一. 解：设 $u(x,t)=X(x)T(t)$，代入方程可得：

$$X''(x)T(t)=T''(t)X(x) \Rightarrow \frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

$\therefore\ X''(x)+\lambda X(x)=0,\qquad T''(t)+\lambda T(t)=0$ ……2'

由边界条件可得 $X'(0)=X'(2)=0$ ……4'

$\therefore$ 固有值为 $\lambda_n=\left(\frac{n\pi}{2}\right)^2,\ n=0,1,2,\cdots$，固有函数 $X_n(x)=\cos\frac{n\pi}{2}x,\ n=0,1,\cdots$

将 $\lambda_n$ 代入 $T(t)$ 的方程：$T''(t)+\left(\frac{n\pi}{2}\right)^2T(t)=0$ ……8'

当 $n=0$ 时，$T_0(t)=C_0+D_0t$

当 $n=1,2,\cdots$ 时，$T_n(t)=C_n\cos\frac{n\pi}{2}t+D_n\sin\frac{n\pi}{2}t$ ……10'

$\therefore\ u_0(x,t)=X_0(x)T_0(t)=C_0+D_0t$

$u_n(x,t)=X_n(x)T_n(t)=\left(C_n\cos\frac{n\pi}{2}t+D_n\sin\frac{n\pi}{2}t\right)\cos\frac{n\pi}{2}x$

由叠加原理，满足方程和边界条件的级数解为：

$$u(x,t)=C_0+D_0t+\sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi}{2}t+D_n\sin\frac{n\pi}{2}t\right)\cos\frac{n\pi}{2}x \quad \cdots 12'$$

最后，由初始条件：$u|_{t=0}=0 \Rightarrow C_0+\sum_{n=1}^{\infty}C_n\cos\frac{n\pi}{2}x=0 \Rightarrow C_n=0$

$u_t|_{t=0}=1+6\pi\cos3\pi x \Rightarrow D_0+\sum_{n=1}^{\infty}D_n\cdot\frac{n\pi}{2}\cos\frac{n\pi}{2}x=1+6\pi\cos3\pi x$

$$\therefore\ D_0=1,\qquad D_n=\begin{cases}0, & n\neq6\\ 2 & n=6\end{cases} \quad \cdots 14'$$

$\therefore\ u(x,t)=t+2\sin3\pi t\cos3\pi x.$ ……15'

二. 解: 相应齐次方程和齐次边界对应的固有函数系为:

$$\{\sin n\lambda x,\ n=1,2,\cdots\} \qquad \cdots \qquad 2'$$

记 $u(x,t)=\sum_{n=1}^{\infty} u_n(t)\sin n\lambda x$, 代入方程得:

$$\sum_{n=1}^{\infty} u_n'(t)\sin n\lambda x = -a^2\sum_{n=1}^{\infty}(n\lambda)^2 u_n(t)\sin n\lambda x + 5\sin 5\lambda x \qquad \cdots 4'$$

$\therefore$ 当 $n\neq 5$ 时, $u_n'(t)+(n\lambda a)^2 u_n(t)=0$

当 $n=5$ 时, $u_5'(t)+(5\lambda a)^2 u_5(t)=5 \qquad \cdots 6'$

由初始条件: $u_n(0)=0$

$\therefore$ 当 $n\neq 5$ 时, $u_n(t)=0$,

当 $n=5$ 时, $u_5(t)=\int_0^t 5\cdot e^{-(5\lambda a)^2(t-\tau)}\,d\tau$

$$=\frac{1}{5\lambda^2a^2}\left(1-e^{-(5\lambda a)^2 t}\right) \qquad \cdots 8'$$

$$\therefore u(x,t)=\frac{1}{5\lambda^2a^2}\left(1-e^{-(5\lambda a)^2 t}\right)\cdot\sin 5\lambda x. \qquad \cdots 10'$$

三.解：∵非齐次项和边界条件(1)都与 $y$ 无关，则可设

$$u(x,y)=V(x,y)+W(x)$$ ···2'

代入方程可得：$V_{xx}+V_{yy}+W_{xx}=\sin x$． ···4'

为使 $V$ 满足齐次方程，则需 $W_{xx}=\sin x \Rightarrow W(x)=-\sin x+C_1x+C_2$

为使 $V$ 满足齐次边界，则 $W(0)=-1$，$W'(\frac{\pi}{2})$

$$\therefore W(x)=-\sin x+x-1$$ ···8'

此时，$V$ 满足
$$\begin{cases} V_{xx}+V_{yy}=0 \\ V|_{x=0}=0,\ V_x|_{x=\frac{\pi}{2}}=0 \\ V_y|_{y=0}=0,\ V|_{y=\pi}=\sin x \end{cases}$$

由分离变量法，满足齐次方程和齐次边界的级数解为

$$V(x,y)=\sum_{n=1}^{\infty}\left(C_n e^{(2n-1)y}+D_n e^{-(2n-1)y}\right)\sin(2n-1)x$$ ··12'

由 $V_y|_{y=0}=0 \Rightarrow C_n=D_n$

$$V|_{y=\pi}=\sin x \Rightarrow \sum_{n=1}^{\infty}C_n\left(e^{(2n-1)\pi}+e^{-(2n-1)\pi}\right)\sin(2n-1)x=\sin x$$

∴当 $n\neq 1$ 时，$C_n=D_n=0$

当 $n=1$ 时，$C_1=D_1=\dfrac{1}{e^{\pi}+e^{-\pi}}$

$$\therefore V(x,y)=\frac{e^{y}+e^{-y}}{e^{\pi}+e^{-\pi}}\sin x \quad \left(\text{或 } \frac{\operatorname{ch} y}{\operatorname{ch}\pi}\sin x\right)$$ ··14'

$$\therefore u(x,y)=V(x,y)+W(x)$$
$$=\frac{\operatorname{ch} y}{\operatorname{ch}\pi}\sin x-\sin x+x-1$$
$$=\left(\frac{\operatorname{ch} y}{\operatorname{ch}\pi}-1\right)\sin x+x-1.$$ ···15'

三.（另解）：设 $w(x)=ax+b$ …2'

由 $w(0)=-1$，$w'(\frac{\pi}{2})=1$ 得 $w(x)=x-1$ …4'

设 $u(x,y)=v(x,y)+x-1$，则 $v(x,y)$ 满足

$$\begin{cases} v_{xx}+v_{yy}=\sin x \\ v|_{x=0}=0,\quad v_x|_{x=\frac{\pi}{2}}=0 \\ v_y|_{y=0}=0,\quad v|_{y=\pi}=0 \end{cases}$$

固有函数系为 $\{\sin(2n-1)x,\ n=1,2,\dots\}$

设 $v(x,y)=\sum\limits_{n=1}^{\infty}u_n(y)\sin(2n-1)x$.

代入方程得： …8'

$$\sum_{n=1}^{\infty}u_n''(y)\sin(2n-1)x-\sum_{n=1}^{\infty}u_n(y)(2n-1)^2\sin(2n-1)x=\sin x$$

$$\therefore u_n''(y)-(2n-1)^2u_n(y)=\begin{cases}0, & n\neq 1\\ 1, & n=1\end{cases}$$

由第一组边界 $u_n'(0)=0$，$u_n(\pi)=0$

$\therefore$ 当 $n\neq 1$ 时，$\because$ 通解 $u_n(y)=C_ne^{(2n-1)y}+D_ne^{-(2n-1)y}$

由条件 $C_n=0$，$D_n=0$. $\therefore u_n(y)=0$

当 $n=1$ 时. $u_1(y)=C_1e^{y}+D_1e^{-y}-1$ …12'

由 $u_n'(0)=0 \Rightarrow C_1=D_1$

由 $u_n(\pi)=0 \Rightarrow C_1(e^{\pi}+e^{-\pi})-1=0$

$$\therefore C_1=D_1=\frac{1}{e^{\pi}+e^{-\pi}} \Rightarrow u_1(y)=\frac{e^{y}+e^{-y}}{e^{\pi}+e^{-\pi}}-1$$

$$\therefore v(x,y)=\left(\frac{e^{y}+e^{-y}}{e^{\pi}+e^{-\pi}}-1\right)\sin x$$ …14'

$$\therefore u(x,y)=\left(\frac{e^{y}+e^{-y}}{e^{\pi}+e^{-\pi}}-1\right)\sin x+x-1.$$ …15'

四. 解: 设 $u(x,t)=f(x+at)+g(x-at)$ $\cdots 2'$

$\because f(x)+g(x)=\varphi(x),\quad af'(x)-ag'(x)=0.\quad (x>0)\cdots 4'$

即当 $x>0$ 时, $f(x)=\frac{1}{2}\varphi(x)+\frac{1}{2}[f(0)-g(0)]$

$$g(x)=\frac{1}{2}\varphi(x)-\frac{1}{2}[f(0)-g(0)]$$

$$\therefore f(x+at)=\frac{1}{2}\varphi(x+at)+\frac{1}{2}[f(0)-g(0)]$$

当 $x\geqslant at$ 时, $g(x-at)=\frac{1}{2}\varphi(x-at)-\frac{1}{2}[f(0)-g(0)]$

即: $u(x,t)=\frac{1}{2}\varphi(x+at)+\frac{1}{2}\varphi(x-at).$ $\cdots 6'$

又由边界条件: $f'(at)+g'(-at)=0.$

$$\therefore f'(x)+g'(-x)=0 \iff f'(x)-[g(-x)]'=0$$

$$\therefore f(x)-g(-x)=f(0)-g(0)$$

$\Rightarrow g(-x)=f(x)-[f(0)-g(0)]$

即当 $x<0$ 时, $g(x)=f(-x)-[f(0)-g(0)]$ $\cdots 8'$

$\therefore$ 当 $x<at$ 时, $g(x-at)=f(at-x)-[f(0)-g(0)]$

$$=\frac{1}{2}\varphi(at-x)+\frac{1}{2}[f(0)-g(0)]-[f(0)-g(0)]$$

$$=\frac{1}{2}\varphi(at-x)-\frac{1}{2}[f(0)-g(0)]$$

$\therefore u(x,t)=\frac{1}{2}\varphi(x+at)+\frac{1}{2}\varphi(at-x).$ $\cdots 10'$

$$\therefore u(x,t)=\begin{cases}\frac{1}{2}[\varphi(x+at)+\varphi(x-at)], & x\geqslant at.\\ \frac{1}{2}[\varphi(x+at)+\varphi(at-x)], & x<at.\end{cases}$$

五. 解：记 $\mathcal{L}[u(x,t)] = U(x,s)$ ··· 2′

则定解问题关于t作 Laplace 变换可得：

$$\begin{cases} s^2 U = U_{xx} + \dfrac{s}{1+s^2} \\ U|_{x=0} = 0, \quad \lim\limits_{x \to +\infty} U_x = 0 \end{cases}$$ ··· 6′

∵方程的一个特解为 $U_* = \dfrac{1}{s(1+s^2)}$ ··· 8′

∴通解 $U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \dfrac{1}{s(1+s^2)}$ ··· 10′

由 $\lim\limits_{x\to+\infty} U_x = 0 \Rightarrow c_1 = 0$

由 $U|_{x=0} = 0 \Rightarrow c_2 = -\dfrac{1}{s(1+s^2)} = \dfrac{s}{1+s^2} - \dfrac{1}{s}$

∴ $U(x,s) = -\dfrac{1}{s(1+s^2)} e^{-sx} + \dfrac{1}{s(1+s^2)}$ ··· 13′

∵ $\mathcal{L}^{-1}\left[\dfrac{1}{s(1+s^2)}\right] = \mathcal{L}^{-1}\left[\dfrac{1}{s} - \dfrac{s}{1+s^2}\right] = 1 - \cos t$

由延迟性质：

$$u(x,t) = \mathcal{L}^{-1}[U(x,s)]$$

$$= -[1-\cos(t-x)]\,u(t-x) + 1 - \cos t$$

$$= \begin{cases} \cos(t-x) - \cos t, & t \geq x, \\ 1 - \cos t, & t < x. \end{cases}$$ ··· 15′

六. 解：记 $F[u(x,t)] = U(\lambda,t)$. $F[\phi(x)] = \Phi(\lambda)$. …2′

则方程关于 $x$ 作 Fourier 变换得：

$$\begin{cases} U_t + (a^2\lambda^2 - 3t^2)U = 0 \\ U|_{t=0} = \Phi(\lambda) \end{cases}$$ … 6′

$\therefore U(\lambda,t) = \Phi(\lambda)\, e^{t^3 - a^2\lambda^2 t}$ … 8′

由卷积定理： $u(x,t) = F^{-1}[U(\lambda,t)]$

$$= e^{t^3} F^{-1}[\Phi(\lambda)\cdot e^{-a^2 t\lambda^2}]$$

$$= e^{t^3}\, \phi(x) * \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$$

$$= e^{t^3}\int_{-\infty}^{+\infty} \phi(y)\cdot \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4a^2 t}}\, dy$$

… 10′

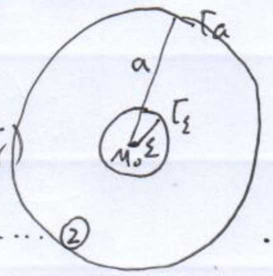
七. 1. 解：由条件 $u$ 为对称解，可设 $u(r)=C_1\ln r+C_2$ ...2'

由 $u|_{r=1}=0 \Rightarrow C_2=0$，$u|_{r=2}=1 \Rightarrow C_1=\frac{1}{\ln 2}$ ...4'

$\therefore u(r)=\frac{\ln r}{\ln 2}$ ...5'

2. 证明：若 $u, v$ 在 $\Omega$ 中二阶连续可微且连续到边界 $\Gamma$，则有如下格林第二公式：$\iiint_\Omega u\Delta v - v\Delta u \, d\Omega = \iint_\Gamma u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \, dS$ ...2'

特殊地，取 $v=1$，则 $\iiint_\Omega \Delta u \, d\Omega = \iint_\Gamma \frac{\partial u}{\partial n} \, dS$ ....① ...3'




在球 $B_a(M_0)$ 中挖去闭球 $B_\varepsilon(M_0)$（$\varepsilon$ 充分小），

此时在 $\Omega_1 = B_a(M_0)\setminus\overline{B_\varepsilon(M_0)}$ 上利用格林第二公式：$(v=\frac{1}{4\pi r})$

$-\iiint_{\Omega_1} \frac{1}{4\pi r}\Delta u \, d\Omega = \iint_{\Gamma_a+\Gamma_\varepsilon} u\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) - \frac{1}{4\pi r}\frac{\partial u}{\partial n} \, dS$ ....② ...5'

在 $\Gamma_\varepsilon$ 上，$\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{4\pi r}\right) = \frac{1}{4\pi r^2}$

在 $\Gamma_a$ 上 $\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{4\pi r}\right) = -\frac{1}{4\pi r^2}$

$\therefore \iint_{\Gamma_a} u\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) - \frac{1}{4\pi r}\frac{\partial u}{\partial n} \, dS = -\frac{1}{4\pi a^2}\iint_{\Gamma_a} u \, dS - \frac{1}{4\pi a}\iint_{\Gamma_a}\frac{\partial u}{\partial n} \, dS$

$\iint_{\Gamma_\varepsilon} u\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) - \frac{1}{4\pi r}\frac{\partial u}{\partial n} \, dS = \frac{1}{4\pi\varepsilon^2}\iint_{\Gamma_\varepsilon} u \, dS - \frac{1}{4\pi\varepsilon}\iint_{\Gamma_\varepsilon}\frac{\partial u}{\partial n} \, dS$ ..7'

代入②式且利用①式：

$\frac{1}{4\pi\varepsilon^2}\iint_{\Gamma_\varepsilon} u \, dS + \iiint_{\Omega_1}\frac{1}{4\pi r}\Delta u \, d\Omega = \frac{1}{4\pi a^2}\iint_{\Gamma_a} u \, dS + \frac{1}{4\pi a}\iiint_{B_a(M_0)}\Delta u \, dS$

$+ \frac{1}{4\pi\varepsilon}\iiint_{B_\varepsilon(M_0)}\Delta u \, dS$

令 $\varepsilon\to 0$，注意到 $u$ 二阶连续可微 $\therefore \Delta u$ 在 $B_\varepsilon$ 上有界 $|\Delta u|\le k$.

$\therefore \left|\frac{1}{4\pi\varepsilon}\iiint_{B_\varepsilon}\Delta u \, dS\right| \le k\cdot\frac{1}{4\pi\varepsilon}\cdot\frac{4}{3}\pi\varepsilon^3 = \frac{k}{3}\varepsilon^2 \to 0$ ..9'

$\therefore u(M_0) = \frac{1}{4\pi a^2}\iint_{\Gamma_a} u \, dS + \iiint_{B_a(M_0)}\left(\frac{1}{4\pi a} - \frac{1}{4\pi r}\right)\Delta u \, dS$ ...10'

$\because r<a$，$\Delta u>0$. $\therefore$ 结论得证.

八. 解：设 $u(r,t)=T(t)R(r)$，代入方程得：

$$T'(t)R(r)=4T(t)\left[R''(r)+\frac{1}{r}R'(r)-\frac{4}{r^2}R(r)\right] \qquad \cdots 1'$$

$$\therefore \frac{T'(t)}{4T(t)}=\frac{R''(r)+\frac{1}{r}R'(r)-\frac{4}{r^2}R(r)}{R(r)}=-\lambda$$

$$\therefore T'(t)+4\lambda T(t)=0$$

$$r^2R''(r)+rR'(r)+(\lambda r^2-4)R(r)=0 \qquad \cdots 3'$$

由边界条件 $R(1)=0$，$|R(0)|<+\infty$ $\qquad \cdots 4'$

$\because$ 关于 $R(r)$ 的方程为二阶 Bessel 方程，$\therefore$ 通解为

$$R(r)=CJ_2(\sqrt{\lambda}r)+DY_2(\sqrt{\lambda}r)$$

由 $|R(0)|<+\infty$，$D=0$

$$R(1)=0 \Rightarrow J_2(\sqrt{\lambda})=0$$

记 $\mu_m^{(2)}$ 为 $J_2(x)$ 的第 m 个正零点，则固有值和固有函数为：

$$\lambda_m=(\mu_m^{(2)})^2,\ m=1,2,\cdots,\quad R_m(r)=J_2(\mu_m^{(2)}r),\ m=1,2,\cdots \qquad \cdots 6'$$

将 $\lambda_m$ 代入到 $T(t)$ 的方程 $T'(t)+4(\mu_m^{(2)})^2T(t)=0$

$$\therefore T_m(t)=C_me^{-4(\mu_m^{(2)})^2t} \qquad \cdots 7'$$

$\therefore$ 满足方程和边界条件的级数解为：

$$u(r,t)=\sum_{m=1}^{\infty}R_m(r)T_m(t)=\sum_{m=1}^{\infty}C_me^{-4(\mu_m^{(2)})^2t}J_2(\mu_m^{(2)}r)$$

由初始条件 $u|_{t=0}=1-r^2 \Rightarrow \sum_{m=1}^{\infty}C_mJ_2(\mu_m^{(2)}r)=1-r^2$

$$\therefore C_m=\frac{\int_0^1 r(1-r^2)J_2(\mu_m^{(2)}r)\,dr}{\frac{1}{2}J_3^2(\mu_m^{(2)})} \qquad \cdots 8'$$

① $\int_0^1 r J_2(\mu_m^{(2)} r)\,dr \xlongequal{x=\mu_m^{(2)} r} \frac{1}{(\mu_m^{(2)})^2} \int_0^{\mu_m^{(2)}} x J_2(x)\,dx$

$= \frac{1}{(\mu_m^{(2)})^2} \int_0^{\mu_m^{(2)}} (x^{-1} J_1(x))' - x^2\,dx$

$= \frac{1}{(\mu_m^{(2)})^2} \left[ -x J_1(x) \Big|_0^{\mu_m^{(2)}} + 2\int_0^{\mu_m^{(2)}} J_1(x)\,dx \right]$

$= \frac{1}{(\mu_m^{(2)})^2} \left[ -x J_1(x) - 2J_0(x) \right] \Big|_0^{\mu_m^{(2)}}$

$= \frac{1}{(\mu_m^{(2)})^2} \left[ -\mu_m^{(2)} J_1(\mu_m^{(2)}) - 2J_0(\mu_m^{(2)}) + 2 \right]$

② $\int_0^1 r^3 J_2(\mu_m^{(2)} r)\,dr \xlongequal{x=\mu_m^{(2)} r} \frac{1}{(\mu_m^{(2)})^4} \int_0^{\mu_m^{(2)}} x^3 J_2(x)\,dx$

$= \frac{1}{(\mu_m^{(2)})^4}\, x^3 J_3(x) \Big|_0^{\mu_m^{(2)}}$

$= \frac{1}{\mu_m^{(2)}} J_3(\mu_m^{(2)})$

$\therefore C_m$ 中的分子 $= -\frac{1}{\mu_m^{(2)}} \left[ J_1(\mu_m^{(2)}) + J_3(\mu_m^{(2)}) \right] + \frac{2}{(\mu_m^{(2)})^2} \left[ 1 - J_0(\mu_m^{(2)}) \right]$

$= \frac{2}{(\mu_m^{(2)})^2} [1 - J_0(\mu_m^{(2)})$ $\quad (\because J_1(\mu_m^{(2)}) + J_3(\mu_m^{(2)}) = 0)$

$\therefore C_m = \frac{4[1 - J_0(\mu_m^{(2)})]}{(\mu_m^{(2)})^2 J_3^2(\mu_m^{(2)})}$

$\therefore u(r,t) = \sum_{m=1}^{\infty} \frac{4[1 - J_0(\mu_m^{(2)})]}{(\mu_m^{(2)})^2 J_3^2(\mu_m^{(2)})} e^{-4(\mu_m^{(2)})^2 t} J_2(\mu_m^{(2)} r)$. $\cdots 10'$