Gomes, Kogan, and Zhang (2003)

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Abstract: DSGE Linking β and Size-B/M

We construct a dynamic general equilibrium production economy to explicitly link expected stock returns to firm characteristics such as firm size and the book-to-market ratio. Stock returns in the model are completely characterized by a conditional capital asset pricing model (CAPM). Size and book-to-market are correlated with the true conditional market beta and therefore appear to predict stock returns. The cross-sectional relations between firm characteristics and returns can subsist even after one controls for typical empirical estimates of beta. These findings suggest that the empirical success of size and book-to-market can be consistent with a single-factor conditional CAPM model.

I. Introduction

1. The one-factor equilibrium embraces the size and book-to-market's ability to explain the cross-section of returns
2. The cross-sectional dispersion is related to the aggregate stock market volatility and business cycle conditions
3. The premia of size and book-to-market are inherently conditional and likely countercyclical

II. The Model

- (1) One representative agent maximizing utility by consumption and (2) multiple heterogeneous firms maximizing firm value by investment
- Mean-reverting market-wide productivity x_t, project-specific (idiosyncratic) productivity ϵ_{it} following a square-root process, time and state are continuous
- New projects are randomly delivered to firms in continuous time—accept if $p(x) \geq e$ (the investment cost), or forgo else
- $p(x)$ as the portion of NPV by aggregate productivity, $\tilde{p}(x)$ as the portion by idiosyncratic counterpart

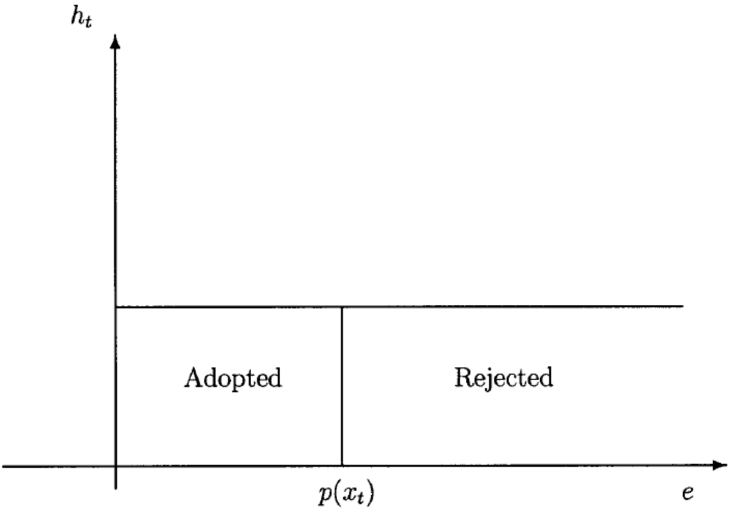
PROPOSITION 1. *Project valuation.*—The value of an existing project i is given by

$$P(x_t, \epsilon_{it}, k_i) = E_t\left[\int_0^\infty e^{-\delta s} M_{t,t+s}(e^{x_{t+s}} \epsilon_{i,t+s} k_i)\, ds\right]$$

$$= k_i[p(x_t) + \tilde{p}(x_t)(\epsilon_{it} - 1)], \qquad (3)$$

Figure 1: h_t as the Arrival Rate of New Projects

Firms Accept Them Only If Profitable, i.e. $p(x) \geq e$



II. The Model (cont.)

- The value of firm's assets-in-place V_{ft}^a—the sum of all the project NPVs—is the difference between the firm value V_{ft} and the growth options V_{ft}^o

  terminology from Berk et al. (1999), V_{ft}^a represents the value of *assets in place*, defined as

$$V_{ft}^a = \sum_{i \in \mathcal{I}_{ft}} P(x_t, \epsilon_{it}, k_i) = \sum_{i \in \mathcal{I}_{ft}} k_i[p(x_t) + \tilde{p}(x_t)(\epsilon_{it} - 1)], \qquad (7)$$

  whereas $V_{ft}^o = V_{ft} - V_{ft}^a$ can be interpreted as the value of *growth options*.

- The pricing kernel $M_{t,t+s} = e^{-\lambda s}\left(\frac{C_t}{C_{t+s}}\right)^{\gamma}$ by households maximizing utility by consumption decisions
- Proposition 1: The competitive equilibrium satisfies (a) optimization and (b) market clearing, and consists of (1) the pricing kernel M, (2) the consumption policy C, and (3) the investment policy $p(x)$—Proposition 2 provides the equilibrium solutions
- Proposition 3 exhibits the equilibrium asset prices

II. The Model (cont.)

- More importantly, Proposition 4 is the beta anatomy—β_{ft} as the market beta, $\tilde{\beta}_t^a$ as the sensitivity through idiosyncratic productivity, β_t^o as the sensitivity through growth options, β_t^a as the sensitivity through aggregate productivity
- β_{ft} is the weighted-average of the other three betas

PROPOSITION 4. *Market betas of individual stocks.*—Firm market betas are described by

$$\beta_{ft} = \boxed{\tilde{\beta}_t^a} + \boxed{\frac{\bar{V}_t^o}{V_{ft}}(\beta_t^o - \tilde{\beta}_t^a)} + \boxed{\frac{1}{p(x_t)}\frac{K_{ft}}{V_{ft}}(\beta_t^a - \tilde{\beta}_t^a)}, \qquad (28)$$

where

$$K_{ft} = \sum_{i \in \mathcal{I}_{ft}} k_i$$

and

$$\beta_t^a = \frac{\partial \log p_t / \partial x}{\partial \log V_t / \partial x}, \qquad \tilde{\beta}_t^a = \frac{\partial \log \tilde{p}_t / \partial x}{\partial \log V_t / \partial x}, \qquad \beta_t^o = \frac{\partial \log V_t^o / \partial x}{\partial \log V_t / \partial x}. \qquad (29)$$

- Therefore, small firms (V) and high book-to-market firms (K/V) have high market betas

Proposition 3: Asset Prices in Equilibrium

PROPOSITION 3. *Equilibrium asset prices.*—The instantaneous risk-free interest rate is determined by

$$r_t = -\frac{E_t[M_{t,t+dt} - 1]}{dt} = \lambda + \gamma[zp(x_t) - \delta] + \gamma\frac{\mathcal{A}[C(x_t, K_t)]}{C_t} - \tfrac{1}{2}\gamma(\gamma+1)\sigma_x^2\left[\frac{\partial \ln C(x_t, K_t)}{\partial x_t}\right]^2. \quad (21)$$

The aggregate stock market value, V_t, can be computed as

$$\begin{aligned} V_t &= E_t\left[\int_0^\infty M_{t,t+s} D_{t+s}\, ds\right] \\ &= E_t\left[\int_0^\infty e^{-\lambda s}\left(\frac{C_t}{C_{t+s}}\right)^\gamma C_{t+s}\, ds\right] \\ &= \{e^{x_t} - \tfrac{1}{2}z[p(x_t)]^2\}^\gamma \psi(x_t) K_t, \end{aligned} \quad (22)$$

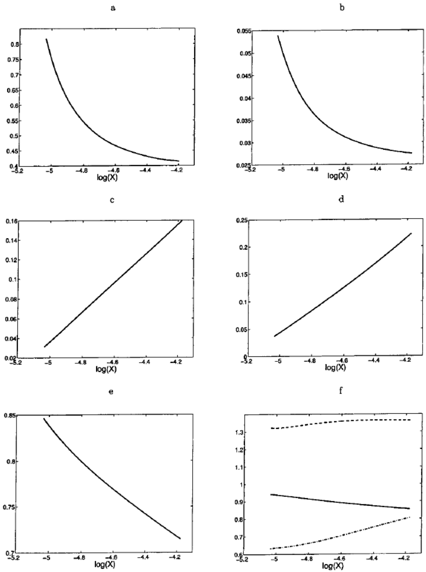
where the function $\psi(x)$ satisfies the differential equation

$$\lambda\psi(x) = \{e^x - \tfrac{1}{2}z[p(x)]^2\}^{1-\gamma} + (1-\gamma)[zp(x) - \delta]\psi(x) + \mathcal{A}[\psi(x)],$$

and $\mathcal{A}[\cdot]$ is defined as in (20).

Figure 2: Comparative Statics by State Variable x

Sharpe Ratio, Consumption Volatility, $p(x)$ or V^a/K, V/K, V^a/V, β^o and β^a and $\tilde{\beta}^a$



III. Aggregate Stock Returns

- Table 1: Calibration with 7 moment conditions
  - First two moments of stock returns
  - First two moments of risk-free rate
  - First two moments of aggregate consumption growth
  - Average level of the investment-to-output ratio

Table 1: Calibration with Seven Moments

	Data		Population		Sample	
	Mean (1)	Standard Deviation (2)	Mean (3)	Standard Deviation (4)	Mean (5)	Standard Deviation (6)
$(C_{t+1}/C_t) - 1$	1.72	3.28	.85	3.22	.84 (.28) [.22 1.33]	3.06 (.26) [2.56 3.50]
r_t	1.80	3.00	1.30	4.33	1.34 (1.30) [−.63 4.23]	3.98 (.85) [2.55 5.73]
$\log R_t - \log r_t$	6.00	18.0	6.00	14.34	5.89 (1.32) [2.97 8.13]	15.28 (1.73) [11.80 18.58]
I_t/Y_t	.19		.23		.23 (.02) [.19 .26]	

Table 2: Simulated Dividend Yield and Book-to-Market and Their Monthly/Annual Relation

Consistent with Pontiff and Schall (1988)

	Source	Mean	Standard Deviation	1 Year	2 Years	3 Years	4 Years	5 Years
Dividend yield	Data	4.267	1.37	.60	.36	.26	.23	.25
	Model	6.407	.97	.69	.46	.31	.19	.11
		(.321)	(.22)	(.08)	(.14)	(.17)	(.18)	(.18)
		[5.789 7.084]	[.61 1.45]	[.51 .82]	[.17 .70]	[−.05 .61]	[−.16 .51]	[−.22 .45]
Book-to-market	Data	.668	.23	.68	.43	.23	.08	.00
	Model	.584	.19	.88	.80	.73	.68	.64
		(.052)	(.04)	(.03)	(.07)	(.09)	(.12)	(.13)
		[.495 .707]	[.12 .28]	[.81 .93]	[.63 .89]	[.48 .86]	[.38 .84]	[.31 .83]

B. Regressions on Book-to-Market

	Data		Model	
	Slope	Adjusted R^2	Slope	Adjusted R^2
Monthly	3.02	.01	1.75	.00
			(.79)	(.00)
			[.68 3.65]	[.00 .01]
Annual	42.18	.16	19.88	.04
			(10.46)	(.04)
			[6.57 46.09]	[.00 .14]

IV. The Cross Section of Stock Returns

- Table 3: 10 size portfolios
- Table 4: 10 B/M portfolios
- Table 5: Fama–MacBeth regressions
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Table 3: The Cross-Section of 10 Size Portfolios

Smalls Earn More, Consistent with Fama and French (1992)

	Portfolio											
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
	A. Historical Data											
Return	1.64	1.16	1.29	1.24	1.25	1.29	1.17	1.07	1.10	.95	.88	.90
β	1.44	1.44	1.39	1.34	1.33	1.24	1.22	1.16	1.08	1.02	.95	.90
$\log(V_f)$	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
$\log(B_f/V_f)$	−.01	−.21	−.23	−.26	−.32	−.36	−.36	−.44	−.40	−.42	−.51	−.65
	B. Simulated Panel											
Return	.73	.72	.71	.70	.69	.70	.68	.67	.66	.64	.61	.55
β	1.05	1.05	1.03	1.02	1.01	1.01	1.00	.99	.97	.95	.89	.89
$\log(V_f)$	4.86	5.04	5.12	5.16	5.20	5.24	5.27	5.32	5.37	5.46	5.58	5.84
$\log(B_f/V_f)$	−.93	−.86	−.85	−.84	−.85	−.86	−.87	−.90	−.97	−1.09	−1.24	−1.49

Table 4: The Cross-Section of 10 B/M Portfolios

Highs Earn More, Consistent with Fama and French (1992)

	Portfolio											
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
	A. Historical Data											
Return	.30	.67	.87	.97	1.04	1.17	1.30	1.44	1.50	1.59	1.92	1.83
β	1.36	1.34	1.32	1.30	1.28	1.27	1.27	1.27	1.27	1.29	1.33	1.35
$\log(V_f)$	4.53	4.67	4.69	4.56	4.47	4.38	4.23	4.06	3.85	3.51	3.06	2.65
$\log(B_f/V_f)$	−2.22	−1.51	−1.09	−.75	−.51	−.32	−.14	.03	.21	.42	.66	1.02
	B. Simulated Panel											
Return	.61	.65	.67	.70	.70	.71	.71	.71	.71	.70	.71	.71
β	.95	.98	1.01	1.02	1.02	1.02	1.03	1.02	1.02	1.02	1.02	1.02
$\log(V_f)$	5.54	5.30	5.18	5.11	5.10	5.09	5.10	5.10	5.12	5.13	5.14	5.16
$\log(B_f/V_f)$	−1.54	−1.29	−1.15	−1.05	−.98	−.92	−.87	−.83	−.78	−.72	−.66	−.59

Table 5: Fama–MacBeth Regressions

β Plays If Precise

	Fama-French (1)	Berk et al. (2)	Benchmark (3)	High Variance (4)	Low Persistence (5)
$\log(V_i)$	−.15	−.035	−.139	−.172	−.141
	(−2.58)	(−.956)	(−2.629)	(−3.016)	(−2.729)
$\log(B_i/V_i)$	.50	...	.082	.107	.103
	(5.71)		(1.955)	(2.274)	(2.341)
$\log(V_i)$	−.11	−.093	−.127	−.156	−.121
	(−1.99)	(−2.237)	(−2.516)	(−2.875)	(−2.446)
$\log(B_i/V_i)$	.35	.393	.045	.053	.052
	(4.44)	(2.641)	(1.225)	(1.261)	(1.340)
β	−.37	.642	1.048	1.193	1.050
	(−1.21)	(2.273)	(2.629)	(2.634)	(2.454)
$\log(V_i)$	−.17	.053	.033	.022	.035
	(−3.41)	(1.001)	(.518)	(.323)	(.524)
β	...	...	.892	1.085	.859
			(2.933)	(3.337)	(2.893)
$\log(B_i/V_i)$	...	...	.014	.020	.024
			(.385)	(.452)	(.604)
β	.15	.377	.916	1.115	.914
	(.46)	(1.542)	(3.079)	(3.432)	(3.106)

Figure 3: t-Stats from Fama–MacBeth Regressions

Size and B/M Play a Role Just Before β

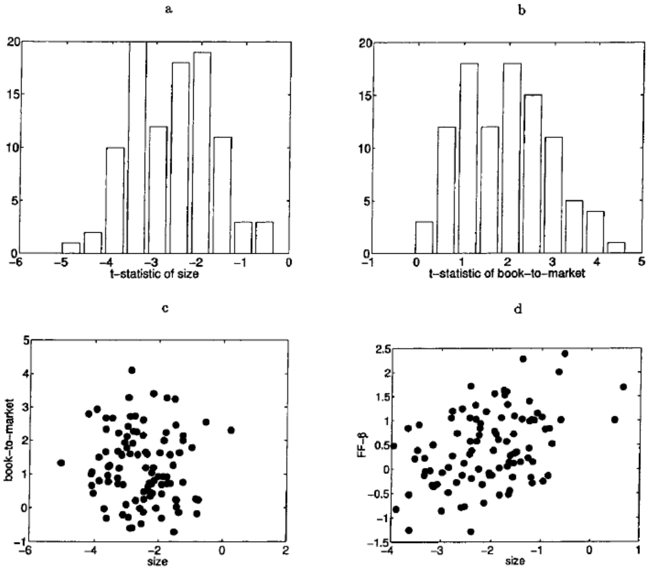

Figure 4: Negative Relation Between B/M and Profitability

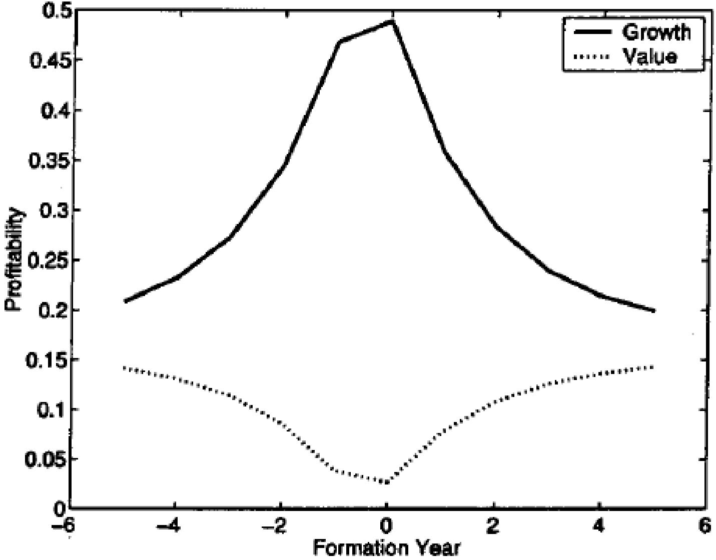

Table 6: Even True β Becomes Silent After Size

	All	Low-β	β-2	β-3	β-4	β-5	β-6	β-7	β-8	β-9	High-β
	A. Average Monthly Returns (Percent) from Fama and French (1992)										
All	1.25	1.34	1.29	1.36	1.31	1.33	1.28	1.24	1.21	1.25	1.14
Small market equity	1.52	1.71	1.57	1.79	1.61	1.50	1.50	1.37	1.63	1.50	1.42
Market equity 2	1.29	1.25	1.42	1.36	1.39	1.65	1.61	1.37	1.31	1.34	1.11
Market equity 3	1.24	1.12	1.31	1.17	1.70	1.29	1.10	1.31	1.36	1.26	.76
Market equity 4	1.25	1.27	1.13	1.54	1.06	1.34	1.06	1.41	1.17	1.35	.98
Market equity 5	1.29	1.34	1.42	1.39	1.48	1.42	1.18	1.13	1.27	1.18	1.08
Market equity 6	1.17	1.08	1.53	1.27	1.15	1.20	1.21	1.18	1.04	1.07	1.02
Market equity 7	1.07	.95	1.21	1.26	1.09	1.18	1.11	1.24	.62	1.32	.76
Market equity 8	1.10	1.09	1.05	1.37	1.20	1.27	.98	1.18	1.02	1.01	.94
Market equity 9	.95	.98	.88	1.02	1.14	1.07	1.23	.94	.82	.88	.59
Large market equity	.89	1.01	.93	1.10	.94	.94	.89	1.03	.71	.74	.56
	B. Average Monthly Returns (Percent) from Simulated Panel										
All	.67	.67	.68	.67	.68	.68	.68	.67	.68	.68	.67
Small market equity	.72	.72	.72	.72	.72	.73	.72	.72	.73	.72	.72
Market equity 2	.71	.70	.71	.71	.71	.70	.72	.71	.70	.71	.70
Market equity 3	.70	.70	.70	.70	.71	.69	.70	.70	.69	.71	.70
Market equity 4	.69	.69	.69	.69	.71	.70	.70	.67	.70	.69	.68
Market equity 5	.70	.70	.72	.70	.70	.71	.71	.69	.70	.69	.68
Market equity 6	.68	.64	.68	.68	.67	.70	.69	.68	.69	.69	.70
Market equity 7	.67	.65	.66	.65	.68	.68	.68	.67	.65	.69	.65
Market equity 8	.66	.64	.67	.65	.67	.68	.66	.66	.64	.67	.65
Market equity 9	.64	.61	.65	.61	.65	.63	.63	.64	.66	.64	.65
Large market equity	.58	.61	.56	.55	.57	.55	.63	.58	.61	.59	.56

Table 7: Characteristics Matter If βs Are Mismeasured

Not the True βs but the Fama–French βs

	Fama-French (1)	Berk et al. (2)	Benchmark (3)	High Variance (4)	Low Persistence (5)
$\log(V_t)$	−.15	−.035	−.139	−.172	−.141
	(−2.58)	(−.956)	(−2.629)	(−3.016)	(−2.729)
$\log(B_t/V_t)$	.50	...	.082	.107	.103
	(5.71)		(1.955)	(2.274)	(2.341)
$\log(V_t)$	−.11	−.093	−.127	−.156	−.121
	(−1.99)	(−2.237)	(−2.516)	(−2.875)	(−2.446)
$\log(B_t/V_t)$	.35	.393	.045	.053	.052
	(4.44)	(2.641)	(1.225)	(1.261)	(1.340)
β	−.37	.642	.133	.178	.214
	(−1.21)	(2.273)	(.429)	(.590)	(.727)
$\log(V_t)$	−.17	.053	−.121	−.151	−.108
	(−3.41)	(1.001)	(−2.057)	(−2.298)	(−1.821)
β	.15	.377	.590	.721	.605
	(.46)	(1.542)	(2.158)	(2.472)	(2.367)

Table 8: Size Versus Fama–French β

Size Rather Than Fama–French β Better Proxies True β

	True β	Fama-French β	$\log(B_t/V_t)$	$\log(V_t)$
True β	1	.598 (.028)	.324 (.022)	−.764 (.012)
Fama-French β		1	.270 (.031)	−.758 (.036)
$\log(B_t/V_t)$			1	−.262 (.019)
$\log(V_t)$				1

Figure 5: Countercyclical Cross-Section (Size, B/M)

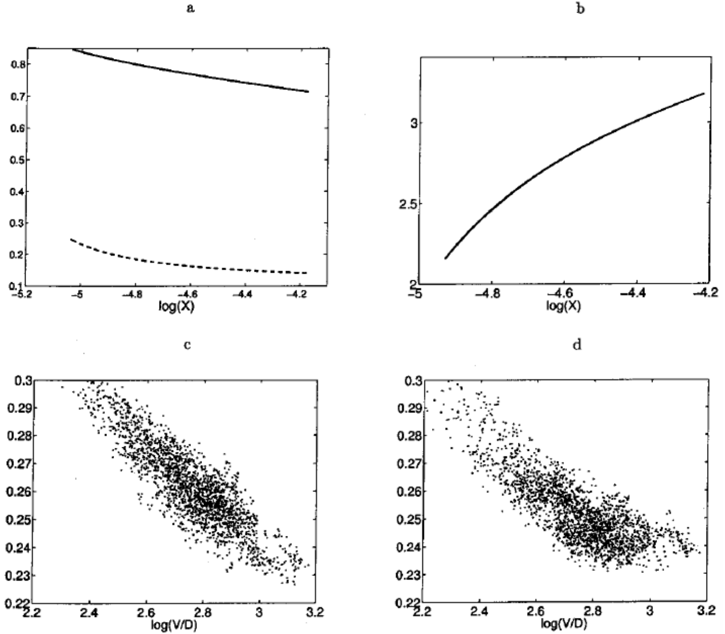

Figure 6: Countercyclical Cross-Section (Returns)

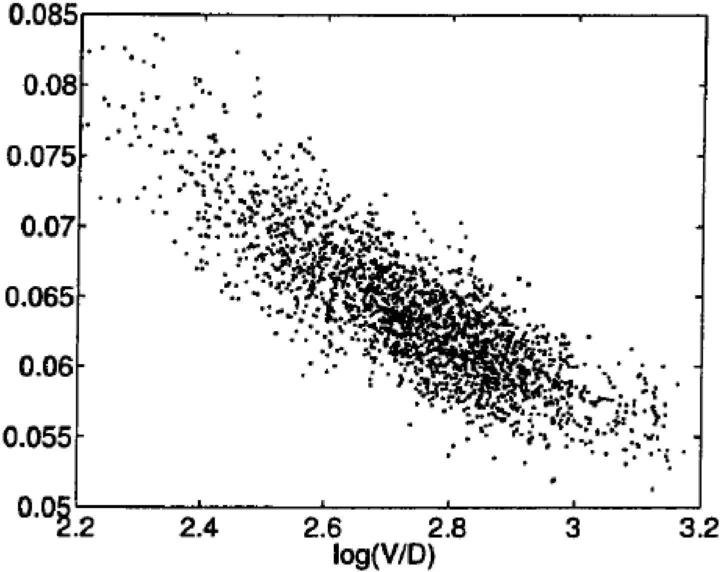

Figure 7: Countercyclical Cross-section (Market σ^2, β)

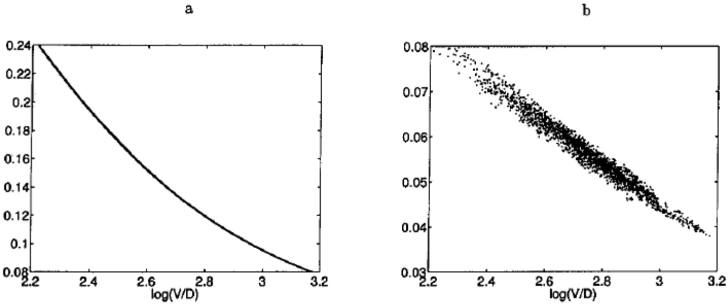

Table 9: Return Dispersion Predicts Market Volatility

b for the Dispersion, *c* for the Lagged Volatility, Considers Asymmetry—Stivers (2003)

	Coefficient				Joint $b_1=0$ $b_2=0$	Joint $c_1=0$ $c_2=0$	R^2 (%)
	b_1	b_2	c_1	c_2			
A. Results from Stivers (2001)							
Full model	$\boxed{.365\ (3.61)}$	.111 (1.40)	−.157 (−2.94)	.221 (1.84)	10.08 (.000)	2.69 (.069)	10.45
B. Simulation Results							
Full model	$\boxed{.918\ (3.408)}$	.016 (.233)	.019 (.285)	.008 (.059)	8.467 (.035)	.949 (.534)	3.72

V. Conclusion

- Theoretically, incorporate firm characteristics into DSGE to explain the size and book-to-market anomalies
  - The true β consists of the sensitivities to (1) aggregate productivity, (2) idiosyncratic productivity, and (3) growth opportunities weighted by size and book-to-market
- Empirically, using simulated panels, exhibit that the reported failure of β is largely due to the measurement errors, so demonstrates that the true β explain the cross-section over size and book-to-market
- Investigates in addition the cross-sectional dispersion of size, B/M, return, volatility, beta—the dispersion is countercyclical, so converges as expands but diverges as contracts