CHAPTER 6

THE LAPLACE TRANSFORM

Content

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6.0 Introduction

- The Laplace transform is a generalization of the continuous-time Fourier transform. It provides us with a representation for signals as linear combinations of complex exponentials of the form e^{st} with $s = \sigma + j\omega$.
- The system function $H(s)$ characterizes LTI systems in a different way.
- With Laplace transform, we expand the applications in which Fourier transform can or can not be used.
- Relationships between the Laplace transform and the continuous-time Fourier transform.

6.1 The Laplace Transform

6.1.1 Introduction of The Laplace Transform

For some signals which is not absolutely integrable, we can *preprocess* them by multiplying with a real exponential $e^{-\sigma t}$ and then calculate the Fourier transform of the product as:

$$\int_{-\infty}^{\infty}\left[x(t)e^{-\sigma t}\right]e^{-j\omega t}dt$$

Let $s = \sigma + j\omega$, and using $X(s)$ to denote this integral, we obtain

$$\boxed{X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt}$$

The *Laplace transform* of $x(t)$

The Laplace transform is a *generalization* of the Fourier transform with the exponential function taking the form e^{st}; the Fourier transform is a *special case* of the Laplace transform when $\sigma = 0$.

6.1.2 Examples

Example 6.1

Consider the signal $x(t) = e^{-\alpha t}u(t)$.

$$\text{Sol}: X(s) = \int_{-\infty}^{\infty} e^{-\alpha t}u(t)e^{-st}\,dt = \int_{0}^{+\infty} e^{-(s+\alpha)t}\,dt$$

$$= -\frac{1}{s+\alpha}e^{-(s+\alpha)t}\Big|_{0}^{\infty} = -\frac{1}{s+\alpha}e^{-(s+\alpha)\lim\limits_{t\to\infty} t} + \frac{1}{s+\alpha}$$

For convergence, we require that Re{$s+\alpha$} > 0, or Re{s} > $-\alpha$

Thus,

$$\boxed{X(s) = \frac{1}{s+\alpha}, \quad \mathrm{Re}\{s\} > -\alpha}$$

region of convergence (*ROC*)

(收斂域)

Example 6.2

Consider the signal $x(t) = -e^{-\alpha t}u(-t)$.

$$\text{Sol}: X(s) = -\int_{-\infty}^{\infty} e^{-\alpha t} e^{-st} u(-t) dt = -\int_{-\infty}^{0} e^{-(s+\alpha)t} dt$$

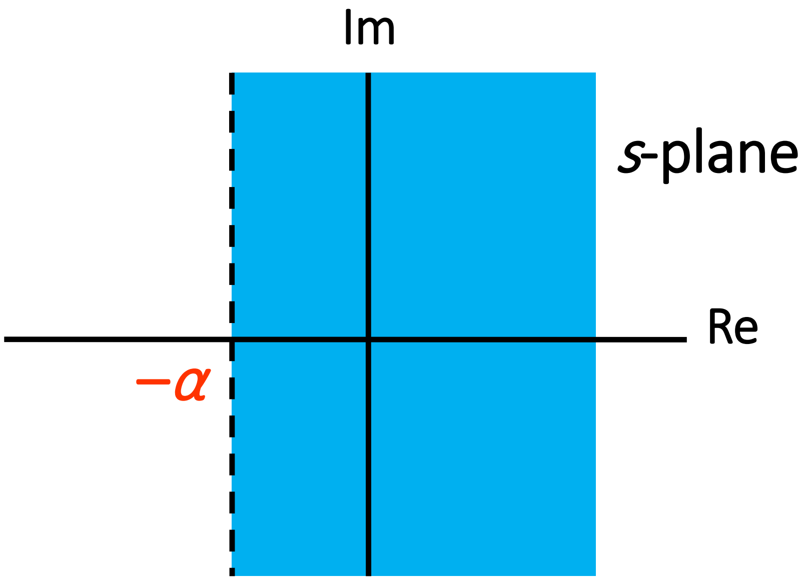
$$= \frac{1}{s+\alpha} e^{-(s+\alpha)t} \Big|_{-\infty}^{0} = \frac{1}{s+\alpha} - \frac{1}{s+\alpha} e^{-(s+\alpha)\lim\limits_{t \to -\infty} t}$$

For convergence, we require that $\mathrm{Re}\{s+\alpha\} < 0$, or $\mathrm{Re}\{s\} < -\alpha$

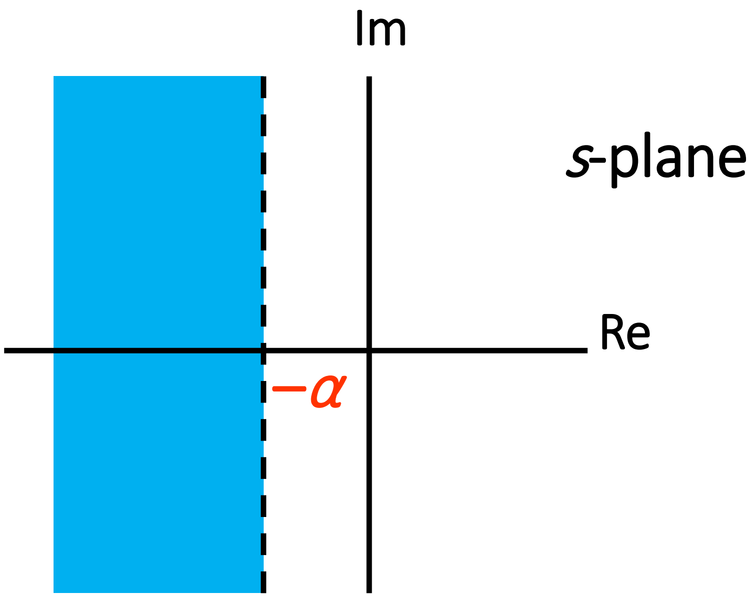
Thus,

$$\boxed{-e^{-\alpha t}u(-t) \xleftrightarrow{LT} \frac{1}{s+\alpha}, \quad \mathrm{Re}\{s\} < -\alpha}$$

6.1 The Laplace Transform



ROC for Example 6.1



ROC for Example 6.2

Example 6.3

Consider the signal $x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$.

Sol: From Euler's relation, we can write

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t} \right] u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-(1-3j)t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+1-3j)t}dt$$

$$= -\frac{1}{s+1-3j}e^{-(s+1-3j)t}\Big|_{0}^{\infty} = -\frac{1}{s+1-3j}e^{-(s+1-3j)\lim\limits_{t\to\infty} t} + \frac{1}{s+1-3j}$$

Thus, $e^{-(1-3j)t}u(t) \xleftrightarrow{LT} \dfrac{1}{s+(1-3j)}, \ \mathrm{Re}\{s\} > -1,$

$e^{-(1+3j)t}u(t) \xleftrightarrow{LT} \dfrac{1}{s+(1+3j)}, \ \mathrm{Re}\{s\} > -1,$ $\quad e^{-2t}u(t) \xleftrightarrow{LT} \dfrac{1}{s+2}, \ \mathrm{Re}\{s\} > -2.$

Consequently,

$$e^{-2t}u(t)+e^{-t}(\cos 3t)u(t) \xleftrightarrow{LT} \frac{1}{s+2}+\frac{1}{2}\left(\frac{1}{s+(1-3j)}\right)+\frac{1}{2}\left(\frac{1}{s+(1+3j)}\right)$$

$$=\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \quad \mathrm{Re}\{s\}>-1$$

Other useful LT pairs:

$$\boxed{\begin{aligned} \cos(\omega t)u(t) &\xleftrightarrow{LT} \frac{s}{s^2+\omega^2}, \quad \mathrm{Re}\{s\}>0 \\ \sin(\omega t)u(t) &\xleftrightarrow{LT} \frac{\omega}{s^2+\omega^2}, \quad \mathrm{Re}\{s\}>0 \end{aligned}}$$

6.1 The Laplace Transform

- Generally, the Laplace transform is *rational*, i.e., it is a ratio of polynomials in the complex variable *s*:

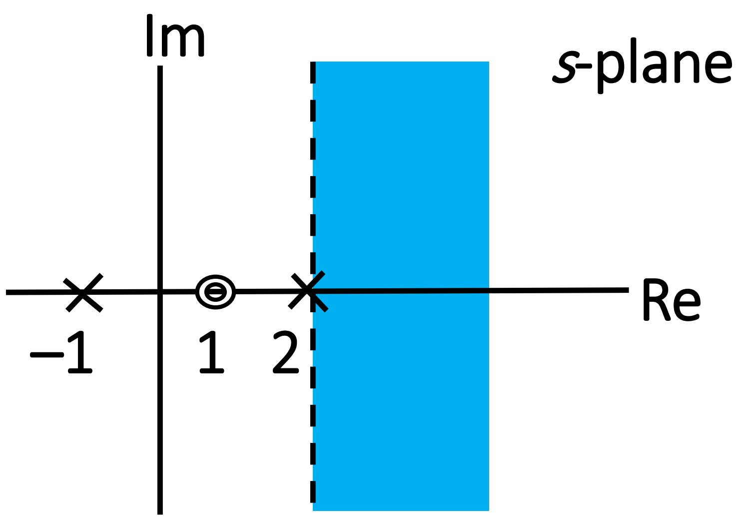
$$\boxed{X(s)=\frac{N(s)}{D(s)}}$$

- The roots of $N(s)$ are referred to as the *zeros* (零点) of $X(s)$; and the roots of $D(s)$ are referred to as the *poles* (极点) of $X(s)$.
- The representation of $X(s)$ through its poles and zeros in the *s*-plane is referred to as the *pole-zero plot* (极零图)of $X(s)$.
- Marking the locations of the roots of $N(s)$ and $D(s)$ in the *s*-plane and indicating the *ROC* provides a convenient *pictorial way of describing the Laplace transform*.
- Except for a scale factor, a complete specification of a rational Laplace transform consists of the pole-zero plot of the transform, together with its *ROC*.

➢ About the *infinity* (无穷远点): In general, if the order of $D(s)$ exceeds the order of $M(s)$ by k, $X(s)$ will have k zeros at infinity. Similarly, if the order of $M(s)$ exceeds the order of $D(s)$ by k, $X(s)$ will have k poles at infinity.

Example 6.4

Consider the signal $x(t)=\delta(t)-\frac{4}{3}e^{-t}u(t)+\frac{1}{3}e^{2t}u(t).$



Pole-zero plot and ROC

$$X(s)=1-\frac{4}{3}\frac{1}{s+1}+\frac{1}{3}\frac{1}{s-2}, \quad \mathrm{Re}\{s\}>2$$

$$=\frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathrm{Re}\{s\}>2$$

6.2 The Region of Convergence For Laplace Transforms

- Property 1: The *ROC* of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

- Property 2: For rational Laplace transforms, the *ROC* *does not* contain any poles.

- Property 3: If $x(t)$ is of *finite duration* and is absolutely integrable, then the *ROC is the entire s-plane*.

Example 6.5

Let $$x(t) = e^{-\alpha t}\left[u(t) - u(t-T)\right]$$

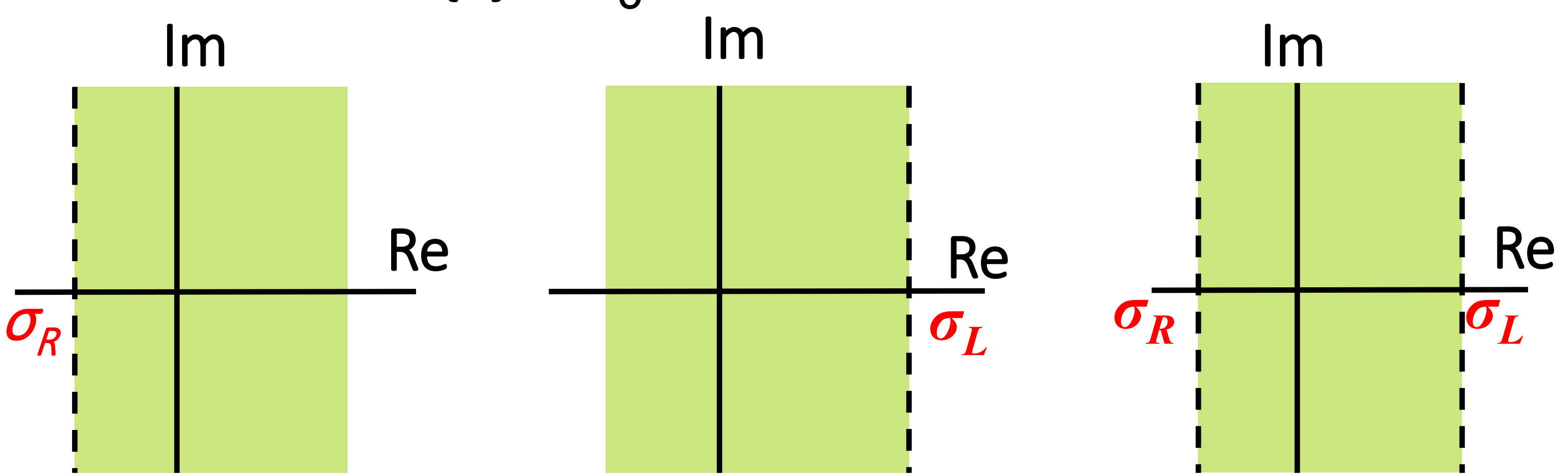
Sol: $$X(s) = \int_0^T e^{-\alpha t} e^{-st} dt = \frac{1}{s+\alpha}\left[1 - e^{-(s+\alpha)T}\right]$$

$$e^{-(s+\alpha)T} = \sum_{n=0}^{\infty} \frac{(-1)^n \left[(s+\alpha)T\right]^n}{n!}$$

The pole at $s = -\alpha$ is removable!

6.2 The Region of Convergence For Laplace Transforms

- Property 4: If $x(t)$ is *right sided*, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}\{s\} > \sigma_0$ will also be in the ROC; and the ROC of a right-sided signal is a *right-half plane*.
- Property 5: If $x(t)$ is *left sided*, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\text{Re}\{s\} < \sigma_0$ will also be in the ROC; and the ROC of a left-sided signal is a *left-half plane*.
- Property 6: If $x(t)$ is *two sided*, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a *strip* in the s-plane that includes the line $\text{Re}\{s\} = \sigma_0$.

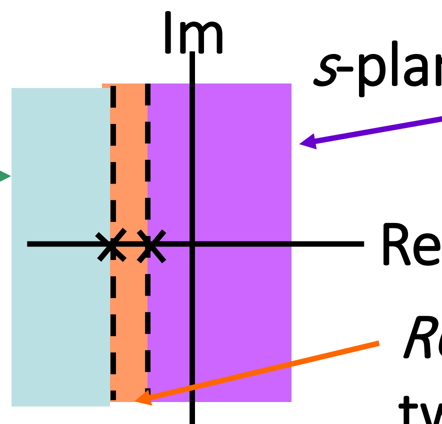


6.2 The Region of Convergence For Laplace Transforms

- Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its *ROC* is bounded by poles or extends to infinity. In addition, *no poles* of $X(s)$ are contained *in the ROC*.
- Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the *ROC* is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the *ROC* is the region in the s-plane to the left of the leftmost pole.

Example 6.6 Let $X(s) = \dfrac{1}{(s+1)(s+2)}$, how many possible *ROCs* relates to this $X(s)$?

ROC corresponding to a left-sided signal



ROC corresponding to a right-sided signal

ROC corresponding to a two-sided signal

6.3 The Inverse Laplace Transform

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{j\omega t}d\omega$$

Multiplying both sides by $e^{\sigma t}$ leading to $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}d\omega$

Changing the variable of this integration from ω to s and using the fact that σ is constant, so that $ds = jd\omega$.

Thus, the basic *inverse Laplace transform* equation is:

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds}$$

- The inverse Laplace transform equation states that *x(t) can be represented as a weighted integral of complex exponentials*.
- The formal evaluation of the integral for a general $X(s)$ requires the use of contour integration(围线积分) in the complex plane.
- For the class of *rational* transforms, the inverse Laplace transform can be determined by using the technique of partial-fraction expansion.

Example 6.7 Let $X(s)=\dfrac{1}{(s+1)(s+2)},\quad -2<\operatorname{Re}\{s\}<-1.$

Sol: Performing the partial-fraction expansion, we obtain

$$X(s)=\frac{1}{(s+1)}-\frac{1}{(s+2)}$$

$$-e^{-t}u(-t)\xleftrightarrow{LT}\frac{1}{s+1},\quad \operatorname{Re}\{s\}<-1$$

$$e^{-2t}u(t)\xleftrightarrow{LT}\frac{1}{s+2},\quad \operatorname{Re}\{s\}>-2$$

$$x(t)=-e^{-t}u(-t)-e^{-2t}u(t)\xleftrightarrow{LT}\frac{1}{(s+1)(s+2)},\quad -2<\operatorname{Re}\{s\}<-1$$

What if $X(s)=\dfrac{1}{s(s+1)(s+2)(s+4)},\quad -2<\operatorname{Re}\{s\}<-1.$?

Example 6.8 Let $X(s)=\dfrac{s+1}{s^2+2},\quad \mathrm{Re}\{s\}>0.$

Sol: From

$$\cos(\omega t)u(t) \xleftrightarrow{LT} \frac{s}{s^2+\omega^2},\quad \mathrm{Re}\{s\}>0$$

$$\sin(\omega t)u(t) \xleftrightarrow{LT} \frac{\omega}{s^2+\omega^2},\quad \mathrm{Re}\{s\}>0$$

By express $X(s)$ as $\quad X(s)=\dfrac{s}{s^2+\left(\sqrt{2}\right)^2}+\dfrac{\sqrt{2}}{s^2+\left(\sqrt{2}\right)^2}\cdot\dfrac{1}{\sqrt{2}}$

We can easily get

$$x(t)=\left(\cos\sqrt{2}t+\frac{1}{\sqrt{2}}\sin\sqrt{2}t\right)u(t)$$

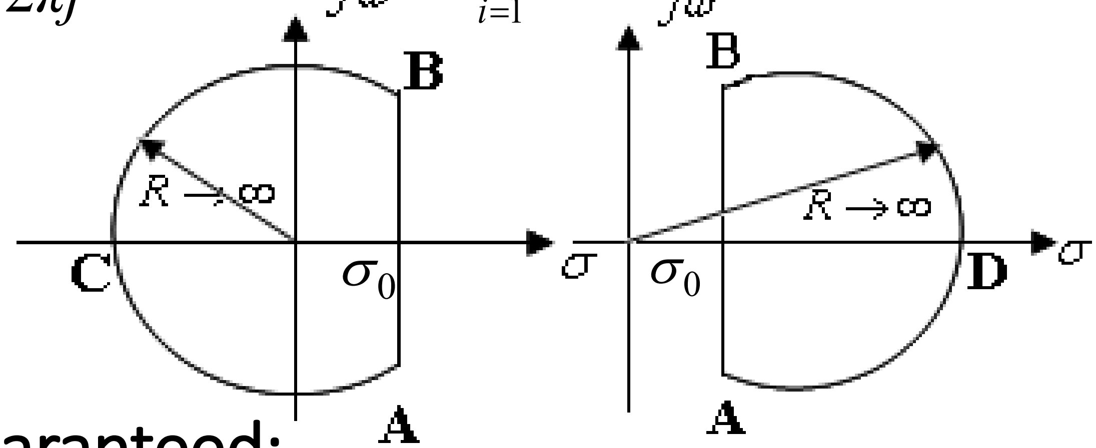
6.3 The Inverse Laplace Transform

➢ Use *Residue Theorem* to calculate contour integration:

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds=\frac{1}{2\pi j}\oint_C X(s)e^{st}ds=\sum_{i=1}^{n}\operatorname{Re}s_i$$

To make $\int_{BCA}X(s)e^{st}ds=0$

or $\int_{BDA}X(s)e^{st}ds=0$



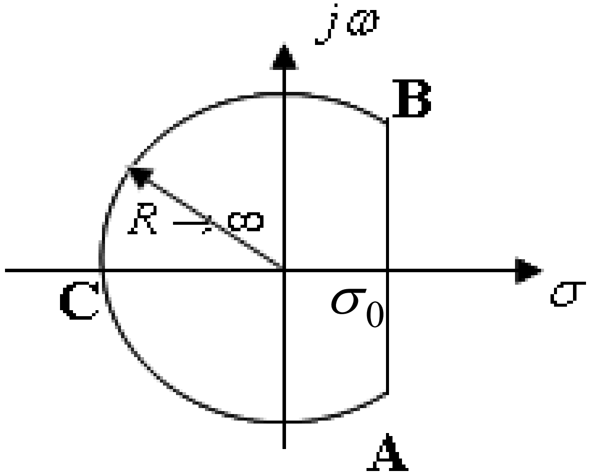
Two conditions must be guaranteed:

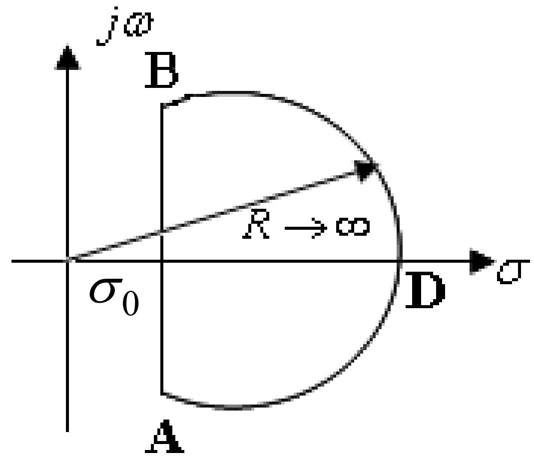
(1) As $|s|=R\to\infty$, $|X(s)|\to 0$ for all *s*;

(2) The exponent of e^{st} has a real part less than $\sigma_0 t$, $\operatorname{Re}(st)=\sigma t<\sigma_0 t$

Condition (1) can be satisfied so long as the order of the denominator exceeds the order of the numerator;

Condition (2) requires that as $t>0$, $\sigma<\sigma_0$; as $t<0$, $\sigma>\sigma_0$.





For $t > 0$, the contour is composed by the straight line AB and its left-side arc BCA; for $t < 0$, the contour is composed by the straight line AB and its right-side arc BDA.

$$x(t) = \frac{1}{2\pi j}\oint_C X(s)e^{st}ds = \begin{cases} \sum\limits_{left-side\quad poles} \mathrm{Re}\, s_l\,, & t > 0 \\ -\sum\limits_{right-side\quad poles} \mathrm{Re}\, s_r\,, & t < 0 \end{cases}$$

Example 6.9 Let $X(s)=\dfrac{s+2}{s(s+3)(s+1)^2}, \quad \operatorname{Re}\{s\}>0.$

Compute the $x(t)$ with contour integration method.

Sol: $X(s)$ has two first-order poles: $s_1=0, \quad s_2=-3.$

and a second-order pole: $s_{3,4}=-1.$

From the Residue Theorem,

$$x(t)=\operatorname{Res}\left[X(s)e^{st},0\right]+\operatorname{Res}\left[X(s)e^{st},-3\right]+\operatorname{Res}\left[X(s)e^{st},-1\right]$$

$$\operatorname{Res}\left[X(s)e^{st},0\right]=\left[sX(s)e^{st}\right]_{s=0}=\left.\frac{s+2}{(s+3)(s+1)^2}e^{st}\right|_{s=0}=\frac{2}{3}$$

$$\operatorname{Res}\left[X(s)e^{st},-3\right]=\left[(s+3)X(s)e^{st}\right]_{s=-3}=\left.\frac{s+2}{s(s+1)^2}e^{st}\right|_{s=-3}=\frac{1}{12}e^{-3t}$$

$$\operatorname{Res}\left[X(s)e^{st},-1\right]=\frac{1}{1!}\frac{d}{ds}\left\{(s+1)^2X(s)e^{st}\right\}_{s=-1}=\frac{d}{ds}\left[\frac{s+2}{s(s+3)}e^{st}\right]_{s=-1}$$

$$=\left[-\frac{s^2+4s+6}{s^2(s+3)^2}e^{st}+t\frac{s+2}{s(s+3)}e^{st}\right]_{s=-1}=-\frac{1}{2}te^{-t}-\frac{3}{4}e^{-t}$$

$$Thus,\quad x(t)=\left[\frac{2}{3}+\frac{1}{12}e^{-3t}-\frac{1}{2}\left(t+\frac{3}{2}\right)e^{-t}\right]u(t)$$

What if $X(s)=\dfrac{s+2}{s(s+3)(s+1)^2},\quad -1<\operatorname{Re}\{s\}<0$?

Or $X(s)=\dfrac{s+2}{s(s+3)(s+1)^2},\quad \operatorname{Re}\{s\}<-3$?

then $x(t)=-\dfrac{2}{3}u(-t)+\left[\dfrac{1}{12}e^{-3t}-\dfrac{1}{2}\left(t+\dfrac{3}{2}\right)e^{-t}\right]u(t)$

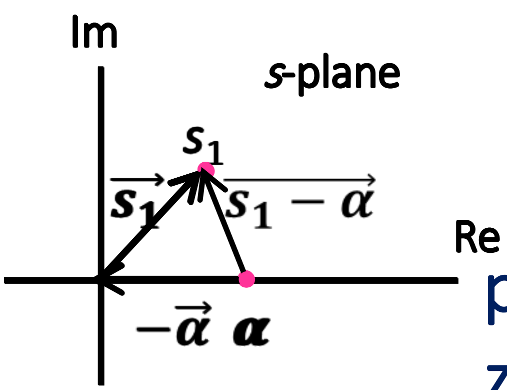
or $x(t)=-\left[\dfrac{2}{3}+\dfrac{1}{12}e^{-3t}-\dfrac{1}{2}\left(t+\dfrac{3}{2}\right)e^{-t}\right]u(-t)$

6.4 Geometric Evaluation of The Fourier Transform From The Pole-Zero Plot

A general rational Laplace transform has the form: $X(s)=\dfrac{N(s)}{D(s)}$

and it can be factored into the form: $$X(s)=M\frac{\prod_{i=1}^{R}(s-\beta_i)}{\prod_{k=1}^{P}(s-\alpha_k)}$$

where β_i, α_j are zeros and poles of $X(s)$, respectively.



$$X(j\omega)=M\frac{\prod_{i=1}^{R}(j\omega-\beta_i)}{\prod_{k=1}^{P}(j\omega-\alpha_k)}\qquad X(j\omega_1)=M\frac{\prod_{i=1}^{R}A_i\cdot e^{j\sum_{i=1}^{R}\theta_i}}{\prod_{k=1}^{P}B_k\cdot e^{j\sum_{k=1}^{P}\varphi_k}}$$

pole vectors (极点向量)

zero vectors (零点向量)

$$\left|X(j\omega_1)\right|=M\frac{\prod_{i=1}^{R}A_i}{\prod_{k=1}^{P}B_k}$$

$$\sphericalangle X(j\omega_1)=\sum_{i=1}^{R}\theta_i-\sum_{k=1}^{P}\varphi_k$$

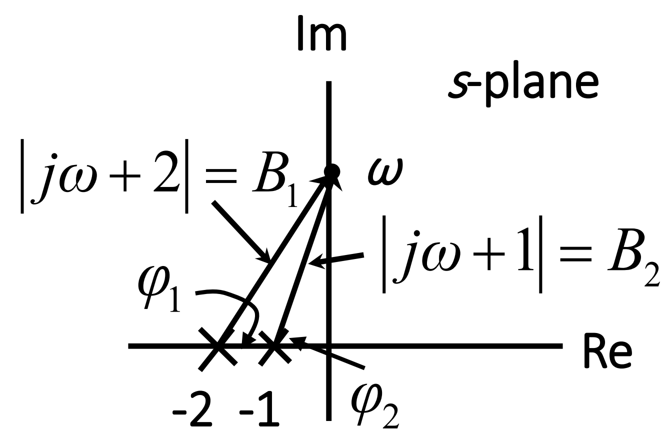
Complex plane representation of the vectors $\vec{s_1}$, $\vec{\alpha}$, and $\overrightarrow{s_1-\alpha}$ representing the complex numbers s_1, α and $s_1-\alpha$ respectively.

6.4 Geometric Evaluation of The Fourier Transform From The Pole-Zero Plot

Given $X(s) = \dfrac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$

Geometrically, we can write

$$|X(j\omega)| = \sqrt{\frac{1}{(\omega^2+2^2)(\omega^2+1^2)}} = \frac{1}{B_1 \cdot B_2}$$



$$\sphericalangle X(j\omega) = -\left(\tan^{-1}\frac{\omega}{2} + \tan^{-1}\omega\right) = -(\varphi_1 + \varphi_2)$$

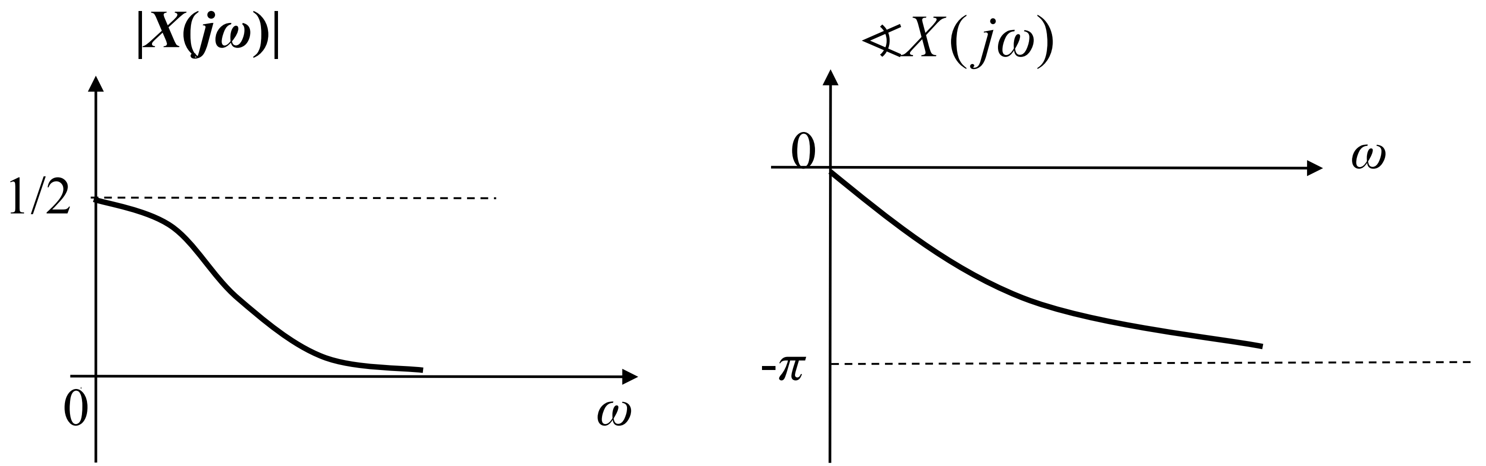
$\omega = 0:\quad B_1 = 2,\ B_2 = 1,\ |X(j\omega)| = \dfrac{1}{2},\ \varphi_1 = 0, \varphi_2 = 0, \sphericalangle X(j\omega) = 0$

$$\omega\uparrow: \begin{cases} B_1\uparrow, B_2\uparrow \Rightarrow |X(j\omega)|\downarrow \\ \varphi_1\uparrow, \varphi_2\uparrow\ (however,\ \varphi_1 < \dfrac{\pi}{2}, \varphi_2 < \dfrac{\pi}{2}) \Rightarrow \sphericalangle X(j\omega) < 0, |\sphericalangle X(j\omega)|\uparrow \end{cases}$$

$\omega \to +\infty:$

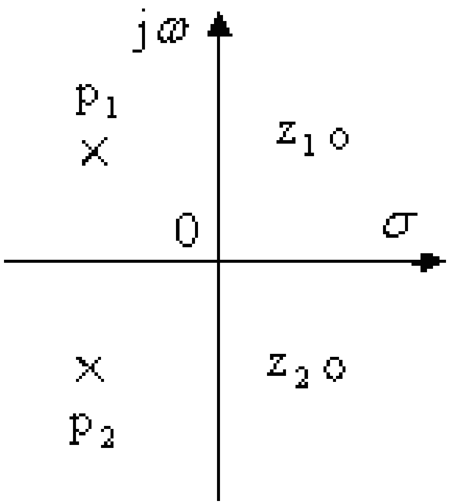
$B_1 \to \infty, B_2 \to \infty,\ |X(j\omega)| \to 0$

$\varphi_1 \to \dfrac{\pi}{2}, \varphi_2 \to \dfrac{\pi}{2}, \sphericalangle X(j\omega) \to -\pi$



|X(jω)|
1/2
0
ω
∢X(jω)
0
ω
-π

All-pass function: A Laplace transform with all of its poles and zeros located on both sides of the $j\omega$- axis symmetrically. And all the poles are on the left of the $j\omega$- axis. All the zeros are on the right of the $j\omega$- axis.



$$p_1 = p_2^* = -z_2 = -z_1^*$$

The products of the magnitudes of all pole vectors

= The products of the magnitudes of all zero vectors

6.5 Properties of The Laplace Transform

6.5.1 Linearity

If $x_1(t) \leftrightarrow X_1(s), \quad ROC = R_1$ and $x_2(t) \leftrightarrow X_2(s), \quad ROC = R_2$

then $ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s), ROC \ \ containing \quad R_1 \cap R_2$

Note: *ROC* is at least the intersection of R_1 and R_2, which could be empty, also can be larger than the intersection.

6.5.2 Time Shifting

If $\quad x(t) \leftrightarrow X(s), \quad ROC = R$

then $\quad x(t - t_0) \leftrightarrow e^{-st_0} X(s), \quad ROC = R$

6.5.3 Shifting in the s-Domain

If $\quad x(t) \leftrightarrow X(s), \quad ROC = R$

then $\quad e^{s_0 t} x(t) \leftrightarrow X(s - s_0), \quad ROC = R + \mathrm{Re}\{s_0\}$

6.5.4 Time Scaling

If $$x(t) \leftrightarrow X(s), \quad ROC = R$$

then $$x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{s}{\alpha}\right), \quad ROC = \alpha R$$

Special case: $$x(-t) \leftrightarrow X(-s), \quad ROC = -R$$

6.5.5 Conjugation

$$x^*(t) \leftrightarrow X^*(s^*), \quad ROC = R$$

If $x(t)$ is real: $$X(s) = X^*(s^*)$$

If $X(s)$ has a pole or zero at $s = s_0$, then $X(s)$ also has a pole or zero at the complex conjugate point $s = s_0^*$.

6.5.6 Convolution Property

If $x_1(t) \leftrightarrow X_1(s), \quad ROC = R_1$ and $x_2(t) \leftrightarrow X_2(s), \quad ROC = R_2$

then $$x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s), ROC \ \ containing \ R_1 \cap R_2$$

6.5.7 Differentiation in the Time Domain

If $$x(t) \leftrightarrow X(s), \quad ROC = R$$

then $$\frac{dx(t)}{dt} \leftrightarrow sX(s), \quad ROC \;\; containing \;\; R$$

6.5. 8 Differentiation in the s-Domain

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds}, \quad ROC = R$$

6.5.9 Integration in the Time Domain

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X(s)}{s}, \quad ROC \;\; containing \; R \cap \left\{ \mathrm{Re}\left\{s\right\} > 0 \right\}$$

6.5.10 The Initial- and Final-Value Theorems（初值和终值定理)

If $x(t)$ is a causal signal, i.e., $x(t) = 0$, for $t < 0$, then

Initial-value theorem: $$x(0^+) = \lim_{s\to\infty} sX(s)$$

Note: If rational $X(s)$ is an improper fraction, rewrite $X(s)$ as a sum of a polynomial of s and a true fraction $X_1(s)$, then take $X_1(s)$ into above limit on the right side.

Final-value theorem: $$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

To guarantee the existence of the final value, no poles of $X(s)$ lie on the right side of $j\omega$-axis. If there is a pole on the $j\omega$-axis, it must be a first-order pole at the origin.

Example 6.10

Consider the signal $x(t) = u(t) - u(t-1)$.

Sol: As we know $u(t) \xleftrightarrow{LT} \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0$

From the time shifting $u(t-1) \xleftrightarrow{LT} \frac{1}{s} e^{-s}, \quad \mathrm{Re}\{s\} > 0$

So that $X(s) = \frac{1}{s} - \frac{1}{s} e^{-s} = \frac{1-e^{-s}}{s}, \quad ROC = entire\ s\ plane.$

Example 6.11

Determine the Laplace transform of $x(t) = te^{-\alpha t} u(t)$.

Sol: Since $e^{-\alpha t} u(t) \xleftrightarrow{LT} \frac{1}{s+\alpha}, \quad \mathrm{Re}\{s\} > -\alpha$

From the differentiation in the s-domain

$$te^{-\alpha t} u(t) \xleftrightarrow{LT} -\frac{d}{ds}\left(\frac{1}{s+\alpha}\right) = \frac{1}{(s+\alpha)^2}, \quad \mathrm{Re}\{s\} > -\alpha$$

Example 6.12

Use the initial-value theorem to determine the initial-value of

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$$

$$e^{-2t}u(t) + e^{-t}(\cos 3t)u(t) \xleftrightarrow{LT} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \quad \operatorname{Re}\{s\} > -1$$

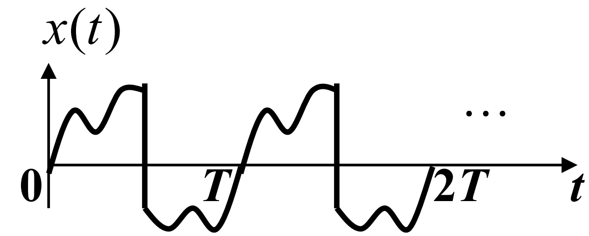
$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{2s^3+5s^2+12s}{s^3+4s^2+14s+20} = 2$$

Example 6.13 Let $X(s) = \dfrac{s+2}{s^2+2s+5}, \quad \operatorname{Re}\{s\} > -1.$

$$X(s) = \frac{s+2}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\cdot\frac{1}{2}$$

$$x(t) = e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)u(t)$$

Example 6.14 Determine the Laplace transform of the *causal periodic signal* $x(t)$ which is depicted in the figure:

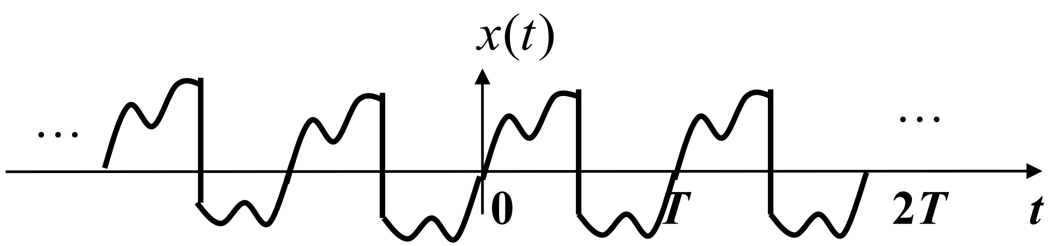


Sol : $x(t) = x_0(t) + x_0(t-T) + x_0(t-2T) + \cdots$

$$X(s) = X_0(s) + X_0(s)\cdot e^{-s\cdot T} + X_0(s)\cdot e^{-s\cdot 2T} + \cdots$$
$$= X_0(s)[1 + e^{-s\cdot T} + e^{-s\cdot 2T} + \cdots]$$
$$= X_0(s)\cdot\frac{1}{1-e^{-sT}} \quad \left(\mathrm{Re}\{s\} > 0\right) \quad \boxed{= \frac{X_0(s)}{1-e^{-sT}}}$$

Consider :

What's the LT of the periodic signal in the following figure?



6.6 Analysis And Characterization of LTI Systems Using Laplace Transform

- The Laplace transforms of the input and the output of an LTI system are related through multiplication by the Laplace transform of the impulse response of the system.

$$Y(s) = H(s)\, X(s)$$

System function、Transfer function

- The *ROC* associated with the system function for a causal system is a right-half plane.
  - ✓ An *ROC* to the right of the rightmost pole does not guarantee that a system is causal.
  - ✓ For a system with a *rational* system function, causality of the system is *equivalent to* the *ROC* being the right-half plane to the right of the rightmost pole.

Example 6.15

Consider a system with impulse response $h(t) = e^{-(1+2j)t}u(t)$.

Sol: Since $h(t) = 0$ for $t < 0$, this system is causal.

It's easy to find the system function: $H(s) = \dfrac{1}{s+1+2j}, \ \mathrm{Re}\{s\} > -1$

The $H(s)$ is rational and the *ROC* is to the right of the rightmost pole, consistent with our statement.

Example 6.16

Consider the system function $H(s) = \dfrac{e^{s}}{s+1}, \quad \mathrm{Re}\{s\} > -1$ irrational !

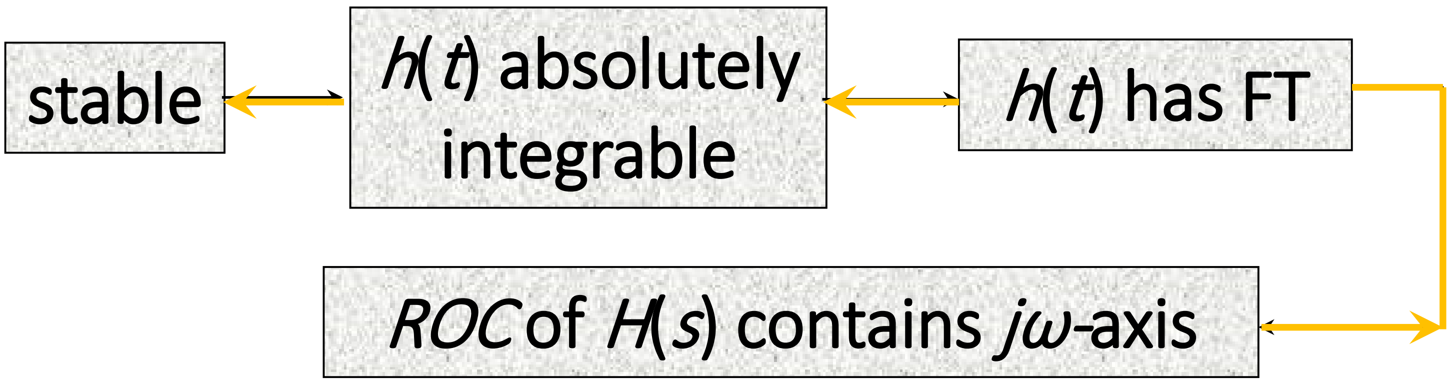
Sol: For this system, the *ROC* is to the right of the rightmost pole.

But the impulse response can be obtained as $h(t) = e^{-(t+1)}u(t+1)$

Obviously this system is not causal.

6.6 Analysis And Characterization of LTI Systems Using Laplace Transform

- An LTI system is stable *if and only if* the *ROC* of its system function $H(s)$ includes the $j\omega$-axis [i.e., Re{s} = 0].



- A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane —i.e., all of the poles have negative real parts.

6.6 Analysis And Characterization of LTI Systems Using Laplace Transform

- For an LTI system which is described by a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Its system function (transfer function) is:

$$\boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}$$

The system function for a system specified by a differential equation is always rational.

Example 6.17 Given the information about an LTI system:

1. The system is causal. 2. The $H(s)$ is rational and has only two poles, at $s = -2$ and $s = 4$. 3. If $x(t) = 1$, then $y(t) = 0$. 4. $\boldsymbol{h(0^+) = 4}$.

Determine the system function of the system.

Sol: From fact 2, we write $H(s) = \dfrac{p(s)}{(s+2)(s-4)} = \dfrac{p(s)}{s^2 - 2s - 8}$

From fact 3, using eigenfunction property

$$x(t) = 1 = e^{0\cdot t} \quad \Rightarrow \quad y(t) = H(0)\cdot e^{0\cdot t} = H(0) = 0$$

$p(s)$ must have a root at $s = 0$ and thus is of the form $p(s) = sq(s)$.

From fact 4 and 1, $\displaystyle \lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{s^2 q(s)}{s^2 - 2s - 8} = 4$

The highest powers in s in both the denominator and the numerator are identical, that is, $q(s)$ must be a constant. We let $q(s) = k$.

It's easy to find that $k = 4$. So that $H(s) = \dfrac{4s}{(s+2)(s-4)}$

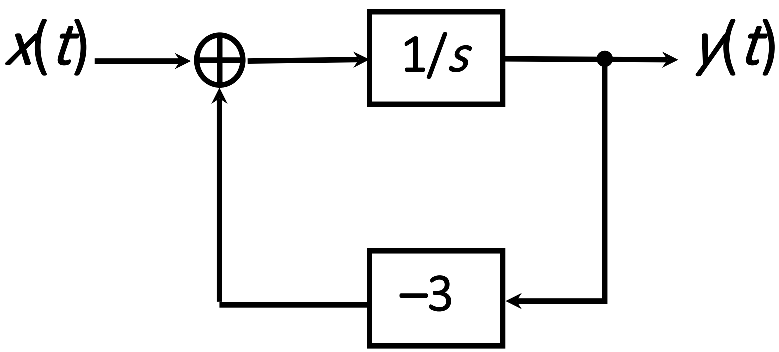
Example 6.18

Consider the causal LTI system with system function

$$H(s) = \frac{1}{s+3}$$

This system can also be described by the differential equation

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$



Block diagram representation of the causal LTI system

$1/s$ is the system function of a system with impulse response $u(t)$, i.e., it is the system function of an integrator.

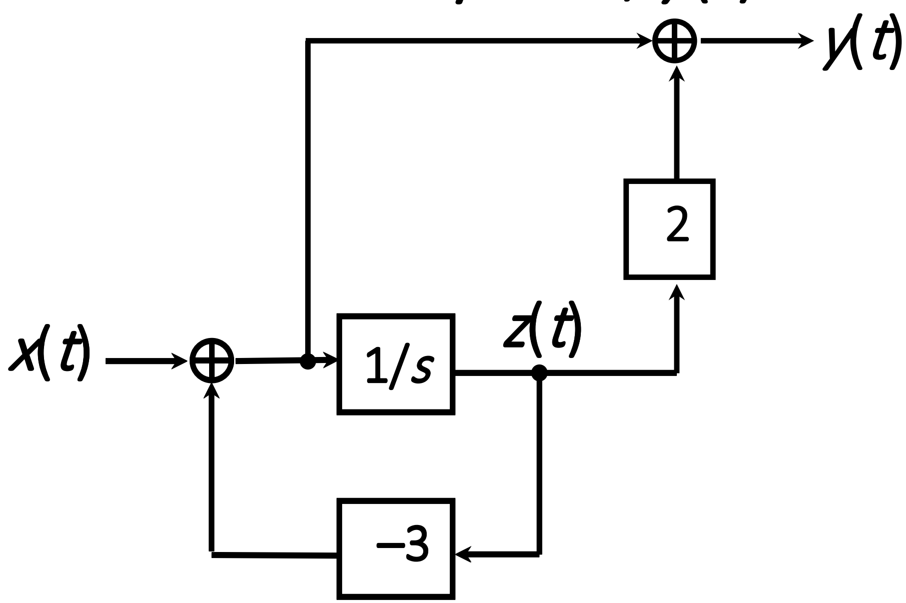
Example 6.19

Consider a causal LTI system with system function $H(s)=\dfrac{s+2}{s+3}$

$$H(s)=\underbrace{\left(\frac{1}{s+3}\right)}_{H_1(s)}\underbrace{(s+2)}_{H_2(s)}$$

Let $z(t)$ be the output of the first subsystem, $y(t)$ is the output of the overall system.

$$y(t)=\frac{dz(t)}{dt}+2z(t)$$



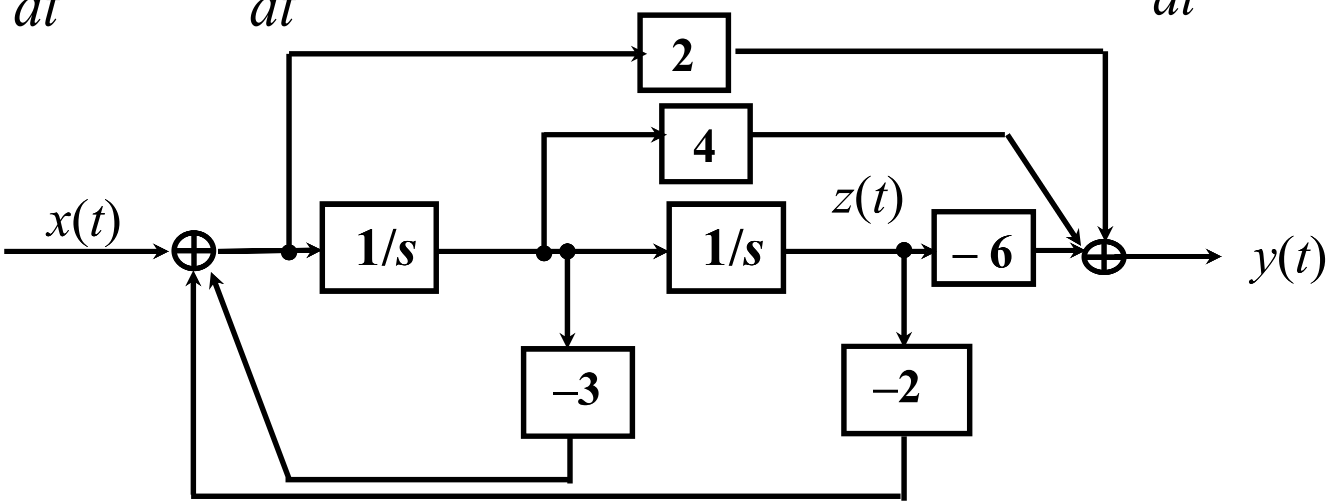
Block diagram representation for the system

Example 6.20 Consider a second-order LTI system with system function

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}$$

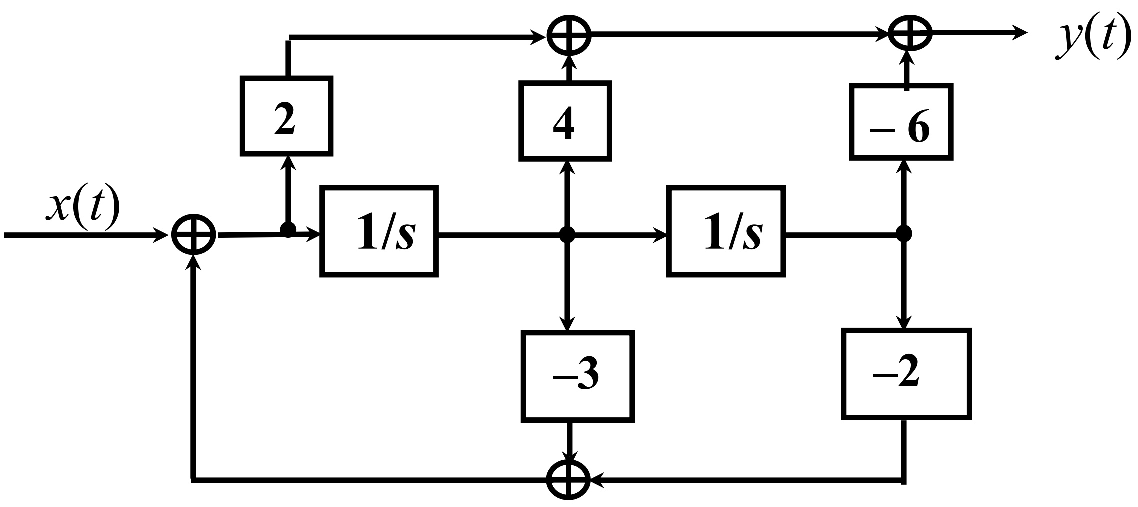
$$H(s) = \underbrace{\left(\frac{1}{s^2 + 3s + 2}\right)}_{H_1(s)} \underbrace{\left(2s^2 + 4s - 6\right)}_{H_2(s)}$$

$H_1(s)$: $\dfrac{d^2z(t)}{dt^2} + 3\dfrac{dz(t)}{dt} + 2z(t) = x(t)$ $\quad$ $H_2(s)$: $y(t) = 2\dfrac{d^2z(t)}{dt^2} + 4\dfrac{dz(t)}{dt} - 6z(t)$



Direct-form (直接型) representation for the system

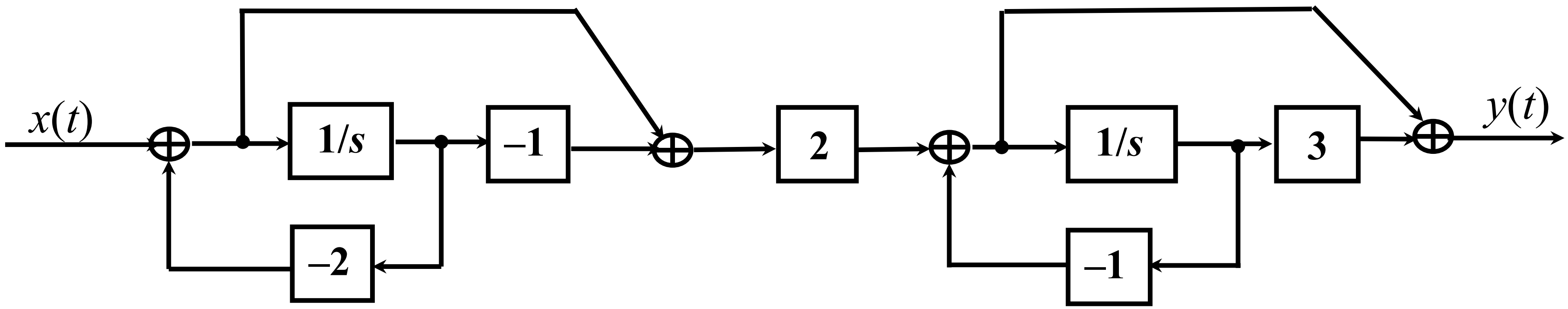
6.7 System Function Algebra And Block Diagram Representations



Direct-form (直接型) representation for the system

6.7 System Function Algebra And Block Diagram Representations

$$H(s) = \left(2 \cdot \frac{s-1}{s+2}\right)\left(\frac{s+3}{s+1}\right)$$



Cascade-form (级联型) representation

What's the cascade-form representation if

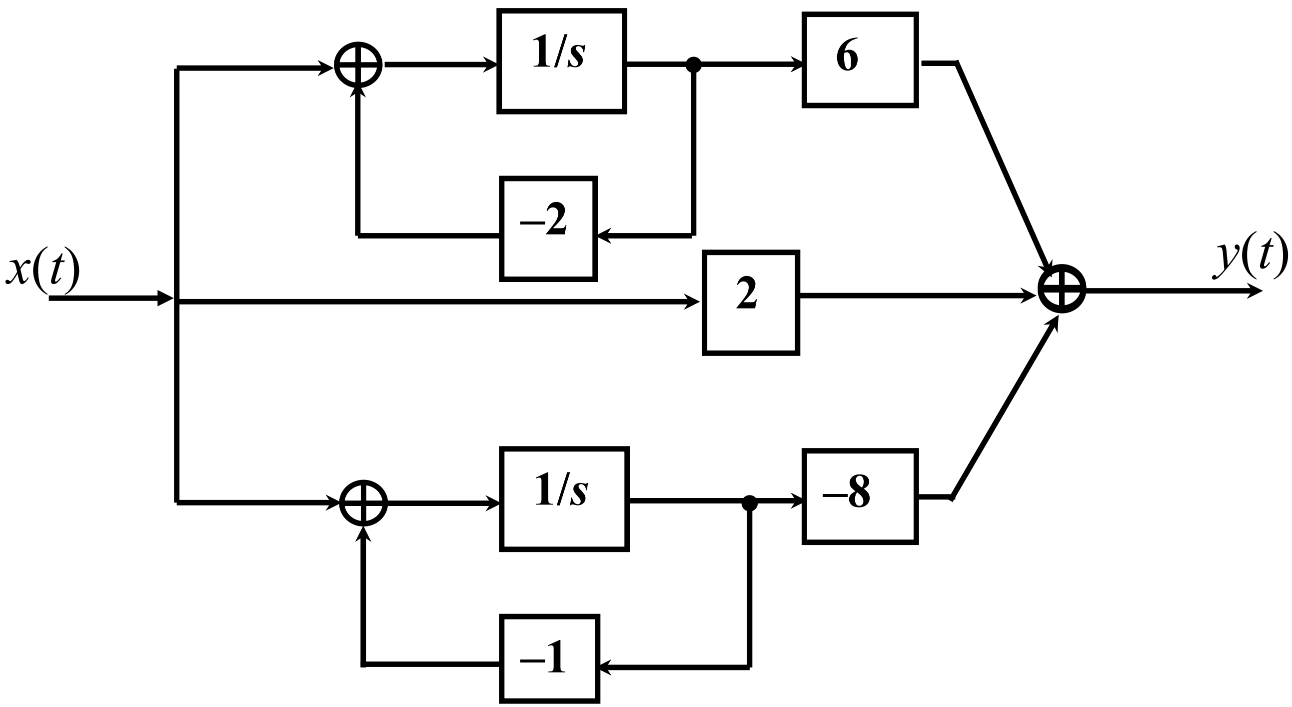
$$H(s) = \left(\frac{2s-2}{s+2}\right)\left(\frac{s+3}{s+1}\right)$$

or

$$H(s) = 2\left(\frac{s+3}{s+2}\right)\left(\frac{s-1}{s+1}\right) \quad ?$$

6.7 System Function Algebra And Block Diagram Representations

$$H(s)=\frac{6}{s+2}+2-\frac{8}{s+1}$$

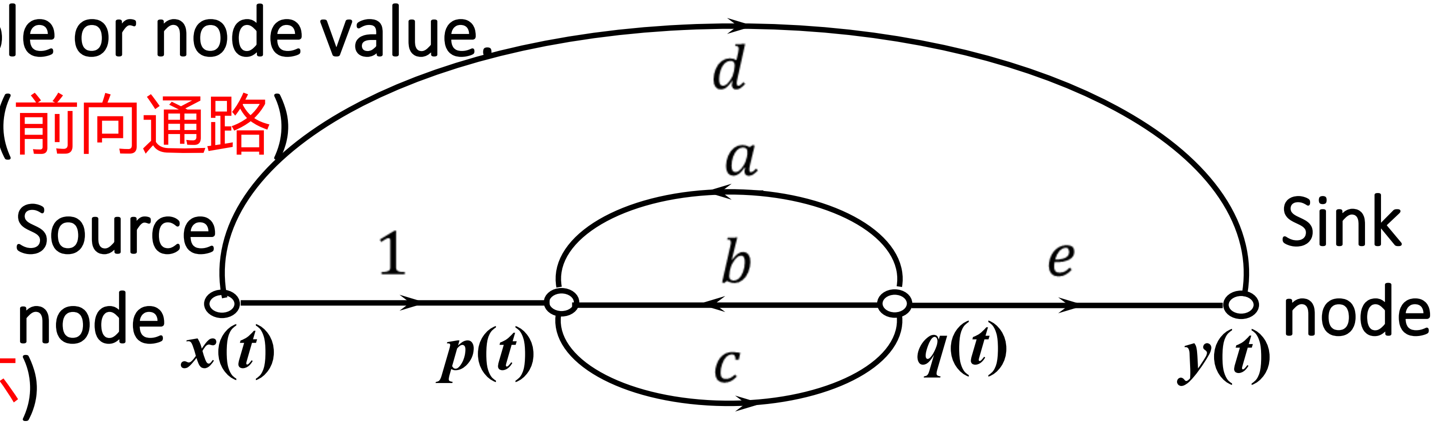


Parallel-form (并联型) representation

6.8 Signal Flow Graph (信号流图) Representations

Formally, a *signal flow graph* is a network of directed branches (有向支路). that connect at nodes (结点). Associated with each node is a variable or node value.

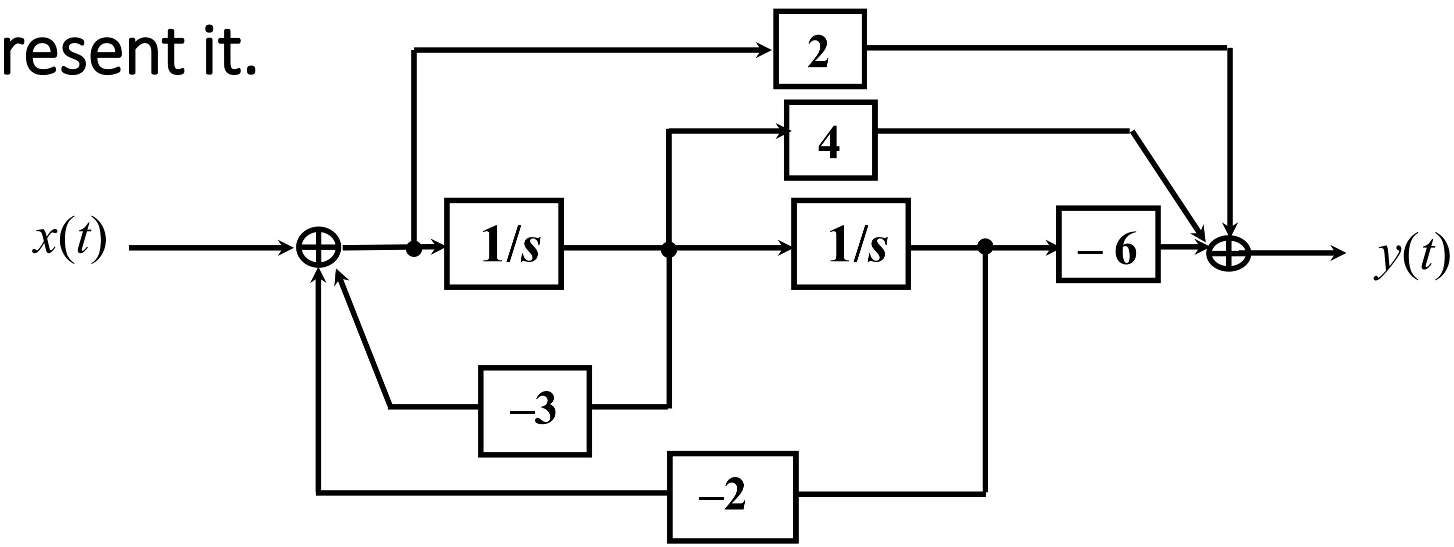
- forward path (前向通路)
- loop (环)
- self-loop (自环)

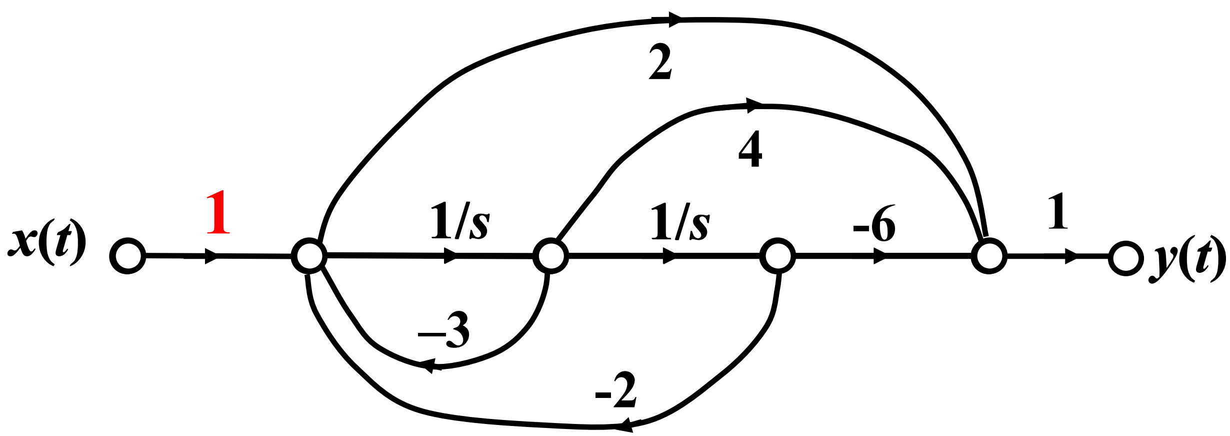


- *Source nodes* (源结点): nodes that have only outgoing branches (出支路), which are used to represent the injection of external inputs or signal sources into a graph.
- *Sink nodes* (汇结点): nodes that have only entering branches (入支路), which are used to extract outputs from a graph.
- mixed node (混合结点)

Example 6.21

Consider again the system in example 6.20. Use signal flow graph to represent it.





$$H(s)=\frac{2s^2+4s-6}{s^2+3s+2}$$

6.8 Signal Flow Graph (信号流图) Representations

Mason's Formula：(梅森公式)

$$\boxed{H=\frac{1}{\Delta}\sum_k G_k\Delta_k}$$

Manson's formula is used to calculate the transfer value (transfer function) between any source node and sink node (or mixed node) in a signal flow graph.

Where $\Delta=1-\sum_i L_i+\sum_{i,j}L_iL_j-\sum_{i,j,k}L_iL_jL_k+\cdots$

is the graph determinant (特征行列式) .

L_i is the gain of each loop.

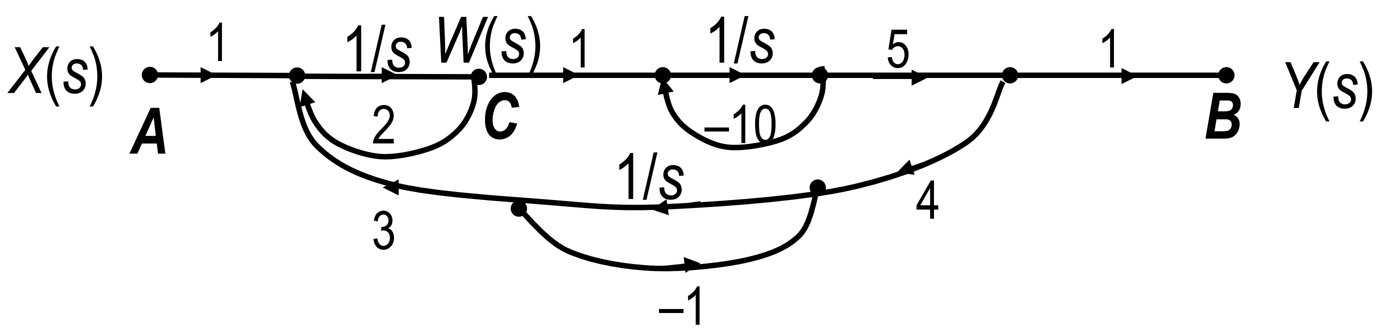
L_iL_j is the product of the gains of two loops which have no shared nodes and branches .

G_k is the gain of the *k*-th forward path between the source node and the sink node (or mixed node).

Δ_k is the graph determinant of the left graph after remove the *k*-th forward path.

Example 6.22

Compute the transfer functions between nodes A and B, and nodes A and C in the following signal flow graph.

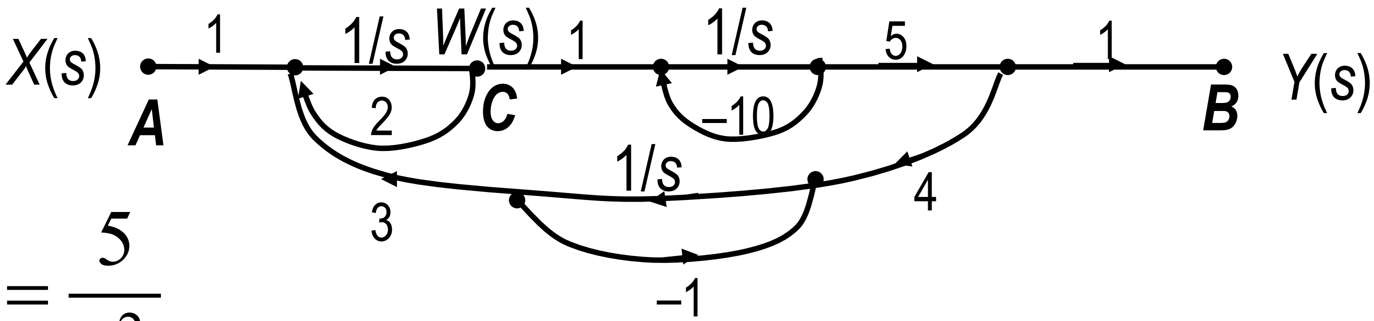


$$\Delta = 1 - \left[\frac{2}{s} - \frac{10}{s} - \frac{1}{s} + \frac{60}{s^3}\right] + \left[\frac{2}{s}\left(-\frac{10}{s}\right) + \frac{2}{s}\left(-\frac{1}{s}\right) + \left(-\frac{10}{s}\right)\left(-\frac{1}{s}\right)\right] - \frac{2}{s}\left(-\frac{10}{s}\right)\left(-\frac{1}{s}\right)$$

$$= 1 + \frac{9}{s} - \frac{12}{s^2} - \frac{80}{s^3}$$

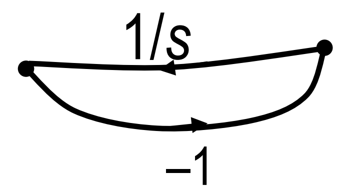
6.8 Signal Flow Graph (信号流图) Representations

A to B:



$$G_1 = \frac{5}{s^2}$$

After removing G_1, the left graph is:



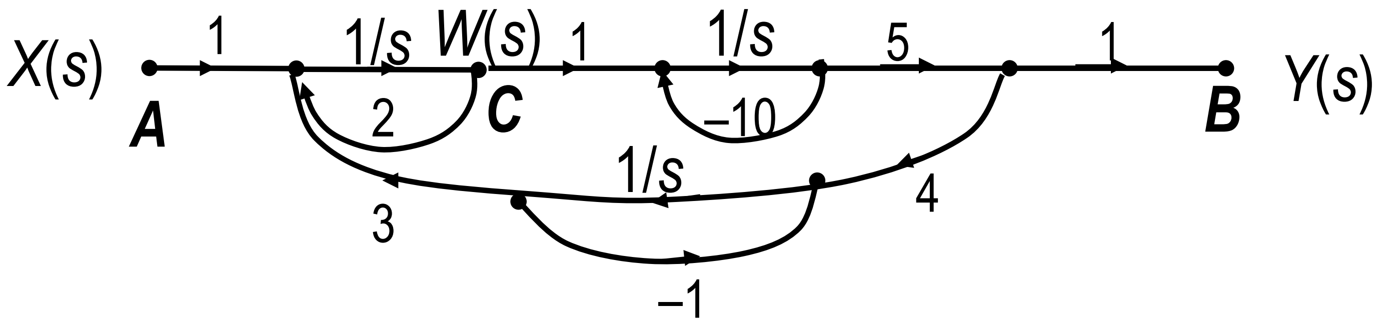
$$\Delta_1 = 1 + \frac{1}{s}$$

Thus,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{5}{s^2}\left(1+\frac{1}{s}\right)}{1+\frac{9}{s}-\frac{12}{s^2}-\frac{80}{s^3}} = \frac{5s+5}{s^3+9s^2-12s-80}$$

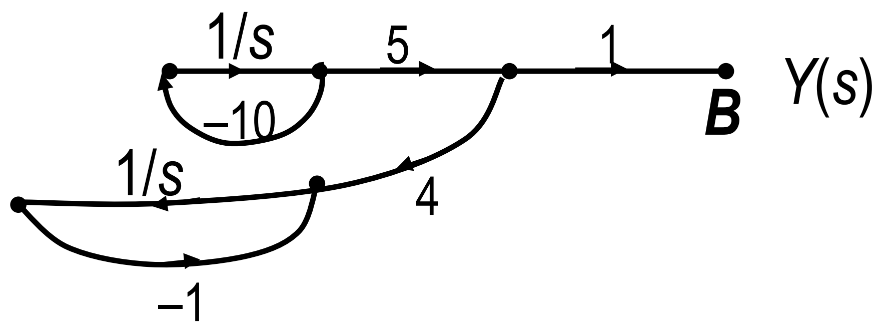
6.8 Signal Flow Graph (信号流图) Representations

A to C:



$$G_1 = \frac{1}{s}$$

After removing G_1 , the left graph is:



$$\Delta_1 = 1 - \left[\left(-\frac{10}{s}\right) + \left(-\frac{1}{s}\right)\right] + \left[\left(-\frac{10}{s}\right)\left(-\frac{1}{s}\right)\right] = 1 + \frac{11}{s} + \frac{10}{s^2}$$

Thus, $$H(s) = \frac{W(s)}{X(s)} = \frac{s^2 + 11s + 10}{s^3 + 9s^2 - 12s - 80}$$

6.9.1 Introduction of the Unilateral Laplace Transform

Bilateral Laplace transform: (双边拉普拉斯变换)

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt$$

Unilateral Laplace transform: (单边拉普拉斯变换)

$$X(s)=\int_{0^-}^{\infty}x(t)e^{-st}dt$$

- The lower limit of integration, $\mathbf{0}^-$, signifies that we include in the interval of integration any impulses or higher order singularity functions (奇异函数) concentrated at t = 0.
- The bilateral transform depends on the entire signal from $t=-\infty$ to $t=+\infty$, whereas the unilateral transform depends only on the signal from $t=\mathbf{0}^-$ to ∞.
- The bilateral transform and the unilateral transform of a causal signal are identical.
- The *ROC* for the unilateral transform is always a right-half plane

6.9 The Unilateral Laplace Transform

- The evaluation of the inverse unilateral Laplace transforms is also the same as for bilateral transforms, with the constraint that the *ROC* for a unilateral transform must always be a right-half plane.

Example 6.23 Consider the signal $x(t) = e^{-\alpha(t+1)}u(t+1)$.

Sol: The bilateral transform *X*(*s*) can be obtained from Example 6.1 and the time-shifting property: $X(s) = \dfrac{e^{s}}{s+\alpha}, \ \mathrm{Re}\{s\} > -\alpha$

By contrast, the unilateral transform is

$$\mathcal{X}(s) = \int_{0^-}^{\infty} e^{-\alpha(t+1)}u(t+1)e^{-st}dt = \int_{0^-}^{\infty} e^{-\alpha}e^{-t(s+\alpha)}dt = e^{-\alpha}\frac{1}{s+\alpha} \quad \mathrm{Re}\{s\} > -\alpha$$

We could recognize $\mathcal{X}(s)$ as the bilateral transform of *x*(*t*)*u*(*t*).

Since $x(t)u(t) = e^{-\alpha}e^{-\alpha t}u(t)$, Thus $\mathcal{X}(s) = e^{-\alpha}\dfrac{1}{s+\alpha}, \quad \mathrm{Re}\{s\} > -\alpha$

6.9 The Unilateral Laplace Transform

Example 6.24 Consider the unilateral LT $X(s)=\dfrac{1}{(s+1)(s+2)}$
Determine the corresponding $x(t)$.

Sol: For the unilateral transform, the ROC must be the right-half plane to the right of the rightmost pole of $X(s)$.

In this case, the ROC consists of all points s with $\mathrm{Re}\{s\} > -1$.

Thus
$$x(t)=(e^{-t}-e^{-2t})u(t)$$

✓unilateral Laplace transform provide us with information about signals only for $t > 0^-$.

Example 6.25 Calculate the inverse of the unilateral LT $X(s)=\dfrac{s^2-3}{s+2}$.

Sol:
$$X(s)=-2+s+\frac{1}{s+2}$$

Taking inverse transforms of each term results in

$$x(t)=-2\delta(t)+\delta'(t)+e^{-2t}u(t)$$

6.9.2 Properties of the Unilateral Laplace Transform

- ✓ Time scaling: $$x(at) \xleftrightarrow{UL} \frac{1}{a} X(\frac{s}{a}), \quad a > 0$$

- ✓ Convolution: assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$.

$$x_1(t) * x_2(t) \xleftrightarrow{UL} X_1(s) X_2(s)$$

- ✓ Differentiation in the time domain :

$$\frac{d}{dt} x(t) \xleftrightarrow{UL} sX(s) - x(0^-)$$

$$\boxed{\frac{d^n}{dt^n} x(t) \xleftrightarrow{UL} s^n X(s) - \sum_{k=0}^{n-1} s^{n-k-1} x^{(k)}(0^-)}$$

Proof of this property for first-derivative of $x(t)$:

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \int_{0^-}^{\infty} e^{-st} dx(t)$$

$$= x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st} dt$$

$$= sX(s) - x(0^-)$$

Similarly, the unilateral Laplace transform of second-derivative of $x(t)$ can be obtained by repeating using the property:

$$\mathcal{UL}\left\{\frac{d^2x(t)}{dt^2}\right\} = s(sX(s) - x(0^-)) - x'(0^-)$$

$$= s^2X(s) - sx(0^-) - x'(0^-).$$

6.9.3 Solving Differential Equations Using the Unilateral Laplace Transform

Example 6.26

Consider the causal system characterized by the differential equation
$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t)$$
with initial conditions $\boldsymbol{y}(\mathbf{0}^-)=\boldsymbol{\beta}, \boldsymbol{y}'(\mathbf{0}^-)=\boldsymbol{\gamma}$ and input signal $\boldsymbol{x}(\boldsymbol{t})=\boldsymbol{\alpha u}(\boldsymbol{t})$. Determine the output signal $y(t)$.

Sol: Applying the unilateral LT to both sides of the differential equation yields

$$s^2Y(s)-sy(0^-)-y'(0^-)+3sY(s)-3y(0^-)+2Y(s)=X(s)$$

or equivalently, $s^2Y(s)-\beta s-\gamma+3sY(s)-3\beta+2Y(s)=\dfrac{\alpha}{s}$

6.9 The Unilateral Laplace Transform

Thus, we obtain $Y(s)=\dfrac{\beta s^2+(\gamma+3\beta)s+\alpha}{s(s+1)(s+2)}$

$$Y(s)=\underbrace{\frac{\beta(s+3)}{(s+1)(s+2)}+\frac{\gamma}{(s+1)(s+2)}}_{zero-input\ response}+\underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{zero-state\ response}$$

Conclusion: The unilateral Laplace transform is of *considerable value* in analyzing causal systems which are specified by linear constant-coefficient differential equations with *nonzero initial conditions*.

6.9.4 Representation of Circuits in s-domain

Unilateral LT is also useful in electric circuits analysis. First we obtain the representations in s-domain for those basic elements in the circuit, then we can have the circuit in the s-domain.

The relationships between currents and voltages in the time domain for *Resistors*, *Inductors*, *Capacitors* are:

$$v_R(t) = Ri_R(t) \qquad v_L(t) = L\frac{di_L(t)}{dt} \qquad i_c(t) = C\frac{dv_c(t)}{dt}$$

Respectively.

Apply unilateral Laplace transform to each equation to obtain

$$V_R(s) = RI_R(s)$$

$$V_L(s) = sLI_L(s) - Li_L(0^-)$$

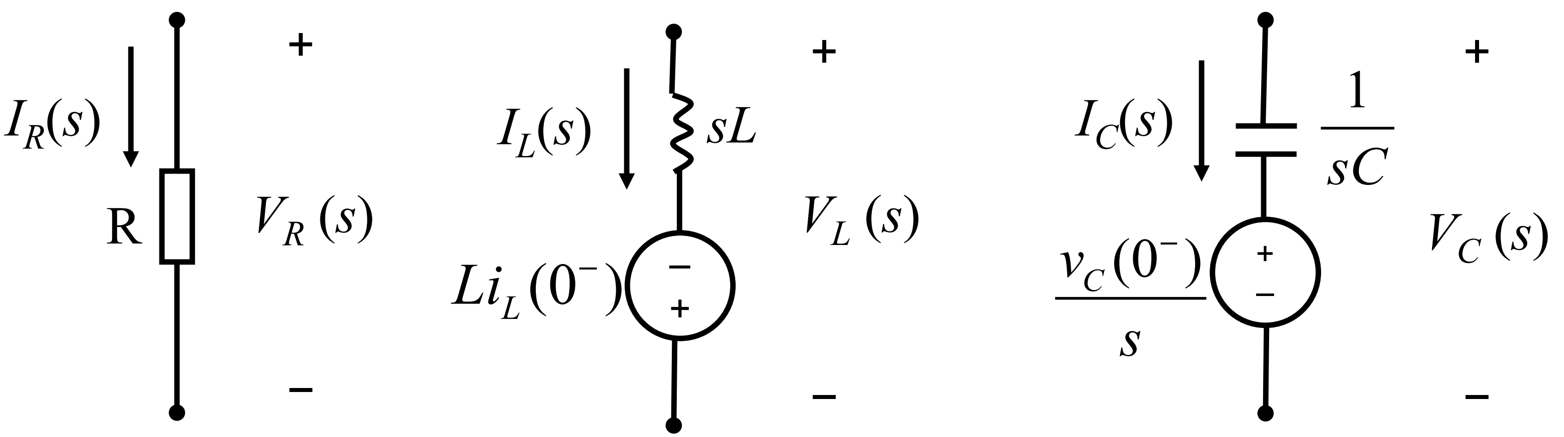
$$V_C(s) = \frac{1}{sC}I_C(s) + \frac{1}{s}v_C(0^-)$$

6.9 The Unilateral Laplace Transform

$$V_R(s) = RI_R(s)$$

$$V_L(s) = sLI_L(s) - Li_L(0^-)$$

$$V_C(s) = \frac{1}{sC} I_C(s) + \frac{1}{s} v_C(0^-)$$



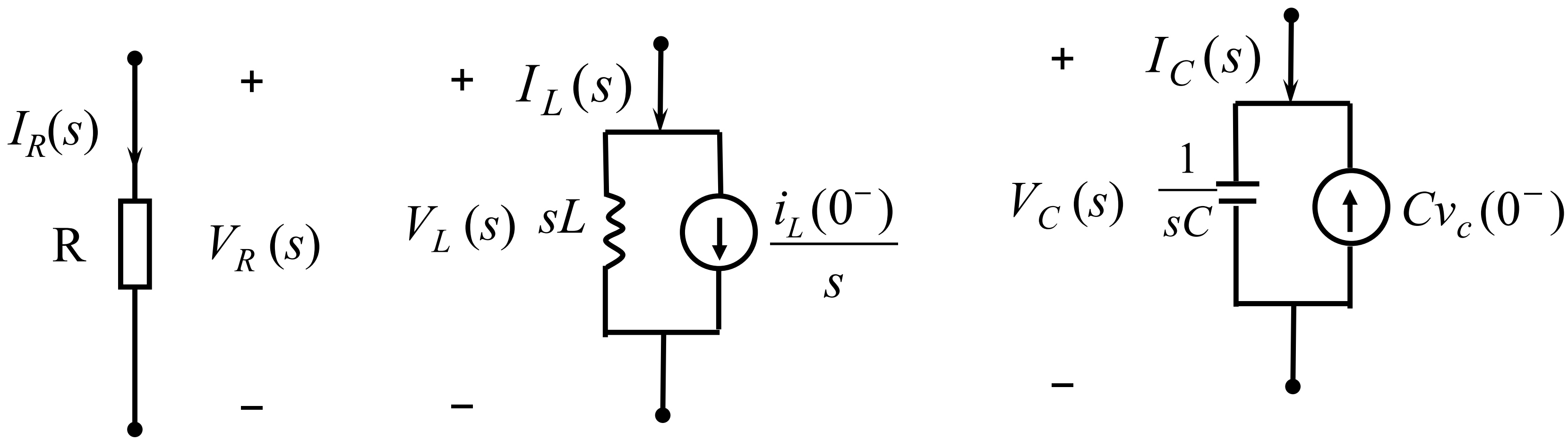
Representation of *Resistors, Inductors, Capacitors* in the s-domain with *initial conditions equivalent to a source voltage.*

- Another expression of the relation between *current* and *voltage* of three basic elements in the s-domain can lead to another model:

$$V_R(s) = RI_R(s)$$

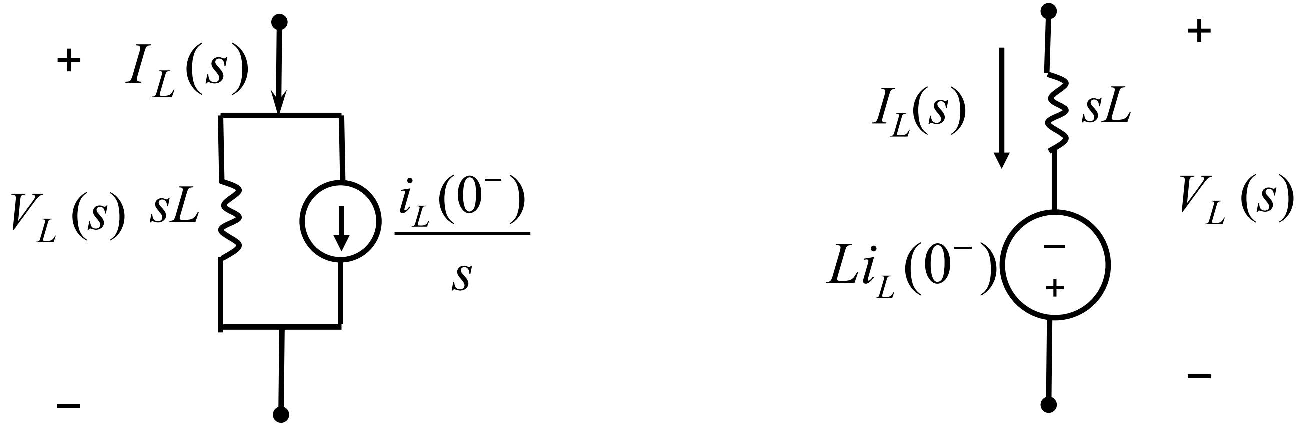
$$I_C(s) = sCV_C(s) - Cv_C(0^-)$$

$$I_L(s) = \frac{1}{sL}V_L(s) + \frac{1}{s}i_L(0^-)$$



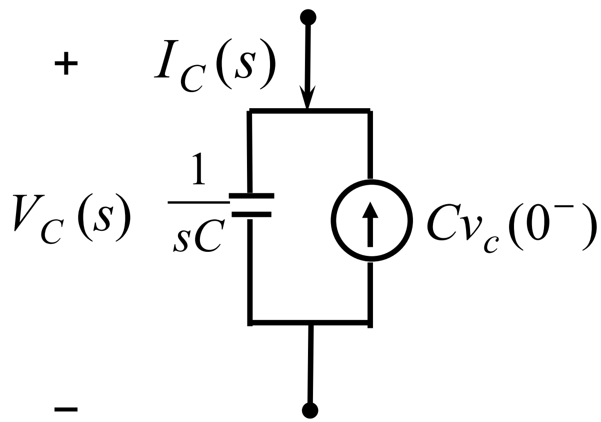
Representation of *Resistors, Inductors, Capacitors* in the s-domain with *initial conditions equivalent to a source current.*

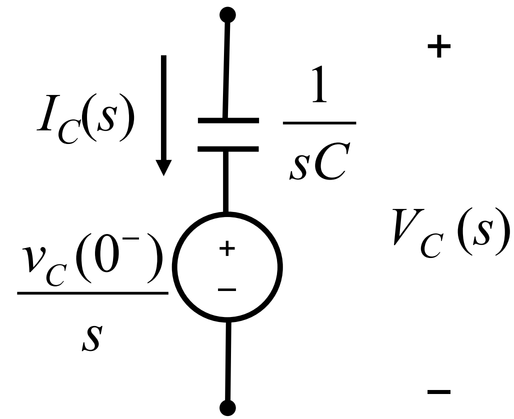
6.9 The Unilateral Laplace Transform



- An inductor with inductance L and initial current $\boldsymbol{i_L(0^-)}$ may be taken as an inductor with inductance L and zero initial current parallelly connected with a step source current with step $\boldsymbol{i_L(0^-)}$ or cascaded with an impulse source voltage with area $\boldsymbol{Li_L(0^-)}$.

6.9 The Unilateral Laplace Transform

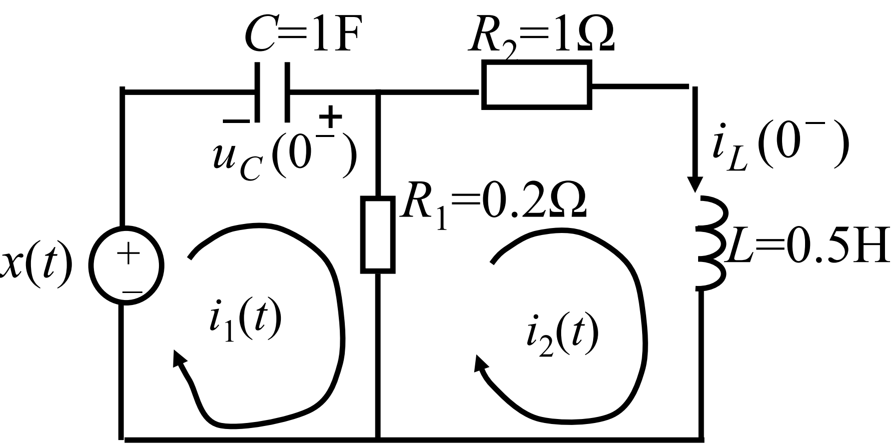




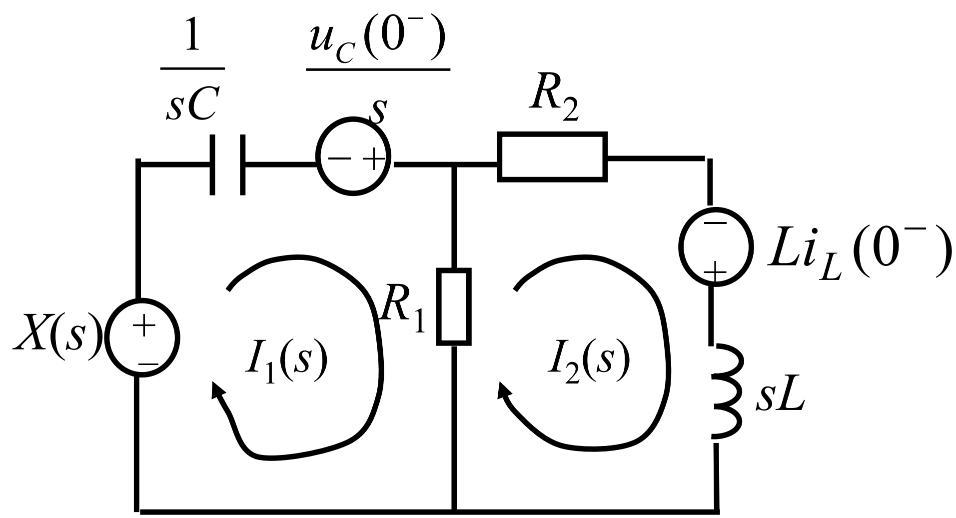
- A capacitor with capacitance C and initial voltage $\boldsymbol{v_C(0^-)}$ may be taken as a capacitor with capacitance C and zero initial voltage parallelly connected with an impulse source current with area $\boldsymbol{Cv_C(0^-)}$ or cascaded with a step source voltage with step $\boldsymbol{v_C(0^-)}$.

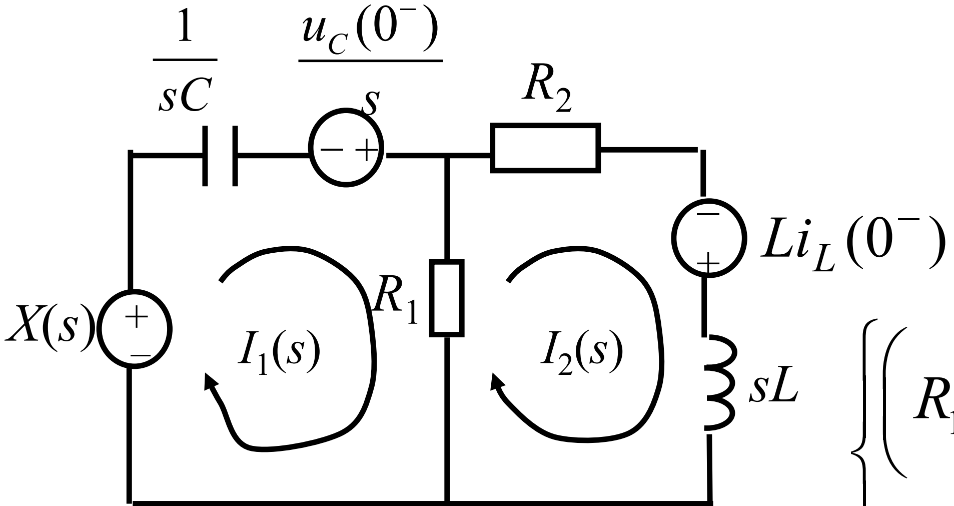
Example 6.27

Determine the current $i_1(t)$ in the following circuit. Given the initial conditions $u_C(0^-) = 5V$, $i_L(0^-) = 4A$, and input $x(t) = 10u(t)V$.



Sol: Drawing the s-domain circuit with all initial conditions equivalent to source voltages.





From *KVL* we can obtain following equations :

$$\begin{cases} \left(R_1 + \frac{1}{sC}\right) I_1(s) - R_1 I_2(s) = X(s) + \frac{u_C(0^-)}{s} \\ -R_1 I_1(s) + \left(R_1 + R_2 + sL\right) I_2(s) = Li_L(0_-) \end{cases}$$

Taking values into above equations to get

$$\begin{cases} \left(0.2 + \frac{1}{s}\right) I_1(s) - 0.2 I_2(s) = \frac{10}{s} + \frac{5}{s} \\ -0.2 I_1(s) + (1.2 + 0.5s) I_2(s) = 2 \end{cases}$$

Thus, $$I_1(s) = \frac{136}{s+4} - \frac{57}{s+3}$$

Consequently, $$i_1(t) = \left(136e^{-4t} - 57e^{-3t}\right) u(t)\ A$$

6.10 SUMMARY

- The bilateral and unilateral Laplace transforms;
- The properties of the *ROC* of LT and the relationship between the *ROC* and the poles;
- Methods to calculate the inverse Laplace transform;
- The properties of the bilateral and unilateral Laplace transforms (note the similarities and the differences);
- Significance of the poles and zeros of LT in characterizing continuous-time signals and systems;
- The computations of the zero-state response and the zero-input response by unilateral Laplace transform.
- The block diagram and signal flow graph representations of continuous-time LTI systems.

Homework

9.21 (a) (d) (g) (j)   9.22 (a) (c) (e) (g)

9.23   9.24   9.25 (b) (d) (f)   9.26

9.27   9.31   9.38   9.40