Advanced Macroeconomic Theory I 2016-09-12
Lecture 1 Junyong Kim

Study: 15:30, Bolton 853

Solow model; why Solow model?

(1) Simple framework that offers proximate cause and the mechanics of growth and cross-country difference
(2) Foundation of Neo-Classical growth theory
(3) (i) Exposure to dynamic framework
  (ii) Steady-state and transitional dynamics
  (iii) Convergence

Harrod-Domar model

(1) Single good economy
(2) Capital and labor are combined in a fixed proportion
(3) $S_t = sY_{t-1}$
(4) (Incremental) Capital labor ratio

$$\frac{K}{Y} = \frac{\Delta K}{\Delta Y} = v, \qquad v \downarrow \Rightarrow \text{technological progress}$$

(5) Closed economy
(6) Population growth=n (so there are 3 parameters in this model; s, v, n)

Capital market equilibrium

$$S_t = I_t$$
$$\Leftrightarrow \text{planned savings} = \text{planned investment}$$

Hence

$$sY_{t-1} = I_t = \Delta K_t = K_t - K_{t-1} = v(Y_t - Y_{t-1})$$
$$\Rightarrow \frac{Y_t - Y_{t-1}}{Y_{t-1}} = g_Y = \frac{s}{v}$$

Also

$$\frac{K}{Y} = \frac{\Delta K}{\Delta Y} \Rightarrow g_Y = \frac{\Delta Y}{Y} = \frac{\Delta K}{K} = g_K = \frac{s}{v}$$

Growth vs labor

$$\text{if } \frac{s}{v} < n \Rightarrow \text{excess supply of labor} = \text{perpetual unemployment}$$
$$\text{if } \frac{s}{v} > n \Rightarrow \text{excess demand for labor}$$

Suppose $s/v<n$→the wage w↓→use more labor and less capital to produce an unemployment→ΔK↓→v↓

$$\text{and hence } \frac{s}{v} = n$$

And hence the equilibrium can be achieved.

Solow model

(a) Closed economy

(b) Households are identical.

(c) ‘s’ is the savings rate.

(d)

$$\begin{aligned}
&Y_t = F(K_t, L_t, A_t) \\
&\frac{\partial F}{\partial K} > 0, \qquad \frac{\partial^2 F}{\partial K^2} < 0, \qquad \frac{\partial F}{\partial L} > 0, \qquad \frac{\partial^2 F}{\partial L^2} < 0 \\
&F(\cdot) \text{ is constant return to scale (CRS) of degree 1 w.r.t. K and L}
\end{aligned}$$

For example,

$$\text{for } F(K, L, A) \text{ if } F(\lambda K, \lambda L, A) = \lambda F(K, L, A) \Rightarrow F \text{ is CRS of degree 1}$$

The function F satisfies the following conditions.

$$\underbrace{\lim_{K\to 0}\frac{\partial F}{\partial K} = \infty, \qquad \lim_{K\to \infty}\frac{\partial F}{\partial K} = 0, \qquad \lim_{L\to 0}\frac{\partial F}{\partial L} = \infty, \qquad \lim_{L\to \infty}\frac{\partial F}{\partial L} = 0}_{\text{Inada condition (in order to guarantee } \exists \text{ steady state)}}$$

Population growth rate: n

Capital depreciation at a rate: δ

$$\begin{aligned}
&\dot{K}_t = I_t - \delta K_t, \qquad \text{for continuous } \dot{K}_t = \frac{dK}{dt}, \qquad \text{for discrete } \dot{K}_t = \Delta K_t \\
&I_t = S_t = sF(K_t, L_t, A_t), \qquad \text{these are the equilibrium condition}
\end{aligned}$$

Then

$$\dot{K}_t = sF(K_t, L_t, A_t) - \delta K_t \Leftrightarrow \frac{\dot{K}_t}{K_t} = s\frac{F(K_t, L_t, A_t)}{K_t} - \delta$$

Define k_t=capital-labor ratio=K_t/L_t, then

$$k_t = K_t/L_t \Rightarrow \log k_t = \log K_t - \log L_t \Rightarrow \underbrace{\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t}}_{\text{differentiate by t}} = \frac{\dot{K}_t}{K_t} - n$$

Then

$$\frac{\dot{k}_t}{k_t} + n = s\frac{F(K_t, L_t, A_t)}{K_t} - \delta \Leftrightarrow \frac{\dot{k}_t}{k_t} = s\frac{F(K_t, L_t, A_t)}{K_t} - (n + \delta)$$

Note that

$$\begin{aligned}
&\frac{F(K_t, L_t, A_t)}{K_t} = \frac{F(K_t, L_t, A_t)/L_t}{K_t/L_t} = \frac{F(K_t/L_t, 1, A_t)}{k_t} = \frac{F(k_t, 1, A_t)}{k_t} \equiv \frac{f(k_t)}{k_t} \\
&\Rightarrow \frac{\dot{k}_t}{k_t} = s\frac{f(k_t)}{k_t} - (n + \delta)
\end{aligned}$$

We've seen that

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (n+\delta), \qquad s \in (0,1)$$

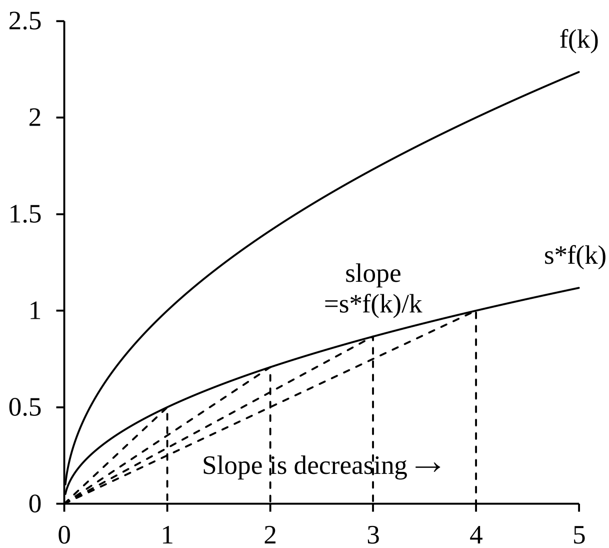
Note technically that (Cobb–Douglas production)

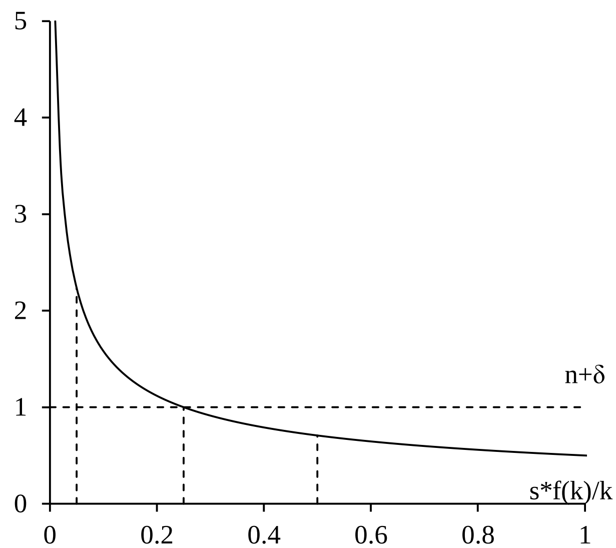
$$Y = AK^{\alpha}L^{1-\alpha} \Leftrightarrow \frac{Y}{L} = AK^{\alpha}L^{-\alpha} = A\left(\frac{K}{L}\right)^{\alpha} \Leftrightarrow y = Ak^{\alpha} = f(k), \qquad \alpha \in (0,1)$$

The very first equation describes the dynamics, so by using the relation we can predict the future behavior of the process k.

$$\{k_0, k_1, k_2, \cdots\} \text{ or } \{k(t) | t \geq 0\}$$

First, think about $f(k)=Ak^{\alpha}$, $sf(k)$ and $s\frac{f(k)}{k}$.





Therefore, by solving the equation below,

$$s\frac{f(k^*)}{k^*} - (n+\delta) = 0 \Rightarrow s\frac{A(k^*)^{\alpha}}{k^*} = n+\delta \Rightarrow k^* = \text{steady state}$$

$$(k^*)^{\alpha-1} = \frac{n+\delta}{sA} \Rightarrow k^* = \left(\frac{n+\delta}{sA}\right)^{\frac{1}{\alpha-1}} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

How can we guarantee the existence of the steady state?
Note: L'Hôpital's rule

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \text{ or } \pm\infty, \qquad \lim_{x\to c}\frac{f'(x)}{g'(x)} = L, \qquad \lim_{x\to c}\frac{f(x)}{g(x)} = L$$

$$\text{Check } \lim_{k\to 0} s\frac{f(k)}{k} = \infty, \qquad \lim_{k\to\infty} s\frac{f(k)}{k} = 0 \text{ as an assignment}$$

Apply L'Hôpital's rule as below.

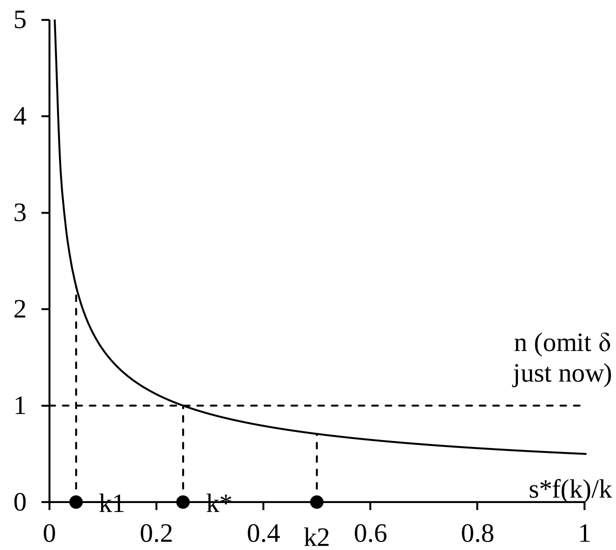
$$s\frac{f(k)}{k} = s\frac{Ak^{\alpha}}{k} \Rightarrow \lim_{k\to 0} sAk^{\alpha} = 0 \ \& \ \lim_{k\to 0} k = 0 \ \& \ \lim_{k\to 0}\frac{sA\alpha k^{\alpha-1}}{1} = \infty \Rightarrow \lim_{k\to 0} s\frac{Ak^{\alpha}}{k} = \infty$$

Similarly,

$$\lim_{k\to\infty} sAk^{\alpha} = \lim_{c\to 0} sAc^{-\alpha} = \infty \ \& \ \lim_{k\to\infty} k = \lim_{c\to 0}\frac{1}{c} = \infty \ \& \ \lim_{k\to\infty}\frac{sA\alpha k^{\alpha-1}}{1} = \lim_{c\to 0} sA\alpha c^{1-\alpha} = 0$$

$$\Rightarrow \lim_{k\to\infty} s\frac{Ak^{\alpha}}{k} = \lim_{c\to 0} s\frac{Ac}{c^{\alpha}} = 0, \qquad \text{hence } s\frac{Ak^{\alpha}}{k} \text{ touch } n+\delta \text{ at least once in } k \in (0,\infty)$$

Therefore, $s\frac{f(k)}{k}$ always intersects n+δ at k∈(0,∞).



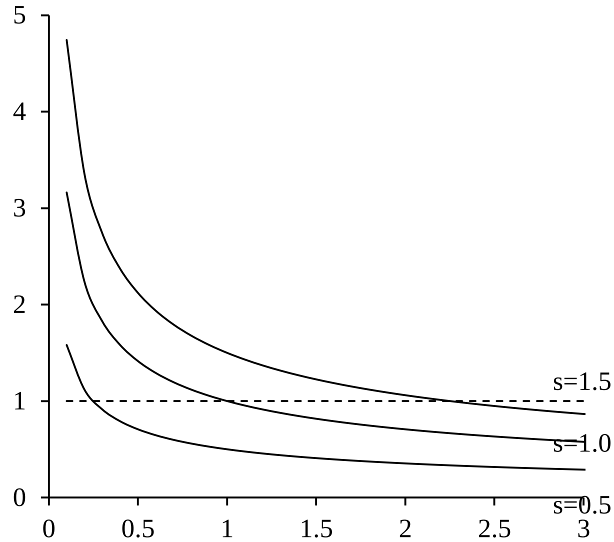
At k=k_1,

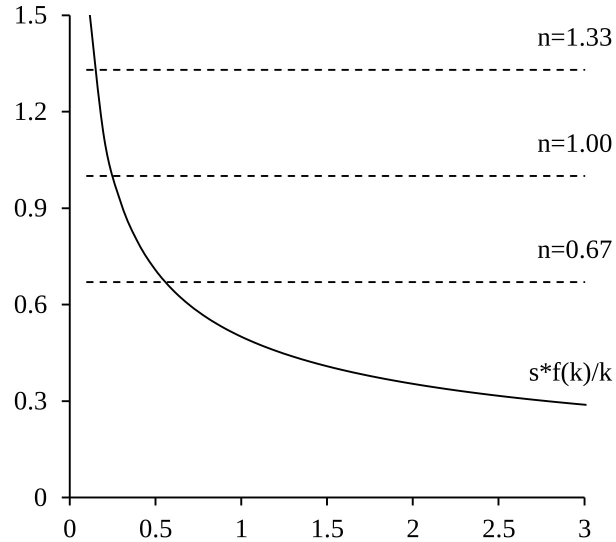
$$s\frac{f(k_1)}{k_1} = \underbrace{\frac{s}{\frac{k_1}{f(k_1)}}}_{\substack{\frac{s}{v}>n \text{ in} \\ \text{Harrod–Domar}}} > n \quad \Rightarrow \frac{\dot{k}}{k} > 0$$

Similarly,

$$\text{at } k = k_2, \qquad s\frac{f(k_2)}{k_2} < n \Rightarrow \frac{\dot{k}}{k} < 0$$

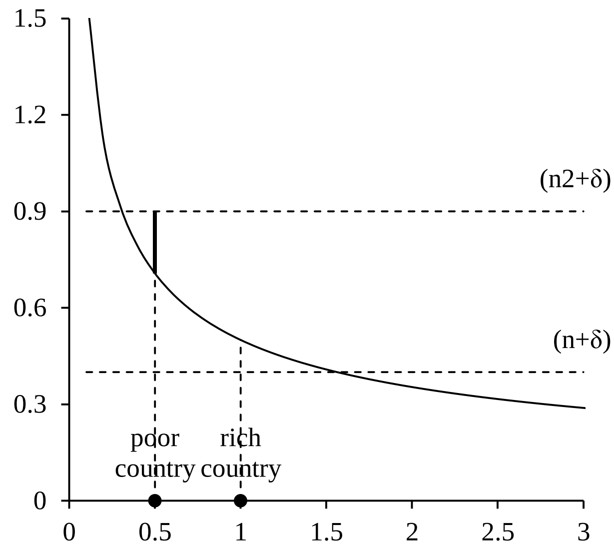
$$\text{at } k = k^*, \qquad s\frac{f(k^*)}{k^*} = n \Rightarrow \frac{\dot{k}}{k} = 0 \text{ hence steady state}$$





$s \uparrow \Rightarrow \frac{\dot{k}}{k} \uparrow$ $n \uparrow \Rightarrow \frac{\dot{k}}{k} \downarrow$

(Check Barro's textbook for empirical works; savings rate vs. growth & population growth & growth)

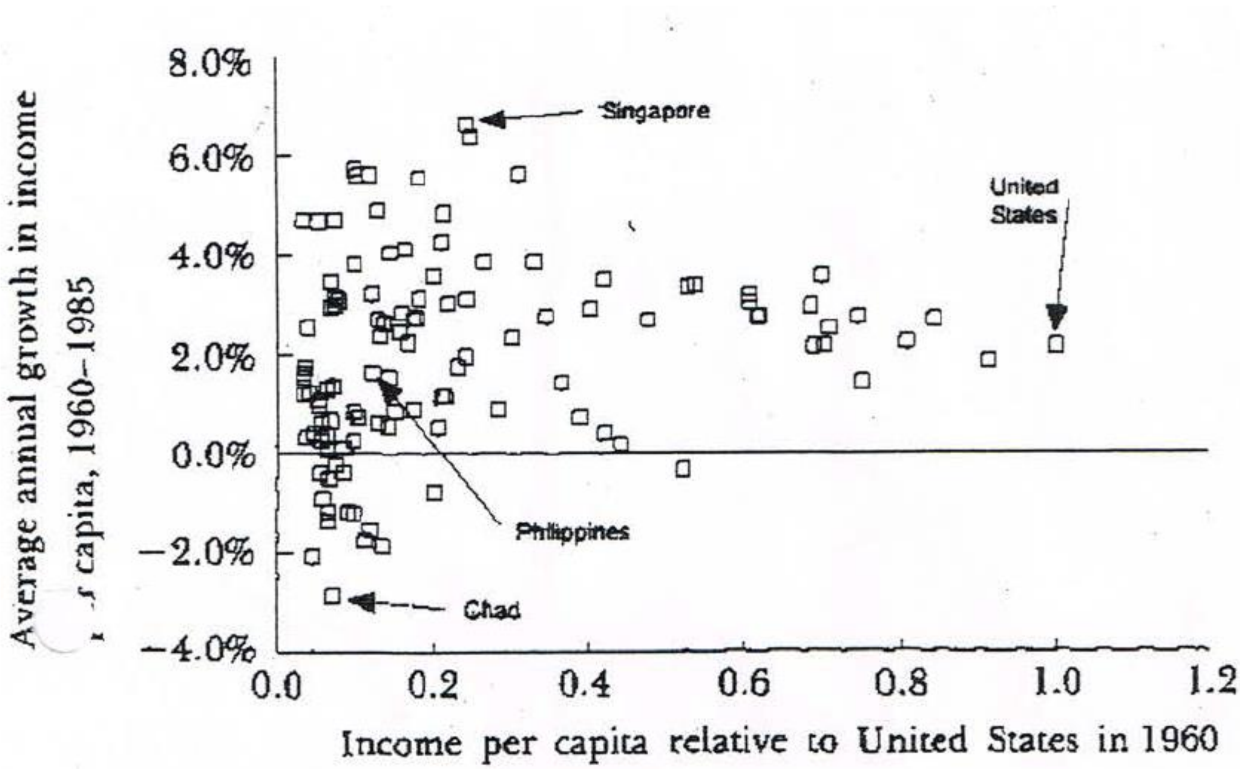


Absolute Convergence: According to Solow model's expectation, the growth rate of a poor country is greater than that of a rich country.

However, data show the different result (should compare countries with identical n, s and δ, but the analysis ignores that).

For example, $\dot{k}/k$ of US can be bigger than that of India because $n_{US}<n_{India}$. Fortunately, the convergence holds when data compares just similar countries.

According to the figure above, the growth rate of the poor is greater than that of the rich if they have identical n, but if the poor has a bigger population growth rate $n_2>n$, then the growth rate of the poor can be even negative.



Absolute Convergence says the negative relation between k and growth, but data show no relation

Now think about the relation between the growth rate $\frac{\dot{k}}{k}$ and the equilibrium k^*.

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (n+\delta) \Rightarrow s\frac{f(k^*)}{k^*} = n + \delta \text{ at equilibrium} \Rightarrow s = (n+\delta)\frac{k^*}{f(k^*)}$$

Hence

$$\frac{\dot{k}}{k} = (n+\delta)\frac{k^*}{f(k^*)}\frac{f(k)}{k} - (n+\delta) = (n+\delta)\left[\frac{f(k)/k}{f(k^*)/k^*} - 1\right]$$

Therefore, the growth rate is determined by k^* as well as k itself, where k^* is its own equilibrium level. This is called as Conditional Convergence.

Now, solve Solow's differential equation

$$\frac{\dot{k}}{k} = s\frac{Ak^\alpha}{k} - (n+\delta) = sAk^{\alpha-1} - (n+\delta), \qquad \Rightarrow \frac{dk}{dt} = sAk^\alpha - (n+\delta)k$$

Bernoulli's method; define y as below.

$$y \equiv k^{1-\alpha}, \qquad \Rightarrow \frac{dy}{dt} = (1-\alpha)k^{-\alpha}\frac{dk}{dt}, \qquad \frac{dk}{dt} = \frac{k^\alpha}{1-\alpha}\frac{dy}{dt}$$

Substitute dk/dt and k as below.

$$\frac{k^\alpha}{1-\alpha}\frac{dy}{dt} = sAk^\alpha - (n+\delta)k, \qquad \Rightarrow \frac{dy}{dt} = (1-\alpha)sA - \underbrace{(1-\alpha)(n+\delta)k^{1-\alpha}}_{Zy}$$

Find the integrating factor.

$$\frac{dy}{dt} + Zy = 0, \qquad \int \frac{1}{y} dy = -Z \int dt, \qquad \ln y = -Zt + c, \qquad y = Ce^{-Zt}, \qquad C = y \underbrace{e^{Zt}}_{\substack{\text{integrating} \\ \text{factor}}}$$

Solve the differential equation of y.

$$\frac{dy}{dt} e^{Zt} + Zye^{Zt} = (1-\alpha)sAe^{Zt}, \qquad \frac{d}{dt}\left(ye^{Zt}\right) = (1-\alpha)sAe^{Zt}, \qquad \int d\left(ye^{Zt}\right) = \int (1-\alpha)sAe^{Zt}dt$$

Integrate both sides.

$$ye^{Zt} = (1-\alpha)sA \int e^{Zt}dt = \frac{(1-\alpha)sA}{Z} e^{Zt} + c = \frac{sA}{n+\delta} e^{(1-\alpha)(n+\delta)t} + c$$

Find a particular solution.

$$y = k^{1-\alpha} = \frac{sA}{n+\delta} + ce^{-(1-\alpha)(n+\delta)t}$$

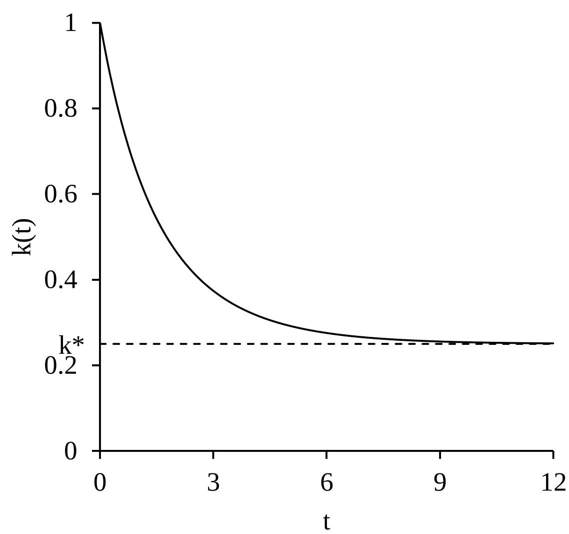
Find c by using the initial condition $k(0)=k_0$.

$$k_0^{1-\alpha} = \frac{sA}{n+\delta} + c, \qquad c = k_0^{1-\alpha} - \frac{sA}{n+\delta}$$

Find a general solution.

$$k^{1-\alpha} = \frac{sA}{n+\delta} + \left(k_0^{1-\alpha} - \frac{sA}{n+\delta}\right) e^{-(1-\alpha)(n+\delta)t}$$

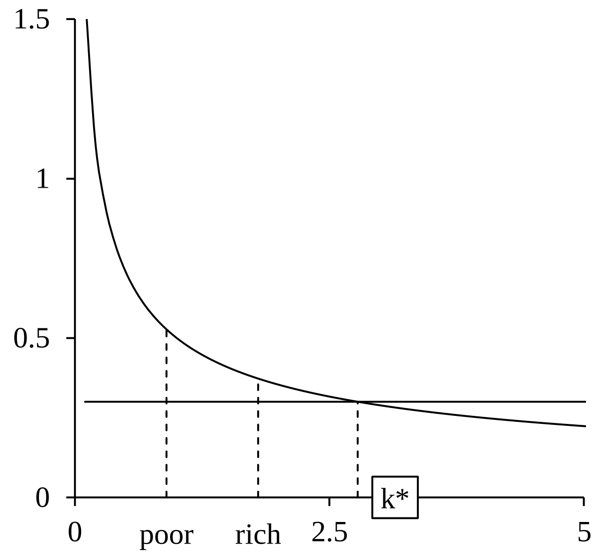
$$k(t) = \left[\frac{sA}{n+\delta} + \left(k_0^{1-\alpha} - \frac{sA}{n+\delta}\right) e^{-(1-\alpha)(n+\delta)t}\right]^{\frac{1}{1-\alpha}}, \qquad \lim_{t\to\infty} k(t) = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}} = k^*$$



Convergence of k(t) to the steady-state (i.e. the long-run equilibrium)

Solow model

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (n+\delta)$$

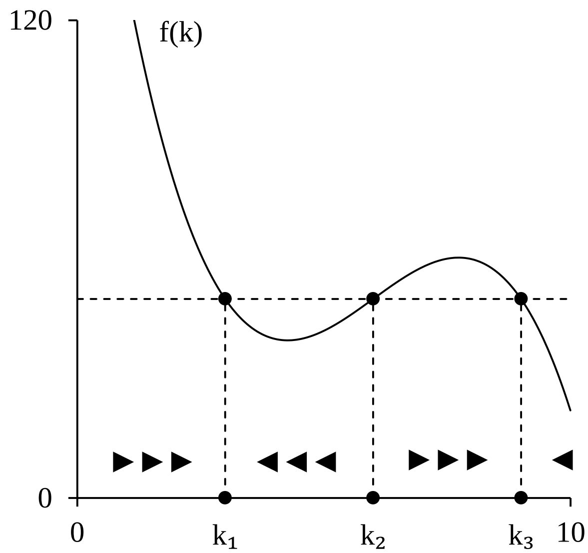


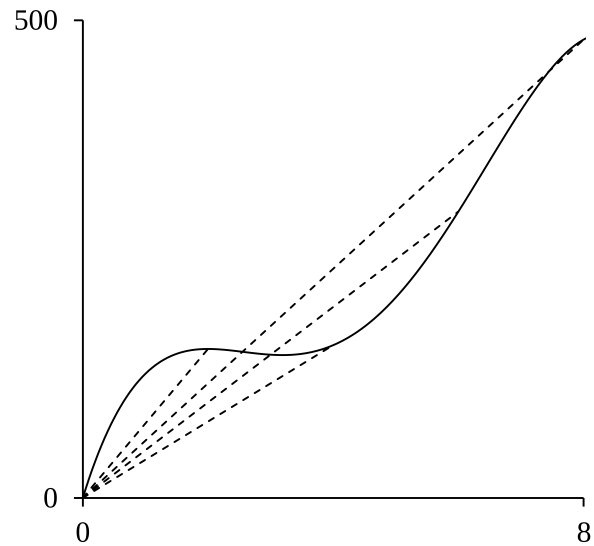
Absolute Convergence: the growth of the poor is faster than that of the rich.

From the last class,

$$\frac{\dot{k}}{k} = (n+\delta)\left[\frac{f(k)/k}{f(k^*)/k^*} - 1\right], \qquad \frac{\dot{k}}{k} \text{ depends } k^* \text{ as well as } k$$

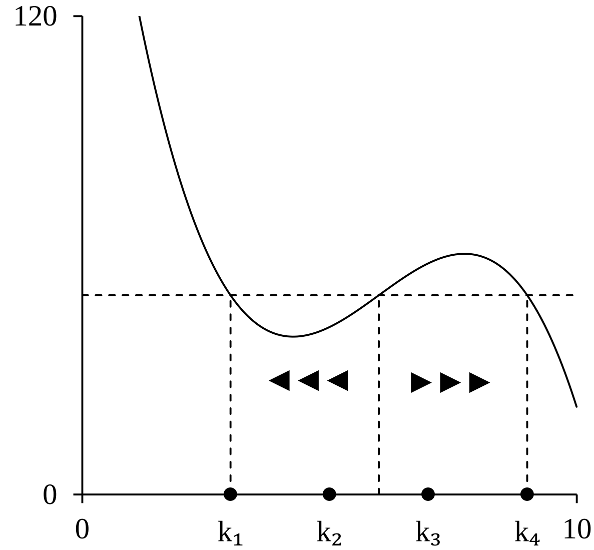
Suppose there is a production function that is not usual.





Then, k_1 and k_3 are stable steady-states (strong to a minute deviation), but k_2 is unstable steady-state (vulnerable to a minute deviation).

Rostow's Big Push model (development trap)

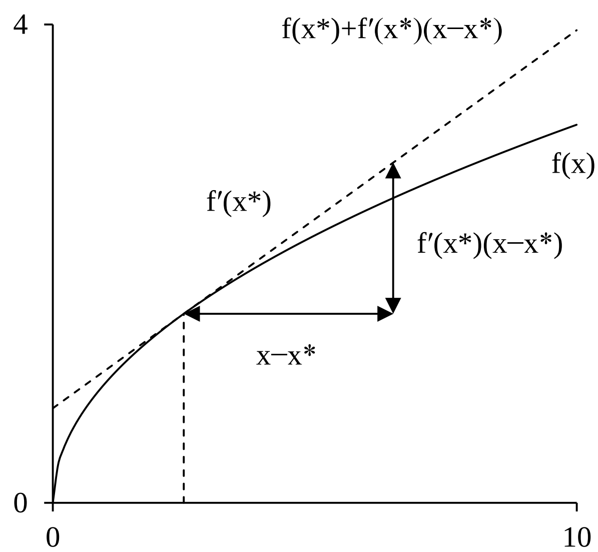


From $k_1 \rightarrow k_2$: k will return back to k_1 (bad)

From $k_1 \rightarrow k_3$: k will go to k_4 (good)

Taylor's rule (or theorem; or first-order approximation)

$$\begin{aligned} f(x) &= f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \cdots \\ &\approx f(x^*) + f'(x^*)(x - x^*) \end{aligned}$$



As $x \rightarrow x^*$, $f(x^*)+f'(x^*)(x-x^*) \rightarrow f(x)$ (i.e. the error decreases)

Now

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (n+\delta) = s\frac{Ak^{\alpha}}{k} - (n+\delta) \equiv g(k)$$

Then

$$\begin{aligned} g(k) &= s\frac{f(k)}{k} - (n+\delta) \\ &= sAk^{\alpha-1} - (n+\delta) \\ &\approx g(k^*) + g'(k^*)(k-k^*), \qquad \text{around the equilibrium } k^* \\ &= g'(k^*)(k-k^*), \qquad \text{since } g(k^*) = 0 \end{aligned}$$

In the equilibrium

$$sA(k^*)^{\alpha-1} = n+\delta, \qquad k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

Hence

$$\begin{aligned} g(k) &\approx g'(k^*)(k-k^*) = sA(\alpha-1)(k^*)^{\alpha-2}(k-k^*) \\ &= sA(\alpha-1)(k^*)^{\alpha-1}\frac{k-k^*}{k^*} \\ &= sA(\alpha-1)\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}\frac{k-k^*}{k^*} \\ &= -\underbrace{(1-\alpha)(n+\delta)}_{\equiv\beta}\frac{k-k^*}{k^*} \\ &= -\beta\frac{k-k^*}{k^*}, \qquad \text{where } \beta \text{ is the speed of convergence} \end{aligned}$$

Also

$$\begin{aligned} &y = Ak^{\alpha}, \qquad \log y = \log A + \alpha\log k, \qquad \Rightarrow \frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} \\ &\log k \approx \log k^* + \frac{1}{k^*}(k-k^*), \qquad \Rightarrow \log k - \log k^* \approx \frac{k-k^*}{k^*} \\ &\log y^* = \log A + \alpha\log k^*, \qquad \Rightarrow \log y - \log y^* = \alpha(\log k - \log k^*) \end{aligned}$$

Hence

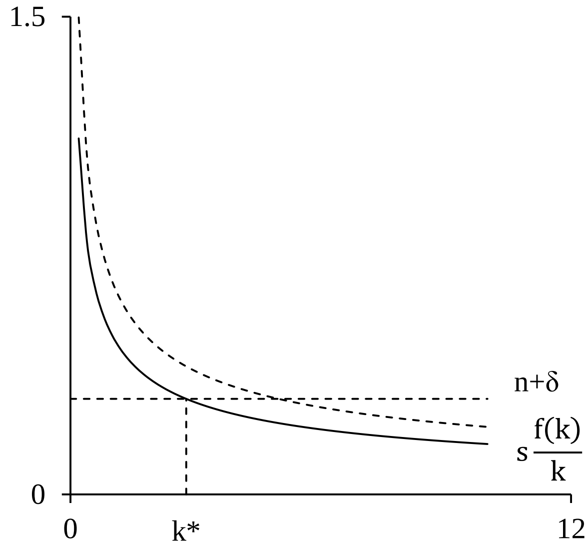
$$g(k) = \frac{\dot{k}}{k} \approx -\beta(\log k - \log k^*), \qquad \Rightarrow \frac{1}{\alpha}\frac{\dot{y}}{y} = -\beta\frac{1}{\alpha}(\log y - \log y^*) \Leftrightarrow \frac{\dot{y}}{y} = -\beta(\log y - \log y^*)$$

Therefore, both k(t) and y(t) have the same speed of convergence.

For Solow model, the technology is Y=F(A,K,L).

$$AF(K,L) = \begin{cases} F(AK, AL), & \text{Hicks neutral} \\ F(K, AL), & \text{Harrod neutral} \\ F(AK, L), & \text{Solow neutral} \end{cases}$$

For g_Y (output growth), g_K (capital growth) and g_C (consumption growth) to be balanced (not necessarily a constant), Harrod neutral condition is required (the next class will cover this).



Suppose you are a governor. In this case, what are you maximizing? Capital growth rate? Maybe the objective will be the welfare and the welfare is determined by consumption.

$$
\begin{aligned}
c^*(s) &= (1-s)f[k^*(s)] \\
sf[k^*(s)] &= (n+\delta)k^*(s), \qquad \text{in equilibrium} \\
c^*(s) &= f[k^*(s)] - sf[k^*(s)] \\
&= f[k^*(s)] - (n+\delta)k^*(s) \\
s^* &= \underset{s\in(0,1)}{\operatorname{argmax}}\, c^*(s) = f[k^*(s)] - (n+\delta)k^*(s) \\
\text{FOC,} \qquad \left.\frac{\partial c^*}{\partial s}\right|_{s=s^*} &= f'[k^*(s^*)](k^*)'(s^*) - (n+\delta)(k^*)'(s^*) = 0 \\
f'[k^*(s^*)](k^*)'(s^*) &= (n+\delta)(k^*)'(s^*)
\end{aligned}
$$

Here s^* is called as <u>Golden-rule Savings Rate</u>.

<u>Balanced Growth Rate</u> (in Solow's model): g_Y (output), g_K (capital), g_C (consumption)

Suppose

$$L_t = L_0 e^{nt}, \qquad Y_t = Y_0 e^{g_Y t}, \qquad K_t = K_0 e^{g_K t}, \qquad C_t = C_0 e^{g_C t}$$

and

$$
\begin{aligned}
Y_t &= F(K_t, L_t, A_t), \qquad \text{CRS} \\
\dot{K}_t &= Y_t - C_t - \delta K_t, \qquad \dot{K}_t = g_K K_t \\
(g_K+\delta)K_t &= Y_t - C_t \\
(g_K+\delta)K_0 e^{g_K t} &= Y_0 e^{g_Y t} - C_0 e^{g_C t} \\
(g_K+\delta)K_0 &= Y_0 e^{(g_Y - g_K)t} - C_0 e^{(g_C - g_K)t}
\end{aligned}
$$

Differentiate both sides by t, then

$$
\begin{aligned}
0 &= (g_Y - g_K)Y_0 e^{(g_Y - g_K)t} - (g_C - g_K)C_0 e^{(g_C - g_K)t} \\
&= (g_Y - g_K)Y_t e^{-g_K t} - (g_C - g_K)C_t e^{-g_K t} \\
&= (g_Y - g_K)Y_t - (g_C - g_K)C_t \\
\Rightarrow (g_Y - g_K)Y_t &= (g_C - g_K)C_t
\end{aligned}
$$

When this relation holds?

1) Trivial: $Y_t=C_t=0$

2) Also trivial: $g_Y=g_K$, $C_t=0$

3) Only meaningful case: $g_Y=g_K=g_C$

Then

$$
\begin{aligned}
Y_t &= F(K_t, L_t, A_t) \\
Y_0 &= F(K_0, L_0, A_0) \\
Y_t e^{-g_Y t} &= F(K_t e^{-g_K t}, L_t e^{-nt}, A_0) \\
Y_t &= e^{g_Y t} F(K_t e^{-g_K t}, L_t e^{-nt}, A_0) \\
&= F\left(K_t e^{(g_Y - g_K)t}, L_t e^{(g_Y - n)t}, A_0\right), \qquad \text{by CRS} \\
&= F(K_t, L_t e^{xt}, A_0), \qquad \text{by } g_Y = g_C = g_K, \qquad \text{where } x = g_Y - n
\end{aligned}
$$

i.e. labor augmented technological progress

$$
\begin{aligned}
Y_t &= F(K_t, A_t \times L_t), \qquad A_t = A_0 e^{xt} \\
\text{Define } \hat{k}_t &= \frac{K_t}{A_t L_t}, \qquad \hat{y}_t = \frac{Y_t}{A_t L_t} = f\left(\hat{k}_t\right) \\
\text{then } \log \hat{k}_t &= \log K_t - \log A_t - \log L_t \\
\text{then } \frac{\dot{\hat{k}}_t}{\hat{k}_t} &= \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{K}_t}{K_t} - x - n \\
\dot{K}_t &= sY_t - \delta K_t \\
\text{hence } \frac{\dot{\hat{k}}_t}{\hat{k}_t} &= s\frac{Y_t}{K_t} - \delta - x - n \\
&= s\frac{Y_t / A_t L_t}{K_t / A_t L_t} - \delta - x - n = s\frac{\hat{y}_t}{\hat{k}_t} - \delta - x - n \\
\text{In addition } \hat{k}_t &= \frac{K_t}{A_t L_t} = k_t A_t^{-1} = k_t (A_0 e^{xt})^{-1} \Rightarrow \log \hat{k}_t = \log k_t - \log A_0 - xt \\
\Rightarrow \frac{\dot{\hat{k}}_t}{\hat{k}_t} &= \frac{\dot{k}_t}{k_t} - x = 0 \text{ when } \frac{\dot{k}_t}{k_t} = x \\
\text{Also } y &= Ak^{\alpha} \Rightarrow \log y = \log A + \alpha \log k \Rightarrow \dot{y}/y = x + \alpha\left(\dot{k}/k\right)
\end{aligned}
$$

Growth of capital will be continued as far as x (exogenous technological progress) is positive (i.e. technology is important for continuation of development).

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$
$$\hat{y}_t = \frac{Y_t}{A_t L_t} = \hat{k}_t^\alpha = f(\hat{k}_t)$$
$$\hat{k}_t = \frac{K_t}{A_t L_t}$$
$$\frac{\dot{\hat{k}}_t}{\hat{k}_t} = s\frac{f(\hat{k}_t)}{\hat{k}_t} - (n + \delta + x)$$
$$\hat{k}^* = \left(\frac{s}{n+\delta+x}\right)^{\frac{1}{1-\alpha}}, \qquad \hat{y}^* = \left(\frac{s}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha}}$$
$$\hat{y}_t = \frac{y_t}{A_t}$$
$$\begin{aligned} y_t &= A_t \left(\frac{s}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha}} \\ &= A_0 e^{xt} \left(\frac{s}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Then compare the US and Ethiopia (under the assumption that n, δ, x are equal for them)

$$\frac{y_t^{US}}{y_t^E} = \left(\frac{s^{US}}{s^E}\right)^{\frac{\alpha}{1-\alpha}}, \qquad \text{assume ceteris paribus}$$

Empirically,

$$\frac{y_{1985}^{US}}{y_{1985}^E} \approx 31 \xRightarrow{\text{suppose } \alpha=\frac{1}{3}} \frac{s_{1985}^{US}}{s_{1985}^E} = \left(\frac{y_{1985}^{US}}{y_{1985}^E}\right)^2 \approx 961, \qquad \text{should be (too big)}$$

Too high savings rate should be observed to justify the yield gap.
Instead, suppose α=2/3, then (s^{US}/s^E)=5.5.
We should give more weight on share of capital α in order to reconcile the observed (y^{US}/y^E) and (s^{US}/s^E).

Go back to y_t then

$$y_t = A_0 e^{xt}\left(\frac{s}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\ln y_t = \ln A_0 + xt + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+\delta+x)$$

Suppose we have cross-sectional data and estimate the regression

$$\ln y_i = a + b_1 \ln s_i + b_2 \ln\left(n^i + \delta^i + x^i\right) + \varepsilon_i$$

According to Romer, Mankiw and Weil (1992, QJE)

$$\hat{a} = 5.48, \qquad \hat{b}_1 = 1.42, \qquad \hat{b}_2 = -1.48, \qquad \overline{R}^2 = .59$$

This result is desirable because (1) $\hat{b}_1$ and $\hat{b}_2$ have opposite signs and (2) their values are close with each other. However, the problem is

$$\text{if } \alpha \in (1/4\,, 1/2), \qquad \text{then } b_1 \in (1/3\,, 1)$$

(Sketch) Then we can change the production with some theories

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \qquad Y_t = A_t K_t^{\alpha} H^{\beta} L_t^{1-\alpha-\beta}$$

Our problem was that we are assigning too much weight on labor. By doing this, we can reduce the weight byβ. This is one way to match Solow's model with empirical observation.

From the last class

$$y_t = A_0 e^{xt} \left(\frac{s}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha}}$$
$$\frac{y_t^{US}}{y_t^{E}} = \left(\frac{s^{US}}{s^{E}}\right)^{\frac{\alpha}{1-\alpha}}$$
$$\log y = \log A + xt + \underbrace{\frac{\alpha}{1-\alpha}}_{=\beta_1}\log s - \underbrace{\frac{\alpha}{1-\alpha}}_{\beta_2}\log(n+\delta+x)$$
$$\hat{\beta}_1 = 1.42$$
$$\hat{\beta}_2 = 1.48$$
$$\text{allowable range for } \alpha \in (.25, .50)$$
$$\text{but we have } \alpha = .58$$

One way to reconcile

$$Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}$$
$$\dot{K}_t = s_K Y_t - \delta K_t$$
$$\dot{H}_t = s_H Y_t - \delta H_t, \qquad \text{two processes for K and H}$$
$$\hat{y}_t = \frac{Y_t}{A_t L_t} = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{-\alpha-\beta} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} \left(\frac{H_t}{A_t L_t}\right)^{\beta} = \hat{k}_t^{\alpha} \hat{h}_t^{\beta}$$
$$\hat{h}_t = \frac{H_t}{A_t L_t}$$

Then

$$\frac{\dot{\hat{k}}_t}{\hat{k}_t} = \frac{\dot{K}_t}{K_t} - x - n = s_K \frac{Y_t}{K_t} - (n+\delta+x)$$
$$= s_K \frac{K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta} / A_t L_t}{K_t / A_t L_t} - (n+\delta+x)$$
$$= s_K \frac{\hat{k}_t^{\alpha} \hat{h}_t^{\beta}}{\hat{k}_t} - (n+\delta+x)$$
$$\frac{\dot{\hat{h}}_t}{\hat{h}_t} = \frac{\dot{H}_t}{H_t} - x - n = s_H \frac{\hat{k}_t^{\alpha} \hat{h}_t^{\beta}}{\hat{h}_t} - (n+\delta+x)$$

In the steady state for $\hat{k}_t$

$$\frac{\dot{\hat{k}}_t}{\hat{k}_t} = 0, \qquad \Rightarrow \hat{k}_t^{\alpha-1}\hat{h}_t^{\beta} = \frac{n+\delta+x}{s_K}$$

By taking the total derivative,

$$(\alpha-1)\hat{k}_t^{\alpha-2}\hat{h}_t^{\beta}d\hat{k}_t + \beta\hat{k}_t^{\alpha-1}\hat{h}_t^{\beta-1}d\hat{h}_t = 0$$

$$\Rightarrow \frac{d\hat{h}_t}{d\hat{k}_t} = \frac{(1-\alpha)\hat{k}_t^{\alpha-2}\hat{h}_t^{\beta}}{\beta\hat{k}_t^{\alpha-1}\hat{h}_t^{\beta-1}} = \frac{1-\alpha}{\beta}\frac{\hat{h}_t}{\hat{k}_t}$$

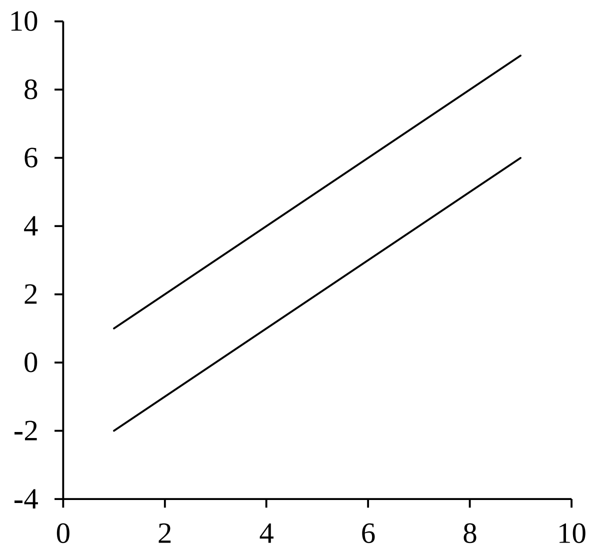
In the steady state for $\hat{h}_t$

$$\frac{\dot{\hat{h}}_t}{\hat{h}_t} = 0, \qquad \Rightarrow \hat{k}_t^{\alpha}\hat{h}_t^{\beta-1} = \frac{n+\delta+x}{s_H}$$

By taking the total derivative,

$$\alpha\hat{k}_t^{\alpha-1}\hat{h}_t^{\beta-1}d\hat{k}_t + (\beta-1)\hat{k}_t^{\alpha}\hat{h}_t^{\beta-2}d\hat{h}_t = 0$$

$$\Rightarrow \frac{d\hat{h}_t}{d\hat{k}_t} = \frac{\alpha\hat{k}_t^{\alpha-1}\hat{h}_t^{\beta-1}}{(1-\beta)\hat{k}_t^{\alpha}\hat{h}_t^{\beta-2}} = \frac{\alpha}{1-\beta}\frac{\hat{h}_t}{\hat{k}_t}$$



If these two functions have identical slopes, then

$$\frac{1-\alpha}{\beta}\frac{\hat{h}_t}{\hat{k}_t} = \frac{\alpha}{1-\beta}\frac{\hat{h}_t}{\hat{k}_t}, \qquad \Rightarrow \frac{1-\alpha}{\beta} = \frac{\alpha}{1-\beta}, \qquad \Rightarrow 1-\alpha-\beta+\alpha\beta = \alpha\beta, \qquad \Rightarrow \alpha+\beta = 1$$

Since $\alpha+\beta$ is smaller than 1 by assumption, the slopes of the two functions are not identical and hence there exists a crossing point between them.

Then, the equilibrium is

$$\begin{cases} \hat{k}_t^{\alpha-1}\hat{h}_t^{\beta} = \dfrac{n+\delta+x}{s_K}, & \Rightarrow \hat{h}_t = \left(\dfrac{n+\delta+x}{s_K}\right)^{\frac{1}{\beta}} \hat{k}_t^{\frac{1-\alpha}{\beta}} \\ \hat{k}_t^{\alpha}\hat{h}_t^{\beta-1} = \dfrac{n+\delta+x}{s_H} & \end{cases}$$

Then

$$\begin{aligned}
\hat{k}_t^{\alpha}\left(\frac{n+\delta+x}{s_K}\right)^{\frac{\beta-1}{\beta}} \hat{k}_t^{\frac{(1-\alpha)(\beta-1)}{\beta}} &= \frac{n+\delta+x}{s_H} \\
\Rightarrow \hat{k}_t^{\frac{\alpha\beta-1+\alpha+\beta-\alpha\beta}{\beta}} &= \left(\frac{n+\delta+x}{s_H}\right)\left(\frac{n+\delta+x}{s_K}\right)^{-\frac{\beta-1}{\beta}} \\
\Rightarrow \hat{k}_t^{\frac{\alpha+\beta-1}{\beta}} &= \left(\frac{s_H}{n+\delta+x}\right)^{-1}\left(\frac{s_K}{n+\delta+x}\right)^{\frac{\beta-1}{\beta}} \\
\Rightarrow \hat{k}_t &= \left(\frac{s_H}{n+\delta+x}\right)^{\frac{\beta}{1-\alpha-\beta}}\left(\frac{s_K}{n+\delta+x}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \\
&= \left(\frac{s_H^{\beta}s_K^{1-\beta}}{n+\delta+x}\right)^{\frac{1}{1-\alpha-\beta}}, \qquad \text{in the steady state}
\end{aligned}$$

And

$$\begin{aligned}
\left(\frac{s_H^{\beta}s_K^{1-\beta}}{n+\delta+x}\right)^{\frac{\alpha-1}{1-\alpha-\beta}} \hat{h}_t^{\beta} &= \frac{n+\delta+x}{s_K} \\
\Rightarrow \hat{h}_t^{\beta} &= \left(\frac{s_K}{n+\delta+x}\right)^{-1}\left(\frac{s_H^{\beta}s_K^{1-\beta}}{n+\delta+x}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \\
&= \left(\frac{s_K}{n+\delta+x}\right)^{\frac{\alpha+\beta-1}{1-\alpha-\beta}}\left(\frac{s_H^{\beta}s_K^{1-\beta}}{n+\delta+x}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \\
&= \left(\frac{1}{n+\delta+x}\right)^{\frac{\beta}{1-\alpha-\beta}} s_K^{\frac{\alpha\beta}{1-\alpha-\beta}} s_H^{\frac{(1-\alpha)\beta}{1-\alpha-\beta}} = \left(\frac{s_K^{\alpha}s_H^{1-\alpha}}{n+\delta+x}\right)^{\frac{\beta}{1-\alpha-\beta}} \\
\Rightarrow \hat{h}_t &= \left(\frac{s_K^{\alpha}s_H^{1-\alpha}}{n+\delta+x}\right)^{\frac{1}{1-\alpha-\beta}}, \qquad \text{in the steady state}
\end{aligned}$$

Then

$$\hat{y}_t^* = (\hat{k}_t^*)^{\alpha}(\hat{h}_t^*)^{\beta} = \left(\frac{s_H^{\beta}s_K^{1-\beta}}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{s_K^{\alpha}s_H^{1-\alpha}}{n+\delta+x}\right)^{\frac{\beta}{1-\alpha-\beta}}, \qquad \text{and } \hat{y}_t = \frac{y_t}{A_t}$$

So

$$y_t^* = A_t\hat{y}_t^* = A_0e^{xt}\left(\frac{s_H^{\beta}s_K^{1-\beta}}{n+\delta+x}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{s_K^{\alpha}s_H^{1-\alpha}}{n+\delta+x}\right)^{\frac{\beta}{1-\alpha-\beta}}$$

By taking the logarithm

$$\log y^* = \log A_0 + xt + \underbrace{\frac{\alpha}{1-\alpha-\beta}}_{=b_1}\log s_K + \underbrace{\frac{\beta}{1-\alpha-\beta}}_{=b_2}\log s_H - \underbrace{\frac{\alpha+\beta}{1-\alpha-\beta}}_{=b_3}\log(n+\delta+x)$$

Results

$$\hat{b}_1 = .69, \qquad \hat{b}_2 = .66, \qquad \hat{b}_3 = -1.73$$

Because of several econometric pitfalls, these analyses are not popular nowadays.

AK model

$$Y_t = Z_tK_t^{\alpha}, \qquad Z_t \equiv AK_t^{\beta}, \qquad \Rightarrow Y_t = A_tK_t^{\beta}K_t^{\alpha} = A_tK_t^{\alpha+\beta}$$

Assumptions
(1) Technology depends on knowledge
(2) Knowledge is non-rivalrous
(3) More activity creates more knowledge
Or other form such as

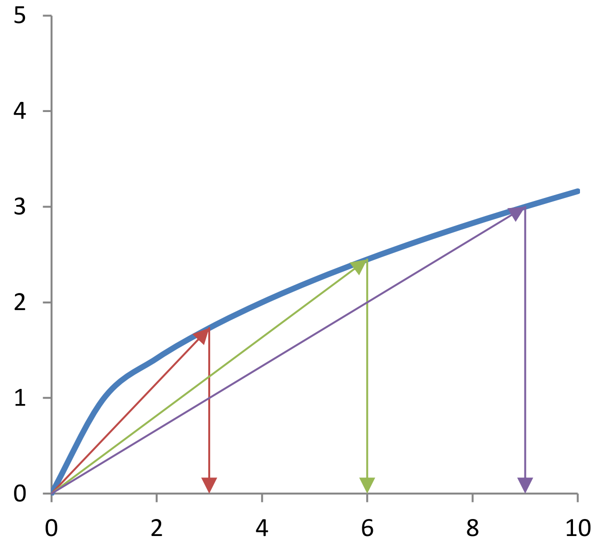
$$Y_t = Z_tK_t^{1-\alpha}L_t^{\alpha}, \qquad \Rightarrow y_t = Z_tk_t^{1-\alpha}, \qquad Z_t \equiv Ak_t^{\beta}, \qquad \Rightarrow y_t = Ak_t^{1-(\alpha-\beta)}$$

If $\alpha=\beta$, then

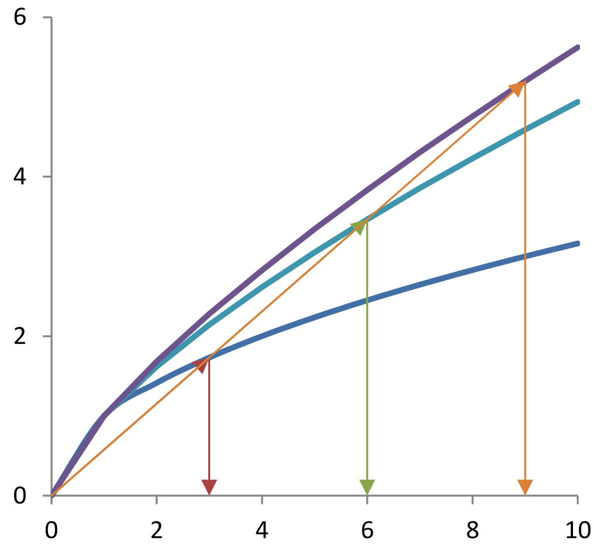
$$y_t = Ak_t$$

For instance, in Solow model

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (n+\delta) = sA - (n+\delta) > 0, \qquad \text{permanent growth}$$



If marginal production is decreasing, return also will be decreasing and hence growth will be diminishing.



However, if there is a technological progress with increasing production then growth will be continued

$$\begin{aligned}
Y_t &= F(A_t, K_t, L_t) \\
\log Y_t &= \log F(A_t, K_t, L_t) \\
\frac{\dot{Y}}{Y} &= \frac{1}{F(\cdot)}\left(F_A\dot{A} + F_K\dot{K} + F_L\dot{L}\right) \\
&= \frac{A \cdot F_A}{F(\cdot)}\frac{\dot{A}}{A} + \frac{K \cdot F_K}{F(\cdot)}\frac{\dot{K}}{K} + \frac{L \cdot F_L}{F(\cdot)}\frac{\dot{L}}{L}
\end{aligned}$$

So

$$\underbrace{\frac{A \cdot F_A}{F(\cdot)}\frac{\dot{A}}{A}}_{(1)} = \underbrace{\frac{\dot{Y}}{Y}}_{(2)} - \frac{K \cdot F_K}{F(\cdot)}\underbrace{\frac{\dot{K}}{K}}_{(3)} - \frac{L \cdot F_L}{F(\cdot)}\underbrace{\frac{\dot{L}}{L}}_{(40}$$

Our objective is measuring (1) and we know $\dot{Y}$, Y & $\dot{K}$, K & $\dot{L}$, L from the data.
However, can we known F_K, F_L...? Suppose the production is

$$\begin{aligned}
F(\cdot) &= Y = AK^{\alpha}L^{1-\alpha} \\
\Rightarrow F_K &= \alpha AK^{\alpha-1}L^{1-\alpha} = r, \qquad \text{return from capital} \\
\Rightarrow K \cdot F_K &= \alpha AK^{\alpha}L^{1-\alpha} = Kr \\
\text{so } r &= \frac{\alpha Y}{K}, \qquad \text{and } \alpha = \frac{rK}{Y}
\end{aligned}$$

Since we know rK (aggregate return from capital) and Y, we know α as well and hence

$$K \cdot F_K = \alpha \cdot Y$$

Then we can calculate the contribution of the technology (1) by calculating others (Solow residual).

What is a huge assumption here is that we are assuming competitive markets by F_K=r (this is true only under the competitive market).

Country	Years	Growth Rate	TFP
Hong Kong	1966–1992	5.7	2.3
Singapore	1966–1990	6.8	0.2
South Korea	1966–1990	6.8	1.7
Taiwan	1966–1990	6.7	2.1

where TFP stands for *Total Factor Productivity*; related papers Young, Solow and Romer.

Data shows constant returns during these periods. According to decreasing marginal producting, return should be decreased. However, constant return implies technological progress (or something missed by model).

Hsieh (2002); R is capital return, W is labor wage.

$$
\begin{aligned}
Y &= RK + WL \\
\log Y &= \log(RK + WL) \\
\frac{\dot{Y}}{Y} &= \frac{1}{RK + WL}\left(\dot{R}K + R\dot{K} + \dot{W}L + W\dot{L}\right) \\
&= \frac{RK}{Y}\left(\frac{\dot{R}}{R} + \frac{\dot{K}}{K}\right) + \frac{WL}{Y}\left(\frac{\dot{W}}{W} + \frac{\dot{L}}{L}\right)
\end{aligned}
$$

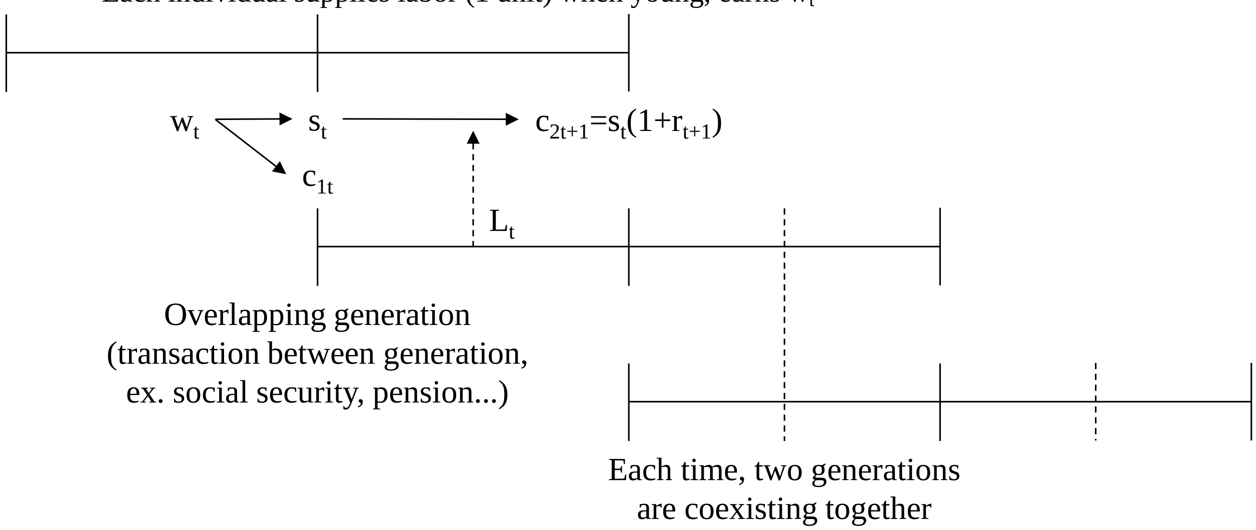
Then by imposing $\dot{K}/K=\dot{L}/L=0$, the contribution of technology is

$$
\frac{\dot{Y}}{Y} = \frac{RK}{Y}\frac{\dot{R}}{R} + \frac{WL}{Y}\frac{\dot{W}}{W}
$$

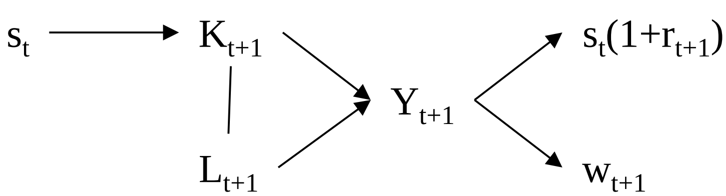
One thing important here is that there is no imposed assumption on market competitiveness.

Overlapping Generation (OLG) model

- Discrete time; t=1,2,...
- Individual lives for 2 periods; young and old
- $L_t=L_0(1+n)^t$
- Each individual supplies labor (1 unit) when young, earns w_t



Big picture



Individual utility

$$\max_{\{c_{1t},c_{2t+1}\}} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1})$$
$$\text{s.t. } c_{1t} + s_t = w_t, \qquad c_{2t+1} = (1 + r_{t+1}) s_t$$

Redefine $c_{1t}=w_t-s_t$

$$\Rightarrow \max_{s_t} u(w_t - s_t) + \frac{1}{1+\rho} u\big(s_t(1 + r_{t+1})\big)$$

FOC

$$\frac{dU}{ds_t} = 0, \qquad \Rightarrow -u'(w_t - s_t) + \frac{u'\big(s_t(1 + r_{t+1})\big)}{1+\rho}(1 + r_{t+1}) = 0$$
$$\Rightarrow u'(c_t) = \frac{1}{1+\rho} u'(c_{t+1})(1 + r_{t+1}), \qquad \cdots \text{Euler equation}$$

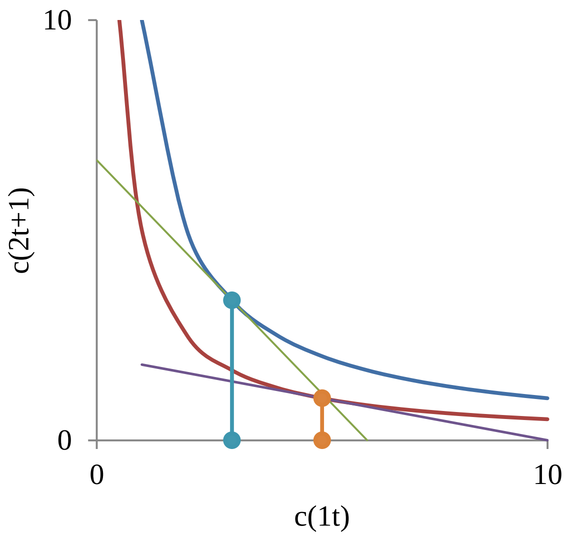
Interpretation

$$\underbrace{u'(c_t)}_{\substack{\text{what you can}\\ \text{get when you}\\ \text{consume today}}} = \underbrace{\frac{1}{1+\rho} u'(c_{t+1})(1 + r_{t+1})}_{\substack{\text{what you can get by}\\ \text{postponing your consumption}\\ \text{from today to tomorrow}}}$$

From Euler equation

$$u'(c_{1t}) = (1+\rho)^{-1}u'(c_{2t+1})(1+r_{t+1})$$

$$\Rightarrow \underbrace{(1+\rho)^{-1}\frac{u'(c_{2t+1})}{u'(c_{1t})}}_{\substack{\text{marginal rate of} \\ \text{intertemporal substitution}}} = \frac{1}{1+r_{t+1}}$$



Slope of budget constraint ($\frac{1}{1+r_{t+1}}$ from c_{2t+1} and $1+r_{t+1}$ from c_{1t})

Rewrite Euler equation

$$u'(w_t - s_t) - \frac{1}{1+\rho}u'\big(s_t(1+r_{t+1})\big)(1+r_{t+1}) = 0$$

By taking total differentiation

$$\left(-u''(c_{1t}) - \frac{1}{1+\rho}u''(c_{2t+1})(1+r_{t+1})^2\right)ds_t + \big(u''(c_{1t})\big)dw_t = 0$$

$$\Rightarrow \frac{ds_t}{dw_t} = \frac{u''(c_{1t})}{u''(c_{1t}) + (1+\rho)^{-1}u''(c_{2t+1})(1+r_{t+1})^2} > 0, \qquad \cdots (*)$$

And

$$\left(-u''(c_{1t}) - \frac{1}{1+\rho}u''(c_{2t+1})(1+r_{t+1})^2\right)ds_t + \left(-\frac{1}{1+\rho}u''(c_{2t+1})c_{2t+1} - \frac{1}{1+\rho}u'(c_{2t+1})\right)dr_{t+1} = 0$$

$$\Rightarrow \frac{ds_t}{dr_{t+1}} = \frac{-\frac{1}{1+\rho}(u'(c_{2t+1}) + u''(c_{2t+1})c_{2t+1})}{u''(c_{1t}) + \frac{1}{1+\rho}u''(c_{2t+1})(1+r_{t+1})^2}, \qquad \cdots (**)$$

We cannot determine the sign of (**) because the sign of $(u'(c_{2t+1}) + u''(c_{2t+1})c_{2t+1})$ is not obvious.
(The matter of INCOME versus SUBSTITUTION EFFECT)

Present Value Budget Constraint

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t$$

$$\text{where } \frac{1}{1+r_{t+1}} = \begin{matrix}\text{discounting}\\ \text{factor}\end{matrix} = \begin{matrix}\text{PRICE of tomorrow's consumption}\\ \text{in terms of today's consumption}\end{matrix}$$

Recall (**)

$$\frac{ds_t}{dr_{t+1}} = s_r = \frac{-\frac{1}{1+\rho}(u'(c_{2t+1}) + u''(c_{2t+1})c_{2t+1})}{\underbrace{u''(c_{1t}) + \frac{1}{1+\rho}u''(c_{2t+1})(1+r_{t+1})^2}_{\equiv\Delta<0}}$$

$$= \frac{1}{1+\rho}u'(c_{2t+1})\left(\underbrace{-\frac{u''(c_{2t+1})}{u'(c_{2t+1})}c_{2t+1}}_{\equiv\theta(c_{2t+1})} - 1\right)\frac{1}{\Delta}$$

$$= \underbrace{\frac{1}{1+\rho}}_{>0}\underbrace{u'(c_{2t+1})}_{>0}(\theta(c_{2t+1}) - 1)\frac{1}{\Delta}$$

Hence

$$s_r > 0 \Leftrightarrow \theta(c_{2t+1}) - 1 < 0 \Leftrightarrow -\frac{u''(c_{2t+1})}{u'(c_{2t+1})}c_{2t+1} < 1$$

$$s_r < 0 \Leftrightarrow \theta(c_{2t+1}) - 1 > 0 \Leftrightarrow -\frac{u''(c_{2t+1})}{u'(c_{2t+1})}c_{2t+1} > 1$$

If $\theta(c_{2t+1})$ is constant and >0 then the function u(.) must be of the form

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \text{when } \theta \neq 1$$

$$= \ln c, \qquad \text{when } \theta = 1$$

Note that CRRA with θ

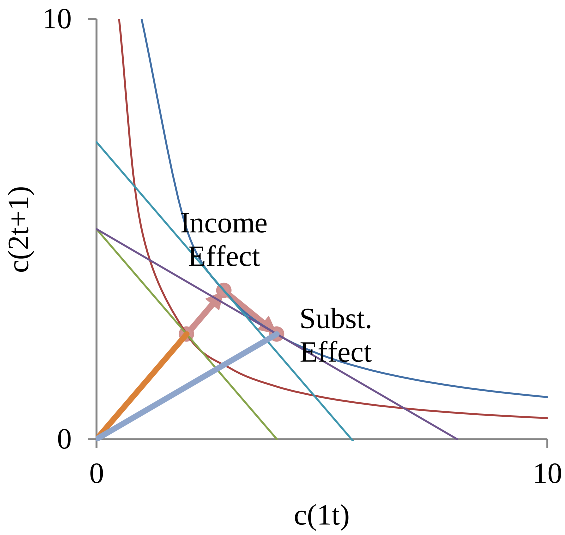
$$-\frac{u''(c)}{u'(c)}c = \theta \Rightarrow \frac{du'}{u'} = -\theta\frac{1}{c} \Rightarrow \ln u' = \ln c^{-\theta} + a_1 \Rightarrow u' = A_1 c^{-\theta}, \qquad \text{with } A_1 \in \mathbb{R}_+$$

$$\Rightarrow du = A_1 c^{-\theta} dc \Rightarrow u(c) = \frac{A_1}{1-\theta}c^{1-\theta} + A_2, \qquad \text{with } A_2 \in \mathbb{R}$$

CRRA with 1

$$du = A_1 c^{-1} dc \Rightarrow u(c) = A_1 \ln c + a_2 = \ln A_2 c^{A_1}, \qquad \text{with } A_1, A_2 \in \mathbb{R}_+$$

Then $\frac{1}{\theta}$ is defined as Intertemporal Elasticity of Substitution



$$R = 1 + r \Rightarrow \frac{\frac{d\left(\frac{c_{2t+1}}{c_{1t}}\right)}{\left(\frac{c_{2t+1}}{c_{1t}}\right)}}{\frac{dR}{R}} \approx \frac{\%\text{ change in consumption ratio}}{\%\text{ change in interest rate}}$$

$$\underbrace{\frac{\dot{c}}{c}}_{\substack{\text{will be related}\\ \text{to growth rate}}} > 0 \Leftrightarrow \frac{\Delta c_{2t+1}}{c_{1t}} > 0 \Leftrightarrow \frac{c_{2t+1}}{c_{1t}} > 1$$

$$Y = F(L, K), \qquad y = f(k) \Rightarrow F(L, K) = Lf(k)$$
$$F_K(K, L) = Lf'(k)\,\partial k/\partial K = Lf'(k)L^{-1} = f'(k)$$
$$F_L(K, L) = f(k) + Lf'(k)\,\partial k/\partial L = f(k) + Lf'(k)(-K/L^2) = f(k) - (K/L)f'(k) = f(k) - kf'(k)$$

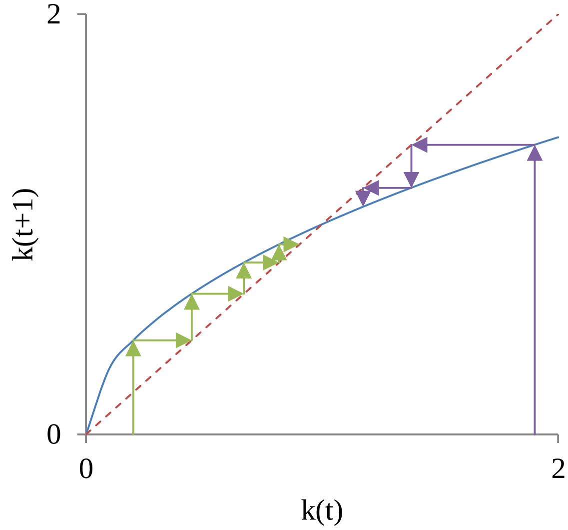
HW #2. Derive this.

Aggregate

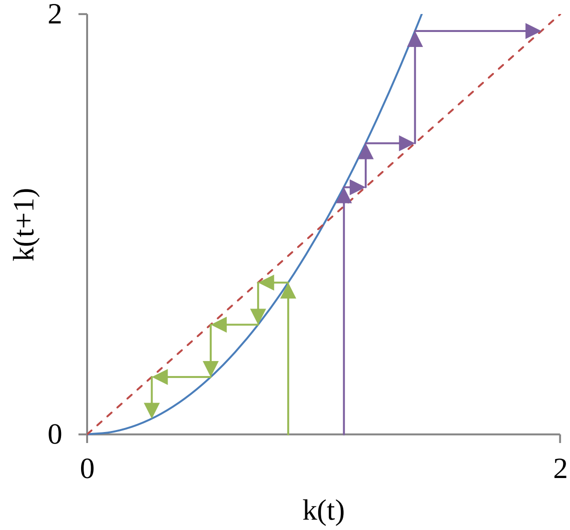
$$S_t = s_t(w_t, r_{t+1})L_t \underbrace{=}_{\substack{\text{by market}\\ \text{clearing}}} K_{t+1}$$

Then

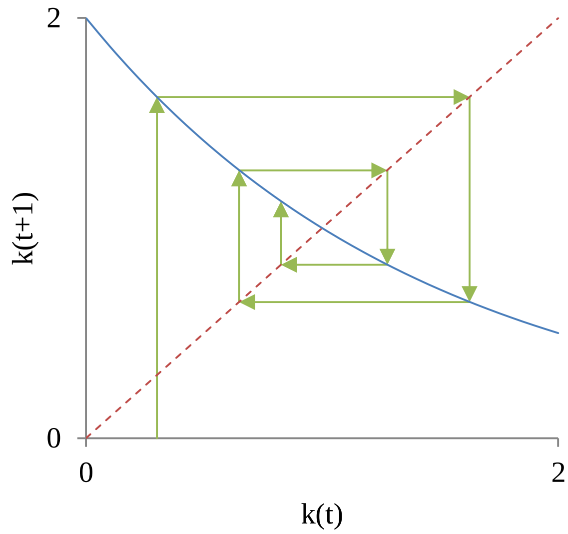
$$K_{t+1} = s_t[w_t(k_t), r_{t+1}(k_{t+1})]L_t$$
$$\Rightarrow k_{t+1} = \phi(k_t)$$



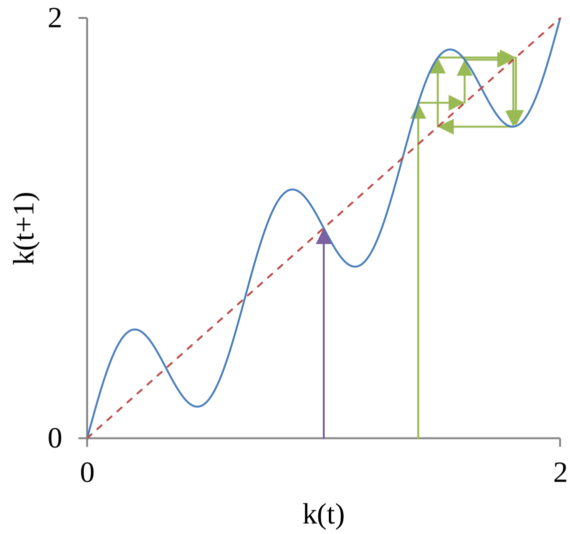
1 Stable Steady State



1 Unstable Steady State



Spiral Convergence (Fluctuation)



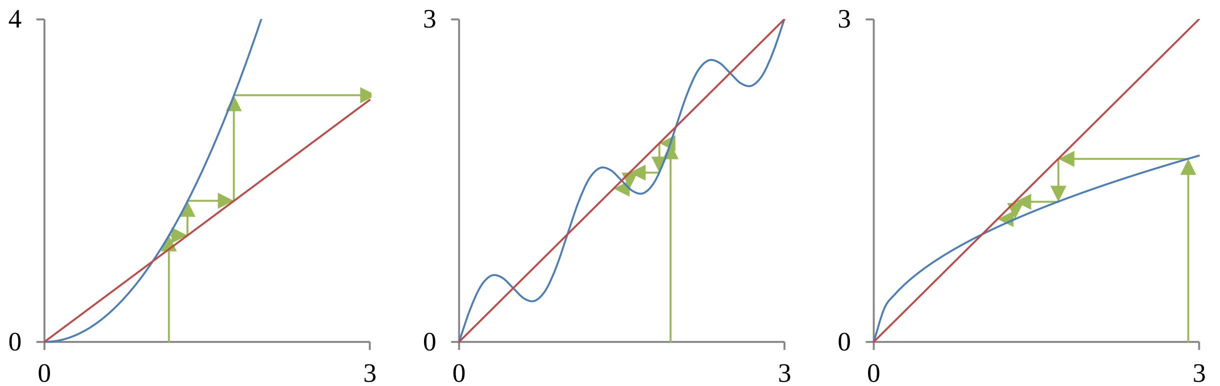
Multiple Equilibria

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+n} s_t[w_t(k_t), r_{t+1}(k_{t+1})], \qquad \cdots (*) \\
w_t &= f(k_t) - k_t f'(k_t) \\
r_{t+1} &= f'(k_{t+1}) \\
k_{t+1} &= \phi(k_t)
\end{aligned}
$$

By using the dynamics ϕ, we can infer about the sequence of capital-labor ratio $\{k_0, k_1, k_2, \ldots\}$. By applying total derivative for (*)

$$
\frac{1}{1+n}\frac{\partial s_t}{\partial w_t}\left(f'(k_t) - k_t f''(k_t) - f'(k_t)\right) dk_t + \left(\frac{1}{1+n}\frac{\partial s_t}{\partial r_{t+1}} f''(k_{t+1}) - 1\right) dk_{t+1} = 0
$$

$$
\Rightarrow \frac{dk_{t+1}}{dk_t} = \frac{s_w[-k_t f''(k_t)]}{(1+n) - s_r f''(k_{t+1})}
$$



1 unstable SS Multiple SS 1 stable SS

Conditions for the unique SS

- $\phi'(0)>1$
- $\phi'(k_*)<1$

With specific utility,

$$
u = \ln c_{1t} + \beta \ln c_{2t+1} \text{ subject to } w_t = c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} \Rightarrow c_{2t+1} = (w_t - c_{1t})(1+r_{t+1})
$$

Then

$$
u = \ln c_{1t} + \beta \ln(w_t - c_{1t})(1+r_{t+1}) \text{ or } L = \ln c_{1t} + \beta \ln c_{2t+1} + \lambda\left(w_t - c_{1t} - \frac{c_{2t+1}}{1+r_{t+1}}\right)
$$

Use Lagrangian

$$\frac{\partial L}{\partial c_{1t}} = \frac{\partial L}{\partial c_{2t+1}} = \frac{\partial L}{\partial \lambda} = 0$$

Or

$$\frac{\partial u}{\partial c_{1t}} = \frac{1}{c_{1t}} - \beta \frac{1 + r_{t+1}}{(w_t - c_{1t})(1 + r_{t+1})} = \frac{1}{c_{1t}} - \frac{\beta}{w_t - c_{1t}} = 0, \qquad \text{at } c_{1t} = c_{1t}^*$$
$$\Rightarrow c_{1t}^* = \frac{w_t - c_{1t}^*}{\beta} \Rightarrow \frac{1+\beta}{\beta} c_{1t}^* = \frac{w_t}{\beta} \Rightarrow c_{1t}^* = \frac{1}{1+\beta} w_t$$
$$\Rightarrow s_t = w_t - c_{1t}^* = \frac{\beta}{1+\beta} w_t$$

Big picture

$$\underbrace{K_{t+1}}_{\substack{\text{capital}\\\text{demand}\\\text{tomorrow}}} = \underbrace{L_t}_{\text{population}} \underbrace{\frac{\beta}{1+\beta} w_t}_{\substack{\text{savings per}\\\text{capita today}}}, \qquad \text{then the market clears}$$
$$\frac{K_{t+1}}{L_{t+1}} = \frac{1}{1+n} \frac{\beta}{1+\beta} w_t, \qquad \text{how can we relate } k_t \text{ here instead } w_t? \cdots (**)$$

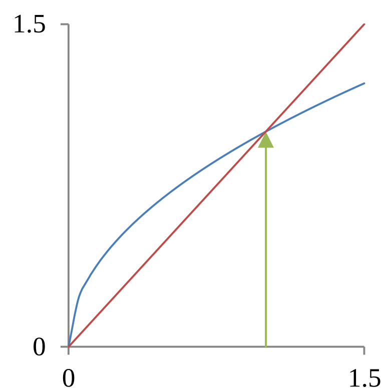
Production function!

$$y_t = Ak_t^\alpha = f(k_t)$$
$$w_t = f(k_t) - k_t f'(k_t) = Ak_t^\alpha - A\alpha k_t^\alpha = (1-\alpha)Ak_t^\alpha$$

From (**)

$$k_{t+1} = \frac{A\beta(1-\alpha)}{(1+n)(1+\beta)} k_t^\alpha \equiv \Omega k_t^\alpha$$

$$\text{at the steady state} \Rightarrow k_* = \Omega k_*^\alpha \Rightarrow k_* = \Omega^{\frac{1}{1-\alpha}} = \left[\frac{A\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}$$



1 stable SS

Two important points!

- $F_L(.)$=<u>considered as "wage"</u> & $F_K(.)$=<u>considered as "capital return"</u>
- $s_t^* = \frac{\beta}{1+\beta} w_t$, which is <u>endogenously</u> determined in the agent's maximization problem based on β, i.e. myopic agent→β ↓→s_t^* ↓ and vice versa (really distinguished characteristics from Solow model; at that time, s(.) was exogenously given)

Government intervention

$$\text{budget constraint} \Rightarrow c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - \underbrace{d}_{\substack{\text{government}\\\text{imposes this}}} + \frac{\overbrace{d(1+r_{t+1})}^{\text{return back at t+1}}}{\underbrace{1+r_{t+1}}_{\text{should be discounted}}} = w_t, \qquad \text{no change!}$$

Hence there is no difference and thus savings would not be changed.

<u>Centralized Economy</u>

$$\underbrace{F(K_t, L_t)}_{\substack{\text{what is}\\\text{produced}\\\text{today}}} = \underbrace{(K_{t+1} - K_t)}_{\substack{\text{can be saved}\\\text{for tomorrow}}} + \underbrace{L_t c_{1t} + L_{t-1} c_{2t}}_{\substack{\text{or can be consumed today}\\\text{by both newbies and oldbies}}}$$

Divide both sides by L_t

$$f(k_t) = (1+n)k_{t+1} - k_t + c_{1t} + \frac{c_{2t}}{1+n}$$
$$f(k_t) + k_t = (1+n)k_{t+1} + c_{1t} + \frac{c_{2t}}{1+n}$$
$$\Rightarrow f(k_*) + k_* = (1+n)k_* + c_{1*} + \frac{c_{2*}}{1+n}$$
$$\Rightarrow f(k_*) - nk_* = c_{1*} + \frac{c_{2*}}{1+n} \equiv c_*$$

Maximizing consumption

$$c_* = f(k_*) - nk_* \Rightarrow \frac{\partial c_*}{\partial k_*} = f'(k_*) - n = 0, \qquad \text{at the maximized point}$$
$$\Rightarrow f'(k_*) = n \Rightarrow k_* = (f')^{-1}(n)$$

Here k_* is the golden rule capital-labor ratio that maximizes welfare

$$f(k) = Ak^{\alpha}$$
$$f'(k) = A\alpha k^{\alpha-1}$$
$$f'(k_G^*) = A\alpha (k_G^*)^{\alpha-1} = n, \qquad \text{should be (in the social optimum)}$$
$$k_G^* = \underbrace{\left(\frac{A\alpha}{n}\right)^{\frac{1}{1-\alpha}}}_{\substack{\text{social}\\\text{optimum}}} \neq \underbrace{\left[\frac{A\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}}_{\substack{\text{selfish decision that}\\\text{agents will achieve instead}}}$$

Modified Golden Rule

$$f'(k^*) = n$$
$$r_{t+1}^* = n, \qquad \text{in competetive market} \Rightarrow \text{Pareto optimal can be achieved}$$

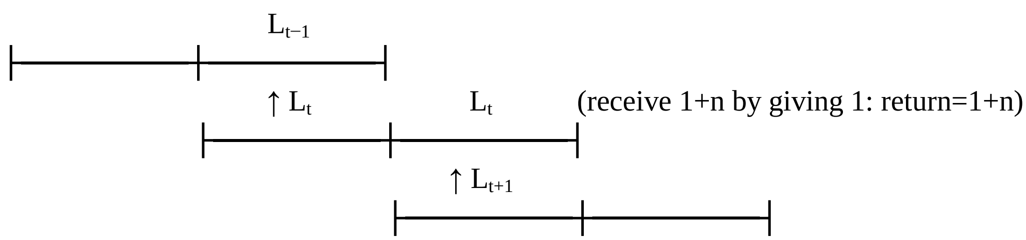
From the last class,

$$u(c_{1t}, c_{2t+1}) = \ln c_{1t} + \beta \ln c_{2t+1}$$
$$\text{s.t. } c_{1t} + \frac{c_{2t+1}}{1 + r_{t+1}} = w_t$$
$$\Rightarrow c_{1t}^* = \frac{w_t}{1+\beta}, c_{2t+1}^* = \frac{\beta}{1+\beta} w_t (1 + r_{t+1})$$

Indirect utility

$$u(c_{1t}^*, c_{2t+1}^*) = \ln\left(\frac{w_t}{1+\beta}\right) + \beta \ln\left[\frac{\beta}{1+\beta} w_t (1 + r_{t+1})\right] = f(w_t, r_{t+1})$$

Overlapping Generation



If L_t people give all things to the older at t, then they receive L_{t+1} at t+1. So the return is n (however, usually this contract is not the case so $r_{t+1}<n$). If this is possible then $r_{t+1}^*=n$, i.e. Pareto optimal.
Social Planner

$$\max_{\{c_{it}\}} \sum_{t=0}^{\infty} u(c_{1t}) + u(c_{2t})$$
$$\text{s.t. } k_t + f(k_t) = (1+n)k_{t+1} + c_{1t} + \frac{c_{2t}}{1+n}, \qquad \text{budget constraint}$$

This is due to

$$F(K_t) = C_{1t} + C_{2t} + K_{t+1} - K_t$$
$$\Rightarrow f(k_t) = c_{1t} + \frac{c_{2t}}{1+n} + (1+n)k_{t+1} - k_t$$

Then

$$\sum_{t=0}^{\infty} u(c_{1t}) + u(c_{2t}) = u(c_{10}) + u(c_{20}) + \cdots + u(c_{1t-1}) + u(c_{2t}) + u(c_{1t}) + u(c_{2t+1}) + \cdots, \qquad \cdots (*)$$

From budget constraint

$$c_{1t-1} = k_{t-1} + f(k_{t-1}) - (1+n)k_t - \frac{c_{2t-1}}{1+n}$$
$$c_{1t} = k_t + f(k_t) - (1+n)k_{t+1} - \frac{c_{2t}}{1+n}$$

Then (*) is

$$\sum_{t=0}^{\infty} u(c_{1t}) + u(c_{2t}) = + \cdots + u\left[\overbrace{k_{t-1} + f(k_{t-1}) - (1+n)k_t - \frac{c_{2t-1}}{1+n}}^{=c_{1t-1}}\right] + u(c_{2t})$$
$$+u\left[\underbrace{k_t + f(k_t) - (1+n)k_{t+1} - \frac{c_{2t}}{1+n}}_{=c_{1t}}\right] + u(c_{2t+1}) + \cdots$$

Then, at the time t corresponding choice variables are c_{2t} and k_t, so

$$\frac{\partial\Sigma}{\partial c_{2t}} = u'(c_{2t}) - \frac{u'(c_{1t})}{1+n} = 0 \Rightarrow u'(c_{1t}) = (1+n)u'(c_{2t})$$
$$\frac{\partial\Sigma}{\partial k_t} = -u'(c_{1t-1})(1+n) + u'(c_{1t}) \times [1 + f'(k_t)] = 0$$
$$\Rightarrow \underbrace{u'(c_{1t-1})}_{\substack{\text{marginal}\\ \text{utility of}\\ \text{people born}\\ \text{at t=1}}} = \frac{1}{1+n} \underbrace{u'(c_{1t})}_{\substack{\text{marginal}\\ \text{utility of}\\ \text{people born}\\ \text{at t}}} [1 + f'(k_t)], \qquad \text{Euler equation}$$

Note that $u(c_{1t-1})$ and $u(c_{1t})$ are NOT INTERTEMPORAL; indeed they are intergenerational.
Then, under the equilibrium, Euler equation is

$$u'(c_1^*) = \frac{1}{1+n} u'(c_1^*)[1 + f'(k^*)]$$
$$\Rightarrow n = f'(k^*), \qquad \Rightarrow \text{Pareto optimal}$$

Pay as you go system

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t, \qquad \text{no restriction}$$
$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - d + \frac{d(1+r_{t+1})}{1+r_{t+1}}, \qquad \text{prompt reimbursement}$$
$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = w_t - d + \frac{d(1+n)}{1+r_{t+1}} = w_t + \frac{n - r_{t+1}}{1+r_{t+1}} d, \qquad \text{pay as you go}$$

If $n>r_{t+1}$ then the overall budget will increase; if $n<r_{t+1}$ then the budget will decrease.

Alarm: Quiz #2 on the next Monday

Galor and Zeira (1993, RES)
Question: Does income distribution matter?

Ecnoomy	Income distribution at t=0	Output at t=0	$t\rightarrow\infty$
Economy A	$\Omega_0{}^A$	$Y_0{}^A$	$\Omega_*{}^A$
Economy B	$\Omega_0{}^B$	$Y_0{}^B$	$\Omega_*{}^B$ ($\neq\Omega_*{}^A$)

(if $\Omega_*{}^A=\Omega_*{}^B$, then we do not need to think about this)

- Small open economy (i.e. the interest rate r is exogenous, can borrow/lend infinitely)
- $\begin{cases} Y_t^s = F(K_t, L_t^s) \\ Y_t^n = w_n L_t^n \end{cases}$
- Two periods OLG
- $u = \alpha \log c + (1-\alpha)\log b$ (though we studied two periods consumptions c_1, c_2 cases in our textbooks, changing this cannot affect the core result; WLOG)
- (for instance, $u = \alpha_1 \log c_1 + \alpha_2 \log c_2 + (1-\alpha_1-\alpha_2)\log b$ is also possible but just messy)
- h>0 is tuition (i.e. cost of being skilled)
- r is the interest rate (or deposit rate)
- $f'(k) = r$ only in the competitive market (r will be market clearing rate), but in reality the capital market has various frictions (by which $f'(k) \neq r$)
- i is the lending rate, and i>r sufficiently
- What individuals have to decide is
  - Education decision (to be skilled or not to be)
  - Consumption-bequest decision (to consume or not to consume)

By comparing u(after skilled) vs. u(remain unskilled), one can make the first decision
The Problem

$$\max_{\{b,c\}} \alpha \log c + (1-\alpha)\log b \text{ subject to } c + b = (\tilde{b} + w_n)(1+r) + w_n$$
$$\Rightarrow b = (\tilde{b} + w_n)(1+r) + w_n - c$$
$$\Rightarrow \max_b \alpha \log c + (1-\alpha)\log[(\tilde{b} + w_n)(1+r) + w_n - c]$$
$$\Rightarrow \frac{\alpha}{c^*} = \frac{1-\alpha}{(\tilde{b} + w_n)(1+r) + w_n - c^*}$$
$$\Rightarrow c^*(1-\alpha) = \alpha[(\tilde{b} + w_n)(1+r) + w_n - c^*]$$
$$= \alpha[(\tilde{b} + w_n)(1+r) + w_n] - \alpha c^*$$
$$\Rightarrow c^* = \alpha[(\tilde{b} + w_n)(1+r) + w_n], \qquad \Rightarrow b^* = (1-\alpha)[(\tilde{b} + w_n)(1+r) + w_n]$$

Indirect Utility (non-skilled)

$$u_n^* = \log\left[\left(\tilde{b} + w_n\right)(1 + r) + w_n\right] + \alpha \log \alpha + (1 - \alpha) \log(1 - \alpha)$$

Indirect Utility (rich-skilled)

$$u_r^* = \log\left[\left(\tilde{b} - h\right)(1 + r) + w_s\right] + \alpha \log \alpha + (1 - \alpha) \log(1 - \alpha)$$

Indirect Utility (poor-skilled)

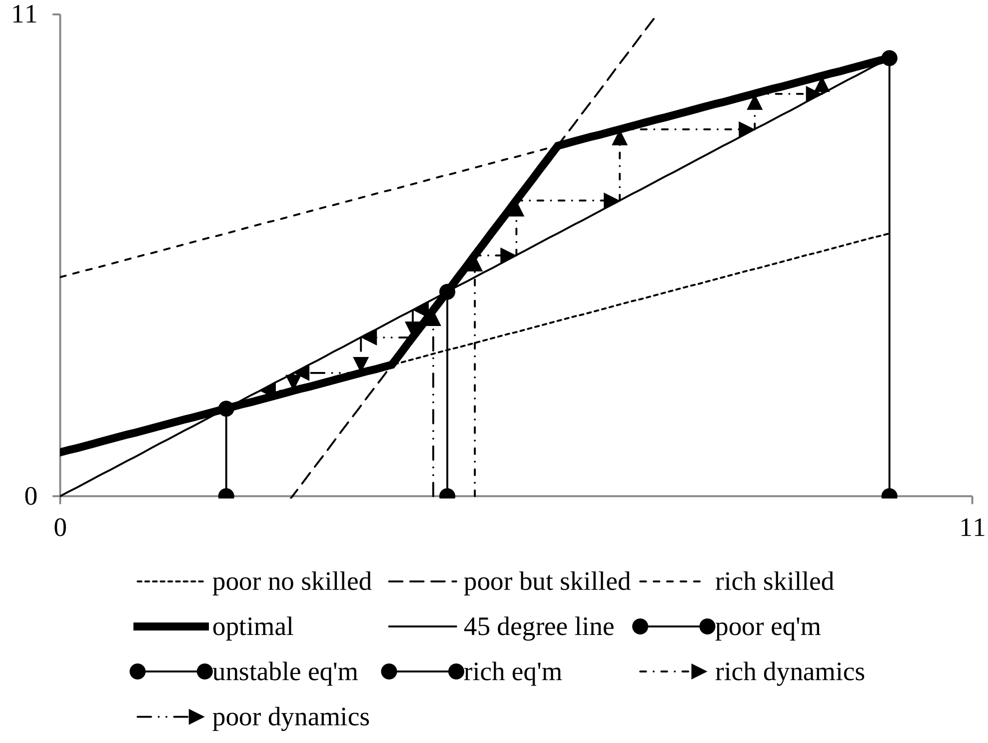
$$u_p^* = \log\left[\left(\tilde{b} - h\right)(1 + i) + w_s\right] + \alpha \log \alpha + (1 - \alpha) \log(1 - \alpha)$$

Hurdle of Attending College: poor guys will go to college only if

$$\begin{aligned} u_r^* \geq u_n^* &\Leftrightarrow \left(\tilde{b} - h\right)(1 + i) + w_s \geq \left(\tilde{b} + w_n\right)(1 + r) + w_n \\ &\Leftrightarrow \tilde{b} - h + \tilde{b}i - hi + w_s \geq \tilde{b} + w_n + \tilde{b}r + w_n r + w_n \\ &\Leftrightarrow \tilde{b}(i - r) \geq w_n(2 + r) + h(1 + i) - w_s \\ &\Leftrightarrow \tilde{b} \geq \underbrace{\frac{w_n(2 + r) + h(1 + i) - w_s}{i - r}}_{\text{bequests enough to go to college}} \equiv f \end{aligned}$$

Bequest Dynamics

$$\tilde{b}_{t+1}\left(\tilde{b}_t\right) = \begin{cases} (1 - \alpha)\left[\left(\tilde{b}_t + w_n\right)(1 + r) + w_n\right], & \tilde{b}_t < f \\ (1 - \alpha)\left[\left(\tilde{b}_t - h\right)(1 + r) + w_s\right], & f \leq \tilde{b}_t < h \\ (1 - \alpha)\left[\left(\tilde{b}_t - h\right)(1 + i) + w_s\right], & \tilde{b}_t \geq h \end{cases}$$



$$\begin{array}{ccccc} & & \text{Model} & & \\ & \swarrow & & \searrow & \\ & \text{Discrete Time} & & \text{Continuous Time} & \\ \swarrow & \searrow & & & \searrow \\ \text{Lagrangian} & & \text{Dynamic Optimization (Bellman)} & & \text{Optimal Control} \end{array}$$

We can choose a consumption stream $\{c_t : t \geq 0\}$
Robinson Crusoe Economy

$$\max_{\{c_t, k_t\}} \sum_{t=0}^{T} \beta^t u(c_t)$$
$$\begin{aligned} \text{s.t. } c_t + k_{t+1} &\leq f(k_t) \\ c_t &\geq 0 \\ k_{t+1} &\geq 0 \\ k_0 &= \text{given} \\ &+\text{concave } u(\cdot), \qquad \text{Inada condition} \end{aligned}$$

Lagrangian

$$\begin{aligned} L = &\sum_{t=0}^{T} \beta^t \{u(c_t) - \lambda_t [c_t + k_{t+1} - f(k_t)] + \mu_t k_{t+1}\} \\ &= \beta^0 \{u(c_0) - \lambda_0 [c_0 + k_1 - f(k_0)] + \mu_0 k_1\} \\ &+ \cdots \\ &+ \beta^t \{u(c_t) - \lambda_t [c_t + k_{t+1} - f(k_t)] + \mu_t k_{t+1}\} \\ &+ \beta^{t+1} \{u(c_{t+1}) - \lambda_{t+1} [c_{t+1} + k_{t+2} - f(k_{t+1})] + \mu_{t+1} k_{t+2}\} \\ &+ \cdots \\ &+ \beta^T \left\{ u(c_T) - \lambda_T [c_T + k_{T+1} - f(k_T)] + \underbrace{\mu_T}_{\substack{\text{shadow} \\ \text{price}}} k_{T+1} \right\} \end{aligned}$$

Hence

$$\frac{\partial L}{\partial c_t} = 0 \Rightarrow \beta^t[u'(c_t) - \lambda_t] = 0, \qquad \forall t = 0, \dots, T, \qquad \cdots (1)$$

$$\frac{\partial L}{\partial k_{t+1}} = 0 \Rightarrow -\beta^t\lambda_t + \beta^t\mu_t + \beta^{t+1}\lambda_{t+1}f'(k_{t+1}) = 0, \qquad \forall t = 0, \dots, T, \qquad \cdots (2)$$

$$\frac{\partial L}{\partial k_{T+1}} = 0 \Rightarrow -\beta^T\lambda_T + \beta^T\mu_T = 0, \qquad \cdots (3)$$

Shadow Price & Kuhn–Tucker Condition

$$\begin{aligned}
&\max u(x,y) \ \text{ subject to } p_x x + p_y y \le M \\
&L = u(x,y) + \lambda\big(M - p_x x - p_y y\big) \\
&\Rightarrow \lambda > 0 \Rightarrow M = p_x x + p_y y, \qquad \text{(binding, no slack)} \\
&\Rightarrow \lambda = 0 \Rightarrow M > p_x x + p_y y, \qquad \text{(not binding, slack)}
\end{aligned}$$

Can we have a consumption stream such as the following?

$$\{c_0 > 0, c_1 > 0, \dots, c_{t-1} > 0, c_t = 0, c_{t+1} = 0, \dots, c_T = 0\}$$

No, because $\frac{\partial u}{\partial c_{t-1}} < \frac{\partial u}{\partial c_t} \to \infty$ (i.e. by delaying one unit consumption from t−1 to t, one can obtain almost infinite utility).
Also, $u'(c_t) = \lambda_t > 0$; hence binding; hence $f(k_t) = c_t + k_{t+1}$ (always)
From (1)

$$u'(c_t) = \lambda_t, \qquad u'(c_{t+1}) = \lambda_{t+1}$$

From (2)

$$-\lambda_t + \beta u'(c_{t+1})f'(k_{t+1}) = 0 \Rightarrow \lambda_t = \beta u'(c_{t+1})f'(k_{t+1})$$

By combining these two

$$\underbrace{u'(c_t)}_{\substack{\text{marginal}\\ \text{utility of}\\ \text{consuming}\\ \text{today}}} = \underbrace{\beta u'(c_{t+1})f'(k_{t+1})}_{\substack{\text{marginal}\\ \text{utility of}\\ \text{delaying that}\\ \text{consumption}}}, \qquad \text{Euler equation}$$

$$\Rightarrow \frac{u'(c_t)}{u'(c_{t+1})} = \beta f'(k_{t+1}) \Rightarrow \begin{cases} f'(k_{t+1}) > \dfrac{1}{\beta} \Rightarrow c_t < c_{t+1} \\ f'(k_{t+1}) = \dfrac{1}{\beta} \Rightarrow c_t = c_{t+1} \\ f'(k_{t+1}) < \dfrac{1}{\beta} \Rightarrow c_t > c_{t+1} \end{cases}$$

Therefore,

$$\begin{aligned}
&u'(c_t) = \beta u'(c_{t+1})f'(k_{t+1}) \Rightarrow f(k_t) = c_t + k_{t+1} \\
&f(k_t) = c_t + k_{t+1} \\
&k_{T+1} = 0
\end{aligned}$$

How about $T \to \infty$ case?

$$
\begin{aligned}
u &= c_0 + \beta c_1 + \beta^2 c_2 + \cdots + \beta^T c_T, && \text{u for different } \{c_t\} \text{ is comparable} \\
&= \sum_{t=0}^{\infty} \beta^t c_t\,, && \text{is not comparable } (\because u \to \infty) \\
&= \sum_{t=0}^{\infty} \beta^t v(c_t)\,, && \text{is bounded utility and also comparable}
\end{aligned}
$$

No-Ponzi-Game

$\tilde{k}$ $\quad \tilde{k}(1+r)$ $\quad \tilde{k}(1+r)^2$ $\quad \ldots \quad$ ∞

$$\lim_{t\to\infty}\left(k_t - \tilde{k}R^t\right) \geq 0$$

(i.e. have to payback someday in the future; No-Ponzi-Game restriction)
Now, optimization problem adds

$$\lim_{t\to\infty} k_t \frac{1}{R^t} \geq 0$$

Nothing different with finite cases, but “TRANSVERSALITY”

Central planner's problem

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t), \qquad \text{s.t. } c_t + k_{t+1} \le f(k_t)$$
$$c_t \ge 0$$
$$k_{t+1} \ge 0\ \forall t \text{ with given } k_0$$

Lagrange function

$$L = \sum_{t=0}^{T} \left(\beta^t u(c_t) + \lambda_t (f(k_t) - c_t - k_{t+1}) + \mu_t k_{t+1}\right)$$
$$\left.\frac{\partial L}{\partial c_t}\right|_{\text{optimum}} = \beta^t u'(c_t^*) - \lambda_t^* = 0, \qquad \forall t = 0,1,2,\dots,T$$
$$\left.\frac{\partial L}{\partial k_{t+1}}\right|_{\text{optimum}} = -\lambda_t^* + \mu_t^* + \lambda_{t+1}^* f'(k_{t+1}^*) = 0, \qquad \forall t = 0,1,2,\dots,T-1$$
$$\left.\frac{\partial L}{\partial k_{T+1}}\right|_{\text{optimum}} = -\lambda_T^* + \mu_T^* = 0 \Rightarrow \lambda_T^* = \mu_T^*, \qquad t = T$$
$$\Rightarrow \mu_T^* = \lambda_T^* = \beta^T u'(c_T^*) > 0, \qquad \therefore \mu_T^* k_{T+1}^* = 0 \Rightarrow k_{T+1}^* = 0$$
$$\Rightarrow \lambda_t^* = \beta^t u'(c_t^*) > 0, \qquad \therefore c_t^* \ne 0, \qquad \forall t$$
$$\Rightarrow k_t^* \ne 0, \qquad t = 1,2,\dots,T$$
$$\Rightarrow \mu_{t-1}^* = 0, \qquad t = 1,2,\dots,T$$

Hence 1)

$$\lambda_t^* = \lambda_{t+1}^* f'(k_{t+1}^*) \Rightarrow \beta^t u'(c_t^*) = \beta^{t+1} u'(c_{t+1}^*) f'(k_{t+1}^*)$$
$$\Rightarrow u'(c_t^*) = \beta u'(c_{t+1}^*) f'(k_{t+1}^*), \qquad \text{Euler equation}$$

2)

$$f(k_t^*) = c_t^* + k_{t+1}^*$$

3) (transversality condition↔non-Ponzi-game condition)

$$\lim_{T\to\infty} \mu_T^* k_{T+1}^* = 0$$

First Welfare Theorem: household

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad \text{s.t. } c_t + \overbrace{a_{t+1}}^{\text{asset}} = a_t \overbrace{R_t}^{\text{interest}} + \overbrace{w_t}^{\text{wage}}$$

(with non-Ponzi-game condition, a_0 given)

Farm

$$F(K_t, L_t) - R_t K_t - w_t L_t$$

After solving the household problem (EXERCISE)

$$\begin{aligned} u'(c_t) &= \beta R_{t+1} u'(c_{t+1}) \\ &= \beta f'(k_{t+1}) u'(c_{t+1}), \qquad \text{in the competitive market} \end{aligned}$$

This is exactly identical to the previous case 1).
Also,

$$F(K_t, L_t) = C_t + K_{t+1}, \qquad \Rightarrow f(k_t) = c_t + k_{t+1}$$

This matches to 2).
In addition,

$$\begin{aligned} &\lim_{T\to\infty} \mu_T a_{T+1} = 0, \qquad \text{and (market clearing) } a_t = k_t \\ \Rightarrow &\lim_{T\to\infty} \mu_T k_{T+1} = 0 \end{aligned}$$

This is identical to 3).
Hence, 1)–3) match with the social planner case; First Welfare Theorem (i.e. under the competitive market, decentralized economy is Pareto optimal and cannot be Pareto improved)
Equilibrium

$$\begin{aligned} &\{R_t^*, w_t^*, C_t^*, K_{t+1}^*\}_{t=0}^{\infty} \\ &C_t^* \;\&\; a_{t+1}^* \text{ solves } K_t^*, L_t^* \\ &F(K_t^*, L_t^*) = C_t^* + a_{t+1}^* \\ &L_t^* = 1 \text{ (normalized)} \\ &C_t^* = K_t^* \end{aligned}$$

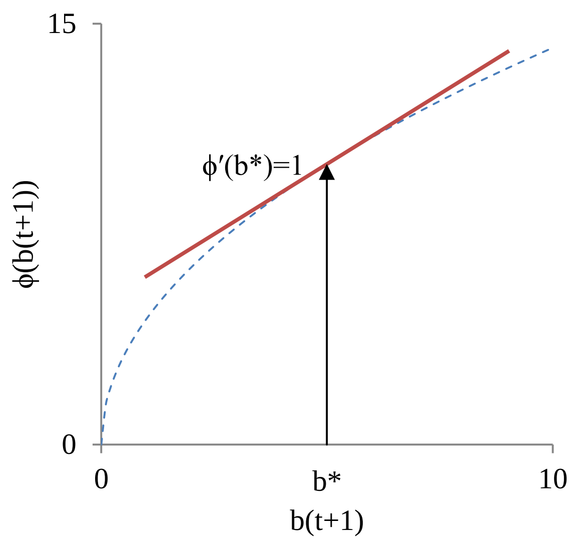
Quiz 2

(1) The problem

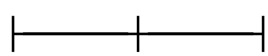
$$\max_{\{c_{t+1},b_{t+1}\}} c_{t+1} + \phi(b_{t+1}) \text{ s.t. } (w_t + b_t)(1+r) = c_{t+1} + b_{t+1}$$

FOC

$$\phi'(b_{t+1}) = 1$$



(2)



$$k_{t+1} = w_t + b = \Omega k_t + b, \qquad k^* = \Omega k^* + b = \frac{b}{1-\Omega}$$

<u>Continuous Time Representation</u>

Household utility

$$U_t = \int_0^\infty u(c_t)e^{nt}e^{-\rho t}dt, \qquad \text{(to bound the utility we need } n < \rho\text{)}$$

$$L_0 = 1, \qquad \text{(scale normalization)}$$

$$A_t = \text{asset of the household}$$

$$L_t w_t = \text{wage earnings}$$

$$\dot{A}_t = r_t A_t + w_t L_t - C_t, \qquad \cdots (*)$$

$$\text{Define } a_t = A_t/L_t, \qquad \text{(asset per capita)}$$

$$\Rightarrow \frac{\dot{a}_t}{a_t} = \frac{\dot{A}_t}{A_t} - n, \qquad \text{(take log and differentiate by t)}$$

Then

$$\dot{a}_t = \frac{\dot{A}_t}{A_t} a_t - n a_t, \qquad \Rightarrow \dot{a}_t + n a_t = \frac{\dot{A}_t}{A_t} a_t = \frac{\dot{A}_t}{A_t}\frac{A_t}{L_t} = \frac{\dot{A}_t}{L_t}$$

From (*)

$$\frac{\dot{A}_t}{L_t} = r_t a_t + w_t - c_t = \dot{a}_t + n a_t, \qquad \Rightarrow \dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad \cdots (1)$$

Define d_t as

$$d_t = \text{debt per individual in the household}$$

Suppose at ‘t’ the household has borrowed B_t

$$d_t = \frac{B_t}{L_t}, \qquad \Rightarrow \frac{\dot{d}_t}{d_t} = \frac{\dot{B}_t}{B_t} - n$$

And

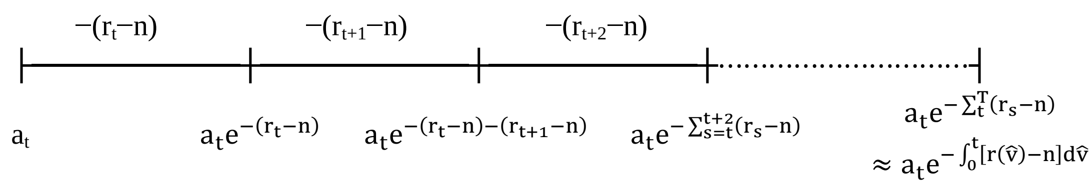
$$\dot{B}_t = rB_t, \qquad \left(\text{or } \frac{\dot{B}_t}{B_t} = r\right), \qquad \Rightarrow \frac{\dot{d}_t}{d_t} = r - n, \qquad \text{note: } -(r-n)?$$

In the discrete case

$$x \to x e^{r_1} e^{r_2} + \cdots = x e^{r_1 + r_2 + \cdots} = x e^{\Sigma r_i} \xrightarrow{\text{for the continuous case}} x e^{\int_0^t r(\hat{v}) d\hat{v}}$$

(2) Non-Ponzi-game condition

$$\lim_{t\to\infty} a_t e^{-\int_0^t [r(\hat{v}) - n] d\hat{v}} \geq 0$$



Hamiltonian (similar to Lagrangian, but different)

$$H = u(c) e^{-(\rho - n)t} + \eta_t [w_t + a_t(r_t - n) - c_t], \qquad \Rightarrow \frac{\partial H}{\partial c_t} = 0, \qquad \frac{\partial H}{\partial a_t} = -\dot{\eta}, \qquad \lim_{t\to\infty} \eta_t a_t = 0$$

Substitution

$$u(c_t) e^{-(\rho - n)t} \equiv v(t), \qquad (r_t - n)a_t + w_t - c_t \equiv g(a_t, c_t)$$

Then, again the problem is

$$\max \int_0^T v(c_t) dt, \qquad \text{subject to } \dot{a}_t = g(a_t, c_t), \qquad a_T e^{-\int_0^T [r(\hat{v}) - n] d\hat{v}} \equiv a_T e^{-\bar{r}(T)} \geq 0$$

$$\xrightarrow{\text{Lagrangian}} L = \int_0^T v(c_t) dt + \int_0^T [\mu_t \{g(a_t, c_t) - \dot{a}_t\}] dt + \gamma a_T e^{-\bar{r}(T)T}$$

Household

$$\max_{\{a_t, c_t\}} \int_0^\infty u(c_t) e^{-(\rho-n)t} dt, \qquad \text{subject to } \dot{a}_t = (r_t - n) a_t + w_t - c_t$$

$$\lim_{t\to\infty} a_t e^{-\int_0^t [r(\hat{v}) - n] d\hat{v}} \geq 0, \qquad \text{non– Ponzi game condition}$$

Hamiltonian

$$H(a_t, c_t) = \underbrace{u(c_t) e^{-(\rho-n)t}}_{\equiv v(c_t)} + \underbrace{\mu_t [w_t + a_t(r_t - n) - c_t]}_{\equiv g(a_t, c_t)}$$

$$\frac{\partial H}{\partial c_t} = 0, \qquad \frac{\partial H}{\partial a_t} = -\dot{\mu}_t, \qquad \lim_{t\to\infty} \mu_t a_t = 0, \qquad \text{transversality condition}$$

After substituting (with finite horizon)

$$\max_{\{a_t, c_t\}} \int_0^T v(c_t) dt, \qquad \text{s.t. } \dot{a}_t = g(a_t, c_t), \qquad a_0 \text{ given}, \qquad a_T e^{-\int_0^T [r(\hat{v}) - n] d\hat{v}} \equiv a_T e^{-\bar{r}(T)T} \geq 0$$

We have

- Control variable: c_t
- State variable: a_t

Lagrangian

$$L = \int_0^T v(c_t) dt + \int_0^T \mu_t \{g(a_t, c_t) - \dot{a}_t\} dt + \gamma a_T e^{-\bar{r}(T)T}$$

$$= \int_0^T \{v(c_t) + \mu_t g(a_t, c_t)\} dt - \underbrace{\int_0^T \mu_t \dot{a}_t dt}_{(*)} + \gamma a_T e^{-\bar{r}(T)T}$$

where $v(c_t) = \text{direct utility}$

$\mu_t = \text{indirect utility; discounted marginal utility of marginal budget}$

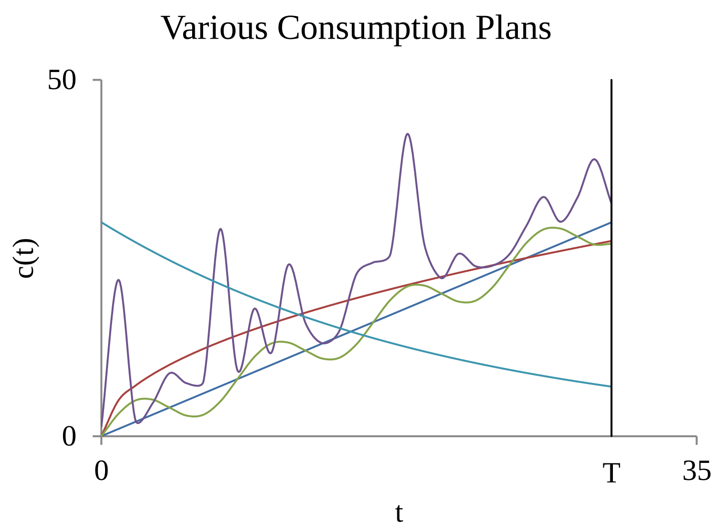
From μ_t, the household obtain the utility indirectly; it is the discounted increments of utility from future asset changes (one can consider it as discounted marginal utility of marginal budget).

For (*)

$$\frac{d}{dt} \mu(t) a(t) = \dot{\mu} a + \mu \dot{a} \Rightarrow d(\mu a) = (\dot{\mu} a + \mu \dot{a}) dt \Rightarrow \int_0^T d(\mu a) = \mu_T a_T - \mu_0 a_0 = \int_0^T \dot{\mu} a dt + \underbrace{\int_0^T \mu \dot{a} dt}_{(*)}$$

Hence

$$L = \int_0^T H(c_t, a_t) dt + \int_0^T \dot{\mu} a dt - \mu_T a_T + \mu_0 a_0 + \gamma a_T e^{-\bar{r}(T)T}$$



Optimal paths: many possibilities

Suppose $\bar{c}_t$ is the optimal path. Then other paths can be expressed by

$$c_t = \bar{c}_t + \varepsilon\phi_1(t), \qquad \text{where } \phi_1(\cdot) \text{ can be any arbitrary function of t}$$
$$\text{similarly } a_t = \bar{a}_t + \varepsilon\phi_2(t)$$

Then

$$L = \int_0^T \{H[c(\cdot,\varepsilon), a(\cdot,\varepsilon)] + \dot{\mu}_t a(\cdot,\varepsilon)\}dt - \mu_T a_T(\cdot,\varepsilon) + \mu_0 a_0 + \gamma e^{-\bar{r}(T)T} a_T(\cdot,\varepsilon)$$

Then what we have to choose here is just ε.

$$\frac{\partial L}{\partial \varepsilon} = \int_0^T \left(\frac{\partial H}{\partial \varepsilon} + \dot{\mu}_t \frac{\partial a}{\partial \varepsilon}\right) dt - \mu_T \frac{\partial a_T}{\partial \varepsilon} + \gamma e^{-\bar{r}(T)T} \frac{\partial a_T}{\partial \varepsilon}$$
$$= \int_0^T \left[\underbrace{\frac{\partial H}{\partial c_t}}_{(1)} \phi_1(t) + \underbrace{\left(\frac{\partial H}{\partial a_t} + \dot{\mu}_t\right)}_{(2)} \phi_2(t)\right] dt + \underbrace{\left(\gamma e^{-\bar{r}(T)T} - \mu_T\right)}_{(3)} \phi_2(T) = 0, \qquad \text{at } \varepsilon^* = \underset{\varepsilon}{\operatorname{argmax}}\, L(\varepsilon)$$

For above function to be equal to 0 for any arbitrary function $\phi_1(t)$, $\phi_2(t)$, (1)=(2)=(3) should be 0. Hence

$$\frac{\partial H}{\partial c_t} = 0, \qquad \frac{\partial H}{\partial a_t} = -\dot{\mu}_t, \qquad \mu_T = \gamma e^{-\bar{r}(T)T} \Rightarrow a_T\mu_T = \gamma a_T e^{-\bar{r}(T)T} \cdots (**)$$

These are identical to Hamiltonian solution; by Kuhn–Tucker complementary slackness condition, (**) equals to 0 at the optimal point. Hence $a_T\mu_T=0$, which is the transversality condition.

If we adopt CRRA utility $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$

$$\frac{\partial H}{\partial c_t} = 0 \Rightarrow \underbrace{u'(c_t)e^{-(\rho-n)t} - \mu_t = 0}_{\text{implicit Euler condition}} \Rightarrow c_t^{-\theta}e^{-(\rho-n)t} = \mu_t \cdots (\dagger)$$

$$\Rightarrow -\theta \ln c_t - (\rho - n)t = \ln \mu_t$$

$$\Rightarrow -\theta \frac{\dot{c}_t}{c_t} - (\rho - n) = \frac{\dot{\mu}_t}{\mu_t}$$

$$\frac{\partial H}{\partial a_t} = -\dot{\mu}_t \Rightarrow \mu_t(r_t - n) = -\dot{\mu}_t$$

$$\Rightarrow -\frac{\dot{\mu}_t}{\mu_t} = r_t - n \cdots (\ddagger)$$

$$\therefore r_t - n = \theta \frac{\dot{c}_t}{c_t} + (\rho - n) \Rightarrow \frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$$

So if $r_t=\rho$, then $\dot{c}_t/c_t=0$ (no consumption growth), which implies consumption smoothing.
From ($\ddagger$)

$$\int_0^t \frac{d\mu}{\mu} = -\int_0^t [r(\hat{v}) - n]d\hat{v} = \ln \frac{\mu(t)}{\mu(0)} \Rightarrow \frac{\mu(t)}{\mu(0)} = e^{-\int_0^t [r(\hat{v})-n]d\hat{v}} \Rightarrow \mu_t = \mu_0 e^{-\int_0^t [r(\hat{v})-n]d\hat{v}}$$

And from ($\dagger$)

$$\mu_0 = c_0^{-\theta}e^{-0} = c_0^{-\theta} \Rightarrow \mu_t = c_0^{-\theta}e^{-\int_0^t [r(\hat{v})-n]d\hat{v}} \Rightarrow a_t\mu_t = c_0^{-\theta}a_t e^{-\int_0^t [r(\hat{v})-n]d\hat{v}}$$

$$\Rightarrow \lim_{t\to\infty} a_t\mu_t = \lim_{t\to\infty} c_0^{-\theta}a_t e^{-\int_0^t [r(\hat{v})-n]d\hat{v}} = c_0^{-\theta} \underbrace{\lim_{t\to\infty} a_t e^{-\int_0^t [r(\hat{v})-n]d\hat{v}}}_{=0 \text{ by nPg condition}} = 0, \qquad \text{i.e. transversality}$$

Remember that

- nPg condition is the condition imposed to the problem
- The transversality condition is what is derived from the optimization

Decentralized Economy: Household

$$\max_{\{a_t, c_t\}} \int_0^\infty u(c_t)e^{-(\rho-n)t}dt$$
$$\text{s.t. } \dot{a}_t = (r_t - n)a_t + w_t - c_t$$
$$+\text{non–Ponzi game condition}$$

Hamiltonian

$$\Rightarrow H = u(c_t)e^{-(\rho-n)t} + \mu_t[w_t + (r_t - n)a_t - c_t]$$
$$\Rightarrow \frac{\partial H}{\partial c_t} = 0, \qquad \frac{\partial H}{\partial a_t} = -\dot{\mu}_t, \qquad \lim_{t\to\infty} a_t\mu_t = 0$$

By solving this,

$$\Rightarrow \frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$$

Production function

$$Y_t = F(K_t, L_t \cdot A_t), \qquad A_t = e^{xt}$$
$$\hat{L}_t = \text{effective labor} = L_t A_t$$
$$\Rightarrow Y_t = F\left(K_t, \hat{L}_t\right) = A_t L_t \cdot F\left(\frac{K_t}{A_t L_t}, 1\right)$$
$$\frac{\partial Y_t}{\partial L_t} = A_t F\left(\frac{K_t}{A_t L_t}, 1\right) + A_t L_t \frac{\partial}{\partial L_t} F\left(\frac{K_t}{A_t L_t}, 1\right)$$
$$= A_t f\left(\hat{k}_t\right) + A_t L_t f'\left(\hat{k}_t\right)\frac{\partial \hat{k}_t}{\partial L_t}$$

Since $\frac{\partial \hat{k}_t}{\partial L_t} = -\frac{K_t}{A_t L_t^2}$

$$\frac{\partial Y_t}{\partial L_t} = A_t\left[f\left(\hat{k}_t\right) - f'\left(\hat{k}_t\right)\hat{k}_t\right] = e^{xt}\left[f\left(\hat{k}_t\right) - f'\left(\hat{k}_t\right)\hat{k}_t\right] = w_t, \qquad \cdots (4)$$

And

$$\frac{\partial Y_t}{\partial K_t} = A_t L_t f'\left(\hat{k}_t\right)\frac{\partial \hat{k}_t}{\partial K_t} = A_t L_t f'\left(\hat{k}_t\right)\frac{1}{A_t L_t} = f'\left(\hat{k}_t\right) = r_t, \qquad \cdots (3)$$

in the competitive economy. Why? Because firms are maximizing their profits

$$\pi = F(K, L) - wL - rK \Rightarrow \frac{\partial \pi}{\partial L} = F_L - w = 0 \Rightarrow F_L = w, \qquad \frac{\partial \pi}{\partial K} = F_K - r = 0 \Rightarrow F_K = r$$

We know that $a_t = k_t$, so

$$\hat{k}_t = \frac{K_t}{A_t L_t} = \frac{k_t}{e^{xt}} = k_t e^{-xt} \Rightarrow k_t = \hat{k}_t e^{xt}, \qquad \cdots (2)$$

$$\hat{c}_t = \frac{C_t}{A_t L_t} = \frac{c_t}{e^{xt}} = c_t e^{-xt}, \qquad \cdots (5)$$

$$\begin{aligned} \Rightarrow \dot{k}_t &= x e^{xt} \hat{k}_t + e^{xt} \dot{\hat{k}}_t \\ &= e^{xt} \left(x\hat{k}_t + \dot{\hat{k}}_t \right), \qquad \cdots (1) \end{aligned}$$

Then by replacing this equation

$$\underbrace{\dot{k}_t}_{(1)} = \left(\underbrace{r_t}_{(3)} - n \right) \underbrace{k_t}_{(2)} + \underbrace{w_t}_{(4)} - \underbrace{c_t}_{(5)}$$

we can obtain

$$\dot{\hat{k}}_t = f(\hat{k}_t) - (x+n)\hat{k}_t - \hat{c}_t$$

Note that

$$\hat{c}_t = c_t e^{-xt} \Rightarrow \frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{\dot{c}_t}{c_t} - x, \qquad \Rightarrow \frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{r_t - \rho}{\theta} - x, \qquad \text{Law of Motion 1}$$

$$\dot{\hat{k}}_t = f(\hat{k}_t) - (x+n)\hat{k}_t - \hat{c}_t, \qquad \text{Law of Motion 2}$$

From the second Hamiltonian partial derivative

$$\frac{\partial H}{\partial a_t} = -\dot{\mu}_t \Rightarrow \frac{\dot{\mu}_t}{\mu_t} = -(r_t - n) \Rightarrow \mu_t = \mu_0 e^{-\int_0^t [r(\hat{v}) - n] d\hat{v}}$$

Then

$$\begin{aligned} \lim_{t\to\infty} \mu_0 a_t e^{-\int_0^t [r(\hat{v}) - n] d\hat{v}} &= \lim_{t\to\infty} \mu_0 k_t e^{-\int_0^t [r(\hat{v}) - n] d\hat{v}} = \lim_{t\to\infty} \mu_0 \hat{k}_t e^{xt} e^{-\int_0^t [r(\hat{v}) - n] d\hat{v}} \\ &= \lim_{t\to\infty} \mu_0 \hat{k}_t e^{-\int_0^t [r(\hat{v}) - n - x] d\hat{v}} = 0 \end{aligned}$$

$$\Rightarrow \lim_{t\to\infty} \hat{k}_t e^{-\int_0^t [r(\hat{v}) - n - x] d\hat{v}} = 0$$

The growth rate at the steady state $\gamma_{\hat{k}}^* = 0$ and $\gamma_{\hat{c}}^* = 0$

Proof Suppose $\gamma_{\hat{k}}^* > 0 \Rightarrow \hat{k} \to \infty \Rightarrow f'(\hat{k}) \to 0 \Rightarrow r \to 0$. Then

$$\Rightarrow \frac{\dot{\hat{c}}}{\hat{c}} < 0, \qquad \text{so } \hat{c} \to 0 \text{ but } u'(\hat{c}) \to \infty, \qquad \text{hence not equilibrium}$$

Suppose $\gamma_{\hat{k}}^* < 0 \Rightarrow \hat{k} \to 0 \Rightarrow f'(\hat{k}) \to \infty \Rightarrow r \to \infty$

$$\Rightarrow \frac{\dot{\hat{c}}}{\hat{c}} > 0, \qquad \text{so } \hat{c} \to \infty, \qquad \text{to consume infinitely from 0 asset,} \qquad \text{one should borrow infinitely}$$

But, the infinite borrowing is restricted by nPg condition, hence the only possible case is $\gamma_{\hat{k}}^* = \gamma_{\hat{c}}^* = 0$

Last time we discussed about

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{1}{\theta}\left[f'\left(\hat{k}_t\right) - \rho - \theta x\right], \qquad \cdots (1)$$

$$\dot{\hat{k}}_t = f\left(\hat{k}_t\right) - \hat{c}_t - (n+x)\hat{k}_t, \qquad \cdots (2)$$

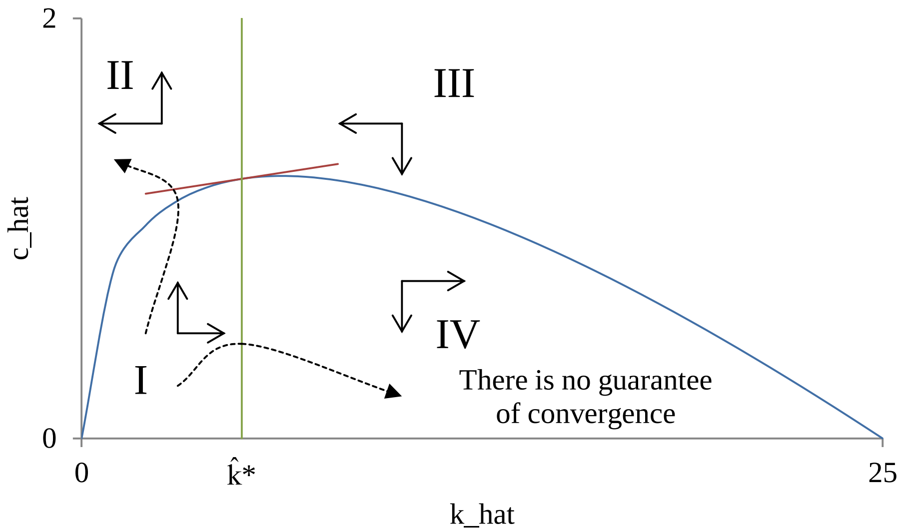
$$\lim_{t\to\infty} \hat{k}_t e^{-\int_0^t [f'(\hat{k}_{\hat{v}}) - n - x] d\hat{v}} = 0, \qquad \cdots (3)$$

Steady state condition

$$\gamma_{\hat{c}}^* = \gamma_{\hat{k}}^* = 0 \Leftrightarrow \begin{array}{l} \hat{k}_t \to \hat{k}^* \\ \hat{c}_t \to \hat{c}^* \end{array}$$

For the steady state, one should solve

$$(1) \to f'\left(\hat{k}_t^*\right) = \rho + \theta x$$
$$(2) \to \hat{c}_t = f\left(\hat{k}_t\right) - (n+x)\hat{k}_t$$

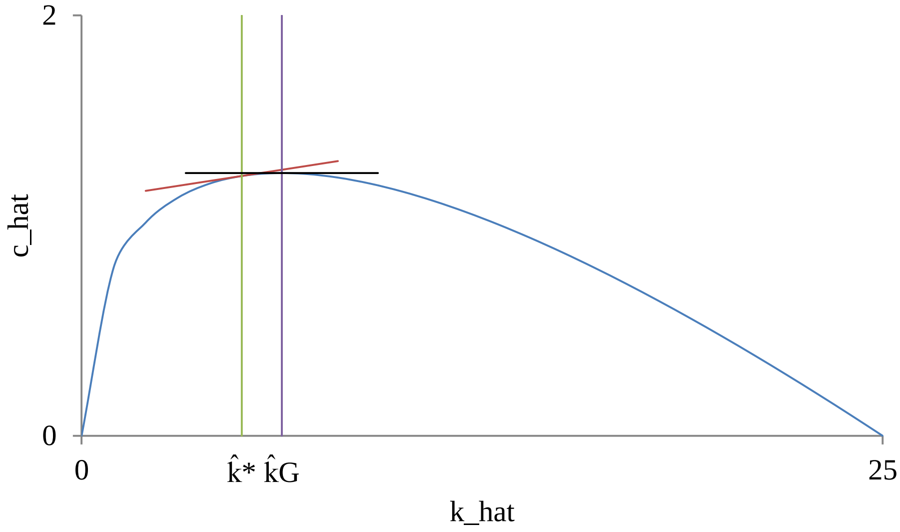


Phase Diagram

ex. If we are in I, then from (1) $f'\left(\hat{k}_t\right) > \rho + \theta x \Rightarrow (1) > 0 \Rightarrow \gamma_{\hat{c}} > 0$

And from (2) $\hat{c}_t < f\left(\hat{k}_t\right) - (n+x)\hat{k}_t \Rightarrow (2) > 0 \Rightarrow r_{\hat{k}} > 0$

(Exercise: Check all the arrows for the quardrants I–IV)



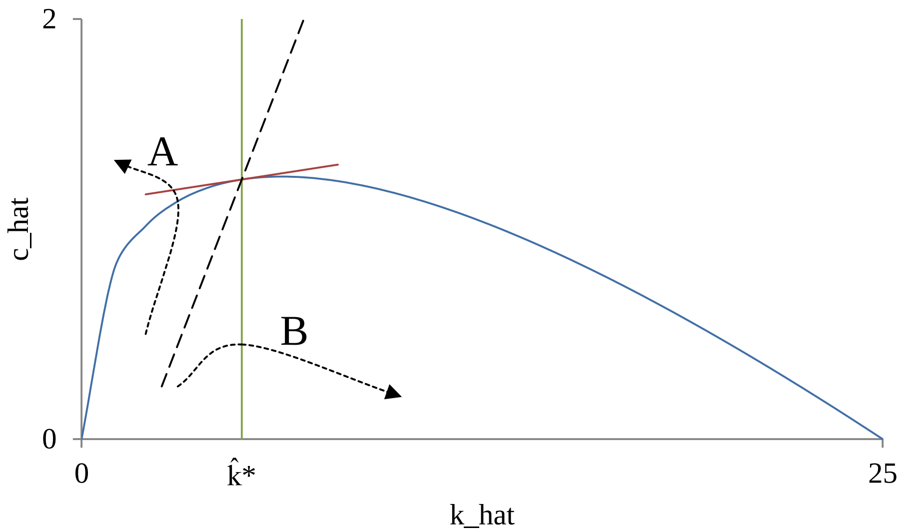
Here $\hat{k}_G$ maximizes $\hat{c}$ from (2).

$$\frac{\partial \hat{c}}{\partial \hat{k}} = f'(\hat{k}_G) - (n + x) = 0 \Rightarrow f'(\hat{k}_G) = n + x$$

In order to satisfy (3)

$$\forall t, \qquad f'(\hat{k}_t) - n - x > 0, \qquad \Rightarrow f'(\hat{k}_t) > n + x = f'(\hat{k}_G)$$
$$\Rightarrow f'(\hat{k}_*) > f'(\hat{k}_G), \qquad \Rightarrow \hat{k}_* < \hat{k}_G$$

According to the phase diagram

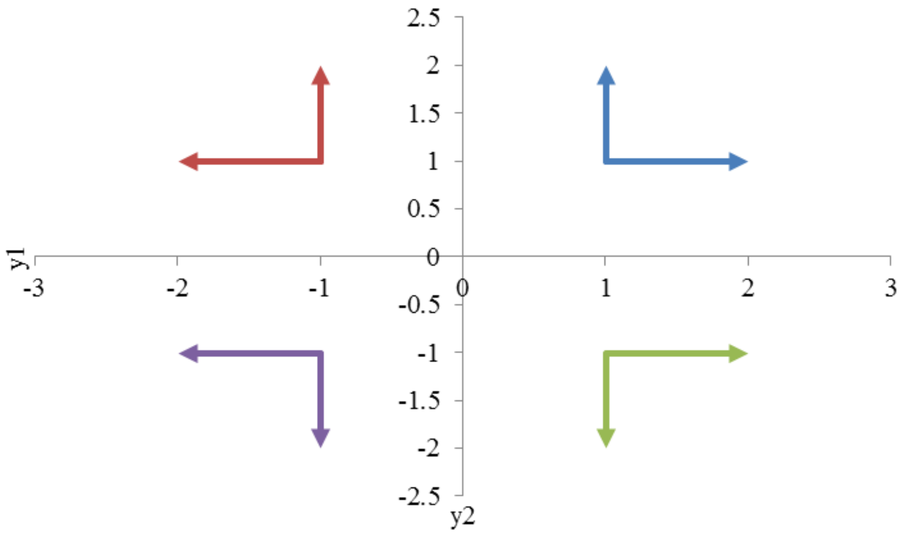


If we are in A, then $\hat{k} \to 0 \Rightarrow f'(\hat{k}) \to \infty$ by INADA $\Rightarrow \dot{\hat{c}}/\hat{c} \to \infty$. This violates transversality condition.
If we are in B, then $\hat{c} \to 0 \Rightarrow u'(\hat{c}) \to \infty$. This does not make sense since not maximizing utility by FOC.

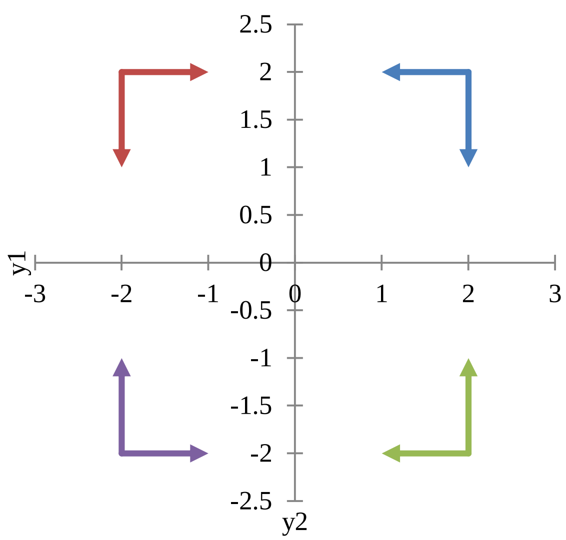
So the only one possible path should go to the equilibrium directly (Saddle Point Theorem).

Think about easy cases.

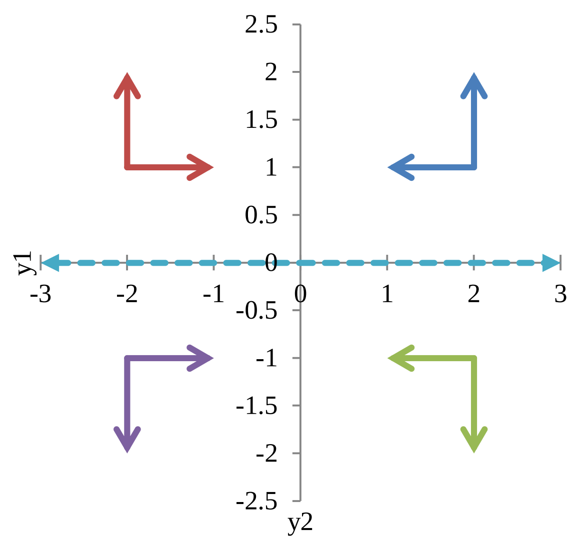
$$\dot{y}_1 = a_1 y_1, \qquad \dot{y}_2 = a_2 y_2$$



$a_1, a_2 > 0$



$a_1, a_2 < 0$



$a_1 < 0$, $a_2 > 0$ (Saddle Point)

Complementary slackness condition

$$\max_{\{x,y\}} u(x,y) \text{ subject to } p_x x + p_y y = M$$
$$L = u(x,y) + \lambda\left(M - p_x x - p_y y\right)$$

- Where λ is the marginal utility of budget
- If λ=0, then there is no reason to consume all (i.e. already satiated) and hence $M>p_x x+p_y y$
- If λ>0, then one should at least consume all the budgets now (i.e. not satiated yet) and hence $M=p_x x+p_y y$

Simultaneous differential equation

$$\dot{y}_1(t) = a_{11}y_1(t)$$
$$\dot{y}_2(t) = a_{22}y_2(t)$$
$$\Rightarrow \begin{pmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\end{pmatrix} = \begin{pmatrix}a_{11} & 0\\ 0 & a_{22}\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}$$

In order for a saddle point to exist, the condition (a_{11}>0 and a_{22}<0) or (a_{11}<0 and a_{22}>0) should be satisfied. i.e. $a_{11}a_{22}<0$. However, what one has here is

$$\dot{\hat{k}}_t = f(\hat{k}_t) - \hat{c}_t - n\hat{k}_t \xRightarrow{\text{assume } x=0} A\hat{k}_t^{\alpha} - \hat{c}_t - n\hat{k}_t, \qquad \text{i.e.}\, A_t = A$$
$$\Rightarrow \frac{\dot{\hat{k}}_t}{\hat{k}_t} = A\hat{k}_t^{-(1-\alpha)} - \frac{\hat{c}_t}{\hat{k}_t} - n$$
$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{1}{\theta}\left[f'(\hat{k}_t) - \hat{c}_t - \theta x\right] \xRightarrow{\text{with } x=0} \frac{1}{\theta}\left(\alpha A\hat{k}_t^{\alpha-1} - \rho\right)$$

Apply these two rules: $\dot{x}/x = \frac{d}{dt}\log x$ and $x = e^{\log x}$

$$\frac{d}{dt}\log\hat{k}_t = Ae^{-(1-\alpha)\log\hat{k}_t} - e^{\log(\hat{c}_t/\hat{k}_t)} - n \equiv f^1(\hat{k}_t,\hat{c}_t)$$
$$\frac{d}{dt}\log\hat{c}_t = \frac{1}{\theta}\left(\alpha Ae^{-(1-\alpha)\log\hat{k}_t} - \rho\right) \equiv f^2(\hat{k}_t,\hat{c}_t)$$

Log-linearization

$$f(x,y) = f(x^*,y^*) + f_x(x^*,y^*)(x-x^*) + f_y(x^*,y^*)(y-y^*)$$
$$+\frac{1}{2!}\left[f_{xx}(x^*,y^*)(x-x^*)^2 + 2f_{xy}(x^*,y^*)(x-x^*)(y-y^*) + f_{yy}(y-y^*)^2\right] + O(\|\mathbf{x}\|^3)$$

Note that $f^1(x^*,y^*) = f^2(x^*,y^*) = 0$, i.e. in the equilibrium, there is no change.

Differential equations

$$\frac{d}{dt}\log\hat{k}_t = Ae^{-(1-\alpha)\log\hat{k}_t} - e^{\log(\hat{c}_t/\hat{k}_t)} - n \equiv f^1(\hat{k}_t,\hat{c}_t)$$
$$\frac{d}{dt}\log\hat{c}_t = \frac{1}{\theta}\left(\alpha Ae^{-(1-\alpha)\log\hat{k}_t} - \rho\right) \equiv f^2(\hat{k}_t,\hat{c}_t)$$

At the equilibrium

$$Ae^{-(1-\alpha)\log\hat{k}^*} - e^{\log(\hat{c}^*/\hat{k}^*)} = n$$
$$\alpha Ae^{-(1-\alpha)\log\hat{k}^*} = \rho$$

Approximation

$$f^1_{\log\hat{k}_t}(\hat{k}^*,\hat{c}^*) = -(1-\alpha)Ae^{-(1-\alpha)\log\hat{k}^*} + e^{\log(\hat{c}^*/\hat{k}^*)} = \rho - n$$
$$f^1_{\log\hat{c}_t}(\hat{k}^*,\hat{c}^*) = -e^{\log(\hat{c}^*/\hat{k}^*)} = n - \frac{\rho}{\alpha}$$
$$f^2_{\log\hat{k}_t}(\hat{k}^*,\hat{c}^*) = -\frac{1}{\theta}(1-\alpha)\alpha Ae^{-(1-\alpha)\log\hat{k}_t} = -\frac{1-\alpha}{\theta}\rho$$
$$f^2_{\log\hat{c}_t}(\hat{k}^*,\hat{c}^*) = 0$$

Hence

$$\begin{pmatrix}\dfrac{d\log\hat{k}_t}{dt}\\ \dfrac{d\log\hat{c}_t}{dt}\end{pmatrix} = \begin{pmatrix} f^1_{\log\hat{k}_t}(\hat{k}^*,\hat{c}^*) & f^1_{\log\hat{c}_t}(\hat{k}^*,\hat{c}^*)\\ f^2_{\log\hat{k}_t}(\hat{k}^*,\hat{c}^*) & f^2_{\log\hat{c}_t}(\hat{k}^*,\hat{c}^*)\end{pmatrix}\begin{pmatrix}\log\dfrac{\hat{k}_t}{\hat{k}^*}\\ \log\dfrac{\hat{c}_t}{\hat{c}^*}\end{pmatrix} = \begin{pmatrix}\rho - n & n-\rho/\alpha\\ -\dfrac{1-\alpha}{\theta}\rho & 0\end{pmatrix}\begin{pmatrix}\log\dfrac{\hat{k}_t}{\hat{k}^*}\\ \log\dfrac{\hat{c}_t}{\hat{c}^*}\end{pmatrix}$$

Diagonalization

$$\mathbf{v} = \begin{pmatrix}v_1\\ v_2\end{pmatrix},\qquad \mathbf{Av} = \alpha\mathbf{v},\qquad (\mathbf{A} - \alpha\mathbf{I}_2)\mathbf{v} = \mathbf{0}$$

- For the matrix **A**, nontrivial **v** that satisfies this relation is an eigenvector (or characteristic vector) and α is an eigenvalue (characteristic value)
- In order for **v** to be not trivial (i.e. $\mathbf{v}\neq\mathbf{0}$), $(\mathbf{A}-\alpha\mathbf{I}_2)$ should have no inverse
- Equivalently, $\det[\mathbf{A}-\alpha\mathbf{I}_2]$ should be 0

$$\mathbf{A} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},\qquad \det[\mathbf{A}-\alpha\mathbf{I}_2] = (a_{11}-\alpha)(a_{22}-\alpha) - a_{12}a_{21} = 0$$

- Since this is quadratic, α can have multiple solutions

$$\alpha = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}}{2},\qquad \text{say } \alpha_1,\alpha_2$$

Then, by using α_1

$$(a_{11}-\alpha_1)v_1 + a_{12}v_2 = 0,\qquad \Rightarrow v_1 = -a_{12}/(a_{11}-\alpha_1)\times v_2$$
$$a_{21}v_1 + (a_{22}-\alpha_1)v_2 = 0,\qquad \Rightarrow v_1 = -(a_{22}-\alpha_1)/a_{21}\times v_2$$

This makes an identity because

$$\frac{a_{12}}{a_{11}-\alpha_1}=\frac{a_{22}-\alpha_1}{a_{21}} \Leftrightarrow (a_{11}-\alpha_1)(a_{22}-\alpha_1)=a_{12}a_{21} \Leftrightarrow \underbrace{(a_{11}-\alpha_1)(a_{22}-\alpha_1)-a_{12}a_{21}}_{\det[\mathbf{A}-\alpha_1\mathbf{I}_2]}=0$$

Normalize $\mathbf{v}$ in order to uniquely determine v_1 and v_2

$$\hat{\mathbf{v}}=\mathbf{v}/\|\mathbf{v}\|, \qquad \text{where } \mathbf{v}=\begin{pmatrix}-\frac{a_{22}-\alpha_1}{a_{21}}v_2 & v_2\end{pmatrix}^\top$$

$$\|\mathbf{v}\|=\sqrt{\mathbf{v}^\top\mathbf{v}}$$

$$\mathbf{v}^\top\mathbf{v}=\left(\frac{a_{22}-\alpha_1}{a_{21}}\right)^2 v_2^2+v_2^2=\left[\left(\frac{a_{22}-\alpha_1}{a_{21}}\right)^2+1\right]v_2^2$$

$$\|\mathbf{v}\|=\sqrt{\left(\frac{a_{22}-\alpha_1}{a_{21}}\right)^2+1}v_2$$

$$\hat{\mathbf{v}}=\begin{pmatrix}-\frac{\frac{a_{22}-\alpha_1}{a_{21}}}{\sqrt{\left(\frac{a_{22}-\alpha_1}{a_{21}}\right)^2+1}} & \frac{1}{\sqrt{\left(\frac{a_{22}-\alpha_1}{a_{21}}\right)^2+1}}\end{pmatrix}^\top$$

Since $\hat{\mathbf{v}}$ is a function of α, one can obtain different $\hat{\mathbf{v}}$ if different α is used; say $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ here.

Decomposition

$$\mathbf{A}\mathbf{v}_1=\alpha_1\mathbf{v}_1, \qquad \mathbf{A}\mathbf{v}_2=\alpha\mathbf{v}_2 \Rightarrow \mathbf{A}(\mathbf{v}_1 \quad \mathbf{v}_2)=(\mathbf{A}\mathbf{v}_1 \quad \mathbf{A}\mathbf{v}_2)=(\alpha_1\mathbf{v}_1 \quad \alpha_2\mathbf{v}_2)=(\mathbf{v}_1 \quad \mathbf{v}_2)\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}$$

$$\Rightarrow \mathbf{A}\mathbf{V}=\mathbf{V}\mathbf{D}, \qquad \text{where } \mathbf{D}=\operatorname{diag}[\alpha_1,\alpha_2] \Rightarrow \mathbf{A}=\mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

By the property of characteristic decomposition, det[**A**]=det[**D**].
If det[**A**]=det[**D**]<0, we can say there is a saddle point.

$$\det[\mathbf{A}]=-\left(\frac{\rho}{\alpha}-n\right)\underbrace{\left(\frac{1-\alpha}{\theta}\rho\right)}_{>0}$$

In order to satisfy the transversality condition

$$\lim_{t\to\infty}\hat{k}_t e^{-\int_0^t[f'(\hat{k}_{\hat{v}})-n-x]d\hat{v}}=0 \Rightarrow f'(\hat{k}_t)-n-x=\rho+\theta x-n-x>0 \Rightarrow \rho+\theta x>n+x$$

So

$$\frac{\rho+\theta x}{\alpha}>\rho+\theta x>n+x \Rightarrow \frac{\rho+\theta x}{\alpha}>n+x \xRightarrow{\text{assume } x=0} \frac{\rho}{\alpha}>n$$

Hence

$$\det[\mathbf{A}]=-(\text{positive})\times(\text{positive})=(\text{negative})<0$$

Hence one can say that there exists a saddle point in these simultaneous differential equations.

Incorporating Tax

$$\max_{\{a_t,c_t\}} \int_0^\infty u(c_t)e^{-(\rho-n)t}dt \text{ subject to } \dot{a}_t = (r_t - n)a_t + w_t - c_t \text{ and nPg}$$

Add tax to wage income

$$\dot{a}_t = (r_t - n)a_t + (1-\tau_w)w_t - c_t, \qquad \cdots (*)$$

Add tax to capital income

$$\dot{a}_t = [(1-\tau_a)r_t - n]a_t + (1-\tau_w)w_t - c_t, \qquad \cdots (\dagger)$$

With CRRA $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, the solution for Hamiltonian is

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$$

From (*)

$$\dot{a}_t = (r_t - n)a_t + w_t - \underbrace{\tau_w w_t}_{g_t} - c_t \xRightarrow{\text{market clearing}} \dot{k}_t = (r_t - n)k_t + w_t - g_t - c_t$$

Instead, from (†)

$$\dot{a}_t = (r_t - n)a_t + w_t - \underbrace{(\tau_a r_t a_t + \tau_w w_t)}_{g_t} - c_t \Rightarrow \dot{k}_t = (r_t - n)k_t + w_t - g_t - c_t$$

And

$$\frac{\dot{c}_t}{c_t} = \frac{(1-\tau_a)r_t - \rho}{\theta}$$

Instead, suppose this dynamics

$$\dot{a}_t = w_t(1-\tau_w) + r_t\underbrace{(b_t + v_t)}_{a_t} - c_t - n\underbrace{(b_t + v_t)}_{a_t}$$

$$\Rightarrow \dot{a}_t - (r_t - n)a_t = w_t(1-\tau_w) - c_t$$

$$\Rightarrow e^{-(r_t-n)t}[\dot{a}_t - (r_t - n)a_t] = e^{-(r_t-n)t}[w_t(1-\tau_w) - c_t]$$

$$\Rightarrow \int_0^\infty d\left[a_t e^{-(r_t-n)t}\right] = \int_0^\infty (1-\tau_w)w_t e^{-(r_t-n)t}dt - \int_0^\infty c_t e^{-(r_t-n)t}dt$$

$$0 - a_0 =$$

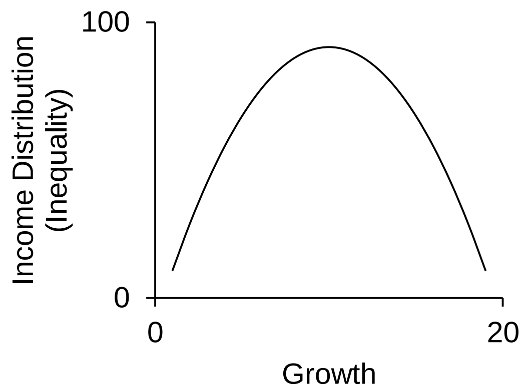
$$\Rightarrow \int_0^\infty c_t e^{-(r_t-n)t}dt = a_0 + \int_0^\infty w_t e^{-(r_t-n)t}dt + \int_0^\infty w_t \tau_w e^{-(r_t-n)t}dt$$

$$= a_0 + \int_0^\infty w_t e^{-(r_t-n)t}dt + \int_0^\infty g_t e^{-(r_t-n)t}dt, \qquad \text{NPG for government}$$

Recardian equivalence (for government)
"Imposing tax or borrowing" is not a matter.

$$\underbrace{\int_0^\infty g_t e^{-(r_t-n)t} dt}_{\substack{\text{PV of government} \\ \text{expenditure}}} = \underbrace{\int_0^\infty \tau_w w_t e^{-(r_t-n)t} dt}_{\text{PV of government tax}}$$

Kuznets Curve



Alesina–Rodrik Model

Alberto Alesina and Dani Rodrik (1994), Distributive Politics and Economic Growth, *Quarterly Journal of Economics* 109 (2), 465–490

Production function

$$\underbrace{y}_{\substack{\text{aggregate}\\\text{production}}} = \underbrace{A}_{\text{technology}} \underbrace{k^\alpha}_{\substack{\text{aggregate}\\\text{capital}}} \underbrace{g^{1-\alpha}}_{\substack{\text{aggregate}\\\text{government}\\\text{expenditure}}} \underbrace{l^{1-\alpha}}_{\substack{\text{aggregate}\\\text{labor}}}, \qquad \text{where } g = \tau k, \qquad \tau \text{ is a tax on capital income}$$

Wage and capital return

$$r = \frac{\partial y}{\partial k} = \alpha A k^{\alpha-1} g^{1-\alpha} l^{1-\alpha} = \alpha A \tau^{1-\alpha} l^{1-\alpha} \xRightarrow{\text{assume inelastic labor supply } l=1} \alpha A \tau^{1-\alpha} \equiv r(\tau)$$
$$w = \frac{\partial y}{\partial l} = (1-\alpha) A k^\alpha g^{1-\alpha} l^{-\alpha} = (1-\alpha) A k \tau^{1-\alpha} l^{-\alpha} \xRightarrow{\text{inelastic labor}} (1-\alpha) A k \tau^{1-\alpha} \equiv w(\tau) k$$

Relative factor endowment

$$\sigma^i = \frac{l^i/l}{k^i/k} = \frac{l^i}{k^i} k \to \begin{cases} \infty, & \text{if labor rich capital poor} \\ 0, & \text{if capital rich labor poor} \end{cases} \Rightarrow \sigma^i k^i = l^i k$$

Earnings of each individual

$$\text{capital income } y^k = [r(\tau) - \tau] k, \qquad \text{labor income } y^l = w(\tau) k l = w(\tau) k$$
$$y^i = y^l + y^k = w(\tau) k l^i + [r(\tau) - \tau] k^i = w(\tau) \sigma^i k^i + [r(\tau) - \tau] k^i$$

Individual optimization

$$\max_{\{c^i(t), k^i(t)\}} \int_0^\infty \log c^i \, e^{-\rho t} dt \;\; \text{subject to } \dot{k}^i = w(\tau) \sigma^i k^i + [r(\tau) - \tau] k^i - c^i$$

Hamiltonian

$$
\begin{aligned}
H &= \log c^i e^{-\rho t} + \lambda\{w(\tau)\sigma^i k^i + [r(\tau)-\tau]k^i - c^i\} \\
\frac{\partial H}{\partial c_i} &= \frac{1}{c^i} e^{-\rho t} - \lambda = 0 \Rightarrow \lambda = \frac{e^{-\rho t}}{c^i} \\
\frac{\partial H}{\partial k^i} &= \lambda\left\{w(\tau)\sigma^i - w(\tau)k^i \frac{l^i k}{(k^i)^2} + [r(\tau)-\tau]\right\} = \lambda[r(\tau)-\tau] = -\dot{\lambda}
\end{aligned}
$$

$$
\lim_{T\to\infty} \lambda_T \geq 0, \qquad \lim_{T\to\infty} \lambda_T k_T^i = 0
$$

So

$$
\frac{e^{-\rho t}}{c^i}[r(\tau)-\tau] = -\left(\frac{-\rho e^{-\rho t}}{c^i} - \frac{e^{-\rho t}}{(c^i)^2}\dot{c}^i\right) = \frac{e^{-\rho t}}{(c^i)^2}\dot{c}^i + \frac{\rho e^{-\rho t}}{c^i} \Rightarrow r(\tau)-\tau = \frac{\dot{c}^i}{c^i} + \rho
$$

$$
\Rightarrow \frac{\dot{c}^i}{c^i} = r(\tau) - \tau - \rho
$$

Homework: Show $\frac{\dot{k}^i}{k^i} = r(\tau) - \tau - \rho$

Proof: From the consumption growth,

$$
\frac{dc^i}{c^i} = (r(\tau)-\tau-\rho)dt \Rightarrow \ln c^i = (r(\tau)-\tau-\rho)t + (\text{constant}) \Rightarrow c^i(t) = c^i(0)e^{(r(\tau)-\tau-\rho)t}
$$

Then the capital growth is

$$
\dot{k}^i = w(\tau)\sigma^i k^i + [r(\tau)-\tau]k^i - c^i = w(\tau)\sigma^i k^i + [r(\tau)-\tau]k^i - c^i(0)e^{(r(\tau)-\tau-\rho)t}
$$

Since the labor supply is inelastic, i.e. l=1 (see Barro and Sala-i-Martin (2004, p.206) as well)

$$
y = Ak^\alpha g^{1-\alpha} l^{1-\alpha} = Ak^\alpha g^{1-\alpha} \Rightarrow w = \frac{\partial y}{\partial l} = 0
$$

Then

$$
\begin{aligned}
\dot{k}^i &= [r(\tau)-\tau]k^i - c^i(0)e^{(r(\tau)-\tau-\rho)t} \\
\Rightarrow e^{-(r(\tau)-\tau)t}\left(\dot{k}^i - (r(\tau)-\tau)k^i\right) &= -c^i(0)e^{-\rho t} \\
\Rightarrow \int d\left(k^i e^{-(r(\tau)-\tau)t}\right) &= -c^i(0)\int e^{-\rho t}dt + (\text{constant}) \\
k^i(t) &= \frac{1}{\rho}c^i(0)e^{(r(\tau)-\tau-\rho)t} + (\text{constant})e^{(r(\tau)-\tau)t} \\
&= \frac{1}{\rho}c^i(t) + (\text{constant})e^{(r(\tau)-\tau)t}
\end{aligned}
$$

According to TV condition,

$$\lim_{T\to\infty} \lambda(T)k^i(T) = \lim_{T\to\infty} \frac{e^{-\rho T}}{c^i(T)}\left(\frac{c^i(T)}{\rho} + (\text{constant})e^{(r(\tau)-\tau)T}\right)$$
$$= \lim_{T\to\infty}\left(\frac{e^{-\rho T}}{\rho} + (\text{constant})e^{(r(\tau)-\tau-\rho)T}\right) = 0$$

The sequence above converges to zero if only if (constant)=0. Hence

$$k^i(t) = \frac{1}{\rho}c^i(t) \Rightarrow \log k^i(t) = -\log\rho + \log c^i(t) \Rightarrow \frac{\dot{k}^i}{k^i} = \frac{\dot{c}^i}{c^i} = r(\tau) - \tau - \rho$$

Growth-maximizing tax rate (the choice of central planner)

$$\frac{\dot{k}^i}{k^i} = r(\tau) - \tau - \rho = \alpha A\tau^{1-\alpha} - \tau - \rho \Rightarrow (1-\alpha)\alpha A\tau_*^{-\alpha} - 1 = 0 \Rightarrow \tau_* = \left((1-\alpha)\alpha A\right)^{1/\alpha}$$

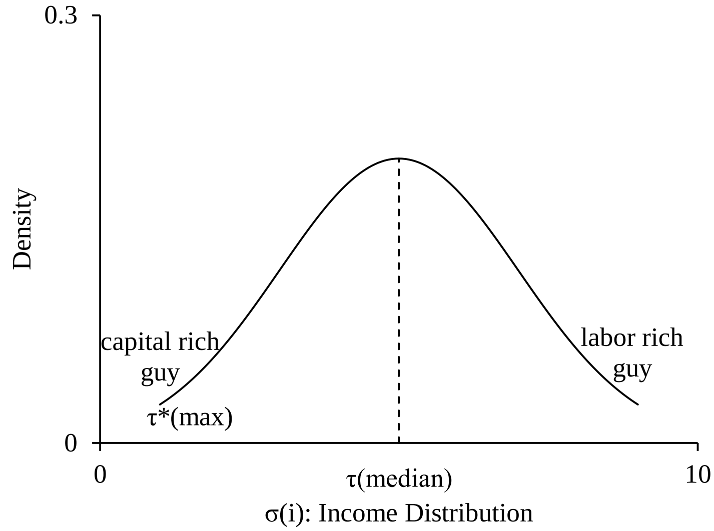
Consumption

$$\dot{k}^i = w(\tau)\sigma^i k^i + [r(\tau)-\tau]k^i - c^i$$
$$\Rightarrow c^i = w(\tau)\sigma^i k^i + [r(\tau)-\tau]k^i - \dot{k}^i = w(\tau)\sigma^i k^i - \rho k^i = \left[w(\tau)\sigma^i - \rho\right]k^i$$

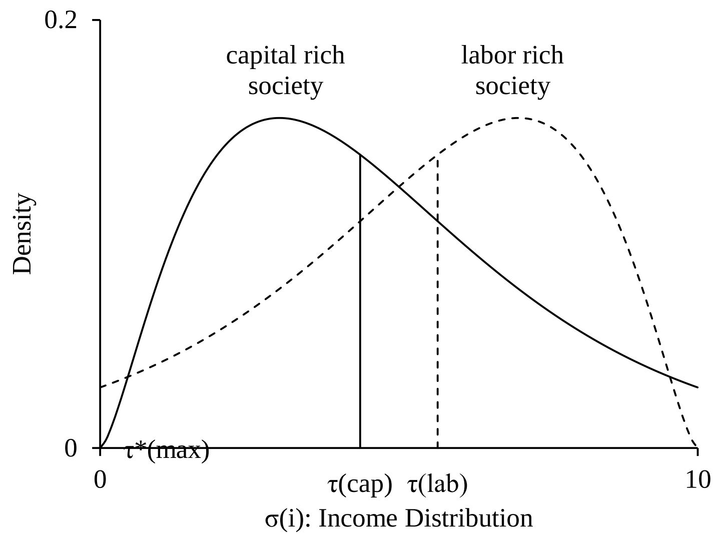
Then for labor-poor guy, $\sigma^i \to 0$, hence $c^i \to \rho k^i$; hence this guy will choose τ that maximizes his consumption growth $\frac{\dot{c}^i}{c^i}$.

$$\tau_*^{\text{labor poor}} = \left((1-\alpha)\alpha A\right)^{1/\alpha}$$

However, if $\sigma^i \neq 0$, then the preferred tax rate τ^i will increase with σ^i; in the extreme, if $\sigma^i \to \infty$, his preferred tax rate τ^i will go to 1.

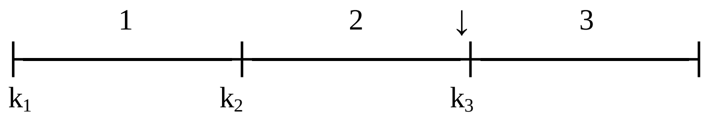


Median voter principle: The preference of “median” guy will be reflected.



Income distribution matters: The τ_{cap} of capital dominated society is closer to the τ_{lab} of labor dominated society; both are distant from τ^* that maximizes the capital growth rate of the entire society.

Assume log utility: $u(c_t)=\log c_t$; consume or save: $c_t=f(k_t)-k_{t+1}$
Suppose there are 3 periods.



In the period 3, the optimal consumption is $c_3=f(k_3)=k_3$ (why $f(k_3)=k_3$?)
In the period 2,

$$\max_{\{c_2,k_3\}} u(c_2)+\beta u(c_3) \text{ subject to } c_2=f(k_2)-k_3 \text{ and } c_3=f(k_3)$$
$$\equiv \max_{k_3} u(c_2)+\beta u[f(k_3)]$$
$$\overset{\text{FOC}}{\Longrightarrow} -u'[f(k_2)-k_3]+\beta u'[f(k_3)]f'(k_3)=0 \Rightarrow -\frac{1}{c_2}+\beta\frac{1}{c_3}=0 \Rightarrow c_3=\beta c_2$$
$$\Rightarrow c_2=f(k_2)-k_3=f(k_2)-c_3=f(k_2)-\beta c_2 \Rightarrow c_2^*=\frac{f(k_2)}{1+\beta}, \qquad c_3^*=\frac{\beta f(k_2)}{1+\beta}$$

Indirect utility

$$U=u\left(\frac{f(k_2)}{1+\beta}\right)+\beta u\left(\frac{\beta f(k_2)}{1+\beta}\right)=V(k_2)$$

In the period 1,

$$\max u(c_1)+\beta u(c_2)+\beta^2 u(c_3) \overset{\text{collapse}}{\Longrightarrow} \max_{c_1} u(c_1)+\beta V(k_2) \equiv \max_{k_2} u[f(k_1)-k_2]+\beta V(k_2)$$

Hence

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ subject to } f(k_t)=c_t+k_{t+1}$$
$$\overset{\text{collapse}}{\Longrightarrow} \max_{c_t} u(c_t)+\beta V(k_{t+1}) \text{ or } \max_{k_{t+1}} u[f(k_t)-k_{t+1}]+\beta V(k_{t+1})$$

Indirect utility

$$\underbrace{V(k_t)}_{\substack{\text{the maximum utility}\\ \text{I can get at time t}}} = \underbrace{u(c_t)}_{\substack{\text{instantaneous}\\ \text{utility at time t}}} + \underbrace{\beta V(k_{t+1})}_{\substack{\text{discounted max}\\ \text{utility at time t+1}}}, \qquad \cdots \text{Bellman equation}$$
$$\Rightarrow V(k_t)=u[f(k_t)-k_{t+1}]+\beta V(k_{t+1}), \qquad \cdots (*)$$

FOC

$$-u'[f(k_t)-k_{t+1}]+\beta V'(k_{t+1})=0$$
$$u'(c_t)=\beta V'(k_{t+1}), \qquad \cdots (\dagger)$$

$$u = f(x,\alpha) \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = 0 \Rightarrow x^* = x^*(\alpha)$$

So

$$v(\alpha) = f[x^*(\alpha),\alpha] \Rightarrow \frac{\partial v}{\partial \alpha} = \underbrace{\frac{\partial f}{\partial x^*}}_{=0}\frac{\partial x^*}{\partial \alpha} + \frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial \alpha}, \qquad \text{Envelope theorem}$$

Therefore, from (*)

$$\begin{aligned} V'(k_t) &= u'[f(k_t) - k_{t+1}]\left[f'(k_t) - \frac{\partial k_{t+1}}{\partial k_t}\right] + \beta V'(k_{t+1})\frac{\partial k_{t+1}}{\partial k_t} \\ &= u'[f(k_t) - k_{t+1}]f'(k_t) - \frac{\partial k_{t+1}}{\partial k_t}\left\{\underbrace{u'[f(k_t) - k_{t+1}] - \beta V'(k_{t+1})}_{=0 \text{ by } (\dagger)}\right\} \\ &= u'[f(k_t) - k_{t+1}]f'(k_t) \end{aligned}$$

Iterate one step further

$$\begin{aligned} V'(k_{t+1}) &= u'[f(k_{t+1}) - k_{t+2}]f'(k_{t+1}) \\ &= \beta V'(k_{t+2})f'(k_{t+1}), \qquad \text{by } (\dagger) \\ u'(c_t) &= \beta u'(c_{t+1})f'(k_{t+1}), \qquad \text{Euler equation} \end{aligned}$$

The importance of V

$$\begin{aligned} &\max \log c_t + \beta\phi \log k_{t+1} \ \text{ subject to } k_{t+1} = f(k_t) - c_t \\ &\Rightarrow \max_{k_{t+1}} \log[f(k_t) - k_{t+1}] + \beta\phi \log k_{t+1} \end{aligned}$$

Questions

- ∃ the functionV(.)?
- ∃! the function V(.)?
- Is V(.) differentiable?

Ex.

$$\begin{aligned} &u(c_t) = \log c_t, \qquad f(k_t) = ak_t, \qquad a > 1 \\ &\Rightarrow \frac{1}{c_t} = \beta\frac{1}{c_{t+1}}a \Rightarrow c_{t+1} = a\beta c_t \Rightarrow c_{t+s} = a^s\beta^s c_t \\ &k_{t+1} = ak_t - c_t \end{aligned}$$

$$k_t = \frac{1}{a}k_{t+1} + \frac{1}{a}c_t = \frac{1}{a^2}k_{t+2} + \frac{1}{a^2}c_{t+1} + \frac{1}{a}c_t = \frac{1}{a^T}k_{t+T} + \frac{1}{a}\sum_{s=0}^{T-1}\frac{1}{a^s}c_{t+s} = \frac{1}{a}\sum_{s=0}^{\infty}\frac{1}{a^s}c_{t+s}$$

$$\Rightarrow ak_t = \sum_{s=0}^{\infty}\frac{1}{a^s}c_{t+s} = \sum_{s=0}^{\infty}\frac{1}{a^s}a^s\beta^s c_t = c_t\sum_{s=0}^{\infty}\beta^s = \frac{c_t}{1-\beta} \Rightarrow c_t = a(1-\beta)k_t, \qquad \text{Policy rule!}$$

*contraction mapping (functional convergence, need concavity of the function)
*How to find V(.)?

- Brute Force method (computer)
- Guess and verify

Maximizing lifetime utility

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad \text{subject to } f(k_t) = c_t + k_{t+1}$$

Bellman equation

$$\max_{c_t} u(c_t) + \beta V(k_{t+1})$$

In order to be valid, V(.) should (1) ∃ (2) ∃! (3) ∃ V′(.). Instead of the convergence of sequence, here focus on the convergence of function (Banach Fixed Point theorem). Instead of $k_{t+1}=f(k_t)$

$$V^{t+1}(.) = cV^t(.), \qquad \text{where c is contraction mapping}$$
$$\xRightarrow{\text{iteration}} V(.) = cV(.), \qquad \text{by iterating over again}$$

Example (Brute Force approach)

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \log c_t, \qquad \text{subject to } a_{t+1} = (1+r)a_t - c_t$$

Since the utility here is a log function, we can guess that the function V(.) will also have the log. Just guess that $V^{t+1}(.)=\log a_{t+1}$ (without any clear clue). By Bellman,

$$\begin{aligned}
\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \log c_t &\Rightarrow \max_{c_t} \log c_t + \beta V(a_{t+1}) \\
&\equiv \max_{c_t} \log c_t + \beta \log[(1+r)a_t - c_t] \\
\text{FOC} &\Rightarrow \frac{1}{c_t} - \beta \frac{1}{a_{t+1}} = 0 \\
&\Rightarrow \beta c_t = a_{t+1} = (1+r)a_t - c_t \\
\text{therefore } &\Rightarrow c_t = \frac{1+r}{1+\beta} a_t, \qquad \text{policy function}
\end{aligned}$$

Then the indirect utility at t

$$\begin{aligned}
V^t(a_t) &= \log\left(\frac{1+r}{1+\beta} a_t\right) + \beta \log\left[(1+r)a_t - \frac{1+r}{1+\beta} a_t\right] \\
&= \log \frac{1+r}{1+\beta} + \log a_t + \beta \log \frac{1+r}{1+\beta} + \beta \log \beta + \beta \log a_t \\
&= \underbrace{\beta \log \beta + (1+\beta) \log \frac{1+r}{1+\beta}}_{\equiv \phi_0} + (1+\beta) \log a_t \\
&= \phi_0 + (1+\beta) \log a_t
\end{aligned}$$

One step backward, by Bellman

$$\max_{c_{t-1}} \log c_{t-1} + \beta[\phi_0 + (1+\beta)\log a_t]$$

Then

$$\text{FOC} \Rightarrow \frac{1}{c_{t-1}} - \beta(1+\beta)\frac{1}{a_t} = 0 \Rightarrow \beta(1+\beta)c_{t-1} = a_t = (1+r)a_{t-1} - c_{t-1}$$

$$\text{policy function} \Rightarrow c_{t-1} = \frac{1+r}{1+\beta+\beta^2}a_{t-1}$$

Indirect utility

$$\log\left(\frac{1+r}{1+\beta+\beta^2}a_{t-1}\right) + \beta\left\{\phi_0 + (1+\beta)\log\left[(1+r)a_{t-1} - \frac{1+r}{1+\beta+\beta^2}a_{t-1}\right]\right\}$$

$$= \underbrace{\beta\phi_0 + (\beta+\beta^2)\log(\beta+\beta^2) + (1+\beta+\beta^2)\log\frac{1+r}{1+\beta+\beta^2}}_{\equiv\phi_1} + (1+\beta+\beta^2)\log a_{t-1}$$

$$= \phi_1 + (1+\beta+\beta^2)\log a_{t-1}$$

$$\xRightarrow{\text{j–th iteration}} \phi_j + \left(\sum_{t=0}^{j}\beta^t\right)\log a_{t-j} \xRightarrow{\text{as } j\to\infty} \phi + \frac{1}{1-\beta}\log a$$

Therefore, by Bellman

$$\max_{c_t} \log c_t + \beta\left(\phi + \frac{1}{1-\beta}\log a_{t+1}\right) \equiv \max_{c_t} \log c_t + \beta\left\{\phi + \frac{1}{1-\beta}\log[(1+r)a_t - c_t]\right\}$$

$$\text{FOC} \Rightarrow \frac{1}{c_t} - \frac{\beta}{1-\beta}\frac{1}{a_{t-1}} = 0$$

$$\Rightarrow \frac{\beta}{1-\beta}c_t = (1+r)a_t - c_t$$

$$\Rightarrow c_t = (1-\beta)(1+r)a_t, \qquad \text{policy function}$$

Another approach (Guess and verify, close to the method of undetermined coefficients)

- Suppose there is a cage in which a female tiger or lion lives. By inputting a male tiger or lion, one can observe an outcome (i.e. an offspring). By checking its characteristics such as stripe, one can guess the type of animal inside the cage

Example (Guess and verify)

$$\max_{\{c_t\}} \sum_{t=0}^{\infty}\beta^t \log c_t\,, \qquad \text{subject to } a_{t+1} = (1+r)a_t - c_t$$

Guess: $V(a_t)=\phi+\theta\log a_t$ (tiger)
Bellman:

$$\max_{c_t} \log c_t + \beta\{\phi + \theta \log[(1+r)a_t - c_t]\}$$

$$\text{FOC} \Rightarrow \frac{1}{c_t} - \beta\theta\frac{1}{a_{t+1}} = 0$$

$$\Rightarrow \beta\theta c_t = (1+r)a_t - c_t$$

$$\Rightarrow c_t = \frac{1+r}{1+\beta\theta}a_t, \qquad \text{policy function (female tiger)}$$

Indirect utility:

$$\log\left(\frac{1+r}{1+\beta\theta}a_t\right) + \beta\left\{\phi + \theta\log\left[(1+r)a_t - \frac{1+r}{1+\beta\theta}a_t\right]\right\}$$

$$= \underbrace{\beta\phi + \beta\theta\log\beta\theta + (1+\beta\theta)\log\frac{1+r}{1+\beta\theta}}_{\equiv\phi'} + \underbrace{(1+\beta\theta)}_{\approx\theta}\log a_t$$

$$= \phi' + (1+\beta\theta)\log a_t, \qquad \text{(offspring)}$$

Therefore, $1+\beta\theta = \theta \Rightarrow \theta = \frac{1}{1-\beta}$

$$V(a) = \phi + \frac{1}{1-\beta}\log a \xRightarrow{\text{Bellman and FOC}} \frac{1}{c_t} - \frac{\beta}{1-\beta}\frac{1}{a_{t+1}} = 0 \xRightarrow{\text{policy function}} c_t = (1-\beta)(1+r)a_t$$

Q: Why backward and forward iterations are equivalent?
A: By Banach fixed point theorem, ∃ V(.) and ∃! V(.).

Consumption function

$$\underbrace{c_i}_{\text{consumption}} = \alpha + \beta \underbrace{y_i}_{\text{income}}$$

Puzzle in empirics

$$\begin{array}{ccc} \hat{\alpha}_t & \hat{\beta}_t & \text{in cross section} \\ \vee & \wedge & \text{vs.} \\ \hat{\alpha}_i & \hat{\beta}_i & \text{in time series} \end{array}$$

Permanent income hypothesis (Milton Friedman)

$$y_i = \underbrace{y_i^P}_{\text{permanent}} + \underbrace{y_i^T}_{\text{transitory}}, \qquad \operatorname{corr}(y_i^P, y_i^T) = 0$$

$$c_i = f(y_i^P), \qquad \text{or } c_i = \gamma_0 + \gamma_1 y_i^P$$

Then, in the cross section

$$\hat{\beta}_t = \frac{\widehat{\operatorname{Cov}}(c_i, y_i)}{\widehat{\operatorname{Var}}(y_i)} \xrightarrow{p} \frac{\operatorname{Cov}(\gamma_0 + \gamma_1 y_i^P, y_i^P + y_i^T)}{\operatorname{Var}(y_i^P) + \operatorname{Var}(y_i^T)} = \frac{\gamma_1 \operatorname{Var}(y_i^P)}{\operatorname{Var}(y_i^P) + \operatorname{Var}(y_i^T)} = \frac{\gamma_1}{1 + \frac{\operatorname{Var}(y_i^T)}{\operatorname{Var}(y_i^P)}}$$

Since $\operatorname{Var}(y_i^T)$ is larger for individuals (i.e. $\operatorname{Var}(y_i^T)>0$), then $\operatorname{plim} \hat{\beta}_t \neq \gamma_1$

Optimization

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad \text{subject to } a_{t+1} = R_{t+1}(a_t + y_t - c_t)$$

where

$$\beta = \frac{1}{1+\rho}, \qquad R_{t+1} = 1 + r_{t+1}$$

Bellman

$$\begin{aligned} &\max_{c_t} u(c_t) + \beta V(a_{t+1}) \\ &\text{FOC} \Rightarrow u'(c_t) - \beta V'(a_{t+1}) R_{t+1} = 0 \\ &\quad \Rightarrow u'(c_t) = \beta V'(a_{t+1}) R_{t+1}, \qquad V'(a_{t+1}) = u'(c_{t+1}) \\ &\quad = \frac{1 + r_{t+1}}{1+\rho} u'(c_{t+1}) \end{aligned}$$

Introduce randomness

$$y_t = \begin{cases} y_t^1, & P[y_t = y_t^1] = p \\ y_t^2, & P[y_t = y_t^2] = 1 - p \end{cases} \Rightarrow a_t = \begin{cases} a_{t+1}^1 = R_{t+1}(a_t + y_t^1 - c_t) \\ a_{t+1}^2 = R_{t+1}(a_t + y_t^2 - c_t) \end{cases}$$

then

$$\begin{aligned}
&\text{optimization: } \max_{\{c_t\}} u(c_0) + \sum_{t=1}^{\infty} \beta^t E[u(c_t)] \\
&\text{Bellman: } \max_{c_t} u(c_t) + \beta[pV(a_{t+1}^1) + (1-p)V(a_{t+1}^2)] = u(c_t) + \beta E[V(a_{t+1})] \\
&\text{FOC: } u'(c_t) - \beta E[V'(a_{t+1})]R_{t+1} = 0, \qquad E[V'(a_{t+1})] = E[u'(c_{t+1})] \\
&\Rightarrow u'(c_t) = \frac{1 + r_{t+1}}{1 + \rho} E[u'(c_{t+1})], \qquad \cdots (*)
\end{aligned}$$

Random walking consumption

$$c_{t+1} = c_t + \varepsilon_{t+1}, \qquad \text{or } \Delta c_{t+1} = \varepsilon_{t+1}, \qquad \text{where } \underbrace{E_t[\varepsilon_{t+1}] = 0 \text{ and } E_t[\varepsilon_{t+1}^2] = \sigma_\varepsilon^2}_{\text{white noise}}$$

then

$$E_t[c_{t+1}] = c_t$$

How to derive?

$$\text{Quadratic utility: } u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2 \Rightarrow u'(c_t) = (\bar{c} - c_t)$$

From (*)

$$\begin{aligned}
\bar{c} - c_t &= \frac{1 + r_{t+1}}{1 + \rho} E_t[\bar{c} - c_{t+1}] \\
&= \frac{1 + r_{t+1}}{1 + \rho}\bar{c} - \frac{1 + r_{t+1}}{1 + \rho} E_t[c_{t+1}] \\
c_t &= \underbrace{\left(1 - \frac{1 + r_{t+1}}{1 + \rho}\right)}_{\equiv 1-\gamma}\bar{c} + \underbrace{\frac{1 + r_{t+1}}{1 + \rho}}_{\equiv \gamma} E_t[c_{t+1}] \\
&= (1 - \gamma) + \gamma E_t[c_{t+1}] \\
\Rightarrow E_t[c_{t+1}] &= \frac{c_t - 1 + \gamma}{\gamma} \approx a + bc_t, \qquad \text{linear in } c_t \\
&= c_t, \qquad \text{if } \rho = r_{t+1}
\end{aligned}$$

Consumption identity

$$\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j c_{t+j} = \sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j y_{t+j} + a_t$$

$$\Rightarrow E_t\left[\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j c_{t+j}\right] = E_t\left[\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j y_{t+j} + a_t\right]$$

$$c_t + \frac{1}{R}c_t + \frac{1}{R^2}c_t + \cdots = E_t\left[\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j y_{t+j} + a_t\right]$$

$$= \frac{c_t}{1 - 1/R}$$

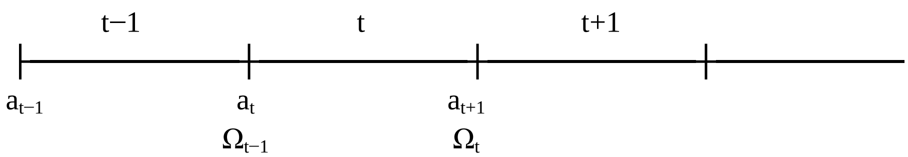
$$= \frac{Rc_t}{R-1}$$

$$= \frac{1+r}{r}c_t$$

$$\Rightarrow c_t = \frac{r}{1+r}E_t\left[\sum_{j=0}^{\infty}\left(\frac{1}{R}\right)^j y_{t+j} + a_t\right]$$

Points

1) Δc_{t+1}, Δc_{t+2}, … are not predictable
2) Consumption decision does not depend on ε_{t+1}, ε_{t+2}, …



Prediction error: $c_{t+1} - E_t[c_{t+1}] = c_{t+1} - c_t$

Assumptions on the income process:

1) $y_t = k + \eta t$
2) AR(1): $y_t = k + \alpha y_{t-1} + \eta_t$

In which process, $c_{t+1} - c_t$ will be smaller?

- In the second case, if $\eta_t\uparrow$, then y_t, y_{t+1}, y_{t+2}, …$\uparrow$
- So c_t, c_{t+1}, c_{t+2}, … will be correlated more
- Hence $c_{t+1} - c_t$ will be smaller

Homework: Can we apply guess-and-verify approach with policy function $c_t = \phi_0 + \phi_1 k_t$ instead of value function $V(a_t) = \phi + \theta \log a_t$?

$$\max_{\{c_t\}} E_t\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j})\right], \qquad \text{subject to } a_{t+1} = R_{t+1}(a_t + y_t - c_t)$$

$$R_{t+1} = 1 + r_{t+1}, \qquad \beta = \frac{1}{1+\rho}$$

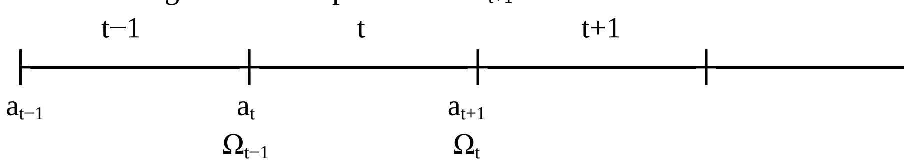
$$\Rightarrow u'(c_t) = \frac{1 + r_{t+1}}{1+\rho} E_t[u'(c_{t+1})]$$

+consumption random walk

$$c_{t+1} = c_t + \varepsilon_{t+1}$$

$$\Rightarrow c_t = \frac{r}{1+r} E_t\left[\underbrace{\sum_{j=0}^{\infty} \frac{1}{R^j} y_{t+j}}_{\equiv H_t} + a_t\right], \qquad \cdots (*)$$

Question 1: The magnitude of consumption response to income shock; y_t↑ then c_t? How much?
Question 2: Where is the origin of consumption shock ε_{t+1}?



$$a_{t+1} = R_{t+1}(a_t + y_t - c_t)$$

Introduce AR(1) process for $\{y_t\}$

$$\begin{aligned} y_{t+1} &= k + \alpha y_t + \eta_{t+1} \\ \Rightarrow y_{t+j} &= k + \alpha y_{t+j-1} + \eta_{t+j} \\ &= k + \alpha(k + \alpha y_{t+j-2} + \eta_{t+j-1}) + \eta_{t+j} \\ &= k\sum_{i=0}^{j-1} \alpha^i + \alpha^j y_t + \sum_{i=0}^{j-1} \alpha^i \eta_{t+j-i}, \qquad \cdots (\|) \end{aligned}$$

From (*)

$$\begin{aligned} \underbrace{E_{t-1}[c_t]}_{=c_{t-1} \text{ by random walk}} &= E_{t-1}\left[\frac{r}{1+r} E_t[H_t + a_t]\right] \\ &= \frac{r}{1+r}(E_{t-1}[H_t] + E_{t-1}[a_t]), \qquad \cdots (\dagger) \\ E_{t-1}[a_t] &= E_{t-1}[R_t(a_{t-1} + y_{t-1} - c_{t-1})] \\ &= R(a_{t-1} + y_{t-1} - c_{t-1}) \\ &= a_t \end{aligned}$$

From (*), (†)

$$c_t = \frac{r}{1+r}(E_t[H_t] + a_t)$$
$$c_{t-1} = \frac{r}{1+r}(E_{t-1}[H_t] + a_t)$$
$$\Rightarrow c_t - c_{t-1} = \frac{r}{1+r}(E_t[H_t] - E_{t-1}[H_t]), \qquad \cdots (\S)$$
$$E_t[H_t] - E_{t-1}[H_t] = \sum_{j=0}^{\infty} \frac{E_t[y_{t+j}] - E_{t-1}[y_{t+j}]}{R^j}, \qquad \cdots (\ddagger)$$

From (‖)

$$E_t[y_{t+j}] = E_t\left[k\sum_{i=0}^{j-1}\alpha^i + \alpha^j y_t + \sum_{i=0}^{j-1}\alpha^i \eta_{t+j-i}\right] = k\sum_{i=0}^{j-1}\alpha^i + \alpha^j y_t$$
$$E_{t-1}[y_{t+j}] = E_{t-1}\left[k\sum_{i=0}^{j-1}\alpha^i + \alpha^j y_t + \sum_{i=0}^{j-1}\alpha^i \eta_{t+j-i}\right] = k\sum_{i=0}^{j-1}\alpha^i + \alpha^j E_{t-1}[y_t]$$
$$\Rightarrow E_t[y_{t+j}] - E_{t-1}[y_{t+j}] = \alpha^j (y_t - E_{t-1}[y_t]) = \alpha^j \eta_t$$

From (‡)

$$E_t[H_t] - E_{t-1}[H_t] = \sum_{j=0}^{\infty}\left(\frac{\alpha}{R}\right)^j \eta_t = \frac{R\eta_t}{R-\alpha}$$

From (§)

$$c_t - c_{t-1} = \frac{r}{1+r}\frac{R\eta_t}{R-\alpha} = \frac{r}{1+r-\alpha}\eta_t$$

If α=0 (no AR process)

$$c_t - c_{t-1} = \frac{r}{1+r}\eta_t, \qquad \frac{\partial \Delta c_t}{\partial \eta_t} = \frac{r}{1+r} \neq 1$$

Q: Why are we only partially ($\frac{r}{1+r}$) increasing our consumption instead of spending 100% of η_t?
A: Consumption smoothing.

If α>0 (some persistence)

$$c_t - c_{t-1} = \frac{r}{1+r-\alpha}\eta_t, \qquad \frac{\partial \Delta c_t}{\partial \eta_t} = \frac{r}{1+r-\alpha} > \frac{r}{1+r}$$

Q: Why in this case (∃AR-ness) do we adjust more than the previous case (∄AR-ness)?
A: My expectation says that today's transitory income shock will last in the future; hence adjust more.

From the last class,

Homework: Can we apply guess-and-verify approach with policy function $c_t=\phi_0+\phi_1 k_t$ instead of value function $V(a_t)=\phi+\theta\log a_t$?

Proof

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t \text{ subject to } a_{t+1} = (1+r)a_t - c_t$$

Bellman

$$\max_{c_t} \ln c_t + \beta V(a_{t+1})$$

FOC

$$\frac{1}{c_t} - \beta V'(a_{t+1}) = 0 \Rightarrow V'(a_{t+1}) = \frac{1}{\beta c_t}$$

Indirect utility at t

$$\begin{aligned}
V(a_t) &= \ln c_t + \beta V'(a_{t+1}) = V\big(a_t, c_t^*(a_t)\big) \\
\Rightarrow \frac{dV}{da_t} &= \frac{\partial V}{\partial a_t} + \frac{\partial V}{\partial c_t}\frac{\partial c_t^*}{\partial a_t} = \frac{\partial V}{\partial a_t}, \qquad \text{Envelope theorem} \\
\Rightarrow V'(a_t) &= \beta V'(a_{t+1})(1+r), \qquad \text{Benveniste–Scheinkman condition} \\
\Rightarrow \frac{1}{\beta c_{t-1}} &= \beta \frac{1}{\beta c_t}(1+r) \Rightarrow c_t = \beta(1+r)c_{t-1}
\end{aligned}$$

Guess: initial guess of policy function, $c_t=\phi_0+\phi_1 k_t$

$$\begin{aligned}
c_{t+1} &= \beta(1+r)c_t \\
&= \phi_0 + \phi_1 k_{t+1} \\
&= \phi_0 + \phi_1 a_{t+1}, \qquad \text{market clearing} \\
&= \phi_0 + \phi_1\big((1+r)a_t - c_t\big) \\
&= \phi_0 + \phi_1(1+r)a_t - \phi_1 c_t \\
\Rightarrow (\beta(1+r)+\phi_1)c_t &= \phi_0 + \phi_1(1+r)a_t \\
\Rightarrow c_t &= \frac{\phi_0}{\beta(1+r)+\phi_1} + \frac{\phi_1(1+r)}{\beta(1+r)+\phi_1} a_t
\end{aligned}$$

Compare the coefficients

$$\begin{aligned}
\phi_1 &= \frac{\phi_1(1+r)}{\beta(1+r)+\phi_1} \Rightarrow \beta(1+r)+\phi_1 = 1+r \Rightarrow \phi_1 = (1-\beta)(1+r) \\
\phi_0 &= \frac{\phi_0}{\beta(1+r)+\phi_1} = \frac{\phi_0}{1+r} \Rightarrow \left(1 - \frac{1}{1+r}\right)\phi_0 = 0 \Rightarrow \phi_0 = 0 \\
\Rightarrow c_t &= (1-\beta)(1+r)a_t, \qquad \text{policy function}
\end{aligned}$$

See you in ECON 804: EMPIRICS!