Information Geometry
https://doi.org/10.1007/s41884-024-00155-y

RESEARCH

# Some computational results on Koszul–Vinberg cochain complexes

**Hanwen Liu** [1,3] · **Jun Zhang** [2,3]

Received: 28 April 2024 / Revised: 6 September 2024 / Accepted: 13 October 2024


**Abstract**
An affine connection is said to be flat if its curvature tensor vanishes identically. Koszul–Vinberg (KV for abbreviation) cohomology has been invoked to study the deformation theory of flat and torsion-free affine connections on tangent bundle. In this Note, we compute explicitly the differentials of various specific KV cochains, and study their relation to classical objects in information geometry, including deformations associated with projective and dual-projective transformations of a flat and torsion-free affine connection. As an application, we also give a simple yet non-trivial example of a left-symmetric algebra of which second cohomology group does not vanish.

**Keywords** Affine structure · Koszul–Vinberg cochain complexes · Conformal and projective transform · Exterior covariant derivative

## 1 Introduction and backgrounds

For each $i \in \{1, \dots, n\}$, denote by $\pi_i : \mathbb{R}^n \to \mathbb{R}$ the projection onto the $i$th coordinate.

Let $M$ be an $n$-dimensional differentiable manifold. As usual, for any chart $(U, \varphi)$ of $M$, the smooth functions $x^1, \dots, x^n \in C^\infty(U)$ defined by

$$x^i : U \to \mathbb{R}, \quad p \mapsto \pi_i(\varphi(p))$$

Communicated by Hiroshi Matsuzoe.

✉ Hanwen Liu
hanwen.liu@warwick.ac.uk

Jun Zhang
junz@umich.edu

1 Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

2 University of Michigan, Ann Arbor, MI 48109, USA

3 Shanghai Institute for Mathematics and Interdisciplinary Sciences, Shanghai, China

are said to be the local coordinates associated to the chart $(U, \varphi)$. An atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ of $M$ is said to be an affine atlas of $M$, if for any $\alpha, \beta \in I$ there exist $A \in GL_n(\mathbb{R})$ and $\mathbf{a} \in \mathbb{R}^n$ such that

$$\varphi_\beta\left(\varphi_\alpha^{-1}(\mathbf{x})\right) = A\mathbf{x} + \mathbf{a}$$

for all $\mathbf{x} \in \varphi_\alpha\left(U_\alpha \cap U_\beta\right)$. For any two affine atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ of $M$, we write $\mathscr{A}_1 \sim \mathscr{A}_2$ if and only if $\mathscr{A}_1 \cup \mathscr{A}_2$ is also an affine atlas of $M$. An affine atlas of $M$ is said to be maximal, if for any affine atlas $\mathscr{A}'$ of $M$ we have that $\mathscr{A}' \sim \mathscr{A}$ implies $\mathscr{A}' \subseteq \mathscr{A}$. Traditionally, a maximal affine atlas on $M$ is also termed an affine structure of $M$.

By Cartan–Ambrose–Hicks theorem, for every flat and torsion-free affine connection $D$ on the tangent bundle $TM \to M$, there exists a maximal affine atlas $\mathscr{A}_D := \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ of $M$, such that

$$Ddx_\alpha^1 = \cdots = Ddx_\alpha^n = 0$$

for all $\alpha \in I$, where $x_\alpha^1, \ldots, x_\alpha^n$ are the local coordinates associated to the chart $(U_\alpha, \varphi_\alpha)$. Moreover, the map $D \longmapsto \mathscr{A}_D$ is a bijection from the set of all flat and torsion-free affine connections on the tangent bundle $TM \to M$ to the collection of all maximal affine atlases of $M$, and the atlas $\mathscr{A}_D$ is termed the adapted affine atlas of $D$. Therefore, to fix an affine structure on $M$ is equivalent to specifying a flat and torsion-free affine connection on the tangent bundle $TM \to M$. In view of this fact, by a slightly abuse of notation we also refer to a flat and torsion-free affine connection on the tangent bundle $TM \to M$ as an affine structure of $M$.

Let $(M, g)$ be a Riemannian manifold, and let $D$ be a flat and torsion-free affine connection on the tangent bundle $TM \to M$. Denote by $\mathscr{A}_D$ the adapted affine atlas of $D$. We say that $(M, g)$ is of Hessian type with respect to $D$, if for any $p \in M$ there exist a chart $(U, \varphi) \in \mathscr{A}_D$ and a smooth function $f \in C^\infty(U)$ such that $p \in U$ and

$$g_{ij} = \frac{\partial f}{\partial x^i \partial x^j}$$

for all $i, j \in \{1, \ldots, n\}$, where $x^1, \ldots, x^n$ are the local coordinates associated to $(U, \varphi)$ and $g_{11}, g_{12}, \ldots, g_{nn}$ are the tensor components of $g$ under basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$.

We shall be interested in the deformation theory of affine structures on a smooth manifold $M$. The deformations of affine structures has been studied by means of the theory of Koszul–Vinberg (KV for abbreviation) cohomology. In [1], Boyom proved a remarkable rigidity theorem for affine structure. In [2], Byande and Boyom related symmetric zeros of Maurer–Cartan polynomial map to the deformation theory of left-symmetric algebra of a flat torsion-free affine connection on tangent bundle. As KV cohomology is so far the main algebraic topological tool that is utilized in the deformation theory of affine structures, a further understanding of its properties seems of interest.

This Note investigates KV cohomology by explicitly calculating the differential of specific KV cohains. After a brief review of left-symmetric algebra associated to a flat torsion-free connection and its cohomology, we provide a few results related to the first and the second KV cochain groups. As specific applications, characterizations of projective transformation and dual-projective transformation of a flat and torsion-free connection are both described in terms of a vanishing condition on differential of their associated 2-cochains. Finally, to give a concrete example of nontrivial deformations of affine structure, we construct a left-symmetric algebra associated to a flat torsion-free connection with non-vanishing second KV cohomology group for the simplest case of a planar domain with its usual Euclidean metric.

The "Appendix" provides a review of exterior covariant derivative and de Rham cohomology twisted by a local system, to allow uninitiated readers a comparison with and appreciation of KV cohomology.

## 2 Brief review of Koszul–Vinberg cohomology

Recall from [3] that, for an arbitrary field $k$, a $k$-vector space $V$ equipped with a bilinear product

$$V \times V \longrightarrow V, \quad (v, w) \mapsto v \cdot w$$

is called a left-symmetric $k$-algebra if the following equality

$$u \cdot (v \cdot w) - (u \cdot v) \cdot w = v \cdot (u \cdot w) - (v \cdot u) \cdot w$$

holds for all $u, v, w \in V$.

We refer to [3] as a survey of various researches on left-symmetric algebras.

Let $M$ be an arbitrary but fixed smooth manifold, and let

$$\nabla \colon \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM), \quad (X, Y) \longmapsto \nabla_X Y$$

be a flat torsion-free affine connection on the tangent bundle $TM \to M$, where $\Gamma(TM)$ is the $\mathbb{R}$-vector space of $C^\infty$ vector fields on $M$. Throughout this article, we view a $(p, q)$-tensor as a multilinear map from the $q$-fold Cartesian product $\Gamma(TM) \times \cdots \times \Gamma(TM)$ to $\Gamma(TM)^{\otimes p}$.

Since the curvature and torsion tensor of $\nabla$ vanish identically, we have

$$\begin{aligned} 0 &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z \\ &= \big(\nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z\big) - \big(\nabla_Y \nabla_X Z - \nabla_{\nabla_Y X} Z\big) \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. This shows that $(\Gamma(TM), \nabla)$ is a left-symmetric $\mathbb{R}$-algebra.

**Definition 2.1** The $\mathbb{R}$-algebra $(\Gamma(TM), \nabla)$ is said to be the left-symmetric algebra of $\nabla$.

To simplify the notation, throughout this article we denote $\mathcal{A} := (\Gamma(TM), \nabla)$.

Recall that a vector field $Z \in \Gamma(TM)$ is termed a Jacobi element of $\mathcal{A}$, if $\nabla_X \nabla_Y Z = \nabla_{\nabla_X Y} Z$ holds for all $X, Y \in \Gamma(TM)$. It is clear that the collection of Jacobi elements in $\mathcal{A}$ forms a $\mathbb{R}$-vector space.

**Definition 2.2** Define $C^0(\mathcal{A})$ to be the subspace of Jacobi elements of $\mathcal{A}$, and for each integer $n \geqslant 1$ define $C^n(\mathcal{A}) := \mathrm{Hom}_{\mathbb{R}}\left(\Gamma(TM)^{\otimes n}, \Gamma(TM)\right)$, or equivalently, the collection of $\mathbb{R}$-multilinear maps from the $n$-fold Cartesian product $\Gamma(TM) \times \cdots \times \Gamma(TM)$ to $\Gamma(TM)$.

Notice that, for positive $n \in \mathbb{N}$, a generic element of $C^n(\mathcal{A})$ is not tensorial.

In addition, we shall introduce the following important notation:

For each $X \in \Gamma(TM)$ and each $\theta \in C^n(\mathcal{A})$ with $n \geqslant 1$, let $\nabla_X \theta$ be the element of $C^n(\mathcal{A})$ satisfying $(\nabla_X \theta)(X_1, \ldots, X_n) := \nabla_X(\theta(X_1, \ldots, X_n)) - \left[\theta(\nabla_X X_1, X_2, \ldots, X_n) + \cdots + \theta(X_1, \ldots, X_{n-1}, \nabla_X X_n)\right]$.

For any integer $n \geqslant 1$, notice that were $\theta \in C^n(\mathcal{A})$ to be a $(1, n)$-tensor on $M$ then the notation above recovers the Leibniz rule.

Next we recall the standard definition of KV differential.

**Definition 2.3** Define endomorphism $d_{\mathrm{KV}} : \bigoplus_{i=0}^{\infty} C^i(\mathcal{A}) \longrightarrow \bigoplus_{i=0}^{\infty} C^i(\mathcal{A})$ of degree +1 of the graded $\mathbb{R}$-vector space $\bigoplus_{i=0}^{\infty} C^i(\mathcal{A})$ as follows:

(i) for each $X \in C^0(\mathcal{A})$, define $d_{\mathrm{KV}} X \in C^1(\mathcal{A})$ by $(d_{\mathrm{KV}} X)(Y) := [X, Y]$;
(ii) for each $\theta \in C^n(\mathcal{A})$ with $n \geqslant 1$, define $d_{\mathrm{KV}}\theta \in C^{n+1}(\mathcal{A})$ by

$$\begin{aligned}(d_{\mathrm{KV}}\theta)(X_1, \ldots, X_{n+1}) := \sum_{i=1}^{n} (-1)^i \Big[ & \left(\nabla_{X_i}\theta\right)\left(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}\right) \\ & + \nabla_{\theta\left(X_1, \ldots, \hat{X}_i, \ldots, X_n, X_i\right)} X_{n+1} \Big]\end{aligned}$$

where the hat on $\hat{X}_i$ indicates that the $X_i$ term is omitted.

**Lemma 2.4** *It holds that* $d_{\mathrm{KV}} \circ d_{\mathrm{KV}} = 0$.

***Proof*** See for example [1] for a detailed proof. □

**Definition 2.5** The differential graded $\mathbb{R}$-vector space $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ is said to be the KV cochain complex of $\mathcal{A}$, and the elements of $C^n(\mathcal{A})$ are termed KV $n$-cochains.

We shall now recall a classical application of the cohomology theory of KV cochain complexes. The following definitions and theorem are from [2].

**Definition 2.6** A family $\left\{\nabla^t \in C^2(\mathcal{A}) \mid t \in \mathbb{R}\right\}$ of flat torsion-free affine connections on the tangent bundle $TM \to M$ is said to be a smooth deformation of $\nabla$, if $\nabla^0 = \nabla$ and for all $X, Y \in \Gamma(TM)$ the mapping

$$M \times \mathbb{R} \longrightarrow TM, \quad (p, t) \longmapsto \left(\nabla^t_X Y\right)(p)$$

is smooth.

**Definition 2.7** A smooth deformation $\left\{\nabla^t \in C^2(\mathcal{A}) \mid t \in \mathbb{R}\right\}$ is said to be trivial, if there exists a one-parameter subgroup

$$\phi\colon \mathbb{R} \longrightarrow \mathrm{Diff}(M), \quad t \longmapsto \phi^t$$

of the group of diffeomorphisms of $M$, such that

$$\nabla^t_X Y = d\phi^t\left(\nabla_{d\phi^{-t}(X)} d\phi^{-t}(Y)\right)$$

for all $X, Y \in \Gamma(TM)$ and $t \in \mathbb{R}$.

In [1], the following rigidity theorem is proven.

**Theorem 2.8** *Suppose that $M$ is compact. If the second cohomology group of $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ vanishes, then all the smooth deformations of $\nabla$ are trivial.*

For more applications, we refer to [4–6].

## 3 A description of the zeroth cochain group

Throughout this section, let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ be the adapted atlas of $\nabla$, and for each $\alpha \in I$ denote by $x^1_\alpha, \ldots, x^n_\alpha$ the local coordinates associated to the chart $(U_\alpha, \varphi_\alpha)$.

The following results stated in [1] completely describes the Jacobi elements in $\mathcal{A}$.

**Theorem 3.1** *Let $Z \in \Gamma(TM)$ be a smooth vector field on $M$. Then $Z \in C^0(\mathcal{A})$ if and only if for each $\alpha \in I$ there exist real numbers $a_{11}, a_{12}, \ldots, a_{nn} \in \mathbb{R}$ and $b_1, \ldots, b_n \in \mathbb{R}$ such that*

$$Z^i = b_i + \sum_{j=1}^{n} a_{ij} x^j_\alpha$$

*for every $i \in \{1, \ldots, n\}$, where $Z^1, \ldots, Z^n$ are the components of $Z$ under basis $\frac{\partial}{\partial x^1_\alpha}, \ldots, \frac{\partial}{\partial x^n_\alpha}$.*

***Proof*** For simplicity we shall omit the subscript $\alpha$ in $x^1_\alpha, \ldots, x^n_\alpha$. Take any $X, Y \in \Gamma(TM)$ and denote by $X^1, \ldots, X^n$ and $Y^1, \ldots, Y^n$ the components of $X$ and $Y$ under basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$. Then computation yields that

$$
\begin{aligned}
\nabla_X Y &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} \\
\nabla_{\nabla_X Y} Z &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial Z^k}{\partial x^j} \frac{\partial}{\partial x^k} \\
\nabla_X \nabla_Y Z &= \nabla_X \left( Y^j \frac{\partial Z^k}{\partial x^j} \frac{\partial}{\partial x^k} \right) \\
&= X^i \frac{\partial}{\partial x^i} \left( Y^j \frac{\partial Z^k}{\partial x^j} \right) \frac{\partial}{\partial x^k} \\
&= \left( X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial Z^k}{\partial x^j} + X^i Y^j \frac{\partial Z^k}{\partial x^i \partial x^j} \right) \frac{\partial}{\partial x^k}
\end{aligned}
$$

and hence we have

$$
\nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z = X^i Y^j \frac{\partial Z^k}{\partial x^i \partial x^j} \frac{\partial}{\partial x^k}
$$

where in all the formulae above we use Einstein's summation convention.

Therefore $\nabla_X \nabla_Y Z = \nabla_{\nabla_X Y} Z$ if and only if

$$
\frac{\partial Z^k}{\partial x^i \partial x^j} = 0
$$

for all $i, j \in \{1, \dots, n\}$. The desired result follows immediately. □

As per usual, for any $C^\infty$ vector field $X$ on $M$, we denote by $\mathcal{L}_X$ the Lie derivative along $X$.

**Lemma 3.2** *Let $Z \in \Gamma(TM)$ be a smooth vector field on $M$. Then for any $X, Y \in \Gamma(TM)$ it holds that $(\mathcal{L}_Z \nabla)(X, Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$.*

***Proof*** Recall that $\nabla$ is flat and torsion-free. Straightforward computation yields that

$$
\begin{aligned}
&(\mathcal{L}_Z \nabla)(X, Y) \\
&\quad = \mathcal{L}_Z (\nabla_X Y) - \nabla_{\mathcal{L}_Z X} Y - \nabla_X (\mathcal{L}_Z Y) \\
&\quad = -\nabla_{[Z,X]} Y + \nabla_X [Y, Z] + [Z, \nabla_X Y] \\
&\quad = (\nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y) + \nabla_X [Y, Z] + [Z, \nabla_X Y] \\
&\quad = \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y + \nabla_X \nabla_Y Z - \nabla_X \nabla_Z Y + \nabla_Z \nabla_X Y - \nabla_{\nabla_X Y} Z \\
&\quad = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z
\end{aligned}
$$

holds for all $X, Y \in \Gamma(TM)$. □

**Corollary 3.3** *Let $X \in \Gamma(TM)$ be a smooth vector field. Then $X \in C^0(\mathcal{A})$ if and only if $\mathcal{L}_X \nabla = 0$.*

***Proof*** This is a direct consequence of Lemma 3.2. □

## 4 Results on the first Cochain group

Let $\theta \in C^1(\mathcal{A}) = \mathrm{End}_{\mathbb{R}}(\Gamma(TM))$ be an arbitrary KV 1-cochain of $\mathcal{A}$. Then by the definition of $d_{\mathrm{KV}}$, we have

$$(d_{\mathrm{KV}}\theta)(X, Y) = -\nabla_X(\theta(Y)) + \theta\,(\nabla_X Y) - \nabla_{\theta(X)}Y.$$

***Example 4.1*** Let $f \in C^\infty(M)$ be a fixed smooth function, and define a KV 1-cochain $\theta \in C^1(\mathcal{A})$ by

$$\theta\colon \Gamma(TM) \longrightarrow \Gamma(TM), \quad Z \longmapsto fZ.$$

Then for any $X, Y \in \Gamma(TM)$, we have that $(d_{\mathrm{KV}}\theta)\,(X, Y) = -\nabla_X(fY) + f\nabla_X Y - \nabla_{fX}Y = -\nabla_X(fY)$.

We identify $\Gamma(TM)$ with the module of derivations of the $\mathbb{R}$-algebra $C^\infty(M)$. Recall that the Poisson bracket $[X, Y] \in \Gamma(TM)$ of two vector fields $X$ and $Y$ on $M$ is defined by

$$[X, Y]\colon\ C^\infty(M) \longrightarrow C^\infty(M), \quad f \longmapsto X(Yf) - Y(Xf)$$

and $\mathfrak{X} := (\Gamma(TM), [\cdot,\cdot])$ is a Lie algebra. Denote by $\mathrm{ad}\colon \mathfrak{X} \longrightarrow \mathrm{Der}(\mathfrak{X})$ the adjoint representation of $\mathfrak{X}$.

***Example 4.2*** Let $Z \in \Gamma(TM)$ be a fixed vector field and define $\theta := \mathrm{ad}_Z \in C^1(\mathcal{A})$. Then, since $\nabla$ is flat and torsion-free, we have that

$$\begin{aligned}(d_{\mathrm{KV}}\theta)\,(X, Y) &= -\nabla_X\,(\mathrm{ad}_Z(Y)) + \mathrm{ad}_Z\,(\nabla_X Y) - \nabla_{\mathrm{ad}_Z(X)}Y\\ &= \nabla_X[Y, Z] - [\nabla_X Y, Z] + \nabla_{[X,Z]}Y\\ &= \nabla_X\,(\nabla_Y Z - \nabla_Z Y) - \nabla_{\nabla_X Y}Z + \nabla_Z\nabla_X Y + \nabla_{[X,Z]}Y\\ &= \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z - \big([\nabla_X, \nabla_Z]\,Y - \nabla_{[X,Z]}Y\big)\\ &= \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z\end{aligned}$$

holds for all $X, Y \in \Gamma(TM)$. In particular, $d_{\mathrm{KV}}\theta$ is a $(1, 2)$-tensor.

**Theorem 4.3** *Let $\theta \in C^1(\mathcal{A})$ be a* KV *1-cochain. Then the following statements are equivalent:*

(i) $(d_{\mathrm{KV}}\theta)\,(X, Y) = (d_{\mathrm{KV}}\theta)\,(Y, X)$ *for all* $X, Y \in \Gamma(TM)$*;*
(ii) *there exists* $X \in \Gamma(TM)$ *such that* $\theta = \mathrm{ad}_X$.

***Proof*** We first prove that $(ii)$ implies $(i)$. Suppose that there exists $Z \in \Gamma(TM)$ such that $\theta = \mathrm{ad}_Z$, then by Example 4.2 we have that

$$(d_{\mathrm{KV}}\theta)\,(X, Y) = \nabla_X\nabla_Y Z - \nabla_{\nabla_X Y}Z$$

holds for all $X, Y \in \Gamma(TM)$. Since $\nabla$ is flat, for any $X, Y \in \Gamma(TM)$ we have

$$\nabla_X \nabla_Y - \nabla_{\nabla_X Y} - \nabla_Y \nabla_X + \nabla_{\nabla_Y X} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = 0$$

and hence $\nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z = \nabla_Y \nabla_X Z - \nabla_{\nabla_Y X} Z$, i.e. $(d_{\mathrm{KV}}\theta)(X, Y) = (d_{\mathrm{KV}}\theta)(Y, X)$.

Now we shall prove that $(i)$ implies $(ii)$. By the very definition of $d_{\mathrm{KV}}$, we have that

$$\begin{aligned}
&(d_{\mathrm{KV}}\theta)(X, Y) - (d_{\mathrm{KV}}\theta)(Y, X) \\
&= \big(-\nabla_X(\theta(Y)) + \theta(\nabla_X Y) - \nabla_{\theta(X)} Y\big) - \big(-\nabla_Y(\theta(X)) + \theta(\nabla_Y X) - \nabla_{\theta(Y)} X\big) \\
&\quad = \theta(\nabla_X Y - \nabla_Y X) - \big(\nabla_{\theta(X)}(Y) - \nabla_Y(\theta(X))\big) - \big(\nabla_X(\theta(Y)) - \nabla_{\theta(Y)}(X)\big) \\
&\quad = \theta([X, Y]) - [\theta(X), Y] - [X, \theta(Y)]
\end{aligned}$$

where the last equality is due to the torsion-freeness of $\nabla$. Suppose that $(i)$ is satisfied, then for any smooth vector fields $X, Y \in \Gamma(TM)$, we have

$$\theta([X, Y]) = [\theta(X), Y] + [X, \theta(Y)].$$

Therefore we conclude that $\theta \in \mathrm{Der}(\mathfrak{X})$. Since by [7] all derivations of $\mathfrak{X}$ are inner, there exists $Z \in \Gamma(TM)$ such that $\theta = \mathrm{ad}_Z$. □

***Remark 4.4*** For an arbitrary $\theta \in C^1(\mathcal{A})$, the equality $(d_{\mathrm{KV}}\theta)(X, Y) = (d_{\mathrm{KV}}\theta)(Y, X)$ does not hold for all $X, Y \in \Gamma(TM)$ in general. For example, if $\theta \in C^1(\mathcal{A})$ is the identity map, then by Example 4.1 $(d_{\mathrm{KV}}\theta)(X, Y) = -\nabla_X Y$ and hence $(d_{\mathrm{KV}}\theta)(X, Y) - (d_{\mathrm{KV}}\theta)(Y, X) = [Y, X]$.

As an application, we give a detailed proof of the following statement proposed in [1].

**Theorem 4.5** *The first cohomology group of* $\big(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\big)$ *always vanishes.*

***Proof*** Take $\theta \in C^1(\mathcal{A})$ and assume that $d_{\mathrm{KV}}\theta = 0$. We shall prove that $\theta \in \mathrm{Im}(d_{\mathrm{KV}})$. Indeed, since

$$(d_{\mathrm{KV}}\theta)(X, Y) = 0 = (d_{\mathrm{KV}}\theta)(Y, X)$$

for all $X, Y \in \Gamma(TM)$, Theorem 4.3 implies that there exists $Z \in \Gamma(TM)$ such that $\theta = \mathrm{ad}_Z$. Furthermore, by Example 4.2, we have that

$$\nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z = (d_{\mathrm{KV}}\theta)(X, Y) = 0$$

for all $X, Y \in \Gamma(TM)$, i.e $Z \in C^0(\mathcal{A})$. Therefore we conclude that $\theta = d_{\mathrm{KV}} Z$. □

We shall close this section with an observation which will be of use later:

**Proposition 4.6** *Let* $\left\{\nabla^t \in C^2(\mathcal{A}) \mid t \in \mathbb{R}\right\}$ *be a trivial smooth deformation of* $\nabla$. *Then it holds that* $\frac{d}{dt}\big|_{t=0} \nabla^t \in \mathrm{Im}\,(d_{\mathrm{KV}})$.

***Proof*** By the definition of a trivial smooth deformation of $\nabla$, there exists a smooth flow

$$\phi\colon \mathbb{R} \longrightarrow \mathrm{Diff}(M), t \mapsto \phi^t$$

on $M$, such that

$$\nabla^t_X Y = d\phi^t \left(\nabla_{d\phi^{-t}(X)} d\phi^{-t}(Y)\right)$$

for all $X, Y \in \Gamma(TM)$ and $t \in \mathbb{R}$. Let $Z \in \Gamma(TM)$ be the velocity field of $\phi$, i.e. the smooth vector field $Z$ on $M$ such that

$$\frac{d}{dt}\phi^t(p) = Z \circ \phi^t(p)$$

for all $p \in M$ and $t \in \mathbb{R}$. By the definition of Lie derivative, we have

$$\lim_{t\to 0} \tfrac{1}{t}(\nabla^t - \nabla^0) = -\mathcal{L}_Z \nabla$$

But $\mathcal{L}_Z \nabla \in \mathrm{Im}(d_{\mathrm{KV}})$ by Lemma 3.2 and Example 4.2. The statement is proven. □

## 5 Results on the second cochain group

Let $\theta \in C^2(\mathcal{A})$ be an arbitrary KV 2-cochain of $\mathcal{A}$. Then by the definition of $d_{\mathrm{KV}}$, we have

$$\begin{aligned}(\mathrm{d}_{KV}\theta)(X, Y, Z) &= -\nabla_X(\theta(Y, Z)) + \theta(\nabla_X Y, Z) + \theta(Y, \nabla_X Z) - \nabla_{\theta(Y,X)} Z \\ &\quad + \nabla_Y(\theta(X, Z)) - \theta(\nabla_Y X, Z) - \theta(X, \nabla_Y Z) + \nabla_{\theta(X,Y)} Z \\ &= (\nabla_Y \theta)(X, Z) - (\nabla_X \theta)(Y, Z) + \nabla_{\theta(X,Y)-\theta(Y,X)} Z.\end{aligned}$$

**Proposition 5.1** *Let* $\mathrm{Id}: \Gamma(TM) \to \Gamma(TM)$ *be the identity map. Then it holds that* $d_{\mathrm{KV}}(-\mathrm{Id}) = \nabla$, *and that* $d_{\mathrm{KV}}\nabla = 0$.

***Proof*** The first assertion follows from Example 4.1. By Lemma 2.4, $d_{\mathrm{KV}}\nabla = d_{\mathrm{KV}}d_{\mathrm{KV}}(-\mathrm{Id}) = 0$. □

**Theorem 5.2** *Let $D$ be a torsion-free affine connection on the tangent bundle* $TM \to M$, *and let* $\theta := D - \nabla \in C^2(\mathcal{A})$. *Then the following properties hold:*

(i) $\theta$ *is a* $(1, 2)$*-tensor;*
(ii) $\theta(X, Y) = \theta(Y, X)$ *for all* $X, Y \in \Gamma(TM)$*;*
(iii) $(d_{\mathrm{KV}}\theta)(X, Y, Z) = (\nabla_Y \theta)(X, Z) - (\nabla_X \theta)(Y, Z)$ *for all* $X, Y, Z \in \Gamma(TM)$, *in particular,* $d_{\mathrm{KV}}\theta$ *is a* $(1, 3)$*-tensor;*

(iv) $\theta \in \mathrm{Im}(d_{\mathrm{KV}})$ *if and only if there exists* $Z \in \Gamma(TM)$ *such that*

$$\theta(X, Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$$

*for all* $X, Y \in \Gamma(TM)$.

***Proof*** Properties $(i)$ and $(ii)$ are proved in [8]. Property $(iii)$ follows from $(ii)$ and the very definition of $d_{\mathrm{KV}}$. Property $(iv)$ is a direct consequence of Example 4.2 and Theorem 4.3. □

As special cases of Theorem 5.2, we now consider several concrete deformation of flat torsion-free connection.

**Definition 5.3** The connection $D$ on tangent bundle $TM \to M$ defined by

$$D_X Y := \nabla_X Y + \omega(X)Y + \omega(Y)X$$

where $\omega$ is a given 1-form, is said to be a projective transformation of $\nabla$.

It is clear that the formula above indeed defines a connection.

**Proposition 5.4** *Suppose that* $\dim M > 1$ *and let* $\omega \in \Omega^1(M)$ *be a 1-form. Let* $\theta \in C^2(\mathcal{A})$ *be the* KV *2-cochain satisfying* $\theta(X, Y) = \omega(X)Y + \omega(Y)X$ *for all* $X, Y \in \Gamma(TM)$. *Then* $d_{\mathrm{KV}}\theta = 0$ *if and only if* $\nabla\omega = 0$.

***Proof*** Take any arbitrary $X, Y, Z \in \Gamma(TM)$. Since $\theta(X, Y) = \theta(Y, X)$ and

$$\begin{aligned}
(\nabla_X\theta)(Y, Z) &= \nabla_X(\omega(Y)Z + \omega(Z)Y) - \omega(\nabla_X Y) Z \\
&\quad - \omega(Z)\nabla_X Y - \omega(Y)\nabla_X Z - \omega(\nabla_X Z) Y \\
&= \nabla_X(\omega(Y)Z) - \omega(\nabla_X Y) Z - \omega(Y)\nabla_X Z \\
&\quad + \nabla_X(\omega(Z)Y) - \omega(\nabla_X Z) Y - \omega(Z)\nabla_X Y \\
&= (\nabla_X\omega)(Y) Z + (\nabla_X\omega)(Z) Y,
\end{aligned}$$

we obtain that

$$\begin{aligned}
(d_{\mathrm{KV}}\theta)(X, Y, Z) &= (\nabla_Y\theta)(X, Z) - (\nabla_X\theta)(Y, Z) + \nabla_{\theta(X,Y)-\theta(Y,X)} Z \\
&= (\nabla_Y\omega)(X) Z + (\nabla_Y\omega)(Z) X - (\nabla_X\omega)(Y) Z - (\nabla_X\omega)(Z) Y.
\end{aligned}$$

Therefore $\nabla\omega = 0$ implies that $d_{\mathrm{KV}}\theta = 0$.

Now assume that $d_{\mathrm{KV}}\theta = 0$. Let $e_1, \ldots, e_n$ be a local frame in the tangent bundle $TM \to M$. Take any arbitrary $i \in \{1, \ldots, n\}$. Since $n = \dim M \geqslant 2$, there exists $j \in \{1, \ldots, n\}$ such that $i \neq j$. Since

$$\begin{aligned}
&\left(\nabla_{e_i}\omega\right)(e_i) e_j + \left[\left(\nabla_{e_i}\omega\right)\left(e_j\right) - 2\left(\nabla_{e_j}\omega\right)(e_i)\right] e_i \\
&\quad = \left(\nabla_{e_i}\omega\right)\left(e_j\right) e_i + \left(\nabla_{e_i}\omega\right)(e_i) e_j - \left(\nabla_{e_j}\omega\right)(e_i) e_i - \left(\nabla_{e_j}\omega\right)(e_i) e_i \\
&\quad = (d_{\mathrm{KV}}\theta)\left(e_j, e_i, e_i\right) \\
&\quad = 0
\end{aligned}$$

in particular we have $\left(\nabla_{e_i}\omega\right)(e_i) = \left(\nabla_{e_i}\omega\right)\left(e_j\right) - 2\left(\nabla_{e_j}\omega\right)(e_i) = 0$. Therefore we obtain that $(\nabla_X\omega)(X) = 0$ for all $X \in \Gamma(TM)$, and hence

$$\begin{aligned}
&(\nabla_X\omega)(Y) + (\nabla_Y\omega)(X)\\
&\quad = (\nabla_X\omega)(X) + (\nabla_Y\omega)(X) + (\nabla_X\omega)(Y) + (\nabla_Y\omega)(Y)\\
&\quad = (\nabla_{X+Y}\omega)(X+Y)\\
&\quad = 0
\end{aligned}$$

i.e. $(\nabla_X\omega)(Y) = -(\nabla_Y\omega)(X)$ for all $X, Y \in \Gamma(TM)$. Since $\left(\nabla_{e_i}\omega\right)\left(e_j\right) = -\left(\nabla_{e_j}\omega\right)(e_i)$, we have

$$\left(\nabla_{e_i}\omega\right)\left(e_j\right) = \tfrac{1}{3}[\left(\nabla_{e_i}\omega\right)\left(e_j\right) - 2\left(\nabla_{e_j}\omega\right)(e_i)] = 0.$$

This proves that $\nabla\omega = 0$. □

**Definition 5.5** The connection $D$ on tangent bundle $TM \to M$ defined by

$$D_XY := \nabla_XY - h(X, Y)V$$

where $h$ is a given pseudo-Riemannian metric and $V$ is a given vector field, is said to be a dual-projective transformation of $\nabla$.

As in Definition 5.3, one can check that the formula above indeed defines a connection.

**Definition 5.6** A pseudo-Riemannian metric $h$ on $M$ is said to be Codazzi-coupled with $\nabla$, if the Codazzi equation

$$(\nabla_Xh)(Y, Z) = (\nabla_Yh)(X, Z)$$

holds for all $X, Y, Z \in \Gamma(TM)$.

**Proposition 5.7** *Let $h$ be a pseudo-Riemannian metric on $M$ and let $V \in \Gamma(TM)$ be a non-vanishing vector field parallel to $\nabla$. Let $\theta \in C^2(\mathcal{A})$ be the* KV *2-cochain satisfying $\theta(X, Y) = -h(X, Y)V$ for all $X, Y \in \Gamma(TM)$. Then $d_{\mathrm{KV}}\theta = 0$ if and only if $h$ is Codazzi-coupled with $\nabla$.*

***Proof*** Take any arbitrary $X, Y, Z \in \Gamma(TM)$. Since $\theta(X, Y) = \theta(Y, X)$ and

$$\begin{aligned}
(\nabla_X\theta)(Y, Z) &= -\nabla_X(h(Y, Z)V) + h\left(\nabla_XY, Z\right)V + h\left(Y, \nabla_XZ\right)V\\
&= -\left(\nabla_Xh\right)(Y, Z)V - h(Y, Z)\nabla_XV\\
&= -\left(\nabla_Xh\right)(Y, Z)V,
\end{aligned}$$

we obtain that

$$\begin{aligned}
(d_{\mathrm{KV}}\theta)(X, Y, Z) &= (\nabla_Y\theta)(X, Z) - (\nabla_X\theta)(Y, Z) + \nabla_{\theta(X,Y)-\theta(Y,X)}Z\\
&= ((\nabla_Xh)(Y, Z) - (\nabla_Yh)(X, Z))V.
\end{aligned}$$

Since $V$ is non-vanishing, we have that $d_{\mathrm{KV}}\theta = 0$ if and only if $(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z)$ for all $X, Y, Z \in \Gamma(TM)$. □

Propositions 5.4 and 5.7 give characterizations of deformation from a flat connection arising from projective and dual-projective transformation in terms of KV cochains.

## 6 Calculations of quantities related to Hessian geometry

Throughout this section, we fix a Riemannian metric $g$ on $M$.

**Definition 6.1** The conjugate connection $\nabla^*$ of $\nabla$ is defined to be the unique affine connection on the tangent bundle $TM \to M$ such that the equation

$$Zg(X, Y) = g(\nabla_Z X, Y) + g\left(X, \nabla^*_Z Y\right)$$

holds for all $X, Y, Z \in \Gamma(TM)$.

See for example [9] for a proof of the fact that $\nabla^*$ is well-defined.

We shall recall some well-known facts about Hessian geometry:

**Theorem 6.2** *The Riemannian manifold $(M, g)$ is of Hessian type with respect to the connection $\nabla$, if and only if the conjugate connection $\nabla^*$ is flat and torsion-free.*

***Proof*** See [10] for a proof of this statement. □

**Lemma 6.3** *If the Riemannian metric $g$ is Codazzi-coupled with $\nabla$, then $(M, g)$ is of Hessian type with respect to $\nabla$, and $\frac{1}{2}(\nabla^* + \nabla)$ is the Levi-Civita connection of $(M, g)$.*

***Proof*** This is proved in [11, 12]. □

For the rest of this Section, we consider only Riemannian manifolds of Hessian type.

As usual, we denote by

$$R^g\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM), \quad (X, Y, Z) \longmapsto R^g(X, Y)Z$$

the Riemann curvature tensor of $(M, g)$, i.e. the curvature of connection $\frac{1}{2}(\nabla^* + \nabla)$.

**Proposition 6.4** *For any $X, Y, Z \in \Gamma(TM)$, it holds that $(d_{\mathrm{KV}}\nabla^*)(X, Y, Z) = -4R^g(X, Y)Z$.*

***Proof*** Since by assumption $(M, g)$ is of Hessian type with respect to $\nabla$, by Theorem 6.2 the affine connections $\nabla$ and $\nabla^*$ are both flat and torsion-free. Define the difference tensor $\theta := \nabla^* - \nabla$. Then by Theorem 5.2 (ii) we have $\theta(X, Y) = \theta(Y, X)$ for all $X, Y \in \Gamma(TM)$.

Now, direct computation yields that

$$\begin{aligned}\nabla_X(\theta(Y,Z)) - \nabla_Y(\theta(X,Z)) &= \left(\nabla_X\nabla^*_Y Z - \nabla_X\nabla_Y Z + \nabla^*_X\nabla^*_Y Z - \nabla^*_X\nabla^*_Y Z\right)\\ &\quad - \left(\nabla_Y\nabla^*_X Z - \nabla_Y\nabla_X Z + \nabla^*_Y\nabla^*_X Z - \nabla^*_Y\nabla^*_X Z\right)\\ &= \nabla^*_{[X,Y]}Z - \nabla_{[X,Y]}Z - \theta\left(X, \nabla^*_Y Z\right) + \theta\left(Y, \nabla^*_X Z\right)\\ &= \theta([X,Y],Z) + \left(\theta\left(Y,\nabla^*_X Z\right) - \theta\left(X, \nabla^*_Y Z\right)\right)\end{aligned}$$

holds for all $X, Y, Z \in \Gamma(TM)$. Therefore for any $X, Y, Z \in \Gamma(TM)$ we have that

$$\begin{aligned}(d_{\mathrm{KV}}\theta)(X,Y,Z) &= -\nabla_X(\theta(Y,Z)) + \theta(\nabla_X Y, Z) + \theta(\nabla_X Z, Y)\\ &\quad + \nabla_Y(\theta(X,Z)) - \theta(\nabla_Y X, Z) - \theta(\nabla_Y Z, X)\\ &= \theta([X,Y],Z) + (\theta(Y,\nabla_X Z) - \theta(X,\nabla_Y Z))\\ &\quad - (\nabla_X(\theta(Y,Z)) - \nabla_Y(\theta(X,Z)))\\ &= \theta(X,\theta(Y,Z)) - \theta(Y,\theta(X,Z))\\ &= -4R^g(X,Y)Z\end{aligned}$$

by [12]. Since by Proposition 5.1 $d_{\mathrm{KV}}\nabla = 0$, we have that $d_{\mathrm{KV}}\nabla^* = d_{\mathrm{KV}}(\nabla^* - \nabla) = d_{\mathrm{KV}}\theta$. This concludes the proof. □

**Theorem 6.5** *It holds that* $d_{\mathrm{KV}}R^g = 0$.

***Proof*** By Proposition 6.4, $R^g = -d_{\mathrm{KV}}\frac{1}{4}\nabla^*$. By Lemma 2.4, $d_{\mathrm{KV}}R^g = -\frac{1}{4}d_{\mathrm{KV}}d_{\mathrm{KV}}\nabla^* = 0$. □

## 7 An example of non-vanishing KV cohomology group

In this section, we construct a non-trivial example of second KV cohomology group. For simplicity, we consider the two dimensional case. Our procedure consists of two steps: First, we study conditions of the vanishing of KV differential $d_{\mathrm{KV}}\theta$ for a KV 2-cochain $\theta$ that results from the conformal transformation of a flat manifold. Second, we construct an explicit example of which the above $\theta$ is not in the image of the KV differential $d_{\mathrm{KV}}$.

**Theorem 7.1** *Suppose that $(M, g)$ is a two dimensional Riemannian manifold with flat Levi-Civita connection $\nabla$. Let $f \in C^\infty(M)$ and let $\theta \in C^2(\mathcal{A})$ be the* KV *2-cochain satisfying*

$$\theta(X,Y) = -g(X,Y)\,\mathrm{grad} f + \{(Xf)Y + (Yf)X\}$$

*for all $X, Y \in \Gamma(TM)$. Then the Levi-Civita connection of $\left(M, e^{2f}g\right)$ is $\nabla + \theta$, and the following statements are equivalent:*

(i) $d_{\mathrm{KV}}\theta = 0$*;*
(ii) $\nabla + \theta$ *is flat;*

(iii) *$f$ is a harmonic function on $(M, g)$.*

***Proof*** Denote by $\Delta$ the Laplace–Beltrami operator on $(M, g)$ and denote $\tilde{g} := e^{2f} g$. The fact that the Levi-Civita connection of $(M, \tilde{g})$ is $\nabla + \theta$ follows from Koszul's formula. According to [13], the Riemann curvature tensor of $(M, \tilde{g})$ is

$$e^{-2f}(K - \Delta f)\tilde{g} \owedge \tilde{g}$$

where $K$ is the Gaussian curvature of $(M, g)$ and $\owedge$ is the Kulkarni–Nomizu product. Since $\nabla$ is flat, we have that $K = 0$. Therefore $\nabla + \theta$ is flat if and only if $\Delta f = 0$. This proves that property $(ii)$ is equivalent to $(iii)$.

We shall now prove that $(i)$ is also equivalent to $(iii)$. For each 1-form $\omega \in \Omega^1(M)$, let $\omega^\# \in \Gamma(TM)$ be the vector field satisfying $\omega(X) = g\left(\omega^\#, X\right)$ for all $X \in \Gamma(TM)$. Recall that for any $X \in \Gamma(TM)$ we have $(\nabla_X df)^\# = \nabla_X \operatorname{grad} f$. Since $\nabla$ is compatible with $g$, the equality

$$\begin{aligned}
(d_{\mathrm{KV}}\theta)(X, Y, Z) = {} & [(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z)] \\
& \operatorname{grad} f + g(Y, Z)\nabla_X \operatorname{grad} f - g(X, Z)\nabla_Y \operatorname{grad} f \\
& + (\nabla_Y df)(X)Z + (\nabla_Y df)(Z)X - (\nabla_X df)(Y)Z \\
& - (\nabla_X df)(Z)Y \\
= {} & (\nabla_Y df)(Z)X - (\nabla_X df)(Z)Y \\
& + [(\nabla_Y df)(X) - (\nabla_X df)(Y)]\, Z \\
& + g(Y, Z)(\nabla_X df)^\# - g(X, Z)(\nabla_Y df)^\#
\end{aligned}$$

holds for all $X, Y, Z \in \Gamma(TM)$. By passing to trivializing neighbourhood if necessary, we can assume w.l.o.g. that there exist $e_1, e_2 \in \Gamma(TM)$ such that $g\left(e_i, e_j\right) = \delta_{ij}$ for all $i, j \in \{1, 2\}$.

Take distinct $i, j \in \{1, 2\}$. By observation $(d_{\mathrm{KV}}\theta)(e_i, e_i, e_i) = (d_{\mathrm{KV}}\theta)\left(e_i, e_i, e_j\right) = 0$. Since the Hessian $\nabla df$ is symmetric, we obtain

$$g\left((d_{\mathrm{KV}}\theta)\left(e_i, e_j, e_i\right), e_i\right) = 2\left(\nabla_{e_j} df\right)(e_i) - \left(\nabla_{e_i} df\right)\left(e_j\right) - \left(\nabla_{e_j} df\right)(e_i) = 0.$$

Also direct computation yields that

$$g\left((d_{\mathrm{KV}}\theta)\left(e_i, e_j, e_i\right), e_j\right) = -\left(\nabla_{e_i} df\right)(e_i) - \left(\nabla_{e_j} df\right)\left(e_j\right) = -\Delta f.$$

Similarly we have $g\left((d_{\mathrm{KV}}\theta)\left(e_i, e_j, e_j\right), e_j\right) = 0$ and $g\left((d_{\mathrm{KV}}\theta)\left(e_i, e_j, e_j\right), e_i\right) = \Delta f$. Therefore $d_{\mathrm{KV}}\theta = 0$ if and only if $\Delta f = 0$. □

Using Theorem 7.1, we now can construct an explicit example of which second KV cohomology does not vanish.

***Example 7.2*** Let $M := \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 0\right\}$ be the punctured plane, and denote by $\Gamma(TM)$ the $\mathbb{R}$-vector space of $C^\infty$ vector fields on $M$. Let $g = dx \otimes dx + dy \otimes dy$ be the standard Euclidean metric on $M$, and denote by $\nabla$ the Levi-Civita connection of $(M, g)$. Then $\nabla$ is flat and torsion free. Denote by $\mathcal{A} := (\Gamma(TM), \nabla)$ the KV algebra of $\nabla$, and denote by $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ its KV cochain complex. Consider the smooth function:

$$f\colon M \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto \frac{1}{2} \ln\left(x^2 + y^2\right)$$

and consider the KV 2-cochain $\theta \in C^2(\mathcal{A})$ satisfying

$$\theta(X, Y) = -g(X, Y)\,\mathrm{grad} f + (Xf)Y + (Yf)X$$

for all $X, Y \in \Gamma(TM)$. Since $f$ is a harmonic function on $(M, g)$, by Theorem 7.1, we have that $d_{\mathrm{KV}}\theta = 0$.

**Claim**. The KV 2-cochain $\theta$ is not an element of Im $(d_{\mathrm{KV}})$.

***Proof*** Assume the contrary, then by Theorem 5.2 there exists $Z \in \Gamma(TM)$ such that the equation

$$\nabla_{\nabla_X Y} Z - \nabla_X \nabla_Y Z = \theta(X, Y) \tag{*}$$

holds for all $X, Y \in \Gamma(TM)$. Take $u, v \in C^\infty(M)$ such that $Z = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$. Then equation $(*)$ implies

$$\begin{aligned}
- u_{xx}\frac{\partial}{\partial x} - v_{xx}\frac{\partial}{\partial y} &= f_y\frac{\partial}{\partial y} - f_x\frac{\partial}{\partial x}\\
- u_{xy}\frac{\partial}{\partial x} - v_{xy}\frac{\partial}{\partial y} &= f_y\frac{\partial}{\partial x} + f_x\frac{\partial}{\partial y}\\
- u_{yy}\frac{\partial}{\partial x} - v_{yy}\frac{\partial}{\partial y} &= f_x\frac{\partial}{\partial x} - f_y\frac{\partial}{\partial y}
\end{aligned}$$

and in particular, $u$ satisfies the following system of partial differential equations on $M$:

$$-\left(x^2 + y^2\right)\mathrm{Hess}(u) = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}.$$

Consider the open subset $\Omega := \left\{(x, y) \in \mathbb{R}^2 \mid y \neq 0\right\}$ of $M$. Direct computation yields that, there exist real numbers $a, b, c \in \mathbb{R}$ such that

$$u(x, y) = \frac{x}{2}\ln\left(x^2 + y^2\right) + \arctan\left(\frac{x}{y}\right) y + ax + by + c$$

for all $(x, y) \in \Omega$. In particular $u(2, t) = \ln(t^2 + 4) + \arctan(2/t)t + b \cdot t + (2a + c)$ for all $t \in \mathbb{R}^\times$. Differentiation yields that $u_y(2, t) = \arctan(2/t) + b$ for all $t \in \mathbb{R}^\times$.

Since $u_y(2,t) \to \frac{\pi}{2} + b$ as $t \to 0+$ and $u_y(2,t) \to -\frac{\pi}{2} + b$ as $t \to 0-$, we obtain that the function

$$\begin{aligned} \mathbb{R}^\times &\longrightarrow \mathbb{R} \\ t &\longmapsto u_y(2,t) \end{aligned}$$

cannot be extended continuously to $\mathbb{R}$. Therefore $u|_\Omega$ cannot be extended to a smooth function defined on $M$, a contradiction. This proves the claim. □

That $d_{\mathrm{KV}}\theta = 0$ but $\theta \notin \mathrm{Im}\,(d_{\mathrm{KV}})$ means that the second cohomology group of $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ does not vanish.

***Remark 7.3*** Under the setting of Example 7.2, for each $t \in \mathbb{R}$ we define $\nabla^t := \nabla + t\theta$. Then since $d_{\mathrm{KV}}\theta = 0$, by Theorem 7.1 $\nabla^t$ is a flat and torsion-free affine connection on the tangent bundle $TM \to M$ for every $t \in \mathbb{R}$. Since moreover for all $X, Y \in \Gamma(TM)$ the mapping

$$M \times \mathbb{R} \longrightarrow TM, \quad (p,t) \longmapsto \left(\nabla^t_X Y\right)(p)$$

is smooth, the family $\left\{\nabla^t \in C^2(\mathcal{A}) \mid t \in \mathbb{R}\right\}$ is by definition a $C^\infty$ deformation of $\nabla$. Since $\theta \notin \mathrm{Im}\,(d_{\mathrm{KV}})$, by Proposition 4.6 the $C^\infty$ deformation $\left\{\nabla^t \in C^2(\mathcal{A}) \mid t \in \mathbb{R}\right\}$ of $\nabla$ is non-trivial.

## Appendix

Let $\pi\colon E \to M$ be a smooth $\mathbb{R}$-vector bundle over a differentiable manifold $M$. Denote by $\Gamma(E)$ the space of $C^\infty$ sections of $\pi$, and $\Gamma(\mathrm{End}E)$ the space of $C^\infty$ sections of the endomorphism bundle of $\pi$. For each $k \in \mathbb{N}$, denote by $\Omega^k(M; E) := \Omega^k(M) \otimes \Gamma(E)$ the $C^\infty(M)$-module of vector-valued $k$-forms on $M$. Let $\nabla$ be a connection on $E \xrightarrow{\pi} M$, i.e. $\nabla\colon \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$ is an $\mathbb{R}$-linear transform such that the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f\nabla s$$

holds for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$. For a given smooth vector field $X$ on $M$, we have the mapping

$$\begin{aligned} \nabla_X\colon \Gamma(E) &\longrightarrow \Gamma(E) \\ s &\longmapsto \nabla_X s := (\nabla s)(X) \end{aligned}$$

induced from $\nabla$. The differential operator $\nabla$ applicable to $\Gamma(E)$ can be extended to a map

$$d^\nabla\colon \Omega^k(M; E) \to \Omega^{k+1}(M; E)$$

for every $k \in \mathbb{N}$, via the formula

$$d^{\nabla}(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

for any $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$.

Straightforward computation yields that

$$\begin{aligned}\left(d^{\nabla}\theta\right)(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i \nabla_{X_i}\left(\theta\left(X_0, \ldots, \hat{X}_i, \ldots, X_k\right)\right) \\ &+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \theta\Big(\left[X_i, X_j\right], X_0, \ldots, \hat{X}_i, \\ &\ldots, \hat{X}_j, \ldots, X_k\Big)\end{aligned}$$

for all $\theta \in \Omega^k(M; E)$ and all smooth vector fields $X_0, \ldots, X_k$ on $M$, where the hats on $\hat{X}_i$ and $\hat{X}_j$ indicate that the $X_i$ and $X_j$ terms are omitted. In particular, the equation above for $k = 2$ reads

$$\left(d^{\nabla}\theta\right)(X, Y) = \nabla_X(\theta(Y)) - \nabla_Y(\theta(X)) - \theta([X, Y])$$

for all $\theta \in \Omega^1(M; E)$ and all smooth vector fields $X, Y$ on $M$.

By the general theory of connections [13], there exists $R^{\nabla} \in \Omega^2(M) \otimes \Gamma(\mathrm{End}E)$ such that $d^{\nabla}\left(d^{\nabla}\theta\right) = R^{\nabla} \wedge \theta$ for all $\theta \in \Omega^k(M; E)$ with $k \in \mathbb{N}$. The matrix-valued 2-form $R^{\nabla}$ is termed the curvature of $\nabla$. To compute $R^{\nabla}$ explicitly, it suffices to notice that

$$\begin{aligned}R^{\nabla}(X, Y)s &= \left(d^{\nabla}\left(d^{\nabla}s\right)\right)(X, Y) \\ &= (d^{\nabla}(\nabla s))(X, Y) \\ &= \nabla_X((\nabla s)(Y)) - \nabla_Y((\nabla s)(X)) - (\nabla s)([X, Y]) \\ &= \nabla_X\left(\nabla_Y s\right) - \nabla_Y\left(\nabla_X s\right) - \nabla_{[X,Y]} s\end{aligned}$$

holds for all $s \in \Gamma(E)$ and all smooth vector fields $X, Y$ on $M$. Therefore

$$R^{\nabla}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for all smooth vector fields $X, Y$ on $M$.

The connection $\nabla$ is said to be flat if its curvature $R^{\nabla}$ vanishes identically. If $\nabla$ is flat, then $d^{\nabla} \circ d^{\nabla} = 0$, and hence $\left(\bigoplus_{i=0}^{\infty} \Omega^i(M; E), d^{\nabla}\right)$ is a differential graded $\mathbb{R}$-vector space, termed the de Rham complex of $M$ twisted by $\nabla$.

In the following, we shall consider only the case that $\pi: E \to M$ is the tangent bundle of $M$, and $\nabla$ is flat and torsion-free. As usual, we denote by $\mathcal{A}$ the KV algebra of $\nabla$, and denote by $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ its KV cochin complex. Since

$\Omega^k(M; E) \subseteq C^k(\mathcal{A})$ for all $k \in \mathbb{N}$, it is natural to compare the cohomology groups of $\left(\bigoplus_{i=0}^{\infty} \Omega^i(M; E), d^{\nabla}\right)$ and $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$. However, it turns out that in general the cohomology of $\left(\bigoplus_{i=0}^{\infty} \Omega^i(M; E), d^{\nabla}\right)$ cannot be embedded into the cohomology of $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ as a graded abelian group, and vice versa.

An example can be constructed as follows.

Let $M := \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 0\right\}$ be the punctured plane. Since the tangent bundle $TM \to M$ is trivial, we have that $\Omega^k(M; TM) = \Omega^k(M) \oplus \Omega^k(M)$ for all $k \in \mathbb{N}$. Let $g = dx \otimes dx + dy \otimes dy$ be the standard Euclidean metric on $M$, and let $\nabla$ be the Levi-Civita connection of $(M, g)$. Then $d^{\nabla} = d \oplus d$, and hence the cohomology of $\left(\bigoplus_{i=0}^{\infty} \Omega^i(M; TM), d^{\nabla}\right)$ is $\bigoplus_{i=0}^{\infty} H^i(M; \mathbb{R})^{\oplus 2}$. Since $M$ is homotopic to the unit circle $\mathbb{S}^1$, we conclude that $H^0(M; \mathbb{R}) = \mathbb{R}$ and $H^2(M; \mathbb{R}) = 0$. Therefore the second cohomology of $\left(\bigoplus_{i=0}^{\infty} \Omega^i(M; TM), d^{\nabla}\right)$ vanishes and the zeroth cohomology of $\left(\bigoplus_{i=0}^{\infty} \Omega^i(M; TM), d^{\nabla}\right)$ does not vanish.

Let $\mathcal{A}$ be the KV algebra of $\nabla$, and $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ its KV cochain complex. By Example 6.2 we have that the second cohomology of $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ does not vanish. We claim that the zeroth cohomology of $\left(\bigoplus_{i=0}^{\infty} C^i(\mathcal{A}), d_{\mathrm{KV}}\right)$ vanishes. Indeed, it suffices to prove the following lemma:

**Lemma** *Let $\Omega$ be a domain in $\mathbb{R}^n$, and let $X$ be a smooth vector field on $\Omega$. If $[X, Y] = 0$ for every smooth vector field $Y$ on $\Omega$, then $X = 0$.*

***Proof*** We shall use Einstein's summation convention. Let $x^1, \ldots, x^n$ be the standard coordinate system on $\Omega$. Then there exist $X^1, \ldots, X^n \in C^{\infty}(\Omega)$ such that $X = X^i \frac{\partial}{\partial x^i}$. Since

$$\frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} = \left[X^i \frac{\partial}{\partial x^i}, -\frac{\partial}{\partial x^j}\right] = 0$$

holds for all $j \in \{1, \ldots, n\}$, we have that $X^1, \ldots, X^n$ are constant functions. Therefore

$$\begin{aligned} X &= X^i \frac{\partial}{\partial x^i} \\ &= X^i \frac{\partial x^j}{\partial x^i} \frac{\partial}{\partial x^j} \\ &= X^i \left[\frac{\partial}{\partial x^i}, x^j \frac{\partial}{\partial x^j}\right] \\ &= \left[X^i \frac{\partial}{\partial x^i}, x^j \frac{\partial}{\partial x^j}\right] \\ &= 0 \end{aligned}$$

as desired. □

**Acknowledgements** The authors want to express their sincere gratitude to the reviewers for useful suggestions and a comment which leads to an improvement of Proposition 5.4.

**Author contributions** Hanwen Liu wrote the main manuscript text and the appendix. Jun Zhang proposed a number of revision suggestions.

**Funding** This research was completed while both authors held visiting positions in and were supported by Shanghai Institute for Mathematics and Interdisciplinary Sciences. All authors certify that they have no affiliations with or involvement in any other organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

**Data availibility** No datasets were generated or analysed during the current study.

## Declarations

**Competing interests** Jun Zhang, Co-Editor-in-Chief of the journal, is not involved in the peer review or handling of the manuscript. On behalf of all authors, Hanwen Liu, the corresponding author, states that there is no other potential conflict of interest to declare.



## References

1. Nguiffo Boyom, M.: The cohomology of Koszul–Vinberg algebras. Pac. J. Math. **225**(1), 119–153 (2006)
2. Boyom, M.N., Byande, P.M.: KV cohomology in information geometry. In: Matrix Information Geometry. KV Cohomology in Information Geometry, pp. 69–92. Springer, Berlin (2012)
3. Burde, D.: Left-symmetric algebras, or left-symmetric algebroids in geometry and physics. Cent. Eur. J. Math. **4**, 323–357 (2006)
4. Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003)
5. Koszul, J.-L.: Déformation des connexions localement plates. Ann. Inst. Fourier **18**, 103–114 (1968)
6. Nijenhuis, A.: Sur une classe de propriétés communes à quelques types différents d'algèbres. Enseign. Math. **2**(14), 225–277 (1968)
7. Takens, F.: Derivations of vector fields. Compos. Math. **26**, 151–158 (1973)
8. Tao, J., Zhang, J.: Transformations and coupling relations for affine connections. Differ. Geom. Appl. **49**, 111–130 (2016). https://doi.org/10.1016/j.difgeo.2016.06.006
9. Amari, S.-i.: Information Geometry and Its Applications. Applied Mathematical Sciences. Springer, Tokyo, (2016)
10. Shima, H.: Geometry of Hessian Structures. In: Nielsen, F., Barbaresco, F. (eds) Geometric Science of Information. GSI 2013. Lecture Notes in Computer Science, vol. 8085. Springer, Berlin. https://doi.org/10.1007/978-3-642-40020-9_4 (2013)
11. Fei, T., Zhang, J.: Interaction of Codazzi couplings with (para-)Kähler geometry. Result Math. **72**, 2037–2056 (2017)
12. Zhang, J.: A note on curvature of $\alpha$-connections of a statistical manifold. AISM **59**, 161–170 (2007)
13. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. 1. Interscience, New York (1963)