

# 华中科技大学数学与统计学院教师备课用纸

2020~2021 复变函数与积分变换  
(参考答案)

一. ADBC BCAC BDDA

二. 解:  $\because u(x, y) + v(x, y) = y^2 + 2xy - x^2 + 2(x - y)$

两边对  $x$  求偏导可得:

$$u_x + v_x = 2y - 2x + 2 \quad \dots \textcircled{1}$$

两边对  $y$  求偏导得:

$$u_y + v_y = 2y + 2x - 2 \quad \dots \textcircled{2}$$

因为  $u, v$  解析, 所以  $u, v$  满足  $-R$  方程

$$u_x = v_y, \quad u_y = -v_x$$

代入  $\textcircled{2}$  式可得:

$$u_x - v_x = 2y + 2x - 2 \quad \dots \textcircled{3}$$

由  $\textcircled{1} \textcircled{3}$  式  $u_x = 2y \dots \textcircled{4}, \quad v_x = -2x + 2 \dots$

对  $x$  作偏积分  $u(x, y) = 2xy + \varphi(y), \quad \varphi(y)$  待定

$$\text{又 } u_y = -v_x = 2x - 2$$

$$\therefore 2x + \varphi'(y) = 2x - 2$$

$$\text{即 } \varphi'(y) = -2$$

$$\therefore \varphi(y) = -2y + C$$

$$\therefore u(x, y) = 2xy - 2y + C$$

$$v(x, y) = y^2 - x^2 + 2x + C$$

$C$  为任意常数.

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三. 解:  $\because$  函数  $f(z)$  在复平面上有两个孤立奇点  $z=2$  和  $z=3$ .

$\therefore$  在  $z_0=4$  展开时, 可分解为如下解析环域

$$\textcircled{1} \quad |z-4| < 1 \quad \textcircled{2} \quad 1 < |z-4| < 2 \quad \textcircled{3} \quad |z-4| > 2$$

$$\text{又 } f(z) = \frac{1}{z-3} - \frac{1}{z-2}$$

$\therefore \textcircled{1}$  当  $|z-4| < 1$  时

$$\frac{1}{z-3} = \frac{1}{z-4+1} = \sum_{n=0}^{+\infty} (-1)^n (z-4)^n$$

$$\frac{1}{z-2} = \frac{1}{z-4+2} = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n} (z-4)^n = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{n+1}} (z-4)^n$$

$$\therefore f(z) = \frac{1}{z-3} - \frac{1}{z-2} = \sum_{n=0}^{+\infty} (-1)^n \left(1 - \frac{1}{2^{n+1}}\right) (z-4)^n$$

$\textcircled{2}$  当  $1 < |z-4| < 2$  时

$$\frac{1}{z-3} = \frac{1}{z-4+1} = \frac{1}{z-4} \frac{1}{1 + \frac{1}{z-4}} = \frac{1}{z-4} \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(z-4)^n}$$

$$\frac{1}{z-2} = \frac{1}{z-4+2} = \frac{1}{2} \frac{1}{1 + \frac{z-4}{2}} = \sum_{n=0}^{+\infty} (-1)^n \cdot \frac{1}{2^{n+1}} (z-4)^n$$

$$\therefore f(z) = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(z-4)^{n+1}} + \sum_{n=0}^{+\infty} (-1)^n \frac{1}{2^{n+1}} (z-4)^n$$

$\textcircled{3}$  当  $|z-4| > 2$  时

$$\frac{1}{z-3} = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{(z-4)^{n+1}}$$

$$\frac{1}{z-2} = \frac{1}{z-4+2} = \frac{1}{z-4} \frac{1}{1 + \frac{2}{z-4}} = \frac{1}{z-4} \sum_{n=0}^{+\infty} (-2)^n \frac{1}{(z-4)^n}$$

$$\therefore f(z) = \sum_{n=0}^{+\infty} [(-1)^n + (-2)^n] \cdot \frac{1}{(z-4)^{n+1}}$$

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四. 解法一. 由高阶导数公式

$$\text{原式} = 2\pi i \cdot \frac{(1+z-e^z)^{(9)} }{9!} \Big|_{z=0} = -\frac{2\pi i}{9!}$$

解法二. 由级数计算.  $\because f(z)$  在  $|z|=1$  内只有一个奇点  $z=0$

$$\text{又 } f(z) = \frac{1+z-e^z}{z^{10}} = \frac{1}{z^{10}} \left( -\frac{z^2}{2!} - \frac{z^3}{3!} \cdots - \frac{z^9}{9!} - \cdots \right)$$

$$\therefore \text{Res}[f(z), 0] = -\frac{1}{9!}$$

$$\therefore \text{原式} = 2\pi i \left( -\frac{1}{9!} \right) = -\frac{2\pi i}{9!}$$

2. 解:  $\because$  函数  $f(z) = \frac{1-\cos z}{\sin^3 z}$  在  $|z|=1$  内只有奇点  $z=0$ .

又  $z=0$  为  $1-\cos z$  的  $2$  阶零点, 是  $\sin^3 z$  的  $3$  阶零点

$\therefore z=0$  为  $f(z)$  的  $-1$  阶极点.

$$\therefore \text{Res}[f(z), 0] = \lim_{z \rightarrow 0} z \cdot \frac{1-\cos z}{\sin^3 z}$$

$$= \lim_{z \rightarrow 0} \frac{1-\cos z}{\sin^2 z} = \lim_{z \rightarrow 0} \frac{\sin z}{2\sin z \cos z} = \frac{1}{2}$$

$$\therefore \text{原式} = \frac{1}{2} \cdot 2\pi i = \pi i$$



五. 1. 解: 被积函数在  $|z| < 2$  中有三个孤立奇点  $z=0, z_{2,3} = \pm 1$ .

$$\text{设 } f(z) = \frac{z}{1-z^2} \cos \frac{1}{z}. \quad |z|$$

$$\text{原式} = 2\pi i \sum_{k=1}^3 \text{Res}[f(z), z_k]$$

$$= -2\pi i \text{Res}[f(z), \infty]$$

$$= 2\pi i \text{Res}\left[f\left(\frac{1}{z}\right) \frac{1}{z^2}, 0\right]$$

$$\therefore f\left(\frac{1}{z}\right) \frac{1}{z^2} = \frac{\frac{1}{z}}{1-\frac{1}{z^2}} \cos z \cdot \frac{1}{z^2} = \frac{\cos z}{z(z^2-1)}$$

$$\therefore \text{Res}\left[f\left(\frac{1}{z}\right) \frac{1}{z^2}, 0\right] = \left. \frac{\cos z}{z^2-1} \right|_{z=0} = -1$$

$$\therefore \text{原式} = -2\pi i$$

注: 也可以用内部的留数计算:

$$\text{Res}[f(z), -1] = -\frac{1}{2} \cos 1,$$

$$\text{Res}[f(z), 1] = -\frac{1}{2} \cos 1,$$

$$\therefore f(z) = z(1+z^2+z^4+\dots)\left(1-\frac{1}{2!z^2}+\frac{1}{4!z^4}+\dots\right)$$

$$\therefore \text{Res}[f(z), 0] = -\frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} = \cos 1 - 1$$

$$2. \text{解: 令 } z = e^{i\theta}, \quad |z| \quad d\theta = \frac{dz}{iz} \quad \sin \theta = \frac{z-z^{-1}}{2i} \quad |z|$$

$$I = \text{原式} = \oint_{|z|=1} \frac{2}{3z^2+10iz-3} dz = \oint_{|z|=1} \frac{2 dz}{3(z+3i)(z+\frac{1}{3})}$$

被积函数在  $|z|=1$  内只有一个一阶极点  $z = -i/3$ , 而

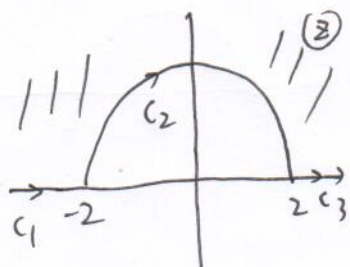
$$\text{Res}[f(z), -\frac{i}{3}] = \left. \frac{2}{3(z+3i)} \right|_{z=-\frac{i}{3}} = -\frac{1}{4}$$

$$\therefore I = 2\pi i \cdot \left(-\frac{1}{4}\right) = -\frac{\pi i}{2}.$$

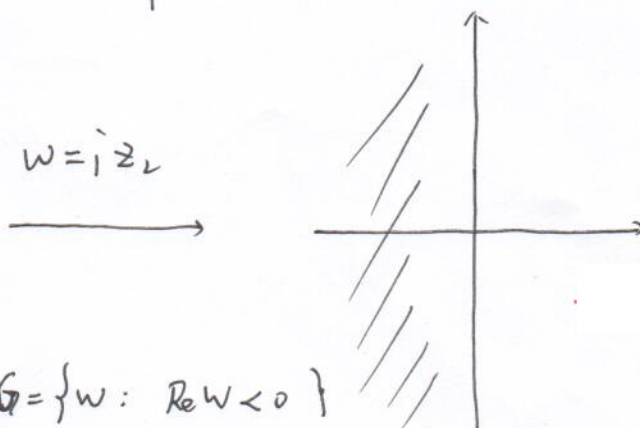
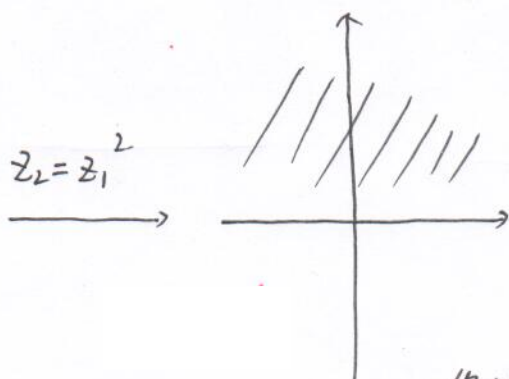
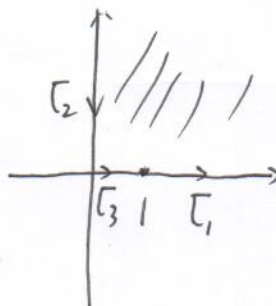
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六. 解. 映射  $w = i \left( \frac{z-2}{z+2} \right)^2$  可分解为如下映射的复合

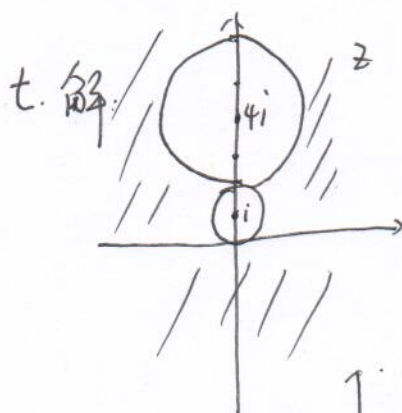
$$z_1 = \frac{z-2}{z+2}, \quad z_2 = z_1^2, \quad w = iz_2$$



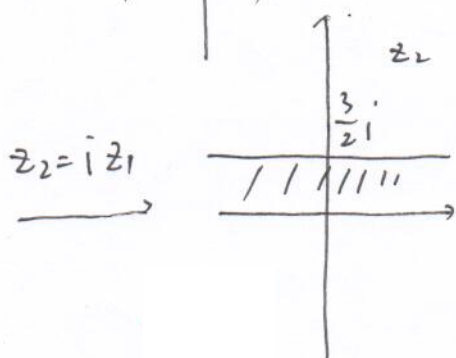
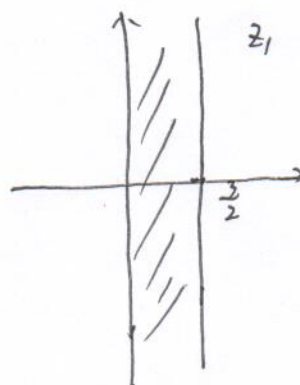
$$z_1 = \frac{z-2}{z+2}$$



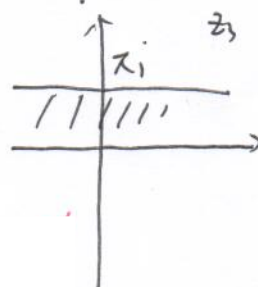
$\therefore$  像域为  $G = \{w : \operatorname{Re} w < 0\}$



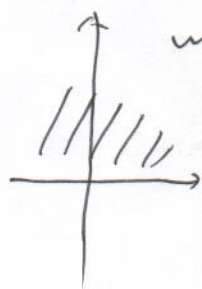
$$z_1 = \frac{z}{z-2i}$$



$$z_3 = \frac{2}{3} z_2$$



$$w = e^{z_3}$$



$$\therefore w = e^{\frac{2}{3} z \frac{iz}{z-2i}}$$

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八. 记  $F(s) = \mathcal{L}[f(t)]$ , 对方程两边作拉氏变换得:

$$s^4 F(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0) - F(s) = \frac{1}{s}$$

$$\text{即 } (s^4 - 1) F(s) = \frac{1}{s} + s + 2 = \frac{(1+s)^2}{s}$$

$$\begin{aligned}\therefore F(s) &= \frac{(s+1)^2}{(s^4-1)s} = \frac{s+1}{(s+1)s \cdot (s^3+1)} \\ &= \frac{1}{s-1} - \frac{1}{s^2+1} - \frac{1}{s}\end{aligned}$$

$$\therefore f(t) = \mathcal{L}^{-1}[F(s)] = e^t - \sin t - 1$$



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九. 证明:  $\because z$  为  $f(z)$  的极点, 且  $f'(z) \neq 0$ .

$\therefore z^0$  为  $f(z)$  的一阶极点, 且不为  $f'(z)$  的极点.

$$\text{对于积分 } \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^2 f'(z)} dz$$

$z=0$  为  $\frac{f(z)}{z^2 f'(z)}$  的一阶极点.

$$\therefore \operatorname{Res} \left[ \frac{f(z)}{z^2 f'(z)}, 0 \right] = \lim_{z \rightarrow 0} \frac{f(z)}{z f'(z)}$$

$$= \lim_{z \rightarrow 0} \frac{f'(z)}{f'(z) + z f''(z)} = \frac{f'(0)}{f'(0)} = 1$$

(也可直接由 Cauchy 积分公式, 留数公式:

$$I_1 = \left( \frac{f(z)}{f'(z)} \right)' \bigg|_{z=0} = \frac{(f'(z))^2 - f(z)f''(z)}{(f'(z))^2} \bigg|_{z=0} = \frac{(f'(0))^2}{(f'(0))^2} = 1$$

)

$$\text{对于积分 } I_2 = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z f'(z)}{f(z)} dz$$

$\because z=0$  为被积函数的可去奇点.

$$\therefore I_2 = 0.$$

$$\therefore I_1 + I_2 = 1. \quad \text{命题得证.}$$