# Monte Carlo Simulation

## Simulate Stochastic Differential Equations (SDE)

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# Simulation of SDE

- Certain SDE's are solvable in closed form, e.g., geometric Brownian Motion:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

Which has an explicit solution

$$S_t = S_0 \exp\left( (r - \frac{1}{2}\sigma^2)t + \sigma W_t \right)$$

- However, most of SDE's are NOT solvable, which requires us to resort to numerical solutions.

# Convergence of SDE solutions

- Suppose we have a general SDE:

$$dX(t) = b(X_t, t)dt + \sigma(X_t, t)dW_t$$

- Suppose we have a solution $\hat{X}_t^{\Delta}$ (via numerical solution through discretization of time step $\Delta$)

- $\hat{X}_t$ has a strong order of convergence of $\alpha$ if

$E[\,|\hat{X}_t^{\Delta} - X_t|\,] \leq C\Delta^{\alpha}$ for some fixed constant C

- $\hat{X}_t$ has a weak order of convergence of $\alpha$ if

$E[\,|f(\hat{X}_t^{\Delta}) - f(X_t)|\,] \leq C\Delta^{\alpha}$ for all sufficiently smooth function $f$ and some fixed constant C

# Convergence of SDE Solutions

- The larger the order $\alpha$, the better quality the solution has.
- Generallly, for any given solution scheme, its strong order of convergence is typically less than its weaker order of convergence.

# Two SDE solution schemes

- Euler scheme

$$X_{i+1} = X_i + b(X_i, t_i)(t_{i+1} - t_i) + \sigma(X_i, t_i)(W_{i+1} - W_i)$$

- Milstein scheme

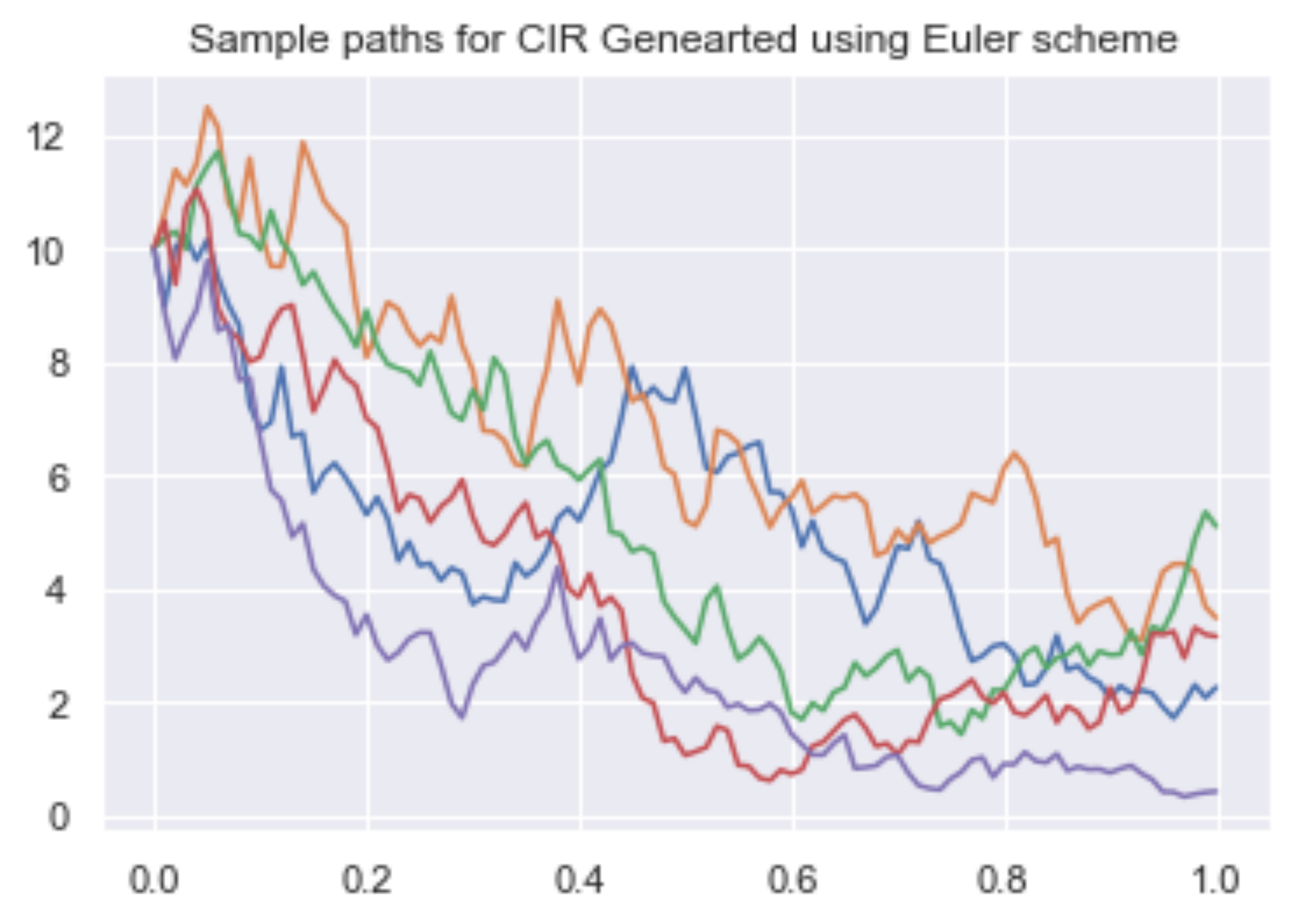
$$X_{i+1} = X_i + b(X_i, t_i)(t_{i+1} - t_i) + \sigma(X_i, t_i)(W_{i+1} - W_i) + \frac{1}{2}\sigma(X_i, t_i)\sigma_x(X_i, t_i)\left((W_{i+1} - W_i)^2 - (t_{i+1} - t_i)\right)$$

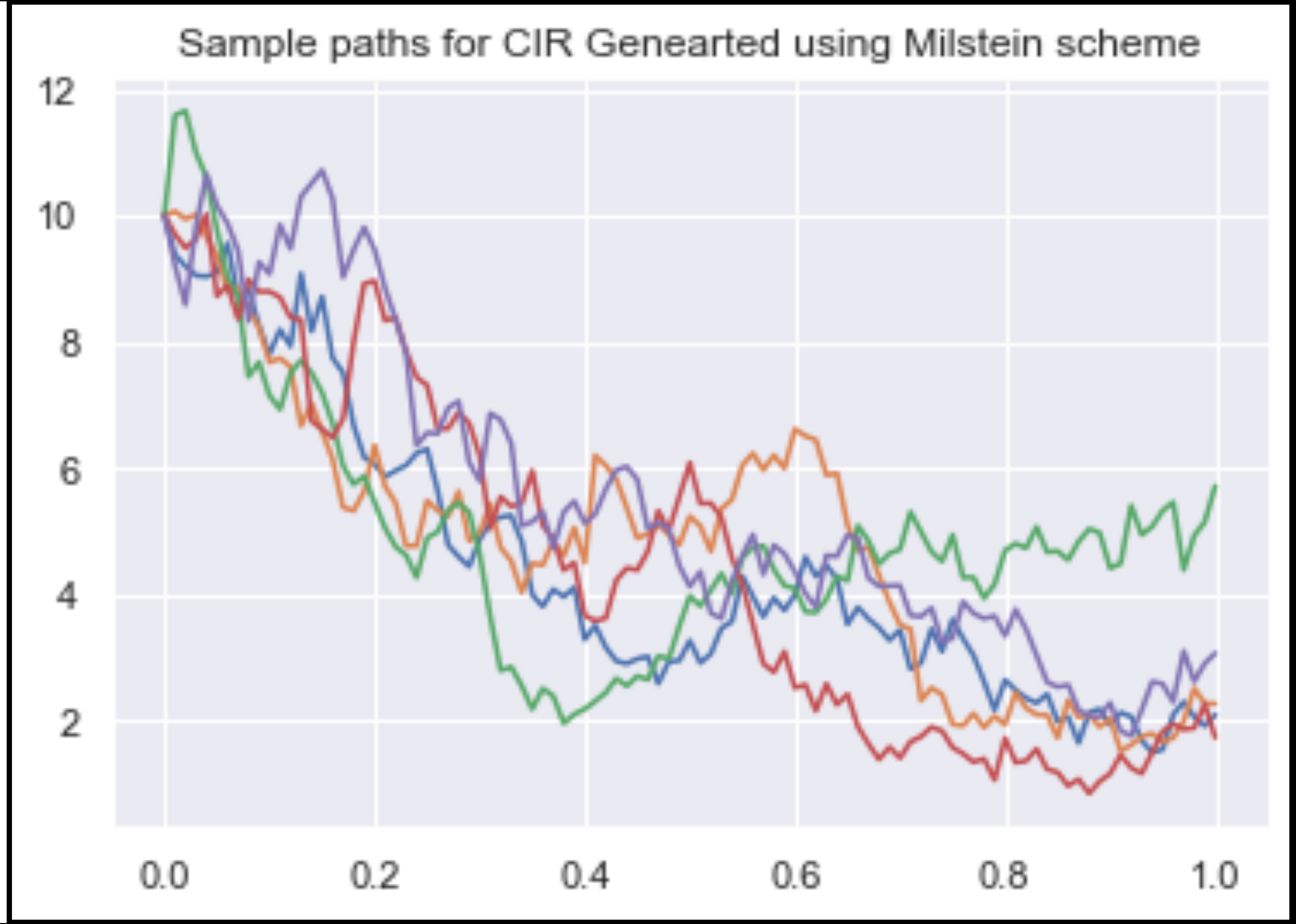
- Euler scheme is more straightforward, but Milstein scheme is more accurate (note that for Milstein scheme derivative of diffusion function with respect to x is required)

# Two SDE solution schemes

- For Euler scheme, strong and weak orders of convergences are 1/2 and 1 respectively
- For Milsten scheme, strong and weak orders of convergences are 1 in both cases.

# SDE simulation trajectories



# Application - Option Pricing with Heston Model

- Heston Stochastic Volatility Model

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dB_t$$

$$d[W_t, B_t] = \rho dt$$

- Using Milstein scheme we get pretty accurate option price (see notebook)