

15. Lagrange interpolation

Last time

- Absolute and relative errors
- Condition number of a problem
- Stability of an algorithm

Goals for today

- Interpolation
- Piecewise interpolation
- Global Lagrange interpolation

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- Heading towards **global function approximation**

Representing data

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- E.g. measurements of a physical or economic system
- These could be discrete **samples** coming from a system with *continuous* output
- How can we **re-construct** a function from a set of discrete samples?

Re-constructing functions from data

- (At least) two different methods to re-construct functions:
 - 1 Fit a “best approximation” to the data from within some class of functions: **approximation theory**
 - 2 A function that *passes through* the data: **interpolation**

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- If in fact the data came from sampling a function, we can compare the original and re-constructed functions

Collaboration I

Interpolation

Suppose we are given data (x_i, y_i) in 2D for $i = 0, \dots, n$ and we want to find a function $f(x)$ that passes through the points.

- 1 Which type of functions f should we try?
- 2 Write down the conditions on f that we want to satisfy.
- 3 What kind of mathematical problem do you get?
- 4 Is this problem solvable?

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- The x_i are **nodes** or **knots**
- We will assume that they are distinct and ordered:

$$a = x_0 < x_1 < \dots < x_n = b$$

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- Find the a_i that solve this system of equations
- What kind of system is it?
- When can we expect to be able to solve it?

Polynomial interpolation II

- Each equation can be written

$$\begin{pmatrix} 1 & x_i & x_i^2 & \cdots & x_i^n \end{pmatrix}^T \mathbf{a} = y_i$$

- $\mathbf{a} = (a_1, a_2, \dots, a_n)^T$ is a vector of the unknown a_i s

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- \mathbf{V} is ill-conditioned; algorithm is expensive and unstable

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- **Iverson bracket notation** (indicator function):

$$[\mathcal{S}] = \begin{cases} 1, & \text{if statement } \mathcal{S} \text{ is correct} \\ 0, & \text{if not} \end{cases}$$

Collaboration II

Line joining two points

Given two points (x_0, y_0) and (x_1, y_1) .

- 1 What degree polynomial interpolates them?
- 2 Find cardinal basis functions ℓ_0 and ℓ_1 satisfying

$$\ell_0(x_0) = 1 \quad \ell_0(x_1) = 0$$

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- 3 Hence find the polynomial interpolating (x_0, y_0) and (x_1, y_1) .

Two points

- Simplest case: Find Line joining (x_0, y_0) and (x_1, y_1)
- We need a degree-1 polynomial ℓ_1 such that $\ell_1(x_1) = 1$ and $\ell_1(x_0) = 0$

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- Instead, since x_0 is a root (zero), $\ell_1(x) = c(x - x_0)$
- So $\ell_1(x_1) = c(x_1 - x_0) = 1$
- Hence $c = \frac{1}{x_1 - x_0}$, giving $\ell_1(x) = \frac{x - x_0}{x_1 - x_0}$
- Symmetry gives ℓ_0 by interchanging labels 0 and 1

Lagrange interpolant

- The **Lagrange interpolant** or **Lagrange polynomial** satisfies

$$L(x_0) = y_0 \quad \text{and} \quad L(x_1) = y_1$$

- Since ℓ_0 and ℓ_1 are cardinal basis functions,

$$L(x) = y_0 \ell_0(x) + y_1 \ell_1(x)$$

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- Hence $\{\ell_0, \ell_1\}$ forms a new **basis** of linear polynomials

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- This points towards **finite-element** methods for solving differential equations

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- Two continuous derivatives
- Impose conditions at each node
- Requires solving a linear system

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- By the Vandermonde argument we know this is possible
- We can solve this by extending the argument from 2 points
- Construct a cardinal basis

Collaboration III

Polynomial interpolation

- 1 How can we make a cardinal basis function that is 1 at x_k and 0 at all other x_j ?
- 2 How can we then make an interpolant of the data?

Global Lagrange interpolation II

- Generalise from 2 to n points:

$$\ell_k(x) = c_k(x - x_0) \cdots \widehat{(x - x_k)} \cdots (x - x_n)$$

where $\hat{\cdot}$ indicates a *missing* term

- We want $\ell_k(x_k) = 1$, so

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- Thus $\ell_k(x) = \prod_{i=0, i \neq k}^n \left[\frac{x - x_i}{x_k - x_i} \right]$

- And $L(x) = \sum_{k=0}^n y_k \ell_k(x)$

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- What can go wrong?
- We will see that global Lagrange interpolation can go *very badly wrong* if we use equally-spaced points
- It turns out to be much better to use points that *cluster* near the endpoints of interval

Summary of Lagrange interpolation

- Given data $(n + 1)$ data points $(x_i, y_i)_{i=0}^n$, the Lagrange interpolant of degree n is

$$L(x) = \sum_{j=0}^n y_j \ell_j(x)$$

- Where

$$\ell_j(x) := \frac{\prod_{k \neq j} (x - x_k)}{\prod_{k \neq j} (x_j - x_k)}$$

Issues with Lagrange interpolation

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- It also turns out to be numerically unstable
- These issues can be solved by reformulating it into **barycentric** Lagrange interpolation

Barycentric Lagrange interpolation

- Let's define the product

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- Also $w_j = 1/\ell'(x_j)$

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- This is the **barycentric form** of Lagrange interpolation

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- Evaluate the interpolant $L(x)$ at x : $\mathcal{O}(N)$ operations
- This algorithm is numerically stable (despite the divisions)

Summary

- Degree- n polynomial **interpolates** $(n + 1)$ data points
- Can construct Lagrange polynomial that **interpolates**
- Given in terms of a new cardinal basis
- The barycentric form gives a practical algorithm