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Modiglian, Franco and Merton H. Miller, 1958, "The Cost of Capital, Corporate Finance and the Theory of Investment," *American Economic Review*, vol. 48, no. 3, pp. 261–297

Proposition 1 In equilibrium,

$$V_j \equiv S_j + D_j = \frac{\bar{X}_j}{\rho_k} \text{ for any firm } j \text{ in class } k$$

Where V=the market value of the firm, S=the market value of its common shares, D=the market value of its debts, $\bar{X}$=the expected return on its assets, ρ_k=the average cost of capital.

Proof Suppose Firm 1 and Firm 2 exist. Firm 1 is unlevered and Firm 2 is levered—thanks to Siyuan (09/26/17). Their expected returns are identical with $\bar{X}$. Then, by investing s_2 into Firm 2,

$$Y_2 = \frac{s_2}{S_2}(\bar{X} - rD_2) = \alpha(\bar{X} - rD_2)$$

Where Y=the return from the portfolio. Meanwhile, by borrowing αD_2 and investing s_2+αD_2 into Firm 1,

$$\begin{aligned} Y_1 = \frac{s_2 + \alpha D_2}{S_1}\bar{X} - r\alpha D_2 &= \frac{\alpha S_2 + \alpha D_2}{S_1}\bar{X} - r\alpha D_2 \\ &= \alpha\left(\frac{S_2 + D_2}{S_1}\bar{X} - rD_2\right) \\ &= \alpha\left(\frac{V_2}{V_1}\bar{X} - rD_2\right) \end{aligned}$$

If $V_2/V_1>1$, then $Y_1>Y_2$ and hence there exists an arbitrage opportunity. Therefore, in order to eliminate this opportunity, $V_2/V_1 \geq 1$.
Similarly, by investing s_1 into Firm 1,

$$Y_1 = \frac{s_1}{S_1}\bar{X} = \alpha\bar{X}$$

Meanwhile, by lending $(D_2/V_2)s_1$ and investing $(S_2/V_2)s_1$ into Firm 2,

$$\begin{aligned} Y_2 &= \frac{(S_2/V_2)s_1}{S_2}(\bar{X} - rD_2) + r(D_2/V_2)s_1 \\ &= \frac{s_1}{V_2}(\bar{X} - rD_2) + r\frac{D_2}{V_2}s_1 \\ &= \frac{\alpha S_1}{V_2}\bar{X} \\ &= \alpha\frac{V_1}{V_2}\bar{X} \end{aligned}$$

If $V_1/V_2>1$, then $Y_2>Y_1$ and hence there exists an arbitrage opportunity. Therefore, in order to eliminate this opportunity, $V_1/V_2\geq1$. Thus, since $V_2/V_1\geq1$ and $V_1/V_2\geq1$, $V_1=V_2$. ■

Proposition 2 For Firm j,

$$i_j = \rho_k + (\rho_k - r)\frac{D_j}{S_j}$$

Where i=the expected rate of return.

Proof The expected rate of return is given by,

$$i_j \equiv \frac{\bar{X}_j - rD_j}{S_j}$$

From Proposition 1,

$$\bar{X}_j = \rho_k(S_j + D_j)$$

Therefore,

$$i_j = \frac{\rho_k S_j + \rho_k D_j - rD_j}{S_j} = \rho_k + (\rho_k - r)\frac{D_j}{S_j} \blacksquare$$

Proposition 3 $S_1 \gtreqless S_0$ (or $W_1 \gtreqless W_0$, $P_1 \gtreqless P_0$) $\Leftrightarrow \rho^* \gtreqless \rho_k$

Where ρ^*=the rate of return on the investment, Proof Firstly, for bonds,

$$V_0 = \bar{X}_0/\rho_k$$
$$S_0 = V_0 - D_0$$

By borrowing I and investing at ρ^*,

$$V_1 = \frac{\bar{X}_0 + \rho^* I}{\rho_k} = V_0 + \frac{\rho^*}{\rho_k} I$$
$$\therefore S_1 = V_1 - (D_0 + I) = V_0 + \frac{\rho^*}{\rho_k} I - D_0 - I = S_0 + \left(\frac{\rho^*}{\rho_k} - 1\right) I$$

If $\rho^*/\rho_k \gtreqless 1$, then $S_1 \gtreqless S_0$.
Secondly, for retained earnings, if the firm acquired I dollars of cash,

$$W_0 = S_0 + I = \frac{\bar{X}_0}{\rho_k} - D_0 + I$$

Where W=the stockholders' wealth. By investing I at ρ^*,

$$W_1 = \frac{\bar{X}_0 + \rho^* I}{\rho_k} - D_0 = S_0 + \frac{\rho^*}{\rho_k} I$$

If $\rho^*/\rho_k \gtreqless 1$, then $W_1 \gtreqless W_0$.
Lastly, for common stock issues,

$$P_0 = S_0/N$$

Where P=the current market price per share of stock, N=the original number of shares. Then, for investing I,

$$M = \frac{I}{P_0}$$

Where M=the number of new shares. Then,

$$S_1 = \frac{\bar{X}_0 + \rho^* I}{\rho_k} - D_0 = S_0 + \frac{\rho^*}{\rho_k} I$$

If $\rho^*/\rho_k \gtreqless 1$, then $S_1 \gtreqless S_0$. Or, equivalently,

$$\begin{aligned} P_1 &= \frac{S_1}{N+M} = \frac{1}{N+M}\left(S_0 + \frac{\rho^*}{\rho_k} I\right) \\ &= \frac{1}{N+M}\left(NP_0 + MP_0 + \frac{\rho^* - \rho_k}{\rho_k} I\right) \\ &= P_0 + \frac{1}{N+M}\frac{\rho^* - \rho_k}{\rho_k} I \end{aligned}$$

If $\rho^* \gtreqless \rho_k$, then $P_1 \gtreqless P_0$. ■

Note The effect of the corporate income tax on investment decisions

$$\frac{(\bar{X}-rD)(1-\tau)+rD}{V}=\underbrace{\rho_k^\tau}_{\text{constant}}$$

$$\therefore \frac{\bar{X}}{V}=\frac{1}{1-\tau}\left(\rho_k^\tau-\frac{\tau rD}{V}\right)=\frac{\rho_k^\tau}{1-\tau}\left(1-\frac{\tau rD}{\rho_k^\tau V}\right)=\begin{cases}\rho_k^S=\dfrac{\rho_k^\tau}{1-\tau}, & \text{if } D/V=0\\ \rho_k^D=\rho_k^S-\dfrac{\tau}{1-\tau}r, & \text{if } D/V=1\end{cases}$$

$$\Rightarrow \frac{d(\bar{X}/V)}{d(D/V)}=-\frac{\tau r}{1-\tau}<0$$

Where ρ_k^S=the required rate of return on a venture with D=0, ρ_k^D=the required rate of return with D=V. For investments financed out of retained earnings,

$$\rho_k^R=\frac{\rho_k^\tau}{1-\tau}\frac{1-\tau_d}{1-\tau_g}=\frac{1-\tau_d}{1-\tau_g}\rho_k^S$$

Where ρ_k^R=the required return for retained earnings, τ_d=the assumed rate of personal income tax on dividends, τ_g=the assumed rate of tax on capital gains.

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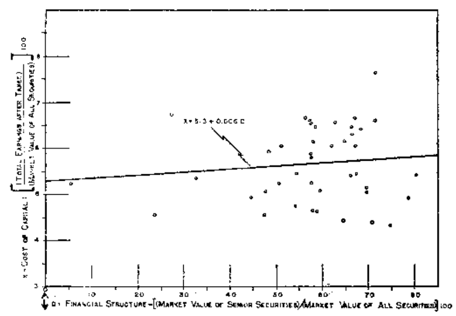

FIGURE 3. COST OF CAPITAL IN RELATION TO FINANCIAL STRUCTURE FOR 43 ELECTRIC UTILITIES, 1947–48

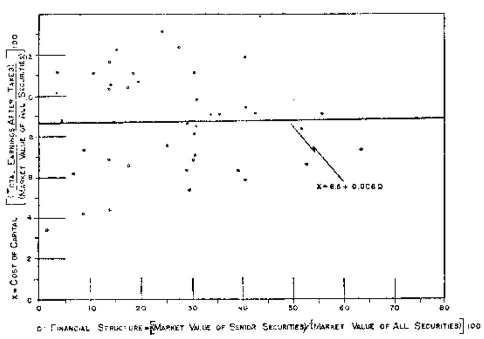

FIGURE 4. COST OF CAPITAL IN RELATION TO FINANCIAL STRUCTURE FOR 42 OIL COMPANIES, 1953

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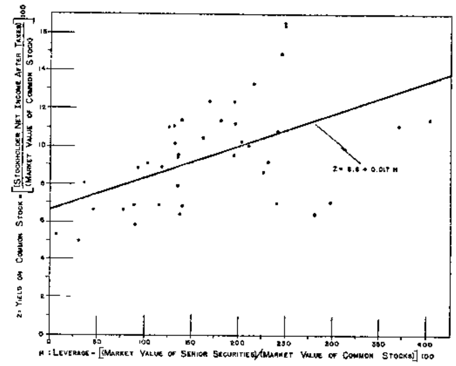

FIGURE 5. YIELD ON COMMON STOCK IN RELATION TO LEVERAGE FOR 43 ELECTRIC UTILITIES, 1947–48

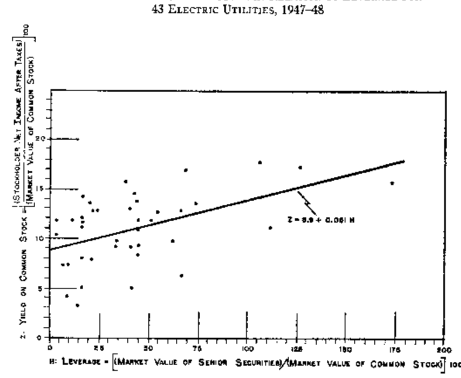

FIGURE 6. YIELD ON COMMON STOCK IN RELATION TO LEVERAGE FOR 42 OIL COMPANIES, 1952–53

ρ_k=WACC: not affected by D/V ratio

$i_j=\rho_k+(\rho_k-r)\times D/S$

Summary

1. If ∄ tax ⇒ firm value and WACC are both independent of capital structure.
2. ⇒ expected rate of return is dependent on capital structure.
3. ⇒ only the rate of return of investment does matter; the way to finance does not matter.