22. Chebyshev methods II

Summary of the previous class

- Convergence of the trapezoid method for periodic integrands
- Mapping Fourier to a non-periodic setting
- Chebyshev polynomials
- Chebyshev interpolation
- Discrete Cosine transform

Goals for today

- Choosing the number of interpolation points
- Operations in Chebyshev representation:
- Derivatives
- Integrals
- Roots

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- $f_j := f(x_j)$ at $(n+1)$ points x_j with $j = 0, \ldots, n$
- Discrete Cosine Transformation (DCT):

$$\sum_k \alpha_k \cos\left(\frac{j\,k\pi}{n}\right) = f_j$$

Choosing the number of interpolation points

- Fundamental idea:

  Represent / approximate function f by Chebyshev interpolant in Chebyshev points

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- Enough that Chebyshev coefficients have decayed to ϵ_{mach}

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- Can reuse: $f_{2j}^{(2n)} = f_j^{(n)}$

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- How can we calculate the derivative f'?

Collaboration I

Differentiating a Chebyshev expansion

Suppose we have $f = \sum_{k=0}^{n} \alpha_k T_k$.

1. How could we calculate the derivative of the expansion directly?
2. What would we need to do so?
3. How could we calculate the derivative indirectly, without differentiating this expansion?

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- This will be polynomial of degree $N - 1$

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- Chapter 6 of Trefethen, *Spectral Methods in MATLAB* has explicit formulae for D_N

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- We get a **banded** matrix if we use the correct bases
- “Differentiating scales the coefficients and changes the basis”

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- Can we do better?
- Clenshaw & Curtis (1960)

Collaboration II

Integration

Suppose we have $f = \sum_{k=0}^{n} \alpha_k T_k$.

1. How can we integrate f over $[-1, +1]$?
2. How accurate should the result be?

Clenshaw–Curtis integration

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  via Fast Cosine Transform
- Double n until last few α_k are of order ϵ_{mach}

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- Now calculate $\int f$ as $\simeq \int f_n$:

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- E.g. exponential (spectral) convergence if f is analytic

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- Find $I_k = \frac{2}{1-k^2}$ if k is even, and 0 if k is odd

Clenshaw–Curtis integration IV

- So we have

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- This is a version of the **Riesz representation theorem** for linear functional

Recurrence relation

- It turns out that there is a 3-term **recurrence relation** relating T_k to T_{k-1} and T_{k_2}:

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

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- Where does this recurrence relation come from?

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- Where $\alpha_{k,j} \propto (xT_k, T_j)$, i.e. equal up to normalization

Recurrence relation II

- We have

$$(xT_k, T_j) = \int_{-1}^{1} xT_k(x)T_j(x)w(x)dx$$

- Change variables using $x = \cos(\theta)$:

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- Use trigonometric relation

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

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- In fact, *any* set of orthogonal polynomials have similar 3-term recurrence

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- $T_m, T_{2n-m}, T_{2n+m}, \ldots$
- This gives a relationship between the coefficients of *truncation* of f and *interpolation* of f
- See Trefethen, *Approx. Theory*, Chap. 4

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- … is p itself!
- For a polynomial written in the monomial basis we get the **companion** matrix
- In the Chebyshev basis it is called the **colleague matrix**

Colleague matrix

- The **colleague matrix** of the polynomial

$$p(x) = \sum_{k=0}^{n} a_k T_k(x), \quad a_n \neq 0$$

is

$$C := \begin{pmatrix} 0 & 1 & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{pmatrix} - \frac{1}{2a_n} \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}$$

Summary

- Spectral convergence gives excellent approximation of function
- Fundamental mathematical operations become “easy” once we have a spectral approximation
- This is (mostly) maintained by operations like differentiation, integration

- 3-term recurrence relation for Chebyshev polynomials
- Clenshaw–Curtis integration
- Spectrally accurate for analytic functions

Further reading

- Boyd, Finding the Zeros of a Univariate Equation: Proxy Rootfinders, Chebyshev Interpolation, and the Companion Matrix, SIAM Review **55**(2), 375–396. https://epubs.siam.org/doi/pdf/10.1137/110838297
- Boyd, **Chebyshev and Fourier Spectral Methods**
- Trefethen, **Spectral Methods in Matlab**
- Trefethen, **Approximation Theory and Approximation Practice**