32. Spectral methods for boundary-value problems

Summary of the previous class

- Choosing the number of interpolation points:
  - Check that the Chebyshev coefficients have decayed
- Using Chebyshev technology:
  - Differentiation
  - Clenshaw–Curtis integration
  - Roots

Goals for today

- Spectral methods
- Boundary-value problems for ODEs
- Eigenvalue problems
- Time evolution

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- Notation: derivative $u_x := u' := \frac{du}{dx}$
- u_{xx} = 2nd derivative $\frac{d^2u}{dx^2}$

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- We will follow Trefethen, *Spectral Methods in MATLAB* (Chaps. 6 + 7)

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- Idea:
  - Choose $N+1$ nodes x_j and solve for $N+1$ *unknown* values $u_j := u(x_j)$
- Impose that the ODE is satisfied at those nodes, i.e. that the residual is exactly 0 there

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- D_N is differentiation matrix

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- E.g. **boundary bordering**:

  set rows 1 and $N+1$ of M to rows of the identity matrix

  to impose $u_0 = u_N = 0$

Chebyshev spectral methods

- The formulation in terms of differentiation matrices can be used with finite differences . . .
- But using Chebyshev methods instead gives the usual advantage: **spectral convergence**

Summary

- Chebyshev spectral methods
- Collocation: Assume ODE is satisfied at interior points