21. Global approximation of non-periodic functions on an interval

Summary of the previous class

- Representing periodic functions as linear combinations of trigonometric basis functions
- Interpolation with trigonometric functions
- Orthogonality of the expansion
- Discrete Fourier transform
- Fast Fourier transform

Goals for today

- Spectral convergence of the trapezoid rule for smooth, periodic functions

- Global approximation of functions on an interval
- Chebyshev polynomials

Convergence rate for the trapezoid rule for integrating periodic functions

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- So if $\hat{f}_n$ decay exponentially fast then so does $|I - S_N|$

Global approximation of non-periodic functions

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- Take $f : [-1, +1] \to \mathbb{R}$
- Other intervals via linear transformation

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- What are suitable functions ϕ_n on the interval $[-1, +1]$?
- Not sinusoids, since not periodic

Collaboration I

Making a periodic function

Suppose we are given a non-periodic function $f : [-1, +1] \to \mathbb{R}$.

1. How could we build a smooth, periodic function g out of f?
2. How can we use that to approximate f?

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- $x := \mathrm{Re}(z)$

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- So $x =$ cos(θ)

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- $x = \cos(\theta)$
- When θ ranges over 0 to 2π

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- g is periodic!

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- g is also **even**: $g(-\theta) = g(\theta)$

Global approximation on an interval IV

- g is periodic and even, so we can approximate it by a Fourier cosine series:

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- Where $\phi_n(x) := \cos(n\theta(x)) = \cos(n \arccos(x))$

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- These do not look like pleasant functions!

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- So we can express $\cos(n\theta)$ in terms of $\cos(\theta)$
- What is the relationship?

Global approximation on an interval VI

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$$\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$$

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- **Chebysev polynomials** (of the first kind), $T_n(x)$

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- Reference: Trefethen, *Spectral Methods in MATLAB*, Chap. 8

Chebyshev polynomials

- We have shown that a function f on $[-1, +1]$ can be written as

$$f(x) = \sum f_n T_n(x)$$

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- Are they orthogonal?

Collaboration II

Finding orthogonality

Suppose $\tilde{f}(\theta) := f(\cos(\theta))$ and $\tilde{g}(\theta) := g(\cos(\theta))$ are orthogonal with respect to the standard inner product.

1. What can you say about $f(x)$ and $g(x)$, where $x := \cos(\theta)$?
2. Can you define a new inner product such that $f(x)$ and $g(x)$ become orthogonal with respect to that new inner product?

Chebyshev polynomials II

- If we compute $\int_{-1}^{1} T_1(x)\, T_2(x) dx$ we do *not* get 0

Chebyshev polynomials II

- If we compute $\int_{-1}^{1} T_1(x)\, T_2(x) dx$ we do *not* get 0
- But an inner product can include a **weight function** w:

$$(f, g) := \int f(x)\, g(x)\, w(x)\, dx$$

- w must be a *positive* function

Chebyshev polynomials III

- T_n *are* orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{1 - x^2}}$$

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- This factor comes from the change of variables $x = \cos(\theta)$ in the integral:

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- This is a consequence of the fact that Chebyshev polynomials satisfy a differential equation of Sturm–Liouville type

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- In the periodic case we found an alternative solution:
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- Can we do the same here?

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- We want to interpolate $f : [-1, +1] \to \mathbb{R}$
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- We need to solve the following for α_k:

$$\sum_{k=0}^{N-1} \alpha_k T_k(t_j) = f_j := f(x_j)$$

- How should we choose the nodes x_j?

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- Its complexity is still $\mathcal{O}(N \log N)$

Summary

- Spectral convergence of trapezoid rule

- By "transplanting" Fourier analysis, we found an orthogonal expansion of $f : [-1, +1] \to \mathbb{R}$
- Basis functions are **Chebyshev polynomials**
- They are orthogonal with respect to a particular inner product
- Chebyshev interpolation of N samples from a general non-periodic function uses the Discrete Cosine Transform
- The Fast Cosine Transform does this in $\mathcal{O}(N \log N)$ operations