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Optimization Review

Let C be a set of choices.
Let f be a function that maps choices to real numbers in the order of the objective.

$$f: C \to \mathbb{R} \; s.t. \; f(c_1) \geq f(c_2) \Leftrightarrow c_1 \succcurlyeq c_2$$

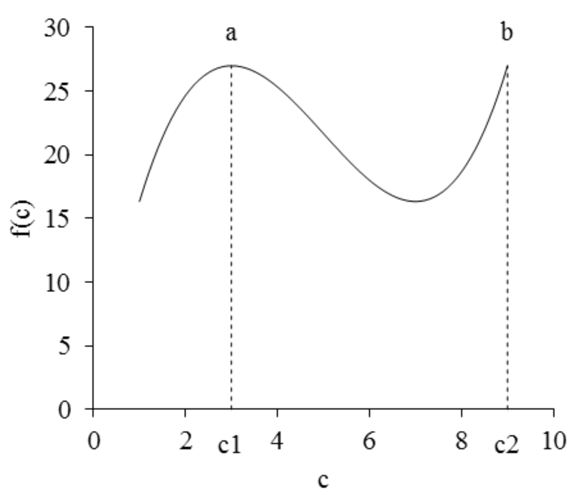
Suppose h:$\mathbb{R} \to \mathbb{R}$ is a strictly increasing function.
Then, $g(c) \equiv h[f(c)]$ ranks the choices in the same order as f does.
⇒ Can say g and f represent equivalent objectives.

Definition *Correspondence* maps points to sets.

Example Two useful correspondences

$$\operatorname*{argmax}_{c \in C} f(c) \equiv \{c \in C | f(c) \geq f(c'), \forall c' \in C\}$$
$$\operatorname*{argmin}_{c \in C} f(c) \equiv \{c \in C | f(c) \leq f(c'), \forall c' \in C\}$$

How do we find optimal choices?
i. Discrete choice set $C=\{c_1, c_2, \cdots, c_N\}$
⇒ Compute $f(c_1)$, $f(c_2)$, $\cdots$, $f(c_N)$.
⇒ Choose the largest/smallest as appropriate.
ii. Continuous choice set: continuous, twice differentiable objective function
a. unconstrained
b. constrained



a. unconstrained case

⇒ Find c_1 such that

$$\frac{\partial f}{\partial c}(c_1) = 0 \text{ i.e. 1st order condition}$$

And only consider c_1 a potential maximum if

$$\frac{\partial^2 f}{\partial c^2}(c_1) \leq 0 \text{ i.e. 2nd order condition}$$

Generalization to multidimensional $\mathbf{c}=\{c_1,c_2,\cdots,c_k\}$

First order condition

$$\frac{\partial f}{\partial c_i} = 0, \forall c_i$$

Second order condition: f is concave (convex for a minimum).

Equivalent

Define Jacobian matrix

$$Df(\mathbf{c}) \equiv \begin{pmatrix} \frac{\partial f}{\partial c_1}(\mathbf{c}) \\ \frac{\partial f}{\partial c_2}(\mathbf{c}) \\ \vdots \\ \frac{\partial f}{\partial c_k}(\mathbf{c}) \end{pmatrix}$$

⇒ $Df(\mathbf{c}^*)=\mathbf{0}$ at a potential optimum.

Define Hessian matrix $D^2f(\mathbf{c})$ s.t.

$$[D^2 f(\mathbf{c})]_{ij} = \underbrace{\frac{\partial^2 f}{\partial c_i \partial c_j}(\mathbf{c})}_{k\times k}$$

Second order condition

$D^2f(\mathbf{c})$ is negative semi-definite at a potential maximum, positive semi-definite at a potential minimum.

Definition A square matrix $\mathbf{M}$ is negative (positive) semi-definite if ∀ conformable column vectors $\mathbf{h}$, $\mathbf{h}^{\mathrm{T}}\mathbf{M}\mathbf{h} \leq 0$ (≥ 0 for positive semi-definite).

Implication ⇒ Consider $\mathbf{h}=(0\ 0\ 1\ 0\ \cdots\ 0)^{\mathrm{T}}$, then, for negative semi-definite D^2f,

$$\mathbf{h}^{\mathrm{T}} D^2 f \mathbf{h} = \frac{\partial^2 f}{\partial c_3^2} \leq 0$$

b. constrained case

⇒ Method of Lagrange Multipliers

⇒ Say our constraints are $g_i(\mathbf{c})=0$ (possibly several constraints, $i=1,2,\cdots,I$).

Define $L(\mathbf{c})$ as below.

$$L(\mathbf{c}) \equiv f(\mathbf{c}) + \sum_{i=1}^{I} \lambda_i g_i(\mathbf{c})$$

Where λ_is are constraints called as *Lagrange Multipliers*. Then,

$$\operatorname*{argmax}_{\mathbf{c}} L(\mathbf{c}) = \operatorname*{argmax}_{\mathbf{c}\text{ satisfies all constraints}} f(\mathbf{c}) \Leftrightarrow L(\mathbf{c}^*) = f(\mathbf{c}^*)$$

Extension: Kuhn–Tucker Condition

⇒ Constraints may or may not bind.

⇒ Write them as $g_i(\mathbf{c})\leq 0$.

Define Lagrange function as below.

$$L(\mathbf{c}) \equiv f(\mathbf{c}) + \sum_{i=1}^{I} \lambda_i g_i(\mathbf{c})$$

⇒ Solve as in Lagrange case.

⇒ Except allow for *Complementary Slackness*; i.e. either

$$\underbrace{g_i(\mathbf{c}) = 0}_{\text{Constraint binds.}} \quad \text{or} \quad \underbrace{\lambda_i = 0}_{\text{It does not bind.}} \quad ,\forall i$$

Once we have found optimal choices $\mathbf{c}^*$, think of this as a function rather than a correspondence. Then, we know

$$\frac{\partial f}{\partial \mathbf{c}^{\mathrm{T}}}(\mathbf{c}^*) \equiv \mathbf{0}$$

1. (Comparative Statics) How does the optimal choice change when some parameter s changes?

⇒ Since

$$\frac{\partial f}{\partial \mathbf{c}^{\mathrm{T}}}(\mathbf{c}^*) \equiv \mathbf{0} \text{ (not just =)}$$

We can take another derivative with respect to s.

$$\frac{\partial^2 f}{\partial \mathbf{c} \partial s}(\mathbf{c}^*) + \frac{\partial^2 f}{\partial \mathbf{c} \partial \mathbf{c}^{\mathrm{T}}} \frac{d\mathbf{c}^*}{ds} = \mathbf{0}$$
$$\Rightarrow \frac{d\mathbf{c}^*}{ds} = -\left(\frac{\partial^2 f}{\partial \mathbf{c} \partial \mathbf{c}^{\mathrm{T}}}\right)^{-1} \frac{\partial^2 f}{\partial \mathbf{c} \partial s}(\mathbf{c}^*)$$

2. How does the maximized value of the objective $M\equiv f(\mathbf{c}^*)$ change as some parameter s changes?

$$\frac{dM}{ds} = \frac{\partial f}{\partial s}(\mathbf{c}^*) + \underbrace{\frac{\partial f}{\partial \mathbf{c}}(\mathbf{c}^*)}_{\mathbf{0}} \frac{d\mathbf{c}^*}{ds} = \frac{\partial f}{\partial s}(\mathbf{c}^*)$$

⇒ The Envelope theorem

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Firms
1. Cost Minimization
2. Profit Maximization
a. with a cost function (explicitly)
b. in one step

Technology of Production
$\mathbf{y} \equiv$ quantities of output; can be a vector for multiproduct firm
$\mathbf{x} \equiv$ quantities of inputs; factors of production

Definition A *Production Function* maps quantities of inputs to quantities of outputs.

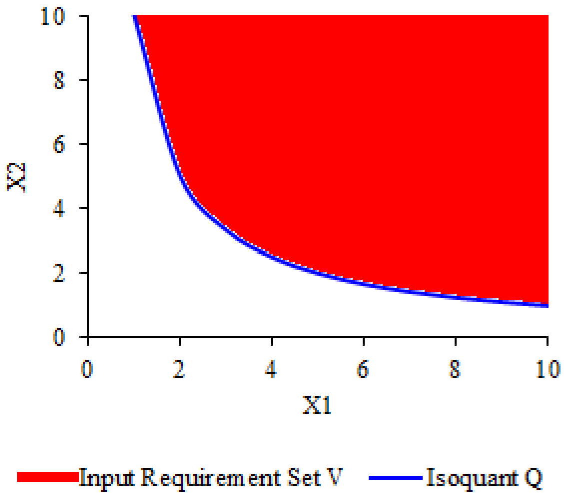
$$\mathbf{f}(\mathbf{x}) = \mathbf{y}$$

Related ⇒ An Input Requirement Set V,

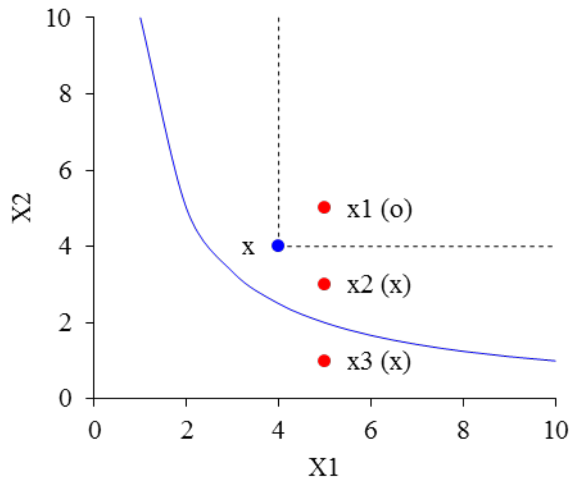
$$V(\mathbf{y}) \equiv \{\mathbf{x}|\mathbf{f}(\mathbf{x}) \geq \mathbf{y}\}$$

An Isoquant Q is,

$$Q(\mathbf{y}) = \{\mathbf{x}|\mathbf{f}(\mathbf{x}) = \mathbf{y}\}$$

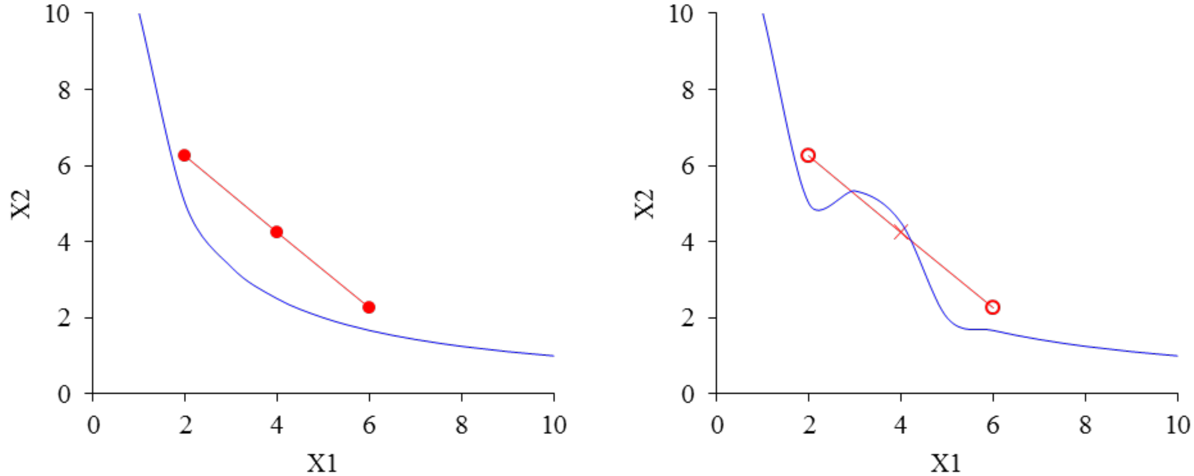


Input Requirement Set and Isoquant

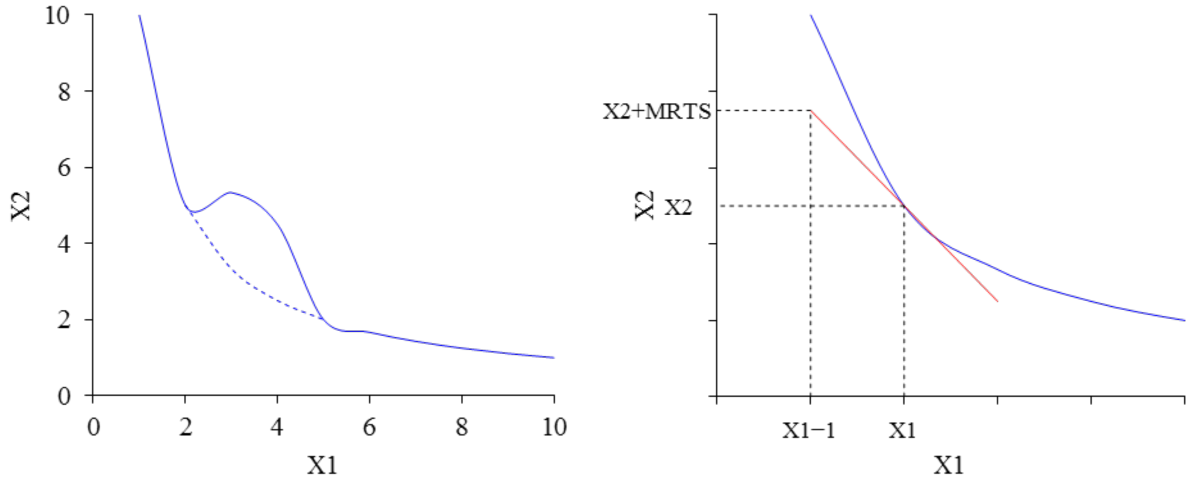


Monotonicity

Assumptions
1. Regularity: $V(\mathbf{y})$ is closed and bounded.
2. Monotonicity: $\forall\ \mathbf{x}^* \geq \mathbf{x}$, if $\mathbf{x} \in V(\mathbf{y})$, then $\mathbf{x}^* \in V(\mathbf{y})$.
3. V(y) is a convex set; if $\mathbf{x}, \mathbf{x}^* \in V(\mathbf{y})$, then $\forall\ \alpha \in [0,1]$, $\alpha\mathbf{x}+(1-\alpha)\mathbf{x}^* \in V(\mathbf{y})$



Convex Non-convex



More Definitions

$$\text{Marginal Product of Input } i \equiv \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x})$$

Marginal Rate of Technical Substitution (for levels of output **y**) is the amount that one input must increase if another input decrease by one unit, in order to continue producing **y**.
Along isoquant $Q(\mathbf{y})$,

$$\mathbf{f}(\mathbf{x}) = \mathbf{y}$$

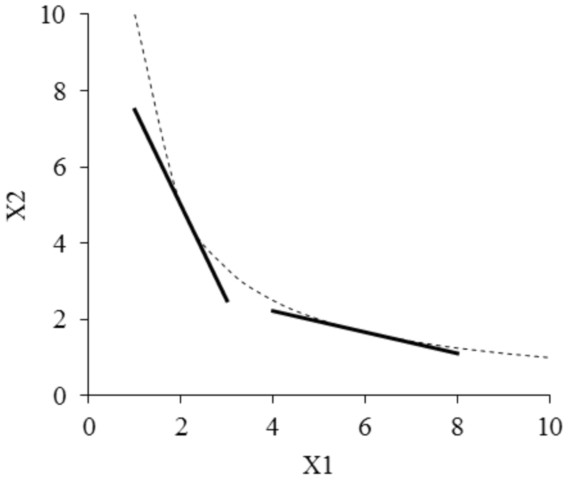
$$\Rightarrow \frac{\partial \mathbf{f}}{\partial x_1} + \frac{\partial \mathbf{f}}{\partial x_2} \underbrace{\frac{dx_2}{dx_1}}_{\text{MRTS}} = \left[\frac{d\mathbf{y}}{dx_1}\right]_{\mathbf{y}\text{ constant}} = \mathbf{0}$$

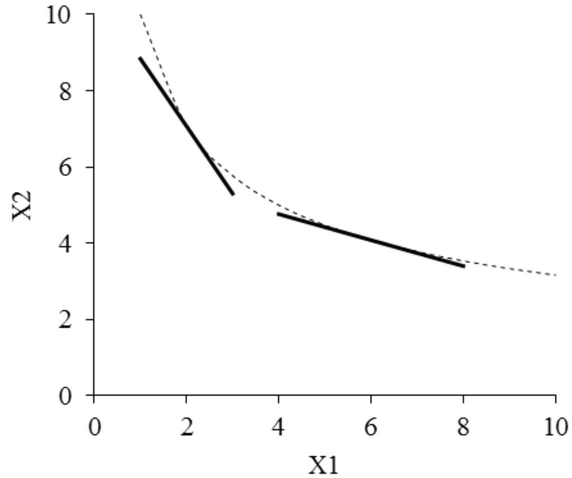
$$\text{if } f \text{ is one dimensional, then } \frac{dx_2}{dx_1} = -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}$$

Definition The *Elasticity of Substitution* σ measures the curvature of the isoquant. This is the percentage change in the input mix (X_2/X_1) that is required to change the [MRTS] by 1%.

$$\sigma = \frac{d\log\left(\frac{X_2}{X_1}\right)}{d\log\left[MRTS_{1,2}\right]}$$

Elasticity of Substitution





Small σ Large σ

Example 1

Production function (the first case)

$$X_1 X_2 = P$$

Elasticity of Substitution

$$\sigma = \frac{d\log\left(\frac{X_2}{X_1}\right)}{d\log MRTS_{1,2}} = \frac{\frac{d(X_2/X_1)}{(X_2/X_1)}}{\frac{dMRTS_{1,2}}{MRTS_{1,2}}}$$

$$MRTS_{1,2} = \frac{MP_1}{MP_2} = \frac{X_2}{X_1} \Rightarrow \sigma = \frac{\frac{d(X_2/X_1)}{(X_2/X_1)}}{\frac{dMRTS_{1,2}}{MRTS_{1,2}}} = \frac{\frac{d(X_2/X_1)}{(X_2/X_1)}}{\frac{d(X_2/X_1)}{(X_2/X_1)}} = 1$$

Example 2

Production function (the second case)

$$\sqrt{X_1} + \sqrt{X_2} = P$$

Then,

$$MRTS_{1,2} = \frac{MP_1}{MP_2} = \frac{2\sqrt{X_2}}{2\sqrt{X_1}} = \sqrt{\frac{X_2}{X_1}} = T \Rightarrow \sigma = \frac{\frac{dT^2}{T^2}}{\frac{dT}{T}} = \frac{T}{T^2}\frac{dT^2}{dT} = \frac{1}{T}2T = 2 > 1$$

This production function has a constant elasticity of substitution σ.

$$f(x_1, x_2, \cdots, x_n) = \left(\sum_{i=1}^{n} \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

Elasticity of Substitution

$$MRTS_{1,2} = \frac{MP_1}{MP_2} = \frac{\frac{\sigma}{\sigma-1}\left(\sum \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} \alpha_1 x_1^{-\frac{1}{\sigma}}}{\frac{\sigma}{\sigma-1}\left(\sum \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} \alpha_2 x_2^{-\frac{1}{\sigma}}} = \frac{\alpha_1 x_1^{-\frac{1}{\sigma}}}{\alpha_2 x_2^{-\frac{1}{\sigma}}} = \underbrace{\frac{\alpha_1}{\alpha_2}}_{\alpha} \underbrace{\left(\frac{x_2}{x_1}\right)^{\frac{1}{\sigma}}}_{t^{1/\sigma}} = \alpha t^{1/\sigma}$$

$$\text{Elasticity of Substitution} = \frac{\frac{dt}{t}}{\frac{d\alpha t^{1/\sigma}}{\alpha t^{1/\sigma}}} = \frac{\alpha t^{1/\sigma}}{t} \frac{dt}{d\alpha t^{1/\sigma}} = \alpha t^{\frac{1-\sigma}{\sigma}} \frac{1}{\frac{\alpha}{\sigma} t^{\frac{1-\sigma}{\sigma}}} = \sigma$$

If $\sigma \to 0$,

$$\begin{aligned} f &\to (\alpha_1 x_1^{-\infty} + \alpha_2 x_2^{-\infty} + \cdots)^0 \\ &\to \left(\alpha_1 \frac{1}{x_1^{\infty}} + \alpha_2 \frac{1}{x_2^{\infty}} + \cdots\right)^0 \\ &\to \min(\alpha_1 x_1, \alpha_2 x_2, \cdots) \Rightarrow \text{Leontief Production} \end{aligned}$$

$$\because MRTS_{1,2} = \frac{\alpha_1}{\alpha_2}\left(\frac{x_2}{x_1}\right)^{\frac{1}{\sigma}} = \begin{cases} 0, & \text{if } x_1 > x_2 \\ \frac{\alpha_1}{\alpha_2}, & \text{if } x_1 = x_2 \\ \infty, & \text{if } x_1 < x_2 \end{cases}$$

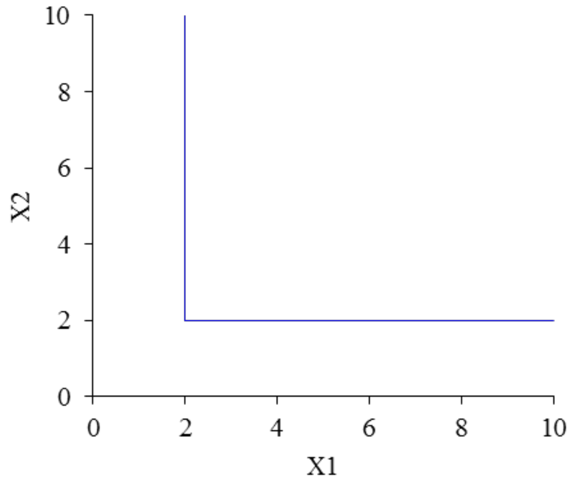
If σ=1,

$$f = x_1^{\alpha_1} x_2^{\alpha_2} \cdots = \prod_{i=1}^{n} x_i^{\alpha_i} \Rightarrow \text{Cobb–Douglas Production}$$

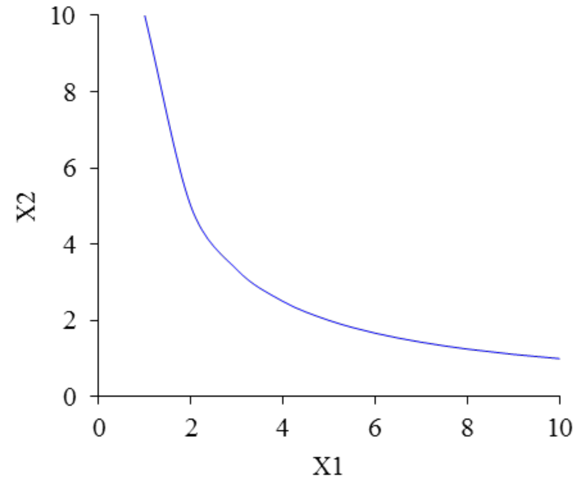
$$\because \text{Cobb–Douglas } MRTS_{1,2} = \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} \cdots}{x_1^{\alpha_1} \alpha_2 x_2^{\alpha_2 - 1} \cdots} = \frac{\alpha_1 x_1^{-1}}{\alpha_2 x_2^{-1}} = \frac{\alpha_1 x_2}{\alpha_2 x_1} = \text{CES } MRTS_{1,2}$$

If $\sigma \to \infty$,

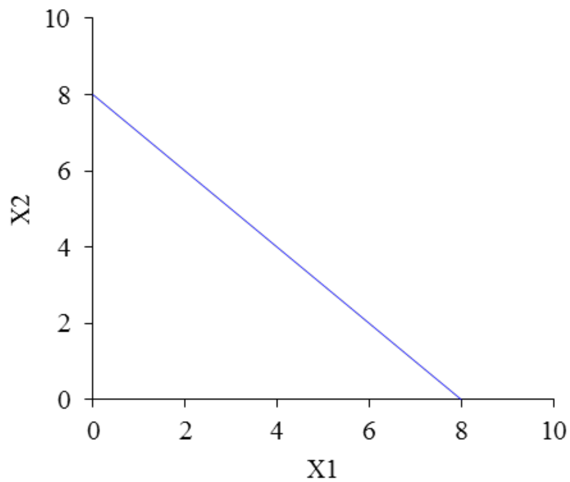
$$\begin{aligned} f &\to (\alpha_1 x_1 + \alpha_2 x_2 + \cdots) \\ &\to \sum_{i=1}^{n} \alpha_i x_i \Rightarrow \text{Perfect Substitution} \end{aligned}$$



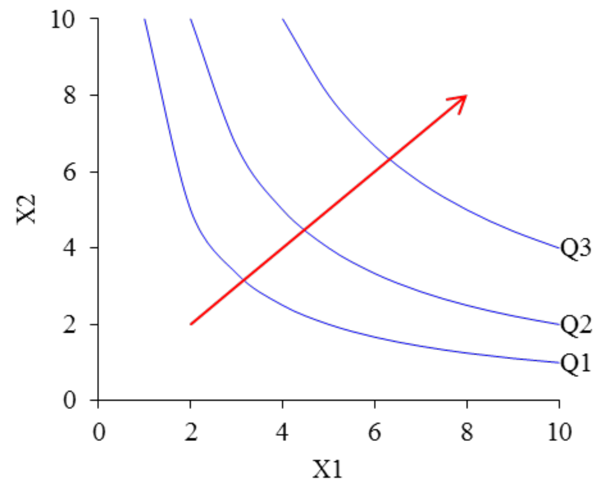
Leontief production



Cobb–Douglas production



Perfect Substitution



Return to Scale

Another Issue: Return to Scale ("Distance between Isoquant")

Definition Let k be a non-negative number and define $\mathbf{y}(k) \equiv \mathbf{f}(k\mathbf{x}) = \mathbf{f}(kx_1, kx_2, \cdots)$ then the *Elasticity of Scale* is defined as

$$\frac{d \log \mathbf{y}(k)}{d \log k}$$

If the elasticity of scale=**1**, then the production function has **constant** returns.
If the elasticity of scale>**1**, then the production function has **increasing** returns.
If the elasticity of scale<**1**, then the production function has **decreasing** returns.

cf. globally-increasing-returns-to-scale production ⇒ might be Marxist?

Definition A function $f(\mathbf{x})$ is *Homogenous of Degree t* if

$$f(k\mathbf{x}) = k^t f(\mathbf{x}) \forall k$$

If so,

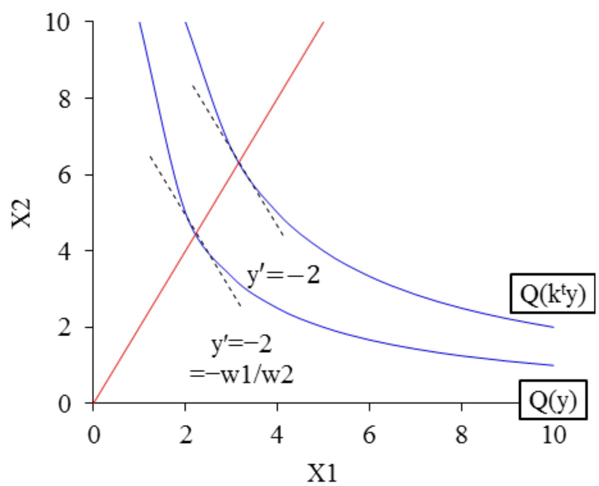
$$\begin{aligned}\frac{d \log f(k\mathbf{x})}{d \log k} &= \frac{\partial}{\partial \log k}[t \log k + \log f(\mathbf{x})] \\ &= t\end{aligned}$$

⇒ f has constant elasticity of scale t.

Corollary *MRTS* is constant along any ray through the origin.

$$MRTS_{1,2}(\mathbf{x}) = -\frac{\dfrac{\partial f}{\partial x_1}(\mathbf{x})}{\dfrac{\partial f}{\partial x_2}(\mathbf{x})}$$

$$\Rightarrow MRTS_{1,2}(kx) = -\frac{\dfrac{\partial f}{\partial x_1}(k\mathbf{x})}{\dfrac{\partial f}{\partial x_2}(k\mathbf{x})} = -\frac{k^t \dfrac{\partial f}{\partial x_1}(\mathbf{x})}{k^t \dfrac{\partial f}{\partial x_2}(\mathbf{x})} = MRTS_{1,2}(\mathbf{x})$$



~~Note that, in the plot, the slope y′ is −w1/w2, but, in my opinion, it seems to be −w2/w1, instead. (?)~~

Definition Let g be a positive monotonic transformation, define $h \equiv g[f(\mathbf{x})]$. Then, if $f(\mathbf{x})$ is homogenous of degree t, then h is called *Homothetic*.

Note that $h(k\mathbf{x}) \neq k^t h(\mathbf{x})$, but $MRTS(k\mathbf{x}) = MRTS(\mathbf{x})$.

Proof

$$\frac{\partial h}{\partial x_1} = g' k^t \frac{\partial f}{\partial x_1}, \frac{\partial h}{\partial x_2} = g' k^t \frac{\partial f}{\partial x_2} \Rightarrow \frac{\partial h/\partial x_1}{\partial h/\partial x_2} = \frac{g' k^t \partial f/\partial x_1}{g' k^t \partial f/\partial x_2} = \frac{\partial f/\partial x_1}{\partial f/\partial x_2}$$

Cost Minimization

Given output **y**, problem is

$$\min_{\mathbf{x}} \mathbf{w}'\mathbf{x} \ \text{s.t.}\ \mathbf{f}(\mathbf{x}) = \mathbf{y}$$

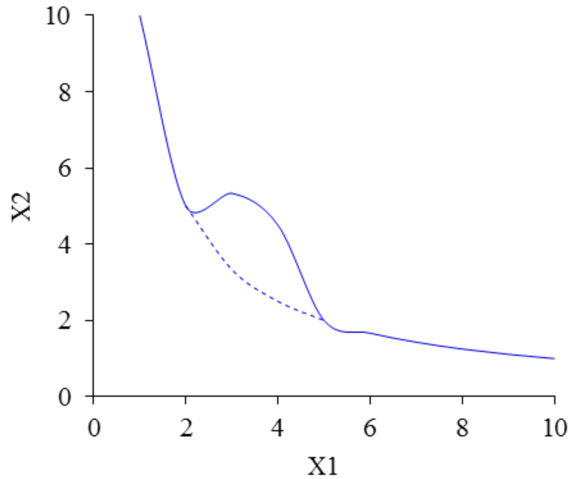
Where **w** is a vector of input prices.

Then, 1st order condition

$$w_i = \lambda \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}^*) \text{ for each } i$$

$$\Rightarrow -\frac{w_1}{w_2} = -\frac{\frac{\partial f}{\partial x_1}(\mathbf{x}^*)}{\frac{\partial f}{\partial x_2}(\mathbf{x}^*)} = MRTS_{x_1,x_2}(\mathbf{x}^*)$$

Note: If Non-convex Production Function,



⇒ Never optimal to choose $\mathbf{x}^*$ from a non-convex portion of $V(\mathbf{y})$

⇒ can always treat the cost minimization as if $V(\mathbf{y})$ were actually its "convex hall"

⇒ If

$$\mathbf{x}^* \in \operatorname*{argmin}_{\mathbf{x} \in V(\mathbf{y})} \mathbf{w}'\mathbf{x}$$

Then, we say that $\mathbf{x}^*$ is on the *Conditional Factor Demand Function*.

$$\mathbf{x}^*(\mathbf{w}, \mathbf{y})$$

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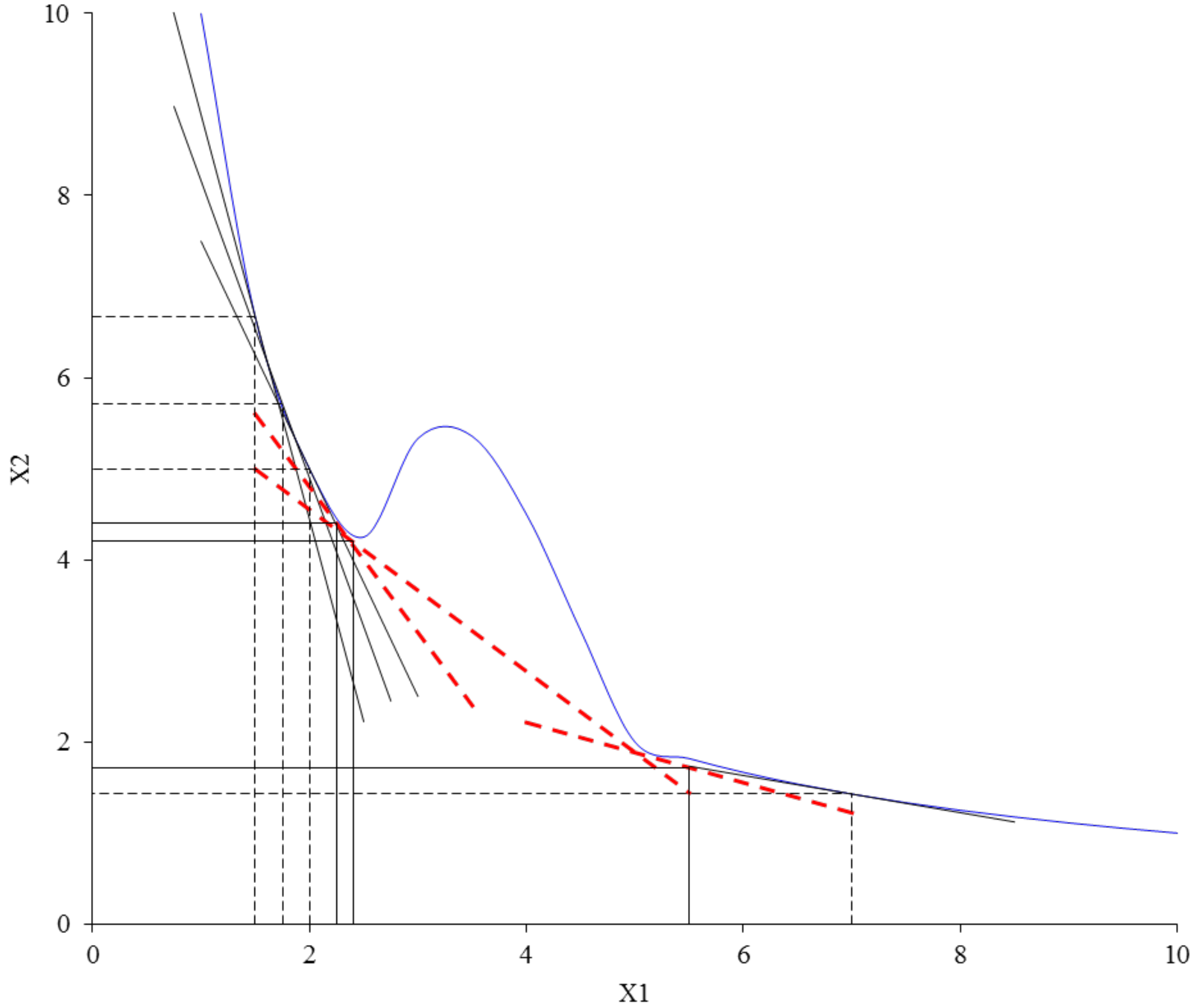
Suppose

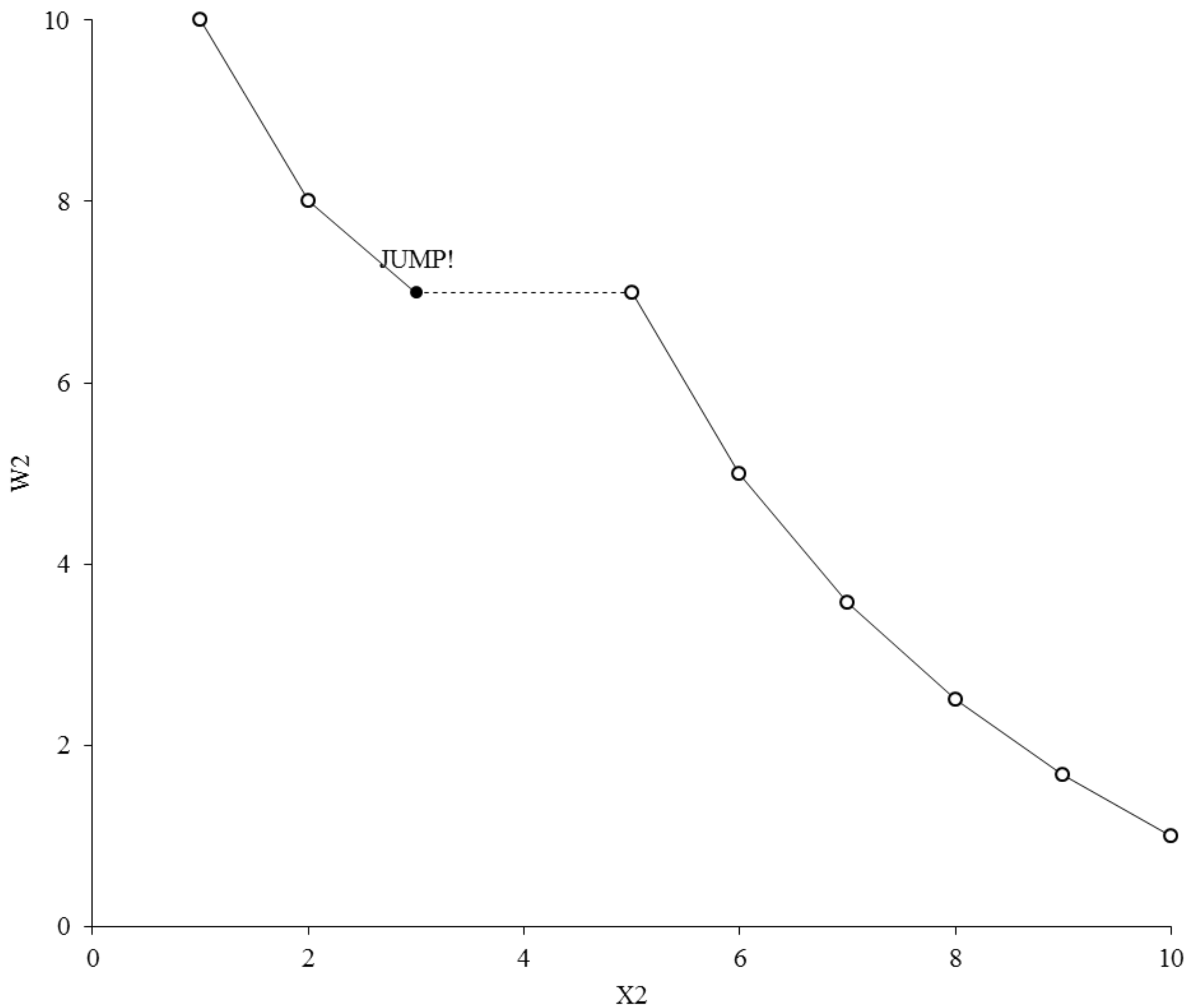
$$\mathbf{x}^* \in \underset{\mathbf{x}\in V(\mathbf{y})}{\operatorname{argmax}}\, \mathbf{w}'\mathbf{x}$$

Then, $\mathbf{x}^*$ is the *Conditional Factor Demand* for inputs $\mathbf{x}$ given input prices $\mathbf{w}$ and output level $\mathbf{y}$.
Write all such $\mathbf{x}$s $\mathbf{x}^*(\mathbf{w},\mathbf{y})$.



The minimized value of $\mathbf{w}'\mathbf{x}^*(\mathbf{w},\mathbf{y})$ is called the cost function

$$C(\mathbf{w},\mathbf{y}) = \min_{\mathbf{x}\in V(\mathbf{y})} \mathbf{w}'\mathbf{x} = \mathbf{w}'\mathbf{x}^*(\mathbf{w},\mathbf{y})$$



(PLEASE ADD EXPLANATION FOR THESE PLOTS!!!)

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Conditional Factor Demand (assume that y is a scalar)

$$\mathbf{x}^*(\mathbf{w}, y) = \underset{\mathbf{x}}{\operatorname{argmin}}\, \mathbf{w}'\mathbf{x} \text{ s.t. } f(\mathbf{x}) = y$$

Cost Function

$$\begin{aligned} C(w, y) &= \min_{\mathbf{x}} \mathbf{w}'\mathbf{x} \text{ s.t. } f(\mathbf{x}) = y \\ &= \mathbf{w}'\mathbf{x}^*(\mathbf{w}, y) \end{aligned}$$

Properties of Cost Functions

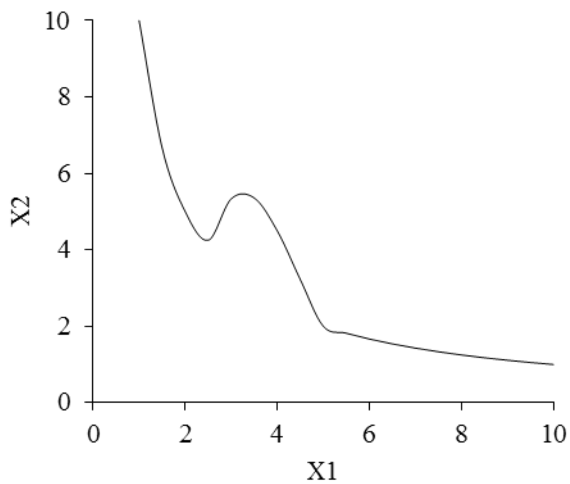
1. Non-decreasing in **w** and y.

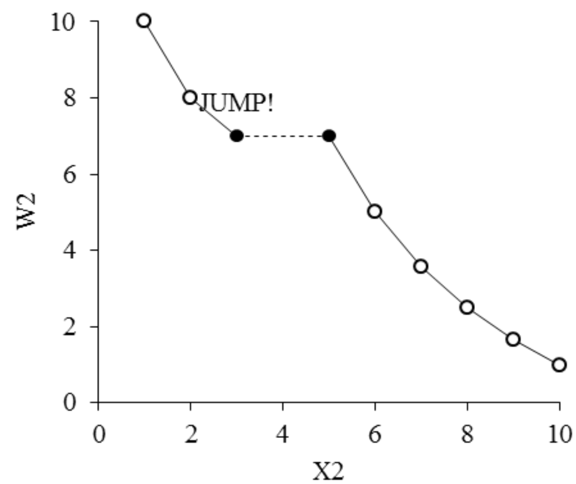
$$C(\mathbf{w}^1, y) \geq C(\mathbf{w}, y) \text{ if } \mathbf{w}^1 \geq \mathbf{w}$$
$$C(\mathbf{w}, y^1) \geq C(\mathbf{w}, y) \text{ if } y^1 \geq y$$

2. Homogenous of degree 1 in **w**.

$$C(k\mathbf{w}, y) = kC(\mathbf{w}, y)$$

3. Continuous in **w** (even if the input requirement set is not convex).





4. Concave in **w**; for all $\alpha \in (0,1]$,

$$C(\alpha\mathbf{w} + (1-\alpha)\mathbf{w}^1, y) \geq \alpha C(\mathbf{w}, y) + (1-\alpha)C(\mathbf{w}^1, y)$$

Proof Let $\mathbf{w}^1$ and $\mathbf{w}^2$ be two sets of prices, and define $\mathbf{w}^3=\alpha\mathbf{w}^1+(1-\alpha)\mathbf{w}^2$, and suppress y.

$$\begin{aligned} C(\mathbf{w}^3) &= (\mathbf{w}^3)' \times \mathbf{x}^*(\mathbf{w}^3) \\ &= \alpha(\mathbf{w}^1)'\mathbf{x}^*(\mathbf{w}^3) + (1-\alpha)(\mathbf{w}^2)'\mathbf{x}^*(\mathbf{w}^3) \\ &\geq \alpha C(\mathbf{w}^1) + (1-\alpha)C(\mathbf{w}^2) \\ &= \alpha(\mathbf{w}^1)'\mathbf{x}^*(\mathbf{w}^1) + (1-\alpha)(\mathbf{w}^2)'\mathbf{x}^*(\mathbf{w}^2) \blacksquare \end{aligned}$$

$$\because (\mathbf{w}^1)'\mathbf{x}^*(\mathbf{w}^3) \geq (\mathbf{w}^1)'\mathbf{x}^*(\mathbf{w}^1) = (\mathbf{w}^1)' \underset{\mathbf{x}\in V(y)}{\operatorname{argmin}} (\mathbf{w}^1)'\mathbf{x}$$

Properties of Conditional Factor Demand

1. $\mathbf{x}^*(\mathbf{w},y)$ is homogenous of degree 0 in $\mathbf{w}$.
2. Shephard's lemma.

$$x_i^*(\mathbf{w},y) = \frac{\partial C(\mathbf{w},y)}{\partial w_i}$$

Proof By Envelope theorem,

$$\begin{aligned} \frac{dC}{dw_i} &= \frac{\partial C}{\partial w_i}(\mathbf{w},\mathbf{y}) + \underbrace{\frac{\partial C}{\partial \mathbf{x}}(\mathbf{x}^*)}_{\mathbf{0}'}\frac{\partial \mathbf{x}}{\partial w_i} \\ &= x_i^*(w,y) \end{aligned}$$

3. Symmetry.

$$\frac{\partial x_i^*}{\partial w_j}(\mathbf{w},y) = \frac{\partial x_j^*}{\partial w_i}(\mathbf{w},y)$$

Proof By Shephard's lemma and Young's theorem,

$$\begin{aligned} &\frac{\partial C}{\partial w_i} = x_i^*(\mathbf{w},y) \\ \Rightarrow &\frac{\partial^2 C}{\partial w_i \partial w_j} = \frac{\partial x_i^*}{\partial w_j} = \frac{\partial^2 C}{\partial w_j \partial w_i} = \frac{\partial x_j^*}{\partial w_i} \end{aligned}$$

4. Negativity.

The matrix $\mathbf{M}$ whose (i,j)th element is

$$\mathbf{M}_{ij} = \frac{\partial x_i^*(w,y)}{\partial w_j} \text{ is a negative semidefinite.}$$

Proof By Shephard's lemma, $\mathbf{M}$ is the hessian of the cost function $C(\mathbf{w},y)$, which is concave. ■

Profit Maximization

A. With a Cost Function (i.e. already solved min $\mathbf{w}'\mathbf{x}$ s.t. $f(\mathbf{x})=y$).

Profit Function (assume that $\mathbf{y}$ is a vector).

$$\Pi(\boldsymbol{p},\boldsymbol{w}) = \max_{\mathbf{y}} \mathbf{p}'\mathbf{y} - C(\mathbf{w},\mathbf{y})$$

First Order Condition

$$\mathbf{p} = \frac{\partial C}{\partial \mathbf{y}'} \Rightarrow \text{Price} = \text{Marginal Cost}$$

Second Order Condition (assume y)

$$-\frac{\partial^2 C}{\partial y^2} \leq 0 \Leftrightarrow \frac{\partial^2 C}{\partial y^2} \geq 0$$

B. Without an Explicit Cost Function (assume $\mathbf{y}$).

$$\Pi(\mathbf{p},\mathbf{w}) = \max_{\mathbf{x}} \mathbf{p}' \underbrace{\mathbf{f}(\mathbf{x})}_{\mathbf{y}} - \underbrace{\mathbf{w}'\mathbf{x}}_{=C(\mathbf{w},\mathbf{y}) \text{ at } \mathbf{x}^*(\mathbf{w},\mathbf{y})}$$

First Order Condition [x_i]

$$p_i \frac{\partial f}{\partial x_i} = w_i \text{ or } p_i \frac{\partial f}{\partial x_i}(\mathbf{x}^*) \equiv w_i$$

Solution: $\mathbf{x}^*(\mathbf{p},\mathbf{w})$=Unconditional Factor Demand (not a function of $\mathbf{y}$).

Relationship between $\mathbf{x}^*(\mathbf{w},\mathbf{y})$ & $\mathbf{x}^*(\mathbf{p},\mathbf{w})$

$$\underbrace{\mathbf{x}^*(\mathbf{p},\mathbf{w})}_{\substack{\text{Unconditional} \\ \text{Factor Demand}}} = \underbrace{\mathbf{x}^*\big(\mathbf{w},\mathbf{y}^*(\mathbf{p},\mathbf{w})\big)}_{\substack{\text{Conditional Factor Demand} \\ \text{at the Some Level of Output}}}$$

Thus,

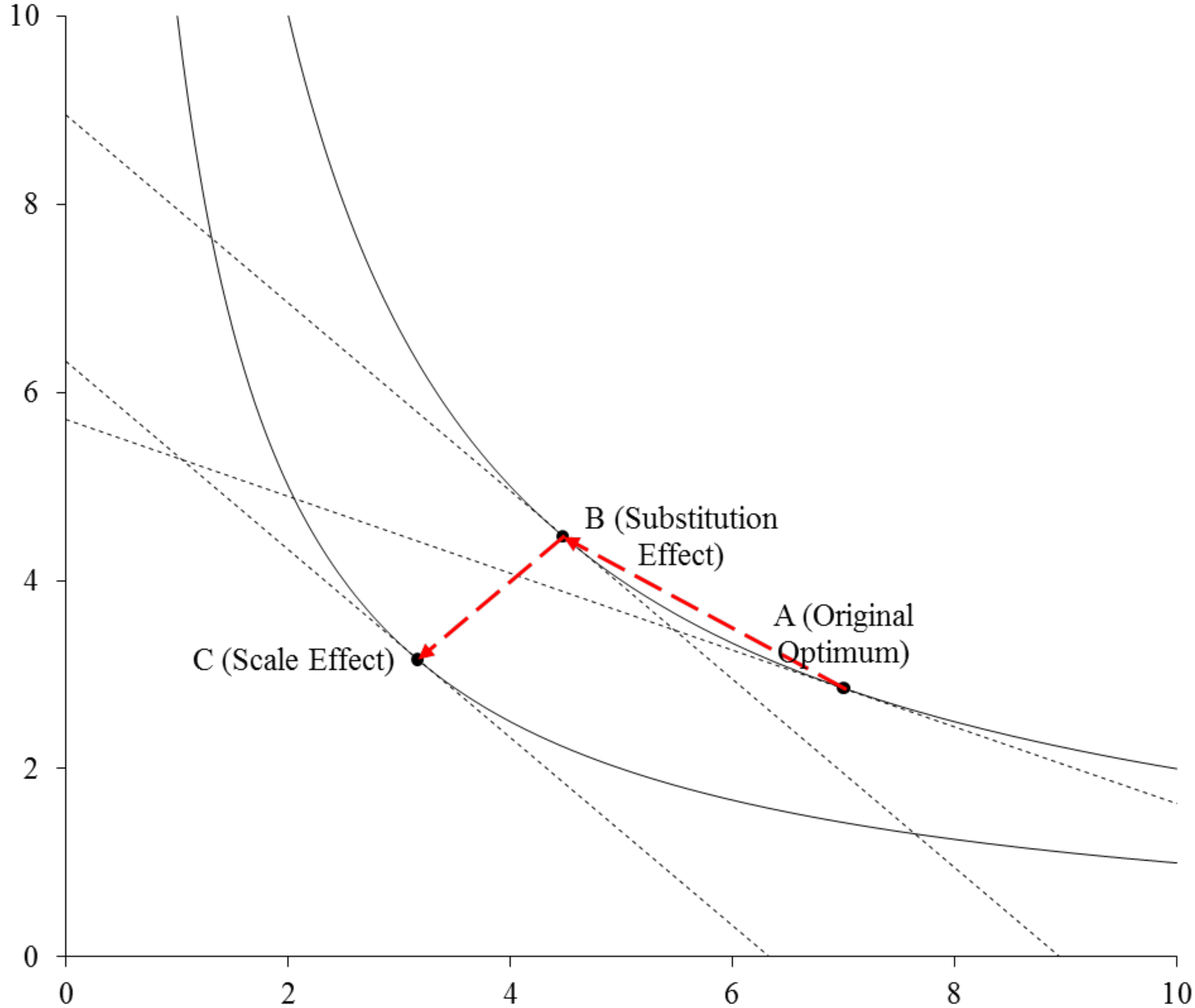
$$\frac{\partial \mathbf{x}^*(\mathbf{p},\mathbf{w})}{\partial w_i} = \underbrace{\frac{\partial \mathbf{x}^*}{\partial w_i}(\mathbf{w},\mathbf{y}^*)}_{\substack{\text{Substitution Effect} \\ (A\to B)}} + \underbrace{\overbrace{\frac{\partial \mathbf{x}^*}{\partial \mathbf{y}}}^{\approx\frac{1}{MP_{\mathbf{x}}}>0} \overbrace{\frac{\partial \mathbf{y}^*}{\partial w_i}}^{<0}}_{\substack{\text{Scale Effect} \\ (B\to C)}}$$

As w_k changes from w_k^1 to w_k^2, if still producing output y^1. ⇒ choose B rather than A (Substitution Effect). Also want to produce a new optional output, y_2, not y_1. ⇒ need corresponding fewer inputs and choose C rather than B (Scale Effect).

Note If,

$$\frac{\partial x_i^*}{\partial w_j}(\mathbf{p},\mathbf{w}) \gtreqless 0$$

Then, determine whether they are respectively gross substitutes or gross complements.


B (Substitution Effect)
A (Original Optimum)
C (Scale Effect)
10
8
6
4
2
0
0
2
4
6
8
10

Properties of the Profit Function

$$\Pi(\mathbf{p},\mathbf{w}) = \max_{\mathbf{x}} \mathbf{p}'\mathbf{f}(\mathbf{x}) - \mathbf{w}'\mathbf{x}$$

1. Non-decreasing in **p**, non-increasing in **w**

$$\text{If } \mathbf{p} \geq \mathbf{p}^1 \text{ and } \mathbf{w} \leq \mathbf{w}^1, \text{then } \Pi(\mathbf{p},\mathbf{w}) \geq \Pi(\mathbf{p}^1,\mathbf{w}^1)$$

2. Homogenous of degree 1

$$\Pi(k\mathbf{p},k\mathbf{w}) = k\Pi(\mathbf{p},\mathbf{w})\ \forall k > 0$$

3. Continuous in (**p**,**w**) for (**p**,**w**)>**0**

4. Convex in (**p**,**w**); for (**p**,**w**)>**0**,

$$\Rightarrow \Pi[\alpha p + (1-\alpha)p^1, \alpha w + (1-\alpha)w^1] \leq \alpha\Pi(p,w) + (1-\alpha)\Pi(p^1,w^1)$$

Properties of Optimal Output $\mathbf{y}^*(\mathbf{p},\mathbf{w})$ and Unconditional Factor Demand $\mathbf{x}^*(\mathbf{p},\mathbf{w})$
1. $\mathbf{y}^*$ is homogenous of degree 0 in (**p**,**w**)
2. Hotelling's lemma

$$\frac{\partial\Pi}{\partial p_i} = y_i^*(\mathbf{p},\mathbf{w}), \frac{\partial\Pi}{\partial w_j} = -x_j^*(\mathbf{p},\mathbf{w})$$

Proof Envelope Theorem
* Remind $\Pi=\mathbf{p}'\mathbf{f}(\mathbf{x})-\mathbf{w}'\mathbf{x}\approx\mathbf{p}'\mathbf{y}-C(\mathbf{w},\mathbf{y})$
3. Symmetry

$$\frac{\partial y_i^*}{\partial p_j} = \frac{\partial y_j^*}{\partial p_i}$$
$$\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}$$
$$\frac{\partial y_i^*}{\partial w_j} = -\frac{\partial x_j^*}{\partial p_i}$$

Proof Young's Theorem

4. Convexity: let **M** be a matrix defined by

$$\mathbf{M} = \begin{pmatrix} \frac{\partial y_1^*}{\partial p_1} & \frac{\partial y_2^*}{\partial p_1} & \cdots & \frac{\partial y_k^*}{\partial p_1} & -\frac{\partial x_1^*}{\partial p_1} & -\frac{\partial x_2^*}{\partial p_1} & \cdots & -\frac{\partial x_n^*}{\partial p_1} \\ \frac{\partial y_1^*}{\partial p_2} & \frac{\partial y_2^*}{\partial p_2} & \cdots & \frac{\partial y_k^*}{\partial p_2} & -\frac{\partial x_1^*}{\partial p_2} & -\frac{\partial x_2^*}{\partial p_2} & \cdots & -\frac{\partial x_n^*}{\partial p_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1^*}{\partial p_k} & \frac{\partial y_2^*}{\partial p_k} & \cdots & \frac{\partial y_k^*}{\partial p_k} & -\frac{\partial x_1^*}{\partial p_k} & -\frac{\partial x_2^*}{\partial p_k} & \cdots & -\frac{\partial x_n^*}{\partial p_k} \\ \frac{\partial y_1^*}{\partial w_1} & \frac{\partial y_2^*}{\partial w_1} & \cdots & \frac{\partial y_k^*}{\partial w_1} & -\frac{\partial x_1^*}{\partial w_1} & -\frac{\partial x_2^*}{\partial w_1} & \cdots & -\frac{\partial x_n^*}{\partial w_1} \\ \frac{\partial y_1^*}{\partial w_2} & \frac{\partial y_2^*}{\partial w_2} & \cdots & \frac{\partial y_k^*}{\partial w_2} & -\frac{\partial x_1^*}{\partial w_2} & -\frac{\partial x_2^*}{\partial w_2} & \cdots & -\frac{\partial x_n^*}{\partial w_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1^*}{\partial w_n} & \frac{\partial y_2^*}{\partial w_n} & \cdots & \frac{\partial y_k^*}{\partial w_n} & -\frac{\partial x_1^*}{\partial w_n} & -\frac{\partial x_2^*}{\partial w_n} & \cdots & -\frac{\partial x_n^*}{\partial w_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \Pi}{\partial \mathbf{p}\partial \mathbf{p}'} & \frac{\partial^2 \Pi}{\partial \mathbf{w}\partial \mathbf{p}'} \\ \frac{\partial^2 \Pi}{\partial \mathbf{p}\partial \mathbf{w}'} & \frac{\partial^2 \Pi}{\partial \mathbf{w}\partial \mathbf{w}'} \end{pmatrix}$$

The **M** is the hessian of the profit function, which is convex.
⇒ **M** is positive semi-definite.
(+ Property #3 says this is symmetric.)

Until now, we assumed **p** was just a vector of numbers, independent of the chosen **y**.
→ Effectively assumes that firms are price-takers; unable to influence prices.
→ But, what if the firm has market power, i.e. $d\mathbf{p}/d\mathbf{y}\neq\mathbf{0}$?
Then, the profit maximization problem is,

$$\max_{\mathbf{y}} \mathbf{p}'(\mathbf{y})\mathbf{y} - C(\mathbf{w},\mathbf{y}) \text{ where } \mathbf{p}(\mathbf{y}) \text{ is the (inverse) demand curve for the firm's product.}$$

Then, the FOC (y_i)

$$\underbrace{p_i + \sum_{j=1}^{k} \frac{\partial p_j}{\partial y_i} y_j}_{\substack{\text{margial revenue} \\ \text{of output } i}} - \underbrace{\frac{\partial C}{\partial y_i}}_{\substack{\text{marginal cost} \\ \text{of output } i}} = 0$$

ex. One output, FOC

$$\begin{aligned} p + \frac{\partial p}{\partial y} y &= \frac{\partial C}{\partial y} \\ \Rightarrow p\left(1 + \frac{\partial p}{\partial y}\frac{y}{p}\right) &= \frac{\partial C}{\partial y} \leftarrow \eta = \frac{\partial \ln y}{\partial \ln p} = \text{elasticity} \\ \Rightarrow p\left(1 + \frac{1}{\eta}\right) &= \frac{\partial C}{\partial y} \\ \Rightarrow p\left(1 - \frac{1}{|\eta|}\right) &= \frac{\partial C}{\partial y} \end{aligned}$$

Without an Explicit Cost Function,

$$\max_{\mathbf{x}} \mathbf{p}'[\mathbf{f}(\mathbf{x})]\mathbf{f}(\mathbf{x}) - C[\mathbf{w}, \mathbf{f}(\mathbf{x})]$$

Conclusion

$$\frac{\partial \mathbf{p}'}{\partial \mathbf{f}}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) + \mathbf{p}'\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial C}{\partial \mathbf{f}}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \mathbf{0} \Rightarrow MR \times MP_j = w_j$$

↑ SUPPLY SIDE
↓ DEMAND SIDE

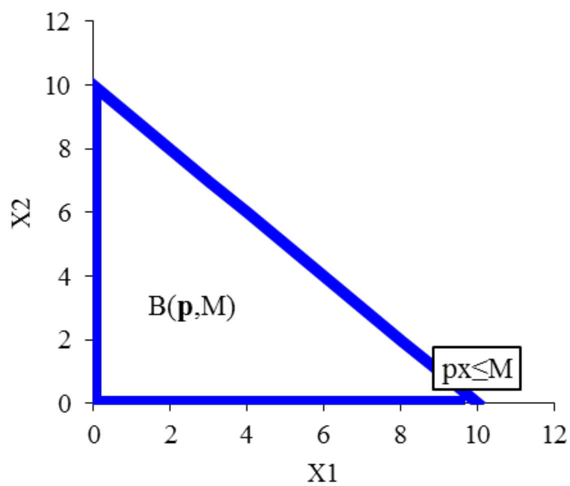
Utility Maximization

Let **x** be a vector representing a market basket of quantities of several goods.
Let $U(\mathbf{x})$ be a utility function (i.e. it provides an ordinal ranking of market baskets).

$$U(\mathbf{x}) \geq U(\mathbf{x}^1) \Leftrightarrow \mathbf{x} \gtrsim \mathbf{x}^1 \text{ with preference relation } \gtrsim$$

Let **p** be a vector of prices of the goods.
Let M be a household's budget.
Define a budget set,

$$B(\mathbf{p}, M) = \{\mathbf{x} | \mathbf{p}'\mathbf{x} \leq M\}$$

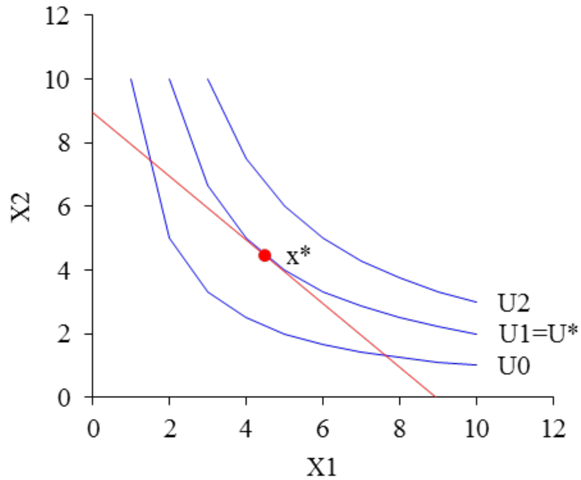


Household's demand for each good is then,

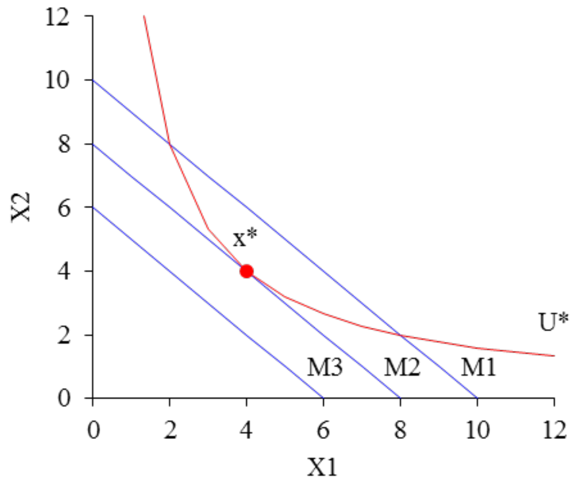
$$\mathbf{x}^*(\mathbf{p}, M) = \underset{\mathbf{x} \in B(\mathbf{p}, M)}{\operatorname{argmax}} U(\mathbf{x})$$

This is called the Marshallian Demand.
Also, define indirect utility function

$$V(\mathbf{p}, M) = \max_{\mathbf{x} \in B(\mathbf{p}, M)} U(\mathbf{x}) = U[\mathbf{x}^*(\mathbf{p}, M)]$$



Given M, maximize U (U_2 is not attainable)



Dual Problem: min $\mathbf{p}'\mathbf{h}$ s.t.$U(\boldsymbol{h})=U^*$

Problem

$$\max_{\mathbf{x}} U(\mathbf{x}) \text{ s.t. } \mathbf{p}'\mathbf{x} \leq M$$

Optimal Marshallian Demand $\mathbf{x}^*(\mathbf{p},M)$
Dual Problem

$$\min_{\mathbf{h}} \mathbf{p}'\mathbf{h} \text{ s.t. } U(\mathbf{h}) = U^*$$

The optimal choice here is called the Hicksian demand, instead.

$$\mathbf{h}(\mathbf{p}, U^*) = \operatorname*{argmin}_{\mathbf{h}} \mathbf{p}'\mathbf{h} \text{ s.t. } U(\mathbf{h}) = U^*$$

And the optimized value (the minimum cost of attaining utility U^* at prices $\mathbf{p}$) is called the expenditure function.

$$\begin{aligned} e(\mathbf{p}, U^*) &= \min_{\mathbf{h}} \mathbf{p}'\mathbf{h} \text{ s.t. } U(\mathbf{h}) = U^* \\ &= \mathbf{p}'\mathbf{h}^*(\mathbf{p}, U^*) \end{aligned}$$

	Given M		Given U^*
Problem	$\max_x U(x)$ s.t. $p'x = M$	Dual Problems ⇔	$\min_h p'h$ s.t. $U(h) = U^*$
Optimal Choice	Marshallian Demand $\mathbf{x}^*(\mathbf{p},M)$	Slutsky Equation ⇔	Hicksian Demand $\mathbf{h}^*(\mathbf{p},U^*)$
Optimized Value	Indirect Utility $V(\mathbf{p},M)$	Inverse Function ⇔	Expenditure Function $e(\mathbf{p},U^*)$

Slutsky Equation

$$\begin{aligned}
\mathbf{h}^*(\mathbf{p},U^*) &= \mathbf{x}^*[\mathbf{p},e(\mathbf{p},U^*)] \\
\Rightarrow \frac{\partial \mathbf{h}^*}{\partial \mathbf{p}} &= \frac{\partial \mathbf{x}^*}{\partial \mathbf{p}} + \frac{\partial \mathbf{x}^*}{\partial M}\underbrace{\frac{\partial e}{\partial \mathbf{p}}}_{=\mathbf{x}^*} \\
&= \frac{\partial \mathbf{x}^*}{\partial \mathbf{p}} + \frac{\partial \mathbf{x}^*}{\partial M}\mathbf{x}^*
\end{aligned}$$

This is important, because, in reality, utility function is invisible; by integrating both side, one can reveal the function "$\mathbf{h}^*(\cdot)$."

1. Observe Marshallian demand $\mathbf{x}^*(\mathbf{p},M)$ ← $\mathbf{p}$ and M are visible
2. Reveal Hicksian demand by using Slutsky equation
3. Expenditure function: $\mathbf{h}^{*\prime}\mathbf{p}=e$
4. Indirect utility: $V[\mathbf{p},e(\mathbf{p},U^*)]=U^*$ ⇒ (Note that $e[\mathbf{p},V(\mathbf{p},M)]=M$)

Indirect Utility Function

$$v(\mathbf{p},m)=\max_{\mathbf{x}} u(\mathbf{x})\ \text{ s.t. } \mathbf{p}'\mathbf{x}\le m$$

Marshallian Demand Function

$$\mathbf{x}(\mathbf{p},m)=\operatorname*{argmax}_{\mathbf{x}} u(\mathbf{x})\ \text{ s.t. } \mathbf{p}'\mathbf{x}\le m$$

Expenditure Function

$$e(\mathbf{p},u^*)=\min_{\mathbf{h}} \mathbf{p}'\mathbf{h}\ \text{ s.t. } u(\mathbf{h})=u^*$$

Hicksian Demand Function

$$\mathbf{h}(\mathbf{p},u^*)=\operatorname*{argmin}_{\mathbf{h}} \mathbf{p}'\mathbf{h}\ \text{ s.t. } u(\mathbf{h})=u^*$$

Identities

$$v[\mathbf{p},m=e(\mathbf{p},u^*)]=u^*$$
$$\mathbf{x}[\mathbf{p},m=e(\mathbf{p},u^*)]=\mathbf{h}(\mathbf{p},u^*)$$

⇒ Slutsky's Equation

$$\frac{\partial\mathbf{x}}{\partial\mathbf{p}}+\underbrace{\frac{\partial\mathbf{x}}{\partial m}\frac{\partial e}{\partial\mathbf{p}}}_{\substack{\text{income}\\\text{effect}}}=\underbrace{\frac{\partial\mathbf{h}}{\partial\mathbf{p}}}_{\substack{\text{substitution}\\\text{effect}}}$$

From before (Cost Minimization),
Cost Function

$$c(\mathbf{w},\mathbf{y})=\min_{\mathbf{x}} \mathbf{w}'\mathbf{x}\ \text{ s.t. } \mathbf{f}(\mathbf{x})>\mathbf{y}$$

Conditional Factor Demand

$$\mathbf{x}(\mathbf{w},\mathbf{y})=\operatorname*{argmin}_{\mathbf{x}} \mathbf{w}'\mathbf{x}\ \text{ s.t. } \mathbf{f}(\mathbf{x})>\mathbf{y}$$

Not surprisingly, $e(\mathbf{p},u^*)$ and $\mathbf{h}(\mathbf{p},u^*)$ share the properties of $c(\mathbf{w},\mathbf{y})$ and $\mathbf{x}(\mathbf{w},\mathbf{y})$, respectively.

Properties of $e(\mathbf{p},u^*)$ ⇒ <u>This is not observable since this involves an utility function.</u>

1. Non-decreasing in u^*, non-increasing in **p**, and increasing at least one element of **p**
2. Homogenous of degree 1 at **p**

$$\forall k > 0, e(k\mathbf{p}, u^*) = ke(\mathbf{p}, u^*)$$

3. Continuous in **p**
4. Concave in **p**

$$\forall \alpha \in [0,1], e[\alpha\mathbf{p} + (1-\alpha)\mathbf{p}^1, u^*] \geq \alpha e(\mathbf{p}, u^*) + (1-\alpha)e(\mathbf{p}^1, u^*)$$

Properties of $\mathbf{h}(\mathbf{p},u^*)$ ⇒ <u>This is not observable since this involves an utility function.</u>

1. Homogenous of degree 0 at **p**

$$\mathbf{h}(k\mathbf{p}, u^*) = \mathbf{h}(\mathbf{p}, u^*)$$

2. Shephard's Lemma

$$\frac{\partial e(\mathbf{p}, u^*)}{\partial p_i} = h_i(\mathbf{p}, u^*)$$

3. Symmetry

$$\frac{\partial h_i(p, u^*)}{\partial p_j} = \frac{\partial h_j(p, u^*)}{\partial p_i}$$

4. Negativity: Define Slutsky matrix **S** such that,

$$[\mathbf{S}]_{ij} = \frac{\partial h_i(\mathbf{p}, u^*)}{\partial p_j}$$

Then,

$$\mathbf{S} = \begin{pmatrix} \frac{\partial h_1(\mathbf{p}, u^*)}{\partial p_1} & \frac{\partial h_1(\mathbf{p}, u^*)}{\partial p_2} & \cdots & \frac{\partial h_1(\mathbf{p}, u^*)}{\partial p_n} \\ \frac{\partial h_2(\mathbf{p}, u^*)}{\partial p_1} & \frac{\partial h_2(\mathbf{p}, u^*)}{\partial p_2} & \cdots & \frac{\partial h_2(\mathbf{p}, u^*)}{\partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n(\mathbf{p}, u^*)}{\partial p_1} & \frac{\partial h_n(\mathbf{p}, u^*)}{\partial p_2} & \cdots & \frac{\partial h_n(\mathbf{p}, u^*)}{\partial p_n} \end{pmatrix}$$

This Slutsky matrix is negative semi-definite.

Properties of $v(\mathbf{p},m)$ ⇒ This is not observable since this is an utility function.

1. Non-decreasing in m, non-decreasing in $\mathbf{p}$, and decreasing in at least one element of $\mathbf{p}$
2. Homogenous of degree 0 in $(\mathbf{p},m)$

$$\forall k > 0, v(k\mathbf{p}, km) = v(\mathbf{p}, m)$$

3. $v(\mathbf{p},m)$ is continuous in $\mathbf{p}$, m
4. Quasi-convex in $\mathbf{p}$, $\forall$ k, the set $\{\mathbf{p}|v(\mathbf{p},m)\leq k\}$ is convex.

In other words,

$$v(\mathbf{p}^1, m) \leq k \;\&\; v(\mathbf{p}^2, m) \leq k \Rightarrow v[\alpha\mathbf{p}^1 + (1-\alpha)\mathbf{p}^2, m] \leq k, \forall\alpha \in [0,1]$$

Properties of $\mathbf{x}(\mathbf{p},m)$ ⇒ This is observable!

1. Adding-up ⇒ This is observable!

$$\mathbf{p}'\mathbf{x}(\mathbf{p}, m) = m$$

2. Homogenous of degree 0 in $(\mathbf{p},m)$ ⇒ This is observable!

$$\forall k > 0, \mathbf{x}(k\mathbf{p}, km) = \mathbf{x}(\mathbf{p}, m)$$

3. Roy's Identity ⇒ This is unobservable because this involves $\partial v/\partial p_i$ and $\partial v/\partial m$ that we do not know.

$$x_i(\mathbf{p}, m) = -\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial m}}$$

Proof Identity: $v[\mathbf{p},e(\mathbf{p},u^*)]=u^*$,

$$\frac{\partial v}{\partial p_i} + \frac{\partial v}{\partial m} \underbrace{\frac{\partial e}{\partial p_i}}_{=x_i(\mathbf{p},m)} = 0$$

Note that

$$x_i(\mathbf{p}, m) \underbrace{=}_{\text{identity}} h_i[\mathbf{p}, v(\mathbf{p}, m)] \overset{\substack{\text{Shephard}\\ \text{lemma}}}{\cong} \frac{\partial e}{\partial p_i} = -\frac{\partial v/\partial p_i}{\partial v/\partial m}$$

4. From Slutsky's Equation, should have a form of Slutsky symmetry ⇒ From data, attainable.

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m}\underbrace{\frac{\partial e}{\partial p_j}}_{=x_j} = \frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial m}\underbrace{\frac{\partial e}{\partial p_i}}_{=x_i} = \frac{\partial h_j}{\partial p_i}$$

5. Likewise, we can use Slutsky's equation to test whether the Slutsky matrix is negative semi-definite.

⇒ Possible; therefore only 4 properties are viable through data.

Argument from Hurwicz and Uzawa (1971): There are no testable implications of *Utility Maximization*.

Point 1
If a utility function is maximized, than it is necessarily true that there is an expenditure function and that

$$\frac{\partial e(\mathbf{p},u)}{\partial p_i} = h_i(\mathbf{p},u) \equiv x_i[\mathbf{p},e(\mathbf{p},u)]$$

So, you have estimated a set of Marshallian demands $\hat{\mathbf{x}}(\mathbf{p},m)$, then those choices can only be justified as the result of a utility maximization problem,

$$\text{if } \exists \text{a function } \hat{e} \text{ such that } \underbrace{\frac{\partial \hat{e}}{\partial p_i} = \hat{x}_i(\mathbf{p},\hat{e})}_{A}$$

Point 2
The system of the differential equation A above necessarily has a solution if, for each i & j,

$$\frac{\partial \hat{x}_i}{\partial m}\hat{x}_j + \frac{\partial \hat{x}_i}{\partial p_j} = \frac{\partial \hat{x}_j}{\partial m}\hat{x}_i + \frac{\partial \hat{x}_j}{\partial p_i}$$

+ Slutsky equation

$$\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m}\underbrace{\frac{\partial e}{\partial p_j}}_{=x_j} = \frac{\partial h_i}{\partial p_j}$$

$$\Rightarrow \frac{p_j}{x_i}\frac{\partial x_i}{\partial p_j} + \frac{m}{m}\frac{p_j}{x_i}\frac{\partial x_i}{\partial m}x_j = \frac{\partial h_i}{\partial p_j}\frac{p_j}{h_i}$$

$$\Rightarrow \underbrace{\frac{\partial \log p_j}{\partial \log x_i}}_{\substack{\text{Marshallian}\\ \text{price elasticity}}} + \underbrace{\frac{\partial \log x_i}{\partial \log m}}_{\substack{\text{Marshallian}\\ \text{income elasticity}}} \underbrace{\omega_j}_{\substack{\text{expenditure}\\ \text{share of } j}} = \underbrace{\frac{\partial \log h_i}{\partial \log p_j}}_{\substack{\text{Hicksian}\\ \text{price elasticity}}}$$

$$\Rightarrow e_{ji} + e_i\omega_j = e_{ji}^*$$

This is Slutsky equation in elasticities.

Elasticity version of Slutsky concepts

1. From the last time: Slutsky equation

$$\underbrace{e_{ij}^*}_{\substack{\text{Hicksian}\\ \text{price elasticity}}} = \underbrace{e_{ij}}_{\substack{\text{Marshallian}\\ \text{price elasticity}}} + \underbrace{e_i}_{\substack{\text{Marshallian}\\ \text{income elasticity}}} \underbrace{\omega_j}_{\substack{\text{expenditure}\\ \text{share of } j}}$$

2. Slutsky symmetry

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$
$$\Rightarrow \frac{h_i}{h_i}\frac{p_i p_j}{e}\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}\frac{p_i p_j}{e}\frac{h_j}{h_j}$$
$$\Rightarrow \omega_i \frac{\partial \log h_i}{\partial \log p_j} = \frac{\partial \log h_j}{\partial \log p_i}\omega_j$$
$$\Rightarrow \omega_i e_{ij}^* = e_{ji}^* \omega_j$$

Stone (1954a)

Model for each good i=1,2,⋯,48=N

$$\log \underbrace{x_i}_{\substack{\text{consumption}\\ \text{of good } i}} = \alpha_i + e_i \log \underbrace{m}_{\text{income}} + \sum_{k=1}^{N} e_{ik} \log \underbrace{p_k}_{\substack{\text{price of}\\ \text{good } k}}$$

Problem: 50 parameters, just 19 observations!

What we cannot give up: α_i, e_i, e_{ii}

Revisit the properties of demand systems

1. Adding-up $\sum p_i x_i = m$
2. Homogenous of degree 0: $\mathbf{x}(t\mathbf{p}, tm) = \mathbf{x}(\mathbf{p}, m)$
3. Slutsky symmetry

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i} \Leftrightarrow \omega_i e_{ij}^* = \omega_j e_{ji}^*$$
$$\Leftrightarrow \omega_i\left(e_{ij} + e_i \omega_j\right) = \omega_j\left(e_{ji} + e_j \omega_i\right)$$

4. Negativity: Slutsky matrix is negative semi-definite.

Stone's work
Homogeneity

$$\log x_i(t\mathbf{p}, tm) = \log x_i(\mathbf{p}, m)$$

Model says,

$$\begin{aligned}\log x_i(t\mathbf{p}, tm) &= \alpha_i + e_i \log tm + \mathbf{e}_i' \log t\mathbf{p} \\ &= \alpha_i + e_i \log m + e_i \log t + \mathbf{e}_i' \log \mathbf{p} + \log t\, \mathbf{e}_i' \mathbf{1} \\ &= \alpha_i + e_i \log m + \mathbf{e}_i' \log \mathbf{p} + \log t \quad \underbrace{(e_i + \mathbf{e}_i'\mathbf{1})}_{\substack{\text{This should be 0 if } \mathbf{x} \\ \text{homogenous degree 0}}}\end{aligned}$$

Then, Slutsky equation

$$\begin{aligned} e_{ij}^* &= e_{ij} + e_i \omega_j \\ \Rightarrow \mathbf{e}_i^* &= \mathbf{e}_i + e_i \boldsymbol{\omega} \\ \Rightarrow \mathbf{1}'\mathbf{e}_i^* &= \mathbf{1}'\mathbf{e}_i + e_i \underbrace{\mathbf{1}'\boldsymbol{\omega}}_{=1} \\ &= 0 \end{aligned}$$

Therefore, the demand model can be transformed as,

$$\begin{aligned}\log x_i &= \alpha_i + e_i \log m + \sum_{j=1}^{n} \overbrace{e_{ij}}^{e_{ij}^* - e_i \omega_j} \log p_j \\ &= \alpha_i + e_i \log m - e_i \sum_{j=1}^{n} \omega_j \log p_j + \sum_{j=1}^{n} e_{ij}^* \log p_j\end{aligned}$$

From the homogeneity condition,

$$\begin{aligned}\sum_{j=1}^{n} e_{ij}^* = 0 \Rightarrow e_{in}^* &= -\sum_{j=1}^{n-1} e_{ij}^* \\ \Rightarrow \sum_{j=1}^{n} e_{ij}^* \log p_j &= \left(\sum_{j=1}^{n-1} e_{ij}^* \log p_j\right) + e_{in}^* \log p_n \\ &= \left(\sum_{j=1}^{n-1} e_{ij}^* \log p_j\right) - \log p_n \sum_{j=1}^{n-1} e_{ij}^* \\ &= \sum_{j=1}^{n} e_{ij}^* (\log p_j - \log p_n) \\ &= \sum_{j=1}^{n} e_{ij}^* \log(p_j / p_n)\end{aligned}$$

Define log$p$ (consumer price index) as

$$\underbrace{\underbrace{\log P}_{\substack{\text{change}\\\text{in index}}} = \sum_{i=1}^{n} \underbrace{\omega_k \underbrace{\log p_k}_{\substack{\text{change}\\\text{in price}}}}_{\substack{\text{weighted}\\\text{average}}}}_{\text{not intuitive but reasonable}}$$

Then,

$$\log \underbrace{x_i}_{\substack{\text{real income}\\\text{like GDP}}} = \alpha_i + e_i \log(m/p) + \sum_{j=1}^{n-1} e_{ij}^{*} \log(p_i/p_n)$$

If results are not favorable,

1. Model problem
2. British people may not be rational.

Stone (1954b)

New strategy: Force the demand system to be consistent with rational choice by deriving it from an assumed utility function.

Uses a Linear Expenditure System: assume

$$u(\mathbf{x}) = \prod_{i=1}^{n} (x_i - \gamma_i)^{\beta_i} \text{ where } \forall i, \beta_i > 0 \text{ and } \sum_{i=1}^{n} \beta_i = 1$$

Implies

$$x_i(\mathbf{p}, m) = \gamma_i + \frac{\beta_i}{p_i}\left(m - \sum_{j=1}^{n} p_j \gamma_j\right) \to A$$

$$h_i(\mathbf{p}, u) = \gamma_i + \frac{\beta_i}{p_i} \delta u \prod_{k=1}^{n} p_k^{\beta_k} \text{ where } \delta = \prod_{k=1}^{n} \beta_k^{-\beta_k}$$

$$v(\mathbf{p}, m) = \frac{m - \sum p_k \gamma_k}{\delta \prod p_k^{\beta_k}}$$

$$e(\mathbf{p}, u) = \sum_{k=1}^{n} p_k \gamma_k + \delta u \prod_{k=1}^{n} p_k^{\beta_k}$$

Then,

$$A \to p_i x_i = \gamma_i p_i + \beta_i m - \beta_i \sum_{j=1}^{n} p_j \gamma_j$$
$$= \sum_{k=1}^{n} \beta_{ik}^* p_k + \beta_i m \text{ where } \beta_{ik}^* = \begin{cases} -\beta_i \gamma_k, & k \neq i \\ (1-\beta_i)\gamma_i, & k = i \end{cases}$$

Q: How many potential price and income elasticity could there be?
A: N goods, for each one N price e_{ik} and 1 income e_i, therefore $N(N+1)=N^2+N$ elasticity.

Q: How parameters are in the linear expenditure system model?
A: N goods, 2 parameters each (γ, β) and subtract 1 because $\sum\beta_k=1$, therefore $2N-1$ total parameters.

Potential Elasticity−Number of Parameters=$N^2+N-2N+1=\underline{N^2-N+1}$
N^2-N+1 is the number of elasticity coefficients that are restricted by the fractional form.

Assumption

$$\frac{\partial x_i}{\partial m} = \frac{\beta_i}{p_i} > 0 \Leftrightarrow \nexists \text{inferior good}$$

$$\text{Also, } p_i x_i = \left(\gamma_i p_i - \beta_i \sum_{j=1}^{n} \gamma_j p_j\right) + \beta_i m \Leftrightarrow \nexists \text{compliment } \left(\because \frac{\partial p_i x_i}{\partial p_j} = -\beta_i \gamma_j < 0 \text{ if } \gamma_j > 0\right)$$

Say we have a system of demand with N goods.
⇒ N income elasticity
⇒ N^2 own and cross price elasticity
⇒ Total N^2+N elasticity

How many restrictions are imposed by the theory of demand?
Only 4 testable constraints

Properties	Number of Constraints
1) Adding-up	1
2) Homogeneity	N
3) Slutsky Symmetry	N(N−1)/2
4) Negativity	$\mathbf{h}'\mathbf{M}\mathbf{h}\leq 0$

⇒ Total $N(N-1)/2+N+1=N(N+1)/2+1$ restrictions

Therefore, # of free parameters is

$$N^2+N-\left[\frac{N(N+1)}{2}+1\right]=\frac{N(N+1)}{2}-1$$

Comparison: Linear Expenditure System with N goods
⇒ For each good, 2 parameters (β, γ)
⇒ One constraint: $\sum\beta_i=1$
⇒ Total $2N-1$ parameters

N	Theory: $N(N+1)/2-1$	LES: $2N-1$
1	0	1
2	2	3
3	5	5
4	9	7
5	14	9

With $N\geq 4$, Linear Expenditure System necessarily imposes more constraints on elasticity coefficients than theory world requires; these are the correction of the algebra of the last class

Note: Slutsky matrix

$$\mathbf{M}=\begin{pmatrix}\frac{\partial h_i}{\partial p_j} & \cdots \\ \vdots & \ddots\end{pmatrix}=\begin{pmatrix}\frac{\partial h_1}{\partial p_1} & \frac{\partial h_2}{\partial p_1} & \cdots & \frac{\partial h_n}{\partial p_1} \\ \frac{\partial h_1}{\partial p_2} & \frac{\partial h_2}{\partial p_2} & \cdots & \frac{\partial h_n}{\partial p_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial p_n} & \frac{\partial h_2}{\partial p_n} & \cdots & \frac{\partial h_n}{\partial p_n}\end{pmatrix}=\frac{\partial^2 e}{\partial\mathbf{p}\partial\mathbf{p}'}$$

Claim: Slutsky symmetry does not hold for aggregate demand curves (except in one special case).

Notation

$x_i^a(\mathbf{p},m^a)$—Marshallian demand for good i by the person a with total expenditure m^a

$\mathbf{m}=(m^1, m^2, \cdots, m^A)$—Vector of total expenditure by person 1, 2, ⋯, A

$m=\mathbf{m}'\mathbf{1}=\sum m^a$—Total sum

Properly specified aggregate demand

$$x_i(\mathbf{p},\mathbf{m})=\sum_{a=1}^{A}x_i^a(\mathbf{p},m^a)$$

Slutsky symmetry: for each person,

$$\frac{\partial x_i^a}{\partial p_j}+\frac{\partial x_i^a}{\partial m^a}x_j^a=\frac{\partial x_j^a}{\partial p_i}+\frac{\partial x_j^a}{\partial m^a}x_i^a$$

Adding this up over people

$$\sum_{a=1}^{A}\left(\frac{\partial x_i^a}{\partial p_j}+\frac{\partial x_i^a}{\partial m^a}x_j^a\right)=\underbrace{\sum_{a=1}^{A}\frac{\partial x_i^a}{\partial p_j}}_{1)}+\underbrace{\sum_{a=1}^{A}\frac{\partial x_i^a}{\partial m^a}x_j^a}_{2)}=\sum_{a=1}^{A}\frac{\partial x_j^a}{\partial p_i}+\sum_{a=1}^{A}\frac{\partial x_j^a}{\partial m^a}x_i^a=\sum_{a=1}^{A}\left(\frac{\partial x_j^a}{\partial p_i}+\frac{\partial x_j^a}{\partial m^a}x_i^a\right)$$

Thus, for 1),

$$\sum_{a=1}^{A}\frac{\partial x_i^a}{\partial p_j}=\frac{\partial\sum x_i^a}{\partial p_j}=\frac{\partial x_i}{\partial p_j}\ \text{and}\ \sum_{a=1}^{A}\frac{\partial x_j^a}{\partial p_j}=\frac{\partial\sum x_j^a}{\partial p_j}=\frac{\partial x_j}{\partial p_j}$$

However, for 2)

$$\sum_{a=1}^{A} \frac{\partial x_i^a}{\partial m^a} x_j^a = ?$$

For one possible case (same income effects), if

$$\frac{\partial x_i^a}{\partial m^a} = \frac{\partial x_i^b}{\partial m^b} \forall a, b \Rightarrow \sum_{a=1}^{A} \frac{\partial x_i^a}{\partial m^a} x_j^a = \frac{\partial x_i^a}{\partial m^a} \sum_{a=1}^{A} x_j^a = \frac{\partial x_i}{\partial m} x_j$$

$$\Rightarrow \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m} x_j = \frac{\partial x_j}{\partial p_i} + \frac{\partial x_j}{\partial m} x_i$$

So, aggregate demand is symmetric if all people have the same income effects for all goods. Otherwise, aggregate demand slope is

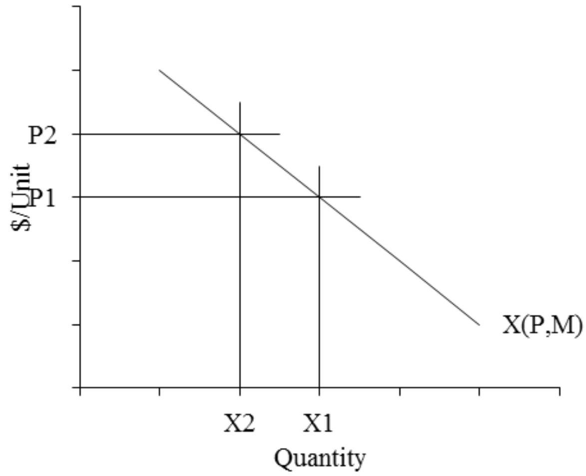
$$\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m} x_j = \underbrace{\sum_{a=1}^{A} \left(\frac{\partial x_i^a}{\partial p_j} + \frac{\partial x_i^a}{\partial m^a} x_j^a \right)}_{\text{This part is symmetric (Slutsky).}} + \underbrace{\sum_{i=1}^{A} \frac{\partial x_i^a}{\partial m^a} \left(\frac{\partial m^a}{\underbrace{\partial m}_{=1?}} x_j - x_j^a \right)}_{\neq 0 \text{ unless } \frac{\partial x_i^a}{\partial m^a} = \frac{\partial x_i^b}{\partial m^b} \forall a,b}$$

More generally, no reason to believe it equals

$$\sum_{a=1}^{A} \frac{\partial x_j^a}{\partial m^a} \left(\frac{\partial m^a}{\partial m} x_i - x_i^a \right)$$

Econ 103: Consumer Surplus

How does welfare change when price change from p_1 to p_2?



$$\Delta CS = \int_{p_1}^{p_2} x(p,m)dp$$

Better: two distinct questions

1. How much would the consumer have been willing to pay as a lump sum in order to avoid the price change?
2. How much would you have to pay the consumer to accept the price change voluntarily?

To answer 1, find E such that

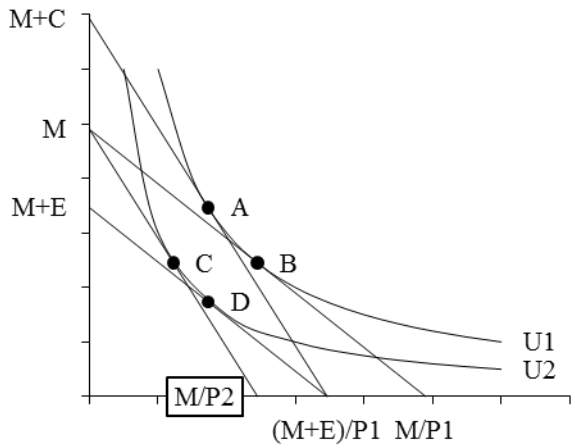
$$v(p_1, m+E) = v(p_2, m) = u_2$$

where $E \equiv$ Equivalent Variation

To answer 2, find C such that

$$v(p_2, m+C) = v(p_1, m) = u_1$$

where $C \equiv$ Compensating Variation



Where

$$
\begin{aligned}
&A: x(p_2, m + C) = x(p_1, m) = h(p_1, u_1)\\
&B: x(p_1, m)\\
&C: x(p_2, m)\\
&D: x(p_1, m + E) = x(p_2, m) = h(p_2, u_2)
\end{aligned}
$$

Inverting the two indirect utility functions,

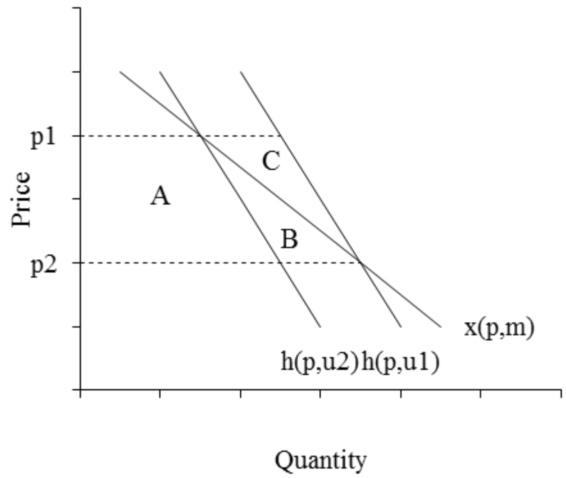
$$
\begin{aligned}
e(p_2, u_2) &= e(p_1, u_2) + E\\
e(p_2, u_1) &= e(p_1, u_1) + C
\end{aligned}
$$

Then, *E* and *C* can be calculated as

$$
\begin{aligned}
E &= \int_{p_1}^{p_2} \frac{\partial e(p, u_2)}{\partial p} dp\\
&= \int_{p_1}^{p_2} h(p, u_2) dp \text{ by Shephard's lemma}\\
C &= \int_{p_1}^{p_2} h(p, u_1) dp
\end{aligned}
$$

Note for normal good

$$
\underbrace{\frac{\partial h}{\partial p}}_{<0} = \underbrace{\frac{\partial x}{\partial p}}_{<0} + \underbrace{\frac{\partial x}{\partial m} x}_{>0 \text{ if normal}}
$$



Area

$$A = \int_{p_1}^{p_2} h(p, u_2)dp = E$$

$$A + B = \int_{p_1}^{p_2} x(p, m)dp = \Delta CS$$

$$A + B + C = \int_{p_1}^{p_2} h(p, u_1)dp = C$$

Where ΔCS: Change in Consumer Surplus

Knightian Uncertainty: random random, unknown unknown.

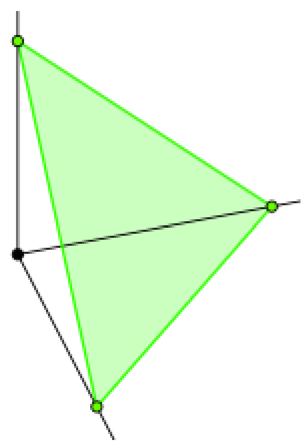
Risk: non-random random, known unknown
Defined by a set of possible outcomes $\mathbf{x}=\{x_1,x_2,\cdots,x_N\}$ and a probability p_i associated with each outcome x_i such that

$$\sum_{i=1}^{N} p_i = 1 \text{ where } p_i \geq 0 \forall i$$

For a continuous set of outcomes (e.g. $X=\mathbb{R}$) we can write the associated probabilities using a probability distribution $f(x)$.

Definition If the set of outcomes is finite then we can define its probability vector as $\mathbf{p}=(p_1,p_2,\cdots,p_N)$ where $\mathbf{p}$ is an element of the "$N-1$" dimensional simplex.

$$\Delta^{N-1} \equiv \left\{\mathbf{p} \in \mathbb{R} | p_i \geq 0 \forall i, \sum p_i = 1\right\}$$



2-dimensional simplex

Lottery 1 $\mathbf{x}_1=\{1,4,7\}$, $\mathbf{p}_1=\{1/10,2/5,1/2\}$, Lottery 2 $\mathbf{x}_2=\{-4,0,27\}$, $\mathbf{p}_2=\{1/4,1/2,1/4\}$
⇒ Could define both over
$\mathbf{x}=\mathbf{x}_1\cup\mathbf{x}_2=\{-4,0,1,4,7,27\}$, $\mathbf{p}_1=\{0,0,1/10,2/5,1/2,0\}$, $\mathbf{p}_2=\{1/4,1/2,0,0,0,1/4\}$

If we use a common set of outcomes $\mathbf{x}$ across all lotteries, a lottery is then defined solely by the probabilities of those outcomes.
⇒ Under general circumstances, we can define a utility function u that represents preferences over those lotteries.
⇒ Any monotonic transformation ranks the lotteries in the same order.

Expected Utility Theorem: There exists a monotonic transformation h such that $h(u)$ is linear in probabilities.

$$h(u) = \sum_{i=1}^{N} p_i v(x_i) \text{ for some function } v(x)$$

or for a continuous set of possible outcomes

$$h(u) = \int_i f(i) v(x_i) di \text{ where } f(i) \text{ is the pdf of the outcomes.}$$

The function $v(x)$ is called the “von Neumann–Morgenstern” utility function.

While any monotonic transformation of u preserves the ranking over lotteries, only an affine transformation $h(u)=A+Bu$ (with $B>0$) preserves the expected utility form.
In general,

$$h(u) = h\left[\sum_i p_i v(x_i)\right]$$

$$\neq \sum_i p_i h[v(x_i)]$$

However, for an affine transformation,

$$h(u) = A + Bu = A + B\sum_i p_i v(x_i) = A\sum_i p_i + \sum_i p_i B v(x_i) = \sum_i p_i[A + Bv(x_i)] = \sum_i p_i h[v(x_i)]$$

ex. If a person has $v(x)=x^{\alpha}$, then, that person’s preferences can also be represented equivalently as

$$v(x) = x^{\alpha} \Rightarrow v^*(x) = A + Bx^{\alpha}$$

$\sum p_i v(x_i)$ ranks in the same order as $\log[\sum p_i v(x_i)]$, but NOT in the same order as $\sum p_i \log[v(x_i)]$.

Risk Aversion
⇒ Conceptually, prefer not to have more variable outcomes that do not improve the mean of outcome.
⇒ Formally,
Let X be any random variable.
Let ε be an independent, mean-zero random variable.
Therefore, $E(X)=E(X+\varepsilon)$
But, $X+\varepsilon$ has more dispersed outcomes than X does. ⇒ Risk averse person prefers a lottery described by X over one described by $X+\varepsilon$.

Concepts of Risk Aversion

Special case: Let X=x_0, i.e. a constant with probability 1. Let $\tilde{\varepsilon}$~$G(\tilde{\varepsilon})$. Then, risk aversion,

$$\underbrace{\int u(x_0+\varepsilon)g(\varepsilon)d\varepsilon}_{=\mathrm{E}[u(X+\tilde{\varepsilon})]} \le \int u(x_0)g(\varepsilon)d\varepsilon = u(x_0)$$

$$= u(x_0+0)$$

$$= u\left[x_0 + \int \varepsilon g(\varepsilon)d\varepsilon\right]$$

$$= \underbrace{u\left[\int (x_0+\varepsilon)g(\varepsilon)d\varepsilon\right]}_{=u[\mathrm{E}(X+\tilde{\varepsilon})]}$$

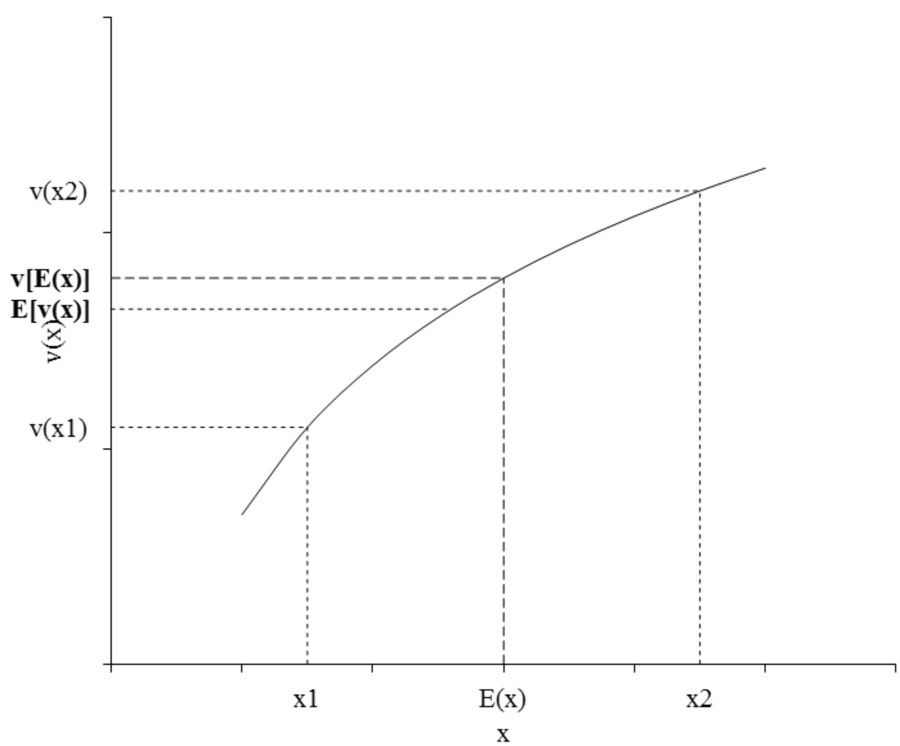
Jensen's Inequality

Let Z be a random variable and $h(z)$ be a strictly concave (convex) function.

Then,

$$\mathrm{E}[h(z)] \le h[\mathrm{E}(z)] \Rightarrow h \text{ concave}$$
$$\mathrm{E}[h(z)] < h[\mathrm{E}(z)] \Rightarrow h \text{ strictly concave}$$
$$\mathrm{E}[h(z)] \ge h[\mathrm{E}(z)] \Rightarrow h \text{ convex}$$
$$\mathrm{E}[h(z)] > h[\mathrm{E}(z)] \Rightarrow h \text{ strictly convex}$$

1. von Neumann–Morgenstern utility function $v(x)$ is concave.



2. Let X be a random variable $X \sim F(X)$. Define a certainty equivalent $c(X)$ by

$$u[c(X)] = \int u(x) f(x) dx$$

Then, for a risk-averse person,

$$c(X) \leq \mathrm{E}(X)$$

3. Let X be a random variable and let ε be another random variable with $\mathrm{E}(\varepsilon)=0$. Define a risk premium $\pi(\varepsilon)$ by

$$\mathrm{E}\{u[X - \pi(\varepsilon)]\} = \mathrm{E}[u(X + \varepsilon)]$$

Then, for a risk-averse person, $\pi(\varepsilon) \geq 0$.

4. Define the "Arrow–Pratt" measure of absolute risk aversion

$$\gamma(x) \equiv -\frac{u''(x)}{u'(x)} \geq 0 \text{ for a risk averse person}$$

5. There is also an "Arrow–Pratt" measure of relative risk aversion

$$\rho(x) \equiv -\frac{u''(x)}{u'(x)} x$$

$$\Rightarrow \gamma \geq 0 \Leftrightarrow \rho \geq 0 \forall x > 0$$

Concepts of Risk-aversion
0. Dislike mean-zero risks
1. Concave von Neumann–Morgenstern utility function
2. Certainty equivalent: $c(X)<\mathrm{E}(X)$
3. Risk premium: $\pi(X)>0$
4. Arrow–Pratt coefficient of ARA (RRA): $r>0$ ($\rho>0$)

How can we say that person A is more risk-averse than person B?
1. There exists a concave function ϕ such that ϕ maps B's von Neumann–Morgenstern utility function v_A to A's v_B:

$$v_A = \phi(v_B)$$

2. For any random variable X, A's certainty equivalent c_A is less than B's (c_B):

$$c_A(X) \le c_B(X), \forall X$$

3. A's risk premium π_A is always greater than B's (π_B):

$$\pi_A \ge \pi_B \text{ for all risks}$$

4. A's coefficient of ARA r_A is always greater than B's one:

$$r_A \ge r_B \text{ or equivalently } \rho_A \ge \rho_B$$

Theorem Pratt's theorem: Criteria 1–4 are equivalent—if one is true, all of them are.

Note Coefficients of ARA r and RRA ρ

$$r(W) = -\frac{v''(W)}{v'(W)} = -\frac{d\log v'}{dv'}\frac{dv'}{dW} = -\frac{d\log v'(W)}{dW} \approx -\frac{\text{percent change of marginal utility of wealth}}{\text{change of wealth}}$$

$$\rho(W) = -\frac{v''(W)}{v'(W)}W = -\frac{d\log v'(W)}{dW}W = -\frac{d\log v'(W)}{d\log W} = -\frac{\text{percent change of marginal utility of wealth}}{\text{percent change of wealth}}$$

Is $r(W)$ constant?
⇒ Change of MU of additional \$1 when you are poor=the Change when you are rich
⇒ Unrealistic

Is $\rho(W)$ constant?
⇒ Change of MU of additional 1% when you are poor=the Change when you are rich
⇒ Quite reasonable

Two Useful von Neumann–Morgenstern Utility Functions

1. Constant ARA (CARA) utility function

$$v(W) = -e^{-rW}$$
$$\Rightarrow r(W) = -\frac{v''}{v'} = -\frac{-\gamma^2 e^{-\gamma W}}{\gamma e^{-\gamma W}} = \gamma$$

2. Constant RRA (CRRA) utility function

$$v(W) = \frac{W^{1-\rho} - 1}{1-\rho} \Rightarrow \lim_{\rho\to 1} v(W) = \lim_{\rho\to 1} \frac{W^{1-\rho} - 1}{1-\rho} = \lim_{\rho\to 1} \frac{-W^{1-\rho}\ln W}{-1} = \ln W$$

Simple model

$$W_{t+1} = W_t + \varepsilon$$
$$\varepsilon \sim N(\mu, \sigma^2)$$
$$v(W) = -e^{-\gamma W}$$

Then,

$$\mathrm{E}[v(W_{t+1})|W_t] = \mathrm{E}[v(W_t + \varepsilon)|W_t] = \mathrm{E}\left[-e^{-\gamma(W_t+\varepsilon)}|W_t\right] = -\mathrm{E}\left[e^{-\gamma(W_t+\varepsilon)}|W_t\right]$$

Since $-\gamma(W_t+\varepsilon)$ follows Normal distribution with mean $-\gamma(W_t+\mu)$ and variance $\gamma^2\sigma^2$, we can use the properties of log-Normal distribution as below.

$$X \sim N(\mu_X, \sigma_X^2) \Rightarrow \mathrm{E}(e^X) = e^{\mu_X + \sigma_X^2/2}, \mathrm{Var}(e^X) = \left(e^{\sigma_X^2} - 1\right)e^{2\mu_X + \sigma_X^2}$$

Therefore,

$$-\mathrm{E}\left[e^{-\gamma(W_t+\varepsilon)}|W_t\right] = -e^{-\gamma(W_t+\mu)+\frac{\gamma^2\sigma^2}{2}} = -e^{-\gamma W_t - \gamma\left(\mu - \frac{\gamma\sigma^2}{2}\right)}$$

Then, the expected utility is strictly increasing in $\mu - \gamma\sigma^2/2$.

Similar model

$$W_{t+1} = W_t R$$
$$\log R \sim N(\mu, \sigma^2)$$
$$v(W) = \frac{W^{1-\rho} - 1}{1-\rho}$$

Similar conclusion

$$E[v(W_{t+1})|W_t] = E[v(W_t R)|W_t] = E\left[\frac{(W_t R)^{1-\rho} - 1}{1-\rho}|W_t\right] = \frac{1}{1-\rho}E[(W_t R)^{1-\rho}|W_t] - \frac{1}{1-\rho}$$

Since $(1-\rho)(\log W_t + \log R)$ follows Normal distribution with mean $(1-\rho)(\log W_t + \mu)$ and variance $(1-\rho)^2\sigma^2$,

$$\begin{aligned}\frac{1}{1-\rho}E[(W_t R)^{1-\rho}|W_t] - \frac{1}{1-\rho} &= \frac{1}{1-\rho}e^{(1-\rho)(\log W_t + \mu) + \frac{(1-\rho)^2\sigma^2}{2}} - \frac{1}{1-\rho} \\ &= \frac{1}{1-\rho}e^{(1-\rho)\log W_t + (1-\rho)\left(\mu - \frac{\rho-1}{2}\sigma^2\right)} - \frac{1}{1-\rho} \\ &= \frac{1}{1-\rho}\left(e^{(1-\rho)\log W_t + (1-\rho)\left(\mu - \frac{\rho-1}{2}\sigma^2\right)} - 1\right)\end{aligned}$$

Hence the expected utility is strictly increasing in $\mu-(\rho-1)\sigma^2/2$.

Definition *States of Nature* are potential outcomes.

Let s be an arbitrary state of nature, and
let S be the set of all possible states,
let $f(s)$ be a pdf describing the probability that states occur,
let $x(s)$ represent the payoff that a person receives in the event that state s occurs.

Normally we would then be interested in an expected utility of the form,

$$u = \int_{s \in S} v[x(s)]f(s)ds$$

However, this can be generalized by allowing the von Neumann–Morgenstern utility function to be “state dependent.”

$$u' = \int_{s \in S} v[x(s), s]f(s)ds$$

Mainly, interesting if

$$\frac{\partial^2 v}{\partial x \partial s} \neq 0$$

Example Two states h (healthy) and i (ill). Let $P(h) \equiv p$, $P(i) \equiv 1-p$,

If state h occurs, get wealth $W(h)$, if state i occurs, get wealth $W(i)<W(h)$.

Fair insurance is available—each unit of insurance pays $W(h)-W(i)$ if state i occurs.

Equilibrium price per unit of insurance=expected payoff; let t be the price,

$$t = (1-p)[W(h) - W(i)]$$

(Definition: price of fair insurance=expected payoff)
Model: Utility NOT State Dependent

Consumer's problem

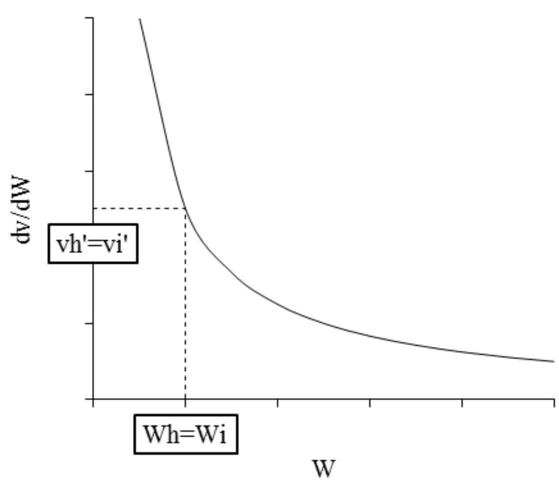
$$\max_n pv[W(h) - tn] + (1-p)v\{W(i) - tn + [W(h) - W(i)]n\}$$

First Order Condition

$$p \times v'[W(h) - tn] \times (-t) + (1 - p) \times v'\{W(i) - tn + [W(h) - W(i)]n\} \times (-t) + [W(h) - W(i)] = 0$$
$$\Rightarrow -pv_h't - (1-p)v_i'[t - W(h) + W(i)] = 0$$

Since $(1-p)(-t)=-t+pt$ and $t=(1-p)[W(h)-W(i)]$,

$$-pv_h't + v_i'(pt - t) + v_i't = 0$$
$$\Rightarrow pv_h't = pv_i't$$
$$\Rightarrow v_h' = v_i'$$



Therefore,

$$W(h) - tn = W(i) - tn + [W(h) - W(i)]n \Rightarrow n^* = 1$$

Model: State Dependent Utility

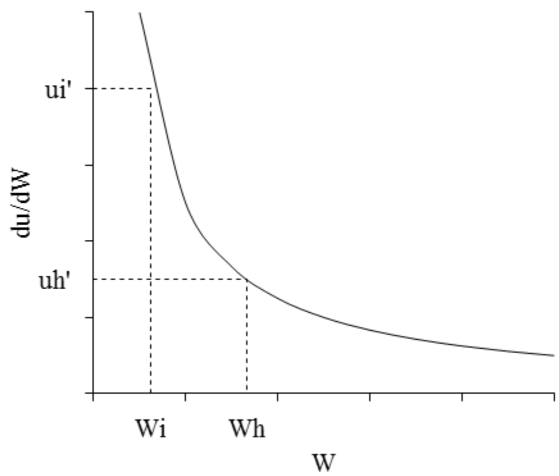
$$v[W(s), s] = \phi(s)u[W(s)] \text{ where } \phi(i) < \phi(h) \equiv 1$$

Problem

$$\max_n pu[W(h) - tn] + (1-p)\phi\,(i)u\{W(i) - tn + [W(h) - W(i)]n\}$$

FOC

$$\begin{aligned} -ptu_h' - (1-p)\phi(i)[t - W(h) + W(i)]u_i' &= 0 \\ \Rightarrow -ptu_h' - \phi(i)tu_i' + \phi(i)ptu_i' + \phi(i)tu_i' &= 0 \\ \Rightarrow -ptu_h' + \phi(i)ptu_i' &= 0 \\ \Rightarrow u_h' &= \phi(i)u_i' \end{aligned}$$



Note What if

$$v[W(s), s] = u[W(s)] + \phi(s)$$

i.e. not for marginal utility, but directly for utility.

$$\max_n p\big(u(W(h) - tn) + \phi(h)\big) + (1-p)\left(u\big(W(i) - tn + \big(W(h) - W(i)\big)n\big) + \phi(i)\right)$$
$$\Rightarrow pu_h'(-t) + (1-p)u_i'(W(h) - W(i) - t) = 0$$
$$\Rightarrow pu_h'(-t) + (1-p)u_i'\big(W(h) - W(i)\big) - (1-p)u_i't = 0$$
$$\Rightarrow pu_h'(-t) + u_i't - (1-p)u_i't = 0$$
$$\Rightarrow pu_h'(-t) + pu_i't = 0 \Rightarrow \therefore u_h' = u_i'$$

Let $\{\varepsilon\}$ be the possible states. $\varepsilon \sim F(\varepsilon)$
Define M to be a person's budget, which they will use to purchase a market basket of amounts of consumption in each state $c(\varepsilon)$.
Let $\pi(\varepsilon)$ be the price of per unit of consumption in state ε.
Let $u[c(\varepsilon),\varepsilon]$ be von Neumann–Morgenstern utility function.
Consumer's problem then

$$\max_{\{c(\varepsilon)\}} \int u(c(\varepsilon),\varepsilon) f(\varepsilon) d\varepsilon \text{ s.t.} \int \pi(\varepsilon) c(\varepsilon) d\varepsilon = M$$

FOC

$$\{c(\varepsilon)\}: \frac{\partial u(c(\varepsilon),\varepsilon)}{\partial c} f(\varepsilon) = \lambda \pi(\varepsilon)$$

Interesting special case: $\pi(\varepsilon)=\alpha f(\varepsilon)$ for all ε, then FOC

$$\frac{\partial u(c(\varepsilon),\varepsilon)}{\partial c} = \alpha\lambda = (\text{constant})$$

If $\partial^2 u/\partial c \partial \varepsilon=0$, (i.e. utility is not state-dependent.)

$$\frac{\partial u(c(\varepsilon),\varepsilon)}{\partial c} = (\text{constant}) \Rightarrow c(\varepsilon) = \bar{c}, \forall \varepsilon$$

Otherwise,

$$\frac{\partial^2 u}{\partial c^2}\frac{dc}{d\varepsilon} + \frac{\partial^2 u}{\partial c \partial \varepsilon} = 0 \Rightarrow \frac{dc}{d\varepsilon} = -\left(\frac{\partial^2 u}{\partial c^2}\right)^{-1} \frac{\partial^2 u}{\partial c \partial \varepsilon} \therefore \text{Sign}\left(\frac{dc}{d\varepsilon}\right) = \text{Sign}\left(\frac{\partial^2 u}{\partial c \partial \varepsilon}\right)$$

Example: $u(c,\varepsilon)=\varphi(c)+\gamma(\varepsilon)$
Intuitive explanation: sick state ⇒ marginal utility of money is high ⇒ will prepare money for the state

Definition A distribution F first-order stochastically dominates another distribution G if for every non-decreasing function u;

$$\mathrm{E}_F[u(X)] = \mathrm{E}_G[u(X)] \Leftrightarrow \int u(x)\frac{\partial F(x)}{\partial x}dx = \int u(x)\frac{\partial G(x)}{\partial x}dx$$

Implication Everyone prefers a lottery with probabilities F over a lottery with probabilities G.
Equivalent Mathematical Condition

$$F(x) \leq G(x)\ \forall x \Leftrightarrow \text{i.e. } 1 - F(x) \geq 1 - G(x)$$

Claim F FOSD $G \Rightarrow \mathrm{E}_F(X) \geq \mathrm{E}_G(X)$
Proof

$$\begin{aligned}\mathrm{E}_F(X) = \int_{-\infty}^{\infty} x\frac{\partial F}{\partial x}dx &= \int_{-\infty}^{0} x\frac{\partial F}{\partial x}dx + \int_{0}^{\infty} x\frac{\partial F}{\partial x}dx\\ &= xF(x)|_{-\infty}^{0} - \int_{-\infty}^{0} F(x)dx + x[F(x)-1]|_0^{\infty} - \int_0^{\infty}[F(x)-1]dx\\ &= -\int_{-\infty}^{0} F(x)dx - \int_0^{\infty}[F(x)-1]dx\end{aligned}$$

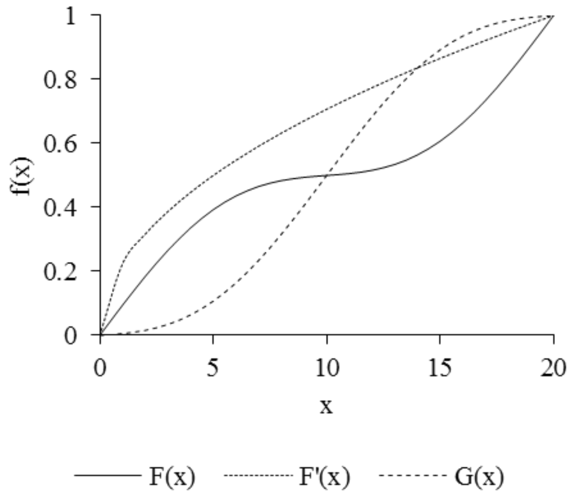
$$\Rightarrow \mathrm{E}_F(X) - \mathrm{E}_G(X) = \int_{-\infty}^{0}[G(x)-F(x)]dx + \int_0^{\infty}[G(x)-F(x)]dx = \int_{-\infty}^{\infty}\underbrace{[G(x)-F(x)]}_{\geq 0\ \forall x \text{ if } F \text{ FOSD } G}dx \geq 0 \ \blacksquare$$

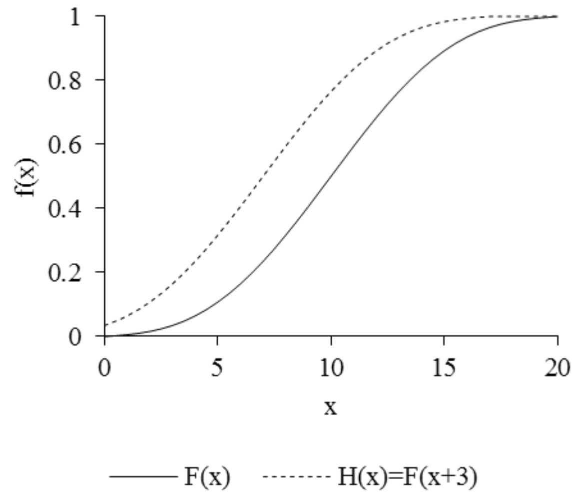
Therefore, if F FOSD G, then $E_F(X) \geq E_G(X)$, but the converse is not always true.
Definition A distribution F second-order stochastically dominates another distribution G if for every non-decreasing concave function u;

$$\mathrm{E}_F[u(X)] = \mathrm{E}_G[u(X)] \Leftrightarrow \int u(x)\frac{\partial F(x)}{\partial x}dx = \int u(x)\frac{\partial G(x)}{\partial x}dx$$

Implication All risk-averse people prefer lotteries with probabilities F over lotteries with probabilities G.
Definition Let $X \sim F(X)$, let ε be a mean-zero random variable, and let the sum $X+\varepsilon \sim G(X+\varepsilon)$. Then, we say that G is a mean-preserving spread of F (maybe iid).
Theorem (Rothschild and Stiglitz) Let F and G be two distributions with the same mean. Then, the followings are equivalent.
1. F SOSD G.
2. G is a mean-preserving spread of F.
3.

$$\int_{-\infty}^{a}[G(x)-F(x)]dx \geq 0\ \forall a$$





F FOSD *F′* and *G* SOSD *F*

H: same shape with *F*, same mean with *G*

Note If *F* and *G* have the same mean,

$$\int_{-\infty}^{\infty}[G(x)-F(x)]dx = 0$$

Problem: What if their means are not the same? i.e. $E_F(X)-E_G(X)=b>0$?

Idea: Define $H(x)=F(x+b)$. Then, *H* has the same mean as *G* know for sure that *F* FOSD *H*.

⇒ Then, if *H* SOSD *G*, we can conclude that *F* SOSD *G*.

About Problem Set

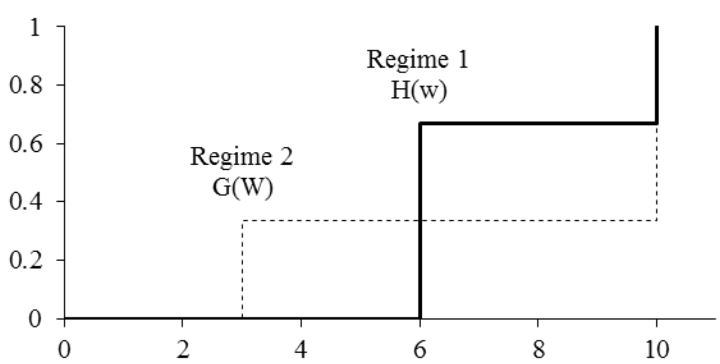
Regime 1: Fine F_1, Probability P_1

Regime 2: $F_2>F_1$, $P_2<P_1$ (High Fine with Low Probability)

Then, potential criminal's problem is,

$$\max_{\{\text{obey law,do not obey}\}}\left\{\mathrm{E}[u(\text{obey})]=v(W), \mathrm{E}[u(\text{not obey})]=(1-P)\left[v(W)+\underbrace{b}_{\substack{\text{benefit}\\\text{of speeding}}}\right]+P[v(W-F)+b]\right\}$$

1. $E[u(\text{speed})]-E[u(\text{obey})]$ under Regime $1=(1-P_1)[v(W)+b]+P_1[v(W-F_1)+b]$
2. $E[u(\text{speed})]-E[u(\text{obey})]$ under Regime $2=(1-P_2)[v(W)+b]+P_2[v(W-F_2)+b]$
3. $1-2=(P_2-P_1)v(W)+P_1v(W-F_1)-P_2v(W-F_2)$: if positive, it is more attractive to speed under Regime 1.



$$\int_{-\infty}^{\infty}(G-H)dw = 0$$

$$\int_{-\infty}^{a}(G-H)dw \geq 0\ \forall a$$

Thus, *H* SOSD *G*.

Speeder prefers regime 1 over regime 2.

If I can speed more by paying a dollar, then everyone will pay it to speed.

Economics 801 LN 13

Situation: Price changes from $p_1 \rightarrow p_2$.
Levels of utility: $u_1 = v(p_1, M)$, $u_2 = v(p_2, M)$

Compensating variation

$$e(p_2, u_1) - e(p_1, u_1) = \int_{p_1}^{p_2} \frac{\partial e}{\partial p}(p, u_1) dp = \int_{p_1}^{p_2} h(p, u_1) dp$$

Equivalent variation

$$e(p_2, u_2) - e(p_1, u_2) = \int_{p_1}^{p_2} \frac{\partial e}{\partial p}(p, u_2) dp = \int_{p_1}^{p_2} h(p, u_2) dp$$

Shephard's lemma

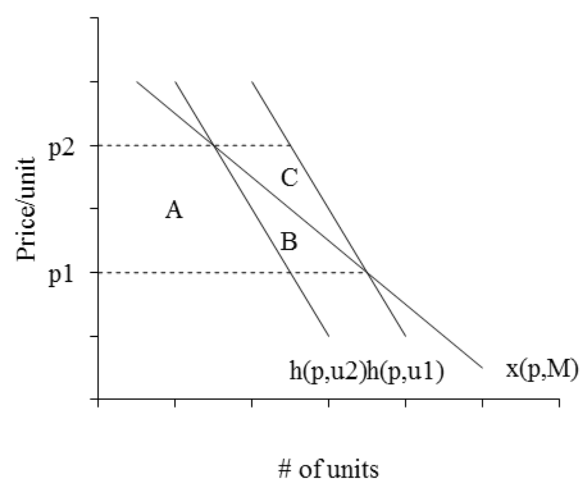
$$\frac{\partial e}{\partial p} = h(p, u)$$

Note: Change of consumer surplus

$$\Delta CS = \int_{p_1}^{p_2} x(p, M) dp$$

Slutsky's theorem

$$\frac{\partial h}{\partial p} = \frac{\partial x}{\partial p} + \frac{\partial x}{\partial M} \underbrace{\frac{\partial e}{\partial p}}_{=h}$$



Hicksian Demand for Normal Good; A=Equivalent Variation
A+B=Change of Consumer Surplus, A+B+C=Compensating Variation

1. Utility Function

$$U(x_1, x_2) = x_1 x_2$$

Budget Constraint

$$p_1 x_1 + p_2 x_2 = y$$

Lagrange Function

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(y - p_1 x_1 - p_2 x_2)$$

(a) Derive the first order and the second order conditions for the consumer equilibrium.
First-order conditions (gradient **g**)

$$\begin{pmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \\ \frac{\partial L}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} x_2 - \lambda p_1 \\ x_1 - \lambda p_2 \\ y - p_1 x_1 - p_2 x_2 \end{pmatrix} = \mathbf{0}$$

Second-order conditions (hessian **H**)

$$\begin{pmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_1} \\ \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial \lambda \partial x_2} \\ \frac{\partial^2 L}{\partial x_1 \partial \lambda} & \frac{\partial^2 L}{\partial x_2 \partial \lambda} & \frac{\partial^2 L}{\partial \lambda^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -p_1 \\ 1 & 0 & -p_2 \\ -p_1 & -p_2 & 0 \end{pmatrix}$$

Determinant $|\mathbf{H}|=0(0+p_1p_2)-1(0-p_1p_2)-p_1(-p_2-0)=2p_1p_2>0$ and hence the utility is maximized.

(b) Derive the ordinary demand functions for x_1 and x_2.

$$x_2 - \lambda p_1 = 0 \Rightarrow \lambda = \frac{x_2}{p_1}$$

$$x_1 - \lambda p_2 = 0 \Rightarrow \lambda = \frac{x_1}{p_2}$$

$$\Rightarrow \frac{x_2}{p_1} = \frac{x_1}{p_2} \Rightarrow p_1 x_1 = p_2 x_2$$

$$y - p_1 x_1 - p_2 x_2 = y - 2p_1 x_1 = 0 \Rightarrow x_1 = \frac{y}{2p_1}, x_2 = \frac{y}{2p_2}$$

(c) Derive the compensated demand functions for x_1 and x_2.

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } \bar{U} = x_1x_2$$

Lagrange Function

$$L = p_1x_1 + p_2x_2 + \lambda(\bar{U} - x_1x_2)$$

First-order conditions

$$p_1 - \lambda x_2 = 0$$
$$p_2 - \lambda x_1 = 0$$
$$\bar{U} - x_1x_2 = 0$$

$$\Rightarrow \lambda = \frac{p_1}{x_2} = \frac{p_2}{x_1} \Rightarrow p_1x_1 = p_2x_2$$
$$\Rightarrow x_2 = \frac{\bar{U}}{x_1} \Rightarrow p_1x_1 = p_2\frac{\bar{U}}{x_1}$$
$$\Rightarrow x_1^2 = \frac{p_2}{p_1}\bar{U} \Rightarrow x_1 = \sqrt{\frac{p_2}{p_1}\bar{U}}, x_2 = \sqrt{\frac{p_1}{p_2}\bar{U}}$$

(d) Derive the indirect utility function.

$$x_1 = \frac{y}{2p_1}, x_2 = \frac{y}{2p_2} \Rightarrow \text{indirect } U = \frac{y}{2p_1}\frac{y}{2p_2} = \frac{y^2}{4p_1p_2}$$

(e) Suppose that p_1=3, p_2=6, and y=180. Find the optimal x_1 and x_2.

$$x_1 = \frac{y}{2p_1} = \frac{180}{2 \times 3} = 30$$
$$x_2 = \frac{y}{2p_2} = \frac{180}{2 \times 6} = 15$$

(f) Suppose that prices are changed to p_1=5 and p_2=5. Find the optimal x_1 and x_2, and compute the income compensation that would leave the consumer on the same level of utility.

$$x_1 = x_2 = \frac{180}{2 \times 5} = 18$$
$$\text{after price change } U = x_1x_2 = 18^2 = 324$$
$$\text{before price change } U = 30 \times 15 = 450$$
$$\Rightarrow U = \frac{y^2}{4 \times 5 \times 5} = 450$$
$$\Rightarrow y^2 = 45000 \Rightarrow y = \sqrt{45000} \approx 212.1320$$

2. Expected Utility Function

$$V(A) = \max_x xP(x, A)$$
$$\text{where } P = \text{probability of correct answer}$$
$$\text{and } \frac{\partial P}{\partial A} > 0, \frac{\partial P}{\partial x} < 0$$

(a) Is it possible that the player's optimal choice x^* is non-positive (i.e., $x^* \leq 0$)? Why or why not?
To maximize expected utility,

$$\frac{\partial V}{\partial x} = P + x\frac{\partial P}{\partial x} = 0$$
$$\Rightarrow x\frac{\partial P}{\partial x} = -P \Rightarrow x^* = -\underbrace{\left(\frac{\partial P}{\partial x}\right)^{-1}}_{\text{negative}} \underbrace{P}_{\text{positive}} \geq 0$$

$\partial P/\partial x<0$ is negative by the given article, and P is positive (or can be equal to zero) by Axioms of Probability. If P is strictly positive, then the optimal choice x^* of the player who maximizes his/her expected utility cannot have a non-positive value.

(b) Show that $dx^*/dA \overset{>}{\underset{<}{=}} 0$ as $\partial^2 \log P/\partial x \partial A \overset{>}{\underset{<}{=}} 0$.

$$x^* = -\frac{P}{\frac{\partial P}{\partial x}} = -\frac{1}{\frac{\partial P}{\partial x}\frac{1}{P}} = -\frac{1}{\frac{\partial P}{\partial x}\frac{\partial \log P}{\partial P}} = -\frac{1}{\frac{\partial \log P}{\partial x}} = -\left(\frac{\partial \log P}{\partial x}\right)^{-1}$$
$$\Rightarrow \frac{dx^*}{dA} = \left(\frac{\partial \log P}{\partial x}\right)^{-2}\frac{\partial^2 \log P}{\partial x \partial A} = \frac{\partial^2 \log P}{\partial x \partial A} \Big/ \left(\frac{\partial \log P}{\partial x}\right)^2$$

Therefore, the sign of dx^*/dA follows the sign of $\partial^2 \log P/\partial x \partial A$.
(c) Prove that players prefer questions in subjects where they know more ($dV/dA>0$). Be clear.
By using x^*, the indirect expected utility can be obtained.

$$V(x|A) = xP(x, A)$$
$$\frac{dV}{dA} = \left[\frac{\partial}{\partial A}xP(x, A)\right]_{x=x^*} + \underbrace{\frac{\partial}{\partial x}xP(x, A)}_{\text{zero}}\frac{dx^*}{dA} \text{ by Envelope theorem}$$
$$= x^*\frac{\partial P}{\partial A} > 0$$

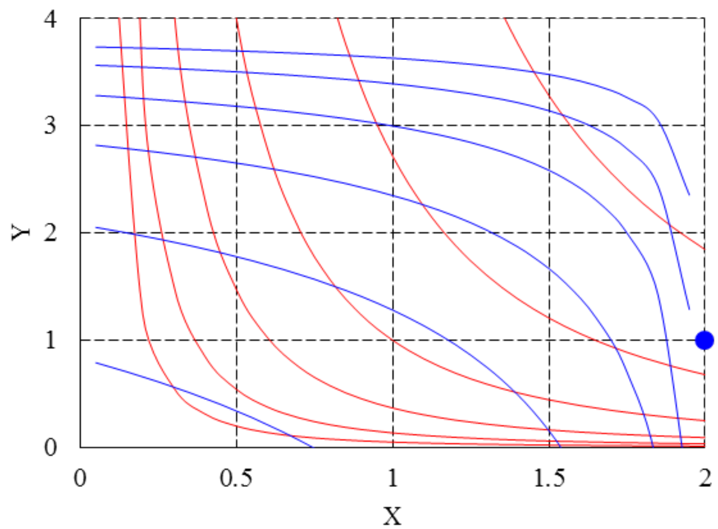
Therefore, dV/dA is positive.

3. Two goods (X,Y) with two consumers (1,2) case
Utility functions

$$U^1(X_1, Y_1) = 2\log X_1 + \log Y_1$$
$$U^2(X_2, Y_2) = \log X_2 + 2\log Y_2$$

(a) Draw the Edgeworth box diagram for this exchange economy.

$$\log Y_1 = \bar{U}^1 - 2\log X_1 \Rightarrow Y_1 = e^{\bar{U}^1 - 2\log X_1}$$
$$2\log Y_2 = \bar{U}^2 - \log X_2 \Rightarrow Y_2 = e^{\frac{\bar{U}^2 - \log X_2}{2}}$$



Edgeworth Box: Endowment and Indifference Curves

(b) What would be the competitive equilibrium?

$$U^1 = 2\ln X_1 + \ln Y_1$$
$$U^2 = \ln X_2 + 2\ln Y_2$$

Constraints

$$M^1 = P_X X_1 + P_Y Y_1 = 2P_X + P_Y$$
$$M^2 = P_X X_2 + P_Y Y_2 = 3P_Y$$

Or simply with P=P_Y/P_X,

$$X_1 + PY_1 = 2 + P$$
$$X_2 + PY_2 = 3P$$

Endowments

$$X_1 + X_2 = 2 + 0 = 2$$
$$Y_1 + Y_2 = 1 + 3 = 4$$

Maximizing utility of the first consumer

$$\frac{MU_X^1}{P_X} = \frac{MU_Y^1}{P_Y} \Rightarrow \frac{2}{X_1 P_X} = \frac{1}{Y_1 P_Y} \Rightarrow 2P_Y Y_1 = P_X X_1 \Rightarrow X_1 = 2PY_1$$

Plugging this into the constraint

$$3PY_1 = 2 + P \Rightarrow Y_1 = \frac{2+P}{3P}$$

Maximizing utility of the second consumer

$$\frac{1}{X_2 P_X} = \frac{2}{Y_2 P_Y} \Rightarrow P_Y Y_2 = 2P_X X_2 \Rightarrow X_2 = \frac{1}{2} PY_2$$

Plugging

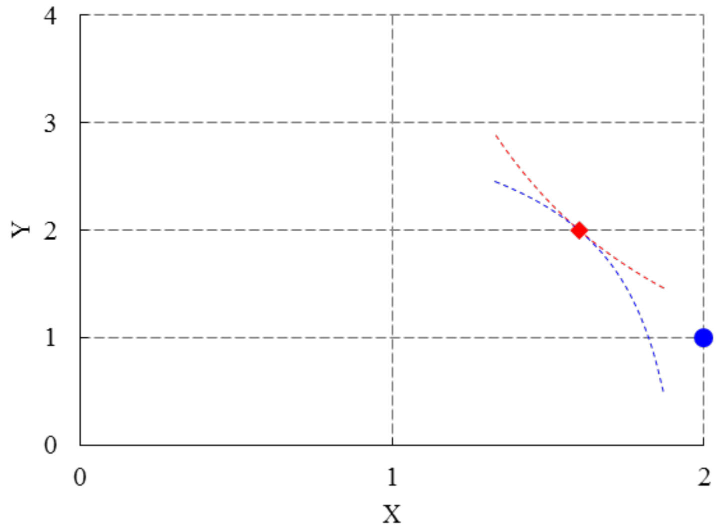
$$\frac{3}{2} PY_2 = 3P \Rightarrow Y_2 = 2 \Rightarrow Y_1 = 4 - 2 = 2$$
$$\Rightarrow X_1 = 2P \times 2 = 4P, X_2 = \frac{1}{2} P \times 2 = P$$

By considering the endowments,

$$X_1 + X_2 = 4P + P = 5P = 2 \Rightarrow P = \frac{2}{5}$$
$$\Rightarrow X_1 = \frac{8}{5}, X_2 = \frac{2}{5}$$

Therefore, the competitive equilibrium is

$$(X_1, X_2, Y_1, Y_2) = \left(\frac{8}{5}, \frac{2}{5}, 2,2\right) \text{ with } P = \frac{P_Y}{P_X} = \frac{2}{5}$$
$$\text{and } U^1 = 2\ln\frac{8}{5} + \ln 2 \approx 1.6332, U^2 = \ln\frac{2}{5} + 2\ln 2 \approx 0.4700$$



Edgeworth Box: Endowments and Competitive Equilibrium

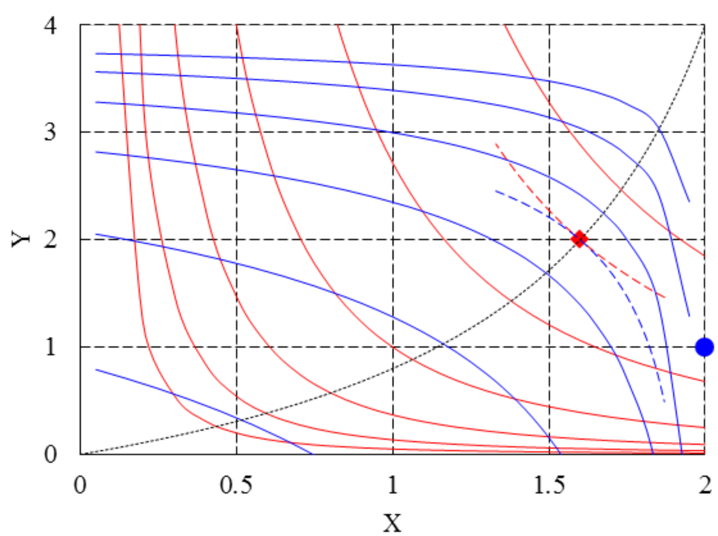
(c) Draw the contract curve in the Edgeworth diagram.

Contract curve can be obtained by below.

$$MRS_{XY}^1 = MRS_{XY}^2$$

$$\Rightarrow -\frac{\frac{\partial U^1}{\partial X_1}}{\frac{\partial U^1}{\partial Y_1}} = -\frac{\frac{\partial U^2}{\partial X_2}}{\frac{\partial U^2}{\partial Y_2}} \Rightarrow \frac{\frac{2}{X_1}}{\frac{1}{Y_1}} = \frac{\frac{1}{X_2}}{\frac{2}{Y_2}} \Rightarrow \frac{2Y_1}{X_1} = \frac{Y_2}{2X_2} = \frac{4 - Y_1}{2(2 - X_1)} \text{ by using endowments}$$

$$\Rightarrow 4Y_1(2 - X_1) = X_1(4 - Y_1) \Rightarrow 8Y_1 - 4X_1Y_1 + X_1Y_1 = 4X_1 \Rightarrow Y_1 = \frac{4X_1}{8 - 3X_1}$$



Edgeworth Box: Contract Curve

Deaton and Muellbauer (1980, AER)

Expenditure Function

$$\log e(\mathbf{p},u) = a(\mathbf{p}) + u \cdot b(\mathbf{p})$$

(a) Show that for any arbitrary expenditure function (not just in the AIDS specification)
From Shepard's Lemma,

$$\begin{aligned} e(\mathbf{p},u) &= \min_{\mathbf{h}} \mathbf{p}'\mathbf{h} \text{ s.t.} u(\mathbf{h}) = u^* \\ &= \mathbf{p}'\mathbf{h}^*(\mathbf{p},u^*) \\ \Rightarrow \frac{\partial e}{\partial p_i} &= h_i^*(\mathbf{p},u^*) \\ \Rightarrow \frac{p_i}{e}\frac{\partial e}{\partial p_i} &= \frac{\partial \log e}{\partial \log p_i} = \frac{p_i h_i}{e} = \omega_i \blacksquare \end{aligned}$$

(b) Use equation (1) and (2), to derive functions $\alpha_i(\mathbf{p})$ and $\beta_i(\mathbf{p})$. Describing a log-linear relationship between ω_i and M:

$$\omega_i = \alpha_i(\mathbf{p}) + \beta_i(\mathbf{p}) \log M$$

Here, the equation (1) is an expenditure function $e(\mathbf{p},u)$. Find an indirect utility function $v(\mathbf{p},M)$ by using this expenditure function.

$$\begin{aligned} \log e(\mathbf{p},u) &= a(\mathbf{p}) + u \cdot b(\mathbf{p}) \\ \Rightarrow v(\mathbf{p},M) &= \frac{\log M}{b(\mathbf{p})} - \frac{a(\mathbf{p})}{b(\mathbf{p})} \end{aligned}$$

By using the equation (2),

$$\begin{aligned} \omega_i = \frac{\partial \log e}{\partial \log p_i} &= \frac{\partial a}{\partial p_i}\underbrace{\frac{\partial p_i}{\partial \log p_i}}_{=p_i} + \underbrace{u}_{=v(\mathbf{p},M)} \times \frac{\partial b}{\partial p_i}\underbrace{\frac{\partial p_i}{\partial \log p_i}}_{=p_i} \\ &= \frac{\partial a}{\partial p_i}p_i + \frac{\partial b}{\partial p_i}p_i\left[\frac{\log M}{b(\mathbf{p})} - \frac{a(\mathbf{p})}{b(\mathbf{p})}\right] \\ &= \underbrace{\left[\frac{\partial a}{\partial p_i} - \frac{\partial b}{\partial p_i}\frac{a(\mathbf{p})}{b(\mathbf{p})}\right]p_i}_{=\alpha_i(\mathbf{p})} + \underbrace{\frac{\partial b}{\partial p_i}\frac{p_i}{b(\mathbf{p})}}_{=\beta_i(\mathbf{p})}\log M \\ &= \alpha_i(\mathbf{p}) + \beta_i(\mathbf{p}) \log M \end{aligned}$$

Therefore,

$$\alpha_i(\mathbf{p}) = \left[\frac{\partial a}{\partial p_i} - \frac{\partial b}{\partial p_i}\frac{a(\mathbf{p})}{b(\mathbf{p})}\right]p_i$$
$$\beta_i(\mathbf{p}) = \frac{\partial b}{\partial p_i}\frac{p_i}{b(\mathbf{p})} \ \blacksquare$$

(c) The AIDS model specifies $a(\mathbf{p})$ and $b(\mathbf{p})$ as follows:

$$a(\mathbf{p}) = \alpha_0 + \sum_{j=1}^{n}\alpha_j \log p_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_j \log p_k$$
$$b(\mathbf{p}) = \beta_0 \prod_{j=1}^{n} p_j^{\beta_j}$$

Explain why this form for $a(\mathbf{p})$ can be viewed as a second-order Taylor approximation in logs to any arbitrary function of prices $f(\mathbf{p})$. (This is called a "trans-log" form, and it is commonly used in a variety of applications.)

For an arbitrary function $f(\mathbf{p})$,

$$f(\mathbf{p}) = f\left(e^{\log \mathbf{p}}\right) = g(\log \mathbf{p})$$

By applying Taylor expansion,

$$\begin{aligned}
g(\log \mathbf{p}) &= g(\mathbf{0}) + \left.\frac{\partial g}{\partial \log \mathbf{p}}\right|_{\log \mathbf{p}=\mathbf{0}} \log \mathbf{p} + \frac{1}{2}(\log \mathbf{p})' \left.\frac{\partial^2 g}{\partial \log \mathbf{p}\, \partial (\log \mathbf{p})'}\right|_{\log \mathbf{p}=\mathbf{0}} \log \mathbf{p} + \cdots \\
&\approx \underbrace{g(\mathbf{0})}_{=\alpha_0} + \sum_{j=1}^{n} \underbrace{\left.\frac{\partial g}{\partial \log p_j}\right|_{\log \mathbf{p}=\mathbf{0}}}_{=\alpha_j} \log p_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \underbrace{\left.\frac{\partial^2 g}{\partial \log p_j\, \partial \log p_k}\right|_{\log \mathbf{p}=\mathbf{0}}}_{=\gamma_{jk}^{*}} \log p_j \log p_k \\
&= \alpha_0 + \sum_{j=1}^{n}\alpha_j \log p_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_j \log p_k \\
&= a(\mathbf{p}) \\
\Rightarrow a(\mathbf{p}) &\approx f(\mathbf{p}) \ \blacksquare
\end{aligned}$$

(d) Define $\gamma_{ij}\equiv(\gamma_{ij}^*+\gamma_{ji}^*)/2$. Show that

$$\omega_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j + \beta_i \log(M/P)$$

Where we have defined

$$\log P \equiv \alpha_j + \sum_{j=1}^{n} \alpha_j \log p_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \gamma_{jk} \log p_j \log p_k = a(\mathbf{p})$$

Note that equation (6) describes a simple regression with fixed parameters, so it is straightforward to estimate.

From (b),

$$\alpha_i(\mathbf{p}) = \left[\frac{\partial a}{\partial p_i} - \frac{\partial b}{\partial p_i}\frac{a(\mathbf{p})}{b(\mathbf{p})}\right] p_i = A_i(\mathbf{p})$$
$$\beta_i(\mathbf{p}) = \frac{\partial b}{\partial p_i}\frac{p_i}{b(\mathbf{p})} = B_i(\mathbf{p})$$

To avoid a possible confusion, I altered $\alpha_i(\mathbf{p})$ and $\beta_i(\mathbf{p})$ by $A_i(\mathbf{p})$ and $B_i(\mathbf{p})$. Then,

$$\frac{\partial a}{\partial p_i} p_i = \frac{\partial a}{\partial \log p_i} = \alpha_i + \frac{1}{2}\sum_{j=1}^{n} \gamma_{ji}^* \log p_j + \frac{1}{2}\sum_{k=1}^{n} \gamma_{ik}^* \log p_k = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j$$
$$\frac{\partial b}{\partial p_i} p_i = \beta_0 \prod_{j\neq i}^{n} p_j^{\beta_j} \cdot \beta_i p_i^{\beta_i - 1} p_i = \beta_0 \beta_i \prod_{j=1}^{n} p_j^{\beta_j} = \beta_i b(\mathbf{p})$$
$$\Rightarrow A_i(\mathbf{p}) = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j - \beta_i b(\mathbf{p}) \frac{a(\mathbf{p})}{b(\mathbf{p})} = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j - \beta_i a(\mathbf{p})$$
$$\Rightarrow B_i(\mathbf{p}) = \beta_i b(\mathbf{p}) \frac{1}{b(\mathbf{p})} = \beta_i$$
$$\begin{aligned} \Rightarrow \omega_i &= A_i(\mathbf{p}) + B_i(\mathbf{p}) \log M \\ &= \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j - \beta_i \underbrace{a(\mathbf{p})}_{\equiv \log P} + \beta_i \log M \\ &= \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j + \beta_i \log(M/P) \ \blacksquare \end{aligned}$$

(e) Show that under assumptions (4) and (5), homogeneity of $e(\mathbf{p},u)$ requires

$$\sum_{j=1}^{n}\alpha_j = 1 \text{ and } \sum_{j=1}^{n}\gamma_{jk} = \sum_{j=1}^{n}\beta_j = 0$$

Homogeneity: $e(t\mathbf{p},u)=te(\mathbf{p},u)\ \forall\ t>0$,

$$\begin{aligned}\log e(t\mathbf{p},u) &= a(t\mathbf{p}) + u\cdot b(t\mathbf{p})\\ &= \log[te(\mathbf{p},u)] \text{ if homogenous}\\ &= \log t + a(\mathbf{p}) + u\cdot b(\mathbf{p})\end{aligned}$$

Then,

$$\begin{aligned}a(t\mathbf{p}) &= \alpha_0 + \sum_{j=1}^{n}\alpha_j \log tp_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log tp_j \log tp_k\\
&= \alpha_0 + \sum_{j=1}^{n}\alpha_j(\log t + \log p_j) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}(\log t + \log p_j)(\log t + \log p_k)\\
&= \alpha_0 + \sum_{j=1}^{n}\alpha_j \log p_j + \log t\sum_{j=1}^{n}\alpha_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\left[(\log t)^2 + \log t\left(\log p_j + \log p_k\right) + \log p_j \log p_k\right]\\
&= \alpha_0 + \sum_{j=1}^{n}\alpha_j \log p_j + \underbrace{\log t\sum_{j=1}^{n}\alpha_j + \frac{(\log t)^2}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*} + \frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}(\log p_j + \log p_k)}_{=\log t \text{ if homogenous}} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_j \log p_k\\
&= \alpha_0 + \sum_{j=1}^{n}\alpha_j \log p_j + \underbrace{\log t\sum_{j=1}^{n}\alpha_j}_{1)} + \underbrace{\frac{(\log t)^2}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}}_{4)} + \underbrace{\frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}(\log p_j + \log p_k)}_{3)} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_j \log p_k\\
&= \alpha_0 + \sum_{j=1}^{n}\alpha_j \log p_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_j \log p_k + \log t = a(\mathbf{p}) + \log t \text{ if homogenous}\end{aligned}$$

And,

$$b(t\mathbf{p}) = \beta_0\prod_{j=1}^{n}(tp_j)^{\beta_j} = \beta_0\,\underbrace{t^{\sum\beta_j}}_{2)}\prod_{j=1}^{n}p_j^{\beta_j} = \beta_0\prod_{j=1}^{n}p_j^{\beta_j} = b(\mathbf{p}) \text{ if homogenous}$$

Therefore,

1) $\sum\alpha_j=1$

$$\log t\sum_{j=1}^{n}\alpha_j = \log t \Rightarrow \sum_{j=1}^{n}\alpha_j = 1$$

2) $\sum\beta_j=0$

$$t^{\sum\beta_j} = 1 \Rightarrow \sum_{j=1}^{n}\beta_j = 0$$

3) $\sum_{k=1}^{n}\gamma_{jk}^{*}=0$, $\sum_{j=1}^{n}\gamma_{jk}^{*}=0$

$$\frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\left(\log p_j + \log p_k\right) = 0$$

$$\Rightarrow \frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\left(\log p_j + \log p_k\right) = \frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_j + \frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_k$$

$$= \underbrace{\frac{\log t}{2}\sum_{j=1}^{n}\log p_j\sum_{k=1}^{n}\gamma_{jk}^{*}}_{3.1)} + \underbrace{\frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_k}_{3.2)}$$

$$3.1) \Rightarrow \frac{\log t}{2}\sum_{j=1}^{n}\log p_j\sum_{k=1}^{n}\gamma_{jk}^{*} = 0$$

$$\forall k \Rightarrow \sum_{k=1}^{n}\gamma_{jk}^{*} = 0$$

$$3.2) \Rightarrow \frac{\log t}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*}\log p_k = \frac{\log t}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\gamma_{jk}^{*}\log p_k$$

$$= \frac{\log t}{2}\sum_{k=1}^{n}\log p_k\sum_{j=1}^{n}\gamma_{jk}^{*} = 0$$

$$\forall j \Rightarrow \sum_{j=1}^{n}\gamma_{jk}^{*} = 0$$

4)

$$\Rightarrow \frac{(\log t)^2}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{jk}^{*} = \frac{(\log t)^2}{2}\sum_{j=1}^{n}0 = 0$$

Hence,

$$\sum_{j=1}^{n}\gamma_{jk} = \sum_{j=1}^{n}\frac{\gamma_{jk}^{*}+\gamma_{kj}^{*}}{2} = \frac{1}{2}\underbrace{\sum_{j=1}^{n}\gamma_{jk}^{*}}_{=0} + \frac{1}{2}\underbrace{\sum_{j=1}^{n}\gamma_{kj}^{*}}_{=0} = 0$$

Therefore,

$$\underbrace{\sum_{j=1}^{n}\alpha_j = 1}_{\text{constraint }\alpha},\underbrace{\sum_{j=1}^{n}\beta_j = 0}_{\text{constraint }\beta},\underbrace{\sum_{j=1}^{n}\gamma_{jk} = 0}_{\text{constraint }\gamma} \Rightarrow \forall t > 0, e(t\mathbf{p},u) = te(\mathbf{p},u)\ \blacksquare$$

(f) What constraints on ω_i fulfill the (i.) adding up, (ii.) homogeneity, and (iii.) symmetry requirements? Compare your answers to the requirements identified in equation (8).

i) Adding up requirements

$$\omega_i = \alpha_i + \sum_{j=1}^{n}\gamma_{ij}\log p_j + \beta_i \log(M/P)$$

$$\begin{aligned}
\Rightarrow \sum_{i=1}^{n}\omega_i &= \sum_{i=1}^{n}\left[\alpha_i + \sum_{j=1}^{n}\gamma_{ij}\log p_j + \beta_i\log(M/P)\right] \\
&= \sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\log p_j + \log(M/P)\sum_{i=1}^{n}\beta_i \\
&= \underbrace{\sum_{i=1}^{n}\alpha_i}_{=1} + \sum_{j=1}^{n}\log p_j \underbrace{\sum_{i=1}^{n}\gamma_{ij}}_{=0} + \log(M/P)\underbrace{\sum_{i=1}^{n}\beta_i}_{=0} \\
&= 1
\end{aligned}$$

All the three constraints, i.e. constraint α, β and γ, fulfill the requirement.

ii) Homogeneity

$$\begin{aligned}
\omega_i &= \alpha_i + \sum_{j=1}^{n}\gamma_{ij}\log p_j + \beta_i\log(M/P) \\
t\mathbf{p} \Rightarrow \omega_i &= \alpha_i + \sum_{j=1}^{n}\gamma_{ij}\log tp_j + \beta_i\log(M/P) \\
&= \alpha_i + \sum_{j=1}^{n}\gamma_{ij}\left(\log t + \log p_j\right) + \beta_i\log(M/P) \\
&= \alpha_i + \log t\underbrace{\sum_{j=1}^{n}\gamma_{ij}}_{=0} + \sum_{j=1}^{n}\gamma_{ij}\log p_j + \beta_i\log(M/P) \\
&= \alpha_i + \sum_{j=1}^{n}\gamma_{ij}\log p_j + \beta_i\log(M/P)
\end{aligned}$$

The third constraint, i.e. constraint γ, fulfills the requirement.

iii) Symmetry

$$\omega_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j + \beta_i \log(M/P)$$
$$\omega_i = \frac{p_i}{e} h_i \Rightarrow h_i = \frac{e}{p_i} \omega_i$$
$$\Rightarrow \frac{\partial \log h_i}{\partial \log p_j} = \frac{\partial \log h_i}{\partial h_i} \frac{\partial h_i}{\partial \omega_i} \frac{\partial \omega_i}{\partial \log p_j} = \frac{e}{h_i p_i} \gamma_{ij} = \frac{e}{\omega_i} \gamma_{ij}$$
$$\Rightarrow \frac{\partial \log h_j}{\partial \log p_i} = \frac{e}{\omega_j} \gamma_{ji}$$

Slutsky Symmetry in Elasticity is,

$$\omega_i \frac{\partial \log h_i}{\partial \log p_j} = \frac{\partial \log h_j}{\partial \log p_i} \omega_j$$
$$\Rightarrow \omega_i \frac{e}{\omega_i} \gamma_{ij} = \frac{e}{\omega_j} \gamma_{ji} \omega_j$$
$$\Rightarrow \gamma_{ij} = \frac{\gamma_{ij}^* + \gamma_{ji}^*}{2} = \frac{\gamma_{ji}^* + \gamma_{ij}^*}{2} = \gamma_{ji}$$

Symmetry constraint, i.e. $\gamma_{ij}=\gamma_{ji}$, fulfills the requirement.
(g) If you ran regression (6), how would you know whether the good is a luxury or a necessity? Why?
Differentiate ω_i by log(M/P) then,

$$\beta_i = \frac{\partial \omega_i}{\partial \log(M/P)}$$

This coefficient relates the expenditure share to the real expenditure; the demand for a luxury good will increase in accordance with the increase of the real expenditure, and vice versa. Therefore,

$$\beta_i \geq 0 \Leftrightarrow i \text{ is a luxury.}$$
$$\beta_i < 0 \Leftrightarrow i \text{ is a necessity.}$$

1. Constant relative risk-aversion (CRRA) utility is

$$u(C) = \frac{C^{1-\rho} - 1}{1-\rho} \Rightarrow -\frac{u''(C)}{u'(C)} C = -\frac{-\rho C^{-\rho-1}}{C^{-\rho}} C = \rho$$

Then, elasticity of substitution is

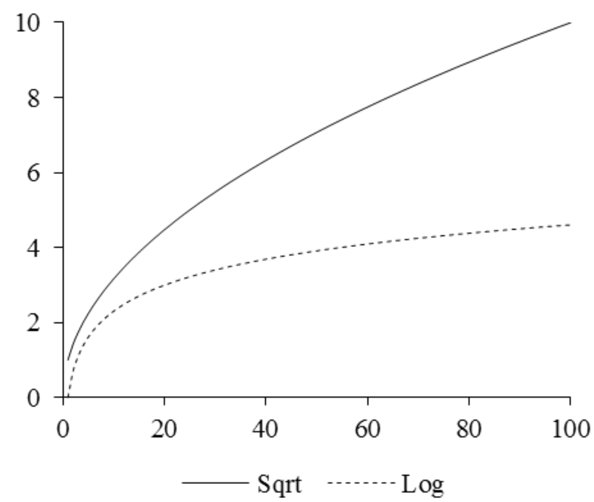
$$\sigma = \frac{d\ln(C_2/C_1)}{d\ln(MU_1/MU_2)} = \frac{d\ln(C_2/C_1)}{d\ln\left(C_1^{-\rho}/C_2^{-\rho}\right)} = \frac{d\ln(C_2/C_1)}{d\rho\ln(C_2/C_1)} = \rho^{-1} \blacksquare$$

Therefore, CRRA utility directly implies constant elasticity of substitution (CES) between two states.

2. The utility function of Sam and that of Terry are,

$$u^s(W) = \alpha W^\beta - \gamma, \text{where } \alpha, \beta, \gamma > 0$$
$$u^t(W) = \delta \log W \text{, where } \delta > 0$$

(a) β should be smaller than 1. If β=1, then Sam will be risk-neutral. If β>1, then he will be risk-loving.
(b) Intuitively, Sam's utility function is similar to the inverse function of X^2 and Terry's utility function is similar to the inverse function of e^X. Since the latter expands faster than the former, Terry's utility function seems to be more risk-averse than Sam's one.



(c) For Sam's utility function,

$$\frac{du^s}{dW}(W) = \alpha\beta W^{\beta-1}, \frac{d^2u^s}{dW^2}(W) = \alpha\beta(\beta-1)W^{\beta-2}$$
$$\Rightarrow r^s(W) = -\frac{\alpha\beta(\beta-1)W^{\beta-2}}{\alpha\beta W^{\beta-1}} = \frac{(1-\beta)}{W}$$
$$\Rightarrow \rho^s(W) = 1-\beta$$

For Terry's utility function,

$$\frac{du^t}{dW}(W) = \frac{\delta}{W}, \frac{d^2u^t}{dW^2}(W) = -\frac{\delta}{W^2}$$
$$\Rightarrow r^t(W) = -\frac{-\delta/W^2}{\delta/W} = \frac{1}{W}\left(> \frac{1-\beta}{W} = r^s\right)$$
$$\Rightarrow \rho^t(W) = 1(> 1-\beta = \rho^s)$$

(d) The bribe is similar to a risk premium π. Since Terry is more risk-averse, Terry will pay more for the opportunity than Sam. (Pratt's theorem)

3. The index of absolute prudence (AP) is,

$$\eta(W) \equiv -\frac{v'''}{v''}$$

(a) Decreasing risk-aversion is,

$$r = -\frac{u''}{u'} \Rightarrow r' = \frac{u''u'' - u'u'''}{(u')^2} < 0$$
$$\Rightarrow \frac{u'}{\underbrace{u''}_{<0}}\frac{u''u'' - u'u'''}{(u')^2} > 0$$
$$\Rightarrow \frac{u''}{u'} - \frac{u'''}{u''} > 0$$
$$\Rightarrow \frac{u''}{u'} > \frac{u'''}{u''}$$
$$\Rightarrow -\frac{u''}{u'} = r < \eta = -\frac{u'''}{u''} \blacksquare$$
$$\text{And } r < \eta \Rightarrow r - \eta < 0$$
$$\Rightarrow \frac{u'''}{u''} - \frac{u''}{u'} < 0$$
$$\Rightarrow \frac{u'u''' - u''u''}{u'u''} < 0$$
$$\Rightarrow \frac{u''}{\underbrace{u'}_{<0}}\frac{u'u''' - u''u''}{u'u''} > 0$$
$$\Rightarrow \frac{u'u''' - (u'')^2}{(u')^2} > 0, \therefore \frac{(u'')^2 - u'u'''}{(u')^2} = r' < 0 \blacksquare$$

Therefore, decreasing risk-aversion means $r(W)<\eta(W)$.

(b) Constant relative risk-aversion is,

$$\rho = -\frac{u''}{u'}W = c \Rightarrow \rho' = -\frac{u''}{u'} + \frac{(u'')^2 - u'u'''}{(u')^2}W = 0$$

$$\Rightarrow \frac{u'u''' - (u'')^2}{(u')^2}W = -\frac{u''}{u'}$$

$$\Rightarrow u'u''' - (u'')^2 = -\frac{u'u''}{W}$$

$$\Rightarrow u''' = \underbrace{\frac{(u'')^2}{u'}}_{>0} - \underbrace{\frac{u''}{W}}_{<0} > 0$$

$$\therefore \eta = -\frac{u'''}{u''} = -\frac{u''}{u'} + \frac{1}{W} \Rightarrow \eta' = \frac{(u'')^2 - u'u'''}{(u')^2} - \frac{1}{W^2}$$

$$= \frac{(u'')^2 - u'\left[\frac{(u'')^2}{u'} - \frac{u''}{W}\right]}{(u')^2} - \frac{1}{W^2}$$

$$= \underbrace{\frac{u''}{u'W}}_{<0} - \underbrace{\frac{1}{W^2}}_{>0} < 0 \blacksquare$$

4. The statement is,

$$c \approx \mu - \frac{r_u}{2}\sigma^2$$

By definition, certainty equivalent is,

$$\mathrm{E}[u(X)] = u[c(X)]$$

$$\int u(x)f(x)dx =$$

$$\int \left[u(\mu) + u'(\mu)(x-\mu) + \frac{1}{2}u''(\mu)(x-\mu)^2 + \cdots\right] f(x)dx = u(\mu) + u'(\mu)[c(X) - \mu] + \cdots$$

$$\Rightarrow u(\mu) + u'(\mu)\mathrm{E}(X-\mu) + \frac{1}{2}u''(\mu)\mathrm{E}[(X-\mu)^2] \approx u(\mu) + u'(\mu)[c(X) - \mu]$$

$$u(\mu) + \frac{1}{2}u''(\mu)\mathrm{Var}(X) \approx$$

$$\Rightarrow u'(\mu)c(X) \approx u'(\mu)\mu + \frac{1}{2}u''(\mu)\mathrm{Var}(X)$$

$$\Rightarrow c(X) \approx \mu + \frac{1}{2}\underbrace{\frac{u''(\mu)}{u'(\mu)}}_{=-r_u(\mu)}\underbrace{\mathrm{Var}(X)}_{=\sigma^2}$$

$$\therefore c \approx \mu - \frac{r_u}{2}\sigma^2 \blacksquare$$

Economics 801 HW 4 | 991253080 Junyong Kim
Advanced Microeconomic Theory I | October 22, 2015

1. (a) Current: p_1, F_1
Legislation: $p_2<p_1$, $F_2>F_1$
Let W_0 be the initial wealth of a person and let a random variable W_1 be the wealth of a disobeying person under the old regime. Then,

$$W_1 = \begin{cases} W_0 - F_1, & \text{with } p_1 \\ W_0, & \text{with } 1-p_1 \end{cases}$$

Similarly, let a random variable W_2 be the wealth of a disobeying person under the new regime. Then,

$$W_2 = \begin{cases} W_0 - F_2, & \text{with } p_2 \\ W_0, & \text{with } 1-p_2 \end{cases}$$

Then, the cumulative distribution function of W_1 and W_2 can be expressed as below.

$$G_1(w) = \begin{cases} 0, & \text{if } w < W_0 - F_1 \\ p_1, & \text{if } W_0 - F_1 \le w < W_0 \\ 1, & \text{if } w \ge W_0 \end{cases}$$
$$G_2(w) = \begin{cases} 0, & \text{if } w < W_0 - F_2 \\ p_2, & \text{if } W_0 - F_2 \le w < W_0 \\ 1, & \text{if } w \ge W_0 \end{cases}$$

Therefore, $G_2(w)=p_2>0=G_1(w)$ if $W_0-F_2\le w<W_0-F_1$ and $G_1(w)=p_1>p_2=G_2(w)$ if $W_0-F_1\le w<W_0$. Hence, W_1 does not FOSD W_2 and vice versa. However,

$$G_2(w)-G_1(w) = \begin{cases} 0, & \text{if } w < W_0 - F_2 \\ p_2, & \text{if } W_0 - F_2 \le w < W_0 - F_1 \\ p_2 - p_1, & \text{if } W_0 - F_1 \le w < W_0 \\ 0, & \text{if } w \ge W_0 \end{cases}$$
$$\int_{-\infty}^{a}[G_2(w)-G_1(w)]dw = \begin{cases} 0, & \text{if } a < W_0 - F_2 \\ p_2(a - W_0 + F_2), & \text{if } W_0 - F_2 \le a < W_0 - F_1 \\ p_2(F_2 - F_1) - (p_1 - p_2)(a - W_0 + F_1), & \text{if } W_0 - F_1 \le a < W_0 \\ 0, & \text{if } a \ge W_0 \end{cases}$$

If $W_0-F_1\le a<W_0$,

$$\begin{aligned} -(p_1-p_2)(a-W_0+F_1) &= p_2F_2 - p_2F_1 - p_1a + p_1W_0 - p_1F_1 + p_2a - p_2W_0 + p_2F_1 \\ &= (p_2-p_1)a - (p_2-p_1)W_0 \\ &= \underbrace{(p_1-p_2)}_{>0}\underbrace{(W_0-a)}_{>0} > 0 \end{aligned}$$
$$\therefore \int_{-\infty}^{a}[G_2(w)-G_1(w)]dw \ge 0\ \forall a \Rightarrow G_1 \overset{\text{FOSD}}{\gtrsim} G_2\ \blacksquare$$

Therefore, if the drivers are risk-averse, then disobeying drivers will decrease.

(b) The expected total amount of fines collected is,

$$\text{Total amount}_1 = \frac{n_1 = \text{Disobeying driver}}{N = \text{Total driver}} \times p_1 \times F_1, \text{Total amount}_2 = \frac{n_2}{N} \times p_2 \times F_2 = \frac{n_2 p_1 F_1}{N}$$

Because $n_1 > n_2$ in the world of risk-averse drivers, the total amount of fines will decrease.

2. The probability density function of Weibull distribution is

$$f(x;\alpha,\beta) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

Where α and β are both greater than 0. The cumulative density function of the distribution is

$$F(x;\alpha,\beta) = \begin{cases} 1 - e^{-\alpha x^{\beta}}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

(a)

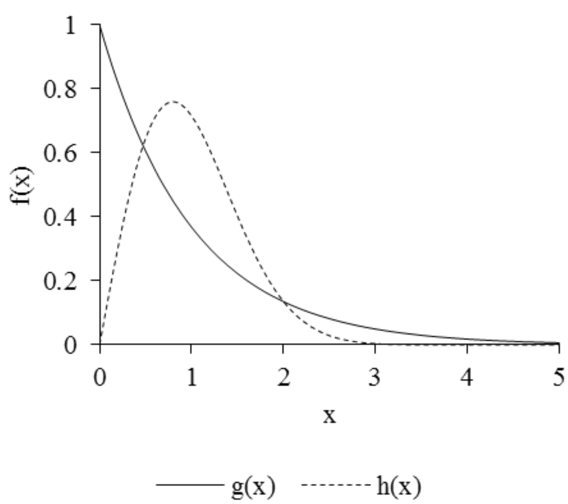
$$\alpha_1 > \alpha_2 > 0 \Leftrightarrow \alpha_1 x^{\beta} > \alpha_2 x^{\beta} \Leftrightarrow -\alpha_1 x^{\beta} < -\alpha_2 x^{\beta} \Leftrightarrow e^{-\alpha_1 x^{\beta}} < e^{-\alpha_2 x^{\beta}} \Leftrightarrow \underbrace{1 - e^{-\alpha_1 x^{\beta}}}_{=F(x;\alpha_1,\beta)=F_1} > \underbrace{1 - e^{-\alpha_1 x^{\beta}}}_{=F_2}$$

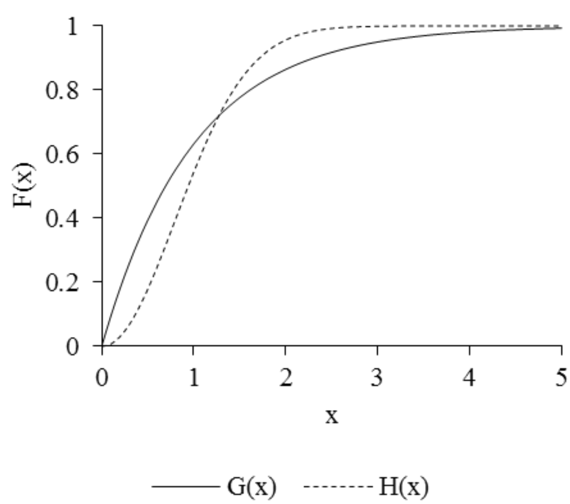
Therefore, if $\alpha_1 > \alpha_2$, then $F_1(x) > F_2(x)$ $\forall x$ and this implies F_2 FOSD F_1; agents will prefer α_2 ($<\alpha_1$). This preference does not depend on the risk-averse assumption; i.e. risk-loving agents will also prefer α_2 over α_1 since F_2 FOSD F_1.

(b)

$$g(x;1) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases} \Rightarrow G(x;1) = \begin{cases} 1 - e^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

$$h(x;\pi/4) = \begin{cases} \frac{\pi}{2} x e^{-\frac{\pi}{4}x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases} \Rightarrow H(x;\pi/4) = \begin{cases} 1 - e^{-\frac{\pi}{4}x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$





i. Since G and H have identical mean,

$$\int_{-\infty}^{\infty}[G(x)-H(x)]dx=\int_{0}^{\infty}[G(x)-H(x)]dx=0$$

And,

$$G(x)-H(x)=e^{-\frac{\pi}{4}x^2}-e^{-x}\geq 0 \Leftrightarrow e^{-\frac{\pi}{4}x^2}\geq e^{-x}\Leftrightarrow -\frac{\pi}{4}x^2\geq -x \Leftrightarrow x-\frac{\pi}{4}x^2\geq 0 \Leftrightarrow x\left(1-\frac{\pi}{4}x\right)\geq 0$$

Therefore, $G(x)-H(x)\geq 0 \Leftrightarrow 0<x\leq 4/\pi$. Then,

$$\int_{0}^{a}[G(x)-H(x)]dx\geq 0\ \forall 0<a\leq\frac{4}{\pi}$$

ii. From the above,

$$\int_{0}^{\infty}[G(x)-H(x)]dx=\int_{0}^{\frac{4}{\pi}}[G(x)-H(x)]dx+\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx=0$$

$$\Rightarrow\int_{0}^{\frac{4}{\pi}}[G(x)-H(x)]dx=-\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx$$

Hence,

$$-\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx\geq -\int_{\frac{4}{\pi}}^{a}[G(x)-H(x)]dx\ \ \forall\frac{4}{\pi}<a<\infty$$

$$\Rightarrow -\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx+\int_{\frac{4}{\pi}}^{a}[G(x)-H(x)]dx\geq 0\ \forall\frac{4}{\pi}<a<\infty$$

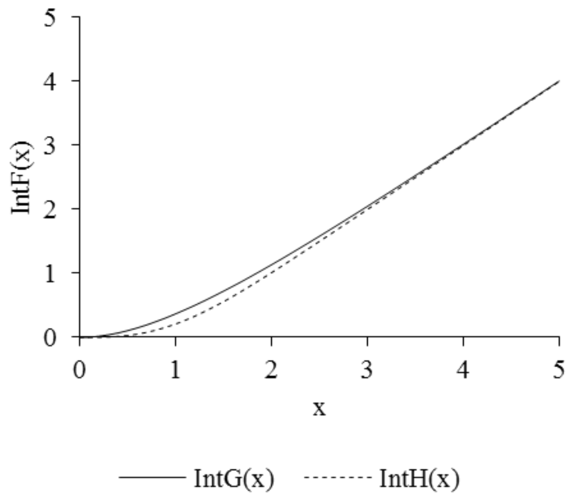
$$\Rightarrow\int_{0}^{\frac{4}{\pi}}[G(x)-H(x)]dx+\int_{\frac{4}{\pi}}^{a}[G(x)-H(x)]dx=\int_{0}^{a}[G(x)-H(x)]dx\geq 0\ \forall\frac{4}{\pi}<a<\infty$$

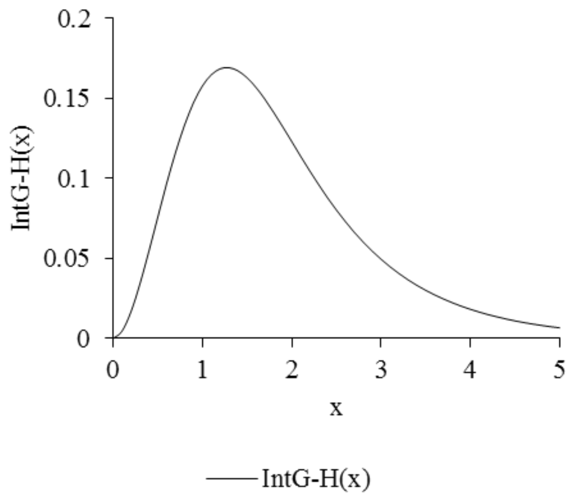
Therefore, from i and ii,

$$\int_{0}^{a}[G(x)-H(x)]dx\geq 0\ \forall a>0\ \blacksquare$$

Note that,

$$\int_0^a G(x)dx = a + e^{-a} - 1, \int_0^a H(x)dx = a - \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}a\right), \int_0^a [G(x) - H(x)]dx = \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}a\right) + e^{-a} - 1$$





3.
(a) i. If $\partial^2 u/\partial c\partial z$>0, then the marginal utility of consumption would be larger when there exist more amenities to enjoy. If $\partial^2 u/\partial c\partial z$=0, then the individuals given y will always consume the same amount.

$$\frac{\partial^2 u}{\partial c \partial z} > 0 \Rightarrow \frac{\partial u}{\partial c}(c, z_h) > \frac{\partial u}{\partial c}(c, z_l)$$

However, if the derivative is positive, then the consumers will spend more because they can take more joy from the environment they are facing with.
ii. For the utility function u,

$$u(y, z_l) = y^b z_l \Rightarrow u' = by^{b-1} z_l \text{ and } u'' = b(b-1)y^{b-2} z_l$$
$$\Rightarrow r(y) = -\frac{b(b-1)y^{b-2} z_l}{by^{b-1} z_l} = \frac{1-b}{y} > 0$$
$$u(y, z_h) = (y - r_h)^b z_h \Rightarrow u' = b(y - r_h)^{b-1} z_h \text{ and } u'' = b(b-1)(y - r_h)^{b-2} z_h$$
$$\Rightarrow r(y) = -\frac{b(b-1)(y - r_h)^{b-2} z_h}{b(y - r_h)^{b-1} z_h} = \frac{1-b}{y - r_h} > 0 \text{ if } y > r_h$$

(b) The individuals will increase r_h until the utility from l and the utility from h are equal.

$$y^b z_l = (y - \bar{r}_h)^b z_h \Leftrightarrow (y - \bar{r}_h)^b = y^b \frac{z_l}{z_h} \Leftrightarrow y - \bar{r}_h = y \sqrt[b]{z_l/z_h} \Leftrightarrow \bar{r}_h(y) = y\left(1 - \sqrt[b]{z_l/z_h}\right)$$

If r_h is given, then

$$\bar{y} = \frac{r_h}{1 - \sqrt[b]{z_l/z_h}} \therefore y > \bar{y} \Rightarrow \text{choose } h \text{ and } y < \bar{y} \Rightarrow \text{choose } l$$

(c)

i. The income is a random variable $Y \sim F(y)$, then

$$k = P(Y > \bar{y}) = 1 - P(Y < \bar{y}) = 1 - F(\bar{y}) = 1 - F\left(\frac{r_h}{1 - \sqrt[b]{z_l/z_h}}\right)$$

$$\Rightarrow F\left(\frac{r_h}{1 - \sqrt[b]{z_l/z_h}}\right) = 1 - k$$

$$\Rightarrow \frac{r_h}{1 - \sqrt[b]{z_l/z_h}} = F^{-1}(1-k)$$

$$\therefore r_h = \left(1 - \sqrt[b]{z_l/z_h}\right) F^{-1}(1-k)$$

$$\Rightarrow \frac{\partial r_h}{\partial k} = -\underbrace{\left(1 - \sqrt[b]{z_l/z_h}\right)}_{>0} \underbrace{\frac{\partial F^{-1}}{\partial (1-k)}}_{>0} < 0$$

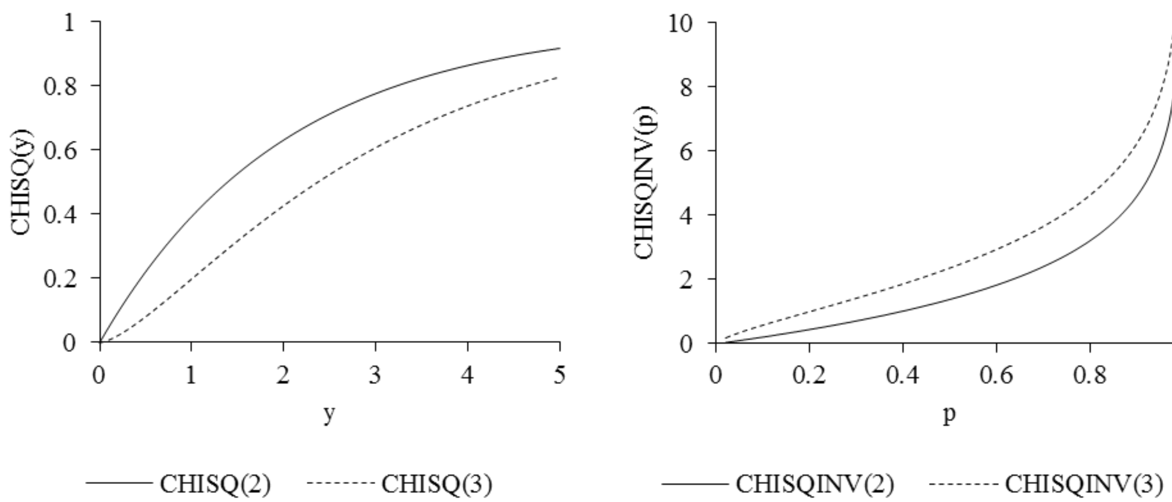
$$\Rightarrow \frac{\partial r_h}{\partial (z_h/z_l)} = \frac{\partial}{\partial (z_h/z_l)} \left[1 - (z_h/z_l)^{-\frac{1}{b}}\right] F^{-1}(1-k)$$

$$= \frac{1}{b}(z_h/z_l)^{-\frac{1+b}{b}} F^{-1}(1-k) = \underbrace{b^{-1}}_{>0} \underbrace{(z_l/z_h)^{\frac{1+b}{b}}}_{>0} \underbrace{F^{-1}(1-k)}_{>0} > 0$$

ii. $G(y)$ FOSD $F(y) \Leftrightarrow F(y) \geq G(y)\ \forall y \Leftrightarrow G^{-1}(p) \geq F^{-1}(p)\ \forall p$. Therefore,

$$r_h^F = \left[1 - (z_l/z_h)^{\frac{1}{b}}\right] F^{-1}(1-k) \leq \left[1 - (z_l/z_h)^{\frac{1}{b}}\right] G^{-1}(1-k) = r_h^G$$

Therefore, r_h will increase; if the income level of society increases, then the amount that people must pay to live in the location h would also increases; under the rich world with G, the rent r_h would be expensive.



Note that, if G FOSD F then $G^{-1}(p) \geq F^{-1}(p)\ \forall 0 \leq p \leq 1$ and $\exists 0 \leq p' \leq 1$ such that $G^{-1}(p') > F^{-1}(p')$.

(d) i.

Decision	Win p	Lose $(1-p)$
Not invest	$\bar{y}$	$\bar{y}$
Invest	$\bar{y}-\phi+\phi/p$	$\bar{y}-\phi$

Then, for the individual with $\bar{y}$, the expected utility of "not invest" is,

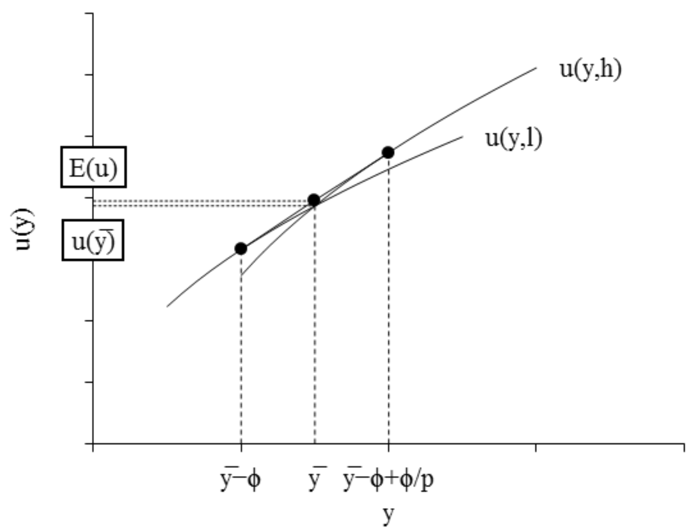
$$\mathrm{E}(u^n) = p\bar{y}^b z_l + (1-p)\bar{y}^b z_l = \bar{y}^b z_l \big(= (\bar{y} - r_h)^b z_h \ \because \text{indifferent}\big)$$

While the indirect utility function of an individual is not conventional as below.

$$u(y) = \max[\bar{y}^b z_l, (\bar{y} - r_h)^b z_h]$$

Hence, the expected utility of "invest" is

$$\mathrm{E}(u^i) = p\left(\bar{y} - \phi + \frac{\phi}{p} - r_h\right)^b z_h + (1-p)(\bar{y} - \phi)^b z_l \geq \bar{y}^b z_l = \mathrm{E}(u^n)$$



For the individual with $\bar{y}$, the expected utility with the investment is greater than that without the investment. Therefore, the individual will exploit the opportunity to maximize the expected utility. An individual whose income lies around $\bar{y}$ will also choose the investment since the expected utility comes from the fair bet exceeds the utility of the status quo. However, an individual whose income is too small to shift their status by investing will not choose to bet because the individual is locally risk-averse. Likewise, an individual whose income is large enough will not choose to gamble.
ii. The investment opportunity both increase agents' (*ex-ante* expected) utility (a. True) and causes a mean-preserving spread of the (ex-post) distribution of income (b. True).
a. For the range around $\bar{y}$, introducing the investment opportunity will improve the agent's expected utility. For the outside of this range, introducing the opportunity will not hurt the agent's utility. Overall, introducing the investment opportunity would have positive effects.
b. The mean of "invest" and that of "not invest" is identical, but the former choice is definitely more risky. Thus, "not invest" SOSD "invest" in this case.
(1) Why those effects are not in conflict? Because the von-Neumann–Morgenstern utility function here is $u(y)=\max[y^b z_l,(y-r_h)^b z_h]$ and this function is not globally concave.

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1.

$$u_A = \sum p_i x_i^{0.5} = \sum p_i v_A(x_i) \Rightarrow r^A = -\frac{v''}{v'} = -\frac{-0.25x^{-1.5}}{0.5x^{-0.5}} = \frac{1}{2x}$$
$$u_B = \sum p_i\left(16.414x_i^{0.5} - 34.25\right) = \sum p_i v_B(x_i) \Rightarrow v_B = \underbrace{16.414v_A - 34.25}_{\text{affine transformation}} \Rightarrow r^B = \frac{1}{2x}$$
$$u_C = \sum p_i(0.25\log x_i + 74) = \sum p_i \underbrace{[0.25\log v_A(x_i) + 74]}_{\text{concave transformation}} \Rightarrow r^C \le r^A = r^B$$

2. a. Both are symmetric.

$x(w,y)$ conditional and $x(p,w)$ unconditional

$\Rightarrow x_i(w,y) = \dfrac{\partial c}{\partial w_i}$ by Shephard's lemma and $\dfrac{\partial x_i(w,y)}{\partial w_j} = \dfrac{\partial^2 c}{\partial w_i \partial w_j} = \dfrac{\partial x_j(w,y)}{\partial w_i}$ by Young's theorem

$\Rightarrow x_i(p,w) = \dfrac{\partial \pi}{\partial w_i}$ by Shephard's lemma and $\dfrac{\partial x_i(p,w)}{\partial w_j} = \dfrac{\partial^2 \pi}{\partial w_i \partial w_j} = \dfrac{\partial x_j(p,w)}{\partial w_i}$ by Young's theorem

b. p=20, y=85, by Hotelling's lemma

$$\frac{\partial \pi}{\partial p} = y \Rightarrow \frac{\partial \pi}{\partial p}dp = 85(20.1-20) = 8.5 \text{ and } \frac{\partial^2 \pi}{\partial p^2} > 0 \Rightarrow \frac{\partial \pi}{\partial p}dp \ge 85(20.2-20)$$

c. $z \lesseqqgtr c \Leftrightarrow F(z) \lesseqqgtr G(z)$

- If F does not FOSD G and G does not FOSD F, then the preference will depend on the taste.
- If $E_F(Z) \geqq E_G(Z)$, then define F' such that $E_{F'}(Z)=E_G(Z) \Rightarrow F$ FOSD F' and F' SOSD G, hence F SOSD G.

d. $y=f(x)$ HOD $t \Rightarrow k^t y=f(kx)$

$$\text{then } \left.\frac{\partial f/\partial x_1}{\partial f/\partial x_2}\right|_x = \left.\frac{\partial f/\partial x_1}{\partial f/\partial x_2}\right|_{kx} = \frac{w_1}{w_2} \text{ for optimal } x$$
$$\text{hence } c(w,y) = wx(w,y) \Rightarrow c(w,ky) = wx(w,ky)$$
$$\Rightarrow ky = f\left(k^{1/t}x\right) \Rightarrow c(w,ky) = k^{1/t}wx(w,y)$$
$$\text{and } \frac{\partial \log f(kx)}{\partial \log k} = \frac{\partial[t\log k + \log f(x)]}{\partial \log k} = t = 1$$
$$\Rightarrow c(kw,ky) = kwx(kw,ky) = kwx(w,ky) = k^{1+\frac{1}{t}}wx(w,y) = k^2 wx(w,y)$$

3. Add-up $\Rightarrow$ check $m=py+qx$ (true); homogeneity $\Rightarrow$ apply tp, tq, tm and cancel out t (true)

$$y(p,q,m) = 400 + 5m/p - 3000q/p \Rightarrow x(p,q,m) = m/q - py/q \Rightarrow 3000 + 4m/5q - 400p/q$$

Symmetry (true)

$$\frac{\partial y}{\partial q} + \frac{\partial y}{\partial m}x = \frac{\partial x}{\partial p} + \frac{\partial x}{\partial m}y \Rightarrow -\frac{3000}{p} + \frac{x}{5p} = -\frac{400}{q} + \frac{4y}{5q} = -80p - 2400q + \frac{4}{25}m$$

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Intertemporal Problem

- Flow of income $\{w_t\}_1^T$
- Will choose how much to consume each period $\{c_t\}_1^T$
- Interest rate r
- Budget constraint

$$\sum \left(\frac{1}{1+r}\right)^t c_t = \sum \left(\frac{1}{1+r}\right)^t w_t \text{ or } \int e^{-rt} c(t)dt = \int e^{-rt} w(t)dt$$

- Objective function

$$\max_{\{c_t\}} u(c_1, \cdots, c_T) \text{ subject to } \sum \left(\frac{1}{1+r}\right)^t (w_t - c_t) = 0$$

- Most often we use

$$u(c_1, \cdots, c_T) = \sum \beta^t v(c_t) \text{ for some function } v \text{ or } u = \int e^{-\delta t} v[c(t)]dt$$

i. Constant discount rate β
ii. Additively separable $\partial^2 u/\partial c_t \partial c_s = 0$ for $t \neq s$
With this standard specification, FOCs are

$$L = \sum \beta^t v(c_t) + \lambda \left[\sum \left(\frac{1}{1+r}\right)^t (w_t - c_t)\right]$$
$$\Rightarrow \frac{\partial L}{\partial c_t} = \beta^t v'(c_t) - \frac{\lambda}{(1+r)^t} = 0 \text{ and } \frac{\partial L}{\partial c_{t+1}} = \beta^{t+1} v'(c_{t+1}) - \frac{\lambda}{(1+r)^{t+1}} = 0$$
$$\Rightarrow v'(c_t) = \beta(1+r) v'(c_{t+1})$$

Hence,

$$\beta(1+r) \gtreqless 1 \Rightarrow v'(c_{t+1}) \gtreqless v'(c_t) \Rightarrow c_t \gtreqless c_{t+1}$$

Therefore, if $\beta = 1/(1+r)$, then $v'(c_t) = v'(c_{t+1})$ and therefore $c_t = c_{t+1}$.

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Intertemporal Problem

$$\max_{\{c_t\}} u(c_0, \cdots, c_T) \text{ subject to } \sum \left(\frac{1}{1+r}\right)^t (w_t - c_t) = 0$$

where $\{w_t\}$ is income stream.

Usual Assumption

$$u(c_0, \cdots, c_T) = \sum \beta^t v(c_t)$$

Note that this form

i. Has a constant rate of time preference $\beta \Rightarrow$ generalize: Hyperbolic discounting

ii. Is additively separable over time, i.e. $\partial^2 u/\partial c_t \partial c_s = 0$ for $t \neq s \Rightarrow$ generalize: Non-separable preferences

Hyperbolic Discounting

$$u(c_0, \cdots, c_T) = \sum (1+\alpha t)^{-\gamma/\alpha} v(c_t) \ \left(\text{note that } \lim_{\alpha \to 0} (1+\alpha t)^{-\frac{\gamma}{\alpha}} = e^{-\gamma t}\right)$$

In the usual case, FOC is

$$\left(\frac{1}{1+r}\right)^k = \frac{\partial u/\partial c_{t+k}}{\partial u/\partial c_t} = \beta^k \frac{v'(c_{t+k})}{v'(c_t)}$$

However, in Hyperbolic Discounting case, FOC is

$$\left(\frac{1}{1+r}\right)^k = \frac{\partial u/\partial c_{t+k}}{\partial u/\partial c_t} = \left[\underbrace{\frac{1+\alpha(t+k)}{1+\alpha t}}_{=1+\frac{\alpha k}{1+\alpha t}}\right]^{-\frac{\gamma}{\alpha}} \frac{v'(c_{t+k})}{v'(c_t)}$$

Implication: Become increasingly patient about the choice between consumption in 2 periods as the first period becomes further from the present.

Simpler Version by Laibson (1997, QJE)

$$u(c_0, \cdots, c_T) = v(c_0) + \delta \sum \beta^t v(c_t)$$

If $\delta \to 1$ then the usual case, but

$$\left.\frac{\partial u/\partial c_{t+k}}{\partial u/\partial c_t}\right|_{t=1} = \beta^k \frac{v'(c_{t+k})}{v'(c_t)} \ (\delta < 1) \text{ but } \frac{\partial u/\partial c_k}{\partial u/\partial c_0} = \delta \beta^k \frac{v'(c_{t+k})}{v'(c_t)} \ (\text{note: Intergeneration Choice?})$$

Time Inconsistency Problem

- Usual case: The decision would not be changed in the future.
- Hyperbolic discount: The decision would be changed when the future is realized. (ex. diet)

Other Examples
- Odyssey: Siren—self-binding to avoid the allure
- Consumer credit: # of credit ↑ ⇒ opportunity to be impulsive ↑ ⇒ would be more myopic
- Penalty for early withdrawal: Binds someone from deviating from original decision
- Automatic savings system: Self-regulation, self-discipline; retirement saving: can stop the automatic saving by one signature, but would go ahead because it is too minor today (retirement is distant future).
- Behavioral issues in Finance
- What is the incentive for banks to help the consumers? With consistent behavior, the depositor would be smarter than before, and would increase their savings in the future.

Rational Addiction

$$u(c_0, \cdots, c_T) = \sum \beta^t v(c_t, y_t, s_t)$$
$c_t =$ consumption of addictive goods
$y_t =$ consumption of other goods
$s_t =$ measure of the stock of part consumption of addictive goods
ex. $s_{t+1} = (1-\delta)s_t + c_t$

Key assumption

$$\frac{\partial^2 v}{\partial c \partial s} \neq 0 \text{ (usually} > 0)$$

Let s_0=current value of s; A=current wealth; r=interest rate; $p(t)$=price of the addictive good at time t (price of y is 1). Income $w(t)$ can depend on s_t.
Then, the consumer solve

$$\max_{\{c_t, y_t\}} u \equiv \sum \beta^t v(c_t, y_t, s_t)$$
$$\text{s.t.} \sum \left(\frac{1}{1+r}\right)^t [w_t(s_t) - p(t)c_t - y_t] + A = 0$$
$$\text{and } s_{t+1} = (1-\delta)s_t + c_t$$

FOCs are

$$\frac{\partial u}{\partial y_t} = \frac{\lambda}{[\beta(1+r)]^t}$$
$$\frac{\partial u}{\partial c_t} = \left.\frac{\partial u}{\partial c_t}\right|_{s_{t+1}, \cdots, s_T} + \sum_{\tau=t+1}^{T} \frac{\partial u}{\partial s_\tau} \frac{ds_\tau}{dc_t}$$
$$= \frac{\lambda}{[\beta(1+r)]^t} p(t) - \underbrace{\sum_{\tau=t+1}^{T} \beta^{\tau-t} \frac{\partial v(c_\tau, y_\tau, s_\tau)}{\partial s_\tau} \underbrace{\frac{\partial s_\tau}{\partial c_t}}_{=(1-\delta)^{\tau-t}}}_{=a(t) \text{ that captures effect on future utility}}$$

What are the implications? (CHECK AND FILL OUT PLEASE!)

$$\max_{\{y(t),c(t)\}} \sum_{t=0}^{T} \beta^t u\big(y(t), c(t), s(t)\big)$$
$$\text{subject to } \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t \big(p(t)c(t) + y(t)\big) = A_0 + \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t w\big(s(t)\big)$$
$$\text{where } s(t+1) = (1+\delta)s(t) + c(t)$$

FOCs are

$$\frac{\partial u}{\partial y(t)} = \frac{\lambda}{\beta^t (1+r)^t}$$
$$\frac{\partial u}{\partial c(t)} = \frac{\lambda p(t)}{\beta^t (1+r)^t} - a(t)$$
$$\text{where } a(t) = \sum_{\tau=t+1}^{T} \left(\beta^{\tau-t} \frac{\partial u}{\partial s(\tau)} \frac{\partial s(\tau)}{\partial c(t)} + \beta^{-t} \frac{\lambda}{(1+r)^\tau} \frac{\partial w}{\partial s(\tau)} \frac{\partial s(\tau)}{\partial c(t)}\right)$$

Intuitions?
1. Today’s consumption decision depends on the future price.
2. Short-run price change: The consumer would not be very sensitive.
3. Long-run price change: The consumer would be quite sensitive.
Ex. Cigarette
- If it is expensive just for this week, then smokers would not quit.
- If there is permanent increase, then smokers would compare the benefit from enjoy and money loss.

Let’s say we have N states of the world; s_1, s_2, ⋯, s_N.

Definition A *Contingent Claim* is the right to receive (something) in a particular state of the world s_i.

Definition An *Asset* is a claim to receive some payment (in money) r_i in each possible state of the world s_i; can write those payoffs for that asset as a vector (r_1, r_2, ⋯, r_N).

Definition An *Arrow–Debreu Security* for state *i* is an asset that pays 1 in state *I* and 0 in all other states (there is a separate A–D security for each of the N states).

Note Any arbitrary asset A with payoff vector $\mathbf{r}_A$=(r_{A1}, r_{A2}, ⋯, r_{AN}) is equivalent to a portfolio containing r_{A1} shares of the first A–D security, r_{A2} shares of the second A–D security, ⋯, and r_{AN} shares of the Nth A–D security.

$$\mathbf{AD1} = (1,0,\cdots,0) \Rightarrow r_{A1}\mathbf{AD1} = (r_{A1},0,\cdots,0)$$
$$\mathbf{AD2} = (0,1,\cdots,0) \Rightarrow r_{A2}\mathbf{AD2} = (0,r_{A2},\cdots,0)$$
$$\vdots$$
$$\mathbf{ADN} = (0,0,\cdots,1) \Rightarrow r_{AN}\mathbf{ADN} = (0,0,\cdots,r_{AN})$$
$$\Rightarrow r_{A1}\mathbf{AD1} + r_{A2}\mathbf{AD2} + \cdots + r_{AN}\mathbf{ADN} = (r_{A1}, r_{A2}, \cdots, r_{AN}) = \text{Asset } A \text{ Payoff}$$

Now say we have K assets, and let's stack their payoffs as rows in a matrix and say the price of asset k is p_k and we define vector $\mathbf{p}$.

$$\underset{K\times N}{\mathbf{R}} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{K1} & r_{K2} & \cdots & r_{KN} \end{pmatrix} \text{ and } \underset{K\times 1}{\mathbf{p}} = (p_1 \quad p_2 \quad \cdots \quad p_K)'$$

Finally, suppose there were A–D securities with prices $\boldsymbol{\pi}$.

$$\underset{N\times 1}{\boldsymbol{\pi}} = (\pi_1 \quad \pi_2 \quad \cdots \quad \pi_N)'$$

Then, there would be no arbitrage only if

$$\mathbf{R}\boldsymbol{\pi} = \mathbf{p}$$

Case 1 K=N, $\mathbf{R}$ full-rank, then

$$\Rightarrow \mathbf{R}\boldsymbol{\pi} = \mathbf{p} \Leftrightarrow \boldsymbol{\pi} = \mathbf{R}^{-1}\mathbf{p}$$

Intuition $\mathbf{R}^{-1}$ is a recipe for making portfolios of existing assets equivalent to the A–D securities.

$$\text{Ex. } \mathbf{R} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \Rightarrow \mathbf{R}^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Claim is that the first A–D security is equivalent to a portfolio containing 2 units of asset 1and −3 units of asset 2, then

$$2(2 \quad 3)' - 3(1 \quad 2)' = (1 \quad 0)'$$

Case 2 K>N, in principle, can find $\boldsymbol{\pi}$ by using only the first N assets.

$$\text{Ex. } \mathbf{R} = \begin{pmatrix} \underset{N\times N}{\mathbf{R}_1} \\ \underset{(K-N)\times N}{\mathbf{R}_2} \end{pmatrix} \text{ and } \mathbf{p} = \begin{pmatrix} \underset{N\times 1}{\mathbf{p}_1} \\ \underset{(K-N)\times 1}{\mathbf{p}_2} \end{pmatrix} \Rightarrow \boldsymbol{\pi} = \mathbf{R}_1^{-1}\mathbf{p}_1 \text{ and then } \mathbf{R}_2\boldsymbol{\pi} = \mathbf{p}_2 \text{ or } \mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{p}_1 = \mathbf{p}_2$$

Alternatively,

$$\mathbf{R}\boldsymbol{\pi} = \mathbf{p} \Rightarrow (\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'\mathbf{R}\boldsymbol{\pi} = (\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'\mathbf{p} \Rightarrow \boldsymbol{\pi} = (\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'\mathbf{p}$$

Case 3 K<N. Now $\mathbf{R}'\mathbf{R}$ is not invertible; i.e. not full rank.
- Cannot create a portfolio of assets that is equivalent to each of the A–D securities (although you may be able to create some of them)
- *Incomplete Market*

$$\text{Ex. } \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \text{the third A– D security is not attainable.}$$

Note *Bundling Problem*: for the employee, his or her capability is not separable. If it is possible, then the market will be more efficient.

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* The price of Arrow–Debreu security does not depend on the preference (utility) of the consumer.

Capital Asset Pricing Model

- A+1 assets where asset 0 pays return R_0 with probability 1 (riskless)
- For others a=1, ⋯, A, let R_{as} be the return on asset a in a state s
- P_s=Probability of a state s. $\sum P_s$=1
- Investors will choose the shares of their budgets that they invest in each of the A+1 assets

$$X_a = \text{share invested in asset } a \text{ and } \sum_{a=0}^{A} X_a = 1$$

Preferences: Among the set of portfolios that have any given expected return $\bar{R}$, the goal is to minimize the variance of the return.

Investor's Problem

$$\min_{\{X_a\}} \sum_a \sum_b X_a X_b \text{Cov}(R_a, R_b) \Leftrightarrow \mathbf{X}'\mathbf{\Sigma}\mathbf{X}$$

$$\text{subject to } [\lambda]: \sum_{a=0}^{A} X_a E(R_a) = \bar{R} \text{ and } [\mu]: \sum_{a=0}^{A} X_a = 1$$

Then,

$$\text{Expected Return} = X_0 R_0 + \sum_{a=1}^{A} \sum_s R_{as} P_s X_a$$

FOCs are

$$[X_a]: 2\sum_{b=0}^{A} X_b \text{Cov}(R_a, R_b) - \lambda \text{E}(R_a) - \mu = 0$$

i) For asset 0 (riskless),

$$-\lambda R_0 - \mu = 0$$

ii) Now consider a special asset e defined as the variance-minimizing portfolio of the A risky assets.

$$2\sum_{b=0}^{A} X_b \text{Cov}(R_e, R_b) - \lambda \text{E}(R_e) - \mu = 0$$

$$2\text{Cov}\left(R_e, \underbrace{\sum_{b=0}^{A} X_b R_b}_{=R_e}\right) - \lambda \text{E}(R_e) - \mu = 0$$

Solve i) and ii) to find

$$\lambda = \frac{2\mathrm{Var}(R_e)}{\mathrm{E}(R_e) - R_0} \text{ and } \mu = -\frac{2\mathrm{Var}(R_e)R_0}{\mathrm{E}(R_e) - R_0}$$

Substituting these into the more general FOC

$$\underbrace{0}_{\substack{\text{at the optimal} \\ \text{portfolio}}} = 2\sum_{b=0}^{A} X_b \mathrm{Cov}(R_a, R_b) - \frac{2\mathrm{Var}(R_e)}{\mathrm{E}(R_e) - R_0}\mathrm{E}(R_a) + \frac{2\mathrm{Var}(R_e)R_0}{\mathrm{E}(R_e) - R_0}$$

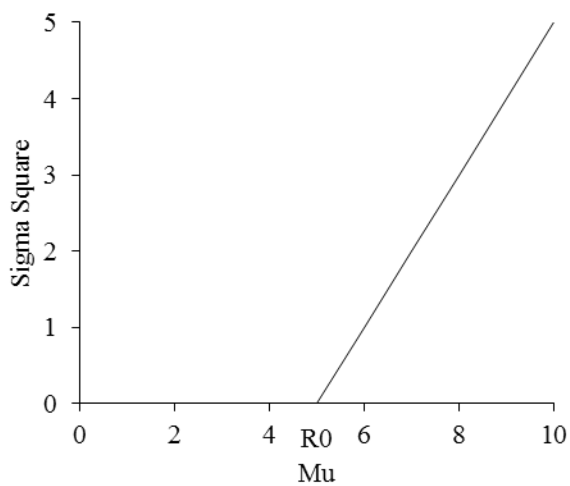
(skip 3 pages of algebra) R_m be the return for the market as a whole (including the riskless asset). Then, the solution is

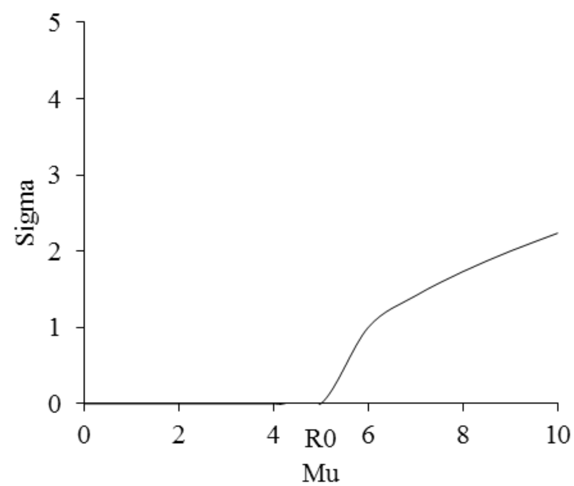
$$\mathrm{E}(R_a) = R_0 + \frac{\mathrm{Cov}(R_m, R_a)}{\mathrm{Var}(R_m)}[\mathrm{E}(R_m) - R_0]$$

$$\mathrm{E}(R_a) - R_0 = \frac{\mathrm{Cov}(R_m, R_a)}{\mathrm{Var}(R_m)}[\mathrm{E}(R_m) - R_0]$$

$$= \sqrt{\frac{\mathrm{Var}(R_a)}{\mathrm{Var}(R_m)}}\mathrm{Corr}(R_m, R_a)[\mathrm{E}(R_m) - R_0]$$

Intuition from our discussion of risk: We said we could often approximate a certainty equivalent to an asset with mean μ and variance σ^2 by $\mu-\rho/2\times\sigma^2=K \Rightarrow \sigma^2=2/\rho\times(\mu-K)$.





Version 1: Machine cost \$100 today, \$120 next year, r=20%, PV=\$100
Two years from now revenue,

$$\pi = \begin{cases} \$250, & \text{probability} = .25 \\ \$150, & \text{probability} = .50 \\ \$50, & \text{probability} = .25 \end{cases}$$

Current ENPV of project

$$\begin{aligned} \text{ENPV} &= \frac{E(\pi)}{(1+r)^2} - \min\left\{100, \frac{120}{1+r}\right\} \\ &= \$4.17 \end{aligned}$$

Version 2: 1 year from now, will receive a signal H with probability .5 and L with probability .5.

$$\text{H} \Rightarrow \text{revised probability of } \pi = \begin{cases} \$250, & \text{with } .5 \\ \$150, & \text{with } .5 \end{cases}$$
$$\text{L} \Rightarrow \text{revised probability of } \pi = \begin{cases} \$150, & \text{with } .5 \\ \$50, & \text{with } .5 \end{cases}$$

Invest today: Same calculations as the version 1
Another possibility: Wait for the signal, and then decide
⇒ As of today, if wait

$$\text{ENPV} = \frac{.5 \times \overbrace{\max\left\{\underbrace{\frac{.5 \times 250 + .5 \times 150}{1+r} - 120}_{\text{Invest into the project}}, \underbrace{0}_{\text{Nothing}}\right\}}^{\text{If the signal is H}} + .5 \times \overbrace{\max\left\{\frac{.5 \times 150 + .5 \times 50}{1+r} - 120, 0\right\}}^{\text{If the signal is L}}}{1+r}$$

$$= \frac{.5 \times \max\left\{\frac{200}{1.2} - 120, 0\right\} + .5 \times \max\left\{\frac{100}{1.2} - 120, 0\right\}}{1.2} = \$19.44 > \$4.17$$

Hence, the tacit cost for today's investment decision is the "giving up" of the option value of waiting for future signals.

Call option on a stock, 2 periods
- Stock price \$10 today ⇒ \$12 at state 1 or \$8 at state 2 next year
- Bond price \$10 today ⇒ \$11 next year

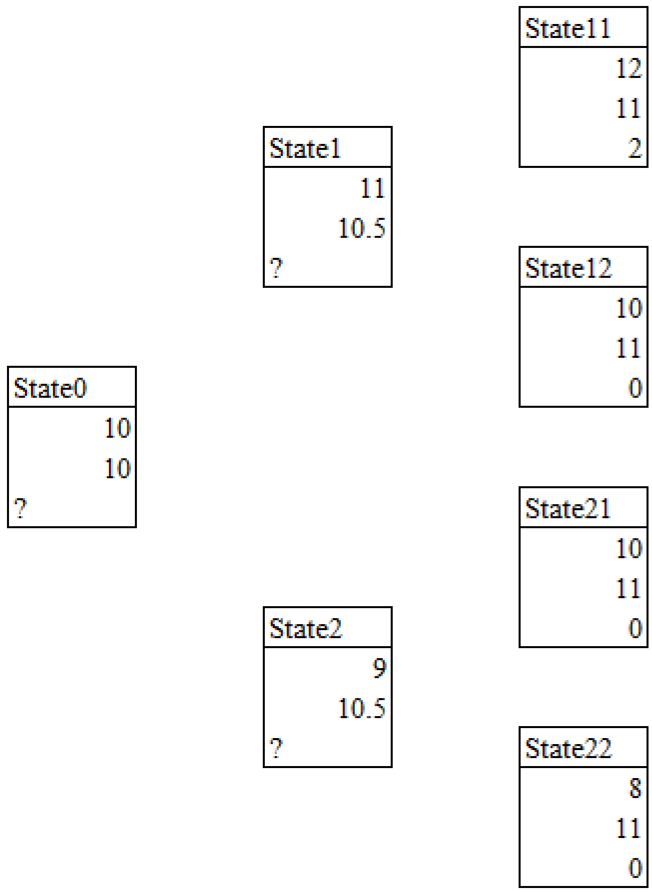
Also there exists a stock option that gives the right to buy 1 share at price \$10 next year. At that point, the option will be worth \$2 at state 1 and 0 at state 2.
Let π_1 and π_2 be the Arrow–Debreu security prices for the two states, then the payoff matrix

$$\begin{pmatrix} 12 & 8 \\ 11 & 11 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \Rightarrow \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{132 - 88} \begin{pmatrix} 11 & -8 \\ -11 & 12 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 15/22 \\ 5/22 \end{pmatrix}$$

The payoff matrix for the option is then (2,0) so the price of the option today must be

$$\begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 15/22 \\ 5/22 \end{pmatrix} = \frac{15}{11} \approx \$1.36 \text{ today}$$

3 periods: Stock price (the first), Bond price (the second), Option value (the third)



For State 1

$$\begin{pmatrix} 12 & 10 \\ 11 & 11 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 10.5 \end{pmatrix} \Rightarrow \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 11 & -10 \\ -11 & 12 \end{pmatrix} \begin{pmatrix} 11 \\ 10.5 \end{pmatrix} = \begin{pmatrix} .7272 \\ .2272 \end{pmatrix} \Rightarrow \text{Option} = 2\pi_1 = \$1.45$$

For State 2

$$\Rightarrow \begin{pmatrix} 10 & 8 \\ 11 & 11 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 10.5 \end{pmatrix} \Rightarrow \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} .6818 \\ .2727 \end{pmatrix} \Rightarrow \underbrace{\text{But Option}}_{\substack{\text{A–D security is} \\ \text{not necessary}}} = 0$$

For State 0

$$\begin{pmatrix} 11 & 9 \\ 10.5 & 10.5 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \Rightarrow \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 10.5 & -9 \\ -10.5 & 11 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} .7143 \\ .2381 \end{pmatrix} \Rightarrow \text{Option} = 2\pi_1 = \$1.04$$

		up @ time 1		down @ time 1	
		stock	option	Stock	option
price	t=0	10	1.04	10	1.04
	t=1	11	1.45	9	0
	return	10%	39.42%	−10%	−100%

* Option prices are more volatile!

Black–Scholes formula: Computing the price of a call option on a share of stock that has exercise date T and exercise price E.

$$P(t) = \text{price of the stock at the time t}$$
$$r = \text{riskless interest rate}$$

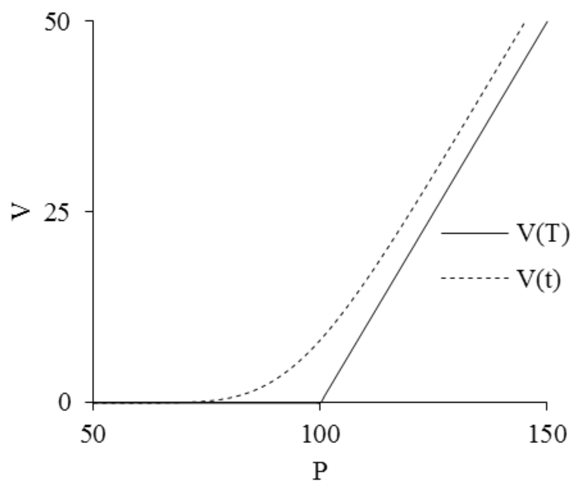
Key assumption: Stock price evolves stochastically according to geometric Brownian motion as $dt \to 0$

$$\frac{P(t+dt)-P(t)}{dt} \sim N(0, \sigma^2[P(t)]^2 dt) \text{ or } \frac{dP}{P} \sim N(0, \sigma^2 dt)$$
$$\Phi(\cdot) = \text{cdf of } N(0,1)$$
$$V(t) = \text{price of the call option at the time } t$$

Then, Black–Scholes formula is

$$V(t) = P(t)\Phi(d) - Ee^{-r(T-t)}\Phi\left(d - \sigma\sqrt{T-t}\right)$$
$$\text{where } d = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{P(t)}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right]$$

Option price is larger than intrinsic value because,
i) you can wait with the option (opportunity to decide in the future, not now)
ii) simply it is convex function of P (Jensen's inequality)
If σ^2 increases, then $V(t)$ will increase as well (more uncertain, more valuable).



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Welfare Economics, General Equilibrium

Notation

- i=1, ⋯, I for individuals, k=1, ⋯, K for commodities
- w_k^i=amount of commodity k endowed to person i
- x_k^i=amount of commodity k ultimately consumed by person i
- $\mathbf{w}^i=(w_1^i, w_2^i, \cdots, w_K^i)$: endowment vector for person i
- $\mathbf{x}^i=(x_1^i, x_2^i, \cdots, x_K^i)$: consumption vector for person i
- $u_i(\mathbf{x}^i)$=utility function for person i

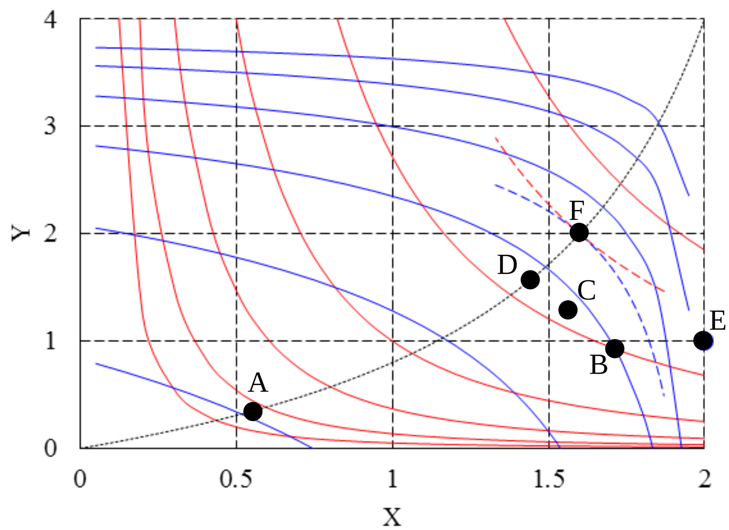
Definition An allocation $\mathbf{X}=(\mathbf{x}^1\ \mathbf{x}^2 \cdots \mathbf{x}^I)$ is a vector specifying each individual's consumption of each commodity. It is feasible if

$$\sum_{i=1}^{I} \mathbf{x}^i \leq \sum_{i=1}^{I} \mathbf{w}^i$$

Definition An allocation **Y** is *Pareto superior* to (or a *Pareto improvement* over) another allocation **X** if

$$u_i(\mathbf{y}^i) \geq u_i(\mathbf{x}^i)\ \forall i \text{ and } > \text{ for some } i$$

Definition An allocation **X** is *Pareto efficient* or *Pareto optimal* if no other feasible allocation is Pareto superior to **X**.



Edgeworth Box

i) A, D, F are all Pareto efficient (not unique).
ii) C is Pareto superior to B, but Pareto inferior to D.
iii) A is not Pareto superior to B or E (initial-condition dependent).
Therefore, Pareto relation is partial ordering. Question: Is this relation transitive?

Now suppose there are competitive markets for each of the K commodities.

Definition $\quad \mathbf{p}=(p_1, \cdots, p_K)$ to be a vector of prices for each of the commodities.

Definition $\quad$ For each individual i, define the budget set $B_i(\mathbf{p})$ as

$$B_i(\mathbf{p}) = \{\mathbf{x}^i | \mathbf{p}'\mathbf{x}^i \leq \mathbf{p}'\mathbf{w}^i\}$$

Definition $\quad$ A *Walasian* (or *Competitive*) equilibrium is a price vector $\mathbf{p}^*$ and allocation $\mathbf{X}^*$ such that

(a) For each individual i,

$$\mathbf{x}^{i*} \in \underset{\mathbf{x}^i \in B_i(\mathbf{p})}{\operatorname{argmax}} u_i(\mathbf{x}^i)$$

And (b) For each k,

$$\sum_{i=1}^{I} x_k^i \leq \sum_{i=1}^{I} w_k^i$$

Two important results

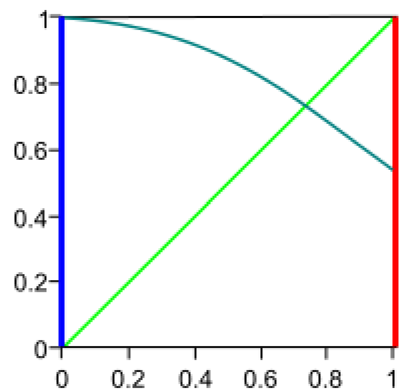
(1) Existence: A Walasian equilibrium exists.

(2) First (Fundamental) Theorem of Welfare Economics: Every Walasian equilibrium involves a Pareto efficient allocation $\mathbf{X}^* \Leftrightarrow (\mathbf{p}^*, \mathbf{X}^*)$

Proof of Existence

(i) Note: Brouwer's fixed point theorem

Let S be non-empty, convex, compact set and define function h:$S \rightarrow S$ to be continuous. Then, h has a fixed point; i.e. $\exists s \in S$ such that $h(s)=s$



(ii) Define $\boldsymbol{\pi}=\mathbf{p}/\mathbf{1}'\mathbf{p}$ then, since $\boldsymbol{\pi}>\mathbf{0}$ and $\mathbf{1}'\boldsymbol{\pi}=1$, $\boldsymbol{\pi} \in \Delta^{k-1}$ (the $k-1$ dimensional simplex) which is a non-empty, convex and compact set. (Note: this will be the S for our application of Brouwer's fixed point theorem)

(iii-a) Define $\mathbf{z}(\mathbf{p})$ to be aggregate excess demand function; i.e. $\mathbf{z}$:$\mathbb{R}_+^k \rightarrow \mathbb{R}_+^k$ such that

$$\mathbf{z}(\mathbf{p}) = \sum_{i=1}^{I} \underbrace{\mathbf{x}^i(\mathbf{p}, \mathbf{p}'\mathbf{w}^i)}_{\text{Marshallian demand}} - \mathbf{w}^i$$

→ Since demand curves are Homogenous of Degree 0 in prices and income, $\mathbf{z}(\boldsymbol{\pi})=\mathbf{z}(\mathbf{p})$

→ We have a Walasian equilibrium if and only if $\mathbf{z}(\boldsymbol{\pi}) \leq \mathbf{0}$

(iii-b) Define $\mathbf{f}(\boldsymbol{\pi})$:$\mathbb{R}_+^k \to \mathbb{R}_+^k$ by $f_j(\boldsymbol{\pi})=\max\{0,z_j(\boldsymbol{\pi})\} \Rightarrow \mathbf{f}(\boldsymbol{\pi})=\mathbf{0} \Leftrightarrow \mathbf{z}(\mathbf{p})\leq\mathbf{0} \Leftrightarrow$ Walasian equilibrium
(iii-c) Define (non-intuitively) $\mathbf{g}:\Delta^{k-1}\to\Delta^{k-1}$ by

$$\mathbf{g}(\boldsymbol{\pi}) = \frac{\boldsymbol{\pi} + \mathbf{f}(\boldsymbol{\pi})}{1 + \mathbf{1}'\mathbf{f}(\boldsymbol{\pi})} \Leftrightarrow g_k(\boldsymbol{\pi}) = \frac{\pi_k + f_k(\boldsymbol{\pi})}{\sum \pi_j + f_j(\boldsymbol{\pi})}$$

(iii-d) Claim

$$\mathbf{f}(\boldsymbol{\pi}) = \mathbf{0} \Leftrightarrow \mathbf{g}(\boldsymbol{\pi}) = \boldsymbol{\pi}$$

Proof

$$[\Rightarrow]\text{: if } \mathbf{f}(\boldsymbol{\pi}) = \mathbf{0} \text{ then } \mathbf{g}(\boldsymbol{\pi}) = \frac{\boldsymbol{\pi} + \mathbf{0}}{1 + \mathbf{1}'\mathbf{0}} = \boldsymbol{\pi}$$
$$[\Leftarrow]\text{: if } \mathbf{g}(\boldsymbol{\pi}) = \boldsymbol{\pi} \text{ then } \boldsymbol{\pi} + \mathbf{f}(\boldsymbol{\pi}) = \boldsymbol{\pi} + \boldsymbol{\pi}\big(\mathbf{1}'\mathbf{f}(\boldsymbol{\pi})\big)$$
$$\Leftrightarrow \mathbf{f}(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{1}'\mathbf{f}(\boldsymbol{\pi})$$
$$\text{where } \boldsymbol{\pi}\mathbf{1}' = \begin{pmatrix} \pi_1 & \cdots & \pi_1 \\ \pi_2 & \cdots & \pi_2 \\ \vdots & \cdots & \vdots \\ \pi_K & \cdots & \pi_K \end{pmatrix} \neq \mathbf{I} \Rightarrow \mathbf{f}(\boldsymbol{\pi}) = \mathbf{0}$$

(iii-e) Walasian equilibrium exists if **g** has a fixed point.
(iii-f) It does by Brouser's fixed point theorem. QED
Implicit assumptions
- Symmetric information, existence of competitive markets for all goods, no transaction cost

First Welfare Theorem

$$(\mathbf{p}, \mathbf{X}) \text{ is a Walasian equilibrium} \Rightarrow \mathbf{X} \text{ is Pareto efficient}$$

Proof (by contradiction): Assume **X** is not Pareto efficient.
Then there exists a different feasible allocation **Y** such that

$$\forall i \ni u_i(\mathbf{y}^i) \geq u_i(\mathbf{x}^i) \text{ and } \exists i \ni u_i(\mathbf{y}^i) > u_i(\mathbf{x}^i)$$

Then for each agent i, either

a. $\mathbf{y}^i \notin B_i(\mathbf{p}) \rightarrow \mathbf{p}'\mathbf{y}^i > \mathbf{p}'\mathbf{w}^i \geq \mathbf{p}'\mathbf{x}^i$ (this must be the case for some i)
b. $u_i(\mathbf{x}^i) = u_i(\mathbf{y}^i)$ and $\mathbf{p}'\mathbf{x}^i \leq \mathbf{p}'\mathbf{y}^i$ $\left(\text{in this case } \mathbf{p}'\mathbf{x}^i = \mathbf{p}'\mathbf{w}^i \because \mathbf{p}'\mathbf{x}^i < \mathbf{p}'\mathbf{w}^i \text{ means satiation}\right)$

So

$$\sum_i \mathbf{p}'\mathbf{y}^i > \sum_i \mathbf{p}'\mathbf{x}^i$$

Also, since **Y** is feasible, for each commodity k,

$$\sum_i y_k^i \leq \sum_i w_k^i \Rightarrow \sum_i p_k y_k^i \leq \sum_i p_k w_k^i, \forall k \because p_k \geq 0, \forall k$$

$$\Rightarrow \underbrace{\sum_k \sum_i p_k y_k^i}_{\sum \mathbf{p}'\mathbf{y}^i} \leq \underbrace{\sum_k \sum_i p_k w_k^i}_{\sum \mathbf{p}'\mathbf{w}^i} \Rightarrow \sum_i \mathbf{p}'\mathbf{w}^i = \sum_i \mathbf{p}'\mathbf{x}^i \xLeftrightarrow{\text{contradiction}} \sum_i \mathbf{p}'\mathbf{y}^i > \sum_i \mathbf{p}'\mathbf{x}^i$$

Walas law

$$p_k > 0 \Leftrightarrow \sum_i \mathbf{w}^i = \sum_i \mathbf{x}^i \text{ so } \underbrace{\sum_k \sum_i p_k w_k^i}_{\sum \mathbf{p}'\mathbf{w}^i} = \underbrace{\sum_k \sum_i p_k x_k^i}_{\sum \mathbf{p}'\mathbf{x}^i}$$

Therefore, no such allocation **Y** exists. ⇒ **X** is Pareto efficient.
The state occurred is already efficient; any regulation is no meaning.
Assumption
i) price-taker assumption, ii) ∃markets for every good, iii) no transaction costs, iv) symmetric information.

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$$X_{it} = \alpha_i + \sum_{j=1}^{N} \beta_{ij} P_{jt} + \gamma_i M_t + \varepsilon_t \text{ (reality)}$$
$$\frac{1}{T}\sum_{t=1}^{T} X_{it} = \alpha_i + \sum_{j=1}^{N} \beta_{ij} \frac{1}{T}\sum_{t=1}^{T} P_{jt} + \gamma_i \frac{1}{T}\sum_{t=1}^{T} M_t + \frac{1}{T}\sum_{t=1}^{T} \varepsilon_{it} \text{ (observable)}$$

* Why an allocation problem is harder to solve than an optimal production problem? Though it is not that hard to estimate β and γ, the policy maker must observe ε_{it} in order to optimize the distribution, but this mission is impossible at the aggregate level (product *i* for *t*; Heyek's argument).

Second Welfare Theorem: Every Pareto efficient allocation is a Walasian equilibrium for some initial endowment allocation (Proof is trivial).
Another Version: Suppose all preferences are "regular" and *W* is an initial allocation. Then every Pareto efficient allocation *X* can be obtained as a Walasian equilibrium by reallocating one good.

(**Reference the figure!!!!!: Edgeworth box**)

(In practice, reallocating requires significant deadweight costs, which reduce the size of Edgeworth box and this reduction is inefficient.)
Advantage of the Second Welfare Theorem: Finding an equilibrium is tedious, but finding a Pareto efficient allocation is straightforward.
Welfare Functions: Let there be *I* individuals.
Definition: A *Social Welfare Function* $W:\mathbb{R}^I \rightarrow \mathbb{R}$ ranks different allocations according to the utility received by each agent.
Example: A benevolent social planner might solve

$$\max_{\{\mathbf{x}^1,\mathbf{x}^2,\cdots,\mathbf{x}^I\}} W\left(u_1(\mathbf{x}^1), u_2(\mathbf{x}^2), \cdots, u_I(\mathbf{x}^I)\right) \text{ subject to constraints}$$

Examples of SWFs are
i) Utilitarian (Benthamite) function

$$W(u_1, \cdots, u_I) = \sum_{i=1}^{I} \lambda_i u_i$$

ii) Rawlsian (max-min) function

$$W(u_1, \cdots, u_I) = \min\{\alpha_1 u_1, \cdots, \alpha_I u_I\}$$

iii) Constant elasticity of substitution

$$W(u_1, \cdots, u_I) = \left(\sum_{i=1}^{I} \beta_i u_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

Theorem: Let W be a SWF such that $\partial W/\partial u_i > 0\ \forall i$ then $X^* \in \operatorname{argmax} W \Rightarrow X^*$ is Pareto efficient.
Theorem: Suppose preferences are "regular" and X^* is an inferior Pareto efficient allocation (i.e. $x_k^{i*}>0$) then X^* maximizes some utilitarian SWF.

$$X^* \in \operatorname{argmax} \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}^i) \text{ subject to } \sum_{i=1}^{I} \mathbf{x}^{i*} = \sum_{i=1}^{I} \mathbf{w}^i$$

Furthermore at X^* $\lambda_i = 1/MUI_i(X^*) = 1/\text{marginal utility of income} = \text{more weight for the rich}$
And the Lagrange multiplier on the kth resource constraint is the price of the kth good in the Walasian equilibrium.
Some intuition: can find a Pareto efficient allocation by solving

$$\max_{x\in X} u_i(\mathbf{x}^i) + \underbrace{\sum_{j\neq i}^{I} \lambda_j\left(u_j(\mathbf{x}^j) - u_j^*\right)}_{\substack{\text{constraints:person } j \text{ is} \\ \text{not worse off then } u_j^*}}$$

$$\overset{\text{same solution}}{\Rightarrow} \max_{x\in X}\left(\underbrace{\lambda_i u_i(\mathbf{x}^i)}_{\substack{\text{multiplied} \\ \text{by constraint}}} + \underbrace{\sum_{j\neq i}^{I} \lambda_j u_j^*}_{\substack{\text{added} \\ \text{constraint}}}\right) + \sum_{j\neq i}^{I} \lambda_j\left(u_j(\mathbf{x}^j) - u_j^*\right) = \max_{x\in X} \sum_{i=1}^{I} \lambda_i u_i(\mathbf{x}^i)$$

Ex. For HW,

$$u_i(\mathbf{x}^i) = \sum_k \alpha_k \log x_k^i$$

Finding Walasian equilibrium ⇒ i) For each agent

$$\max_{\{\mathbf{x}^i\}} \sum_k \alpha_k \log x_k^i \text{ subject to } \sum_k p_k x_k^i = \sum_k p_k w_k^i$$

$$\text{FOC:} \frac{\alpha_k}{x_k^i} = \theta_i p_k \ K \text{ equations for every } I \text{ and } \sum_k p_k x_k^i = \sum_k p_k w_k^i \ \text{ 1 equation for every } I$$

Solve to find $x_k^{i*}(\mathbf{p},\mathbf{p}'\mathbf{w}^i)$ and θ_i and ii) solve

$$\sum_i x_k^{i*}(\mathbf{p},\mathbf{p}'\mathbf{w}^i) = \sum_i w_k^i$$

Social Planner (convenient)

$$\max_{\{\mathbf{x}^i\}} \sum_i \lambda_i \left(\sum_k \alpha_k \log x_k^i\right) \text{ subject to } \sum_i \left(w_k^i - x_k^i\right) \geq 0\ \forall k$$

$$\text{FOC } x_k^i \Rightarrow \frac{\lambda_i \alpha_k}{x_k^i} = \mu_k \text{ and FOC } x_k^j \Rightarrow \frac{\lambda_j \alpha_k}{x_k^j} = \mu_k \Rightarrow \frac{\lambda_i}{x_k^i} = \frac{\lambda_j}{x_k^j} \Leftrightarrow \frac{\lambda_i}{\lambda_j} = \frac{x_k^i}{x_k^j}$$

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Social Planner's Problem

$$W(\mathbf{X}) = \sum_i \lambda_i u_i(\mathbf{x}^i) + \sum_k \mu_k \left[\sum_i (w_k^i - x_k^i)\right]$$

$$\Rightarrow \max_{\{x_k^i\}} W(\mathbf{X}) \Rightarrow \left.\frac{\partial L}{\partial x_k^i}\right|_{\max} = \lambda_i \left.\frac{\partial u_i}{\partial x_k^i}\right|_{\max} - \mu_k = 0$$

$$\Rightarrow \left.\frac{\partial L}{\partial x_m^i}\right|_{\max} = \lambda_i \left.\frac{\partial u_i}{\partial x_m^i}\right|_{\max} - \mu_m = 0$$

$$\Rightarrow \underbrace{\frac{\frac{\partial u_i}{\partial x_k^i}}{\frac{\partial u_i}{\partial x_m^i}}}_{\substack{\text{marginal rate}\\ \text{of substitution}}} = \underbrace{\frac{\mu_k}{\mu_m}}_{\substack{\text{then this should}\\ \text{be price ratio}}}$$

$$\Rightarrow \underbrace{\left.\frac{\partial u_i}{\partial x_k^i}\right|_{\max}}_{\substack{\text{when } i \text{ buy } k \text{ he}\\ \text{will get this utility}}} = \frac{\mu_k}{\lambda_i} = \underbrace{\underbrace{p_k}_{\text{price}} \underbrace{MUI_i}_{\substack{\text{marginal utility}\\ \text{of income}}}}_{\substack{\text{opportunity cost: when he}\\ \text{does not buy then he will}\\ \text{save this much}}}$$

$$\Rightarrow \left.\frac{\partial L}{\partial x_k^j}\right|_{max} = \lambda_j \frac{\partial u_j}{\partial x_k^j} - \mu_k = 0 \Rightarrow \underbrace{\lambda_j \frac{\partial u_j}{\partial x_k^j} = \mu_k = \lambda_i \frac{\partial u_i}{\partial x_k^i}}_{\substack{\text{here testable implication:}\\ \text{it is not in reality}}}$$

With, uncertainty, let s represent states

$$x(s) = \text{consumption in state } s$$
$$p(s) = \text{probability of state } s$$
$$v(x) = \text{von Neumann} - \text{Morgenstern utility function}$$

Then, the planner's problem is

$$\max_{\{x_k^i(s)\}} \sum_i \lambda_i u_i + \sum_s \sum_k \mu_k(s) \left[\sum_i \left(w_k^i(s) - x_k^i(s)\right)\right]$$

$$\text{where } u_i = \mathrm{E}\big(v_i(s)\big) = \sum_s p(s) v_i\left(\mathbf{x}^i(s)\right)$$

$$\Rightarrow \max_{\{x_k^i(s)\}} \sum_i \sum_s \lambda_i p(s) v_i\left(\mathbf{x}^i(s)\right) + \sum_s \sum_k \sum_i \mu_k(s)\left(w_k^i(s) - x_k^i(s)\right)$$

FOC

$$
\begin{aligned}
&\frac{\partial L}{\partial x_k^i(s)} \Rightarrow \lambda_i p(s) \left.\frac{\partial v_i}{\partial x_k^i(s)}\right|_{\max} = \mu_k(s) \\
&\Rightarrow \lambda_i \frac{\partial v_i}{\partial x_k^i(s)} = \lambda_j \frac{\partial v_j}{\partial x_k^j(s)} \text{ in every state } \left(\begin{array}{c}\text{we do not have to assume} \\ \text{identical utility over agents}\end{array}\right) \\
&\Rightarrow \frac{\partial v_i}{\partial x_k^i(s)} = \frac{\lambda_j}{\lambda_i} \frac{\partial v_j}{\partial x_k^j(s)} \\
&\Rightarrow \frac{\partial^2 v_i}{\partial x_k^i(s)\partial z} = \frac{\lambda_j}{\lambda_i} \frac{\partial^2 v_j}{\partial x_k^j(s)\partial z} \text{ they have same } \partial v \text{ and } \partial^2 v \text{ (not exactly but roughly)}
\end{aligned}
$$

Recall Walasian equilibrium

i)

$$
\begin{aligned}
\mathbf{x}^{i*} \in \underset{x^i \in B_i(\mathbf{p})}{\operatorname{argmax}}\, u_i(\mathbf{x}^i) &\xRightarrow[\text{equilibrium directly}]{\text{finding Walasian}} \left(\begin{array}{c}\text{(1) Find each agent's Marshallian demand function} \\ \max_{\{\mathbf{x}_i\}} u_i(\mathbf{x}^i) \text{ subject to } \mathbf{p}'\mathbf{w}^i \le \mathbf{p}'\mathbf{x}^i\end{array}\right) \\
&\Rightarrow \text{Solution} \quad \underbrace{\mathbf{x}^{i*}(\mathbf{p}, \mathbf{p}'\mathbf{w}^i)}_{\substack{\text{I×K functions} \\ \text{of K prices} \\ \text{(1 can be a numaire)}}} \\
&\Rightarrow \left(\begin{array}{c}\text{(2) Find equilibrium prices } \mathbf{p}^* \text{ by solving} \\ \sum_i \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^{*\prime}\mathbf{w}^i) = \sum_i \mathbf{w}^i\end{array}\right) \\
&\Rightarrow \left(\begin{array}{c}\text{(3) Find equilibrium allocation by plugging } \mathbf{p}^* \text{ into each } \mathbf{x}^{i*} \text{ function} \\ \{\mathbf{x}^{i*}(\mathbf{p}^*, \mathbf{p}^{*\prime}\mathbf{w}^i)\}\end{array}\right)
\end{aligned}
$$

To get this, (1) and (2) should be solved simultaneously.

⇔ Solving the social planner's problem is more convenient and tractable way.

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Definition *Private* benefits and costs accrue to decision makers.
Definition *External* benefits and costs accrue to other people who are not making the decisions.
Definition *Social*=*Private*+*External*

Let $B^P(x)$ be the Private Benefits associated with activity at level x.
$C^P(x)$→Private Costs, $B^E(x)$→External Benefits, $C^E(x)$→External Costs
Decision maker solves

$$\max_x B^P(x) - C^P(x)$$

FOC

$$\frac{d(B^P - C^P)}{dx} = 0$$

In contrast, a utilitarian social planner with equal λ-weights solves

$$\max_x B^P(x) - C^P(x) + B^E(x) - C^E(x)$$

FOC

$$\frac{d(B^P - C^P)}{dx} + \frac{d(B^E - C^E)}{dx} = 0$$

Coasian view: Takes into account possible payments between decision makers and external parties → If external parties offered upto $B^E(x)-C^E(x)$ in order to influence the decision maker, the decision makers problem would be

$$\max_x B^P(x) - C^P(x) + B^E(x) - C^E(x)$$

Ex. Carbon emission trading market: Right to contaminate, but must by the right (Coasian view)

E_1=endowment if smoker has the right to smoke
E_2=endowment if non-smoker has the right to clean air (both can achieve Pareto efficiency)

(**Refer the figure!!!!!: Edgeworth box**)

Some more severe barriers (markets will degenerate due to the reasons below; be careful in interpreting)

1. Hold-out problem: many people have to agree to sell their rights before the outcome can change: Profits generated due to the absence of trading, close to monopoly → market failure (ex. Eminent Domain, speculation) → Despite of the existence of Pareto improvement, there exists a possibility of no negotiation unfortunately.
2. Free-rider problem: possible for some external parties to benefit from payments made by other external parties: ex. Flut-shot
3. Assignment problem: do not know which decision makers' choices are affecting a given external party: ex. Global warming?

A. C. Pigou (eminent in externality researches)

Pigouvian taxes/subsidies: change the decision maker's problem by adding a marginal cost/benefit equal to the marginal external net cost.

The idea (example with a negative externality):

$$\text{say } \frac{d}{dx}\left(B^E(x) - C^E(x)\right) < 0$$

Imagine changing a per unit tax of

$$t = -\frac{d}{dx}\left(B^E(x) - C^E(x)\right)$$

Decision maker solves

$$\max_x B^P(x) - C^P(x) - tx$$

$$\text{FOC} \Rightarrow \frac{d}{dx}\left(B^P(x) - C^P(x)\right) - t = 0$$

$$\Leftrightarrow \frac{d}{dx}\left(B^P(x) - C^P(x)\right) + \frac{d}{dx}\left(B^E(x) - C^E(x)\right) = 0$$

(**Refer the figure!!!!!: related to Pigouvian tax**)

* Even with the Pigouvian tax, it is difficult to directly achieve the social optimum (requires trial & error).

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San Francisco Bridge Problem

$$\begin{aligned} q &= \text{cars in bridge} \\ N - q &= \text{cars in go around} \\ tq &= \text{time to cross the bridge} \\ t_0 &= \text{time to go around} \end{aligned}$$

$$\Rightarrow q_p \text{ solves } tq_p = t_0 \Rightarrow q_p = \frac{t_0}{t}$$

Social planner

$$\min_q q \times tq + (N - q)t_0$$

$$\text{FOC} \Rightarrow 2tq_s - t_0 = 0 \Rightarrow q_s = \frac{t_0}{2t} = \frac{q_p}{2}$$

Pigouvian toll: s dollars; now bridge costs $tq+s$ whereas around still costs t_0.

$$tq(s) + s = t_0 \Rightarrow q(s) = \frac{t_0 - s}{t}$$

Know we want

$$q = \frac{t_0}{2t}$$

So find s such that

$$\frac{t_0 - s}{t} = \frac{t_0}{2t} \Rightarrow s = \frac{t_0}{2}$$

Monopolist's problem: choose r to maximize

$$rq(r) = r\frac{t_0 - r}{t} \Rightarrow \frac{t_0}{t} - \frac{2}{t}r^* = 0 \Rightarrow r^* = \frac{t_0}{2} = s\ (OMG!)$$

Definition A good is *Rival* in consumption if one person's consumption of the good makes it harder or impossible for someone else to consume it.
Definition A good is *Excludable* if it is possible to prevent someone from consuming it if they do not pay for it.

	Excludable	Non-excludable
Rival	Private goods	Commons
Non-rival	Club goods	Public goods

* Commons: Fishery; free-rider problem
* Club goods: Superstar, television; originally lighthouse was, but monopolized so (then Public goods)
* Human Capital (example of missing market): high return, student loan (non-transferable collateral), high-risk (firms can refund their debt by selling their factories, but can one sell one's brain to refund?)

Thomas Veblen: Theory of Leisure Class, *Conspicuous Consumption*
* Altruism: child's utility function, hyperbolic discount
* Patent race: two incentives

i) Invention (socially valuable)

ii) To be the first (rank matters; individually valuable) → over R&D expenditure
* Herding behavior: running zebra (why? maybe lion, importance of idiosyncratic events ↓), stock bubble
* Crime (the poor get poorer): criminal cases ↑→police resource occupation ↑→likelihood of arrest ↓
* Teenager's peer effect: do not know what's going on, new haircut trends
* New financial assets: buy because they are just 'brand-new'

Let *x* be one good; *y* be another good; *s*=index of other people's consumption of *x*.
Utility function: maximize by choosing *x* and *y*, taking *s* as given.

$$u(x,y;s) \text{ subject to } px+y=I$$

Lagrangian function

$$L(x,y,\lambda)=u(x,y;s)+\lambda(I-px-y)$$

FOC

$$\frac{\partial L}{\partial x}=\frac{\partial u}{\partial x}-p\lambda^*=0, \qquad \frac{\partial L}{\partial y}=\frac{\partial u}{\partial y}-\lambda^*=0, \qquad \frac{\partial L}{\partial \lambda}=I-px^*-y^*=0$$

$$\Rightarrow \frac{\partial u}{\partial x}(x^*,y^*;s)-p\frac{\partial u}{\partial y}(x^*,y^*;s)=0$$

$$\text{As } s \text{ changes} \Rightarrow \frac{\partial}{\partial x}\left(u_x-pu_y\right)\frac{dx^*}{ds}+\frac{\partial^2 u}{\partial x\partial s}-p\underbrace{\frac{\partial^2 u}{\partial y\partial s}}_{=0}=0$$

$$\Rightarrow \frac{dx^*}{ds}=-\frac{\frac{\partial^2 u}{\partial x\partial s}}{\underbrace{\frac{\partial}{\partial x}\left(u_x-pu_y\right)}_{<0 \text{ by SOC}}} \Rightarrow \operatorname{sign}\left(\frac{dx^*}{ds}\right)=\operatorname{sign}\left(\frac{\partial^2 u}{\partial x\partial s}\right)$$

Assumption

$$\frac{\partial^2 u}{\partial x\partial s}>0, \qquad \frac{\partial^2 u}{\partial y\partial s}=0$$

Possible multiple equilibria: ex. (Professor Drewianka's PhD Dissertation)
Define *S*=% NOT married; δ=separation rate of existing marriages (exogenous); $\mu(S)$=marriage rate among unmarried people; Assumption $\frac{\partial \mu}{\partial s}<0$
Steady State Equilibrium $S_{t+1}=S_t$
Find it by using the dynamics of the problem

$$S_{t+1}=S_t+(1-S_t)\delta-\mu(S_t)S_t$$

Brouser's Fixed Point theorem

$$S_{t+1}=S_t \Leftrightarrow \underbrace{\delta-S_t\left(\delta+\mu(S_t)\right)=}_{\equiv S_{t+1}-S_t}0$$

(**Attach the figure here!!!!!: Multiple Equilibrium**)
Possible that there are multiple equilibria if

$$\frac{d}{dS}\left(\delta-S\left(\delta+\mu(S)\right)\right)>0 \text{ for some } S \Rightarrow \underbrace{-(\delta+\mu)}_{-}\underbrace{-S\frac{\partial \mu}{\partial S}}_{+}$$

Social Multiplier (herding behavior, spill-over, cultural spread)
Let

$$x_i = \text{person i's consumption of some good}$$
$$S = \frac{1}{I}\sum_{i=1}^{I} x_i \ \text{(average person's consumption)}$$

Model

$$x_i = a - bp + cS \Rightarrow \frac{\partial x_i}{\partial p} = -b, \qquad \text{where } p = \text{price}$$
$$\Rightarrow S = \frac{1}{I}\sum_{i=1}^{I} x_i = a - bp + cS = \frac{a-bp}{1-c} \Rightarrow \left|\frac{dS}{dp}\right| = \underbrace{\left|-\frac{b}{1-c}\right|}_{\text{Social Multiplier}} > |-b|$$

If c=0.2, then $\frac{\partial S}{\partial p}$ will be 25% larger than $\frac{\partial x_i}{\partial p}$. If $c \to 1$, then $\frac{\partial S}{\partial p}$ will be explosive.
Ex. x crime rate, p punishment.
→ If c is large, then $-\frac{b}{1-c}$ will be larger in absolute value. (Reflection problem by Charles Manski)

(**HW Solution PASS!!!!!**)

Ex. of outcome that depends on rank: patent race
Two identical firms competing to be first to invent a product and thus get a patent
π=monopoly profit if product is invented, c=consumer surplus if product is invented
I_j=investment in R&D by firm j=1 or 2
$P_j=P(I_j)$=probability that firm j succeeds in developing the product
P_1=probability that firm 1 succeeds, P_2=probability that firm 2 succeeds
$P_1+P_2-P_1P_2$=probability that anyone develops the product
Social planner who values everyone equally

$$\max_{I_1,I_2}(\pi + c)\big(P(I_1) + P(I_2) - P(I_1)P(I_2)\big) - (I_1 + I_2) \Rightarrow \text{FOC: } \underbrace{(\pi + c)\Big(P'(I_1)\big(1 - P(I_2)\big)\Big)}_{\substack{\text{marginal social benefit of} \\ \text{\$1 investment RnD at firm 1}}} = 1$$

$$\text{if } c = 0 \text{ and } \pi = 1 \Rightarrow P'(I_1) = \frac{1}{1 - P(I_2)}$$

Firm 1's own choice

$$\max_{I_1} \pi\left(P(I_1)\left(1 - \underbrace{\frac{P(I_2)}{2}}_{\substack{\text{if both firms succeed probability} \\ \text{that firm 1 is the first=1/2}}}\right)\right) - I_1$$

$$\text{FOC: } \underbrace{\pi\left(P'(I_1)\left(1 - \frac{P(I_2)}{2}\right)\right)}_{\text{marginal benefit for firm 1}} = 1 \Rightarrow \text{if } c = 0 \text{ and } \pi = 1 \Rightarrow P'(I_1) = \frac{2}{2 - P(I_2)} < \text{the former}$$

1. (a) Current: p_1, F_1

Legislation: $p_2<p_1$, $F_2>F_1$

Let W_0 be the initial wealth of a person and let a random variable W_1 be the wealth of a disobeying person under the old regime. Then,

$$W_1=\begin{cases} W_0-F_1, & \text{with } p_1 \\ W_0, & \text{with } 1-p_1 \end{cases}$$

Similarly, let a random variable W_2 be the wealth of a disobeying person under the new regime. Then,

$$W_2=\begin{cases} W_0-F_2, & \text{with } p_2 \\ W_0, & \text{with } 1-p_2 \end{cases}$$

Then, the cumulative distribution function of W_1 and W_2 can be expressed as below.

$$G_1(w)=\begin{cases} 0, & \text{if } w<W_0-F_1 \\ p_1, & \text{if } W_0-F_1\le w<W_0 \\ 1, & \text{if } w\ge W_0 \end{cases}$$

$$G_2(w)=\begin{cases} 0, & \text{if } w<W_0-F_2 \\ p_2, & \text{if } W_0-F_2\le w<W_0 \\ 1, & \text{if } w\ge W_0 \end{cases}$$

Therefore, $G_2(w)=p_2>0=G_1(w)$ if $W_0-F_2\le w<W_0-F_1$ and $G_1(w)=p_1>p_2=G_2(w)$ if $W_0-F_1\le w<W_0$. Hence, W_1 does not FOSD W_2 and vice versa. However,

$$G_2(w)-G_1(w)=\begin{cases} 0, & \text{if } w<W_0-F_2 \\ p_2, & \text{if } W_0-F_2\le w<W_0-F_1 \\ p_2-p_1, & \text{if } W_0-F_1\le w<W_0 \\ 0, & \text{if } w\ge W_0 \end{cases}$$

$$\int_{-\infty}^{a}[G_2(w)-G_1(w)]dw=\begin{cases} 0, & \text{if } a<W_0-F_2 \\ p_2(a-W_0+F_2), & \text{if } W_0-F_2\le a<W_0-F_1 \\ p_2(F_2-F_1)-(p_1-p_2)(a-W_0+F_1), & \text{if } W_0-F_1\le a<W_0 \\ 0, & \text{if } a\ge W_0 \end{cases}$$

If $W_0-F_1\le a<W_0$,

$$\begin{aligned} -(p_1-p_2)(a-W_0+F_1) &= p_2F_2-p_2F_1-p_1a+p_1W_0-p_1F_1+p_2a-p_2W_0+p_2F_1 \\ &= (p_2-p_1)a-(p_2-p_1)W_0 \\ &= \underbrace{(p_1-p_2)}_{>0}\underbrace{(W_0-a)}_{>0}>0 \end{aligned}$$

$$\therefore \int_{-\infty}^{a}[G_2(w)-G_1(w)]dw\ge 0\ \forall a \Rightarrow G_1 \overset{\text{FOSD}}{\gtrsim} G_2\ \blacksquare$$

Therefore, if the drivers are risk-averse, then disobeying drivers will decrease.

(b) The expected total amount of fines collected is,

$$\text{Total amount}_1 = \frac{n_1 = \text{Disobeying driver}}{N = \text{Total driver}} \times p_1 \times F_1, \text{Total amount}_2 = \frac{n_2}{N} \times p_2 \times F_2 = \frac{n_2 p_1 F_1}{N}$$

Because $n_1 > n_2$ in the world of risk-averse drivers, the total amount of fines will decrease.

2. The probability density function of Weibull distribution is

$$f(x;\alpha,\beta) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

Where α and β are both greater than 0. The cumulative density function of the distribution is

$$F(x;\alpha,\beta) = \begin{cases} 1 - e^{-\alpha x^\beta}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

(a)

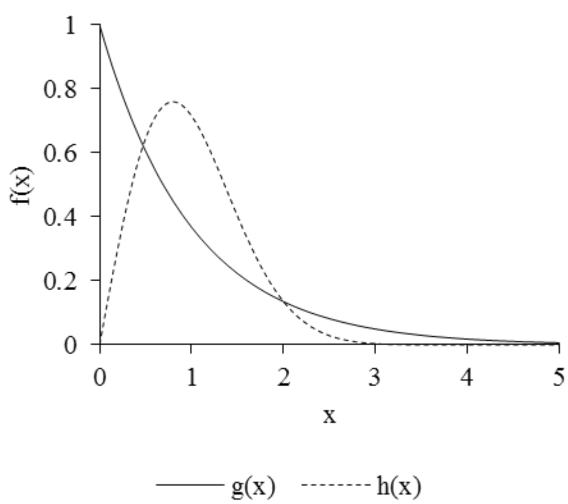
$$\alpha_1 > \alpha_2 > 0 \Leftrightarrow \alpha_1 x^\beta > \alpha_2 x^\beta \Leftrightarrow -\alpha_1 x^\beta < -\alpha_2 x^\beta \Leftrightarrow e^{-\alpha_1 x^\beta} < e^{-\alpha_2 x^\beta} \Leftrightarrow \underbrace{1 - e^{-\alpha_1 x^\beta}}_{=F(x;\alpha_1,\beta)=F_1} > \underbrace{1 - e^{-\alpha_1 x^\beta}}_{=F_2}$$

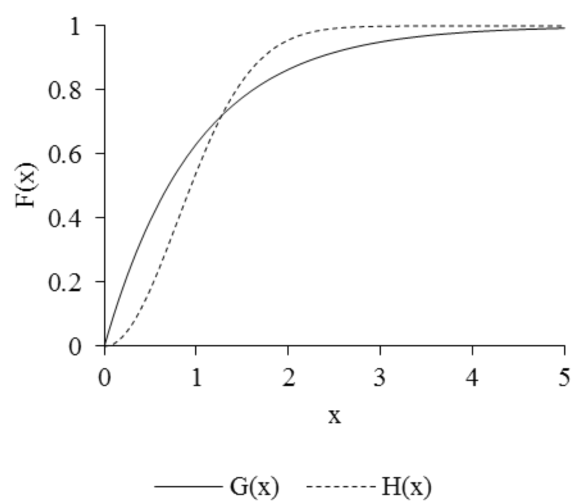
Therefore, if $\alpha_1 > \alpha_2$, then $F_1(x) > F_2(x)$ $\forall x$ and this implies F_2 FOSD F_1; agents will prefer α_2 ($<\alpha_1$). This preference does not depend on the risk-averse assumption; i.e. risk-loving agents will also prefer α_2 over α_1 since F_2 FOSD F_1.

(b)

$$g(x;1) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \Rightarrow G(x;1) = \begin{cases} 1 - e^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

$$h(x;\pi/4) = \begin{cases} \frac{\pi}{2} x e^{-\frac{\pi}{4}x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \Rightarrow H(x;\pi/4) = \begin{cases} 1 - e^{-\frac{\pi}{4}x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$





i. Since G and H have identical mean,

$$\int_{-\infty}^{\infty}[G(x)-H(x)]dx=\int_{0}^{\infty}[G(x)-H(x)]dx=0$$

And,

$$G(x)-H(x)=e^{-\frac{\pi}{4}x^2}-e^{-x}\geq 0 \Leftrightarrow e^{-\frac{\pi}{4}x^2}\geq e^{-x} \Leftrightarrow -\frac{\pi}{4}x^2\geq -x \Leftrightarrow x-\frac{\pi}{4}x^2\geq 0 \Leftrightarrow x\left(1-\frac{\pi}{4}x\right)\geq 0$$

Therefore, $G(x)-H(x)\geq 0 \Leftrightarrow 0<x\leq 4/\pi$. Then,

$$\int_{0}^{a}[G(x)-H(x)]dx\geq 0\ \forall 0<a\leq\frac{4}{\pi}$$

ii. From the above,

$$\int_{0}^{\infty}[G(x)-H(x)]dx=\int_{0}^{\frac{4}{\pi}}[G(x)-H(x)]dx+\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx=0$$

$$\Rightarrow\int_{0}^{\frac{4}{\pi}}[G(x)-H(x)]dx=-\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx$$

Hence,

$$-\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx\geq-\int_{\frac{4}{\pi}}^{a}[G(x)-H(x)]dx\ \ \forall\frac{4}{\pi}<a<\infty$$

$$\Rightarrow-\int_{\frac{4}{\pi}}^{\infty}[G(x)-H(x)]dx+\int_{\frac{4}{\pi}}^{a}[G(x)-H(x)]dx\geq 0\ \forall\frac{4}{\pi}<a<\infty$$

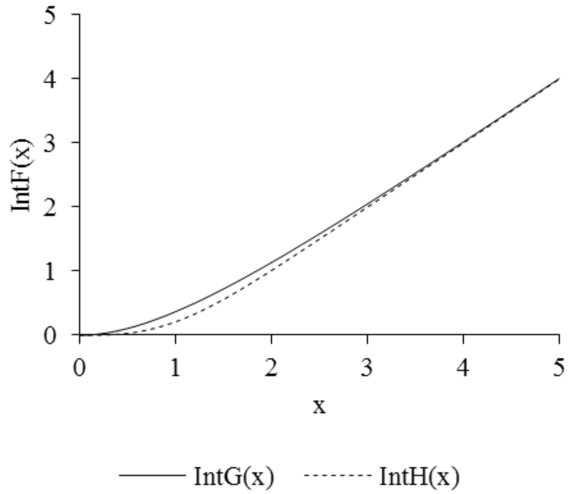
$$\Rightarrow\int_{0}^{\frac{4}{\pi}}[G(x)-H(x)]dx+\int_{\frac{4}{\pi}}^{a}[G(x)-H(x)]dx=\int_{0}^{a}[G(x)-H(x)]dx\geq 0\ \forall\frac{4}{\pi}<a<\infty$$

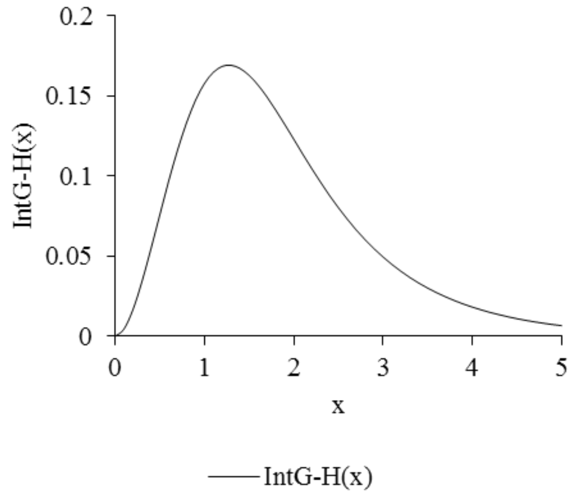
Therefore, from i and ii,

$$\int_{0}^{a}[G(x)-H(x)]dx\geq 0\ \forall a>0\ \blacksquare$$

Note that,

$$\int_0^a G(x)dx = a + e^{-a} - 1, \int_0^a H(x)dx = a - \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}a\right), \int_0^a [G(x) - H(x)]dx = \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}a\right) + e^{-a} - 1$$





3.

(a) i. If $\partial^2 u/\partial c\partial z>0$, then the marginal utility of consumption would be larger when there exist more amenities to enjoy. If $\partial^2 u/\partial c\partial z=0$, then the individuals given y will always consume the same amount.

$$\frac{\partial^2 u}{\partial c \partial z} > 0 \Rightarrow \frac{\partial u}{\partial c}(c, z_h) > \frac{\partial u}{\partial c}(c, z_l)$$

However, if the derivative is positive, then the consumers will spend more because they can take more joy from the environment they are facing with.

ii. For the utility function u,

$$u(y, z_l) = y^b z_l \Rightarrow u' = by^{b-1}z_l \text{ and } u'' = b(b-1)y^{b-2}z_l$$

$$\Rightarrow r(y) = -\frac{b(b-1)y^{b-2}z_l}{by^{b-1}z_l} = \frac{1-b}{y} > 0$$

$$u(y, z_h) = (y - r_h)^b z_h \Rightarrow u' = b(y - r_h)^{b-1} z_h \text{ and } u'' = b(b-1)(y - r_h)^{b-2} z_h$$

$$\Rightarrow r(y) = -\frac{b(b-1)(y - r_h)^{b-2} z_h}{b(y - r_h)^{b-1} z_h} = \frac{1-b}{y - r_h} > 0 \text{ if } y > r_h$$

(b) The individuals will increase r_h until the utility from l and the utility from h are equal.

$$y^b z_l = (y - \bar{r}_h)^b z_h \Leftrightarrow (y - \bar{r}_h)^b = y^b \frac{z_l}{z_h} \Leftrightarrow y - \bar{r}_h = y \sqrt[b]{z_l/z_h} \Leftrightarrow \bar{r}_h(y) = y\left(1 - \sqrt[b]{z_l/z_h}\right)$$

If r_h is given, then

$$\bar{y} = \frac{r_h}{1 - \sqrt[b]{z_l/z_h}} \ \therefore y > \bar{y} \Rightarrow \text{choose } h \text{ and } y < \bar{y} \Rightarrow \text{choose } l$$

(c)

i. The income is a random variable $Y \sim F(y)$, then

$$k = P(Y > \bar{y}) = 1 - P(Y < \bar{y}) = 1 - F(\bar{y}) = 1 - F\left(\frac{r_h}{1 - \sqrt[b]{z_l/z_h}}\right)$$

$$\Rightarrow F\left(\frac{r_h}{1 - \sqrt[b]{z_l/z_h}}\right) = 1 - k$$

$$\Rightarrow \frac{r_h}{1 - \sqrt[b]{z_l/z_h}} = F^{-1}(1-k)$$

$$\therefore r_h = \left(1 - \sqrt[b]{z_l/z_h}\right) F^{-1}(1-k)$$

$$\Rightarrow \frac{\partial r_h}{\partial k} = -\underbrace{\left(1 - \sqrt[b]{z_l/z_h}\right)}_{>0} \underbrace{\frac{\partial F^{-1}}{\partial(1-k)}}_{>0} < 0$$

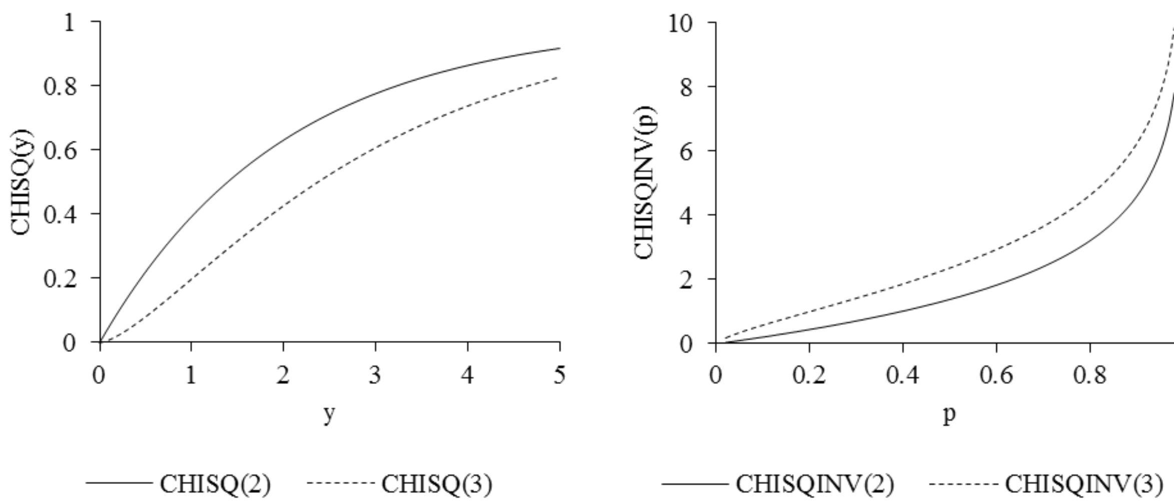
$$\Rightarrow \frac{\partial r_h}{\partial(z_h/z_l)} = \frac{\partial}{\partial(z_h/z_l)}\left[1 - (z_h/z_l)^{-\frac{1}{b}}\right] F^{-1}(1-k)$$

$$= \frac{1}{b}(z_h/z_l)^{-\frac{1+b}{b}} F^{-1}(1-k) = \underbrace{b^{-1}}_{>0} \underbrace{(z_l/z_h)^{\frac{1+b}{b}}}_{>0} \underbrace{F^{-1}(1-k)}_{>0} > 0$$

ii. $G(y)$ FOSD $F(y) \Leftrightarrow F(y) \geq G(y)\ \forall y \Leftrightarrow G^{-1}(p) \geq F^{-1}(p)\ \forall p$. Therefore,

$$r_h^F = \left[1 - (z_l/z_h)^{\frac{1}{b}}\right] F^{-1}(1-k) \leq \left[1 - (z_l/z_h)^{\frac{1}{b}}\right] G^{-1}(1-k) = r_h^G$$

Therefore, r_h will increase; if the income level of society increases, then the amount that people must pay to live in the location h would also increases; under the rich world with G, the rent r_h would be expensive.



Note that, if G FOSD F then $G^{-1}(p) \geq F^{-1}(p)\ \forall 0 \leq p \leq 1$ and $\exists 0 \leq p' \leq 1$ such that $G^{-1}(p') > F^{-1}(p')$.

(d) i.

Decision	Win p	Lose $(1-p)$
Not invest	$\bar{y}$	$\bar{y}$
Invest	$\bar{y}-\phi+\phi/p$	$\bar{y}-\phi$

Then, for the individual with $\bar{y}$, the expected utility of "not invest" is,

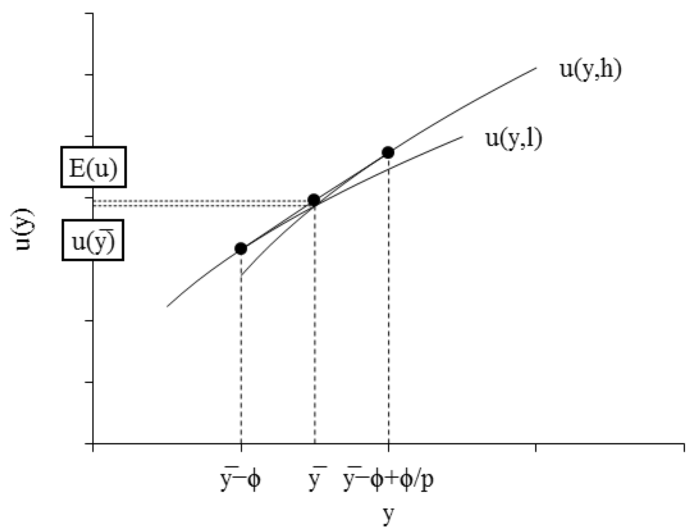
$$\mathrm{E}(u^n) = p\bar{y}^b z_l + (1-p)\bar{y}^b z_l = \bar{y}^b z_l \big(= (\bar{y}-r_h)^b z_h \ \because \text{indifferent}\big)$$

While the indirect utility function of an individual is not conventional as below.

$$u(y) = \max[\bar{y}^b z_l, (\bar{y}-r_h)^b z_h]$$

Hence, the expected utility of "invest" is

$$\mathrm{E}(u^i) = p\left(\bar{y}-\phi+\frac{\phi}{p}-r_h\right)^b z_h + (1-p)(\bar{y}-\phi)^b z_l \geq \bar{y}^b z_l = \mathrm{E}(u^n)$$



For the individual with $\bar{y}$, the expected utility with the investment is greater than that without the investment. Therefore, the individual will exploit the opportunity to maximize the expected utility. An individual whose income lies around $\bar{y}$ will also choose the investment since the expected utility comes from the fair bet exceeds the utility of the status quo. However, an individual whose income is too small to shift their status by investing will not choose to bet because the individual is locally risk-averse. Likewise, an individual whose income is large enough will not choose to gamble.

ii. The investment opportunity both increase agents' (*ex-ante* expected) utility (a. True) and causes a mean-preserving spread of the (ex-post) distribution of income (b. True).

a. For the range around $\bar{y}$, introducing the investment opportunity will improve the agent's expected utility. For the outside of this range, introducing the opportunity will not hurt the agent's utility. Overall, introducing the investment opportunity would have positive effects.

b. The mean of "invest" and that of "not invest" is identical, but the former choice is definitely more risky. Thus, "not invest" SOSD "invest" in this case.

(1) Why those effects are not in conflict? Because the von-Neumann–Morgenstern utility function here is $u(y)=\max[y^b z_l,(y-r_h)^b z_h]$ and this function is not globally concave.

1.

(a) If the agent is risk-neutral, then the agent's von Neumann–Morgenstern utility function should be neither concave nor convex; hence the function is globally linear. Therefore, the agent's expected utility will depends only on the expected return. In other words, the investor will not consider the degree of the uncertainty of outcome.

$$
\begin{aligned}
& v(\pi) = a + b\pi \text{ where } b > 0 \\
\Rightarrow\ & u(\pi) = \mathrm{E}\big(v(\pi)\big) \\
& \quad = a + b\mathrm{E}(\pi) \neq f(\mathrm{Var}(\pi), \mathrm{Skew}(\pi), \mathrm{Kurt}(\pi), \cdots)
\end{aligned}
$$

(b)

$$
\begin{aligned}
V(\mathbf{p}) &= \max_{\mathbf{x}} u(\pi) = \max_{\mathbf{x}}\big(a + b\mathrm{E}(\pi)\big) \overset{\text{equivalent}}{\approx} \max_{\mathbf{x}} \mathrm{E}(\pi) \\
&= \mathbf{p}' \underbrace{\boldsymbol{\pi}^*}_{\text{optimal solution given } \mathbf{p}} = \mathbf{p}'\mathbf{R} \underbrace{\mathbf{x}^*}_{\text{optimal solution given } \mathbf{p}} \\
\text{define } \mathbf{p}_3 &= \alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2 \text{ where } \alpha \in (0,1) \\
V(\mathbf{p}_3) &= \mathbf{p}_3'\boldsymbol{\pi}_3^* \\
&= \mathbf{p}_3'\mathbf{R}\mathbf{x}_3^* \\
&= (\alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2)\mathbf{R}\mathbf{x}_3^* \\
&= \alpha\mathbf{p}_1\mathbf{R}\mathbf{x}_3^* + (1-\alpha)\mathbf{p}_2\mathbf{R}\mathbf{x}_3^* \\
&\leq \alpha\mathbf{p}_1\mathbf{R} \underbrace{\mathbf{x}_1^*}_{\text{optimal solution given } \mathbf{p}_1} + (1-\alpha)\mathbf{p}_2\mathbf{R} \underbrace{\mathbf{x}_2^*}_{\text{optimal solution given } \mathbf{p}_2} \\
&= \alpha V(\mathbf{p}_1) + (1-\alpha)V(\mathbf{p}_2) \\
\Rightarrow V(\alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2) &\leq \alpha V(\mathbf{p}_1) + (1-\alpha)V(\mathbf{p}_2) \text{ hence convex } \blacksquare
\end{aligned}
$$

(c) In the equation above, the left-hand-side means the indirect utility that can be achieved under the uncertainty, and the right-hand-side means the indirect utility that can be achieved under the certainty (this does not mean that the situation is certain). The meaning of "possessing" options is that the agent can make the best decision state-by-state (in this case, $\mathbf{x}_1^*$, $\mathbf{x}_2^*$, etc.) after certain state is realized. However, if there is no option to choose, then the agent should make decision that may or may not be the best decision (in this case, $\mathbf{x}_3^*$) before the true state is revealed (here α and $1-\alpha$ can be regarded as the respective probability of Bernoulli random environment).

(d) If another agent is more risk-averse then the risk-neutral agent above, then he or she will more hate this randomness and, if it is possible, that risk-averse agent will "bribe" or something to avoid that unfavorable status. Hence the options will be more valuable for the agent (V will be still convex).

On the other hand, the third risk-seeking agent may enjoy that stochastic circumstances and hence will not try to escape from the conditions he or she is facing with. Therefore, this agent will more under-evaluate the value of options than the first and the second agents (V will be concave). Indeed, this risk-loving agent will pay to enjoy the gamble (i.e. the agent will provide the options to the others).

(a) Given Cobb–Douglas utility function is

$$U_i(\mathbf{x}^i) = \sum_{k=1}^{K} \alpha_k \log x_k^i \text{ where } \sum_{k=1}^{K} \alpha_k = 1$$

$$\Rightarrow U_i(\mathbf{x}^i) = \alpha_1 \log x_1^i + \cdots + \alpha_K \log x_K^i$$

$$= \log(x_1^i)^{\alpha_1} + \cdots + \log(x_K^i)^{\alpha_K}$$

$$= \log(x_1^i)^{\alpha_1} \cdots (x_K^i)^{\alpha_K}$$

$$\Rightarrow \max_{\mathbf{x}^i} U_i \approx \max_{\mathbf{x}^i} \underbrace{(x_1^i)^{\alpha_1} \cdots (x_K^i)^{\alpha_K}}_{\text{Cobb–Douglas Form}}$$

(b) Budget constraint

$$\sum_{k=1}^{K} p_k x_k^i = \sum_{k=1}^{K} p_k e_k^i = m^i$$

Lagrangian function; to avoid any possible notational confliction, I will hereafter use ‘hat’ instead of ‘asterisk’ to denote maximization.

$$L(\mathbf{x}^i, \lambda) = U_i(\mathbf{x}^i) + \lambda(m^i - \mathbf{p}'\mathbf{x}^i)$$

$$\Rightarrow \text{first order condition } \left.\frac{\partial L}{\partial x_k^i}\right|_{\max} = \frac{\alpha_k}{\hat{x}_k^i} - \hat{\lambda} p_k = 0 \Rightarrow \hat{\lambda} = \frac{\alpha_k}{\hat{x}_k^i p_k} = \frac{\alpha_1}{\hat{x}_1^i p_1} = \cdots = \frac{\alpha_K}{\hat{x}_K^i p_K}$$

$$\Rightarrow \hat{x}_k^i p_k = \frac{\alpha_k}{\hat{\lambda}} \Rightarrow \sum_{k=1}^{K} \hat{x}_k^i p_k = \frac{1}{\hat{\lambda}} \underbrace{\sum_{k=1}^{K} \alpha_k}_{=1} = m^i \Rightarrow \hat{\lambda} = \frac{1}{m^i}$$

$$\Rightarrow \text{Marshallian demand } \hat{x}_k^i = \frac{\alpha_k}{\hat{\lambda} p_k} = \frac{\alpha_k}{p_k} m^i$$

$$\text{indirect utility } \widehat{U}_i = \sum_{k=1}^{K} \alpha_k \log\left(\frac{\alpha_k}{p_k} m^i\right) = \sum_{k=1}^{K} \alpha_k \log \frac{\alpha_k}{p_k} + \log m^i \sum_{k=1}^{K} \alpha_k$$

$$= \log m^i + \sum_{k=1}^{K} \alpha_k \log \alpha_k - \sum_{k=1}^{K} \alpha_k \log p_k$$

$$\Rightarrow \text{marginal utility of income } \frac{\partial \widehat{U}_i}{\partial m^i} = \frac{1}{m^i}$$

(c) Walasian equilibrium is a pair $(\hat{\mathbf{p}}, \hat{\mathbf{X}})$ such that

$$\hat{\mathbf{x}}^i \in \underset{\mathbf{x}^i \in B_i(\hat{\mathbf{p}})}{\operatorname{argmax}} u_i(\mathbf{x}^i) \text{ and } \sum_{k=1}^{K} \hat{\mathbf{x}}^i \leq \sum_{k=1}^{K} \mathbf{e}^i \ \forall i$$

Parameters:

i) Preference: K number of 'α's for all i=1, ⋯, I. Total K×I.

ii) Endowment: K number of 'e's for all i=1, ⋯, I. Total K×I.

⇒ Total 2KI parameters are required.

Variables:

i) Marshallian demand: K number of '$\hat{x}$'s for all i=1, ⋯, I. Total K×I.

ii) the Market price: K number of '$\hat{p}$,' Total K.

⇒ Total (I+1)K variables are required.

Systems of equations:

First order condition: Every individual should solve the following Lagrangian function.

$$L(\mathbf{x}^i, \lambda) = u_i(\mathbf{x}^i) + \lambda(\hat{\mathbf{p}}'\mathbf{e}^i - \hat{\mathbf{p}}'\mathbf{x}^i)$$

Alternative 'Utilitarian Social Welfare' function approach

$$W(u_1, \cdots, u_I) = \sum_{i=1}^{I} \lambda_i u_i \text{ subject to } \sum_{i=1}^{I} x_k^i \leq e_k$$

$$L(u_1(\mathbf{x}^1), \cdots, u_I(\mathbf{x}^I), \mu_1, \cdots, \mu_K) = \sum_{i=1}^{I} \lambda_i u_i(x_1^i, \cdots x_K^i) + \sum_{k=1}^{K} \mu_k \left(e_k - \sum_{i=1}^{I} x_k^i \right)$$

(d)

$$\left.\frac{\partial L}{\partial x_k^i}\right|_{\max} = \lambda_i \frac{\alpha_k^i}{\hat{x}_k^i} - \hat{\mu}_k = 0 \Rightarrow \hat{x}_k^i = \frac{\lambda_i \alpha_k^i}{\hat{\mu}_k}, \hat{x}_k^j = \frac{\lambda_j \alpha_k^j}{\hat{\mu}_k} \Rightarrow \frac{\hat{x}_k^j}{\hat{x}_k^i} = \frac{\lambda_j \alpha_k^j}{\lambda_i \alpha_k^i} = \frac{\lambda_j}{\lambda_i} \text{ under identical preference}$$

(e)

$$\frac{\hat{x}_k^j}{\hat{x}_k^i} = \frac{\lambda_j}{\lambda_i} \Rightarrow \sum_{j=1}^{I} \frac{\hat{x}_k^j}{\hat{x}_k^i} = \frac{e_k}{\hat{x}_k^i} = \frac{\sum \lambda_j}{\lambda_i} \Rightarrow \hat{x}_k^i = \underbrace{\frac{\lambda_i}{\sum \lambda_j}}_{=\beta_i} e_k = \beta_i e_k$$

(f)

$$\hat{x}_k^i = \beta_i e_k \Rightarrow m^i = \sum_{k=1}^{K} p_k \hat{x}_k^i = \beta_i \underbrace{\sum_{k=1}^{K} p_k e_k}_{=m} = \beta_i m = \frac{\lambda_i}{\sum \lambda_j} m \Rightarrow \beta_i = \frac{m^i}{m} = \frac{\lambda_i}{\sum \lambda_j}$$

(f) (Cont'd)

$$\Rightarrow \hat{u}_i = \sum_{k=1}^{K} \alpha_k \log \hat{x}_k^i = \sum_{k=1}^{K} \alpha_k (\log \beta_i + \log e_k) = \log \beta_i \underbrace{\sum_{k=1}^{K} \alpha_k}_{=1} + \sum_{k=1}^{K} \alpha_k \log e_k$$

$$= \log \beta_i + \sum_{k=1}^{K} \alpha_k \log e_k = \log m^i - \log m + \sum_{k=1}^{K} \alpha_k \log e_k$$

$$\Rightarrow \frac{\partial \hat{u}_i}{\partial m^i} = \frac{1}{m^i}$$

(g)

$$\hat{x}_k^i = \frac{\lambda_i \alpha_k}{\hat{\mu}_k}$$

$$\Rightarrow \hat{\mu}_k = \frac{\lambda_i \alpha_k}{\hat{x}_k^i} = \frac{\lambda_i \alpha_k}{\beta_i e_k} = \frac{\sum \lambda_j}{\lambda_i} \frac{\lambda_i \alpha_k}{e_k} = \frac{\alpha_k \sum \lambda_j}{e_k}$$

$$\Rightarrow m = \sum_{k=1}^{K} p_k e_k = \underbrace{e_1 + \sum_{k=2}^{K} p_k^* e_k}_{\text{imposing } p_1 = 1} \Rightarrow m - e_1 = p_k^* e_k + \sum_{j \neq k} p_j^* e_j \Rightarrow p_k = \frac{m - e_1 - \sum p_j^* e_j}{e_k}$$

$$\Rightarrow e_k = \frac{m - e_1 - \sum p_j^* e_j}{p_k^*}$$

$$\Rightarrow \hat{\mu}_k = \frac{\alpha_k \sum \lambda_j}{m - e_1 - \sum p_j^* e_j} p_k^*$$

$$\propto p_k^* = \frac{p_k}{p_1}$$

Economics 801 HW 7 | 991253080 Junyong Kim
Advanced Microeconomic Theory I | December 10, 2015

1.
(a) The 'Club' method would be more efficient (because potential thieves will not attempt to steal). It would reduce the total rate of auto theft.
(b) If there are many locked cars, then the thieves will attempt to steal the other cars without the lock; those externalities are negative; thus, that 'locking' behavior can initiate trends (other drivers will also choose the 'Club' method as well). +If there are many cars with 'Lojack,' then the thieves will give up their attempts and this is a positive externality.
(c) ~~Policy makers should encourage all people to participate in the policy and, at the same time, give information about the riskiness of ignoring that policy.~~ For policy makers, 'Lojack' is better.

2. q=# of cars crossing the bridge
$t\times q$=times required to cross it
t_0=times required to drive around (the cost=$w\times t_0$)
w=wage in San Francisco
(a)

$$\text{costs with } q-1 \text{ other cars} = wtq$$
$$wt\hat{q}_p = wt_0 \text{ in equilibrium} \Rightarrow \hat{q}_p = \frac{t_0}{t}$$

(b)

$$\min_{q_s} \underbrace{wtq_s^2}_{\text{costs of crossing drivers}} + \underbrace{wt_0(q-q_s)}_{\text{costs of rounding drivers}} \Rightarrow 2wt\hat{q}_s - wt_0 = 0 \Rightarrow \hat{q}_s = \frac{wt_0}{2wt} = \frac{t_0}{2t} = \frac{\hat{q}_p}{2}$$

(c) To achieve social optimum, some people have to take costs (e.g. time loss), voluntarily. However, they will avoid take that 'personal' expenses.
(d)

$$\text{crossing costs} = wtq + s$$
$$wt\hat{q}_p + s = wt_0 \text{ in equilibrium} \Rightarrow \hat{q}_p = \frac{wt_0 - \hat{s}}{wt} = \frac{t_0}{2t} \text{ for optimum} \Rightarrow 2(wt_0 - \hat{s}) = wt_0 \Rightarrow s = \frac{wt_0}{2}$$

(e) The company will maximize its profit.

$$\max_r rq(r) = \max_r r\frac{wt_0 - r}{wt} \Rightarrow \frac{d}{dr}\left(\frac{wt_0}{wt}r - \frac{1}{wt}r^2\right) = \frac{t_0}{t} - \frac{2}{wt}r = 0 \Rightarrow r = \frac{wt_0}{2} = s$$

So r=s and this is because of Coase theorem; if there is a traffic congestion, the firm will take all the costs of the congestion. And to avoid that costs, the firm will price the toll by which the costs can be minimized. The (equilibrium) price the firm can pay for the bridge is equal to the amount the firm can collect from it.

$$\frac{wt_0}{2}\left(\frac{wt_0 - wt_0/2}{wt}\right) = \frac{wt_0}{2}\frac{wt_0}{2wt} = \frac{wt_0^2}{4t} = \text{Price of the bridge}$$

If the price is smaller than that amount, then the firm will buy that bridge.

1.

(a) If

$$50 - p_1 + 50S + \epsilon_j > 75 - p_2 + 50(1 - S) \Rightarrow \text{choose VHS over Beta}$$
$$\Leftrightarrow \epsilon_j > 75 + p_1 - p_2 - 100S \text{ or } S > \frac{75}{100} + \frac{1}{100}(p_1 - p_2) - \epsilon_j$$

(b)

$$\Pr\left(\epsilon_j > 75 + p_1 - p_2 - 100S^*\right) = \underbrace{1 - \frac{100 + 75 + p_1 - p_2 - 100S^*}{200} = S^*}_{\text{in equilibrium}}$$
$$\Rightarrow 200 - 100 - 75 - p_1 + p_2 + 100S^* = 200S^*$$
$$\Rightarrow 100S^* = 25 - p_1 + p_2$$
$$\Rightarrow S^* = \frac{25}{100} - \frac{1}{100}p_1 + \frac{1}{100}p_2$$

(c)

$$\text{Sales for VHS Producer} = S \times p_1 \text{ and Costs} = cS$$
$$\Rightarrow \text{Profit for VHS Producer} = Sp_1 - cS$$
$$\Rightarrow \max_{p_1} \pi_1(p_1, p_2) = \max_{p_1} S(p_1, p_2)p_1 - cS(p_1, p_2)$$
$$\text{Sales for Beta Producer} = (1 - S) \times p_2 \text{ and Costs} = k(1 - S)$$
$$\Rightarrow \text{Profit for Beta Producer} = (1 - S)p_2 - k(1 - S)$$
$$\Rightarrow \max_{p_2} \pi_2(p_1, p_2) = \max_{p_2}\left(1 - S(p_1, p_2)\right)p_2 - k\left(1 - S(p_1, p_2)\right)$$

FOC

$$\text{i) } \frac{dS}{dp_1}p_1 + S - c\frac{dS}{dp_1} = 0$$
$$\Rightarrow -\frac{1}{100}p_1 + \left(\frac{25}{100} - \frac{1}{100}p_1 + \frac{1}{100}p_2\right) + \frac{1}{100}c = 0$$
$$\Rightarrow -p_1 + 25 - p_1 + p_2 + c = 0$$
$$\Rightarrow 25 + c - 2p_1 + p_2 = 0$$
$$\Rightarrow 2p_1 - p_2 = 25 + c$$

(Continued)

$$\text{ii)}\ -\frac{dS}{dp_2}p_2 + (1-S) + k\frac{dS}{dp_2} = 0$$

$$\Rightarrow -\frac{1}{100}p_2 + \left(\frac{75}{100} + \frac{1}{100}p_1 - \frac{1}{100}p_2\right) + \frac{1}{100}k = 0$$

$$\Rightarrow -p_2 + 75 + p_1 - p_2 + k = 0$$

$$\Rightarrow 75 + k + p_1 - 2p_2 = 0$$

$$\Rightarrow p_1 - 2p_2 = -75 - k$$

$$\text{iii)}\ 3p_1 = 125 + 2c + k$$

$$\Rightarrow p_1^* = \frac{125 + 2c + k}{3}$$

$$\text{iv)}\ \frac{125 + 2c + k}{3} - 2p_2 = -75 - k$$

$$\Rightarrow -2p_2 = -75 - \frac{125 + 2c + k}{3} - k$$

$$\Rightarrow 2p_2 = \frac{225}{3} + \frac{125 + 2c + k}{3} + \frac{3k}{3}$$

$$\Rightarrow p_2^* = \frac{175 + c + 2k}{3}$$

By solving two FOCs (two best response functions) simultaneously,

$$\therefore S^* = \frac{25}{100} - \frac{1}{100}\left(\frac{125 + 2c + k}{3}\right) + \frac{1}{100}\left(\frac{175 + c + 2k}{3}\right)$$

$$= \frac{75}{300} - \frac{125 + 2c + k}{300} + \frac{175 + c + 2k}{300}$$

$$= \frac{125 + k - c}{300}$$

$$\text{and } 1 - S^* = \frac{175 + c - k}{300}$$

(d) To determine 'socially optimal' share *S*, apply the planner who determines that.

$$\max_S \underbrace{S\big(50 - p_1 + 50S + \mathrm{E}(\epsilon|\epsilon > \epsilon^*)\big)}_{\text{benefit of VHS buyers}} + \underbrace{(1-S)(75 - p_2 + 50 - 50S)}_{\text{benefit of Beta buyers}} + \underbrace{(p_1 - c)S}_{\text{VHS profits}} + \underbrace{(p_2 - k)(1-S)}_{\text{Beta profits}}$$

$$\Rightarrow \frac{100 + \epsilon^*}{200} = S \Rightarrow \epsilon^* = 200S - 100 \Rightarrow \mathrm{E}(\epsilon|\epsilon > \epsilon^*) = \frac{100 - \epsilon^*}{2} = \frac{200 - 200S}{2} = 100 - 100S$$

$$\Rightarrow \max_S S(150 - p_1 - 50S) + (1-S)(125 - p_2 - 50S) + p_1S - cS + p_2(1-S) - k(1-S)$$

$$= \max_S 150S - Sp_1 - 50S^2 + 125 - p_2 - 50S - 125S + Sp_2 + 50S^2 + p_1S - cS + p_2 - p_2S - k + kS$$

$$= \max_S (k - c - 25)S + (125 - k) \Rightarrow \text{FOC: } k - c - 25 = 0 \Rightarrow S^* = \begin{cases} 0, & \text{if } k - c - 25 < 0 \\ 1, & \text{if } k - c - 25 > 0 \\ \forall \in [0,1], & \text{if } k - c - 25 = 0 \end{cases}$$

If $k - c - 25 \neq 0$, then the share S^* determined by the market is not socially optimal.

(e) k, the cost to produce Beta is 100 more expensive than c, the cost to produce VHS.

$$S^* = \frac{125 + k - c}{300} = \frac{75}{100} = \frac{225}{300} \Rightarrow k - c = 100 \Rightarrow k = c + 100$$

2. (a) Expected disutility of avoiding the rest is

$$\sqrt{S} \times 1 + \left(1 - \sqrt{S}\right) \times 0 = \sqrt{S}$$

Expected disutility of getting sufficient rest is

$$\frac{1}{2}\sqrt{S} \times 1 + \left(1 - \frac{1}{2}\sqrt{S}\right) \times 0 = \frac{1}{2}\sqrt{S}$$

Then,

$$\frac{1}{2}\sqrt{S} + e_j < \sqrt{S} \Leftrightarrow e_j < \sqrt{S}/2 \Rightarrow \text{he or she will take a rest}$$
$$\Rightarrow \Pr\left(e < \sqrt{S}/2\right) = \sqrt{S}/2 = R(S)$$

(b)

$$R \times \frac{\sqrt{S}}{2} + (1 - R) \times \sqrt{S}$$
$$\Rightarrow \underbrace{\frac{\sqrt{S}}{2} \times \frac{\sqrt{S}}{2}}_{\substack{\text{catch a cold} \\ \text{with getting rest}}} + \underbrace{\left(1 - \frac{\sqrt{S}}{2}\right) \times \sqrt{S}}_{\substack{\text{catch a cold} \\ \text{without getting rest}}} = \frac{1}{4}S + \sqrt{S} - \frac{1}{2}S = \sqrt{S} - \frac{S}{4}$$
$$\Rightarrow S = \sqrt{S} - \frac{S}{4} \Rightarrow \frac{5S}{4} = \sqrt{S} \Rightarrow \frac{25S^2}{16} = S \Rightarrow \left(\frac{25}{16}S - 1\right)S = 0$$
$$\therefore S = 0 \text{ or } \frac{16}{25} \Rightarrow R = 0 \text{ or } \frac{2}{5}$$

(c) The total cost (-1 for each) of the sickness is

$$S(R) = R \times \frac{\sqrt{S(R)}}{2} + (1 - R) \times \sqrt{S(R)} \Rightarrow \sqrt{S(R)} = 1 - \frac{R}{2} \Rightarrow S(R) = 1 - R + \frac{R^2}{4}$$

The total cost of the rest is

$$\Pr(e < e^*) = e^* = R \Rightarrow \frac{0 + R}{2} = \frac{R}{2} \text{ for each} \Rightarrow \text{total costs} = \frac{R^2}{2}$$

Hence the social planner's problem is

$$\min_R 1 - R + R^2/4 + R^2/2 = \min_R 1 - R + 3R^2/4 \Rightarrow \text{FOC: } 1.5R^* - 1 = 0 \Rightarrow R^* = 2/3$$

Here R^*=2/3, which is larger than the natural optimal R^*=2/5.
(d) If there is no transaction cost (the costs of lawsuit here; in reality, it is expensive), then this legislation will internalize the externality and hence the social optimum will be achieved (Coase theorem).
(e) To improve the welfare, Pigouvian tax can be imposed to people who do not want to take a rest. Then, they will consider the tax in their decision. But it is a progressive tax, so makes the distribution wider.