15. Lagrange interpolation

Last time

- Absolute and relative errors
- Condition number of a problem
- Stability of an algorithm

Goals for today

- Interpolation
- Piecewise interpolation
- Global Lagrange interpolation

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- Heading towards **global function approximation**

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- These could be discrete **samples** coming from a system with *continuous* output
- How can we **re-construct** a function from a set of discrete samples?

Re-constructing functions from data

- (At least) two different methods to re-construct functions:
  1. Fit a “best approximation” to the data from within some class of functions: **approximation theory**
  2. A function that *passes through* the data: **interpolation**

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- If in fact the data came from sampling a function, we can compare the original and re-constructed functions

Collaboration I

Interpolation

Suppose we are given data (x_i, y_i) in 2D for $i = 0, \dots, n$ and we want to find a function $f(x)$ that passes through the points.

1. Which type of functions f should we try?
2. Write down the conditions on f that we want to satisfy.
3. What kind of mathematical problem do you get?
4. Is this problem solvable?

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- The x_i are **nodes** or **knots**
- We will assume that they are distinct and ordered:

$$a = x_0 < x_1 < \cdots < x_n = b$$

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- Find the a_i that solve this system of equations
- What kind of system is it?
- When can we expect to be able to solve it?

Polynomial interpolation II

- Each equation can be written

$$\begin{pmatrix} 1 & x_i & x_i^2 & \cdots & x_i^n \end{pmatrix}^{\mathsf{T}} \mathbf{a} = y_i$$

- $\mathbf{a} = (a_1, a_2, \ldots, a_n)^{\mathsf{T}}$ is a vector of the unknown a_is

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- V is ill-conditioned; algorithm is expensive and unstable

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- **Iverson bracket notation** (indicator function):

$$[\mathcal{S}] = \begin{cases} 1, & \text{if statement } \mathcal{S} \text{ is correct} \\ 0, & \text{if not} \end{cases}$$

Collaboration II

Line joining two points

Given two points (x_0, y_0) and (x_1, y_1).

1. What degree polynomial interpolates them?
2. Find cardinal basis functions ℓ_0 and ℓ_1 satisfying

$$\begin{aligned} \ell_0(x_0) &= 1 \qquad & \ell_0(x_1) &= 0 \\ \ell_1(x_1) &= 1 \qquad & \ell_1(x_0) &= 0 \end{aligned}$$

3. Hence find the polynomial interpolating (x_0, y_0) and (x_1, y_1).

Two points

- Simplest case: Find Line joining (x_0, y_0) and (x_1, y_1)
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- So $\ell_1(x_1) = c(x_1 - x_0) = 1$
- Hence $c = \frac{1}{x_1 - x_0}$, giving $\ell_1(x) = \frac{x - x_0}{x_1 - x_0}$
- Symmetry gives ℓ_0 by interchanging labels 0 and 1

Lagrange interpolant

- The **Lagrange interpolant** or **Lagrange polynomial** satisfies

$$L(x_0) = y_0 \quad \text{and} \quad L(x_1) = y_1$$

- Since ℓ_0 and ℓ_1 are cardinal basis functions,

$$L(x) = y_0\ell_0(x) + y_1\ell_1(x)$$

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- Hence $\{\ell_0, \ell_1\}$ forms a new **basis** of linear polynomials

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- This points towards **finite-element** methods for solving differential equations

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- Impose conditions at each node
- Requires solving a linear system

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- We can solve this by extending the argument from 2 points
- Construct a cardinal basis

Collaboration III

Polynomial interpolation

1. How can we make a cardinal basis function that is 1 at x_k and 0 at all other x_j?
2. How can we then make an interpolant of the data?

Global Lagrange interpolation II

- Generalise from 2 to n points:

  $$\ell_k(x) = c_k(x - x_0)\cdots\widehat{(x - x_k)}\cdots(x - x_n)$$

  where $\widehat{\cdot}$ indicates a *missing* term

- We want $\ell_k(x_k) = 1$, so

$$c_k = \frac{1}{(x_k - x_0)\cdots\widehat{(x_k - x_k)}\cdots(x - x_n)}$$

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- Thus $\ell_k(x) = \prod_{i=0, i\neq k}^{n} \left[\right] \frac{x - x_i}{x_k - x_i} \Big]$
- And $L(x) = \sum_{k=0}^{n} y_k \ell_k(x)$

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- What can go wrong?
- We will see that global Lagrange interpolation can go *very badly wrong* if we use equally-spaced points
- It turns out to be much better to use points that *cluster* near the endpoints of interval

Summary of Lagrange interpolation

- Given data $(n+1)$ data points $(x_i, y_i)_{i=0}^{n}$, the Lagrange interpolant of degree n is

$$L(x) = \sum_{j=0}^{n} y_j \ell_j(x)$$

- Where

$$\ell_j(x) := \frac{\prod_{k \neq j}(x - x_k)}{\prod_{k \neq j}(x_j - x_k)}$$

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- These issues can be solved by reformulating it into **barycentric** Lagrange interpolation

Barycentric Lagrange interpolation

- Let's define the product

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- Also $w_j = 1/\ell'(x_j)$

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- This is the **barycentric form** of Lagrange interpolation

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- This algorithm is numerically stable (despite the divisions)

Summary

- Degree-n polynomial **interpolates** $(n+1)$ data points
- Can construct Lagrange polynomial that **interpolates**
- Given in terms of a new cardinal basis

- The barycentric form gives a practical algorithm