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Tangent Groupoid and Information Geometry

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Abstract. For a smooth manifold M, the tangent groupoid "glues" the set $M \times M$ with TM as two underlying pieces in smooth transition from one to the other. We show that any contrast function defined on $M \times M$ naturally leads to a Riemannian metric and a pair of torsion-free conjugate connections (so-called "statistical structure") that are objects defined for sections of TM. This is achieved through smooth "extension" of the contrast function and its anti-symmetrized version on $M \times M$ to, respectively, a quadratic and a cubic function on TM. We recovered the standard formulae [1,4–6] linking contrast functions to statistical structure through differentiation of the former by two and three vector fields to obtain the metric and the connections, respectively.

Keywords: divergence function · yoke · statistical structure

1 Introduction

Information geometry models parametric families of probability density (or mass) functions as differentiable manifolds, and characterizes their invariant properties using geometric notions. Central to information geometry is the idea that the so-called "statistical structure" of such manifold, namely the Riemannian metric with a pair of torsion-free conjugate connections, can be induced from the divergence function (a.k.a. contrast functions) defined for pairs of points on the manifold (see [1,4–6]). It is also known that the converse is not unique: given a statistical structure, there can be many divergence/contrast functions that induce it [8] – the standard Fisher-Rao metric along with the family of α-connections can be induced by a multitude of contrast functions. Hence, a question arises as to the exact information encoded by an arbitrary contrast function that is captured on TM. In this paper, using the method of tangent groupoid, we extend the contrast function $f(u_a, u_b)$ between two points u_a, u_b on M to a quadratic function on TM and the anti-symmetrized contrast function, $f(u_a, u_b) - f(u_b, u_a)$, to a trilinear function on TM. We then show how these extensions precisely recover the Riemannian metric and the pair of torsion-free conjugate functions on sections of TM.

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2 Tangent Groupoid

The notion of groupoid generalizes that of a group. Roughly speaking, a groupoid can be thought of as a "group with many different specifications/identities." Below is the precise definition in the language of Category Theory.

Definition 1. *A groupoid (G_0, G) consists of the following data:*

- $G^{(0)} \equiv G_0$ *a set of objects (sometimes called the unit space);*
- $G^{(1)} \equiv G$ *a set of morphisms, mapping objects of* $G_0 \to G_0$*;*
- $G^{(2)} = \{(\gamma, \eta) \in G \times G \mid s(\gamma) = r(\eta)\}$ *the set of composable pairs;*
- $m : G^{(2)} \to G$ *the multiplication map which describe the composition of morphisms. The multiplication map is usually denoted by* $m(\gamma, \eta) \equiv \gamma \circ \eta$*;*
- $s : G \to G^{(0)}$ *the source map;*
- $r : G \to G^{(0)}$ *the range map;*
- $\varepsilon : G^{(0)} \to G$ *the unit map satisfies*

$$\varepsilon(r(\gamma)) \circ \gamma = \gamma \circ \varepsilon(s(\gamma)) = \gamma, \quad \forall \gamma \in G;$$

- $\iota : G \to G$ *the inverse map satisfies*

$$\iota(\gamma) \circ \gamma = \varepsilon(s(\gamma)), \quad \gamma \circ \iota(\gamma) = \varepsilon(r(\gamma)), \quad \forall \gamma \in G.$$

The inverse map is sometimes denoted by $\iota(\gamma) = \gamma^{-1}$.

Each morphism (element of G) takes in an element of G_0 to produce another element of G_0; the action by a morphism is reflected as the "source" and "target" maps. On the other hand, the set of morphisms G *almost* form a group (with group composition m and inverse ι), almost in the sense that composition of morphisms is allowed only for those pairs (γ, η) of morphisms satisfying $s(\gamma) = r(\eta)$.

Now take the set of objects to be a manifold M with dimension $n = \dim(M)$ (where each object is just a point in M). With respect to a smooth manifold M, the tangent groupoid $(M \times [0,1], \mathbb{T}M)$ is a groupoid whose set of objects is $M \times [0,1]$ and set of morphisms is

$$\mathbb{T}M = TM \sqcup M \times M \times (0,1].$$

Here TM denotes the tangent bundle, and $\sqcup$ denotes disjoint union as sets, and $M \times [0,1]$ can be viewed a series of manifolds M_t indexed by $t \in [0,1]$. The source and range maps of $\mathbb{T}M$ are given by projecting onto the second and the first M variables, respectively, in the $M \times M \times (0,1]$ part and the bundle projection $TM \to M$ in the TM part. More precisely, the source and range maps send $(p_a, p_b, t) \in M \times M \times (0,1]$ to (p_b, t) and (p_a, t) respectively and send $(p, X) \in TM$ to $p \in M$.

So in a tangent groupoid, the set of morphisms and the set of objects are both smooth manifolds. Indeed, let $M \supset U \xrightarrow{\varphi} \mathbb{R}^n$ be a local coordinate chart of M. The smooth structure of $\mathbb{T}M$ is given by local coordinate charts of the form

$$\mathbb{T}M \supset \mathbb{T}U \xrightarrow{\Phi} \mathbb{R}^{2n+1} \tag{1}$$

which

- sends $(p_1, p_2, t) \in M \times M \times (0,1]$ to $\left(\varphi(p_1), \frac{\varphi(p_2)-\varphi(p_1)}{t}, t\right)$ if $t \neq 0$;
- sends $(p, X) \in TM$ to $(\varphi(p), d\varphi(X))$, where $d\varphi(X) \in \mathbb{R}^n$ is the image of $X \in T_pM$ under the differential of φ.

It is straightforward to check that the source and range maps are submersions and all the other structure maps (multiplication, unit and the inverse maps) are all smooth. See [2] for the detailed treatment of tangent groupoid.

An important concept for understanding tangent groupoid $\mathbb{T}M$ is so-called normal cone deformation, which we review below. Let $M \subset V$ be a submanifold. With respect to the tangent bundle TM, the normal bundle N_MV of M is an object that is inside V but "transversal" to TM. Formally, it is a fiber bundle with M as base and each of its fiber consists of equivalent class of elements of TV, that is, $N_MV = TV/TM$ where the equivalence relation $\sim$ in each fiber T_uV over $u \in M$ is defined by $v_1 \sim v_2$ (for $v_1, v_2 \in T_uV$) if and only if there exists an element $x \in T_uM$ such that $v_1 = v_2 + x$.

Deformation to the normal cone associated to the pair (M, V), by definition, is a smooth manifold whose underlying set is

$$\mathbb{N}_MV = N_MV \sqcup V \times (0,1].$$

Let $V \supset U \xrightarrow{\varphi} \mathbb{R}^{\dim(V)}$ be a local coordinate chart where $U \cap M$ is the preimage of $\mathbb{R}^{\dim(V)-\dim(M)} \times \{0\} \subset \mathbb{R}^{\dim(V)}$ under φ. Then the smooth structure of $\mathbb{N}_MV$ is given by the local coordinate charts of the form

$$\mathbb{N}_MV \supset \mathbb{N}_{U\cap M}U \xrightarrow{\Phi} \mathbb{R}^{\dim(V)+1}$$

which

- sends $(p,t) \in V \times (0,1]$ to $\left(\frac{\varphi(p)}{t}, t\right)$ if $t \neq 0$;
- sends $(p, X) \in N_{U\cap M}U$ to $(\varphi(p), d\varphi(X))$, where $d\varphi(X) \in \mathbb{R}^{\dim(V)}$ is the image of X under the differential of φ.

A special case is when $V = M \times M$ and $M \hookrightarrow V$ is its diagonal embedding $M \subset M \times M$. Then, apart from an isomorphism between oriented bundles, $N_M(M \times M)$ can be identified with TM, and the associated deformation to the normal cone is precisely the tangent groupoid $\mathbb{T}M$:

$$\mathbb{N}_M(M \times M) \simeq \mathbb{T}M.$$

This means that the smooth structure of the tangent groupoid $\mathbb{T}M$ arises as normal cone deformation. Moreover, take $V = M \times M \times M$ and $M \hookrightarrow V$ as the diagonal embedding, the associated deformation to the normal cone is the composable pairs $\mathbb{T}^{(2)}M$ of the tangent groupoid, with

$$\mathbb{T}^{(2)}M = TM \oplus TM \sqcup M \times M \times M \times (0,1].$$

Another mathematical object closely related to the tangent groupoid $\mathbb{T}M$ is the Lie groupoid, defined as a groupoid (M, G) where the set of objects M

and the set of morphisms G are both manifolds while the source and target maps are both submersions. While both Lie groupoid and tangent groupoid are groupoids with smooth manifold structures, they are linked through the notion of "adiabatic groupoid" [3]. Adiabatic groupoid of a groupoid G is obtained by the deformation to the normal cone construction corresponding to the inclusion $M \subset G$. It represents a smooth deformation of G into its Lie algebroid $\mathfrak{A}_G \to M$. In the special case of a Lie groupoid (M, G) with $G = M \times M$, the corresponding adiabatic groupoid is precisely the tangent groupoid $\mathbb{T}M$.

The relevance of tangent groupoid $\mathbb{T}M$ to information geometry is through the concept of normal bundle and normal cone, arising out of diagonal embedding $\Delta : M \hookrightarrow M \times M$; here $(M, M \times M)$ is both a "paired groupoid" (where $G = G_0 \times G_0$) and Lie groupoid (where both G_0 and G are manifolds). Previously, [7] studied information geometry using a general Lie groupoid approach. The present work studies information geometry using a tangent groupoid approach.

3 Divergence (Contrast) Function and Yoke

We revisit the construction of statistical structure on M, namely a Riemannian metric and a pair of torsionfree conjugate connections, from contrast function [4–6] or yoke [1].

Let $f(u_a, u_b)$ denote a divergence function (a.k.a. contrast function, yoke) in the sense of information geometry. Namely, for any two points $(\varphi^{-1}(u_a), \varphi^{-1}(u_b)) \in M \times M$ under a local coordinate system $\phi : M \to R^n$, and for any tangent vector $X \in T_{\varphi^{-1}}(u)M$, $f \in C^3(\phi(M) \times \phi(M))$ satisfies

i) $f(u, u) = 0$;
ii) $X_a f(u, u) = X_b f(u, u) = 0$;
iii) $-X_a X_b f(u, u) \equiv -X_b X_a f(u, u)$ is non-negative.

Notation: the subscript a, b of a tangent vector $X = X^i \frac{\partial}{\partial u^i}$ refers to the variable u_a, u_b of $f(u_a, u_b)$ that the differential is acting on; the superscript i, ranging from 1 to $\dim(M) = n$, refers to the vector component, which is often suppressed. Einstein summation convention is always observed with a pair of repeated sub/superscripts. For instance, for vector field X, Y, Z,

$$X_a f(u, u) := X^i \frac{\partial f(u_a, u)}{\partial u_a^i}\bigg|_{u_a = u}, \quad X_b f(u, u) := X^i \frac{\partial f(u, u_b)}{\partial u_b^i}\bigg|_{u_b = u},$$

$$(X_a)^2 f(u, u) = X^i X^j \frac{\partial^2 f(u_a, u)}{\partial u_a^i \partial u_a^j}\bigg|_{u_a = u}, \quad X_a Y_b Z_a f(u, u) = X^i Y^j Z^k \frac{\partial^3 f(u_a, u_b)}{\partial u_a^i \partial u_b^j \partial u_a^k}\bigg|_{u_a = u_b = u},$$

Here $u_a = u_b = u$ amounts to restricting to the diagonal $\Delta : M \hookrightarrow M \times M$.

More generally, we adopt the notation

$$T^{m,n}(\underbrace{X, \cdots, Y}_{m \text{ slots}}, \underbrace{Z, \cdots, W}_{n \text{ slots}})(u)$$

$$= X^{i_1} \cdots Y^{i_m} Z^{i_{m+1}} \cdots W^{i_{m+n}} \frac{\partial^{m+n} f(u_a, u_b)}{\partial u_a^{i_1} \cdots \partial u_a^{i_m} \partial u_b^{i_{m+1}} \cdots \partial u_b^{i_{m+n}}}\bigg|_{u_a = u_a = u}$$

that is, the m vectors starting from X till Y operate on the first variable while the n vectors starting from Z until the last one W operate on the second variable. So $T^{m,n}$ are $(m+n)$-forms, taking in $m+n$ tangent vectors to produce a function of u. Clearly, $T^{m,n}$ is symmetric in the first m slots as well as in the last n slots, so they are *symmetric forms*.

Using this notation, divergence function can be succinctly defined as

i) $T^0(u) = 0$;
ii) $T^{1,0}(X)(u) = T^{0,1}(X)(u) = 0$;
iii) $T^{1,1}(X, X)(u)$ is non-negative.

The following Lemma will be useful for the rest of the paper.

Lemma 1.

$$\begin{aligned} & V\big(T^{m,n}(\underbrace{X,\cdots,Y}_{m\ slots},\underbrace{Z,\cdots,W}_{n\ slots})(u)\big) \\ = \; & T^{m+1,n}(\underbrace{V,X,\cdots,Y}_{m+1\ slots},\underbrace{Z,\cdots,W}_{n\ slots})(u) + T^{m,n+1}(\underbrace{X,\cdots,Y}_{m\ slots},\underbrace{Z,\cdots,W,V}_{n+1\ slots})(u). \end{aligned}$$

The above formula reads that, fixing $m+n$ vector fields $\underbrace{X,\cdots,Y,Z,\cdots,W}_{m+n}$, differentiating $T^{m,n}(u)$ as a function of u by tangent vector V yields two terms, one term $T^{m+1,n}(u)$ plus another term $T^{m,n+1}(u)$. Both are $(m+n+1)$-forms, taking in vector field V along with $(m+n)$ vector fields $X,\cdots,Y,Z,\cdots,W$ to lead to a function of u.

3.1 Second-Order Relation

Differentiating property ii) of a divergence function yields:

$$Y_aX_af(u,u) + Y_bX_af(u,u) = Y_aX_bf(u,u) + Y_bX_bf(u,u) = 0, \quad (2)$$
$$X_aY_af(u,u) + X_bY_af(u,u) = X_aY_bf(u,u) + X_bY_bf(u,u) = 0. \quad (3)$$

In terms of the bilinear forms of X, Y:

$$\begin{aligned} T^{2,0}(X,Y)(u) &:= X_aY_af(u,u), \\ T^{1,1}(X,Y)(u) &:= X_aY_bf(u,u), \\ T^{0,2}(X,Y)(u) &:= X_bY_bf(u,u), \end{aligned}$$

the identities (2) and (3) further lead to

$$T^{2,0}(X,Y) = T^{0,2}(X,Y) = -T^{1,1}(X,Y) = -T^{1,1}(Y,X) =: g_{XY}(u). \quad (4)$$

Here we use $g_{XY}(u)$ to express the above four quantities T obtained by differentiating f using the two vector fields X, Y, with $g_{XY}(u) = g_{YX}(u)$. We remark that $g_{XY}(u)$ is nothing but the Riemannian metric $\langle , \rangle$ since, writing $X = X^ie_i$ and $Y = Y^je_j$ with $e_i = \frac{\partial}{\partial u_i}$, then

$$\langle X,Y\rangle_u = X^iY^j\langle e_i,e_j\rangle_u = X^iY^jg_{ij}(u) = g_{XY}(u)$$

where $\langle e_i, e_j\rangle = g_{ij}(u)$ denotes the basis $e_i \otimes e_j$ of the metric tensor $g_{XY}(u)$.

3.2 Third-Order Relation

Introduce tri-linear (homogeneous) functions in X, Y, Z

$$\begin{aligned} T^{3,0}(X,Y,Z)(u) &:= X_a Y_a Z_a f(u,u), \\ T^{2,1}(X,Y,Z)(u) &:= X_a Y_a Z_b f(u,u), \\ T^{1,2}(X,Y,Z)(u) &:= X_a Y_b Z_b f(u,u), \\ T^{0,3}(X,Y,Z)(u) &:= X_b Y_b Z_b f(u,u). \end{aligned}$$

Here, both $T^{3,0}$ and $T^{0,3}$ are symmetric in all three variables X, Y, Z, while

$$T^{2,1}(X,Y,Z) = T^{2,1}(Y,X,Z), \quad T^{1,2}(X,Y,Z) = T^{1,2}(X,Z,Y).$$

The trilinear functions arise upon differentiating, by the tangent vector Z, the four expressions in (4) representing $g_{XY}(u) = \langle X, Y \rangle_u$, that is,

$$Z(T^{2,0}(X,Y)) = T^{3,0}(X,Y,Z) + T^{2,1}(X,Y,Z), \tag{5}$$

$$Z(T^{1,1}(X,Y)) = T^{2,1}(Z,X,Y) + T^{1,2}(X,Y,Z), \tag{6}$$

$$Z(T^{0,2}(X,Y)) = T^{0,3}(X,Y,Z) + T^{1,2}(Z,X,Y). \tag{7}$$

Because the left-hand-side of the first and last equations are both equal to g_{XY},

$$T^{1,2}(Z,X,Y) - T^{2,1}(X,Y,Z) = T^{3,0}(X,Y,Z) - T^{0,3}(X,Y,Z) \tag{8}$$

$$=: C(X,Y,Z), \tag{9}$$

so the difference on the left-hand side must be symmetric in X, Y, Z. Hence

$$T^{1,2}(Z,X,Y) - T^{2,1}(X,Y,Z) = T^{1,2}(X,Z,Y) - T^{2,1}(Z,Y,X) = T^{1,2}(Y,X,Z) - T^{2,1}(X,Z,Y).$$

The two quantities $T^{2,1}$ and $T^{1,2}$ deserve particular attention—they can be identified with a pair of affine connections. Since

$$\begin{aligned} T^{LC}(X,Y,Z) := &\ \frac{1}{2}\left(X(T^{1,1}(Z,Y)) + Y(T^{1,1}(X,Z)) - Z(T^{1,1}(X,Y))\right) \\ = &\ \frac{1}{2}\big(T^{2,1}(X,Z,Y) + T^{1,2}(Z,Y,X) + T^{2,1}(Y,X,Z) + T^{1,2}(X,Z,Y) \\ &\ -T^{2,1}(Z,X,Y) - T^{1,2}(X,Y,Z)\big) \\ = &\ \frac{1}{2}\left(T^{1,2}(Z,X,Y) + T^{2,1}(X,Y,Z)\right), \end{aligned} \tag{10}$$

solving for $T^{1,2}$ and $T^{2,1}$ from (9) and (10) leads to

$$T^{1,2}(Z,X,Y) = T^{LC}(X,Y,Z) + \frac{1}{2}C(X,Y,Z),$$

$$T^{2,1}(X,Y,Z) = T^{LC}(X,Y,Z) - \frac{1}{2}C(X,Y,Z).$$

Here T^{LC} is the (covariant form of the) Levi-Civita connection, which is related to the contravariant form ∇^{LC} by

$$T^{LC}(X, Y, Z) = \langle \nabla_X^{LC} Y, Z \rangle.$$

Since $C(X, Y, Z)$ is homogeneous of degree 3, we can define, for vector field X, K_X to be an endomorphism of sections of TM that satisfy

$$C(X, Y, Z) = \langle K_X Y, Z \rangle.$$

Total symmetry of C requires K operator to satisfy (i) symmetry $\langle K_X Y, Z \rangle = \langle Y, K_X Z \rangle$ and (ii) torsion-freeness $K_X Y = K_Y X$. Then $\nabla^1 = \nabla^{\text{LC}} + K$ and $\nabla^2 = \nabla^{\text{LC}} - K$ are two torsion-free connections that satisfy

$$Z\langle X, Y \rangle = \langle \nabla_Z^1 X, Y \rangle + \langle X, \nabla_Z^2 Y \rangle, \ \ \frac{1}{2}(\nabla^1 + \nabla^2) = \nabla^{LC}$$

which is a re-write of (6) and (10); they are conjugate with respect to $\langle , \rangle$.

4 Smooth Extensions of Divergence Function on $M \times M$ to Functions on TM

This Section will apply tools of tangent groupoid to extend the lowest order approximations to the divergence function, and its anti-symmetric version, to functions on TM. Those functions extended to TM lead to construction of the Riemannian metric and C-tensor associated to the underlying manifold M.

4.1 Second-Order Extension: Constructing Riemannian Metric

Let $P_2(X)$ be a quadratic function of X defined by

$$P_2(X)(u) := g_{XX}(u) = (X_a)^2 f(u, u) = (X_b)^2 f(u, u) = -X_a X_b f(u, u).$$

The identity

$$XY = \frac{1}{2}\big((X + Y)^2 - X^2 - Y^2\big)$$

leads to an expression for g_{XY} defined by (4)

$$g_{XY}(u) = \frac{1}{2}\big(P_2(X + Y)(u) - P_2(X)(u) - P_2(Y)(u)\big).$$

Proposition 1. *For any divergence function f, the assignment*

$$h_0 : (\varphi^{-1}(u_a), \varphi^{-1}(u_b), t) \mapsto t^{-2} f(u_a, u_b)$$

defines a smooth function on $M \times M \times (0, 1]$ and this function h_0 extends smoothly to whole tangent groupoid $\mathbb{T}M$. Moreover, denote the extension by capital letter $H_0 \in C^\infty(\mathbb{T}M)$. Then

$$H_0(\varphi^{-1}(u), X, 0) = \frac{1}{2} X_b^2 f(u, u) = \frac{1}{2} P_2(X)(u)$$

for all $\varphi^{-1}(u) \in M$ and $X = x^i \frac{\partial}{\partial u^i} \in T_{\varphi^{-1}(u)} M$.

Proof. Due to the smooth structure (1) of the tangent groupoid $\mathbb{T}M$, we have

$$h_0 \circ \Phi^{-1}(u, x, t) = t^{-2} f\,(u, u + tx)\,.$$

According to L'Hopoital rule, the above equation has limit, as $t \to 0$:

$$\frac{1}{2} X_b^2 f(u, u).$$

Notice that here $X = x^i \frac{\partial}{\partial u^i}$ is a tangent vector of M at $\varphi^{-1}(u) \in M$ and the differential is taken with respect to the second variable of $f(\cdot, \cdot)$ as the subscript b indexes. Therefore, the value of H_0 at $(\varphi^{-1}(u), X) \in TM$ is given by $H_0(\varphi^{-1}(u), X, 0) = \frac{1}{2}(X_b)^2 f(u, u)$. This, by definition, is $\frac{1}{2} P_2(X)(u)$.

The following Proposition links $\langle \cdot, \cdot \rangle$ to the H_0 function on TM.

Proposition 2. *The Riemannian metric $\langle \cdot, \cdot \rangle$ of M is given by*

$$\langle X, Y \rangle = H_0(u, X + Y, 0) - H_0(u, X, 0) - H_0(u, Y, 0).$$

Proof. Due to the assumption iii) of a divergence function f, we can set the Riemannian metric on M to be given by $(X, Y) \mapsto X_a Y_b f(u, u)$, or equivalently $X_b Y_a f(u, u)$:

$$\langle X, Y \rangle_u = -X_a Y_b f(u, u) = X_a Y_a f(u, u).$$

Since

$$\begin{aligned} X_a Y_b f(u, u) &= \frac{1}{2} \big((X_a + Y_b)^2 f(u, u) - (X_a)^2 f(u, u) - (Y_b)^2 f(u, u) \big) \\ &= \frac{1}{2} \big(P_2(X + Y)(u) - P_2(X)(u) - P_2(Y)(u) \big), \end{aligned}$$

an application of Proposition 1 yields the desired relation.

So from the perspective of tangent groupoid, the Riemannian metric is derived from the H_0 function on TM, which is a smooth extension of h_0, obtained from the divergence function $f(u_a, u_b)$.

4.2 Third-Order Extension: Constructing *C*-Tensor

Let $P_3(X)$ be a cubic function of X defined by

$$P_3(X)(u) := C(X, X, X)(u) = (X_a)^3 f(u, u) - (X_b)^3 f(u, u).$$

The identity

$$\begin{aligned} C(X, Y, Z) = \frac{1}{6} \Big(& P_3(X + Y + Z) \\ & - P_3(X + Y) - P_3(X + Z) - P_3(Y + Z) + P_3(X) + P_3(Y) + P_3(Z) \Big) \end{aligned}$$

leads to the homogeneous trilinear function $C(X, Y, Z)$ expressible by $P_3(\cdot)$. And so will $T^{3,0}(X, Y, Z)$ and $T^{0,3}(X, Y, Z)$. Next we show how P_3 can be computed from the divergence function f.

Proposition 3. *For any divergence function f, the assignment*

$$h_1 : (\varphi^{-1}(u_a), \varphi^{-1}(u_b), t) \mapsto t^{-3} \left(f(u_a, u_b) - f(u_b, u_a) \right)$$

defines a smooth function on $M \times M \times (0, 1]$ and this function h_1 extends smoothly to whole tangent groupoid $\mathbb{T}M$. Moreover, denote the extension by capital letter $H_1 \in C^\infty(\mathbb{T}M)$. Then

$$H_1(\varphi^{-1}(u), X, 0) = \frac{1}{6} \left((X_a)^3 f(u, u) - (X_b)^3 f(u, u) \right) = \frac{1}{6} P_3(X)(u)$$

for all $\varphi^{-1}(u) \in M$ and $X = x^i \frac{\partial}{\partial u^i} \in T_{\varphi^{-1}(u)}M$.

Proof. Due to the smooth structure (1) of the tangent groupoid $\mathbb{T}M$, we have

$$h_1 \circ \Phi^{-1}(u, x, t) = t^{-3} \left(f(u, u + tx) - f\left(u + tx, u\right) \right).$$

According to L'Hopital rule, the above equation has limit, as $t \to 0$

$$\frac{1}{6} \left((X_a)^3 f(u, u) - (X_b)^3 f(u, u) \right),$$

where $X = x^i \frac{\partial}{\partial u^i}$ is a tangent vector at $T_{\varphi^{-1}(u)}M$ and the subscript a, b denotes that the differential is acting, respectively, on the 1st and 2nd variable of $f(\cdot, \cdot)$.

So from the perspective of tangent groupoid, the C-tensor can be written in terms of H_1 function on TM, which itself is a smooth extension of h_1 obtained from the anti-symmetrized divergence function $f(u_a, u_b) - f(u_b, u_a)$.

5 Conclusion

We constructed a natural extension of the second-order approximation of a divergence (contrast) function on $M \times M$ to a quadratic function on TM, capturing the Riemannian metric g of M (Proposition 1), and of the anti-symmetrized divergence (contrast) function on M to a trilinear function on TM, capturing the C-tensor (Proposition 3) that equivalently expresses a pair of conjugate connections. Note that this construction of statistical structure on M slightly differs from the standard procedure [1,4–6] in that only one vector field (and hence a section of TM) is used, rather than 2 or 3 vector fields for constructing, respectively, the Riemannian metric g and C-tensor—the latter are viewed as symmetrized bilinear and trilinear functions on TM.

Our paper is related to an earlier work by [7], which extended the notion of divergence (contrast) function on $M \times M$ to that on a general Lie groupoid (M, G), and established the analogous relationship for two-forms and three-forms on the corresponding Lie algebroid $\mathfrak{A}_G \to M$. Recall that adiabatic groupoid [3] of G is the groupoid obtained by the deformation-to-normal-cone construction corresponding to the inclusion $M \subset G$; it represents a smooth deformation of the Lie groupoid (M, G) into its Lie algebroid $\mathfrak{A}_G$. When the Lie groupoid (M, G) is taken to be the pair groupoid $(M, M \times M)$, that is, when taking $G = M \times M$ in [7], the corresponding adiabatic groupoid is precisely the tangent groupoid $\mathbb{T}M$ we use in this paper, with the classical result of information geometry recovered. Viewed in this way, the tangent groupoid method proposed here is an alternative way to the Lie groupoid method proposed by [7], both recovering the statistical structure from the contrast function. How to extend our technique beyond tangent groupiod to general adiabatic groupoid in order to recover the construction of [7] and how to generalize our methods to further capture the curvatures of the pair of conjugate connections remain open questions for future research.

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