华中科技大学2021-2022学年第一学期

《复变函数与积分变换》A卷参考答案及评分标准

一. 单选题. (每题2分, 共24分)

DBCC　BABA　DCAC

二. 解 (1)　$u_{xx} = 6x + 2ay$,　$u_{yy} = 2bx + 6y$

$\because$ u 为调和函数　$\therefore$ $u_{xx} + u_{yy} = 0$

即: $(6+2b)x + (2a+6)y = 0$

$\therefore$ $a = -3$, $b = -3$

(2)　$\because u(x,y) = x^3 - 3x^2y - 3xy^2 + y^3$

$\therefore u_x = 3x^2 - 6xy - 3y^2 \cdots\cdots ①$

$u_y = -3x^2 - 6xy + 3y^2 \cdots\cdots ②$

由 C-R方程: $u_x = v_y$

$\therefore v_y = 3x^2 - 6xy - 3y^2$

偏积分得: $v(x,y) = 3x^2y - 3xy^2 - y^3 + \varphi(x)$.

又 $u_y = -v_x$,　$\therefore$ $-3x^2 - 6xy + 3y^2 = -[6xy - 3y^2 + \varphi'(x)]$

$\Rightarrow \varphi'(x) = 3x^2$,　$\therefore$ $\varphi(x) = x^3 + C$

$\therefore v(x,y) = 3x^2y - 3xy^2 - y^3 + x^3 + C$

代入 $f(z) = u(x,y) + i\,v(x,y)$ 即可.

三. 解: (1) $f(z)=\frac{1}{z^2(z-i)} = -\frac{1}{iz^2}\frac{1}{(1-\frac{z}{i})}$

$$= -\frac{1}{iz^2}\sum_{n=0}^{+\infty}\left(\frac{z}{i}\right)^n$$

$$= -\sum_{n=0}^{+\infty}\frac{1}{i^{n+1}}z^{n-2}$$

(2) $\because \frac{1}{z} = \frac{1}{z-i+i} = \frac{1}{z-i}\frac{1}{1-(-\frac{i}{z-i})}$

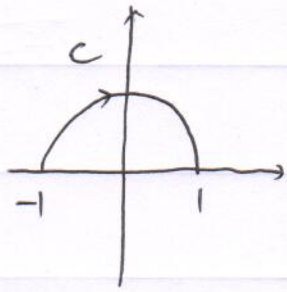
$$= \frac{1}{z-i}\sum_{n=0}^{+\infty}\left(-\frac{i}{z-i}\right)^n$$

$$= \sum_{n=0}^{+\infty}(-i)^n(z-i)^{-(n+1)}$$

$\therefore \frac{1}{z^2} = -\left(\frac{1}{z}\right)' = \sum_{n=0}^{+\infty}(-i)^n(n+1)(z-i)^{-(n+2)}$

$\therefore f(z) = \frac{1}{z^2(z-i)} = \sum_{n=0}^{+\infty}(-i)^n(n+1)(z-i)^{-(n+3)}$

四. 1. $\int_C \frac{3z}{\bar z} + \sin\frac{\pi z}{2}\,dz$



$= \int_C \frac{3z^2}{\bar z\cdot z} + \sin\frac{\pi z}{2}\,dz$

$= \int_C 3z^2 + \sin\frac{\pi z}{2}\,dz$

$= z^3 + (-\frac{2}{\pi}\cos\frac{\pi}{2}z)\Big|_{-1}^{1}$ （由被积函数解析性）

$= 2$

注：第一项的积分也可由曲线的参数方程化为定积分.

2. 由高阶导数定理. 原式 $= 2\pi i\,(e^{iz} - e^{-iz})'\Big|_{z=0}$

$= 2\pi i\,(i - 2i)$

$= 2\pi$.

另解：也可由留数定理计算，$z=0$ 是一阶极点

$\therefore$ 原式 $= 2\pi i\ \mathrm{Res}[f(z), 0]$

$= 2\pi i \lim\limits_{z\to 0} z\cdot\frac{e^{iz} - e^{i2z}}{z^2}$

$= 2\pi i \lim\limits_{z\to 0} \frac{e^{iz} - e^{i2z}}{z}$

$= 2\pi i \lim\limits_{z\to 0} (ie^{iz} - 2ie^{i2z})$

$= 2\pi i\,(-i)$

$= 2\pi$.

五. 1. 原式 $= -2\pi i \, \mathrm{Res}[f(z), \infty]$

$$= 2\pi i \, \mathrm{Res}[f(\tfrac{1}{z}) \cdot \tfrac{1}{z^2}, 0]$$

$$= 2\pi i \, \mathrm{Res}\left[\frac{1}{(1-z^2)(1+z^8)z^5}, 0\right]$$

$$\because \frac{1}{(1-z^2)(1+z^8)z^5} = \frac{1}{z^5}(1+z^2+z^4+z^6+\cdots)(1-z^8+z^{16}-\cdots)$$

$$= \cdots + \frac{1}{z} + \cdots$$

$$\therefore \mathrm{Res}\left[\frac{1}{(1-z^2)(1+z^8)z^5}, 0\right] = 1$$

$\therefore$ 原式 $= 2\pi i$

2. 原式 $= \dfrac{1}{2}\displaystyle\int_{-\infty}^{+\infty} \frac{1}{(x^2+3)(x^2+5)} dx$

设 $f(z) = \dfrac{1}{(z^2+3)(z^2+5)}$，则 $f(z)$ 在上半平面仅

有两个一阶极点 $\sqrt{3}i$ 和 $\sqrt{5}i$

$$\therefore \mathrm{Res}[f(z), \sqrt{3}i] = \left.\frac{1}{(z+\sqrt{3}i)(z^2+5)}\right|_{z=\sqrt{3}i} = \frac{1}{4\sqrt{3}i}$$

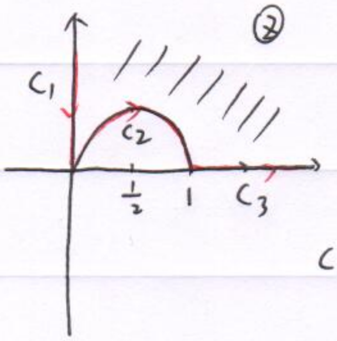
$$\mathrm{Res}[f(z), \sqrt{5}i] = \left.\frac{1}{(z+\sqrt{5}i)(z^2+3)}\right|_{z=\sqrt{5}i} = \frac{1}{-4\sqrt{5}i}$$

$$\therefore 原式 = \frac{1}{2} \cdot 2\pi i \left(\frac{1}{4\sqrt{3}i} + \frac{1}{-4\sqrt{5}i}\right)$$

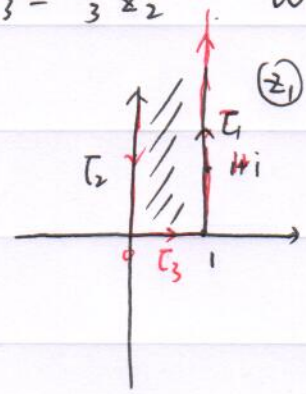
$$= \frac{\pi}{4}\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right) = \frac{\pi(\sqrt{5}-\sqrt{3})}{4\sqrt{15}}$$

六、解 $w=e^{\frac{\pi}{3}i(1-\frac{1}{z})}$ 可分解为如下几种映射的复合

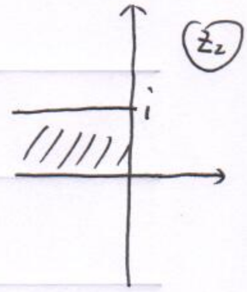
$z_1=\frac{z-1}{z}$, $z_2=iz_1$, $z_3=\frac{\pi}{3}z_2$, $w=e^{z_3}$



$z_1=\frac{z-1}{z}$



$z_2=iz_1$

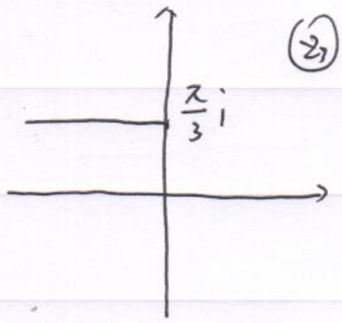


$$c_1:\begin{cases}\infty\to 1\\ i\to 1+i\\ 0\to\infty\end{cases}$$

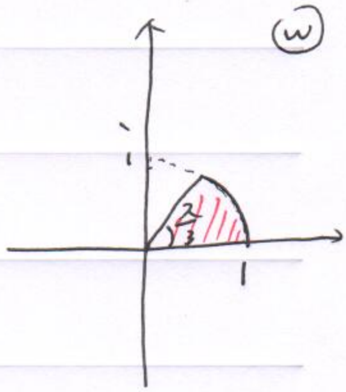
$$c_3\begin{cases}1\to 0\\ 2\to\frac{1}{2}\\ \infty\to 1\end{cases}$$

c_2 利用与 c_3 的保角性

$z_3=\frac{\pi}{3}z_2$

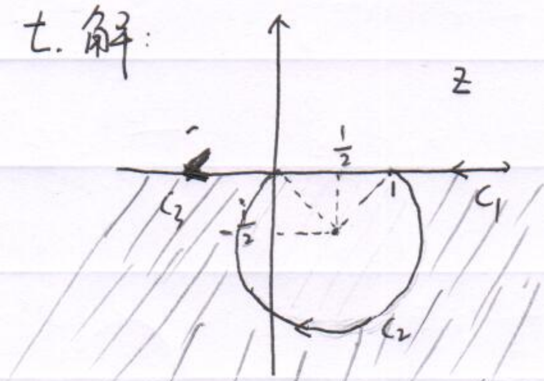


$w=e^{z_3}$



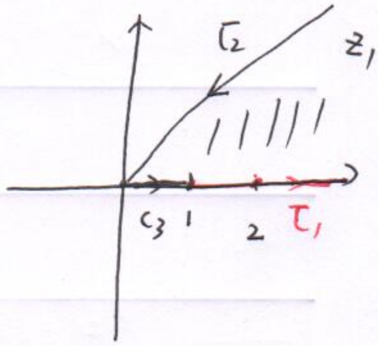
像区域为 $\{w: |w|<1,\ 0<\arg w<\frac{\pi}{3}\}$.

七. 解:



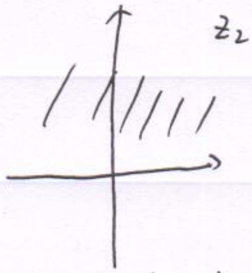
$$z_1=\frac{z}{z-1}$$

$c_1:\begin{cases}\infty\to 1\\ 2\to 2\\ 1\to\infty\end{cases}$



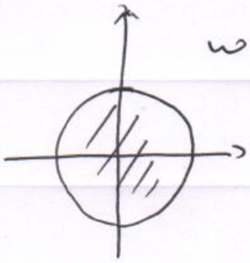
$c_3:\begin{cases}0\to 0\\ -1\to \frac{1}{2}\\ \infty\to 1\end{cases}$

$$z_2=z_1^4$$



c_2 利用保角性

$$w=\frac{z_2-i}{z_2+i}$$



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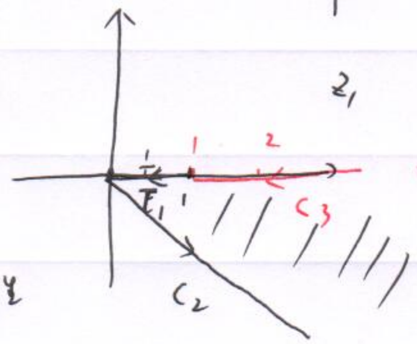
第一步里面若选 $z_1=\frac{z-1}{z}$

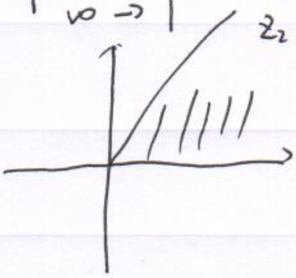


$c_1:\begin{cases}\infty\to 1\\ 2\to \frac{1}{2}\\ 1\to 0\end{cases}$

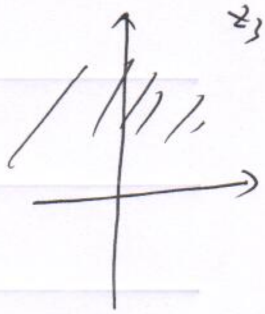
c_2 保角性

$c_3:\begin{cases}0\to \infty\\ -1\to 2\\ \infty\to 1\end{cases}$

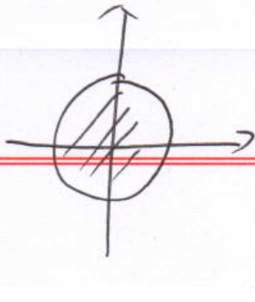
$$z_2=e^{\frac{\pi}{4}i}z_1$$



$$z_3=z_2^4$$



$$w=\frac{z_3-i}{z_3+i}$$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八. 解: 记 $X(s)=\mathcal{L}(x(t))$, $Y(s)=\mathcal{L}(y(t))$, 对方程组左右两边作 Laplace 变换得:

$$\begin{cases} sX(s)-1+Y(s)=2\cdot\frac{1}{s-1} \\ sY(s)-X(s)=-\frac{1}{s^2} \end{cases}$$

解得: $Y(s)=\frac{1}{(s-1)s}=\frac{1}{s-1}-\frac{1}{s}$

$$X(s)=\frac{1}{s^2}+\frac{1}{s-1}$$

$$\therefore y(t)=\mathcal{L}^{-1}[Y(s)]=e^t-1$$

$$x(t)=\mathcal{L}^{-1}[X(s)]=e^t+t$$

九：证明：$\because f(z)$ 在 $|z|\le 2$ 上解析

设 $g(z)=f'(z)$，则 $|g(z)-1|\le |z|$，且 $g(z)$ 在 $|z|\le 2$ 上解析.

$$\therefore f''(1)=g'(1)=\frac{1}{2\pi i}\oint_{|z-1|=r}\frac{g(z)}{(z-1)^2}\,dz \qquad (r\le 1)$$

$$\therefore |f''(1)|\le \frac{1}{2\pi}\oint_{|z-1|=r}\frac{|g(z)|}{|z-1|^2}\,ds$$

$$\le \frac{1}{2\pi}\oint_{|z-1|=r}\frac{|g(z)-1|+1}{|z-1|^2}\,ds$$

$$\le \frac{1}{2\pi}\oint_{|z|=r}\frac{|z|+1}{|z-1|^2}\,ds$$

$$\le \frac{1}{2\pi}\oint_{|z|=r}\frac{|z-1|+2}{|z-1|^2}\,ds$$

$$=\frac{1}{2\pi}\,\frac{r+2}{r^2}\cdot 2\pi r$$

$$=1+\frac{2}{r}$$

取 $r=1$. 则 $|f''(1)|\le 3$.