CHAPTER 8

STATE MODEL REPRESENTATION

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8.0 Introduction

- There are two models for systems: *input-output representation* and *state-variable representation*. The former describes the input/output behavior of systems. The latter describes the internal behavior of systems.

- Objectives: grasp basic concepts and methods of constructing and analyzing state model for both continuous- and discrete-time systems.

8.1 State Model (状态模型)

For a single-input single-output causal continuous-time system,

input : $v(t)$ output: $y(t)$

Question:

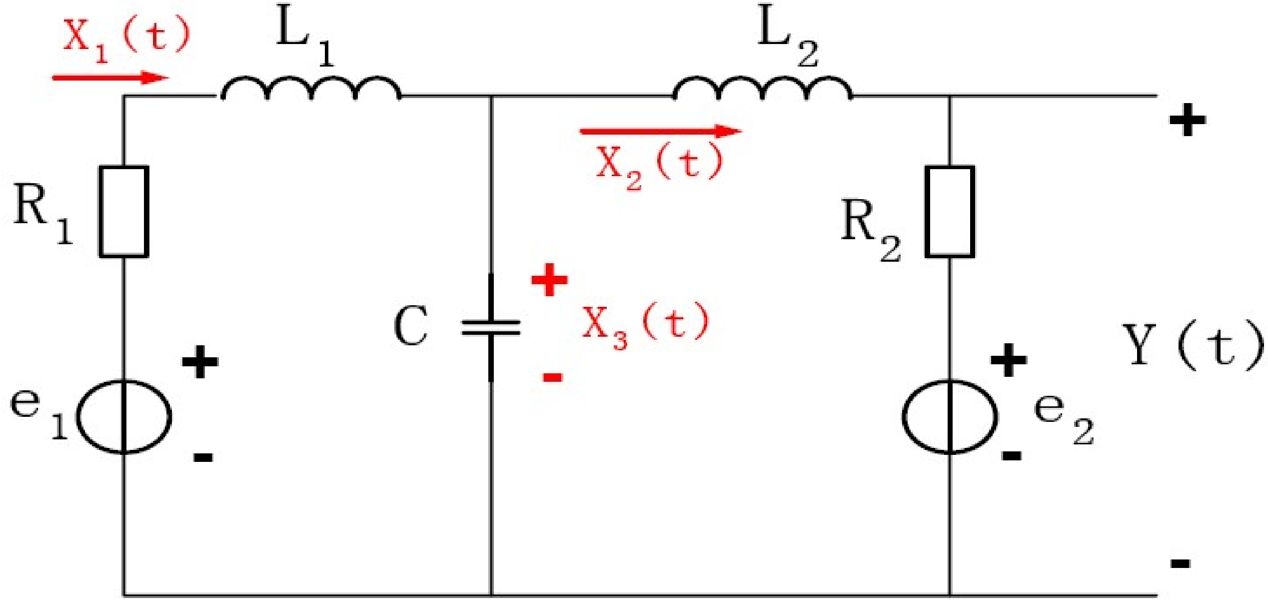
At the time of t_1, is it possible to compute the output response $y(t)$ from only the knowledge of the input $v(t)$ for $t \geq t_1$?

Obviously it is not. The reason is that the application of the input $v(t)$ for $t < t_1$ may put energy into the system that affects the output response for $t \geq t_1$.

- For any time point t_1, the state $x(t)$ of the system at time $t = t_1$ is defined to be that portion of the past history $t \le t_1$ of the system required to determine the output response $y(t)$ for all $t \ge t_1$ given the input $v(t)$ for $t \ge t_1$. A nonzero state $x(t_1)$ at time t_1 indicates the presence of energy in the system at time t_1.

- If the system is zero at t_1, $y(t)$ can be computed from $v(t)$ for $t \ge t_1$.

- If the system is not zero at t_1, knowledge of the state is necessary to be able to compute the output $y(t)$.



Example 8.1
Consider the circuit in the right-side figure.

Try to determine the currents of inductors L_1 and L_2, and the voltage of capacitor C, besides the output signal $y(t)$.

Sol: Let x_1 be the current through L_1,
x_2 be the current through L_2,
x_3 be the voltage on C,

$$KVL: \quad \begin{cases} L_1\dot{x}_1 + x_3 + R_1 x_1 = e_1 \\ L_2\dot{x}_2 + R_2 x_2 - x_3 = -e_2 \end{cases}$$

$$KCL: \quad C\dot{x}_3 = x_1 - x_2$$

$$y = R_2 x_2 + e_2$$

8.1 State Model

Rewrite the former equations, respectively, as

$$\dot{x}_1 = -\frac{R_1}{L_1}x_1 - \frac{1}{L_1}x_3 + \frac{1}{L_1}e_1$$

$$\dot{x}_2 = -\frac{R_2}{L_2}x_2 + \frac{1}{L_2}x_3 - \frac{1}{L_2}e_2$$

$$\dot{x}_3 = \frac{1}{C}x_1 - \frac{1}{C}x_2$$

$$y = R_2 x_2 + e_2$$

Matrix form representation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-\frac{R_1}{L_1} & 0 & -\frac{1}{L_1}\\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2}\\ \frac{1}{C} & -\frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}\frac{1}{L_1} & 0\\ 0 & -\frac{1}{L_2}\\ 0 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}$$

$$y=\begin{bmatrix}0 & R_2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}$$

If we have x_1, x_2, x_3, we can get the state of the system at arbitrary time. So they are necessary and enough.

8.1 State Model

From the example, if the given system is *N*-dimensional, the state $\boldsymbol{x}(t)$ of the system at time *t* is an *N*-element column vector given by:

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

The components $x_1(t)$, $x_2(t)$, …., $x_N(t)$ are called the *state variable* (状态变量) of the system.

8.2 State Equations (状态方程)

For a single-input single-output N-dimensional continuous-time system with state $\boldsymbol{x}(t)$ given by :

$$\boldsymbol{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

It can be modeled by the state equations given by :

derivative of the state vector

$$\dot{\boldsymbol{x}}(t)=f\left(\boldsymbol{x}(t),v(t),t\right)$$

$$y(t)=g\left(\boldsymbol{x}(t),v(t),t\right)$$

output equation

Here, both f and g are generally vector-valued function（矢量方程）of state $\boldsymbol{x}(t)$ at time t, the input $v(t)$ at time t, and time t.

- The above two equations comprise the *state model* of the system.
- The state equation describes the state response resulting from the application of an input $v(t)$ with initial state.
- The output equation gives the output response as a function of the state and input.

The two parts correspond to a cascade decomposition of the system as illustrated in Figure 1.

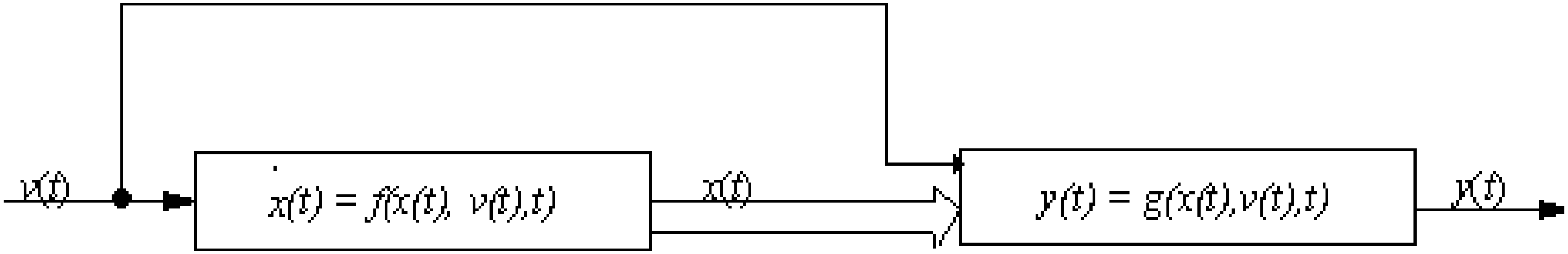


Figure 1 Cascade structure corresponding to state model

8.2 State Equations

If *f* and *g* are both linear, the state equations can be written in the form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)v(t)$$

$$y(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + D(t)v(t)$$

- $\boldsymbol{A}(t)$ is a $N \times N$ matrix whose entries are functions of time *t*;
- $\boldsymbol{B}(t)$ is a *N*-element column vector whose components are functions of *t*;
- $\boldsymbol{C}(t)$ is a *N*-element row vector with time-varying components;
- $D(t)$ is a real-valued function of time;
- The number *N* of state model variables is called the *dimension* of the state model (or system).

If the system is time invariant, then the state model is given by:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}v(t) \qquad (1)$$

$$y(t) = \boldsymbol{C}\,\boldsymbol{x}(t) + Dv(t) \qquad (2)$$

In this case, $\boldsymbol{A}(t)$, $\boldsymbol{B}(t)$, $\boldsymbol{C}(t)$ and $D(t)$ are constant.

With a_{ij} equal to the *ij* entry of $\boldsymbol{A}$ and b_i equal to the *i*th component of $\boldsymbol{B}$, (1) can be written in the expanded form:

$$
\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1N}x_N(t) + b_1 v(t) \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2N}x_N(t) + b_2 v(t) \\
&\vdots \\
\dot{x}_N(t) &= a_{N1}x_1(t) + a_{N2}x_2(t) + \cdots + a_{NN}x_N(t) + b_N v(t)
\end{aligned}
$$

With $c = [c_1 \quad c_2 \quad \cdots \; c_N]$, the expanded form of (2) is:

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t) + dv(t)$$

From the expanded form of the state equations, it is seen that the derivative $\dot{x}_i(t)$ of the *i*th state variable and the output *y*(*t*) are equal to linear combinations of all the state variables and the input.

8.3 Construction of State Models

Consider a single-input single-output continuous-time system given by the first-order input/output differential equation:

$$\dot{y}(t) = f\big(y(t), v(t), t\big)$$

Defining the state $x(t)$ of the system to be equal to $y(t)$ results in the state model:

$$\dot{x}(t) = f\big(x(t), v(t), t\big)$$

$$y(t) = x(t)$$

If the given system is LTI so that:

$$\dot{y}(t) = -ay(t) + bv(t)$$

a and *b* are constants, then the state model is :

$$\dot{x}(t) = -ax(t) + bv(t)$$

$$y(t) = x(t)$$

8.3 Construction of State Models

Suppose that the system has the second-order input/output differential equation:

$$\ddot{y}(t) = f\big(y(t),\ \dot{y}(t),\ v(t),\ t\big)$$

Defining the state variables by :

$$x_1(t) = y(t), \qquad x_2(t) = \dot{y}(t)$$

yields the state model:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = f\big(x_1(t),\ x_2(t),\ v(t),\ t\big)$$
$$y(t) = x_1(t)$$

8.3 Construction of State Models

Example 8.2 Consider a continuous-time second-order LTI system described by the following input-output equation:

$$\ddot{y}(t) = -a_1\dot{y}(t) - a_0 y(t) + b_0 v(t)$$

Construct its state model.

Sol: Let $x_1(t) = y(t), x_2(t) = \dot{y}(t)$ to obtain:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + b_0 v(t)$$

Thus,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

8.3 Construction of State Models

The defining of state variables in terms of the output and derivatives of the output extends to any system given by the *N*th-order input/output differential equation:

$$y^{(N)}(t) = f\left(y(t), y^{(1)}(t), \cdots, y^{(N-1)}(t), v(t), t\right)$$

with the state variables defined by

$$x_i(t) = y^{(i-1)}(t), \qquad i = 1, 2, \cdots, N$$

The resulting state equations are:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\vdots$$

$$\dot{x}_{N-1}(t) = x_N(t)$$

$$\dot{x}_N(t) = f\left(x_1(t), x_2(t), \cdots, x_N(t), v(t), t\right)$$

$$y(t) = x_1(t)$$

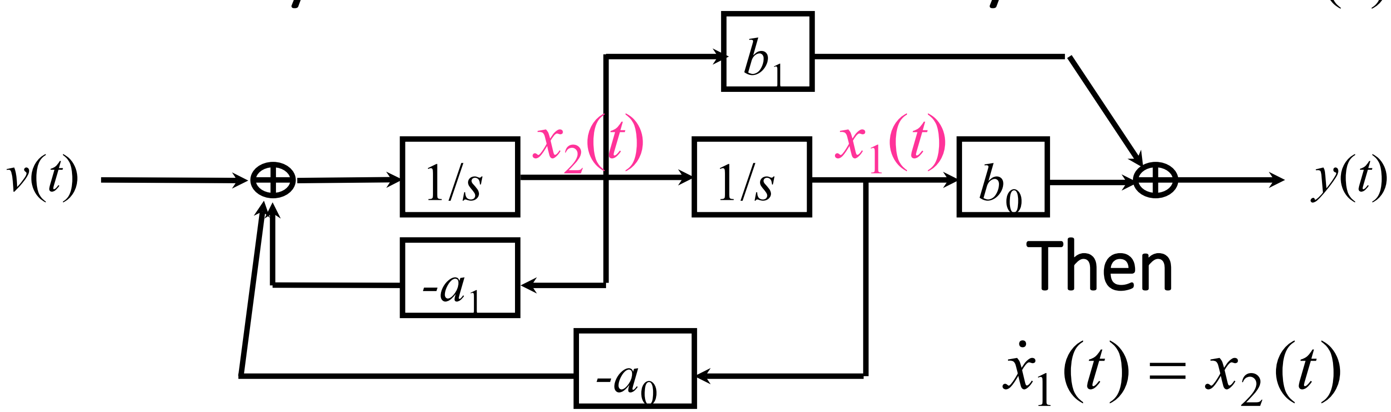
8.3 Construction of State Models

Example 8.3 If the input-output equation for a system is:

$$\ddot{y}(t)+a_1\dot{y}(t)+a_0 y(t)=b_1\dot{v}(t)+b_0 v(t)$$

Construct its state model.

Sol: The system function for the system is: $H(s)=\dfrac{b_1 s+b_0}{s^2+a_1 s+a_0}$



Then

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-a_1x_2(t)-a_0x_1(t)+v(t)$$

The state model is:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}v(t)$$

$$y(t)=b_1x_2(t)+b_0x_1(t)$$

$$y(t)=\begin{bmatrix}b_0 & b_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

8.3 Construction of State Models

Rewrite the system function as

$$H(s) = H_1(s)H_2(s) = \frac{1}{s^2 + a_1 s + a_0}\left(b_1 s + b_0\right)$$

Let $x_1(t) = z(t), x_2(t) = \dot{z}(t)$, where $z(t)$ is the output of $H_1(s)$.
Then

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + v(t)$$

And

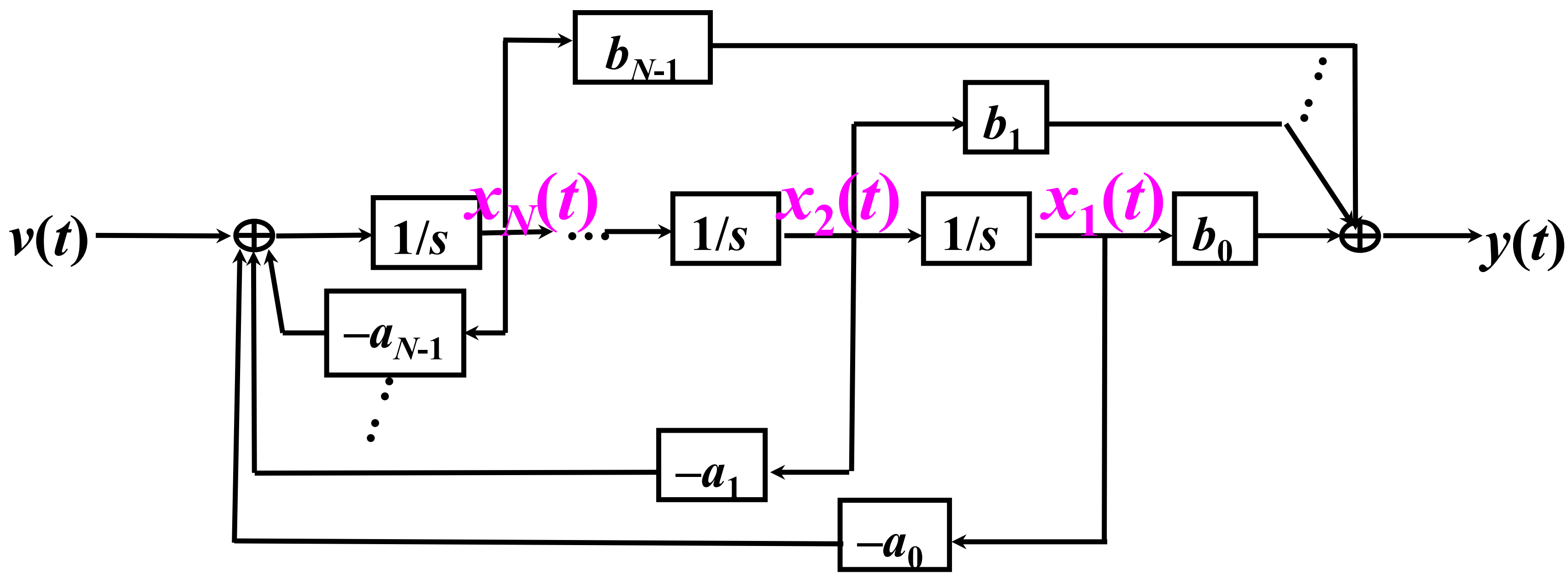
$$\begin{aligned} y(t) &= b_1 \dot{z}(t) + b_0 z(t) \\ &= b_1 x_2(t) + b_0 x_1(t) \end{aligned}$$

8.3 Construction of State Models

For a general LTI system given by the *N*th-order input/output differential equation:

$$y^{(N)}(t) + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = \sum_{i=0}^{N-1} b_i v^{(i)}(t)$$

Its block diagram representation is:



This system has the *N*-dimensional state model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}v(t), \quad y(t) = \boldsymbol{C}\,\boldsymbol{x}(t)$$

where :

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} b_0 \; b_1 \cdots b_{N-1} \end{bmatrix}$$

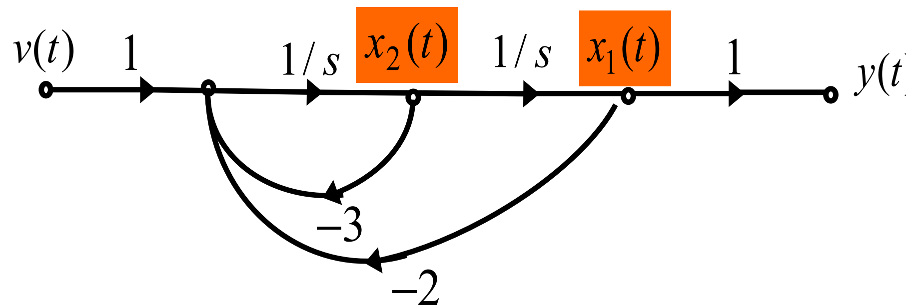
8.3 Construction of State Models

Example 8.4 Consider a continuous-time LTI system with transfer function

$$H(s)=\frac{1}{(s+1)(s+2)}$$

Draw the direct-, cascade- and parallel form signal flow graph of the system, respectively. And construct the state models of the system based on the signal flow graph, respectively.

Direct-form: $H(s)=\dfrac{1}{s^2+3s+2}$



State model:

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-2x_1(t)-3x_2(t)+v(t)$$

$$y(t)=x_1(t)$$

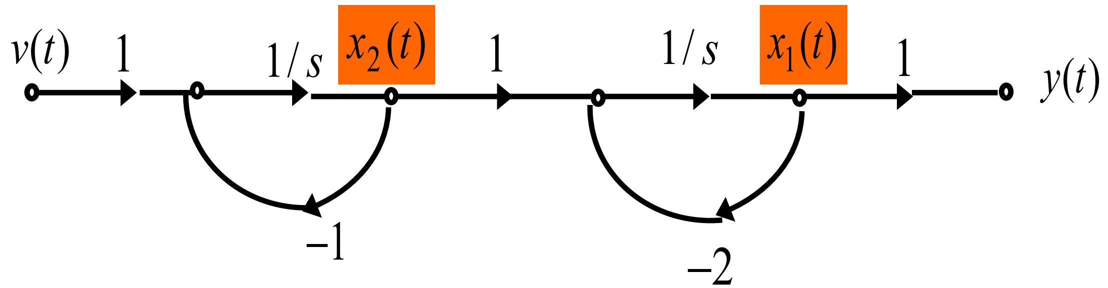
Matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}v \qquad y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

8.3 Construction of State Models

Cascade-form:

$$H(s)=\frac{1}{s+1}\cdot\frac{1}{s+2}$$



State model:

$$\dot{x}_1(t)=-2x_1(t)+x_2(t)$$

$$\dot{x}_2(t)=-x_2(t)+v(t)$$

$$y(t)=x_1(t)$$

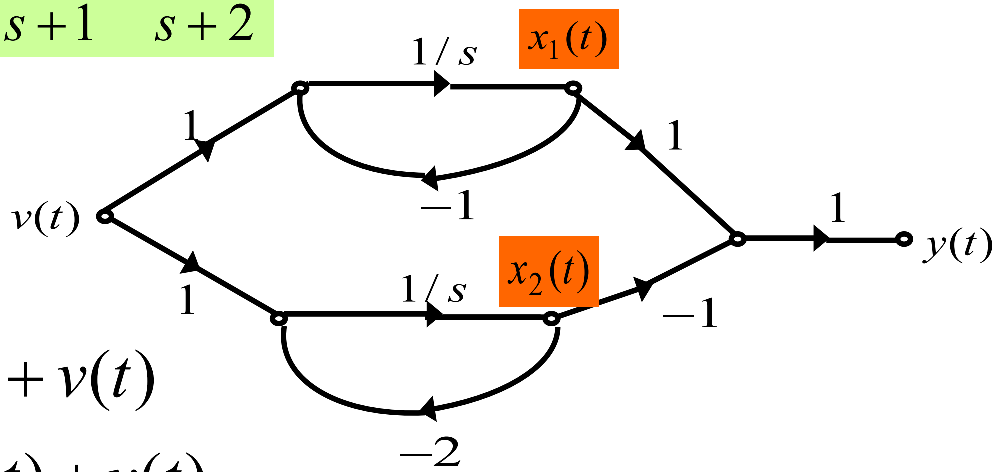
Matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}v \qquad y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

8.3 Construction of State Models

Parallel-form:

$$H(s) = \frac{1}{s+1} + \frac{-1}{s+2}$$



State model:

$$\dot{x}_1(t) = -x_1(t) + v(t)$$

$$\dot{x}_2(t) = -2x_2(t) + v(t)$$

$$y(t) = x_1(t) - x_2(t)$$

Matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v \qquad y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

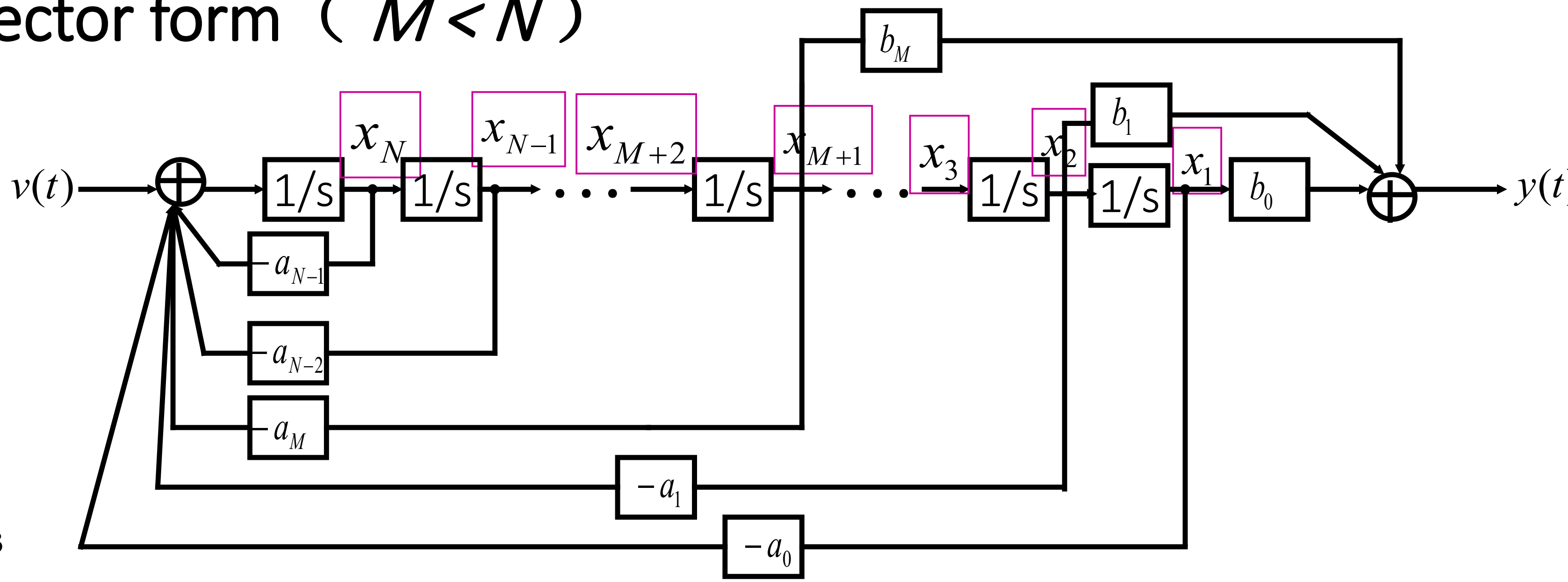
You may obtain different state equations depending on the different choice of state variables!

Summary on the general form of the state model:

Nth-order differential equation(*Scalar 标量*) $\rightarrow$ ***First-order differential equations in N-dimensional space***(*Vector 矢量*)

$$H(s)=\frac{b_M s^M + b_{M-1}s^{M-1}+\cdots+b_1 s+b_0}{s^N + a_{N-1}s^{N-1}+\cdots+a_1 s+a_0}$$

① Director form（ $M<N$ ）



$$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \cdots \\ \dot{x}_{N-1} = x_N \\ \dot{x}_N = -a_{N-1}x_n - a_{N-2}x_{n-1} - \cdots - a_1 x_2 - a_0 x_1 + v \end{cases}$$

$$y = b_0 x_1 + b_1 x_2 + \cdots + b_M x_{M+1}$$

8.3 Construction of State Models

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{N-1} \\ \dot{x}_N \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-2} & -a_{N-1} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} v$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 & \cdots & b_M & 0 & \cdots & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}$$

② Parallel form

$$H(s) = \frac{k_1}{s-\lambda_1} + \frac{k_2}{s-\lambda_2} + \cdots + \frac{k_N}{s-\lambda_N}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{N-1} \\ \dot{x}_N \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}}_{B} v$$

$$y = \underbrace{\begin{bmatrix} k_1 & k_2 & \cdots & k_{N-1} & k_N \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}$$

Example 8.5 *Integrator Realization*

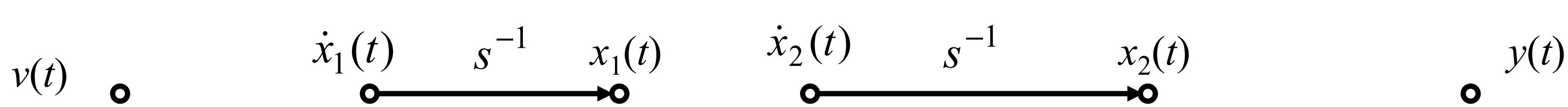
Consider a two-dimensional state model with arbitrary coefficients; that is,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

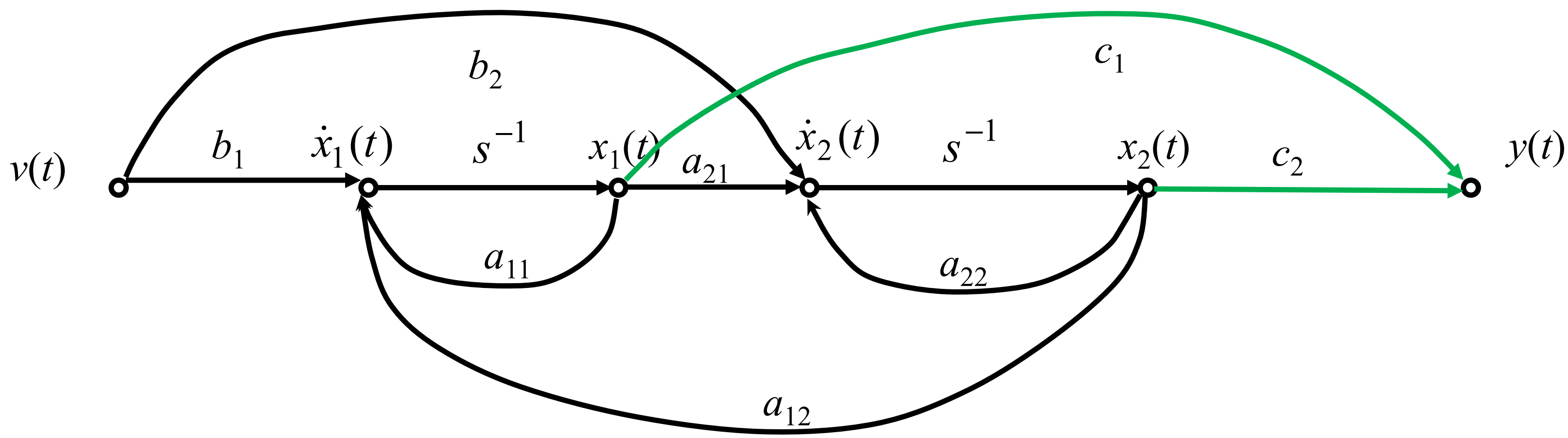
Draw the signal flow graph of the system.

Sol: Step 1: Define the output of each integrator in the interconnection to be a state variable. Then if the output of the ith integrator is $\dot{x}_i(t)$, the input to this integrator is $x_i(t)$.



Step 2: Realize the state equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} v(t)$$



Step 3: Realize the output equation

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

8.4 Multi-Input Multi-Output Systems

The state model of a p-input r-output LTI Nth-order continuous-time system is given by

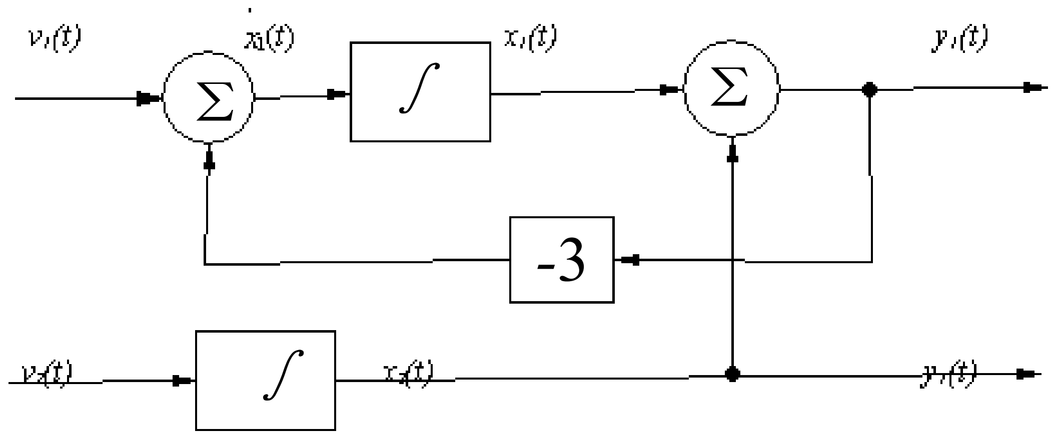
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{v}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{v}(t)$$

Where now $\boldsymbol{B}$ is a $N \times p$ matrix of real numbers, $\boldsymbol{C}$ is a $r \times N$ matrix of real numbers, and $\boldsymbol{D}$ is a $r \times p$ matrix.

8.4 Multi-Input Multi-Output Systems

Example 8.6 *Two-Input Two-Output System*

A two-input two-output system is shown in the following figure



Sol: From the figure,

$$\dot{x}_1(t) = -3y_1(t) + v_1(t) \qquad y_1(t) = x_1(t) + x_2(t)$$

$$\dot{x}_2(t) = v_2(t) \qquad y_2(t) = x_2(t)$$

Inserting the expression for $y_1(t)$ into the expression for $\dot{x}_1(t)$ gives

$$\dot{x}_1(t) = -3\left[x_1(t) + x_2(t)\right] + v_1(t)$$

Putting these equations in matrix from results in the state model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Matrix Exponential e^{At} (矩阵指数函数):

For each real value of *t*, e^{At} is defined by the matrix power series:

$$e^{At} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \frac{\boldsymbol{A}^4 t^4}{4!} + \cdots$$

Where $\boldsymbol{I}$ is the $N \times N$ identity matrix.

Properties of e^{At} :

- For any real numbers t and λ , $e^{A(t+\lambda)} = e^{At} \bullet e^{A\lambda}$
- e^{At} always has an inverse, which is equal to the matrix e^{-At}

$$e^{At} \bullet e^{-At} = e^{A(t-t)} = \boldsymbol{I}_N$$

- The derivative of the matrix exponential is

$$\frac{d}{dt} e^{At} = \boldsymbol{A} + \boldsymbol{A}^2 t + \frac{\boldsymbol{A}^3 t^2}{2!} + \frac{\boldsymbol{A}^4 t^3}{3!} + \cdots = \boldsymbol{A}\left(\boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \cdots \right)$$

$$= \boldsymbol{A} \bullet e^{At} = e^{At} \bullet \boldsymbol{A}$$

8.5 Solution of State Equations

From the derivative property of e^{At}, we have that the solution of $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), t > 0$ is:

$$\boldsymbol{x}(t) = e^{At} \bullet \boldsymbol{x}(0), \qquad t \geq 0$$

It is seen that the state $\boldsymbol{x}(t)$ at time $\boldsymbol{t}$ resulting from state $\boldsymbol{x}(0)$ at time $t = 0$ with no input applied for $t \geq 0$ can be computed by multiplying $\boldsymbol{x}(0)$ by the matrix e^{At} .

As a result of this property, The matrix e^{At} is called the *state-transition matrix* (状态转移矩阵, 状态过渡矩阵) of the system.

8.5 Solution of State Equations

For the state eqution $\boxed{\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{v}(t)}$,

Multiplying both sides on the left by e^{-At} and rearranging terms yields:

$$e^{-At}\left[\dot{\boldsymbol{x}}(t) - \boldsymbol{A}\boldsymbol{x}(t)\right] = e^{-At}\boldsymbol{B}\boldsymbol{v}(t)$$

From the derivative property we can get

$$\frac{d}{dt}\left[e^{-At}\boldsymbol{x}(t)\right] = e^{-At}\boldsymbol{B}\boldsymbol{v}(t)$$

$$e^{-At}\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t e^{-A\lambda}\boldsymbol{B}\boldsymbol{v}(\lambda)d\lambda$$

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\lambda)}\boldsymbol{B}\boldsymbol{v}(\lambda)d\lambda, \qquad t \geq 0$$

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + e^{At}\boldsymbol{B} * \boldsymbol{v}(t), \qquad t \geq 0$$

This is the complete solution of the state equation resulting from initial state $\boldsymbol{x}(0)$ and input $\boldsymbol{v}(t)$ applied for $t \geq 0$.

8.6 Output Response

From $\boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{v}(t)$ and the solution for the state equations, we can get:

$$\boldsymbol{y}(t)=\boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0)+\int_0^t \boldsymbol{C}e^{\boldsymbol{A}(t-\lambda)}\boldsymbol{B}\boldsymbol{v}(\lambda)d\lambda+\boldsymbol{D}\boldsymbol{v}(t), \qquad t\geq 0$$

From the definition of the unit impulse,we can rewrite the former equation as:

$$\boldsymbol{y}(t)=\boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0)+\int_0^t \left\{\boldsymbol{C}e^{\boldsymbol{A}(t-\lambda)}\boldsymbol{B}\boldsymbol{v}(\lambda)+\boldsymbol{D}\delta\left(t-\lambda\right)\boldsymbol{v}(\lambda)\right\}d\lambda, \qquad t\geq 0$$

Where the *zero-input response* and the *zero-state response* are:

$$\boldsymbol{y}_{zi}(t)=\boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

$$\boldsymbol{y}_{zs}(t)=\int_0^t \left\{\boldsymbol{C}e^{\boldsymbol{A}(t-\lambda)}\boldsymbol{B}\boldsymbol{v}(\lambda)+\boldsymbol{D}\delta\left(t-\lambda\right)\boldsymbol{v}(\lambda)\right\}d\lambda=\left[\boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{B}+\boldsymbol{D}\delta(t)\right]*\boldsymbol{v}(t)$$

The *impulse response matrix* is : $\boldsymbol{h}(t)=\boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{B}+\boldsymbol{D}\delta\left(t\right), \quad t\geq 0$

8.7 Solution via The Laplace Transform

Taking the Laplace transform of the equation $\dot{x}(t) = Ax(t) + Bv(t)$ gives:

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{V}(s)$$

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{V}(s)$$

From this we can get:

$e^{At} = inverse$ Laplace transform of $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$

Where $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ is the Laplace transform of the state-transition matrix e^{At}.

8.7 Solution via The Laplace Transform

Taking the Laplace transform of the output equation

$\boldsymbol{y}(t) = \boldsymbol{C}\,\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{v}(t)$ yields:

$$\boldsymbol{Y}(s) = \boldsymbol{C}\,\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{V}(s)$$

From the Laplace transform solution for state variable $\boldsymbol{x}(t)$, we can get:

$$\boldsymbol{Y}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + [\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}]\boldsymbol{V}(s)$$

If $\boldsymbol{x}(0) = 0$ then $\boldsymbol{Y}(s) = \boldsymbol{Y}_{zs}(s) = \boldsymbol{H}(s)\boldsymbol{V}(s)$

Where $\boldsymbol{H}(s)$ is the *transfer function matrix* of the system given by

$$\boldsymbol{H}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

8.7 Solution via The Laplace Transform

Example 8.7: Consider the two-input three-output two-dimensional system with state model $\dot{x}(t) = Ax(t) + Bv(t),\ y(t) = Cx(t)$, where

$$A = \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 1 & -1 \end{bmatrix}$$

If the initial state $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and input $v(t) = \begin{bmatrix} u(t) \\ e^{-t}u(t) \end{bmatrix}$, compute the output $y(t)$.

Sol: First compute the state-transition matrix. Since

$$(sI - A)^{-1} = \begin{bmatrix} s+3 & -1 \\ 2 & s+1 \end{bmatrix}^{-1} = \frac{1}{s^2+4s+5}\begin{bmatrix} s+1 & 1 \\ -2 & s+3 \end{bmatrix} = \frac{1}{(s+2)^2+1}\begin{bmatrix} s+1 & 1 \\ -2 & s+3 \end{bmatrix}$$

The state-transition matrix

$$e^{At} = e^{-2t}\begin{bmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{bmatrix} u(t)$$

8.7 Solution via The Laplace Transform

The state response $\boldsymbol{x}(t)$ resulting from the initial state $\boldsymbol{x}(0)$ with zero input is given by $\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0),\ t \ge 0$, so

$$\boldsymbol{x}_{zi}(t) = e^{-2t}\begin{bmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix}, \quad t \ge 0$$

The state response $\boldsymbol{x}(t)$ resulting from the input $\boldsymbol{v}(t) = \begin{bmatrix} u(t) \\ e^{-t}u(t) \end{bmatrix}$ is to be computed as follows:

Since $V(s) = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+1} \end{bmatrix}$, From $\boldsymbol{X}_{zs}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}V(s)$, we have

$$\boldsymbol{X}_{zs}(s) = \begin{bmatrix} s+3 & -1 \\ 2 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+1} \end{bmatrix} = \frac{1}{s^2+4s+5}\begin{bmatrix} s+1 & 1 \\ -2 & s+3 \end{bmatrix}\begin{bmatrix} \frac{5s+3}{s(s+1)} \\ \frac{3s+2}{s(s+1)} \end{bmatrix}$$

$$= \frac{1}{[(s+2)^2+1]s(s+1)}\begin{bmatrix} 5s^2+11s+5 \\ 3s^2+s \end{bmatrix}$$

Taking the inverse Laplace transform of $\boldsymbol{X}_{zs}(s)$ yields

$$\boldsymbol{x}_{zs}(t) = \begin{bmatrix} e^{-2t}(-1.5\cos t + 2.5\sin t) + 1 + 0.5e^{-t} \\ e^{-2t}(\cos t + 4\sin t) - e^{-t} \end{bmatrix} u(t)$$

Then the state variables are

$$\boldsymbol{x}(t) = \boldsymbol{x}_{zi}(t) + \boldsymbol{x}_{zs}(t) = \begin{bmatrix} e^{-2t}(-0.5\cos t + 2.5\sin t) + 1 + 0.5e^{-t} \\ e^{-2t}(2\cos t + 3\sin t) - e^{-t} \end{bmatrix} u(t)$$

The output response

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \begin{bmatrix} e^{-2t}(3.5\cos t + 8.5\sin t) + 1 - 1.5e^{-t} \\ e^{-2t}(5\cos t + \sin t) - 2 - 3e^{-t} \\ e^{-2t}(-2.5\cos t - 0.5\sin t) + 1 + 1.5e^{-t} \end{bmatrix} u(t)$$

Example(13.15): A two-input two-output LTI system has the transfer function matrix

$$\boldsymbol{H}(s)=\begin{bmatrix}\dfrac{1}{s+1} & 0\\ \dfrac{1}{s+1} & \dfrac{1}{s+2}\end{bmatrix}$$

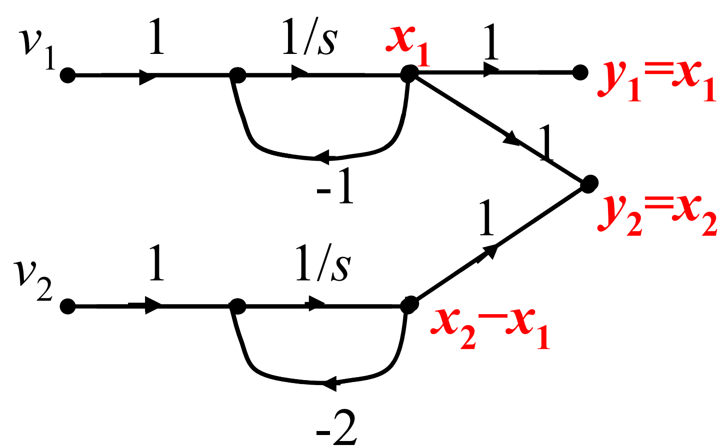
Find the state model of the system with the state variables defined to be $x_1(t)=y_1(t)$, $x_2(t)=y_2(t)$, where $y_1(t)$ is the first system output and $y_2(t)$ is the second system output.

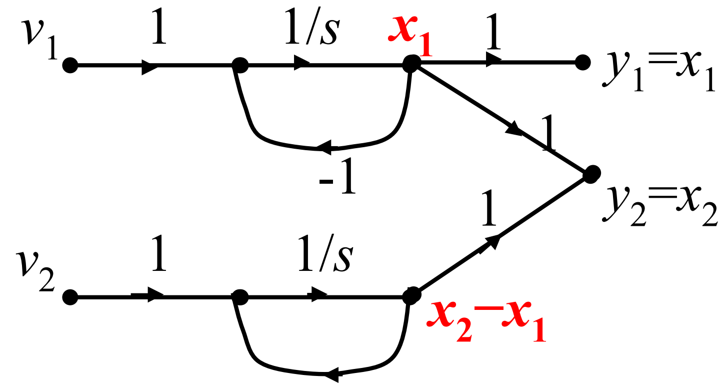
Sol: Suppose $v_1(t)$ and $v_2(t)$ are the inputs, from the transfer function matrix, we have

Drawing the diagram <u>step by step</u>:

$$\left.\frac{Y_1(s)}{V_1(s)}\right|_{V_2=0}=\frac{1}{s+1},$$

$$\left.\frac{Y_2(s)}{V_1(s)}\right|_{V_2=0}=\frac{1}{s+1},\quad \left.\frac{Y_2(s)}{V_2(s)}\right|_{V_1=0}=\frac{1}{s+2}$$





From the signal flow graph, we can write

$$\dot{x}_1 = -x_1 + v_1 \qquad y_1 = x_1$$

$$\dot{x}_2 - \dot{x}_1 = -2(x_2 - x_1) + v_2 \qquad y_2 = x_2$$

Matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

8.8 Discrete-Time Systems

A p-input r-output finite-dimensional linear time-invariant discrete-time system can be modeled by the state model:

$$\boldsymbol{x}[n+1] = \boldsymbol{A}\boldsymbol{x}[n] + \boldsymbol{B}\boldsymbol{v}[n]$$
$$\boldsymbol{y}[n] = \boldsymbol{C}\,\boldsymbol{x}[n] + \boldsymbol{D}\boldsymbol{v}[n]$$

The state vector $\boldsymbol{x}[n]$ is the N-element column vector:

$$\boldsymbol{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix}$$

The input $\boldsymbol{v}[n]$ and output $\boldsymbol{y}[n]$ are the column vectors:

$$\boldsymbol{v}[n] = \begin{bmatrix} v_1[n] \\ v_2[n] \\ \vdots \\ v_p[n] \end{bmatrix}, \quad \boldsymbol{y}[n] = \begin{bmatrix} y_1[n] \\ y_2[n] \\ \vdots \\ y_r[n] \end{bmatrix}$$

The matrix $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are $N \times N$, $N \times p$, $r \times N$ and $r \times p$ respectively.

8.9 Construction of State Models

For a single-input single-output LTI discrete-time system with the input/output difference equation:

$$y[n+N]+\sum_{i=0}^{N-1} a_i y[n+i]=\sum_{i=0}^{N-1} b_i v[n+i]$$

The system function is:

$$H(z)=\frac{\sum_{i=0}^{N-1} b_i z^i}{z^N+\sum_{i=0}^{N-1} a_i z^i}$$

Rewrite it as:

$$H(z)=H_1(z)H_2(z)=\frac{1}{z^N+\sum_{i=0}^{N-1} a_i z^i}\sum_{i=0}^{N-1} b_i z^i$$

Defining the state variables as

$$x_{i+1}[n]=f[n+i], \qquad i=0,1,2,\ldots,N-1$$

Where $f[n]$ is the output of the first sub-system $H_1(z)$.

8.9 Construction of State Models

Then $$x_1[n+1] = x_2[n]$$

$$x_2[n+1] = x_3[n]$$

$$\vdots$$

$$x_{N-1}[n+1] = x_N[n]$$

$$x_N[n+1] = -a_{N-1}x_N[n] - a_{N-2}x_{N-1}[n] - \cdots - a_0 x_1[n] + v[n]$$

$$y[n] = b_{N-1}x_N[n] + b_{N-2}x_{N-1}[n] + \cdots + b_1 x_2[n] + b_0 x_1[n]$$

Thus, the state model is: $$\boldsymbol{x}[n+1] = \boldsymbol{A}\boldsymbol{x}[n] + \boldsymbol{B}\boldsymbol{v}[n]$$

$$\boldsymbol{y}[n] = \boldsymbol{C}\,\boldsymbol{x}[n] + \boldsymbol{D}\boldsymbol{v}[n]$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C} = [b_0 \quad b_1, \quad \cdots \quad b_{N-1}], \quad D = 0$$

8.10 Solution of State Equations

Consider the *p*-input *r*-output discrete-time system with the state model:

$$\boldsymbol{x}[n+1] = \boldsymbol{A}\boldsymbol{x}[n] + \boldsymbol{B}\boldsymbol{v}[n] \qquad (1)$$

$$\boldsymbol{y}[n] = \boldsymbol{C}\boldsymbol{x}[n] + \boldsymbol{D}\boldsymbol{v}[n] \qquad (2)$$

Setting *n*=0 in (1) gives $\boldsymbol{x}[1] = \boldsymbol{A}\boldsymbol{x}[0] + \boldsymbol{B}\boldsymbol{v}[0]$

Setting *n*=1 in (1) gives

$$\begin{aligned}\boldsymbol{x}[2] &= \boldsymbol{A}\boldsymbol{x}[1] + \boldsymbol{B}\boldsymbol{v}[1] \\ &= \boldsymbol{A}\left[\boldsymbol{A}\boldsymbol{x}[0] + \boldsymbol{B}\boldsymbol{v}[0]\right] + \boldsymbol{B}\boldsymbol{v}[1] \\ &= \boldsymbol{A}^2\boldsymbol{x}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{v}[0] + \boldsymbol{B}\boldsymbol{v}[1]\end{aligned}$$

If this process is continued, for any integer value of $n \geq 1$,

$$\boxed{\boldsymbol{x}[n] = \boldsymbol{A}^n\boldsymbol{x}[0] + \sum_{i=0}^{n-1}\boldsymbol{A}^{n-i-1}\boldsymbol{B}\boldsymbol{v}[i], \qquad n \geq 1}$$

$$\boldsymbol{x}[n] = \boldsymbol{A}^n\boldsymbol{x}[0]u[n] + \sum_{i=0}^{n-1}\boldsymbol{A}^{n-i-1}\boldsymbol{B}\boldsymbol{v}[i]u[n-1] = \boldsymbol{A}^n\boldsymbol{x}[0]u[n] + \boldsymbol{A}^{n-1}\boldsymbol{B}u[n-1] * \boldsymbol{v}[n]$$

8.10 Solution of State Equations

The right-hand side of the former equation is the state response resulting from initial state $\boldsymbol{x}(0)$ and input $\boldsymbol{v}[n]$ applied for $n \geq 0$. Note that if $\boldsymbol{v}[n] = 0$ for $n \geq 0$, then

$$\boldsymbol{x}[n] = \boldsymbol{A}^n \boldsymbol{x}[0], \qquad n \geq 0$$

It is seen that the state transition from initial state $\boldsymbol{x}(0)$ to state $\boldsymbol{x}[n]$ at time $\boldsymbol{n}$ (with no input applied) is equal to $\boldsymbol{x}(0)$ times the matrix $\boldsymbol{A}^n$.

Therefore, in the discrete-time case the state-transition matrix is the matrix $\boldsymbol{A}^n$.

8.10 Solution of State Equations

Taking the former equation into the output equation gives:

$$\boldsymbol{y}[n]=\boldsymbol{C}\boldsymbol{A}^{n}\boldsymbol{x}[0]+\sum_{i=0}^{n-1}\boldsymbol{C}\boldsymbol{A}^{n-i-1}\boldsymbol{B}\boldsymbol{v}[i]+\boldsymbol{D}\boldsymbol{v}[n], \qquad n\geq 1$$

Where the term $\boldsymbol{y}_{zi}[n]=\boldsymbol{C}\boldsymbol{A}^{n}\boldsymbol{x}[0], \qquad n\geq 0$

is the *zero-input* response, and the term

$$\boldsymbol{y}_{zs}[n]=\sum_{i=0}^{n-1}\boldsymbol{C}\boldsymbol{A}^{n-i-1}\boldsymbol{B}\boldsymbol{v}[i]+\boldsymbol{D}\boldsymbol{v}[n], \qquad n\geq 1$$

$$=\left[\boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{B}u[n-1]+\boldsymbol{D}\delta[n]\right]*\boldsymbol{v}[n]$$

is the *zero-state* response.

With the sample response $$\boxed{\boldsymbol{h}[n]=\begin{cases}\boldsymbol{D}, & n=0\\ \boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{B}, & n\geq 1\end{cases}}$$

8.11 Solution via The z-Transform

Taking the z-transform of the vector difference equation gives :

$$z\boldsymbol{X}(z) - z\boldsymbol{x}[0] = \boldsymbol{A}\boldsymbol{X}(z) + \boldsymbol{B}\boldsymbol{V}(z)$$

Then

$$\boldsymbol{X}(z) = (z\boldsymbol{I} - \boldsymbol{A})^{-1} z\boldsymbol{x}[0] + (z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{V}(z)$$

Where $(z\boldsymbol{I} - \boldsymbol{A})^{-1} z$ is the z-transform of the state-transition matrix.

Thus $\boldsymbol{A}^n$ = inverse z-transform of $(z\boldsymbol{I} - \boldsymbol{A})^{-1} z$

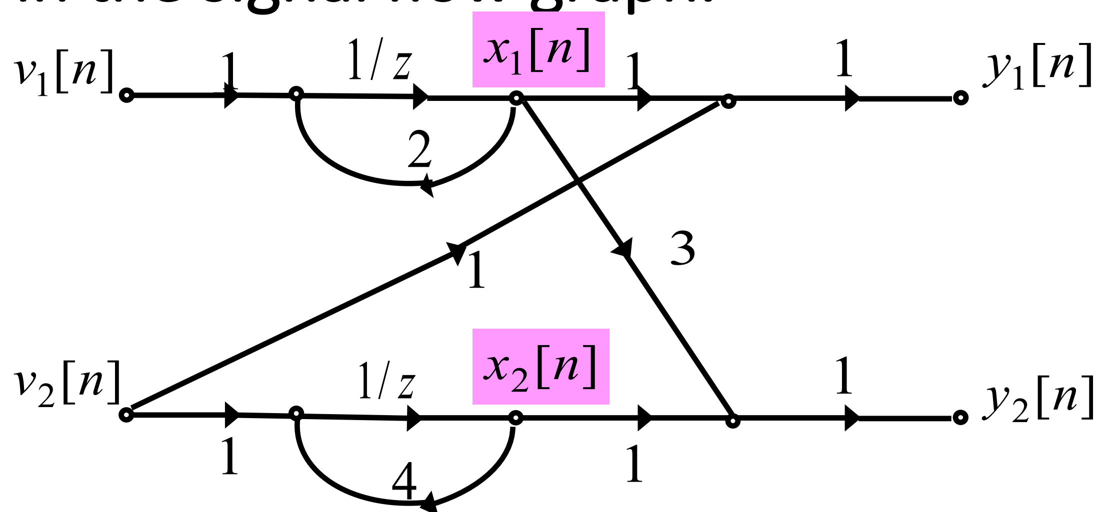
Taking the solution for state variable into the output equation to obtain:

$$\boldsymbol{Y}(z) = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1} z\boldsymbol{x}[0] + [\boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}]\boldsymbol{V}(z)$$

And the transfer function matrix $\boldsymbol{H}(z) = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$

8.11 Solution via The z-Transform

Example 8.8 Consider the two-input two-output two-dimensional system shown in the signal flow graph.



Construct the state equations and compute the state-transition matrix $\boldsymbol{A}^n$ and the transfer function matrix $\boldsymbol{H}(z)$.

Sol: From the signal flow graph, we can construct the following equations:

$$\begin{cases} x_1[n+1] = 2x_1[n] + v_1[n] \\ x_2[n+1] = 4x_2[n] + v_2[n] \end{cases} \qquad \begin{cases} y_1[n] = x_1[n] + v_2[n] \\ y_2[n] = 3x_1[n] + x_2[n] \end{cases}$$

8.11 Solution via The z-Transform

Matrix form:

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix}$$

$$\begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix}$$

Thus,

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad \boldsymbol{D} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The state-transition matrix:

$$\boldsymbol{A}^n = \mathcal{Z}^{-1}\{(\boldsymbol{I} - z^{-1}\boldsymbol{A})^{-1}\} = \mathcal{Z}^{-1} \begin{bmatrix} \dfrac{1}{1-2z^{-1}} & 0 \\ 0 & \dfrac{1}{1-4z^{-1}} \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix} u(n)$$

The transfer function matrix:

$$\boldsymbol{H}(z) = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} z-2 & 0 \\ 0 & z-4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{z-2} & 1 \\ \dfrac{3}{z-2} & \dfrac{1}{z-4} \end{bmatrix}$$

Example(13.23): Consider the discrete-time system with state model $\boldsymbol{x}[n+1]=\boldsymbol{A}\boldsymbol{x}[n]+\boldsymbol{B}v[n]$, $\boldsymbol{y}[n]=\boldsymbol{C}\boldsymbol{x}[n]$, where

$$\boldsymbol{A}=\begin{bmatrix}1 & 0\\ 0.5 & 1\end{bmatrix},\qquad \boldsymbol{B}=\begin{bmatrix}2\\ 1\end{bmatrix},\qquad \boldsymbol{C}=\begin{bmatrix}2 & -1\\ 1 & 0\\ 1 & -1\end{bmatrix}$$

The following parts are independent.

(a) Compute $y[0]$, $y[1]$, and $y[2]$ when $x[0]$ = [-1 2]', and the input $v[n]$ = sin(π/2)n;

(b) Suppose that $x[3]$ = [1 -1]'. Compute $x[0]$ assuming that $v[n]$ =0 for n=0,1,2,…;

(c) Suppose that $y[3]$ = [1 2 -1]'. Compute $x[3]$.

Sol: (a) From $x[0]$ = [-1 2]' and $\boldsymbol{y}[n]=\boldsymbol{C}\boldsymbol{x}[n]$ we get

$$\boldsymbol{y}[0]=\boldsymbol{C}\boldsymbol{x}[0]=\begin{bmatrix}2 & -1\\ 1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix}=[-4 \quad -1 \quad -3]'$$

Then from $\boldsymbol{x}[1]=\boldsymbol{A}\boldsymbol{x}[0]+\boldsymbol{B}v[0]$ and $v[0]=\sin(\pi/2)\cdot 0=0$ we get

$$\boldsymbol{x}[1]=\boldsymbol{A}\boldsymbol{x}[0]+\boldsymbol{B}v[0]=\begin{bmatrix}1 & 0\\ 0.5 & 1\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix}=\begin{bmatrix}-1 & 1.5\end{bmatrix}'$$

Thus $$\boldsymbol{y}[1]=\boldsymbol{C}\boldsymbol{x}[1]=\begin{bmatrix}2 & -1\\ 1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}-1\\ 1.5\end{bmatrix}=[-3.5 \quad -1 \quad -2.5]'$$

In the same way we can get

$$\boldsymbol{x}[2]=\boldsymbol{A}\boldsymbol{x}[1]+\boldsymbol{B}v[1]=\begin{bmatrix}1 & 0\\ 0.5 & 1\end{bmatrix}\begin{bmatrix}-1\\ 1.5\end{bmatrix}+\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}1 & 2\end{bmatrix}'$$

and

$$\boldsymbol{y}[2]=\boldsymbol{C}\boldsymbol{x}[2]=\begin{bmatrix}2 & -1\\ 1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=[0 \quad 1 \quad -1]'$$

(b) Since $u[n]$ =0 for n=0,1,2,…, then for n ≥ 0, $x[n] = A^n \boldsymbol{x}[0]$

So $\boldsymbol{x}[0] = \boldsymbol{A}^{-3}\boldsymbol{x}[3] = \left(\boldsymbol{A}^{-1}\right)^3 \boldsymbol{x}[3]$

By computing $\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}$ to get

$$\boldsymbol{x}[0] = \begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -2.5 \end{bmatrix}'$$

(c) Let $x[3] = [a \quad b]'$. From $\boldsymbol{y}[n] = \boldsymbol{C}\boldsymbol{x}[n]$ we have $\boldsymbol{y}[3] = \boldsymbol{C}\boldsymbol{x}[3]$, i.e.,

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a-b \\ a \\ a-b \end{bmatrix}$$

Solving above equations we get $a = 2, \quad b = 3$

That is $\boldsymbol{x}[3] = [2 \quad 3]'$

8.12 SUMMARY

- Concepts of state model, state variable, state equation, output equation, state transition matrix;
- Construction methods of state model for both continuous- and discrete-time systems;
- Time domain solutions of state model for both continuous- and discrete-time LTI systems;
- Laplace transform solution of state model for continuous-time LTI systems;
- Z-transform solution of state model for discrete-time LTI systems.

Homework

13.1 13.3 13.7 13.8 13.15

13.16 13.19 13.22 13.23