CHAPTER 5

THE DISCRETE-TIME FOURIER TRANSFORM

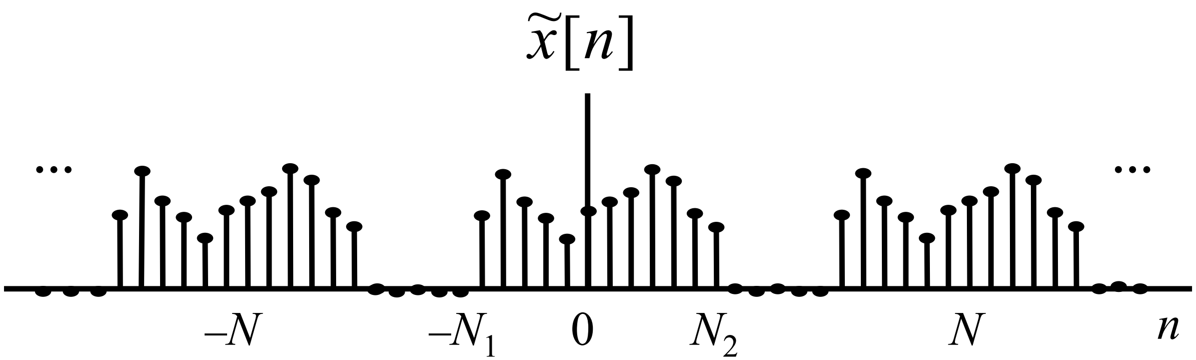
Content

5.0 Introduction

- Discrete-time Fourier transform (DTFT) and inverse Fourier transform
- Application of DTFT in discrete-time LTI systems analysis
- Similarities and differences between continuous-time and discrete-time Fourier transforms

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

5.1.1 Fourier Transform and Inverse Fourier Transform



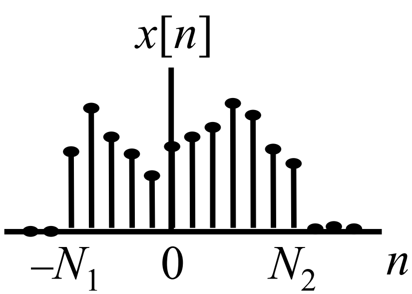
$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk(2\pi/N)n}$$

As $N \rightarrow \infty$,

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} \tilde{x}[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk(2\pi/N)n}$$



Defining a function $$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}$$

We see that $a_k = \frac{1}{N} X(e^{jk\omega_0})$

discrete-time Fourier transform

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

Taking the new expression for a_k into synthesis equation yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

As $N \to \infty$,

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega}$$

discrete-time inverse Fourier transform

$$\omega = k\omega_0 = k \cdot \frac{2\pi}{N}, \quad As \quad N \to \infty,$$

$$for\ k = 0, \qquad k\omega_0 = 0$$

$$for\ k = N-1, \qquad k \cdot \frac{2\pi}{N} \to 2\pi;$$

$$for\ k = -\frac{N}{2} (N\ even), \qquad k \cdot \frac{2\pi}{N} = -\pi$$

$$for\ k = \frac{N}{2} - 1, \qquad k \cdot \frac{2\pi}{N} \to \pi$$

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

Taking the new expression for a_k into synthesis equation yields

$$\tilde{x}[n]=\sum_{k=\langle N\rangle}\frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}=\frac{1}{2\pi}\sum_{k=\langle N\rangle}X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0$$

As $N\rightarrow\infty$,

$$\boxed{x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega}$$

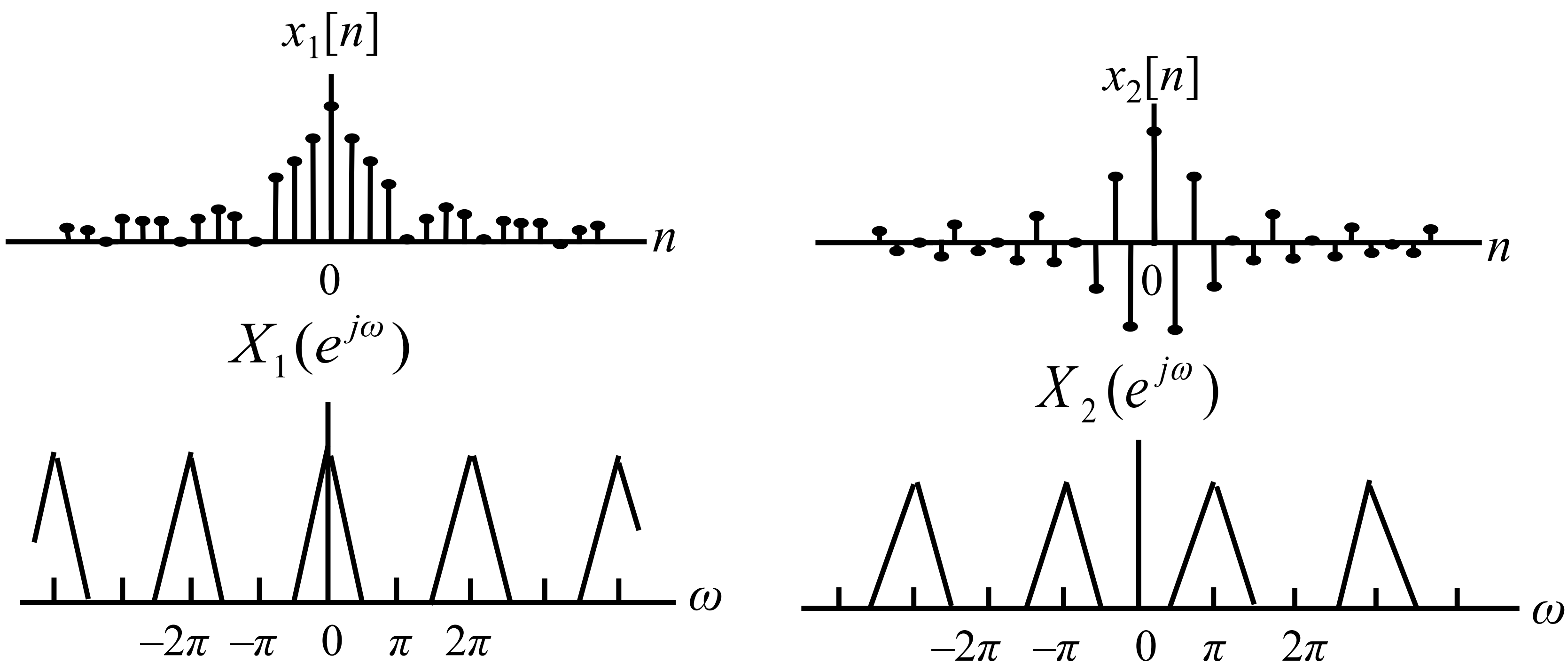
discrete-time inverse Fourier transform

- The synthesis equation is an integration over a *finite interval*.
- As $N\rightarrow\infty$, $k\omega_0\rightarrow\omega$, a continuous variable, so $X(e^{j\omega})$ is continuous as function of ω, and *periodic* with period 2π.
- From $a_k=\frac{1}{N}X(e^{j\omega})\Big|_{\omega=k\omega_0}$, a_k is proportional to the samples of $X(e^{j\omega})$, and ω_0 is the sampling period.

Two differences between CTFT and DTFT !

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

Question: In discrete time what values of frequency can be referred to as high or low frequencies ($X(e^{j\omega})$ is periodic)?



- In discrete time,

Low frequencies are the values of ω near *even multiple of* π;
high frequencies are those values of ω near *odd multiples of* π.

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Additional Information about frequencies in CTFT and DTFT:

$$x_p(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Applying FT on both sides yields

$$X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega \cdot nT}$$, Ω is analog frequency

Comparing with the analysis equation in DTFT leads to

$$\boxed{X(e^{j\omega})\Big|_{\omega=\Omega T} = X_p(j\Omega)}$$ ($\mathbf{I}$), ω is digital frequency

✓ $\omega = \Omega T$ tells that the unit of digital frequency ω is *rad* rather than *rad/sec*, which is the unit of analog frequency Ω.

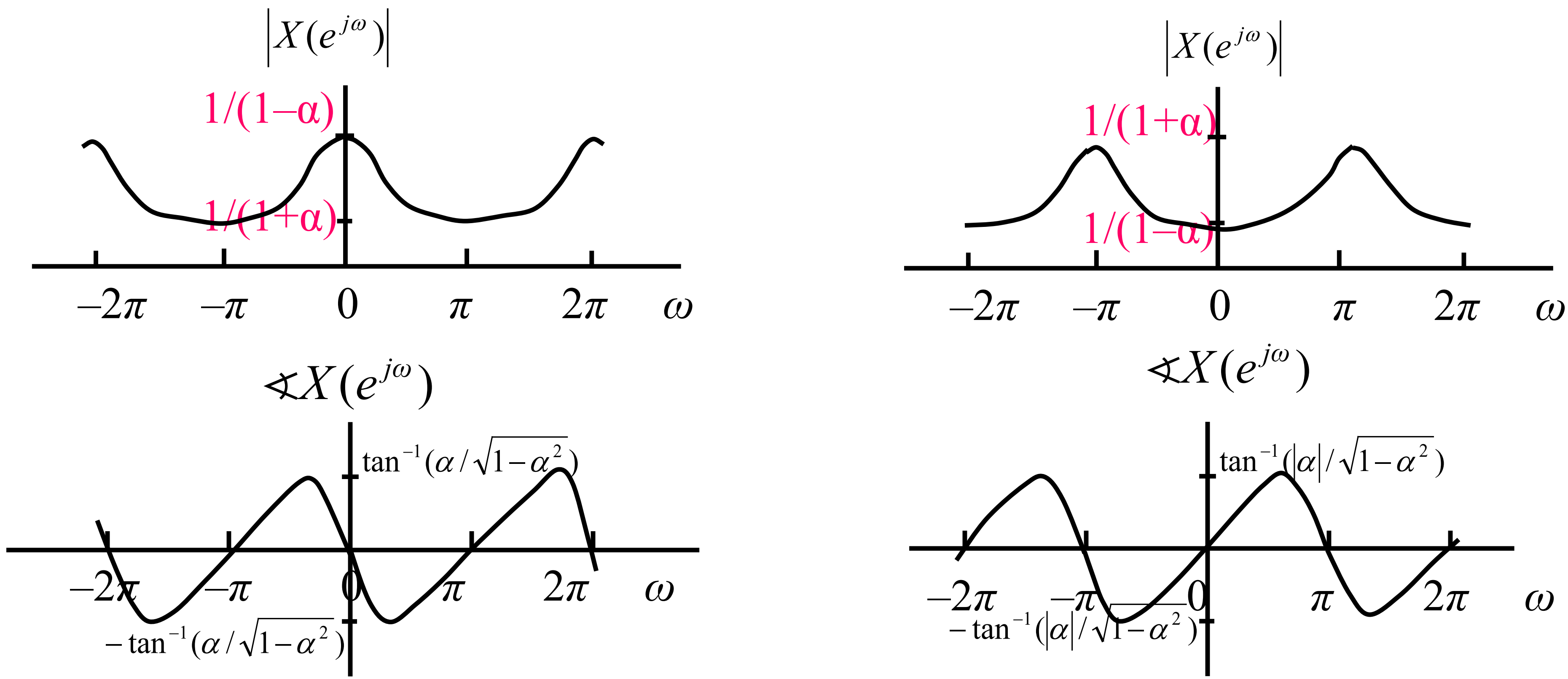
✓ Since $T = \frac{2\pi}{\Omega_s}$, ΩT maps Ω_s to 2π, and one period of $X_p(j\Omega)$ to an interval with length 2π (period).

✓ $\omega = \Omega T$ is called *frequency normalization*. Equation ($\mathbf{I}$) says $X(e^{j\omega})$ can be obtained by normalizing the frequency of $X_p(j\Omega)$.

5.1.2 Examples

Example 5.1 Consider the signal $x[n]=\alpha^{n}u[n], \quad |\alpha|<1.$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}\alpha^{n}u[n]e^{-j\omega n}=\sum_{n=0}^{\infty}(\alpha e^{-j\omega})^{n}=\frac{1}{1-\alpha e^{-j\omega}}$$



$1>\alpha>0$ $\qquad$ $0>\alpha>-1$

Example 5.2

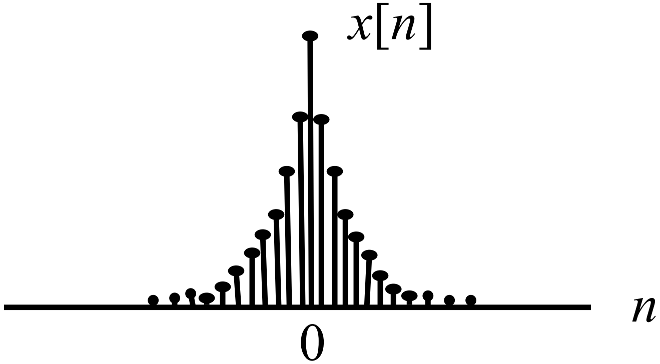
Consider the signal $x[n] = \alpha^{|n|}, \qquad |\alpha| < 1.$

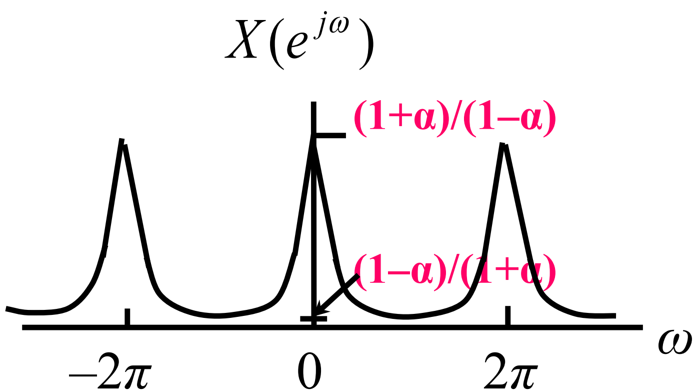
$$\text{Sol}: X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} \alpha^{-n} e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n + \sum_{m=1}^{\infty} (\alpha e^{j\omega})^m = \frac{1}{1-\alpha e^{-j\omega}} + \frac{\alpha e^{j\omega}}{1-\alpha e^{j\omega}}$$

$$= \frac{1-\alpha^2}{1-2\alpha\cos\omega + \alpha^2}$$

for $0 < \alpha < 1$,

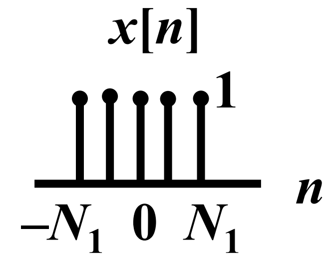




5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

Example 5.3

Consider the rectangular pulse $x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases}$

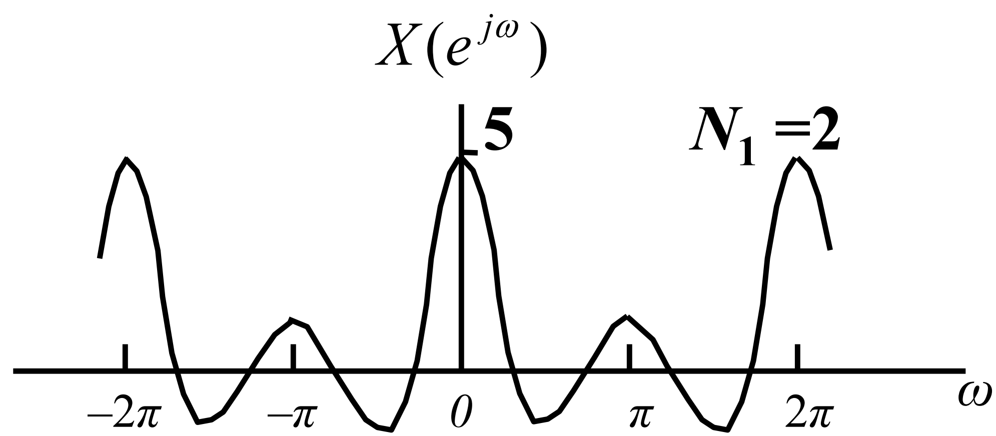


$$\text{Sol}: X(e^{j\omega})=\sum_{n=-N_1}^{N_1} e^{-j\omega n}=\frac{e^{j\omega N_1}(1-e^{-j\omega(2N_1+1)})}{1-e^{-j\omega}}$$

$$=\frac{e^{j\omega N_1}-e^{-j\omega N_1}e^{-j\omega}}{1-e^{-j\omega}}=\frac{e^{j\omega N_1}e^{j\omega(1/2)}-e^{-j\omega N_1}e^{-j\omega(1/2)}}{e^{j\omega(1/2)}-e^{-j\omega(1/2)}}$$

$$=\frac{2j\sin\omega(N_1+\frac{1}{2})}{2j\sin(\omega/2)}$$

$$\boxed{=\frac{\sin\omega(N_1+\frac{1}{2})}{\sin(\omega/2)}}$$



5.1.3 Convergence Issues of the Discrete-Time Fourier Transform

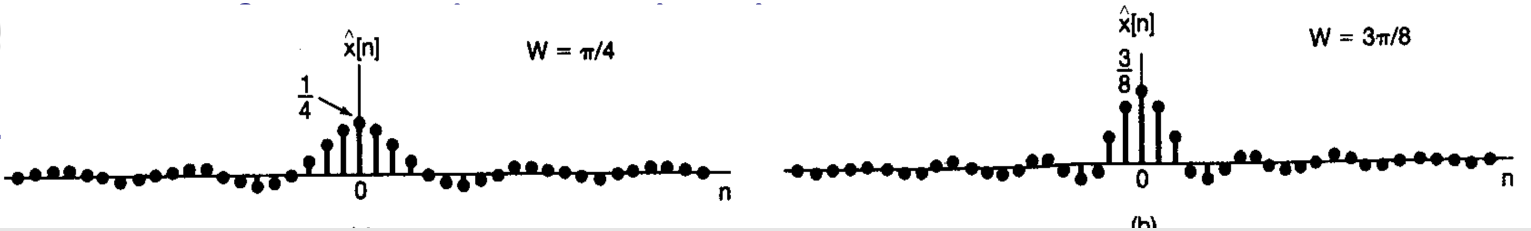
If $x[n]$ is an infinite duration signal, we must consider the question of convergence of the infinite summation in the analysis equation.

- The analysis equation will converge either if $x[n]$ is absolutely summable; that is $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ or if $x[n]$ has finite energy, that is $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$.
- In contrast to the situation for the analysis equation, there are generally *no convergence issues* associated with the *synthesis equation*.

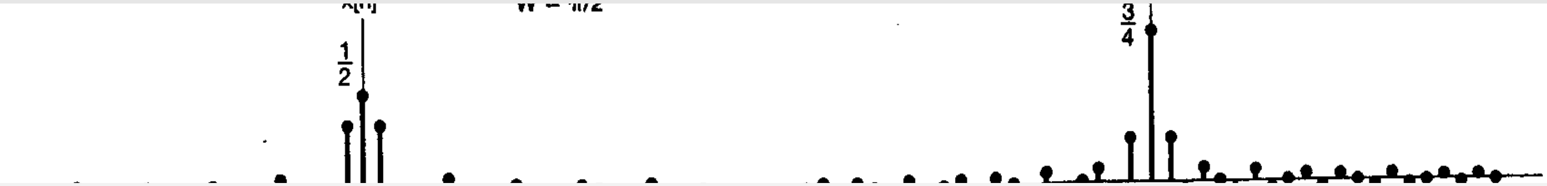
Consider the FT pair $\delta[n] \xleftrightarrow{FT} 1$

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} 1 \cdot e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}$$

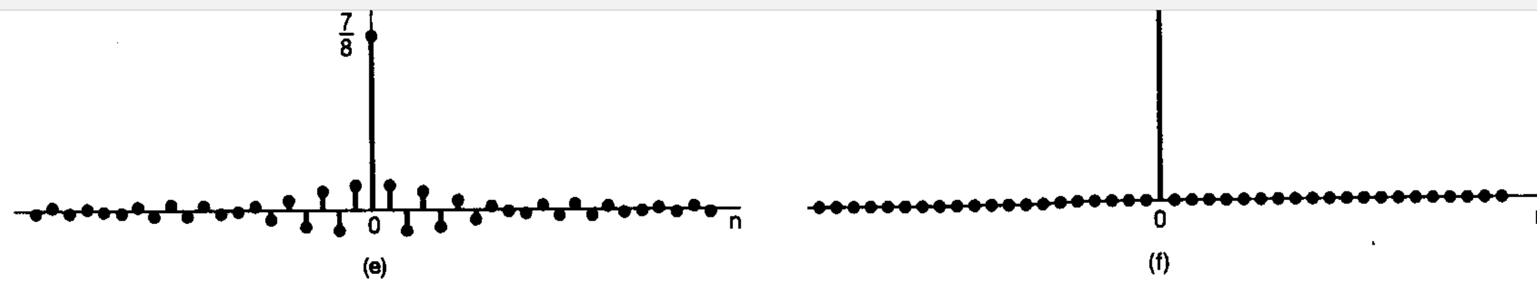
5.1 Represe Transform



- ✓ As *W* is increased, the oscillation frequency of the approximation also increase.



- ✓ As *W* is increased, the amplitude of these oscillations decreases relative to the magnitude of $\hat{x}[0]$, and the oscillations disappear entirely for *W* = π. No Gibbs phenomenon happens!



Approximation to the unit sample using complex exponentials with frequencies $|\omega| \leq W$: (a) *W* = π/4; (b) *W* = 3π/8; (c) *W* = π/2; (d) *W* = 3π/4; (e) *W* = 7π/8; (f) *W* = π. Notice that for *W* = π, $\hat{x}[n] = \delta[n]$

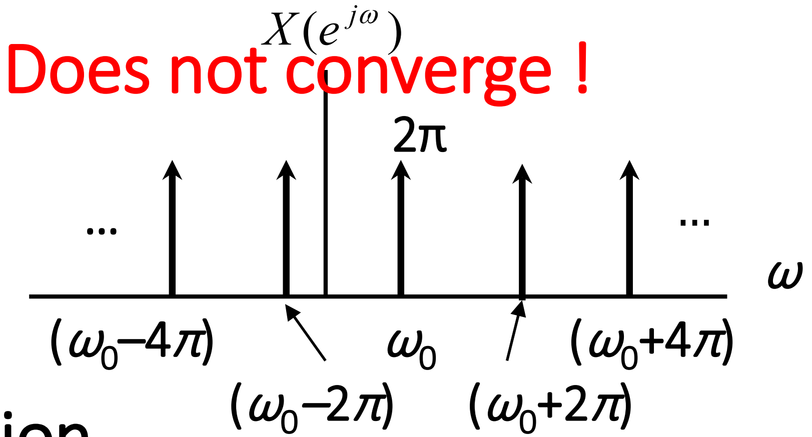
5.2 The Fourier Transform for Periodic Signals

First consider the Fourier transform of $x[n] = e^{j\omega_0 n}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega_0 - \omega)n}$$

Does not converge !

$$\boxed{X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)}$$



To check the validity of this expression,

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}d\omega = e^{j(\omega_0 + 2\pi r)n} = e^{j\omega_0 n}$$

For an arbitrary periodic sequence $x[n]$ with period N and with the Fourier series representation $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$.

The Fourier transform is

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)}$$

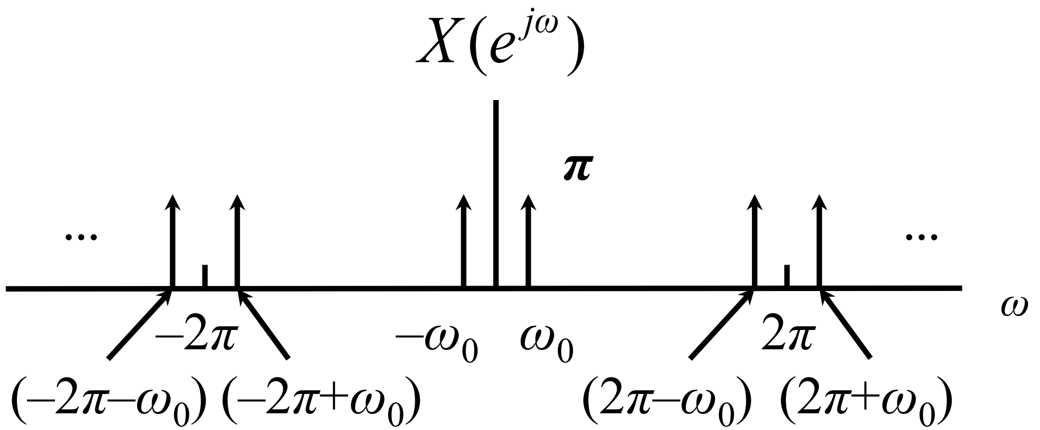
Example 5.4

Consider the periodic signal $x[n] = \cos\omega_0 n$, with $\omega_0 = \dfrac{2\pi}{5}$

Sol: $$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi\delta(\omega - \frac{2\pi}{5} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi\delta(\omega + \frac{2\pi}{5} - 2\pi l)$$

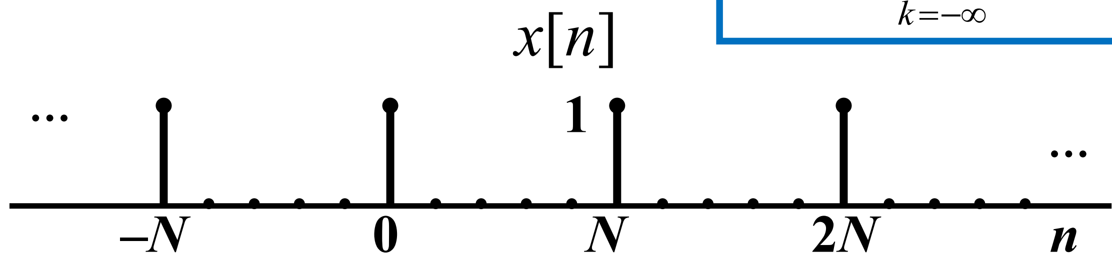
Or $$X(e^{j\omega}) = \pi\delta(\omega - \frac{2\pi}{5}) + \pi\delta(\omega + \frac{2\pi}{5}), \qquad -\pi \le \omega < \pi$$



5.2 The Fourier Transform for Periodic Signals

Example 5.5

Consider the periodic sample train $$x[n]=\sum_{k=-\infty}^{\infty}\delta[n-kN]$$



Sol:

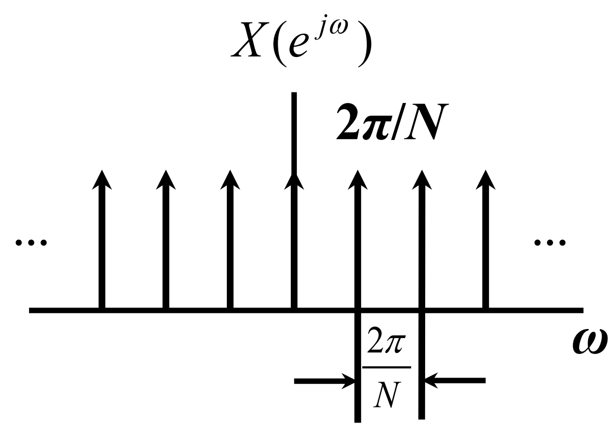
$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle}x[n]e^{-jk(2\pi/N)n}$$

Choosing the interval of summation as $0\le n\le N-1$, we have

$$a_k=\frac{1}{N}$$

Then

$$X(e^{j\omega})=\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$$



5.3 Properties of The Discrete-Time Fourier Transform

5.3.1 Periodicity of the Discrete-Time Fourier Transform

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

5.3.2 Linearity

If $x_1[n] \xleftrightarrow{FT} X_1(e^{j\omega})$ and $x_2[n] \xleftrightarrow{FT} X_2(e^{j\omega})$

then $ax_1[n] + bx_2[n] \xleftrightarrow{FT} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

5.3.3 Time Shifting and Frequency Shifting

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$

then $x[n-n_0] \xleftrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega})$ and $e^{j\omega_0 n} x[n] \xleftrightarrow{FT} X(e^{j(\omega-\omega_0)})$

5.3.4 Conjugation and Conjugate Symmetry

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$ then $x^*[n] \xleftrightarrow{FT} X^*(e^{-j\omega})$

- For real valued *x*[*n*],

$X(e^{j\omega}) = X^*(e^{-j\omega})$ ⇨ $\mathrm{Re}\left\{X(e^{j\omega})\right\}$、$\left|X(e^{j\omega})\right|$ are even

$\mathrm{Im}\left\{X(e^{j\omega})\right\}$、$\measuredangle X(e^{j\omega})$ are odd

$x_e[n] \xleftrightarrow{FT} \mathrm{Re}\left\{X(e^{j\omega})\right\}$ $\quad x_o[n] \xleftrightarrow{FT} j\,\mathrm{Im}\left\{X(e^{j\omega})\right\}$

5.3 Properties of The Discrete-Time Fourier Transform

5.3.5 Differencing and Accumulation

First-difference: $x[n]-x[n-1] \overset{FT}{\longleftrightarrow} (1-e^{-j\omega})X(e^{j\omega})$

Accumulation: $\sum_{m=-\infty}^{n} x[m] \overset{FT}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

5.3.6 Time Reversal

$x[-n] \overset{FT}{\longleftrightarrow} X(e^{-j\omega})$

5.3.7 Time Expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & n \text{ is a multiple of } k \\ 0, & n \text{ is not a multiple of } k \end{cases}$$

If $x[n] \overset{FT}{\longleftrightarrow} X(e^{j\omega})$

then $x_{(k)}[n] \overset{FT}{\longleftrightarrow} X(e^{jk\omega})$ $k:$ a positive integer

5.3.8 Differentiation in Frequency

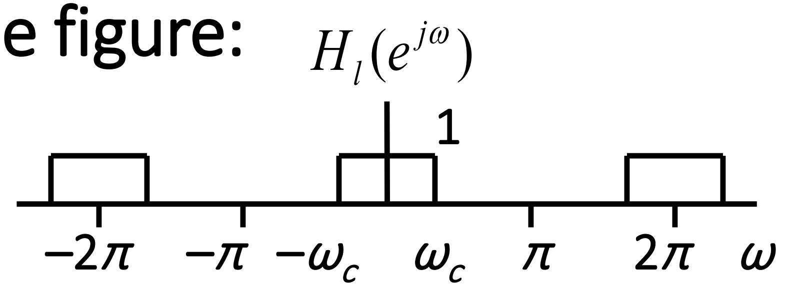
$nx[n] \overset{FT}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$

5.3.9 Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 \, d\omega$$

Example 5.6

The frequency response of a discrete-time low-pass filter with cutoff frequency ω_c is illustrated in the figure:

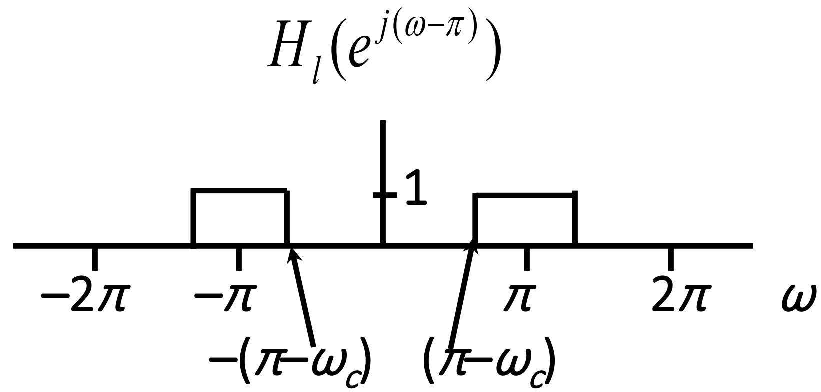


If we shift $H_l(e^{j\omega})$ by one-half period, i.e., by π, we obtain

$$H_h(e^{j\omega}) = H_l(e^{j(\omega-\pi)})$$

By frequency-shifting property,

$$h_h[n] = h_l[n]e^{j\pi n} = (-1)^n h_l[n]$$



Example 5.7

Determine the Fourier transform of the unit step $u[n]$.

Sol: Since $$u[n]=\sum_{m=-\infty}^{n}\delta[m]$$

and $$g[n]=\delta[n]\xleftrightarrow{FT}G(e^{j\omega})=1$$

Thus, $$\mathcal{F}\{u[n]\}=\frac{1}{1-e^{-j\omega}}G(e^{j\omega})+\pi G(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

$$=\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

$$\boxed{u[n]\xleftrightarrow{FT}\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}$$

5.4 The Convolution Property

If $y[n] = x[n] * h[n]$, then $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

- The *convolution property* represents that the Fourier transform of the response of an LTI system to a non-periodic input are simply the Fourier transform of the input multiplied by the system's frequency response evaluated at the corresponding frequencies.
- The convolution property maps the convolution operation of two time signals to the multiplication operation of their Fourier transforms.
- The frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier transform of the input at each frequency ω.

Example 5.8

Consider an LTI system with sample response $h[n] = \delta[n - n_0]$.

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

Thus, for any input $x[n]$, the Fourier transform of the output is

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Consequently,

$$y[n] = x[n - n_0]$$

$$H(e^{j\omega}) = e^{-j\omega n_0} \quad \Longrightarrow$$

Magnitude = 1 at all frequencies

phase = $-\omega n_0$

Example 5.9

Consider an LTI system with sample response $h[n] = \alpha^n u[n], |\alpha| < 1$

The input to this system is $x[n] = \beta^n u[n], |\beta| < 1$. The output $y[n]$ = ?

$$\text{Sol}: H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}, \quad X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

If $\alpha \neq \beta$,

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$A = \frac{\alpha}{\alpha-\beta}, \quad B = -\frac{\beta}{\alpha-\beta}$$

$$y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] = \frac{1}{\alpha-\beta}\left\{\alpha^{n+1} - \beta^{n+1}\right\} u[n]$$

5.4 The Convolution Property

If $\alpha = \beta$ $$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2$$

$$(Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right))$$

$$\alpha^n u[n] \xleftrightarrow{FT} \frac{1}{1-\alpha e^{-j\omega}}$$

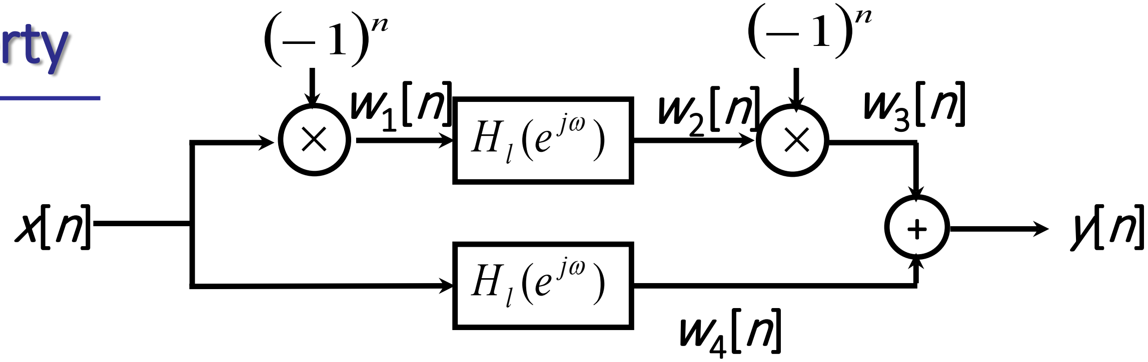
$$n\alpha^n u[n] \xleftrightarrow{FT} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$(n+1)\alpha^{n+1} u[n+1] \xleftrightarrow{FT} je^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$y[n] = (n+1)\alpha^n u[n]$$

Example 5.10

Consider the system



What is the frequency response of the overall system?

Where $H_l(e^{j\omega})$ is an ideal low-pass filter with cutoff frequency $\pi/4$ and unity gain in the passband.

Sol: The key step: $(-1)^n = e^{j\pi n}$

Then $\quad w_1[n] = e^{j\pi n}x[n] \longleftrightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$

Since, $\quad w_3[n] = e^{j\pi n}w_2[n] \longleftrightarrow W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)})$

$$= H_l(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)})$$

$$W_4(e^{j\omega}) = H_l(e^{j\omega})X(e^{j\omega})$$

$$= H_l(e^{j(\omega-\pi)})X(e^{j\omega})$$

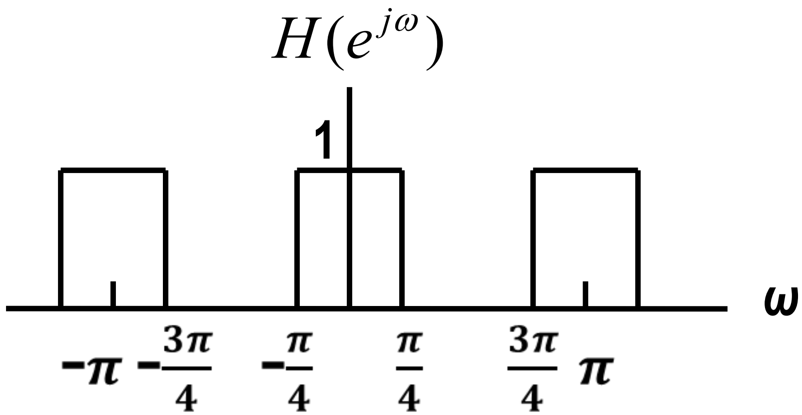
Consequently,

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = \left[H_l(e^{j(\omega-\pi)}) + H_l(e^{j\omega}) \right] X(e^{j\omega})$$

From the convolution property, the overall system has the frequency response:

$$H(e^{j\omega}) = H_l(e^{j(\omega-\pi)}) + H_l(e^{j\omega})$$

$$= H_h(e^{j\omega}) + H_l(e^{j\omega})$$



Ideal ***band-stop*** *filter*

Stopband: $\pi/4 < |\omega| < 3\pi/4$

5.5 The Multiplication Property

Consider the Fourier transform of $\boldsymbol{y[n] = x_1[n]x_2[n]}$, where $\boldsymbol{x_1[n] \leftrightarrow X_1(e^{j\omega})}$, $\boldsymbol{x_2[n] \leftrightarrow X_2(e^{j\omega})}$.

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\omega n}$$

Since

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left[\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta$$

$$X_2(e^{j(\omega-\theta)})$$

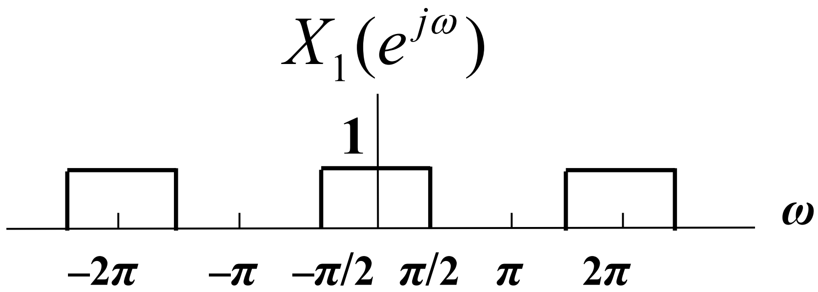
$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta}$$

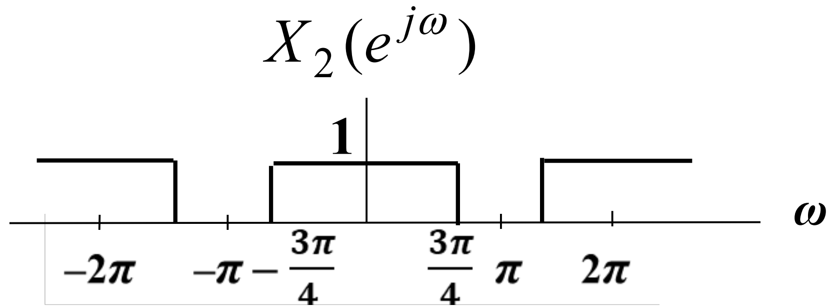
periodic convolution

Example 5.11

Find the Fourier transform $X(e^{j\omega})$ of a signal $x[n]$ which is the product of $x_1[n]=\dfrac{\sin(\pi n/2)}{\pi n}$ and $x_2[n]=\dfrac{\sin(3\pi n/4)}{\pi n}$.

Sol:

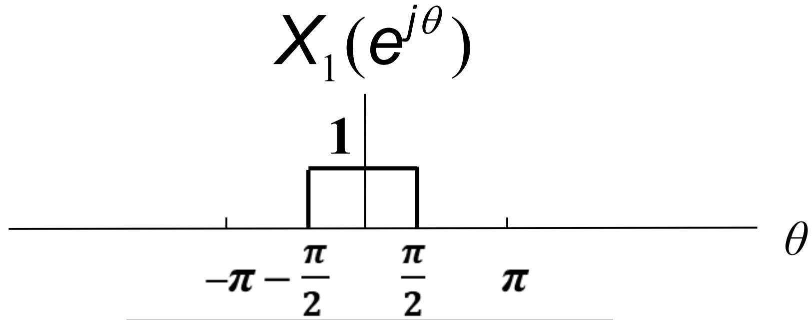




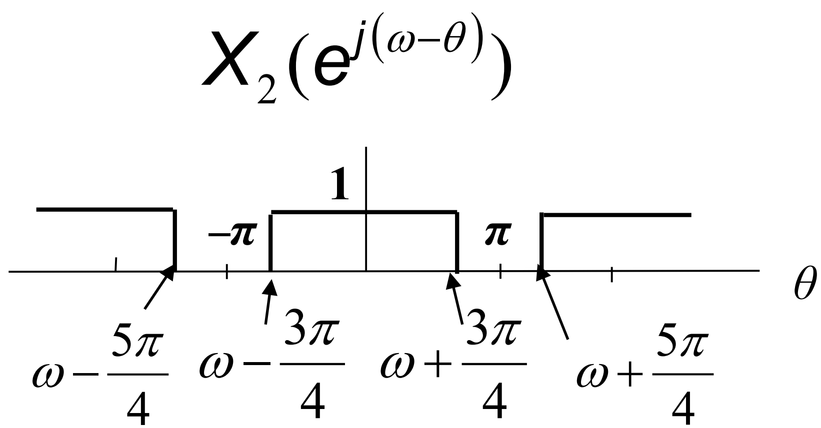
From the multiplication property,

$$X(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

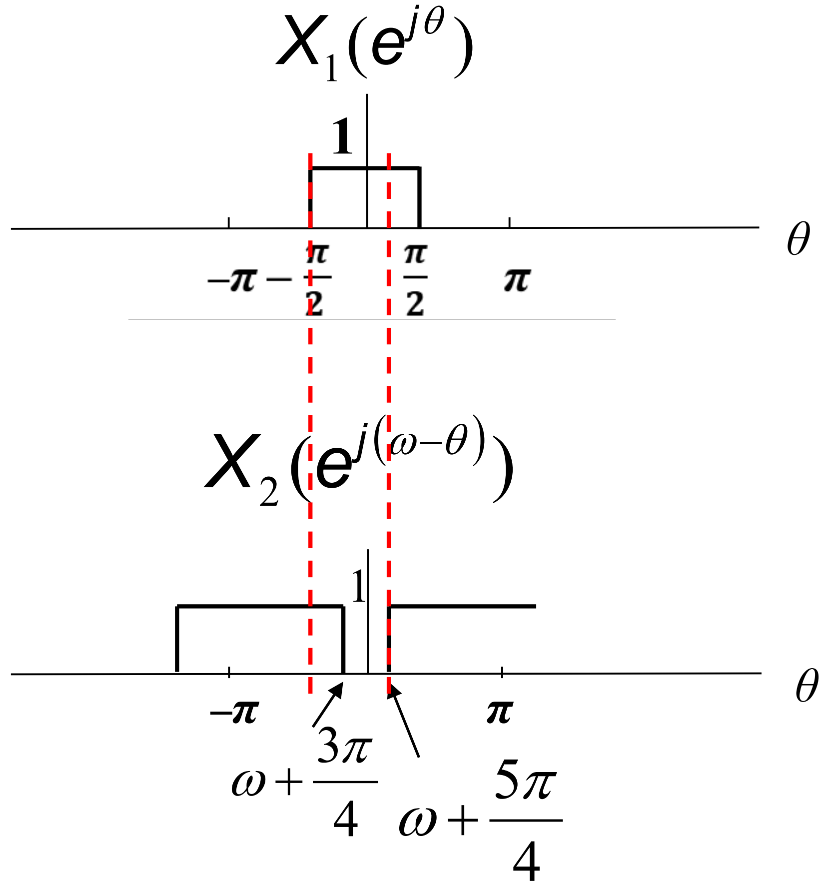
5.5 The Multiplication Property



$when \quad \omega = 0$



5.5 The Multiplication Property

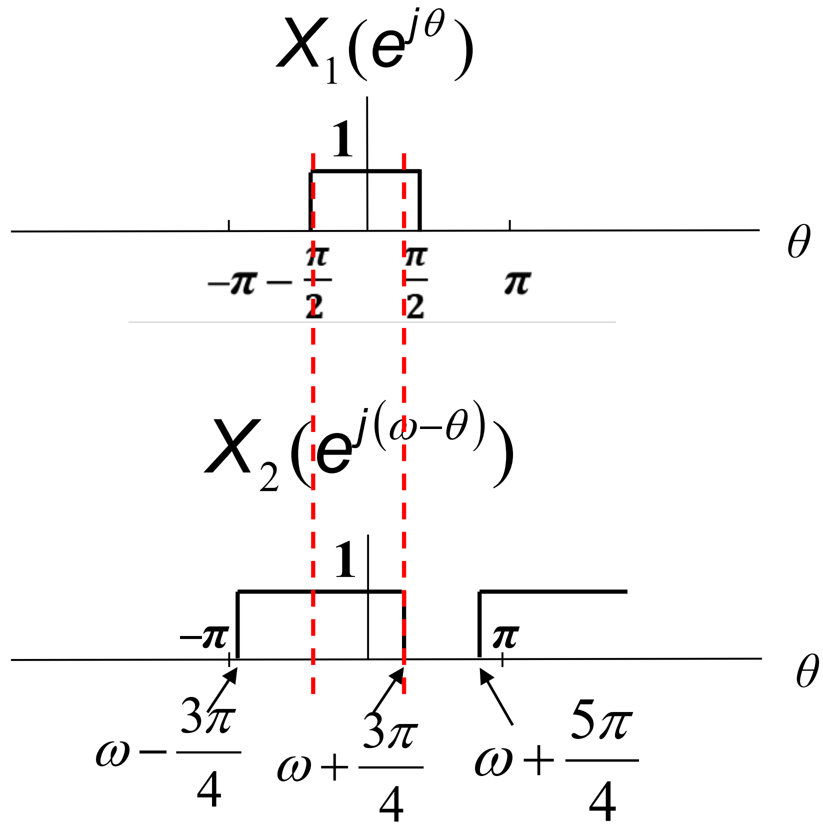


For $-\pi \le \omega < -\frac{3\pi}{4}$,

$$\left(\omega+\frac{5\pi}{4}<\frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\omega+\frac{3\pi}{4}} d\theta+\int_{\omega+\frac{5\pi}{4}}^{\frac{\pi}{2}} d\theta\right]=\frac{1}{2\pi}\cdot\frac{\pi}{2}=\frac{1}{4}$$

5.5 The Multiplication Property

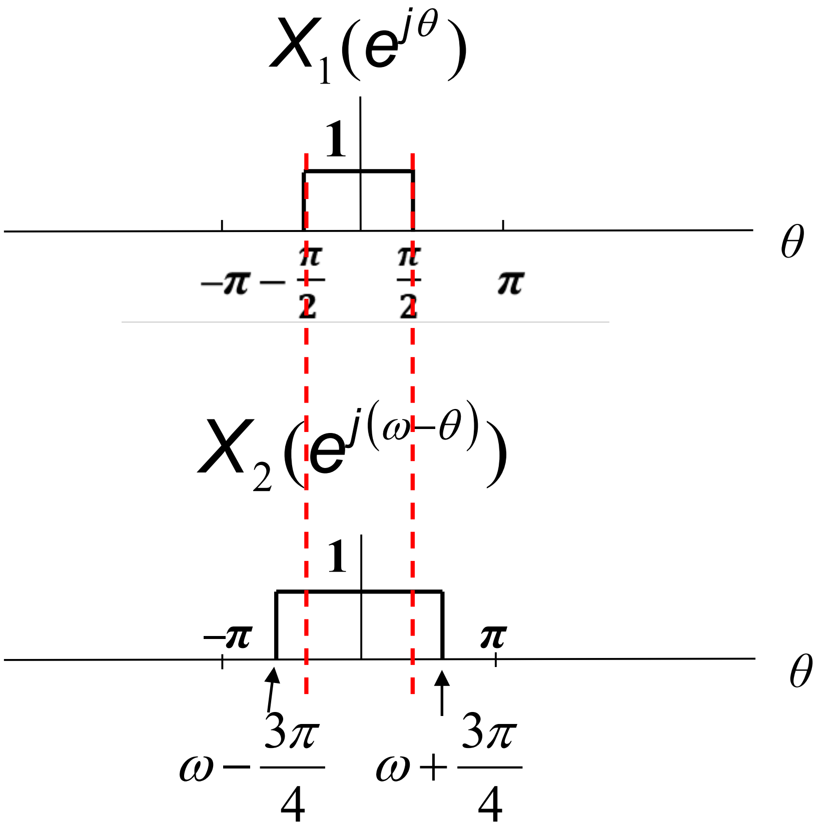


For $-\frac{3\pi}{4} \le \omega < -\frac{\pi}{4}$,

$$\left(\omega+\frac{5\pi}{4} \ge \frac{\pi}{2},\ \omega+\frac{3\pi}{4} < \frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\omega+\frac{3\pi}{4}} d\theta = \frac{1}{2\pi}\left(\omega+\frac{5\pi}{4}\right)=\frac{\omega}{2\pi}+\frac{5}{8}$$

5.5 The Multiplication Property

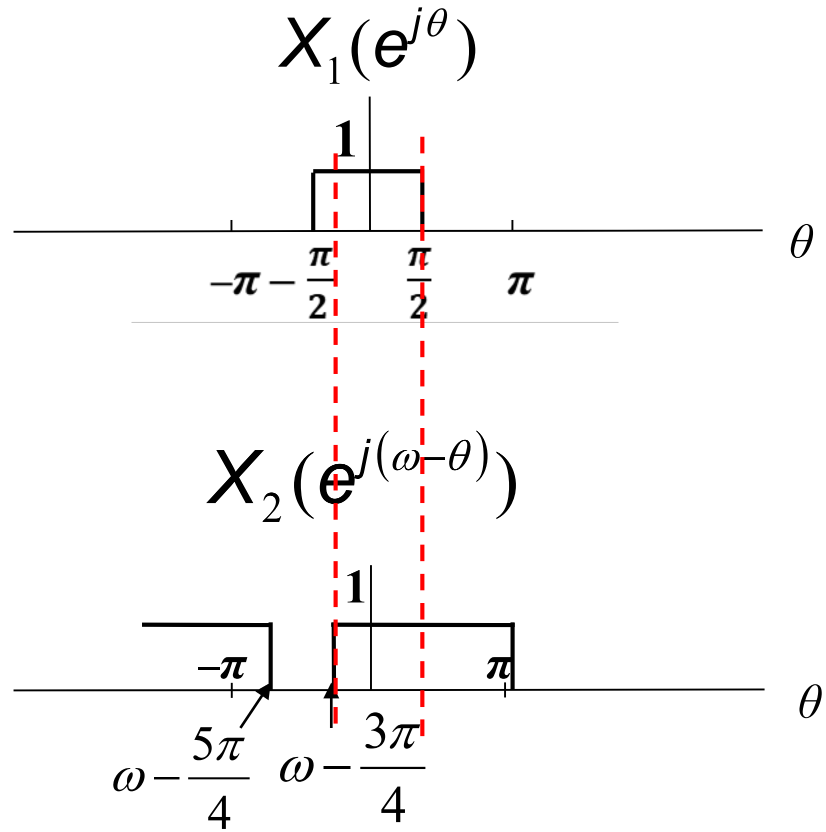


For $-\frac{\pi}{4} \le \omega < \frac{\pi}{4}$,

$$\left(\omega+\frac{3\pi}{4} \ge \frac{\pi}{2},\ \omega-\frac{3\pi}{4} < -\frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta=\frac{1}{2\pi}\cdot\pi=\frac{1}{2}$$

5.5 The Multiplication Property

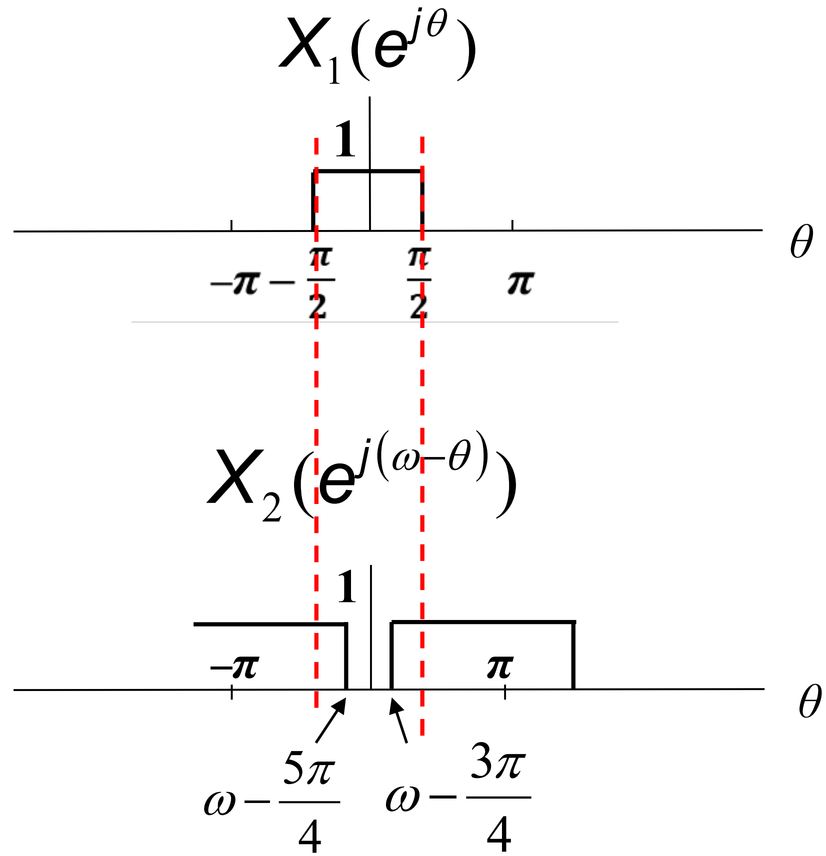


For $\frac{\pi}{4} \le \omega < \frac{3\pi}{4}$,

$$\left(\omega - \frac{3\pi}{4} \ge -\frac{\pi}{2}, \omega - \frac{5\pi}{4} < -\frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{\omega-\frac{3\pi}{4}}^{\frac{\pi}{2}} d\theta = \frac{1}{2\pi}\left(\frac{\pi}{2} - \omega + \frac{3\pi}{4}\right) = -\frac{\omega}{2\pi} + \frac{5}{8}$$

5.5 The Multiplication Property

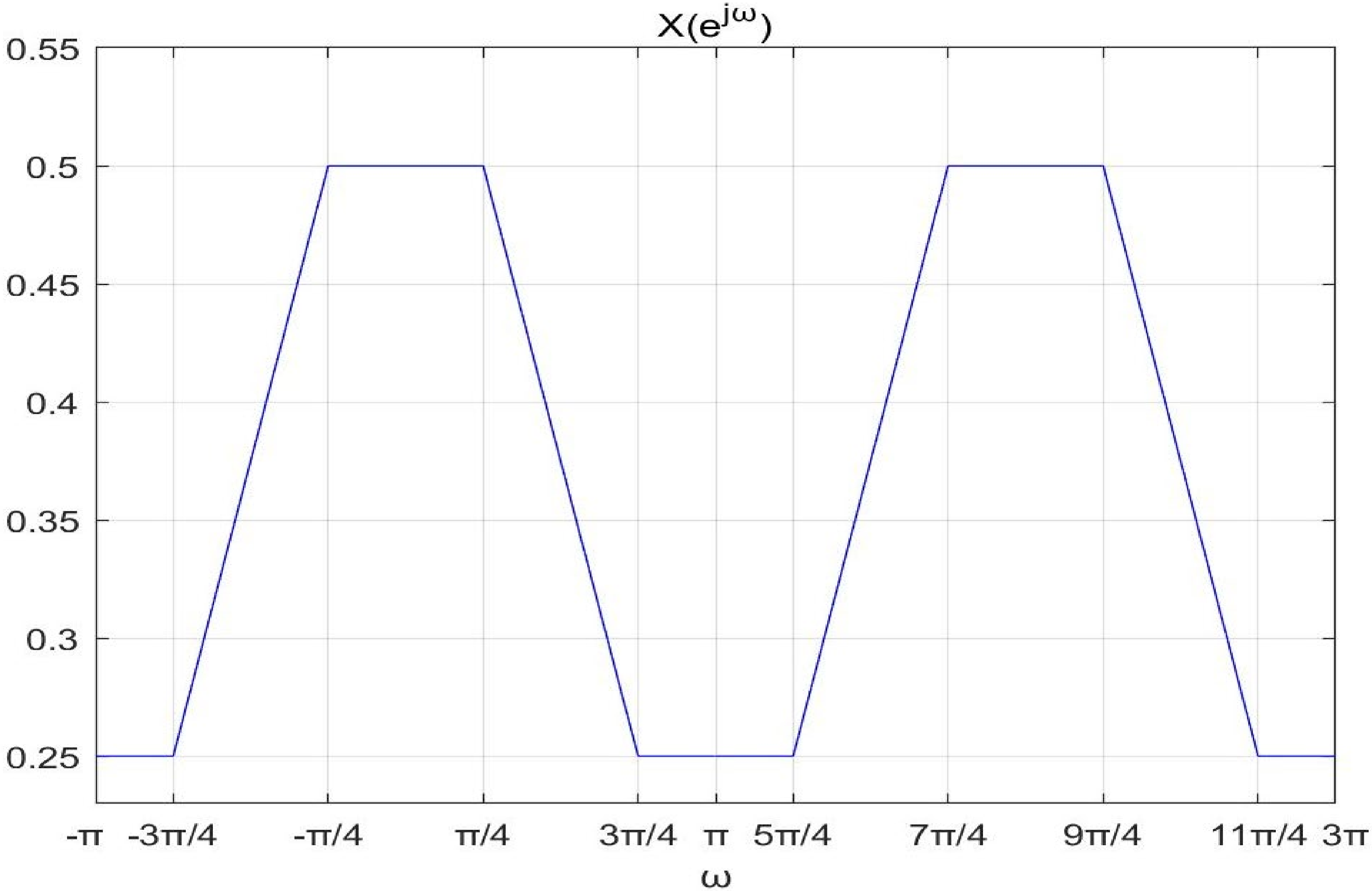


For $\frac{3\pi}{4} \le \omega < \pi,$

$$\left(\omega-\frac{5\pi}{4} \ge -\frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\omega-\frac{5\pi}{4}} d\theta+\int_{\omega-\frac{3\pi}{4}}^{\frac{\pi}{2}} d\theta\right]=\frac{1}{2\pi}\cdot\frac{\pi}{2}=\frac{1}{4}$$

5.5 The Multiplication Property



5.6 Systems Characterized By Linear Constant-Coefficient Difference Equations

A general *N*th-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}}$$

- $H(e^{j\omega})$ is a ratio of polynomials in the variable $e^{-j\omega}$.
- Coefficients of the *numerator polynomial* = Coefficients appearing on the *right side* of the difference equation.
- Coefficients of the *denominator polynomial* = Coefficients appearing on the *left side* of the difference equation.

Example 5.13

Consider a causal LTI system that is characterized by the difference equations $y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$, and let the input to this system be $x[n]=(1/4)^n u[n]$. Determine the output $y[n]$.

Sol:

$$Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})=\left[\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$=\left[\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

The form of the partial-fraction expansion in this case is

$$Y(e^{j\omega})=\frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}}+\frac{B_{12}}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}+\frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}$$

5.6 Systems Characterized By Linear Constant-Coefficient Difference Equations

$$B_{12} = \left[\left(1 - \frac{1}{4} e^{-j\omega}\right)^2 Y(e^{j\omega}) \right]\Bigg|_{e^{-j\omega}=4} = -2,$$

$$B_{21} = \left[\left(1 - \frac{1}{2} e^{-j\omega}\right) Y(e^{j\omega}) \right]\Bigg|_{e^{-j\omega}=2} = 8,$$

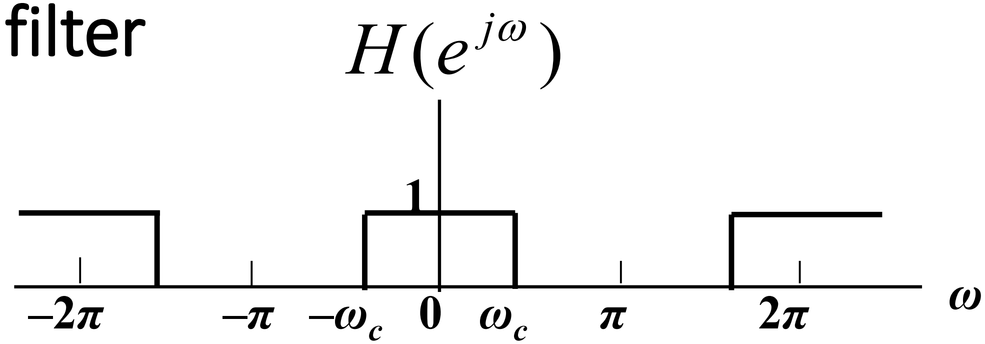
$$B_{11} = (-4)\left[\frac{d}{d(e^{-j\omega})} \left(1 - \frac{1}{4} e^{-j\omega}\right)^2 Y(e^{j\omega}) \right]\Bigg|_{e^{-j\omega}=4} = -4.$$

So that, $$Y(e^{j\omega}) = \frac{-4}{1 - \frac{1}{4} e^{-j\omega}} + \frac{-2}{\left(1 - \frac{1}{4} e^{-j\omega}\right)^2} + \frac{8}{1 - \frac{1}{2} e^{-j\omega}}$$

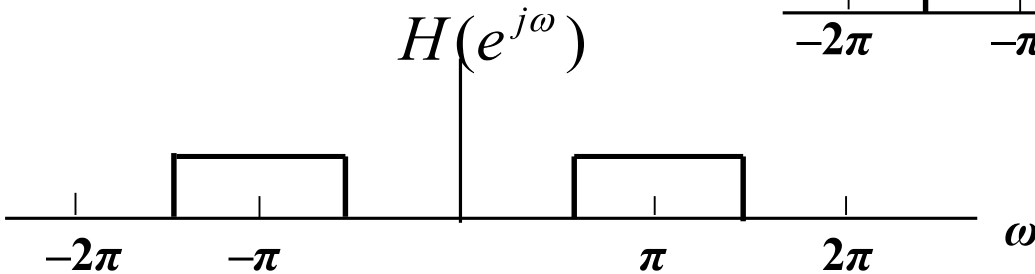
Consequently, $$y[n] = \left\{ -4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n \right\} u[n]$$

5.7 Discrete-Time Frequency-Selective Filters

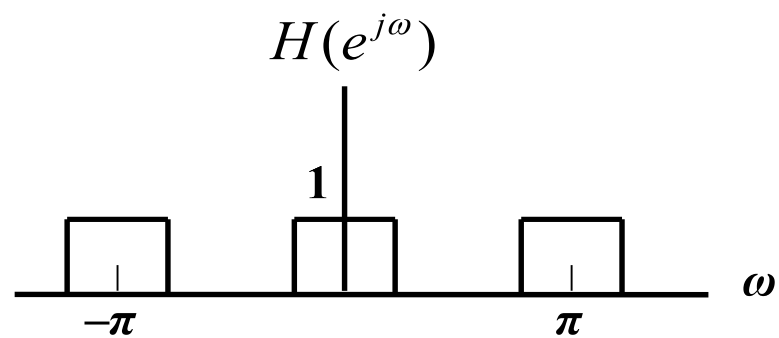
Ideal low-pass filter



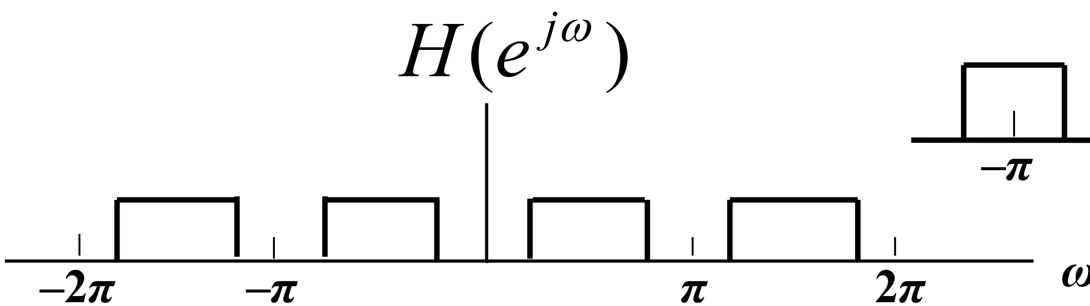
Ideal high-pass filter



Ideal band-stop filter



Ideal band-pass filter



5.8 SUMMARY

- The Fourier transform for non-periodic and periodic discrete-time signals;
- The differences between the DTFT and the CTFT (Especially the periodicity of DTFT);
- The properties of the Fourier transform (relationships between characteristics of a discrete-time signal in time and frequency domain);
- Fourier analysis (Frequency domain analysis) for discrete-time LTI systems including both characteristics of systems and responses to some input signals;
- Frequency response and the way to obtain it.

Homework

5.21 (a) (g) (h) (i)   5.22 (a) (b) (d) (f)

5.26   5.29 (i) in (a)、(ii) in (b)

5.30 (a) 、(ii) (iii) in (b)、(ii) in (c)