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Barabási, Albert-László, Réka Albert and Hawoong Jeong, 1999, "Mean-field Theory for Scale-free Random Networks," *Physica A*, vol. 272, pp. 173–187

1. Claim

Claim For arbitrary node in Barabási–Albert network, the probability density that the node is linked to other k nodes in the network follows a power rule below.

$$P(k) \propto k^{-\gamma}$$

Where γ is a parameter that has to be computed.

2. Assumption

Assumption There exists a fully-integrated network with m_0 nodes.

Assumption For each time increment, a node with m links is added to the network.

Assumption Preferential Attachment: for each time increment, the node i is linked with the newly-appeared node with the probability Π_i that can be denoted as below.

$$\Pi_i = k_i \Big/ \sum_{j \in \mathrm{Network}} k_j$$

Where k_i is the number of links possessed by the node i.

3. Equation

Since the model incorporates the flow of time, k_i is also a function of time; i.e. $k_i=k_i(t)$. Along with this time flow, k_i changes proportionately to m and Π_i. Therefore,

$$\begin{aligned}\frac{dk_i}{dt} &= m\Pi_i \\ &= m \times k_i \Big/ \sum_j k_j \\ &= \frac{mk_i}{2mt} \\ &= \frac{k_i}{2t}\end{aligned}$$

By solving the equation, the analytic solution for $k_i(t)$ can be obtained.

$$
\begin{aligned}
\frac{1}{k_i}dk_i &= \frac{1}{2t}dt \\
\Rightarrow \int \frac{1}{k_i}dk_i &= \int \frac{1}{2t}dt + C \\
\Rightarrow \ln k_i &= \frac{1}{2}\ln t + C \\
&= \frac{1}{2}\ln t + \frac{1}{2}\ln C^* \\
&= \frac{1}{2}\ln tC^* \\
\Rightarrow k_i(t) &= \sqrt{tC^*}
\end{aligned}
$$

Since each node starts with initial m links, the initial condition $k_i(t_i)=m$ can be applied. Where t_i means the starting time for node i.

$$
\begin{aligned}
k_i(t_i) &= \sqrt{t_iC^*} = m \\
\Rightarrow t_iC^* &= m^2 \\
\Rightarrow C^* &= m^2/t_i
\end{aligned}
$$

By imposing the initial condition, corresponding particular solution can be obtained.

$$
k_i(t) = \sqrt{t\frac{m^2}{t_i}} = m\sqrt{\frac{t}{t_i}}
$$

4. Cumulative Density Function

For the given time t, the probability Pr that the given node i has its links smaller than arbitrary constant k can be written as below.

$$
\begin{aligned}
\Pr[k_i(t) < k] &= \Pr\left(m\sqrt{\frac{t}{t_i}} < k\right) \\
&= \Pr\left(m^2\frac{t}{t_i} < k^2\right) \\
&= \Pr\left(t_i > t\frac{m^2}{k^2}\right)
\end{aligned}
$$

At the given time t, there exist m_0+t nodes. Thus, for the given node i, the probability that it is added at specific time t_i can be written as below.

$$
\begin{aligned}
\Pr(t_i = 0) &= \frac{m_0}{m_0 + t} \\
\Pr(t_i = 1) &= \frac{1}{m_0 + t} = \Pr(t_i = 2) = \cdots = \Pr(t_i = t)
\end{aligned}
$$

$$\sum_{j=0}^{t} \Pr(t_i = j) = \frac{m_0 + 1 + 1 + \cdots + 1}{m_0 + t} = \frac{m_0 + t}{m_0 + t} = 1$$

Hence,

$$\begin{aligned}
\Pr\left(t_i > t\frac{m^2}{k^2}\right) &= 1 - \Pr\left(t_i \le t\frac{m^2}{k^2}\right) \\
&= 1 - \frac{m_0 + t\frac{m^2}{k^2}}{m_0 + t} \\
&= \Pr[k_i(t) < k]
\end{aligned}$$

5. Probability Density Function

Above cumulative density function can be changed into the probability density function as below.

$$\begin{aligned}
\Pr[k_i(t) = k] &= \frac{d}{dk}\Pr[k_i(t) < k] \\
&= \frac{2tm^2}{(m_0 + t)k^3}
\end{aligned}$$

Consequently, as $t \to \infty$,

$$\begin{aligned}
\lim_{t\to\infty} \Pr[k_i(t) = k] &= \lim_{t\to\infty}\left[\frac{2tm^2}{(m_0 + t)k^3}\right] \\
&= \frac{2m^2}{k^3}
\end{aligned}$$

As a result,

$$\begin{aligned}
\Pr(k_i = k) &\coloneqq P(k) \\
&= 2m^2k^{-3} \text{ as } t \to \infty \\
&\propto k^{-\gamma} \text{ where } \gamma = 3 \ \blacksquare
\end{aligned}$$

Reference

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