



## CHAPTER 3

# FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS

# Content

0

Introduction

1

The Response of LTI Systems to Complex Exponentials

2

Fourier Series Representation of Continuous-Time Periodic Signals

3

Convergence of the Fourier Series

4

Fourier Series Representation of Discrete-Time Periodic Signals

5

Fourier Series and LTI Systems

## 3.0 Introduction

---

- Eigenfunction property of LTI systems
- Representation of continuous-time and discrete-time **periodic** signals — Fourier Series
- Frequency spectrum of periodic signals
- Responses of LTI systems to periodic signals

## 3.1 The Response of LTI Systems to Complex Exponentials

- Defining two Functions  $H(s)$  and  $H(z)$  respectively:

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \qquad H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

Here  $H(s)$  and  $H(z)$  are functions of the complex variable  $s$  or  $z$ .

- A continuous-time LTI system with the impulse response  $h(t)$  responds to a complex exponential input  $e^{s_0 t}$  —  $s_0$  a complex constant — by

$$y(t) = e^{s_0 t} * h(t) = \int_{-\infty}^{\infty} h(\tau) e^{s_0(t-\tau)} d\tau = e^{s_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-s_0 \tau} d\tau = H(s_0) e^{s_0 t}$$

- A discrete-time LTI system with the impulse response  $h[n]$  responds to a complex exponential input  $z_0^n$  —  $z_0$  a complex constant — by

$$y[n] = z_0^n * h[n] = \sum_{k=-\infty}^{\infty} h[k] z_0^{n-k} = z_0^n \sum_{k=-\infty}^{\infty} h[k] z_0^{-k} = H(z_0) z_0^n$$

## 3.1 The Response of LTI Systems to Complex Exponentials

- Eigenfunction (特征函数): an input signal for which the system output is a constant times the input.  
For continuous-time systems:  $e^{s_0 t} \rightarrow H(s_0)e^{s_0 t}$   
For discrete-time systems:  $z_0^n \rightarrow H(z_0)z_0^n$
- $e^{st}$  is the eigenfunction of continuous-time LTI systems.  $z^n$  is the eigenfunction of discrete-time LTI systems. This is referred to as the **eigenfunction property** of LTI systems.
- If the input to an LTI system is represented as a linear combination of complex exponentials as:

$$x(t) = \sum_k a_k e^{s_k t} \quad \text{or} \quad x[n] = \sum_k a_k z_k^n$$

From **eigenfunction property** and **superposition property**, the output will be:

$$y(t) = \sum_k a_k H(s_k) e^{s_k t} \quad \text{or} \quad y[n] = \sum_k a_k H(z_k) z_k^n$$

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

### 3.2.1 Complex Exponential Fourier Series (指数型傅里叶级数)

Given periodic  $x(t)$  with fundamental period  $T$ , its complex exponential Fourier series is :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\omega_0 = 2\pi / T$   
↑  
Fundamental frequency  
(基频)

Coefficients  $a_k$  is a complex function of  $k\omega_0$ .

$a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$  : fundamental component or the first harmonic component  
(基波分量) (一次谐波分量)

$a_N e^{jN\omega_0 t} + a_{-N} e^{-jN\omega_0 t}$  : the  $N$ th harmonic components (N次谐波分量)

By taking use of orthogonal property of functions  $e^{jk\omega_0 t}$ ,  $k = 0, \pm 1, \dots$  we can derive the equation to determine the Fourier series coefficients:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

*synthesis* equation:  
(综合式)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

*analysis* equation:  
(分析式)

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt$$

*Fourier series coefficients* or *spectral coefficients* of  $x(t)$ :  $\{a_k\}$   
(傅里叶级数系数) (频谱系数)

*magnitude spectrum*:  $|a_k|$     *phase spectrum*:  $\angle a_k$   
(幅度频谱) (相位频谱)

constant component or *dc* of  $x(t)$ :  $a_0 = \frac{1}{T} \int_T x(t) dt$   
(常数或直流分量)

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

### 3.2.2 Trigonometric Fourier Series (三角型傅里叶级数)

For real  $x(t)$ :

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$$

or

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

The relationships between  $a_k$  and  $B_k$ ,  $C_k$ ,  $A_k$ ,  $\theta_k$  are:

$$a_k = B_k + jC_k$$

$$a_k = A_k e^{j\theta_k}$$

Note:  $\{a_k\}$  are generally complex, while  $B_k$ ,  $C_k$ ,  $A_k$ ,  $\theta_k$  are real, so trigonometric FS is for **real** periodic signals.



## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

### Example 3.1

Consider a real periodic signal  $x(t)$ , with fundamental frequency  $2\pi$ , that is expressed in the complex exponential Fourier series as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

where  $a_0 = 1$ ,  $a_1 = a_{-1} = 1/4$ ,  $a_2 = a_{-2} = 1/2$ ,  $a_3 = a_{-3} = 1/3$

Use the trigonometric form to express the signal  $x(t)$ .

Sol: Collecting the harmonic components which have same frequency, we have

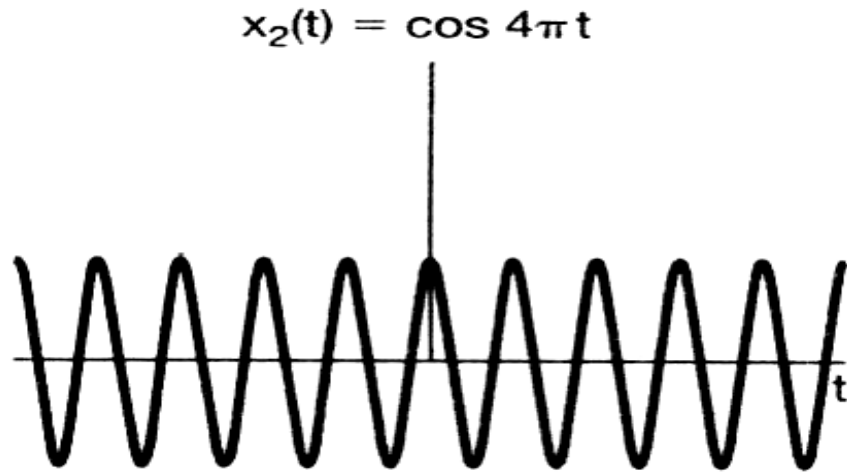
$$x(t) = 1 + \frac{1}{4} (e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2} (e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3} (e^{j6\pi t} + e^{-j6\pi t})$$

From Euler's relation

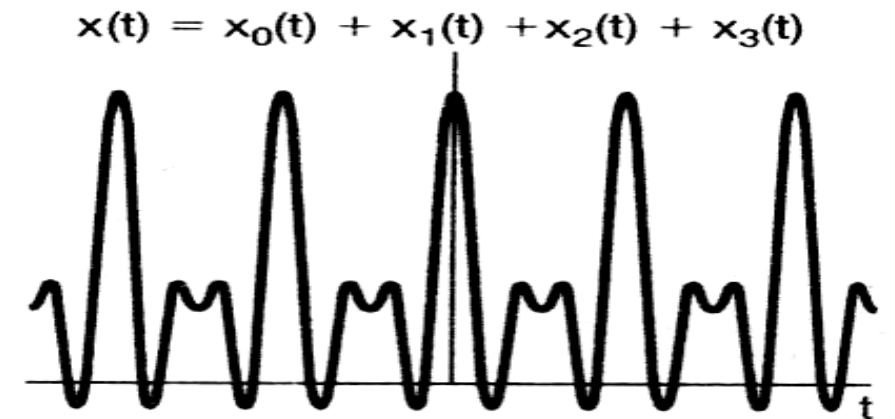
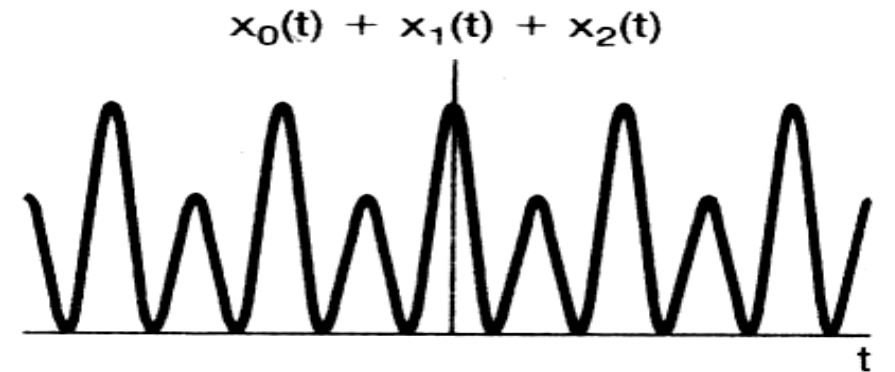
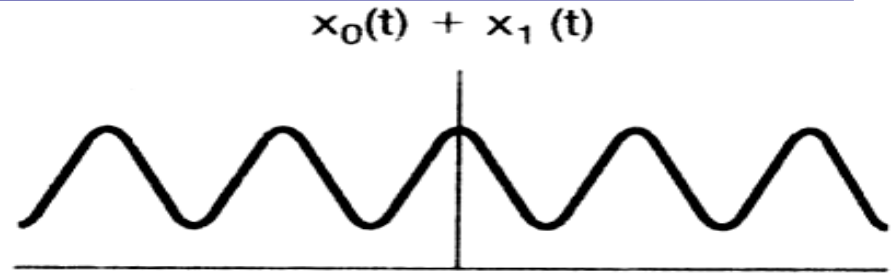
$$x(t) = 1 + \frac{1}{4} \cdot 2 \cos 2\pi t + \frac{1}{2} \cdot 2 \cos 4\pi t + \frac{1}{3} \cdot 2 \cos 6\pi t$$

$$= 1 + \frac{1}{2} \cos 2\pi t + \cos 4\pi t + \frac{2}{3} \cos 6\pi t$$

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals



$x_3(t) = \frac{2}{3} \cos 6\pi t$



Construction of the signal  $x(t)$  as a **linear combination of the harmonically related sinusoidal signals**

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

### Example 3.2

Consider the signal

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$$

Draw the graphs of magnitude and phase spectrums of  $x(t)$ .

Sol: Complex exponential form can be easily obtained as

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}e^{j(\pi/4)}e^{j2\omega_0 t} + \frac{1}{2}e^{-j(\pi/4)}e^{-j2\omega_0 t}$$

Thus, the Fourier series coefficients for this example are :

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j = \frac{\sqrt{5}}{2}e^{-j\arctan(1/2)},$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j = \frac{\sqrt{5}}{2}e^{j\arctan(1/2)},$$

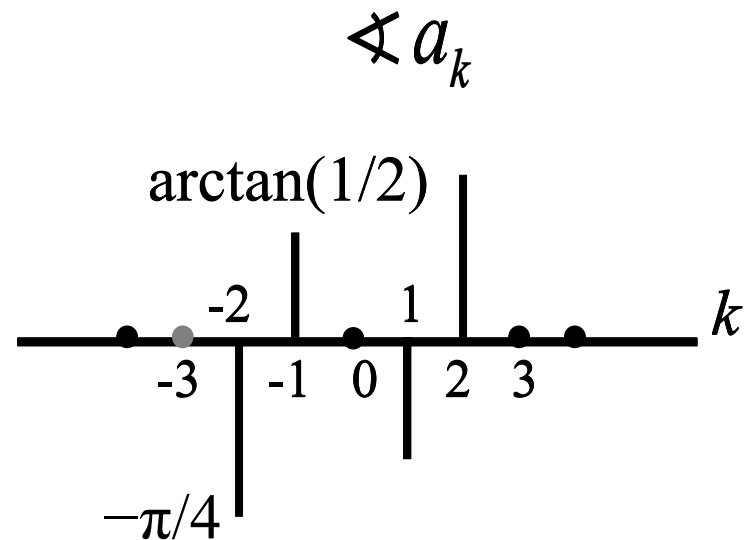
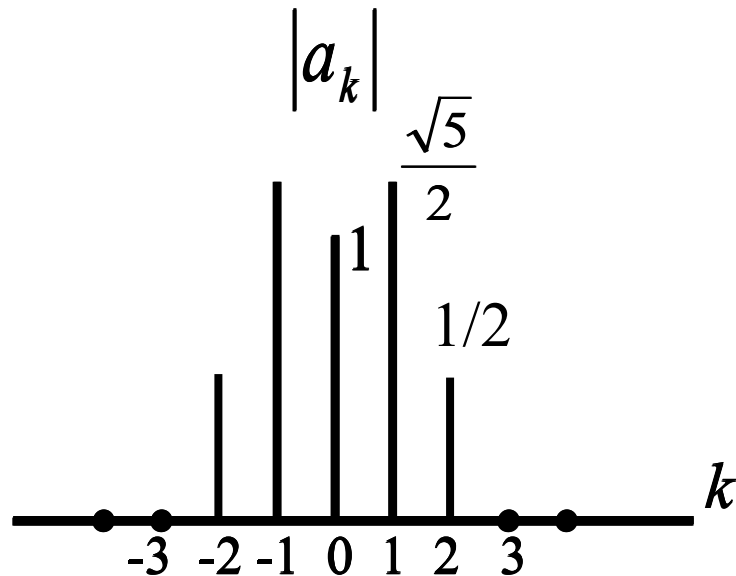
$$a_2 = \frac{1}{2}e^{j(\pi/4)},$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)},$$

$$a_k = 0, \quad |k| > 2.$$

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

$$a_0 = 1, \quad a_1 = \frac{\sqrt{5}}{2} e^{-j \arctan(1/2)}, \quad a_{-1} = \frac{\sqrt{5}}{2} e^{j \arctan(1/2)},$$
$$a_2 = \frac{1}{2} e^{j(\pi/4)}, \quad a_{-2} = \frac{1}{2} e^{-j(\pi/4)}$$



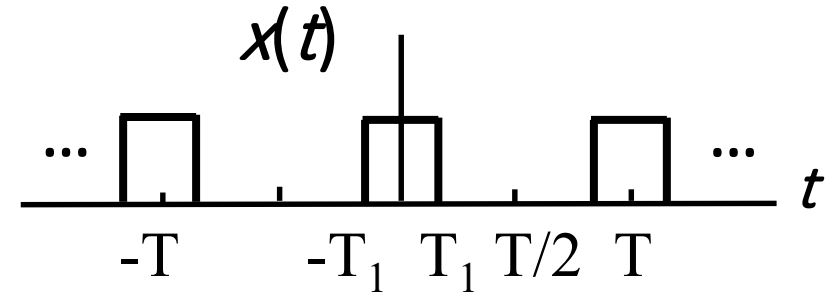
Plots of the magnitude and phase of Fourier coefficients of  $x(t)$

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

### Example 3.3

The periodic square wave is defined over one period as:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



Do some research into its frequency spectrum.

Sol:  $a_0 = \frac{1}{T} \int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$  ← Duty Circle (占空比)

$$a_k = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1} = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$

$$a_k = \frac{\sin\left(k\pi \frac{2T_1}{T}\right)}{k\pi}$$

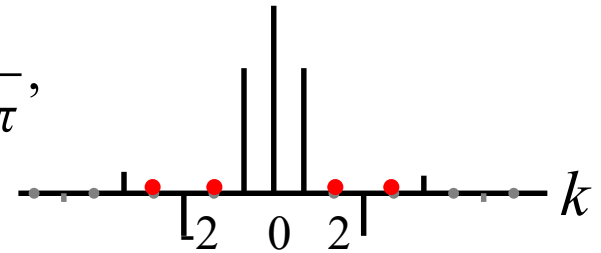
## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

$$a_k = \frac{\sin\left(k\pi \frac{2T_1}{T}\right)}{k\pi}$$

For  $T=4 T_1$ , duty circle=50%, the coefficients are:

$$a_0 = \frac{1}{2}, \quad a_1 = a_{-1} = \frac{1}{\pi}, \quad a_3 = a_{-3} = -\frac{1}{3\pi}, \quad a_5 = a_{-5} = \frac{1}{5\pi},$$

$$\vdots$$

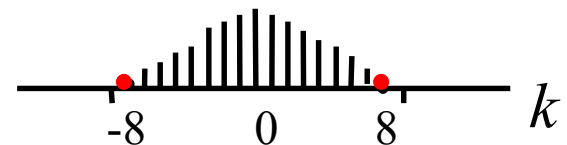
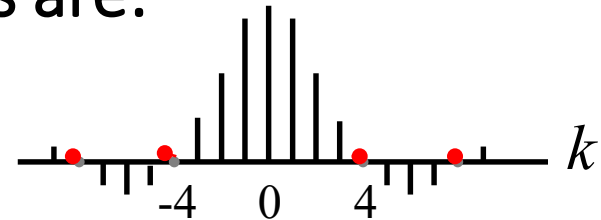


For  $T=8 T_1$ , duty circle=25%, coefficients are:

$$a_0 = \frac{1}{4}, \quad a_1 = a_{-1} = \frac{\sqrt{2}}{2\pi}, \quad a_2 = a_{-2} = \frac{1}{2\pi},$$

$$a_3 = a_{-3} = \frac{\sqrt{2}}{6\pi}, \quad a_4 = a_{-4} = 0, \quad a_5 = a_{-5} = -\frac{\sqrt{2}}{10\pi},$$

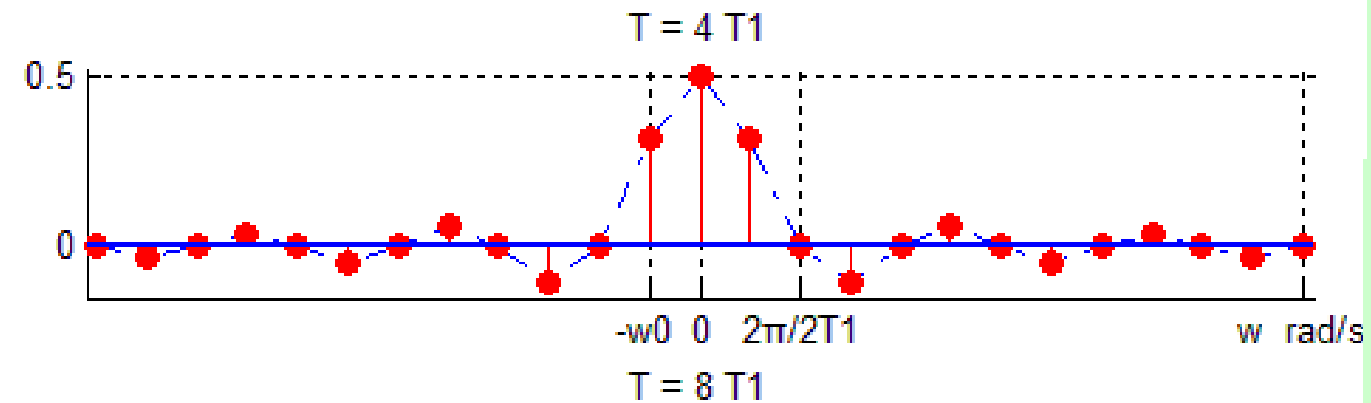
$$\vdots$$



Plots of the Fourier Series coefficients  $a_k$  for the periodic square wave with  $T_1$  fixed and for several values of  $T$ :

(a)  $T=4 T_1$ ; (b)  $T=8 T_1$ ; (c)  $T=16 T_1$ .

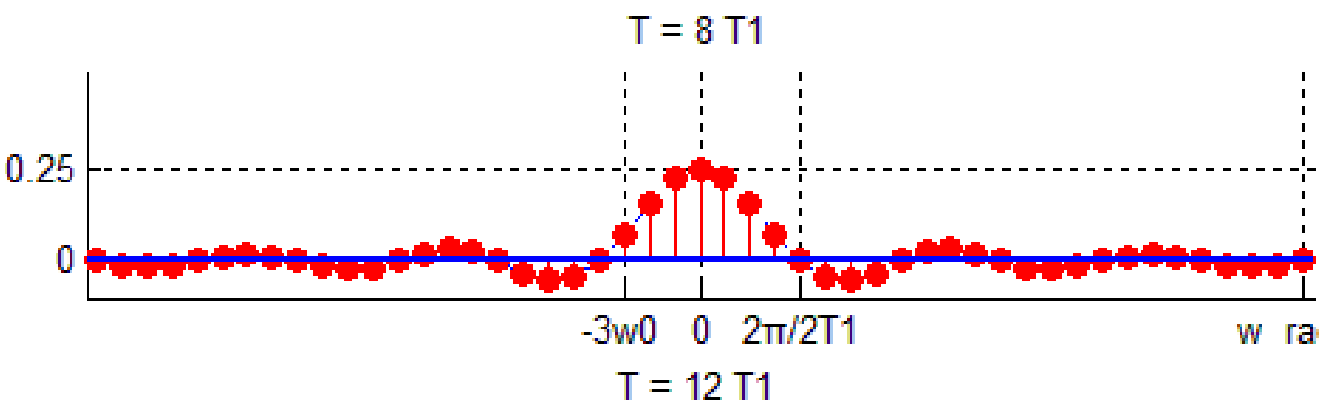
## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals



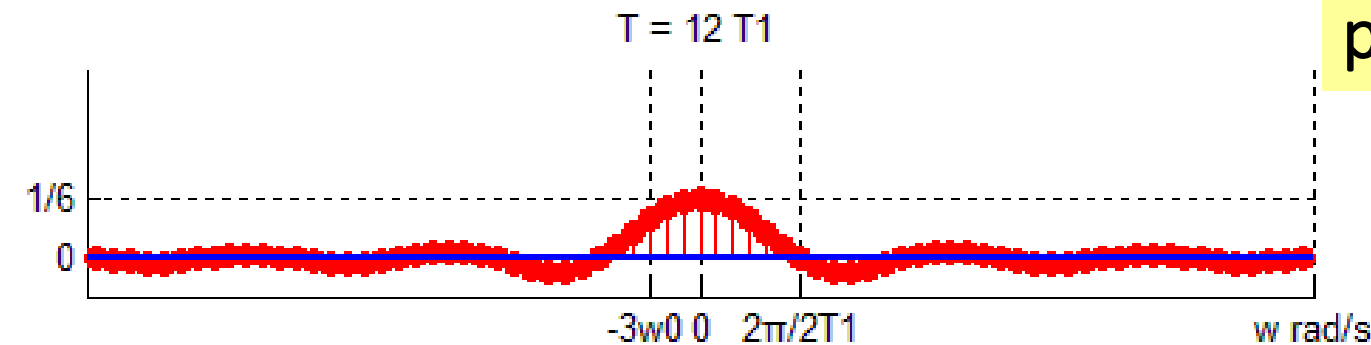
$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$$k\omega_0 T_1 = m \cdot \pi$$

$$\Rightarrow k\omega_0 = \frac{m \cdot \pi}{T_1} = m \frac{2\pi}{2T_1}$$



Properties of  
frequency  
spectrum of  
periodic signals :



Discrete  
Harmonic  
Convergent

Spectrum graph of periodic square wave

Consider: If  $T$  fixed,  $2 T_1$  decreases (duty circle decreases),  
what are the changes of the spectrum?

## 3.2 Fourier Series Representation of Continuous-Time Periodic Signals

---

### ➤ Frequency bandwidth

- ✓ If the spectral function is a *Sinc* (or *Sa*), the frequency bandwidth is the width of the range of positive frequencies from zero to the first zero point of *Sinc*.
- ✓ In engineering, *3-dB* or *half-power* bandwidth is generally used, which is the width of the range of positive frequencies where a peak value at zero is attenuated to 0.707 the value at the peak.

### ➤ Two conclusions obtained from the frequency spectrum analysis of periodic square wave :

- ✓ The width of pulse in time domain is inverse proportional to the width of frequency band in frequency domain.
- ✓ Signals changing faster in time domain must have wider frequency band width. (Thus contain more high frequencies in the frequency band width. )





### 3.3 Convergence of the Fourier Series

---

- The Dirichlet conditions (狄里赫利条件) (sufficient not necessary conditions):
  - ✓ Condition 1: Over any period,  $x(t)$  must be *absolutely integrable*; that is 
$$\int_T |x(t)| dt < \infty$$
  - ✓ Condition 2: In any finite interval of time,  $x(t)$  is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.
  - ✓ Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.
- If  $x(t)$  is continuous everywhere, then the series converges absolutely and uniformly; The infinite series equals  $x(t)$  at every continuity point and equals the average  $0.5[x(t+0_+) + x(t+0_-)]$  at every discontinuity point.

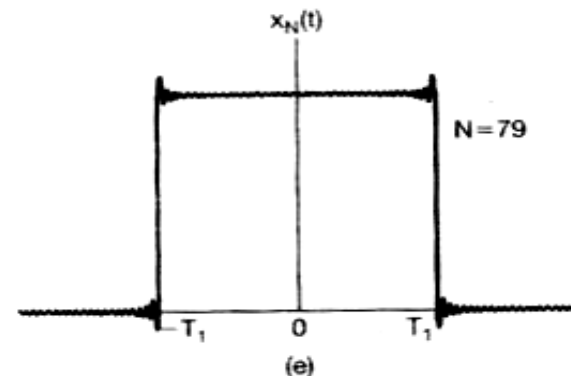
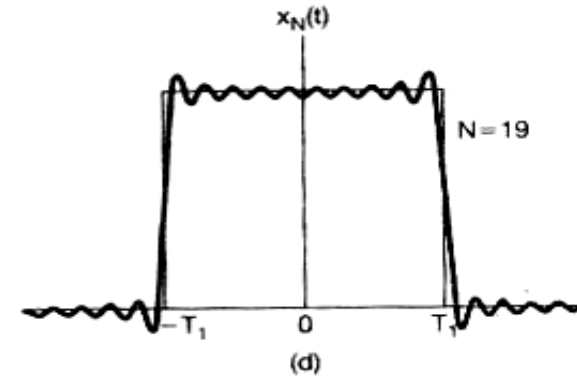
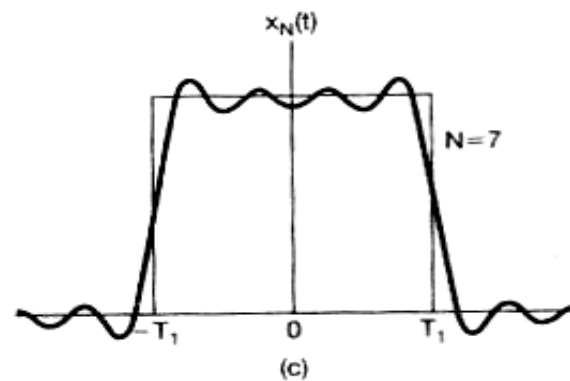
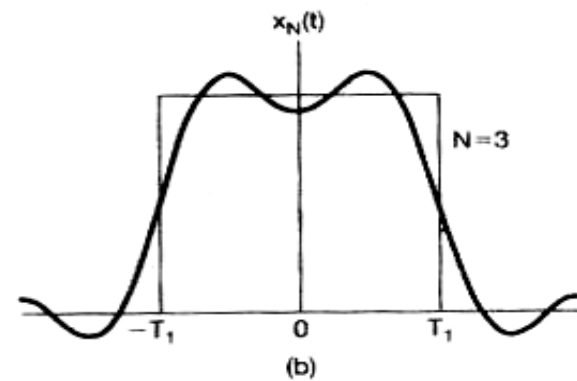
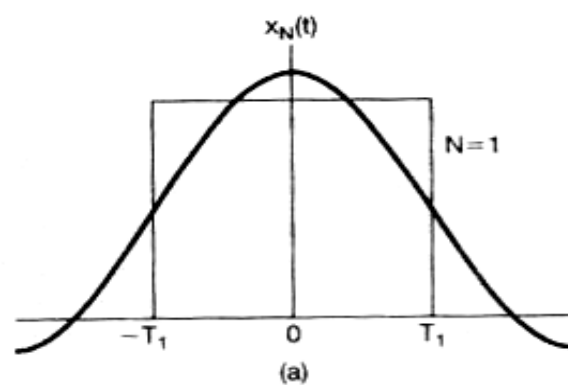
### 3.3 Convergence of the F

The smoother the sign  
with a Fourier series w

Consider an approxim

$$x_N(t) = \sum_{k=-N}^N a_k e^{jk\omega_0 t}$$

Convergence of the  
Fourier series  
representation of a  
square wave:  
an illustration of the  
Gibbs phenomenon  
(吉布斯现象).



## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

### 3.4.1 Linear Combination of Harmonically Related Complex Exponentials

Given periodic  $x[n]$  with fundamental period  $N$ , its Fourier series takes the form:

$$x[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n} \quad k = 0, \pm 1, \pm 2, \dots$$

Since

$$\begin{array}{ccccccc} e^{j0(2\pi/N)n} & = & e^{jN(2\pi/N)n} & = & \dots & = & e^{jrN(2\pi/N)n} \\ e^{j(2\pi/N)n} & = & e^{j(N+1)(2\pi/N)n} & = & \dots & = & e^{j(rN+1)(2\pi/N)n} \\ e^{j2(2\pi/N)n} & = & e^{j(N+2)(2\pi/N)n} & = & \dots & = & e^{j(rN+2)(2\pi/N)n} \\ \vdots & & \vdots & & & & \vdots \\ e^{j(N-1)(2\pi/N)n} & = & e^{j(2N-1)(2\pi/N)n} & = & \dots & = & e^{j(rN+N-1)(2\pi/N)n} \end{array}$$

## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

$e^{jk(2\pi/N)n}$  is periodic thus there are only  $N$  distinct signals in

$$\left\{ e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \dots \right\}$$

The summation need **only include terms over a range of  $N$  successive values of  $k$** . We use  $k = \langle N \rangle$  to indicate this. Then,

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

*finite series*

## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

### 3.4.2 Determination of the Fourier Series Representation of a Periodic Signal

Multiplying both sides of the discrete-time Fourier series equation by  $e^{-jr(2\pi/N)n}$  and summing over  $N$  terms, we obtain

$$\sum_{n=\langle N \rangle} x[n] e^{-jr(2\pi/N)n} = \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j(k-r)(2\pi/N)n}$$

Interchanging the order of summation on the right side yields

$$\sum_{n=\langle N \rangle} x[n] e^{-jr(2\pi/N)n} = \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-r)(2\pi/N)n}$$

$$\sum_{n=\langle N \rangle} e^{j(k-r)(\frac{2\pi}{N})n} = \frac{1 - \left[ e^{j\frac{2\pi}{N}(k-r)} \right]^N}{1 - e^{j\frac{2\pi}{N}(k-r)}} = \begin{cases} 0, & k \neq r + mN \\ N, & k = r + mN \end{cases}$$

### 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

The Fourier series coefficients are determined by equation:

$$a_r = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jr(2\pi/N)n}$$

*synthesis* equation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

*analysis* equation:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

↑  
*periodic*

$$a_k = a_{k+N}$$

Discreteness  $\leftrightarrow$  Periodicity

## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

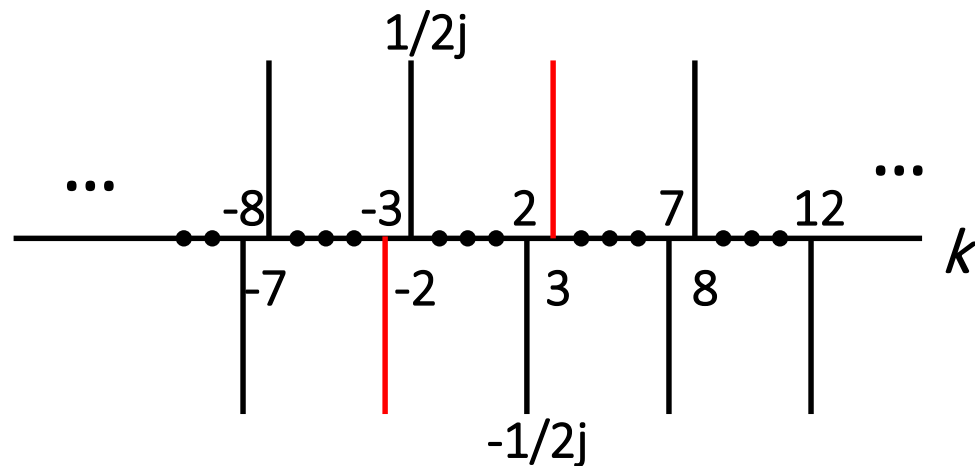
### Example 3.8

Consider the signal  $x[n] = \sin 3(2\pi/5)n$ , draw the graph of coefficients.

Sol: This signal is periodic with period  $N=5$ .

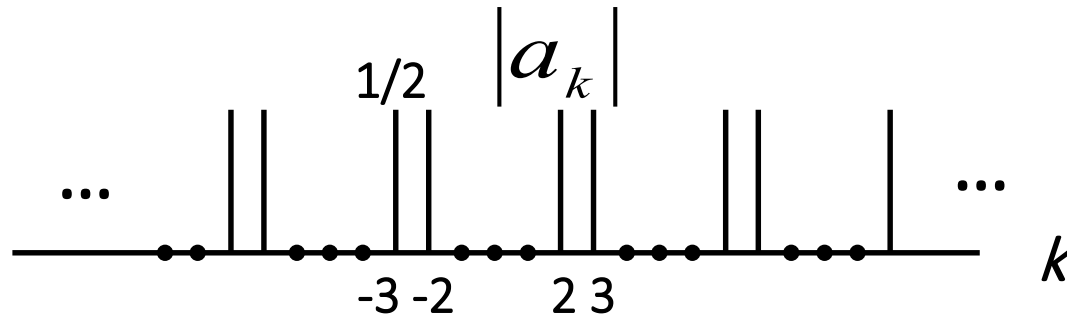
$$x[n] = \frac{1}{2j} e^{j3(2\pi/5)n} - \frac{1}{2j} e^{-j3(2\pi/5)n}$$

$$a_3 = \frac{1}{2j}, \quad a_{-3} = -\frac{1}{2j}$$

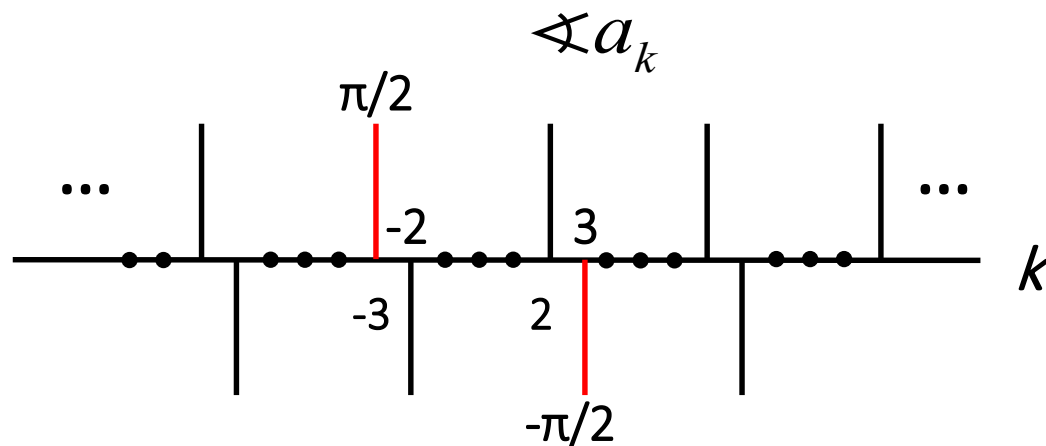


Fourier coefficients for  $x[n] = \sin 3(2\pi/5)n$ .

## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals



Magnitude of the coefficients



Phase of the coefficients

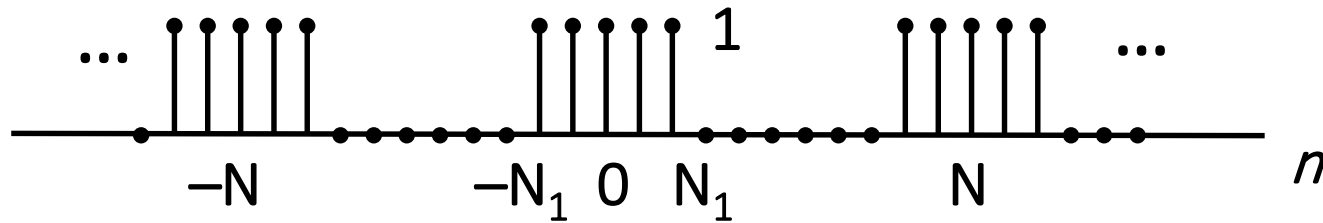




## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

### Example 3.9

Consider the discrete-time periodic square wave:



$$a_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j0(\frac{2\pi}{N})n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{2N_1 + 1}{N} = a_N = a_{-N} = a_{2N} = a_{-2N} = \dots$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(\frac{2\pi}{N})n} \quad \underline{\underline{m = n + N_1}} \quad \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(\frac{2\pi}{N})(m-N_1)} = \frac{1}{N} e^{jk(\frac{2\pi}{N})N_1} \sum_{m=0}^{2N_1} e^{-jk(\frac{2\pi}{N})m}$$

$$= \frac{1}{N} \frac{\sin \left[ 2k\pi \left( \frac{N_1 + 1/2}{N} \right) \right]}{\sin(\frac{k\pi}{N})}, \quad k \neq 0, \pm N, \pm 2N, \dots$$



## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

$$a_k = \frac{1}{N} \frac{\sin\left[2k\pi\left(\frac{N_1 + 1/2}{N}\right)\right]}{\sin\left(\frac{k\pi}{N}\right)}$$

$$k \neq 0, \pm N, \pm 2N, \dots$$

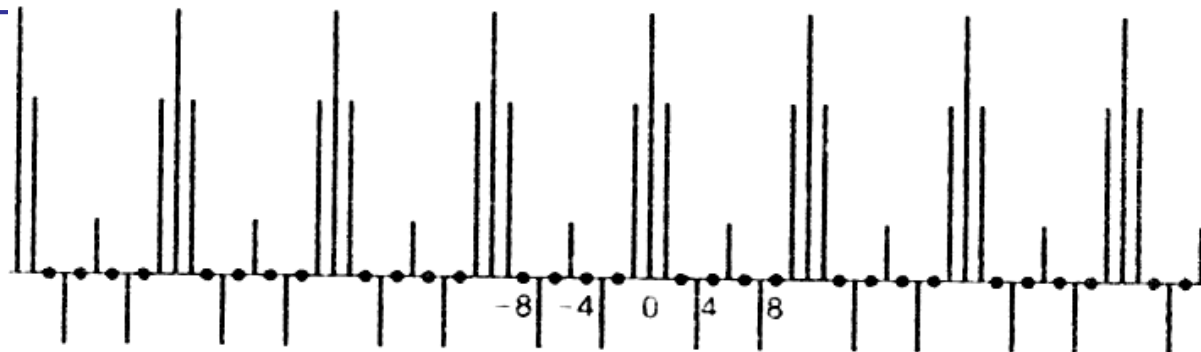
Fourier series coefficients for the periodic square wave of Example 3.9; plots of

$Na_k$  for  $2N_1 + 1 = 5$  and

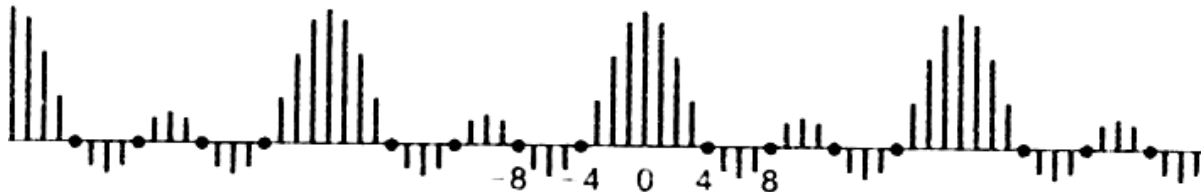
(a)  $N = 10$ ;

(b)  $N = 20$ ;

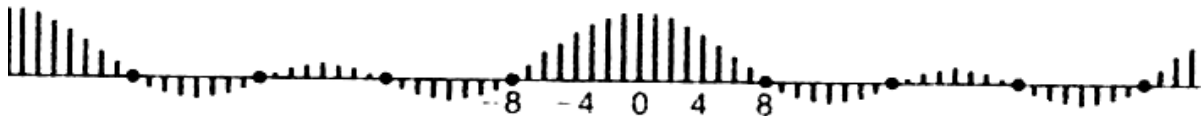
(c)  $N = 40$



(a)



(b)



(c)

## 3.4 Fourier Series Representation of Discrete-Time Periodic Signals

---

➤ In contrast to CTFS, there are no convergence issues with the DTFS in general, because any discrete-time periodic sequence  $x[n]$  is completely specified by a *finite* number  $N$  of parameters. So for discrete-time square wave we will not observe Gibbs phenomenon as the number of FS terms increases.

## 3.5 Fourier Series and LTI Systems

- Reviewing **eigenfunction property** of LTI systems:

$$x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}, -\infty < t < \infty$$

*system function*

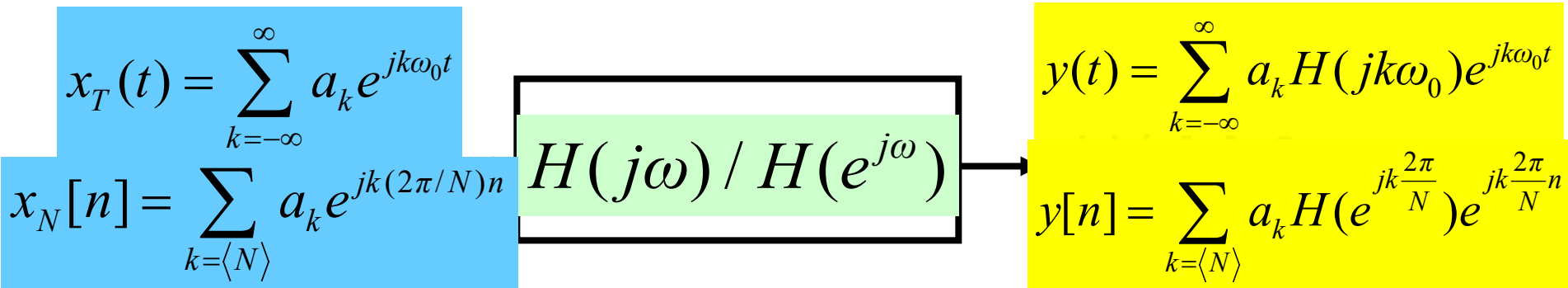
$$x[n] = z^n \rightarrow y[n] = H(z)z^n, -\infty < n < \infty$$

**(系统函数)**

- If  $\text{Re}\{s\} = 0, s = j\omega$ ,  $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$  *frequency response*

- If  $|z| = 1, z = e^{j\omega}$ ,  $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$  **(频率响应)**

- If the input signal of an LTI system is periodic, ...





## 3.5 Fourier Series and LTI Systems

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} \quad y[n] = \sum_{k=\langle N \rangle} a_k H(e^{j2k\pi/N}) e^{jk(2\pi/N)n}$$

➤ This results show that the response of an LTI system to a periodic signal is still *periodic*, and the output signal has **same period** as that of the input signal; furthermore, relationships between the FS coefficients of input and those of output are

$$x_T(t) \leftrightarrow a_k \rightarrow y_T(t) \leftrightarrow b_k = a_k H(jk\omega_0) \quad \text{Steady response}$$

$$x_N[n] \leftrightarrow a_k \rightarrow y_N[n] \leftrightarrow b_k = a_k H(e^{j2k\pi/N}) \quad (\text{稳态响应})$$

➤ The effect of the LTI system is to modify individually each of the Fourier coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

## 3.5 Fourier Series and LTI Systems

### Example 3.12

Input periodic signal  $x(t) = \sum_{k=-3}^3 a_k e^{jk2\pi t}$  as in example 3.1 to an LTI system with impulse response  $h(t) = e^{-t}u(t)$ , determine the Fourier series coefficients  $b_k$  of the output  $y(t)$ .

Sol: First compute the frequency response,

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau = -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau} \bigg|_0^{\infty} = \frac{1}{1+j\omega}$$

Since  $b_k = a_k H(jk2\pi)$

$$a_0 = 1, \quad a_1 = a_{-1} = 1/4,$$

$$a_2 = a_{-2} = 1/2,$$

$$a_3 = a_{-3} = 1/3$$

$$b_1 = \frac{1}{4} \left( \frac{1}{1+j2\pi} \right),$$

$$b_2 = \frac{1}{2} \left( \frac{1}{1+j4\pi} \right),$$

$$b_3 = \frac{1}{3} \left( \frac{1}{1+j6\pi} \right),$$

$$b_0 = a_0 \cdot \frac{1}{1+j0} = 1,$$

$$b_{-1} = \frac{1}{4} \left( \frac{1}{1-j2\pi} \right),$$

$$b_{-2} = \frac{1}{2} \left( \frac{1}{1-j4\pi} \right),$$

$$b_{-3} = \frac{1}{3} \left( \frac{1}{1-j6\pi} \right).$$

## 3.5 Fourier Series and LTI Systems

### Example 3.13

Consider an LTI system with sample response  $h[n] = \alpha^n u[n]$ ,  $-1 < \alpha < 1$ , and the input  $x[n] = \cos\left(\frac{2\pi}{N}n\right)$ , find the steady-response  $y[n]$ .

Sol: First compute the frequency response:

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = y[n] = r \cos\left(\frac{2\pi}{N}n + \theta\right)$$

Writing  $x[n]$  as

$$x[n] = \frac{1}{2} e^{j(2\pi/N)n} + \frac{1}{2} e^{-j(2\pi/N)n}$$

$$y[n] = \frac{1}{2} H(e^{j2\pi/N}) e^{j(2\pi/N)n} + \frac{1}{2} H(e^{-j2\pi/N}) e^{-j(2\pi/N)n}$$

$$r e^{j\theta} = \frac{1}{2} \left( \frac{1}{1 - \alpha e^{-j2\pi/N}} \right) e^{j(2\pi/N)n} + \frac{1}{2} \left( \frac{1}{1 - \alpha e^{j2\pi/N}} \right) e^{-j(2\pi/N)n} = r e^{-j\theta}$$

## 3.6 SUMMARY

- Eigenfunction property of LTI systems;
- Fourier series representations for both continuous-time and discrete-time periodic signals—i.e., weighted sum of harmonically related complex exponentials that share a common period;
- Different characteristics of signals reflected in their Fourier series coefficients (properties of Fourier series);
- Steady response of LTI systems — when input signal is periodic.



# Homework

3.22 (d) in (a)、(c) 3.25

3.27 3.28 (b) (c)

3.31 3.34 (a) (b)