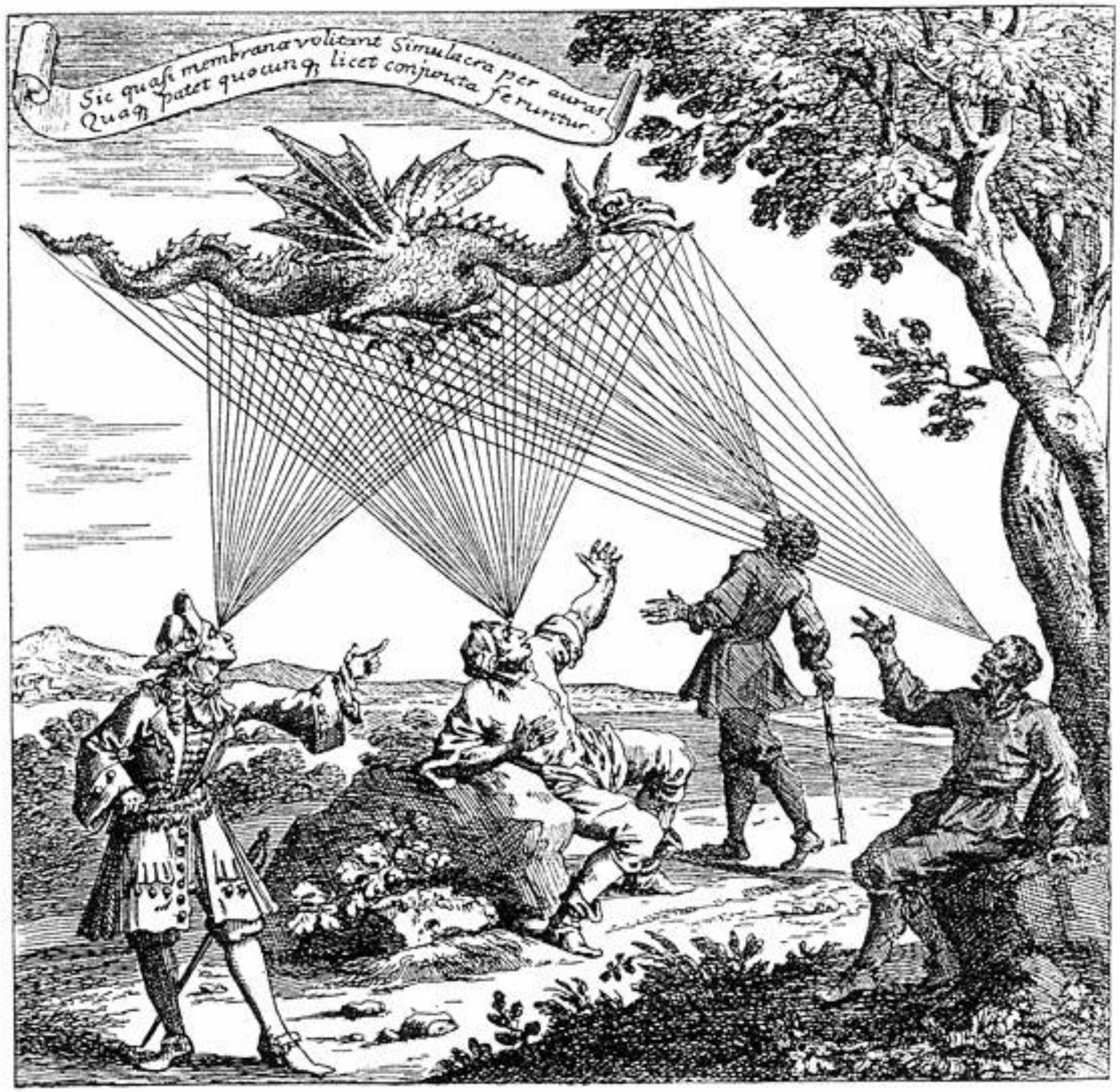
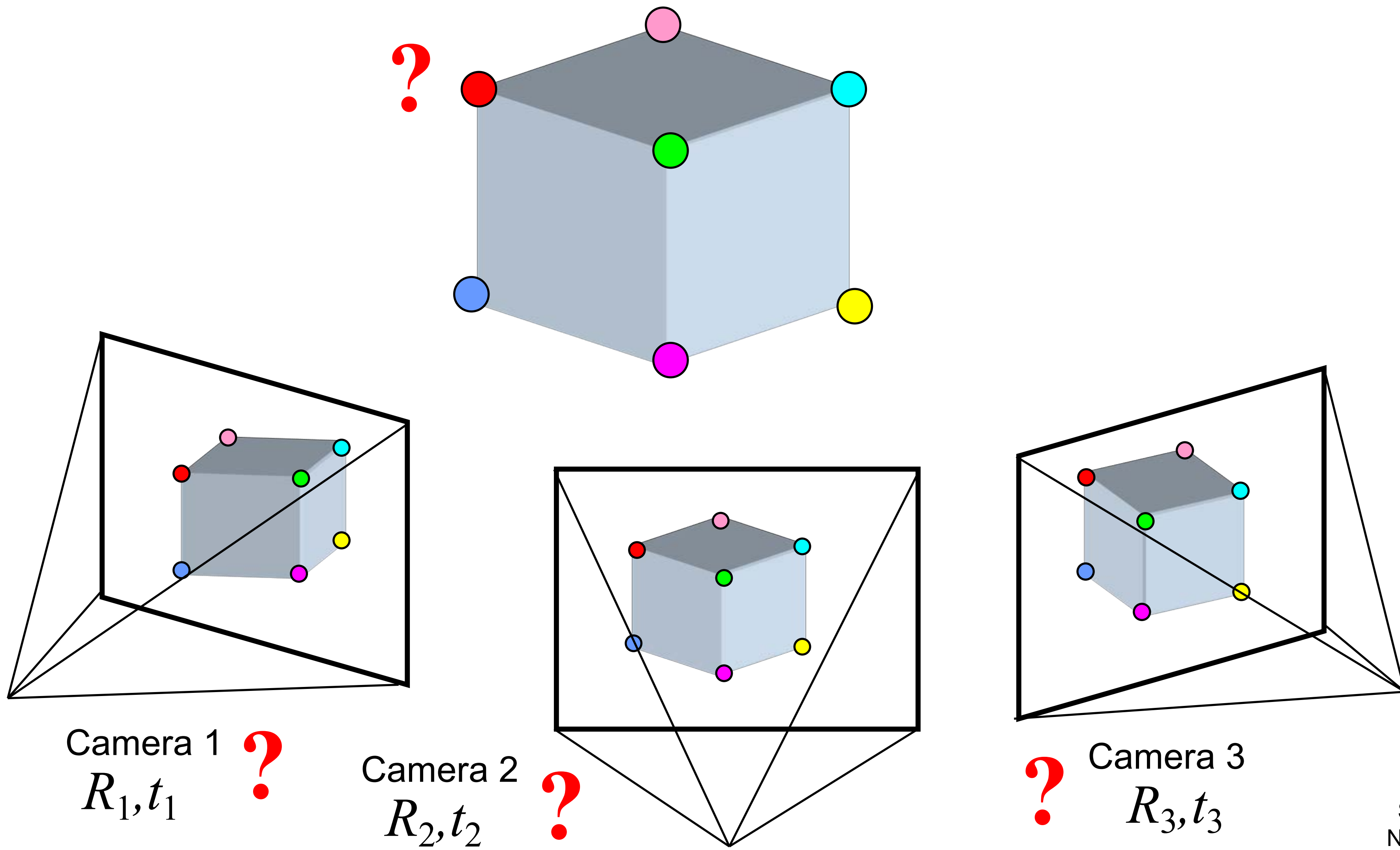
Structure from motion



Драконъ, видимый подъ различными углами зрѣнія
По гравюрѣ на мѣди изъ „Oculus artificialis teledioptricus" Цана. 1702 года.

Structure from motion

- Given a set of corresponding points in two or more images, compute the camera parameters and the 3D point coordinates



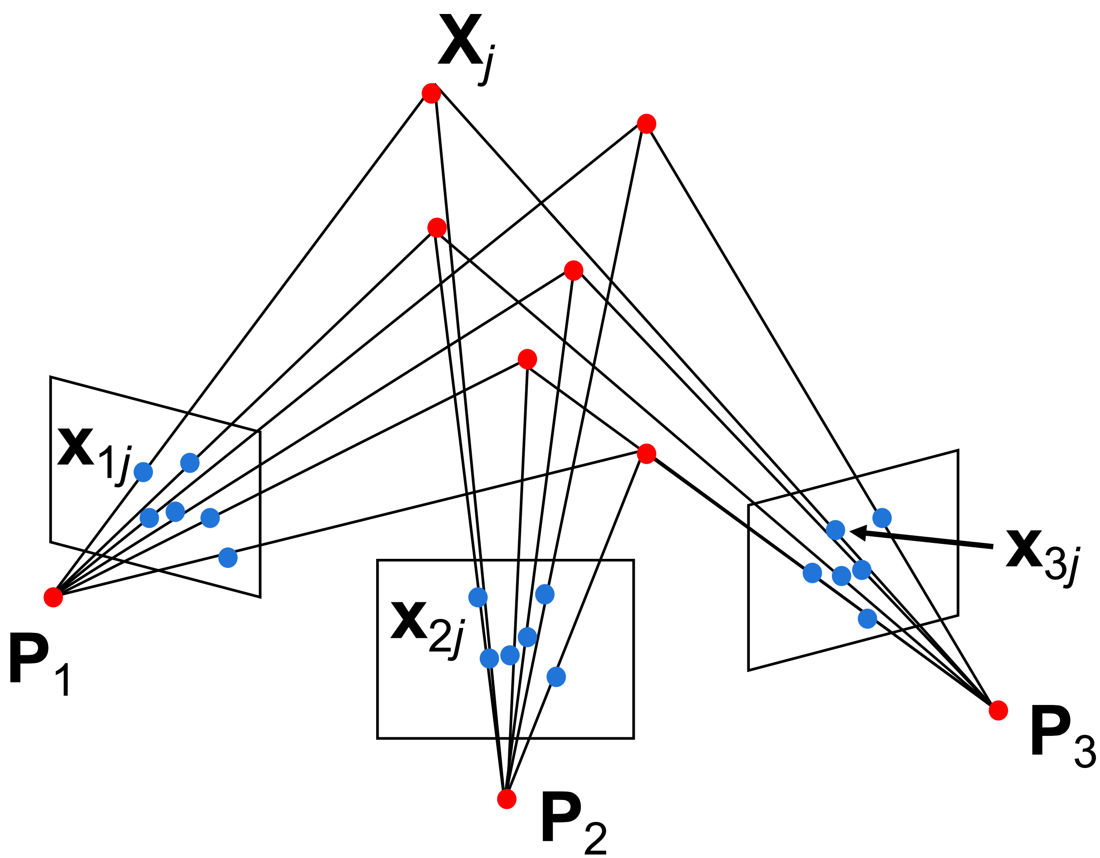
Slide credit: Noah Snavely

Structure from motion

- Given: *m* images of *n* fixed 3D points

$$\lambda_{ij}\mathbf{x}_{ij} = \mathbf{P}_i \mathbf{X}_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

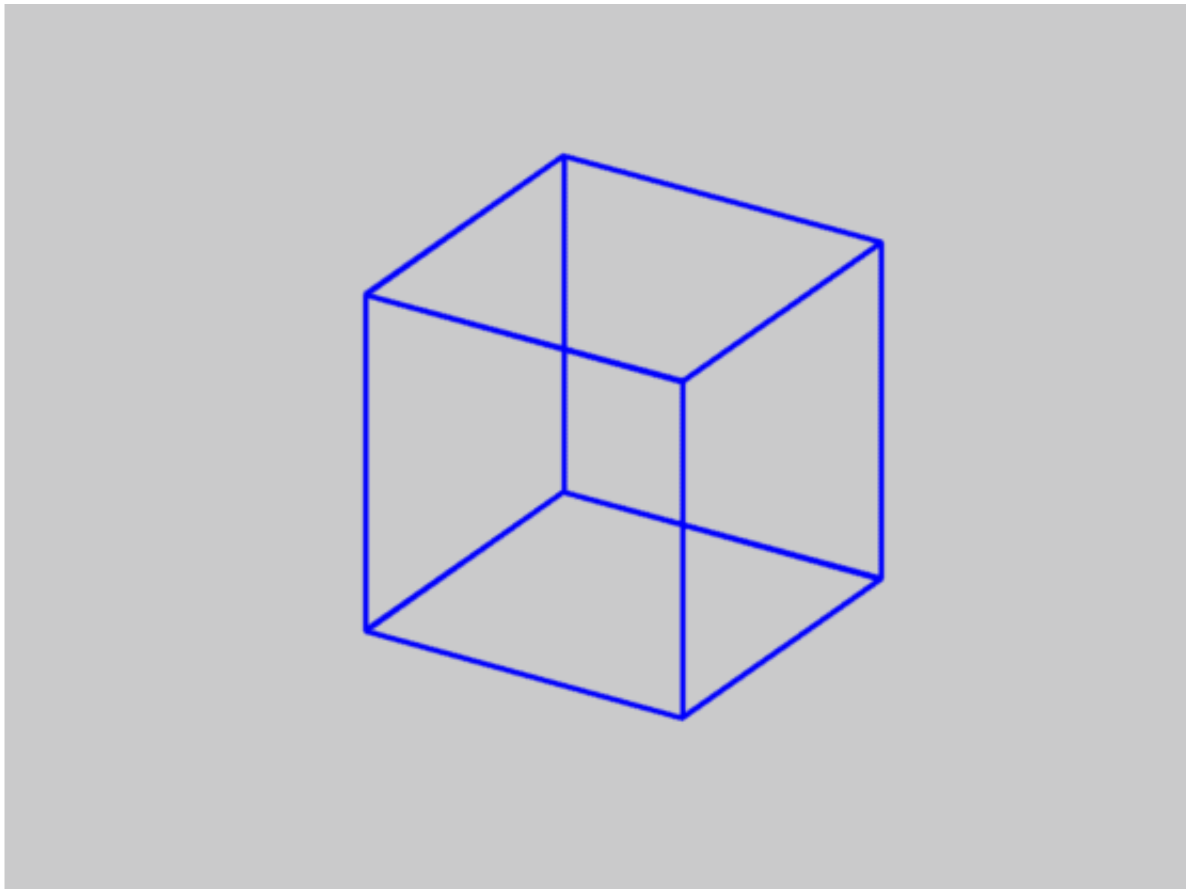
- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* correspondences $\mathbf{x}_{ij}$



Outline

- Reconstruction ambiguities
- Affine structure from motion
  - Factorization
- Projective structure from motion
  - Bundle adjustment
  - Modern structure from motion pipeline

Is SFM always uniquely solvable?

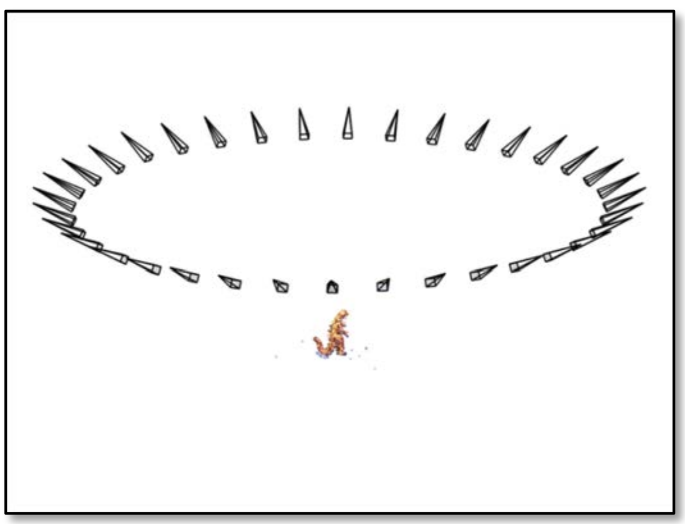


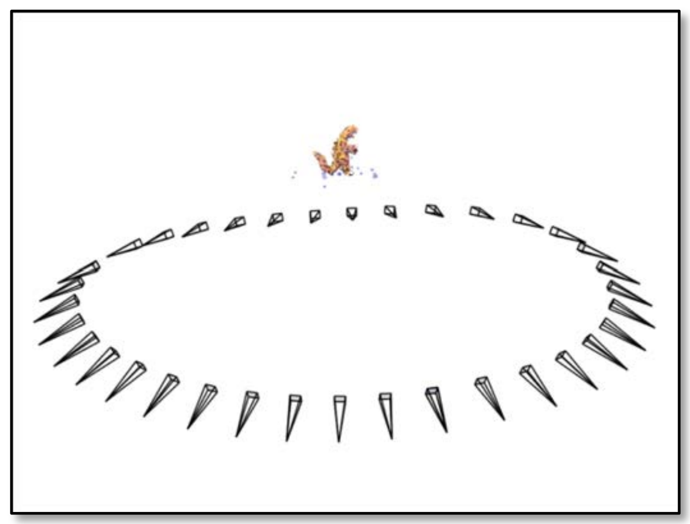
[Necker cube]

Source: N. Snavely

Is SFM always uniquely solvable?

- Necker reversal





Source: N. Snavely

Structure from motion ambiguity

- If we scale the entire scene by some factor *k* and, at the same time, scale the camera matrices by the factor of 1/*k*, the projections of the scene points in the image remain exactly the same:

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\frac{1}{k}\mathbf{P}\right)(k\,\mathbf{X})$$

It is impossible to recover the absolute scale of the scene!

Structure from motion ambiguity

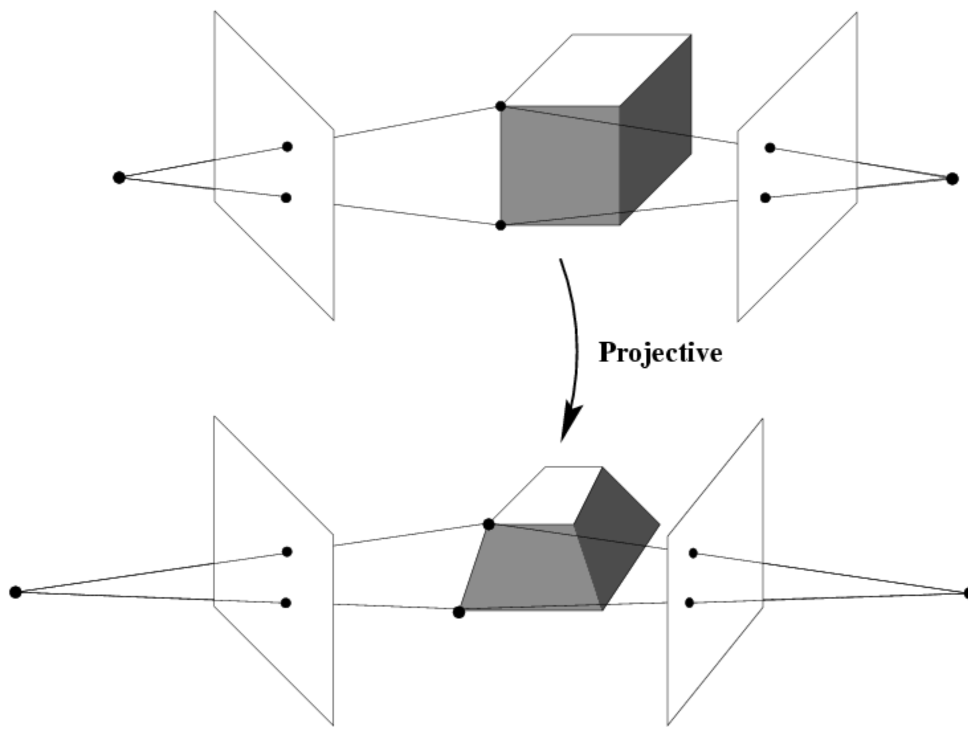
- If we scale the entire scene by some factor *k* and, at the same time, scale the camera matrices by the factor of 1/*k*, the projections of the scene points in the image remain exactly the same

- More generally, if we transform the scene using a transformation **Q** and apply the inverse transformation to the camera matrices, then the images do not change:

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}^{-1}\right)\left(\mathbf{Q}\mathbf{X}\right)$$

Projective ambiguity

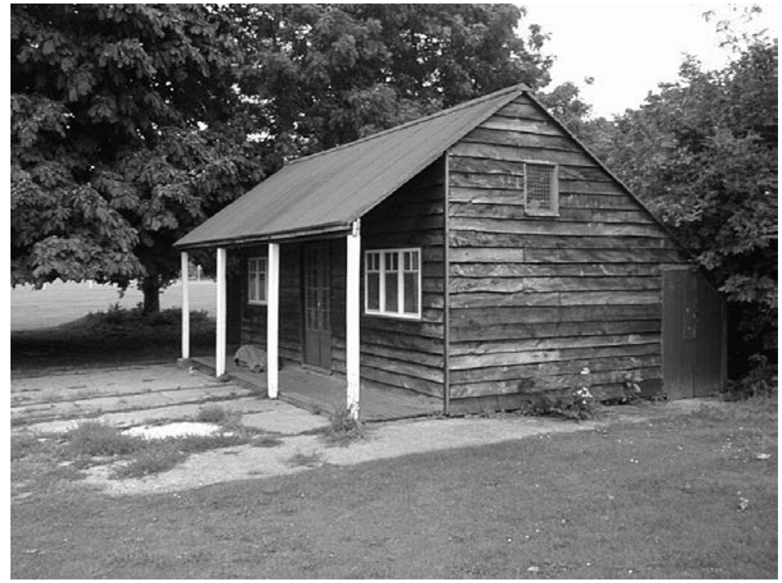
- With no constraints on the camera calibration matrix or on the scene, we can reconstruct up to a *projective* ambiguity

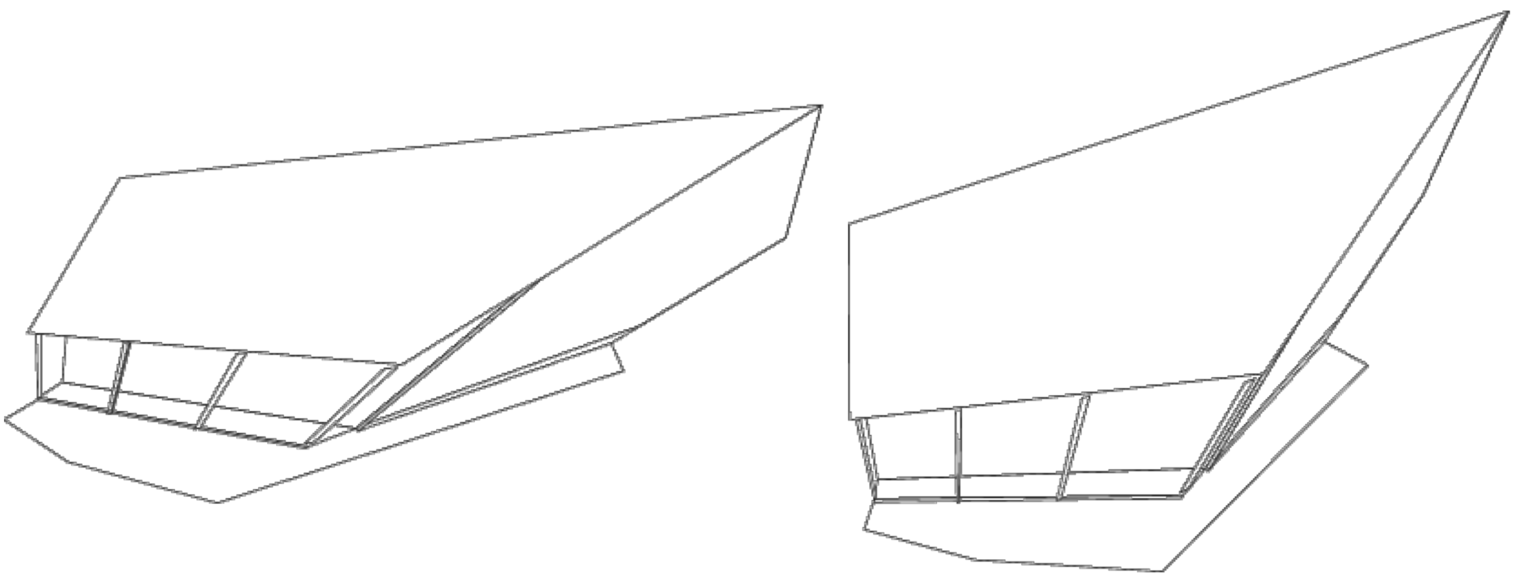


$$\mathbf{Q_p} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix}$$

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}_{\mathbf{P}}^{-1}\right)\left(\mathbf{Q}_{\mathbf{P}}\,\mathbf{X}\right)$$

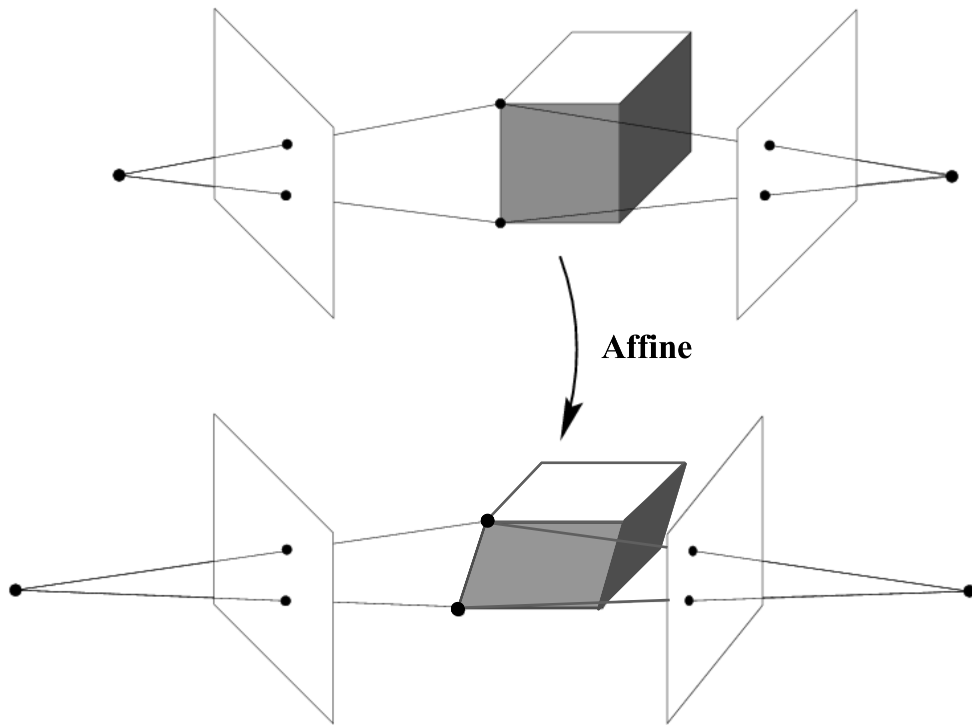
Projective ambiguity





Affine ambiguity

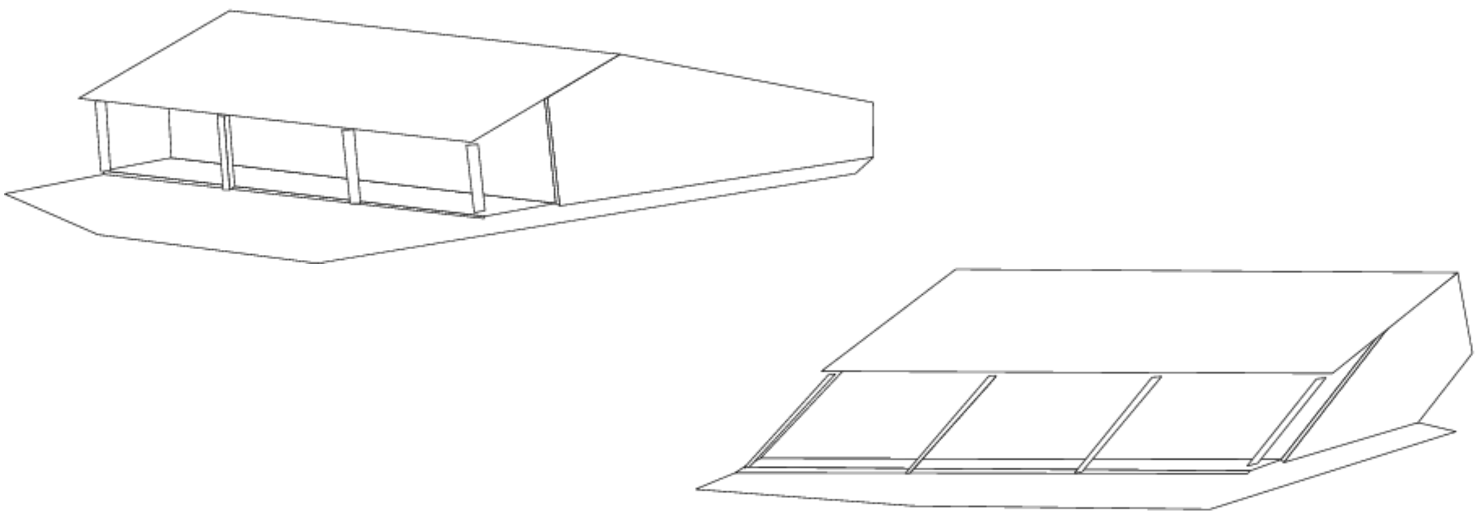
- If we impose parallelism constraints, we can get a reconstruction up to an *affine* ambiguity



$$\mathbf{Q_A} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$$

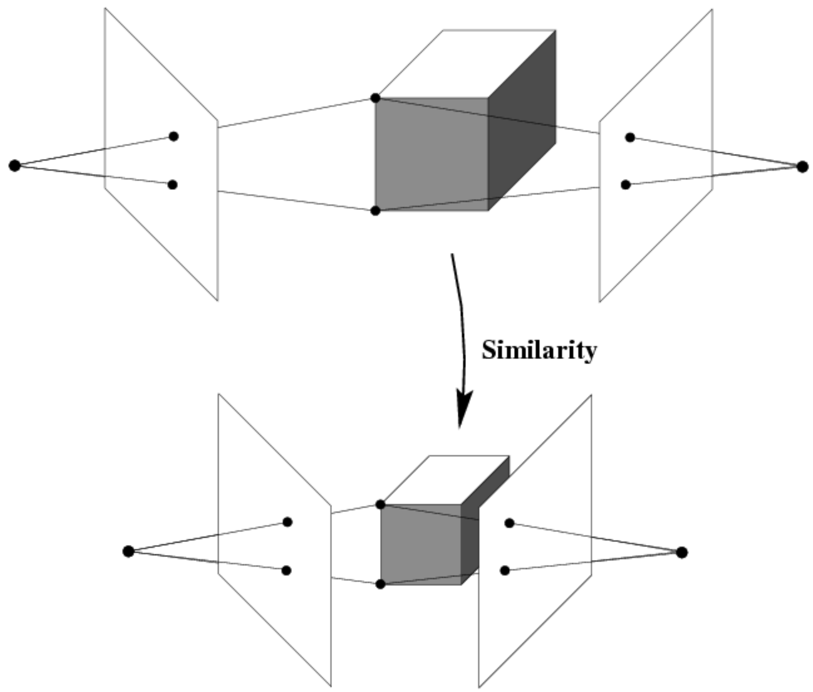
$$\mathbf{x} = \mathbf{PX} = \left(\mathbf{PQ_A^{-1}}\right)\left(\mathbf{Q_A X}\right)$$

Affine ambiguity



Similarity ambiguity

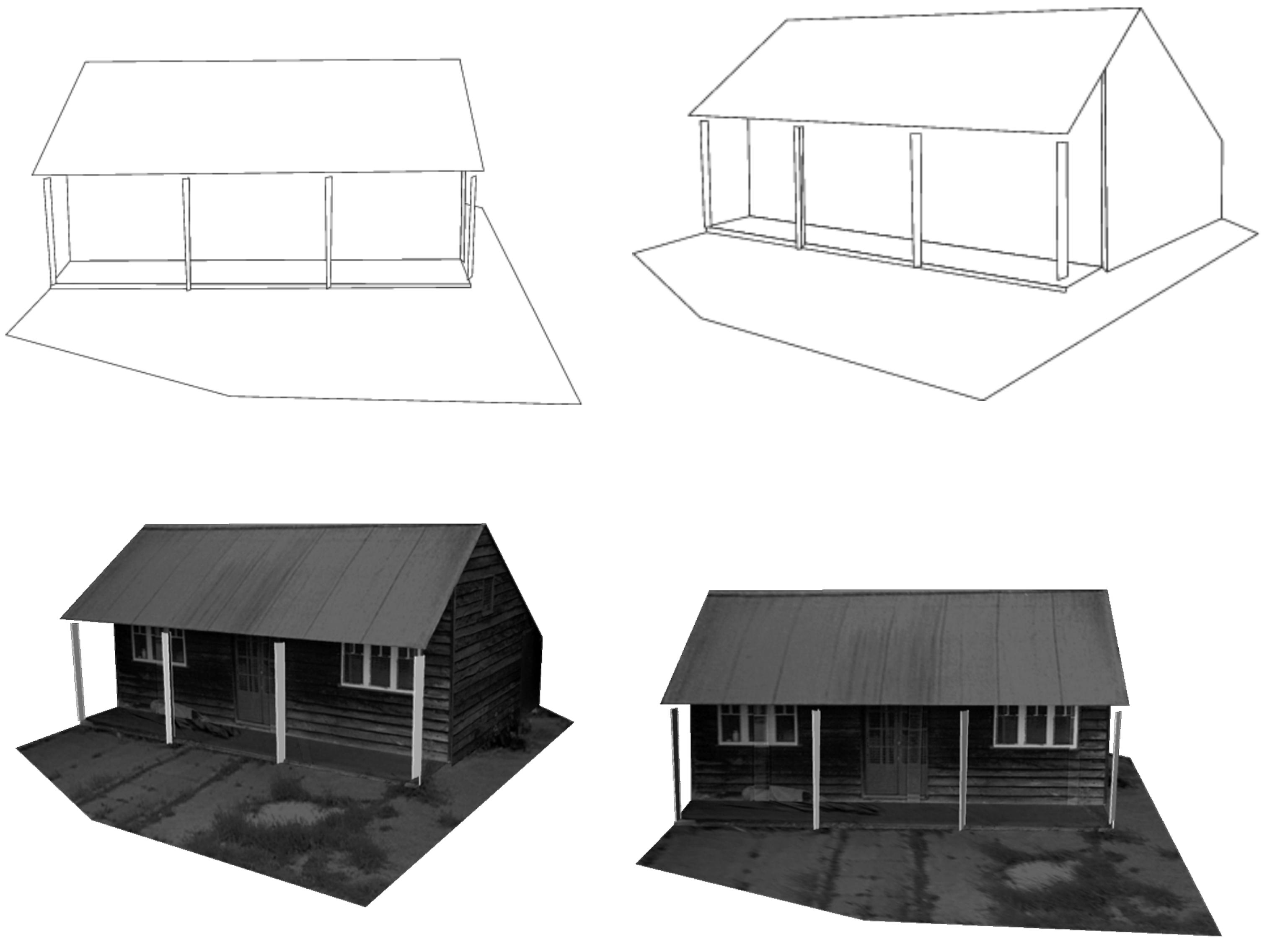
- A reconstruction that obeys orthogonality constraints on camera parameters and/or scene



$$\mathbf{Q}_s = \begin{bmatrix} s\mathrm{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$$

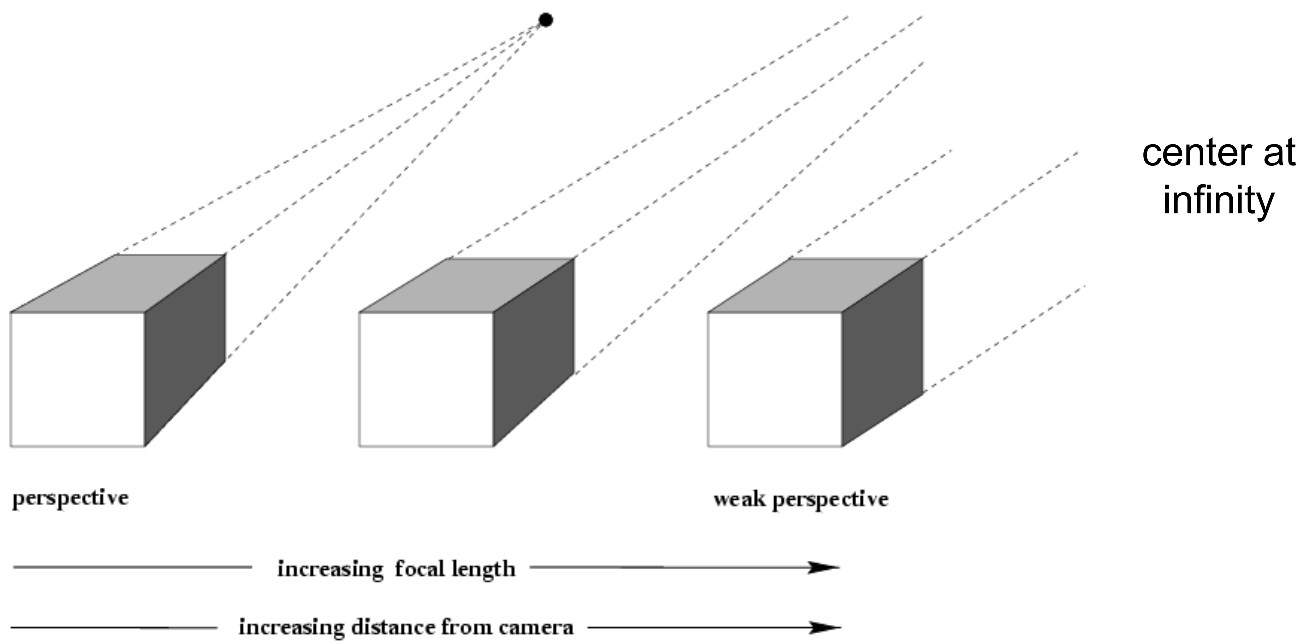
$$\mathbf{x} = \mathbf{PX} = \left(\mathbf{PQ}_S^{-1}\right)\left(\mathbf{Q}_S\mathbf{X}\right)$$

Similarity ambiguity

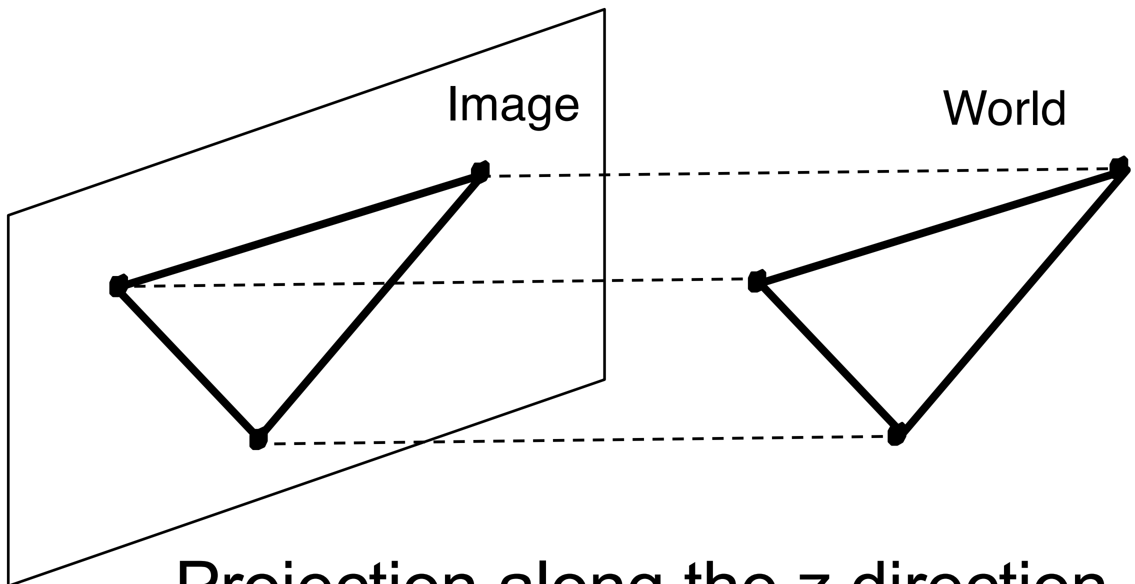


Affine structure from motion

- Let's start with *affine* or *weak perspective* cameras (the math is easier)



Recall: Orthographic Projection

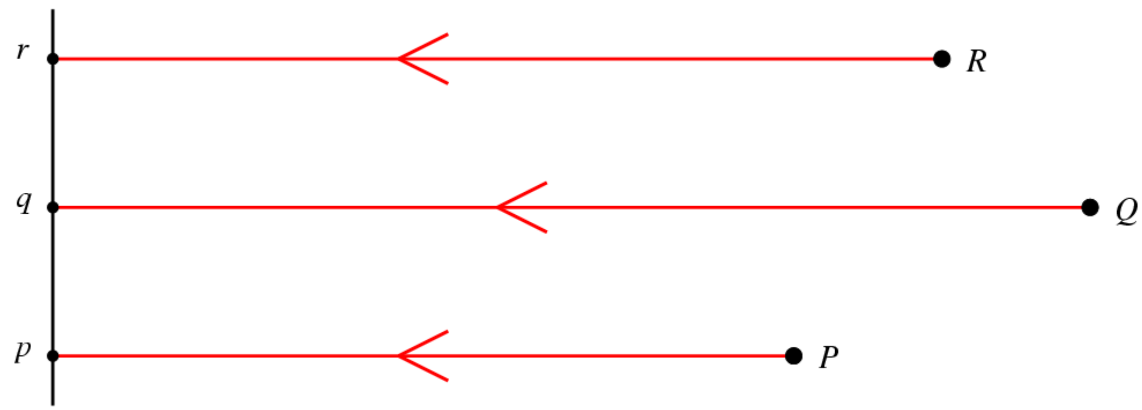


Projection along the z direction

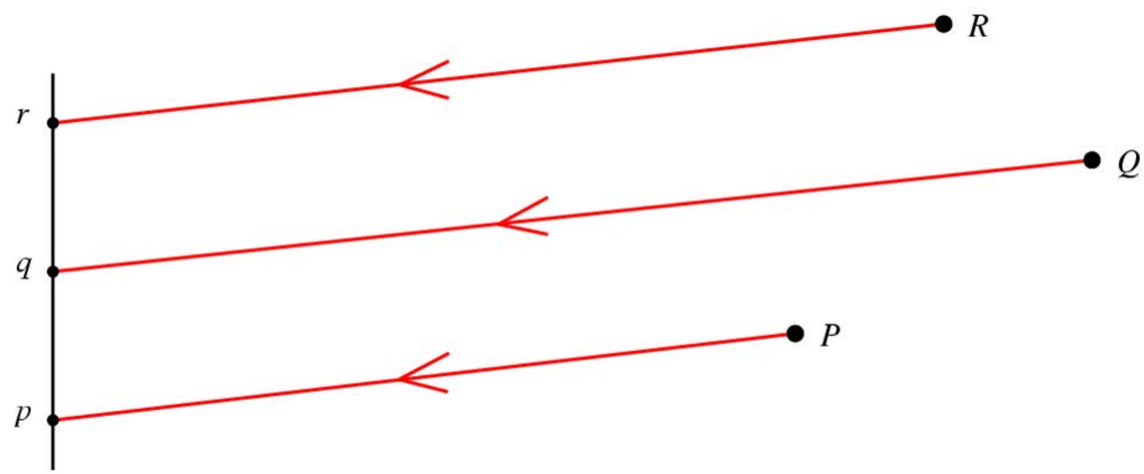
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

Affine cameras

Orthographic Projection



Parallel Projection

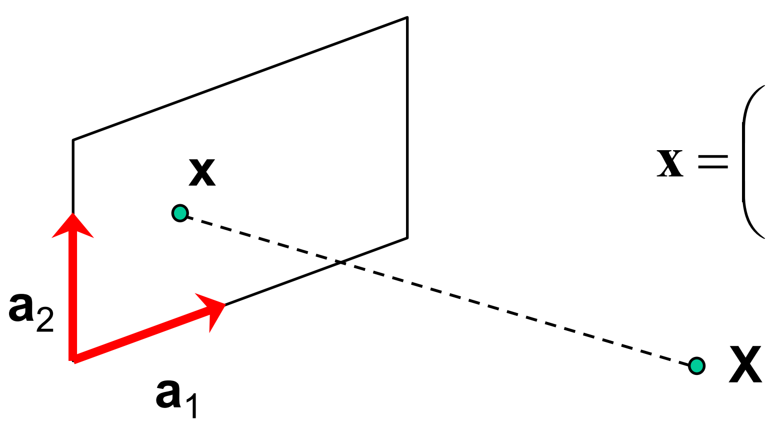


Affine cameras

- A general affine camera combines the effects of an affine transformation of the 3D space, orthographic projection, and an affine transformation of the image:

$$\mathbf{P} = [3\times 3\,\text{affine}]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}[4\times 4\,\text{affine}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Affine projection is a linear mapping + translation in non-homogeneous coordinates



$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{A}\mathbf{X} + \mathbf{b}$$

Projection of world origin

Affine structure from motion

- Given: *m* images of *n* fixed 3D points:

$$\mathbf{x}_{ij} = \mathbf{A}_i \mathbf{X}_j + \mathbf{b}_i, \qquad i = 1, \ldots, m, \;\; j = 1, \ldots, n$$

- Problem: use the *mn* correspondences $\mathbf{x}_{ij}$ to estimate *m* projection matrices $\mathbf{A}_i$ and translation vectors $\mathbf{b}_i$, and *n* points $\mathbf{X}_j$
- The reconstruction is defined up to an arbitrary *affine* transformation **Q** (12 degrees of freedom):

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{Q}^{-1}, \qquad \begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix} \rightarrow \mathbf{Q} \begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix}$$

- We have 2*mn* knowns and 8*m* + 3*n* unknowns (minus 12 dof for affine ambiguity)
- Thus, we must have 2*mn* >= 8*m* + 3*n* – 12
- For two views, we need four point correspondences

Affine structure from motion

- Centering: subtract the centroid of the image points in each view

$$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_j + \mathbf{b}_i - \frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{A}_i\mathbf{X}_k + \mathbf{b}_i\right)$$

$$= \mathbf{A}_i\left(\mathbf{X}_j - \frac{1}{n}\sum_{k=1}^{n}\mathbf{X}_k\right) = \mathbf{A}_i\hat{\mathbf{X}}_j$$

- For simplicity, set the origin of the world coordinate system to the centroid of the 3D points
- After centering, each normalized 2D point is related to the 3D point $\mathbf{X}_j$ by

$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i\mathbf{X}_j$$

Affine structure from motion

- Let's create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix}$$

cameras ($2m$)

points (n)

C. Tomasi and T. Kanade. Shape and motion from image streams under orthography: A factorization method. *IJCV*, 9(2):137-154, November 1992.

Affine structure from motion

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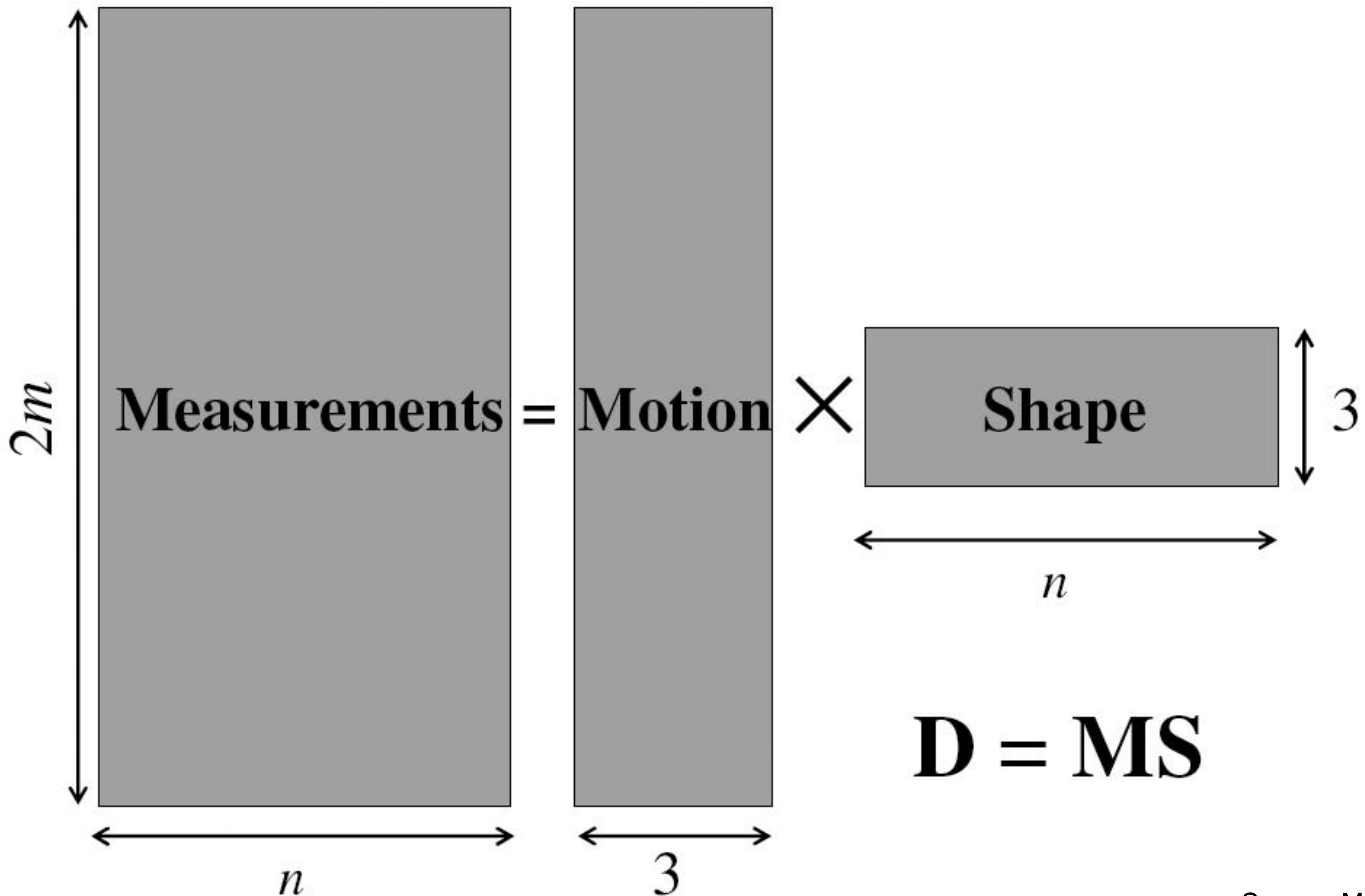
The measurement matrix $\mathbf{D} = \mathbf{MS}$ must have rank 3!

C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.]() *IJCV*, 9(2):137-154, November 1992.

Factorizing the measurement matrix

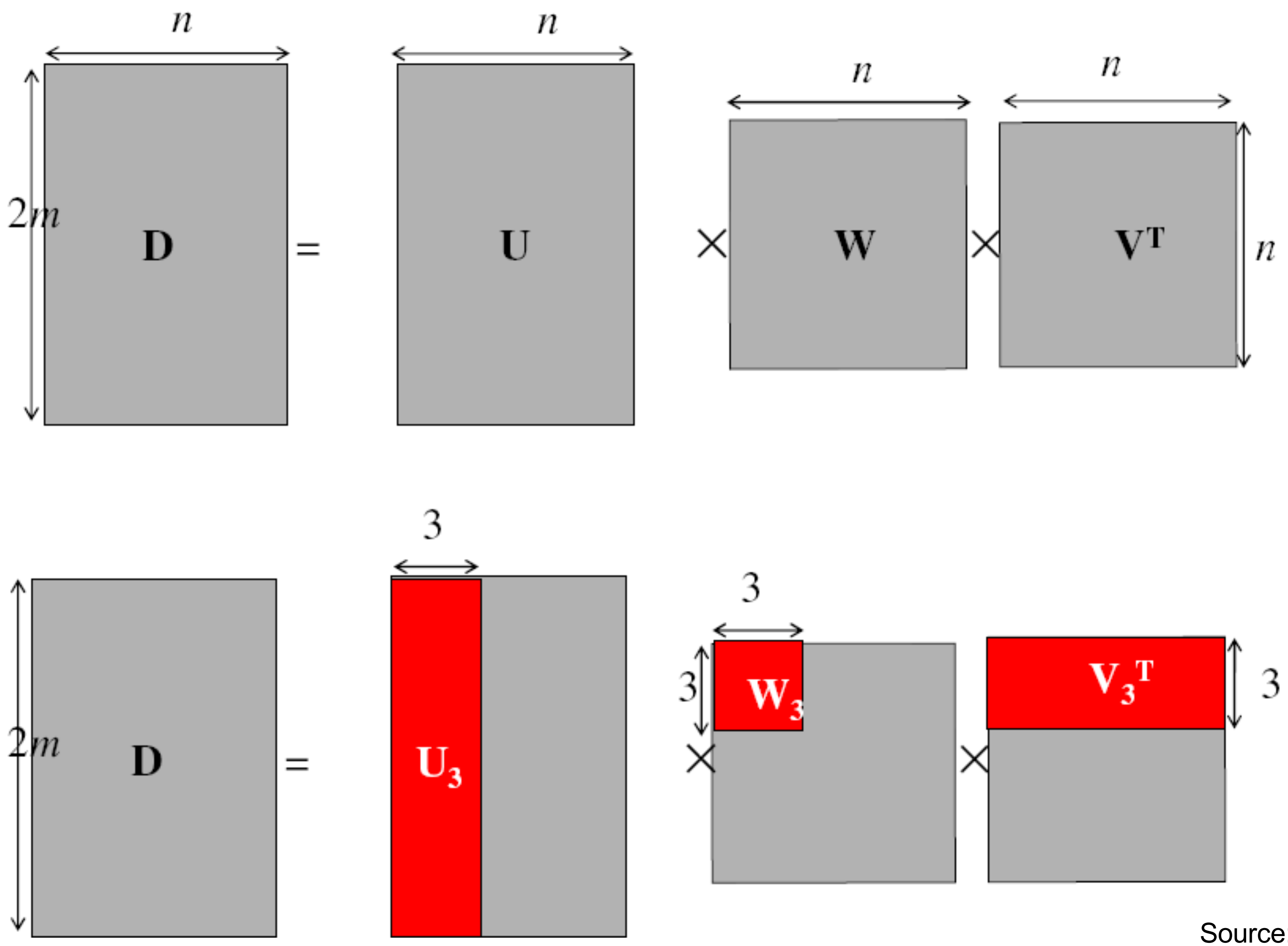


$$\mathbf{D} = \mathbf{MS}$$

Source: M. Hebert

Factorizing the measurement matrix

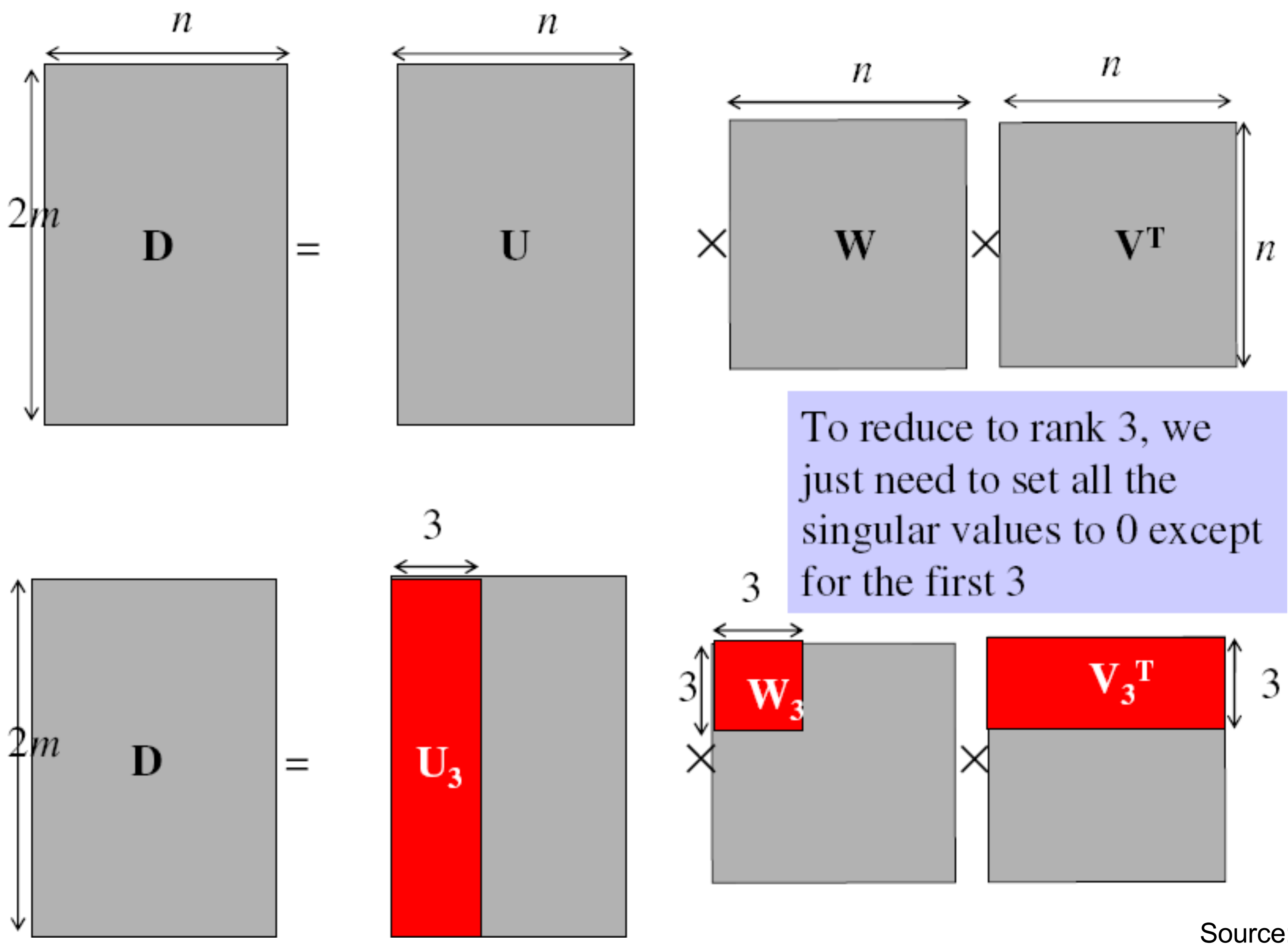
- Singular value decomposition of D:



Source: M. Hebert

Factorizing the measurement matrix

- Singular value decomposition of D:

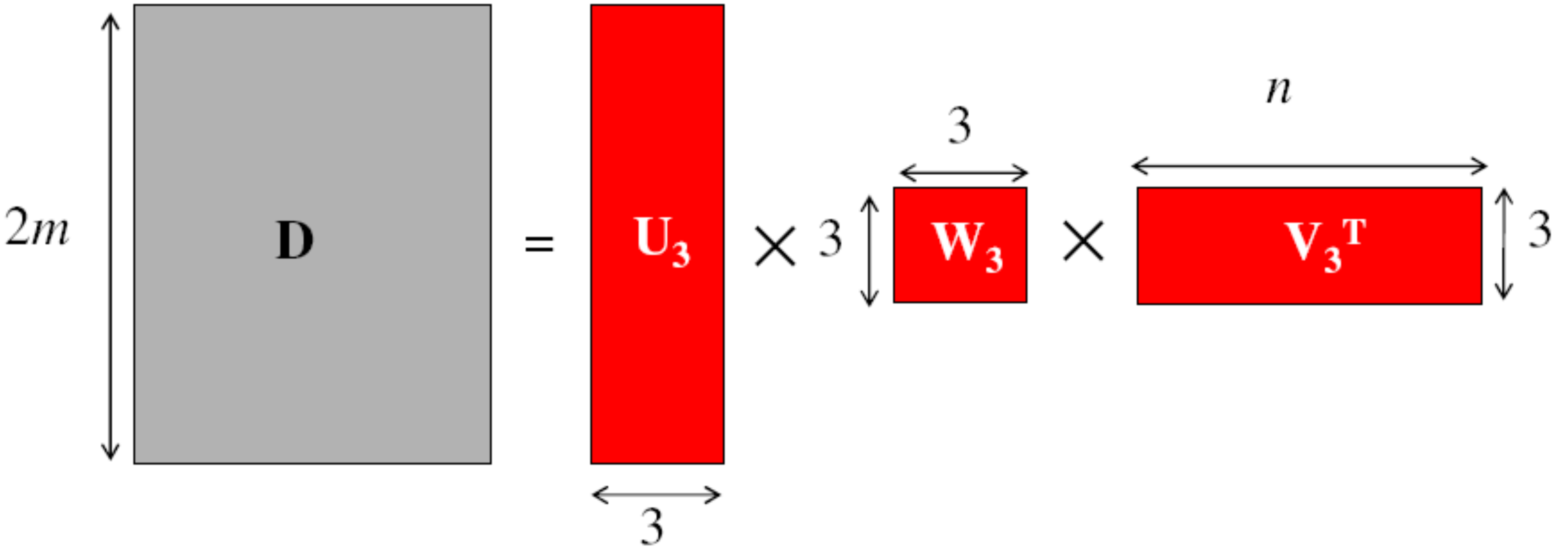


To reduce to rank 3, we just need to set all the singular values to 0 except for the first 3

Source: M. Hebert

Factorizing the measurement matrix

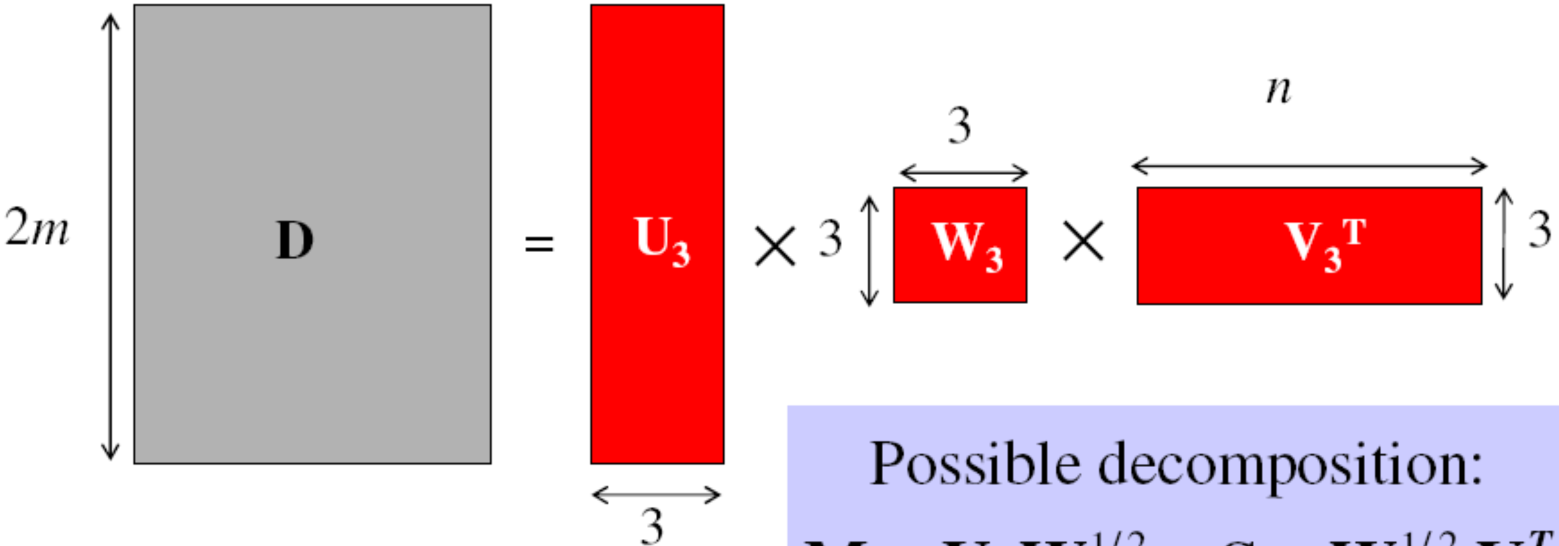
- Obtaining a factorization from SVD:



Source: M. Hebert

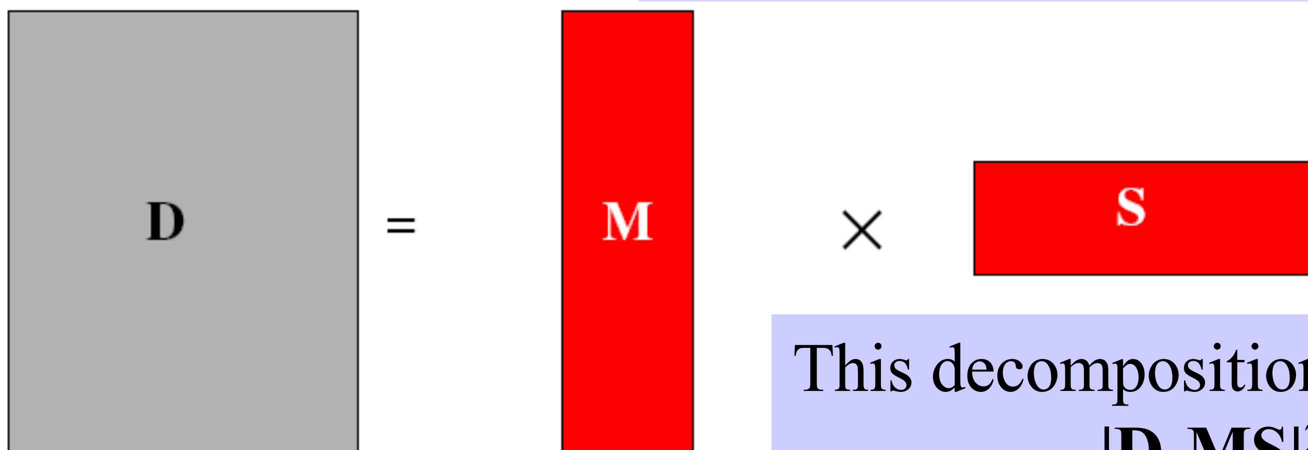
Factorizing the measurement matrix

- Obtaining a factorization from SVD:



Possible decomposition:

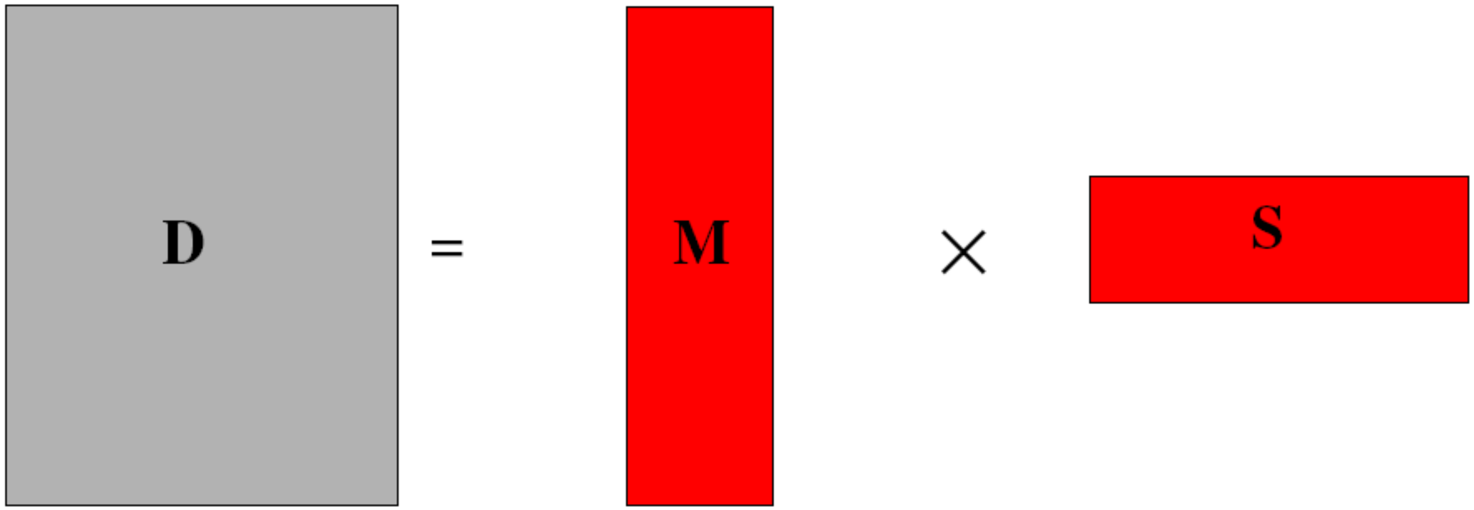
$$\mathbf{M} = \mathbf{U}_3 \mathbf{W}_3^{1/2} \qquad \mathbf{S} = \mathbf{W}_3^{1/2}\,\mathbf{V}_3^T$$



This decomposition minimizes $|\mathbf{D}\text{-}\mathbf{MS}|^2$

Source: M. Hebert

Affine ambiguity

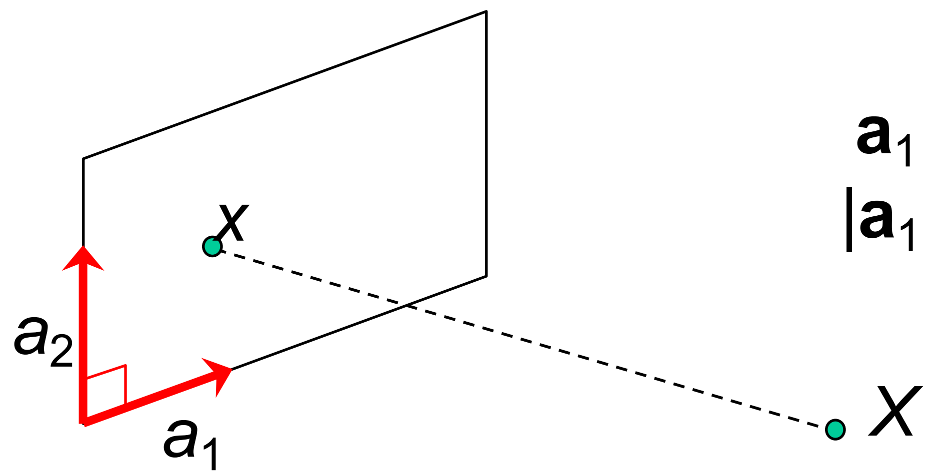


- The decomposition is not unique. We get the same **D** by using any 3×3 matrix **C** and applying the transformations **M** → **MC**, **S** → $\mathbf{C}^{-1}\mathbf{S}$
- That is because we have only an affine transformation and we have not enforced any Euclidean constraints (like forcing the image axes to be perpendicular, for example)

Source: M. Hebert

Eliminating the affine ambiguity

- Transform each projection matrix A to another matrix AC to get orthographic projection
  - Image axes are perpendicular and scale is 1



$$\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$$
$$|\mathbf{a}_1|^2 = |\mathbf{a}_2|^2 = 1$$

- This translates into 3*m* equations:

$$(\mathbf{A}_i\mathbf{C})(\mathbf{A}_i\mathbf{C})^T = \mathbf{A}_i(\mathbf{C}\mathbf{C}^T)\mathbf{A}_i = \mathbf{Id}, \qquad i = 1, \ldots, m$$

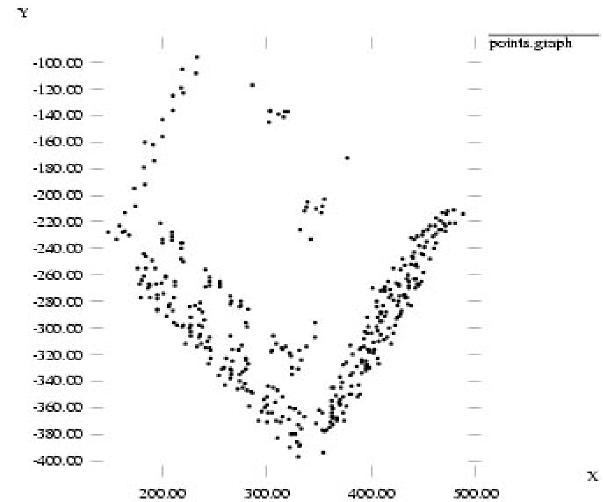
  - Solve for $\mathbf{L} = \mathbf{C}\mathbf{C}^T$
  - Recover **C** from **L** by Cholesky decomposition: $\mathbf{L} = \mathbf{C}\mathbf{C}^T$
  - Update **M** and **S**: $\mathbf{M} = \mathbf{MC}$, $\mathbf{S} = \mathbf{C}^{-1}\mathbf{S}$

Source: M. Hebert

Reconstruction results

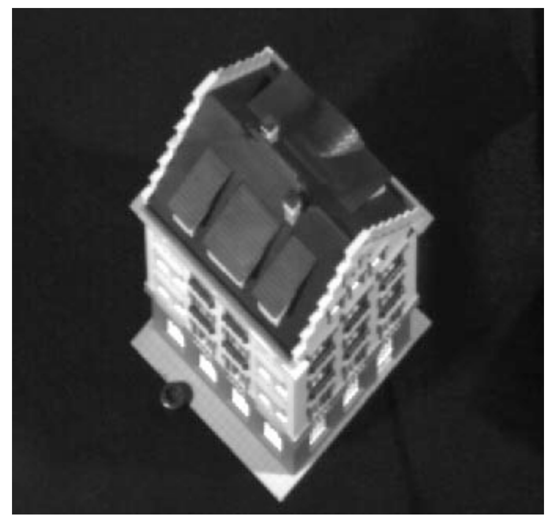
1

60



120

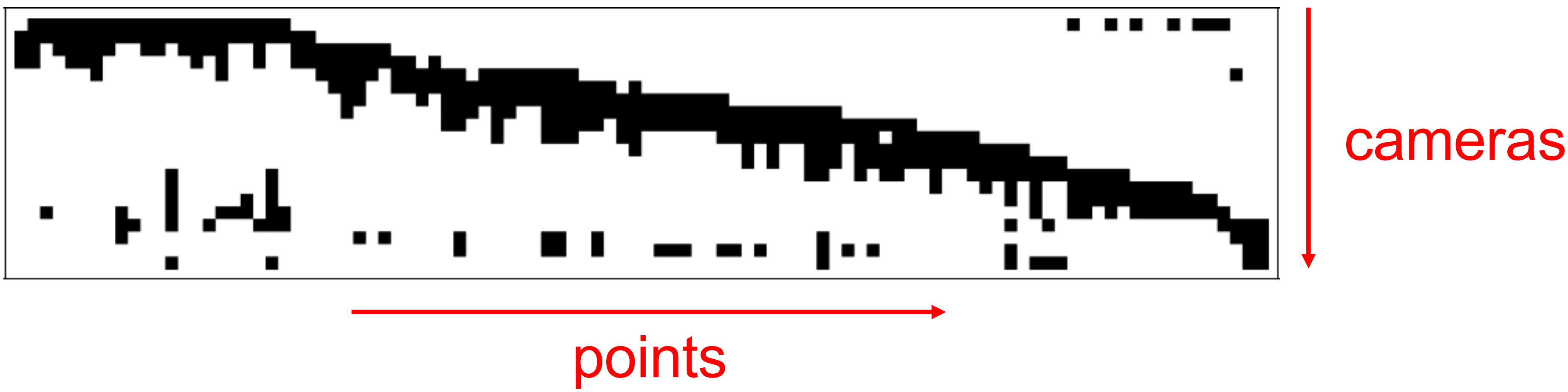
150



C. Tomasi and T. Kanade, [Shape and motion from image streams under orthography: A factorization method](), IJCV 1992

Dealing with missing data

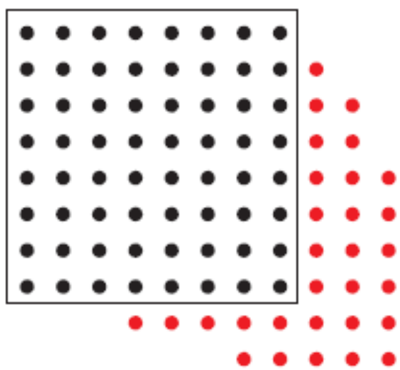
- So far, we have assumed that all points are visible in all views
- In reality, the measurement matrix typically looks something like this:



- Possible solution: decompose matrix into dense sub-blocks, factorize each sub-block, and fuse the results
  - Finding dense maximal sub-blocks of the matrix is NP-complete (equivalent to finding maximal cliques in a graph)

Dealing with missing data

- Incremental bilinear refinement



(1) Perform factorization on a dense sub-block

(2) Solve for a new 3D point visible by at least two known cameras (*triangulation*)

(3) Solve for a new camera that sees at least three known 3D points (*calibration*)

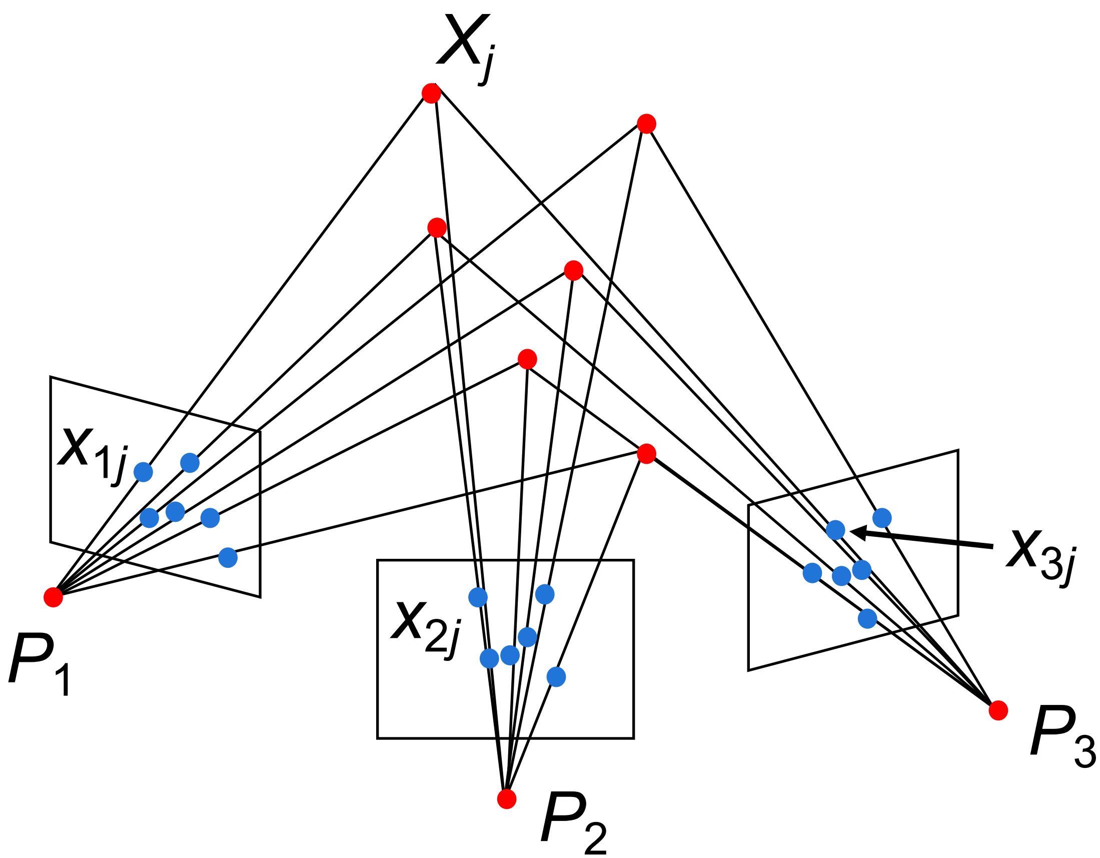
F. Rothganger, S. Lazebnik, C. Schmid, and J. Ponce. [Segmenting, Modeling, and Matching Video Clips Containing Multiple Moving Objects](). PAMI 2007.

Projective structure from motion

- Given: *m* images of *n* fixed 3D points

$$\lambda_{ij}\,\mathbf{x}_{ij} = \mathbf{P}_i\,\mathbf{X}_j\,, \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* correspondences $\mathbf{x}_{ij}$



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$$\lambda_{ij}\mathbf{x}_{ij} = \mathbf{P}_i \mathbf{X}_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* correspondences $\mathbf{x}_{ij}$
- With no calibration info, cameras and points can only be recovered up to a 4x4 projective transformation **Q**:

$$\mathbf{X} \rightarrow \mathbf{QX},\ \mathbf{P} \rightarrow \mathbf{PQ}^{-1}$$

- We can solve for structure and motion when

$$2mn >= 11m + 3n - 15$$

- For two cameras, at least 7 points are needed

Projective SFM: Two-camera case

- Compute fundamental matrix $\mathbf{F}$ between the two views
- First camera matrix: $[\mathbf{I} \mid \mathbf{0}]$
- Second camera matrix: $[\mathbf{A} \mid \mathbf{b}]$
- Then $\mathbf{b}$ is the epipole ($\mathbf{F}^T\mathbf{b} = 0$), $\mathbf{A} = -[\mathbf{b}_\times]\mathbf{F}$

F&P sec. 8.3.2

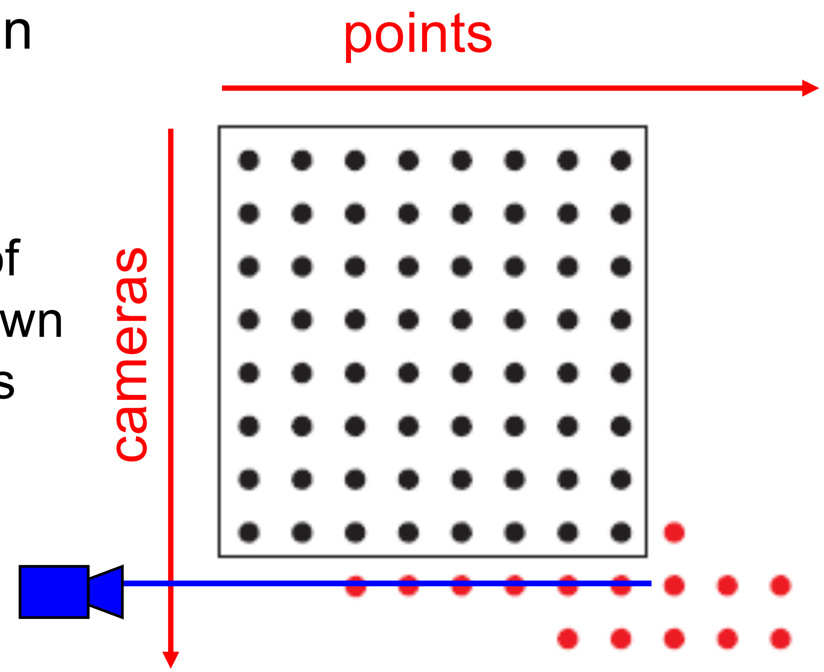
Incremental structure from motion

•Initialize motion from two images using fundamental matrix

•Initialize structure by triangulation

•For each additional view:

- Determine projection matrix of new camera using all the known 3D points that are visible in its image – *calibration*



Incremental structure from motion

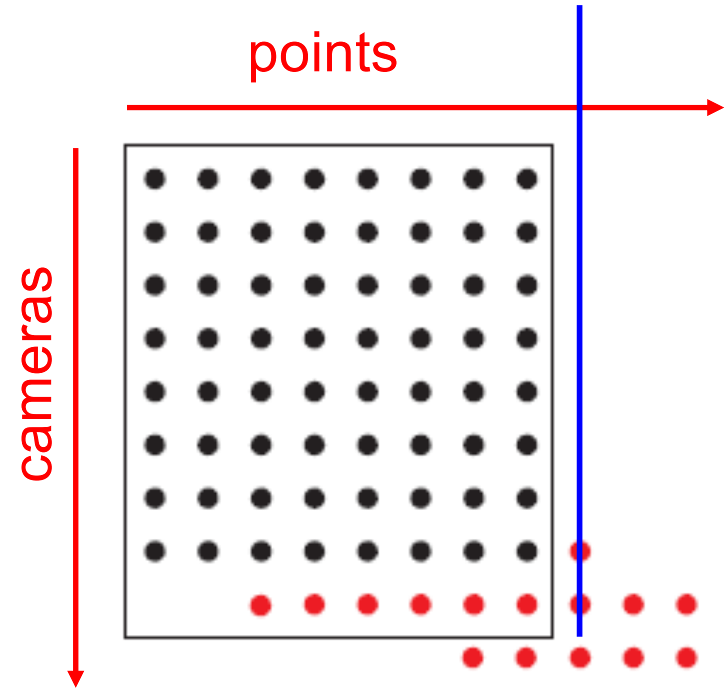
•Initialize motion from two images using fundamental matrix

•Initialize structure by triangulation

•For each additional view:

- Determine projection matrix of new camera using all the known 3D points that are visible in its image – *calibration*

- Refine and extend structure: compute new 3D points, re-optimize existing points that are also seen by this camera – *triangulation*



Incremental structure from motion

- Initialize motion from two images using fundamental matrix

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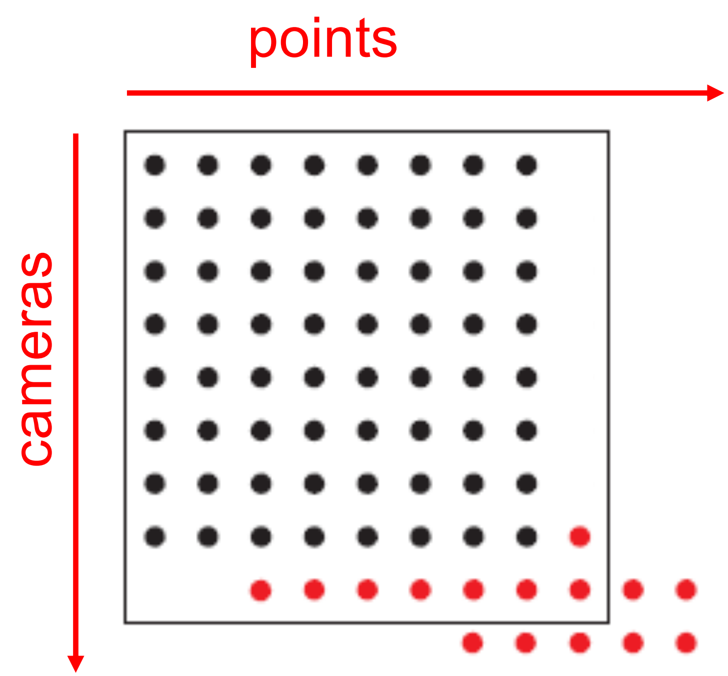
- For each additional view:
  - Determine projection matrix of new camera using all the known 3D points that are visible in its image – *calibration*

  - Refine and extend structure: compute new 3D points, re-optimize existing points that are also seen by this camera – *triangulation*

- Refine structure and motion: bundle adjustment

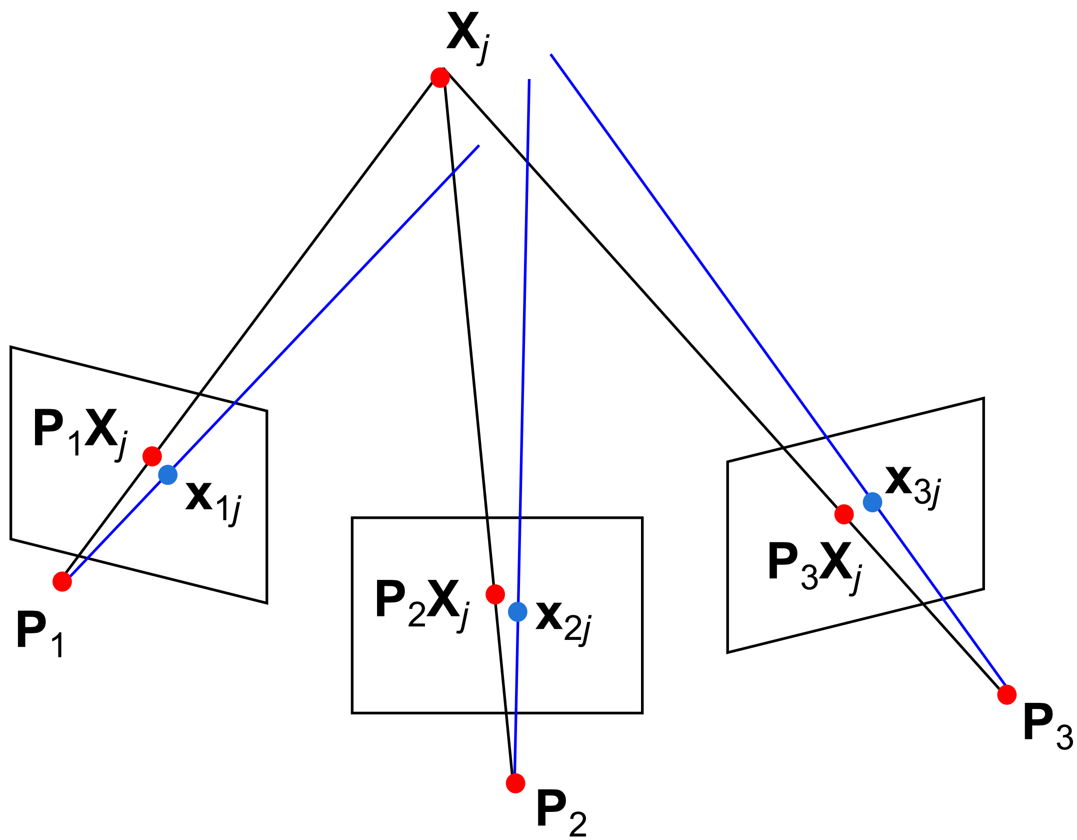


Bundle adjustment

- Non-linear method for refining structure and motion
- Minimize reprojection error

$$\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij} \left\| \mathbf{x}_{ij} - \frac{1}{\lambda_{ij}} \mathbf{P}_i \mathbf{X}_j \right\|^2$$

visibility flag: is point j visible in view i?

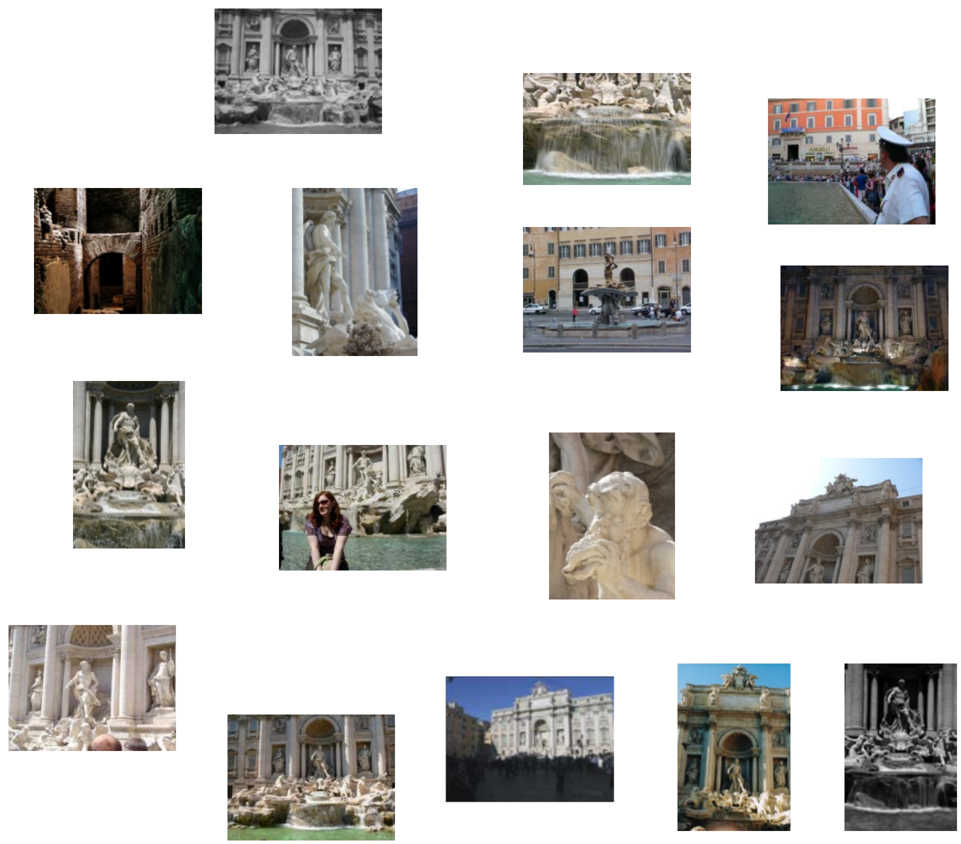


Representative SFM pipeline

N. Snavely, S. Seitz, and R. Szeliski, [Photo tourism: Exploring photo collections in 3D](), SIGGRAPH 2006.
http://phototour.cs.washington.edu/

Feature detection

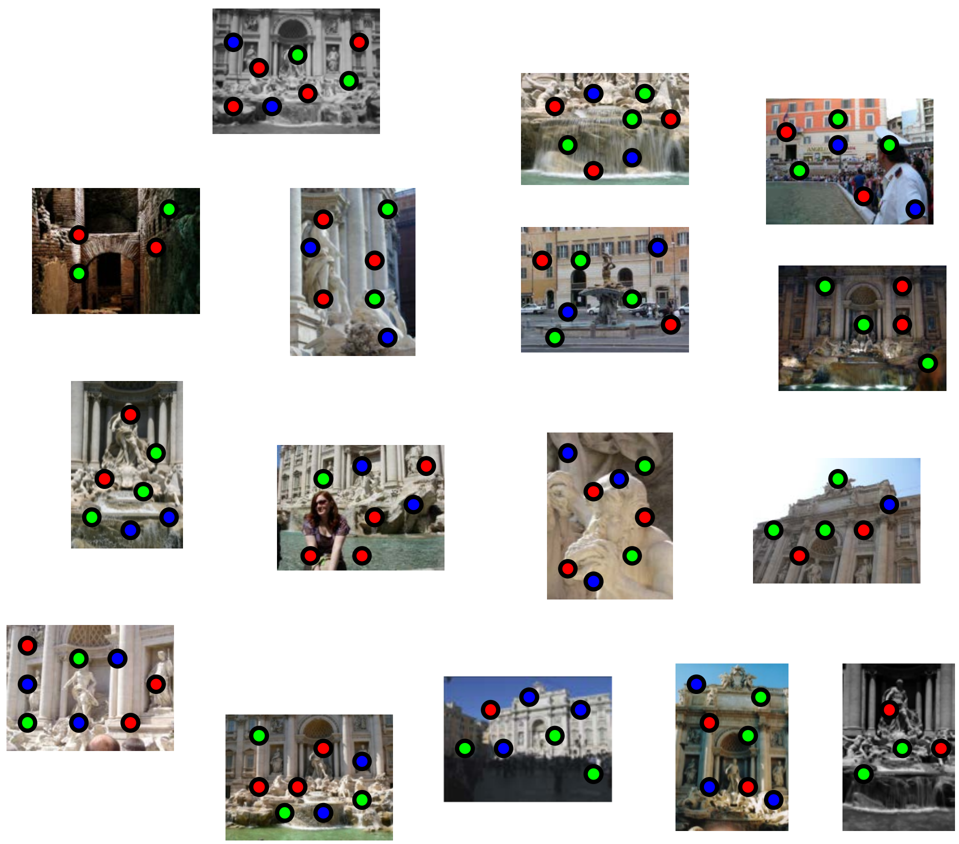
Detect SIFT features



Source: N. Snavely

Feature detection

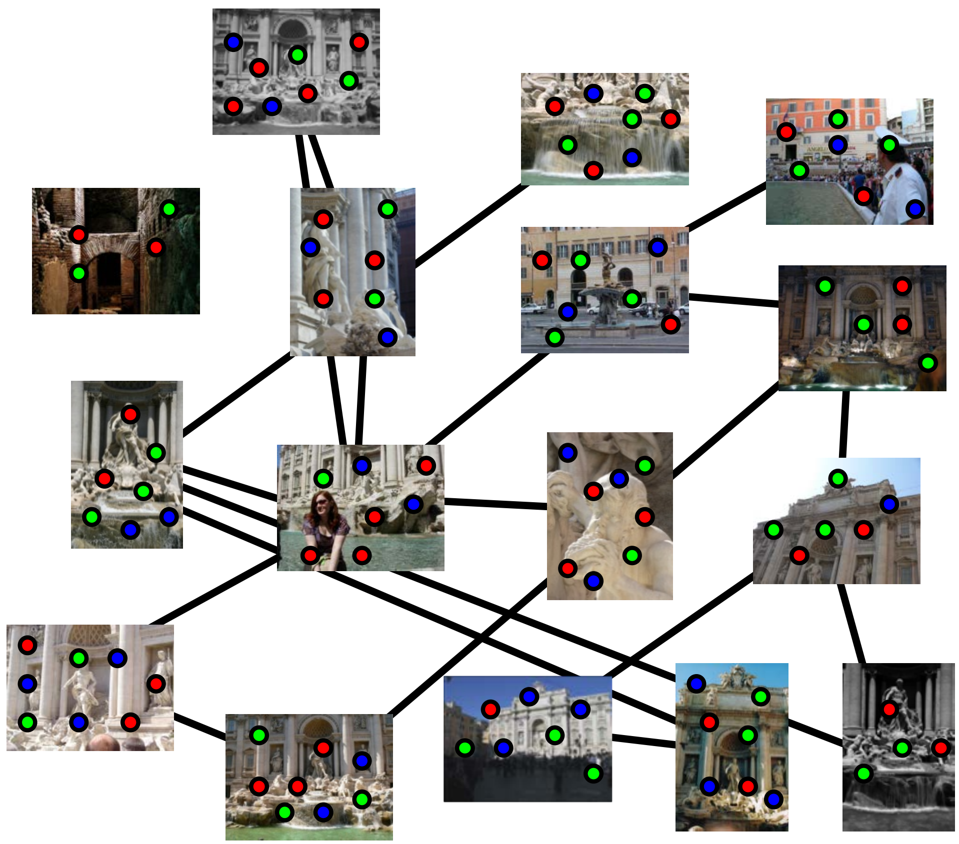
Detect SIFT features



Source: N. Snavely

Feature matching

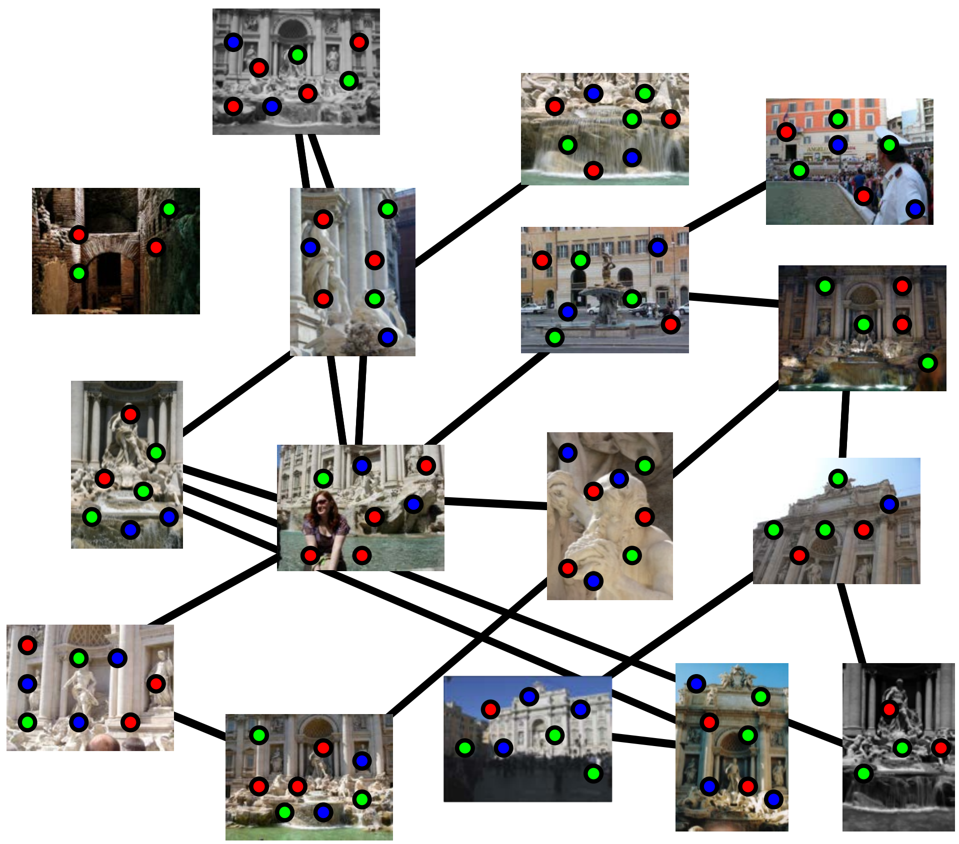
Match features between each pair of images



Source: N. Snavely

Feature matching

Use RANSAC to estimate fundamental matrix between each pair



Source: N. Snavely

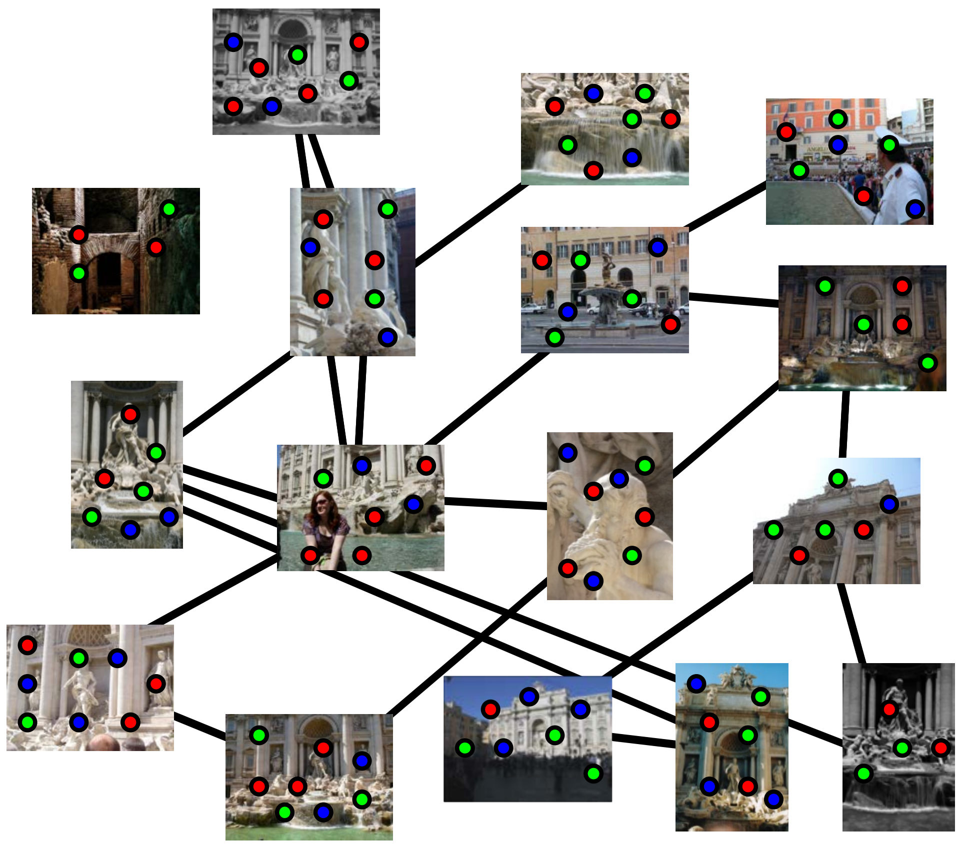
Feature matching

Use RANSAC to estimate fundamental matrix between each pair


Image source

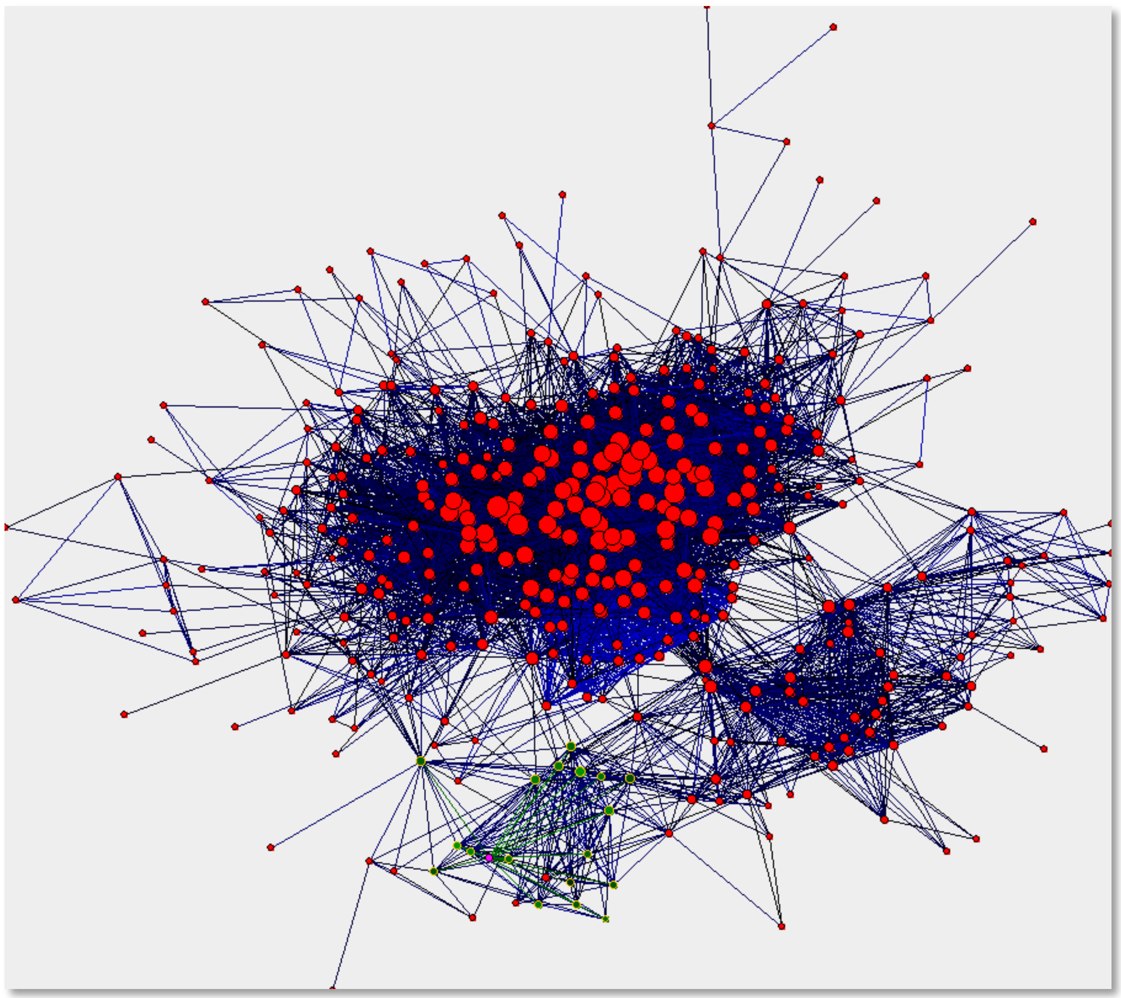
Feature matching

Use RANSAC to estimate fundamental matrix between each pair



Source: N. Snavely

Image connectivity graph



(graph layout produced using the Graphviz toolkit: http://www.graphviz.org/)

Source: N. Snavely

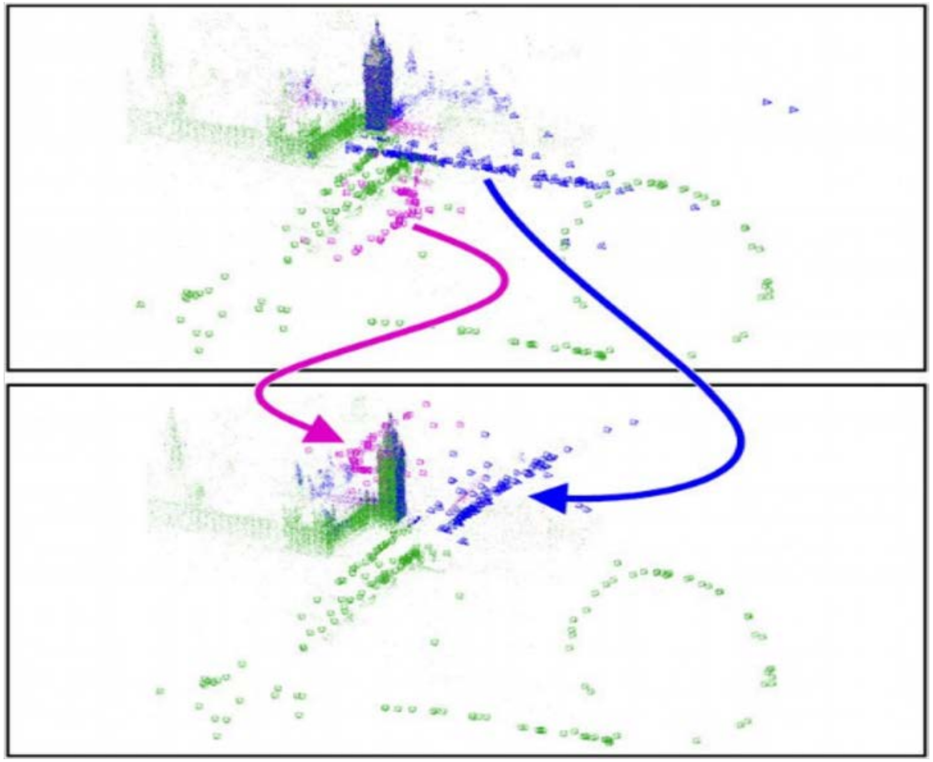
Incremental SFM

- Pick a pair of images with lots of inliers (and preferably, good EXIF data)
  - Initialize intrinsic parameters (focal length, principal point) from EXIF
  - Estimate extrinsic parameters (**R** and **t**) using five-point algorithm
  - Use triangulation to initialize model points
- While remaining images exist
  - Find an image with many feature matches with images in the model
  - Run RANSAC on feature matches to register new image to model
  - Triangulate new points
  - Perform bundle adjustment to re-optimize everything

The devil is in the details

- Handling degenerate configurations (e.g., homographies)
- Eliminating outliers
- Dealing with repetitions and symmetries

Repetitive structures



https://demuc.de/tutorials/cvpr2017/sparse-modeling.pdf

The devil is in the details

- Handling degenerate configurations (e.g., homographies)
- Eliminating outliers
- Dealing with repetitions and symmetries
- Handling multiple connected components
- Closing loops
- Making the whole thing efficient!
  - See, e.g., Towards Linear-Time Incremental Structure from Motion

SFM software

- Bundler
- OpenSfM
- OpenMVG
- VisualSFM
- See also Wikipedia's list of toolboxes

Review: Structure from motion

- Ambiguity
- Affine structure from motion
  - Factorization
- Dealing with missing data
  - Incremental structure from motion
- Projective structure from motion
  - Bundle adjustment
  - Modern structure from motion pipeline