Problem 1 - Pricing European option if Stock price follows a normal distribution

Suppose the stock price S obeys a normal (instead of a lognormal) distribution:

$$S_T \sim N(S_0 e^{rT}, \sigma\sqrt{T})$$

We'd like to calculate the price of an option which pays $max(S_T - K, 0)$ at expiry T.

The way to do it is to calculate the following expectation value:

$$C = e^{-rT}E^Q[max(S-K,0)] = e^{-rT}\int_{-\infty}^{\infty} max(S_0e^{rT} + \sigma\sqrt{T}x - K, 0)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = e^{-rT}\int_{d}^{\infty}(S_0e^{rT} + \sigma\sqrt{T}x - K)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

Where $d = \dfrac{K - S_0e^{rT}}{\sigma\sqrt{T}}$.

Your task is to find the option price by solving the above integral.

Hint: you can express your final answer with the following function: $N(z) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\infty}^{z} e^{\frac{-x^2}{2}}dx$

Notice that $N(-z) = 1 - N(z) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{z}^{\infty} e^{\frac{-x^2}{2}}dx$

Solution:

All you need to do is to calculate the following

$$C = e^{-rT}\int_{d}^{\infty}\left(S_0e^{rt} + \sigma\sqrt{T}x - K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

Which can be transformed to

$$C = \left(S_0 - Ke^{-rT}\right)\int_{d}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + e^{-rT}\sigma\sqrt{T}\int_{d}^{\infty}x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= \left(S_0 - Ke^{-rT}\right)N(-d) - e^{-rT}\sigma\sqrt{T}\int_{d}^{\infty}\frac{1}{\sqrt{2\pi}}de^{-\frac{x^2}{2}}$$

$$= \left(S_0 - Ke^{-rT}\right)N(-d) + e^{-rT}\sigma\sqrt{T}n(d)$$

Where

$$n(d) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}$$

Problem 2 Monte Carlo Simulation

Use Monte Carlo Simulation to find the expected number of times for flipping a coin (which contains 2 faces: head and tail) until the first time you get two consecutive heads.

Hint: represent the sequence of coin tosses as a series of heads (H) and tails (T). Every time simulate a random variable X which is uniformly distributed between 0 and 1 (via np.random.rand() as done in the class), and take the experiment result as 'H' if X>0.5 and 'T' otherwise (so with probability 1/2 you get either H or T each time). Repeat the experiment many (e.g., 10000) times and count the average number of times it takes to get 2 consecutive heads (HH).

Solution - see the uploaded Jupiter notebook. Theoretical answer (which you can prove) is 6.