_1}{\partial u^2}du^2\wedge du^3\wedge du^1+\frac{\partial a_3h_1h_2}{\partial u^3}du^3\wedge du^1\wedge du^2\right)\\
&= \frac{1}{h_1h_2h_3}\left(\frac{\partial a_1h_2h_3}{\partial u^1}+\frac{\partial a_2h_3h_1}{\partial u^2}+\frac{\partial a_3h_1h_2}{\partial u^3}\right)
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
&\nabla\times\boldsymbol{a} = \star\, d\left(h_1a_1du^1+h_2a_2du^2+h_3a_3du^3\right)\\
&= \star\left(\frac{\partial h_1a_1}{\partial u^2}du^2\wedge du^1+\frac{\partial h_1a_1}{\partial u^3}du^3\wedge du^1\right.\\
&\quad+\frac{\partial h_2a_2}{\partial u^3}du^3\wedge du^2+\frac{\partial h_2a_2}{\partial u^1}du^1\wedge du^2\\
&\quad\left.+\frac{\partial h_3a_3}{\partial u^1}du^1\wedge du^3+\frac{\partial h_3a_3}{\partial u^2}du^2\wedge du^3\right)\\
&= \frac{h_1}{h_2h_3}\left(\frac{\partial h_3a_3}{\partial u^2}-\frac{\partial h_2a_2}{\partial u^3}\right)du^1+\frac{h_2}{h_3h_1}\left(\frac{\partial h_1a_1}{\partial u^3}-\frac{\partial h_3a_3}{\partial u^1}\right)du^2+\frac{h_3}{h_1h_2}\left(\frac{\partial h_2a_2}{\partial u^1}-\frac{\partial h_1a_1}{\partial u^2}\right)du^3\\
&= \frac{1}{h_2h_3}\left(\frac{\partial h_3a_3}{\partial u^2}-\frac{\partial h_2a_2}{\partial u^3}\right)\hat{e}_1+\frac{1}{h_3h_1}\left(\frac{\partial h_1a_1}{\partial u^3}-\frac{\partial h_3a_3}{\partial u^1}\right)\hat{e}_2+\frac{1}{h_1h_2}\left(\frac{\partial h_2a_2}{\partial u^1}-\frac{\partial h_1a_1}{\partial u^2}\right)\hat{e}_3
\end{aligned} \tag{1.5}$$

SimpleNote[1] 修改自 jsk-lecnotes，是一个简单的 Typst 模板。本模板主要适用于编写课程笔记，默认页边距为 2.5cm，默认使用的中文字体为 Noto Sans CJK SC，英文字体为 Linux Libertine，字号为 12pt（小四），你可以根据自己的需求进行更改，如需使用**伪粗体**或伪斜体，可以使用外部包 cuti。封面图片由 Tabbied 生成。

*默认模板文件由主要以下六部分组成*：

- main.typ 主文件
- template.typ 文档格式控制，包括一些基础的设置、函数
- refs.bib 参考文献
- content 正文文件夹，存放正文章节
- fonts 字体文件夹
- figures 图片文件夹

使用模板首先需配置 main.typ，设置标题、描述、作者等信息。如需要进一步更改文档格式，请修改 template.typ。

# 2. 使用示例

## 2.1. 特殊记号

你可以 Typst 的语法对文本进行特殊标记，我们为如下标记设定了样式：

1. **突出**
2. *强调*
3. 引用 小节 2.3
4. 脚注[1]

## 2.2. 代码

行内代码使用例 `println!("Hello, typst!")`，下面是代码块使用例：

```
fn main() {
    println!("Hello, typst!");
}
```

代码 2.1　代码块插入例

## 2.3. 图片

[1]脚注例

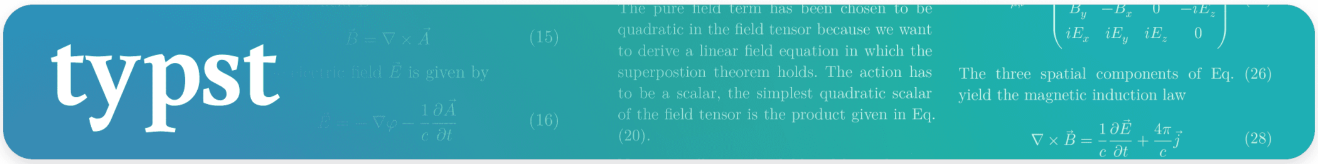


图 2.2 图片插入例

## 2.4. 表格

| 表头 1 | 表头 2 | 表头 3 |
| --- | --- | --- |
| 单元格 1 | 单元格 2 | 单元格 3 |
| 单元格 1 | 单元格 2 | 单元格 3 |
| 单元格 1 | 单元格 2 | 单元格 3 |

表 2.1 表格插入例

## 2.5. 引用块

> 引用块例
>
> > 二级引用块

## 2.6. 定理环境

**定义 2.6.1（定义）**

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aeque doleamus animo, cum corpore dolemus, fieri. ♣

**例题 2.6.1 (示例)**：Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aeque doleamus animo, cum corpore dolemus, fieri.

提示 提示 **2.6.1**

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**注意** 注意 **2.6.1**

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**引用** 引用 **2.6.1**

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**定理 2.6.1（定理）**

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**命题 2.6.1（命题）**

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# 参考文献

[1] AKKIRI. SimpleNote[EB/OL]. (2023-12-19). https://github.com/a-kkiri/SimpleNote.