Why CMLL Has 42 Cases: Reachability, Parity, and Burnside's Lemma

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Abstract

We give a group-theoretic derivation of the number of CMLL cases arising in the Roux method for the Rubik's Cube. Under the standard CMLL constraints—fixing the bottom-layer corners and a specified set of six edges—we show that the space of top-layer corner states compatible with the cube-group reachability invariants has cardinality $4! \cdot 3^3 = 648$. In particular, no even-permutation restriction is imposed on the corner permutation, as parity may be absorbed by the remaining unsolved edges.

We then quotient this corner state space by a natural symmetry action and apply Burnside's lemma to compute the resulting orbit count. The calculation yields exactly 42 equivalence classes. Here "admissible" refers to global reachability (existence of a globally reachable cube state satisfying the fixed-piece constraints), not reachability via move sequences that preserve the Roux blocks throughout.

1 Introduction

In the Roux method for solving the Rubik's Cube, the *CMLL* step completes the four top-layer corners while preserving two already-built blocks. A recurring point of confusion is whether the corner-permutation parity forces one to "divide by 2" (i.e. restrict to even corner permutations) when counting CMLL cases.

The resolution is that *CMLL is typically not the final step of the Roux solve*: several edges (notably in the M-slice) remain unsolved and can absorb permutation parity. Consequently, the set of attainable top-corner states under the CMLL constraints has size $4! \cdot 3^3 = 648$ rather than $12 \cdot 3^3 = 324$. After establishing this state space, we then classify these 648 states up to a standard symmetry relation and obtain exactly 42 equivalence classes via Burnside's lemma.

We proceed in two steps: first we justify the 648 admissible top-corner states under the usual CMLL constraints; then we quotient by the standard symmetries and apply Burnside's lemma to obtain 42.

2 Preliminaries and State Model

We model the Rubik's Cube using the standard *cubie model*. Let

$$C = \{1, \dots, 8\} \quad \text{(corners)}, \qquad E = \{1, \dots, 12\} \quad \text{(edges)}.$$

A cube state is represented by a quadruple

$$(\sigma_c, t_c, \sigma_e, t_e),$$

where:

- $\sigma_c \in S_8$ is the corner permutation;
- $t_c = (t_1, \ldots, t_8) \in (\mathbb{Z}/3\mathbb{Z})^8$ is the corner orientation vector;
- $\sigma_e \in S_{12}$ is the edge permutation;
- $t_e = (f_1, \ldots, f_{12}) \in (\mathbb{Z}/2\mathbb{Z})^{12}$ is the edge orientation vector.

We take the solved cube as the identity state.

Remark 1. The particular conventions used to assign corner/edge orientations do not affect the statements below, provided they satisfy the standard global constraints described in Theorem 1.

3 Classical Reachability Conditions

The following theorem is classical in the mathematical analysis of the Rubik's Cube.

Theorem 1 (Reachability Invariants)**.** *A state $(\sigma_c, t_c, \sigma_e, t_e)$ is reachable from the solved cube by legal face turns if and only if the following three conditions hold:*

1. ***Parity:*** $\operatorname{sgn}(\sigma_c) = \operatorname{sgn}(\sigma_e)$*;*
2. ***Corner orientation:*** $\sum_{i=1}^{8} t_i \equiv 0 \pmod 3$*;*
3. ***Edge orientation:*** $\sum_{j=1}^{12} f_j \equiv 0 \pmod 2$*.*

Remark 2. The necessity direction can be verified by checking that each generator U, D, L, R, F, B preserves these three quantities. The sufficiency direction is nontrivial in general. In this note we only require Theorem 1 as a black box and give a fully constructive argument in the constrained setting below.

4 Constrained Setting and Statement

Fix the following constraints, motivated by partially built cube states:

- The *bottom four corner positions* (say positions $5, 6, 7, 8$) are solved:

$$\sigma_c(i) = i, \quad t_i = 0 \qquad (i = 5, 6, 7, 8).$$

- A prescribed subset $B \subset E$ of *six edges* is solved in both position and orientation:

$$\sigma_e(k) = k, \quad f_k = 0 \qquad (k \in B).$$

  Let $R := E \setminus B$ be the remaining set of *unconstrained* edges, so $|R| = 6$.

Let $\tau := \sigma_c|_{\{1,2,3,4\}} \in S_4$ denote the induced permutation of the top four corner positions, and let (t_1, t_2, t_3, t_4) denote their orientations.

Definition 1 (Admissible top-corner state)**.** A pair $(\tau, (t_1, t_2, t_3, t_4))$ with $\tau \in S_4$ and $(t_1, t_2, t_3, t_4) \in (\mathbb{Z}/3\mathbb{Z})^4$ satisfying $t_1 + t_2 + t_3 + t_4 \equiv 0 \pmod 3$ is called *admissible* if it occurs as the top-corner component of some cube state that satisfies the fixed bottom corners and fixed edge set B and is globally reachable from the solved cube.

Theorem 2 (Corner state space under six fixed edges)**.** *Under the constraints above, every top-corner configuration consisting of an arbitrary permutation* $\tau \in S_4$ *and an orientation vector* $(t_1, t_2, t_3, t_4) \in (\mathbb{Z}/3\mathbb{Z})^4$ *satisfying*

$$t_1 + t_2 + t_3 + t_4 \equiv 0 \pmod 3$$

is admissible. Consequently, the number of admissible top-corner states is $4! \cdot 3^3 = 648$.

Remark 3. Theorem 2 is a global consistency statement: it asserts existence of a globally reachable cube state realizing the specified top-corner data under the fixed-piece constraints. It does not address reachability via move sequences that preserve the Roux blocks throughout.

5 Proof of Theorem 2

We prove realizability by constructing compatible edge data that satisfy the reachability invariants of Theorem 1.

5.1 Corner orientation constraint

Lemma 1. *If* (t_1, t_2, t_3, t_4) *satisfies* $\sum_{i=1}^{4} t_i \equiv 0 \pmod 3$ *and* $t_5 = \cdots = t_8 = 0$*, then* $\sum_{i=1}^{8} t_i \equiv 0 \pmod 3$.

Proof. Immediate:

$$\sum_{i=1}^{8} t_i = \sum_{i=1}^{4} t_i + \sum_{i=5}^{8} t_i \equiv 0 + 0 \equiv 0 \pmod 3.$$

□

5.2 Edge orientation constraint

Lemma 2. *There exists an edge orientation assignment* $(f_1, \ldots, f_{12})$ *such that* $f_k = 0$ *for all* $k \in B$ *and* $\sum_{j=1}^{12} f_j \equiv 0 \pmod 2$.

Proof. Set $f_k = 0$ for $k \in B$. On the remaining set R (which has size 6), freely choose the values of f on any five elements of R. Define the sixth value so that the total sum over R is 0 modulo 2. Then $\sum_{j=1}^{12} f_j = \sum_{k \in B} f_k + \sum_{r \in R} f_r \equiv 0 + 0 \equiv 0 \pmod 2$. □

5.3 Parity matching via the remaining edges

Lemma 3. *For any* $\tau \in S_4$ *there exists a permutation* $\pi \in S_6$ *such that* $\operatorname{sgn}(\pi) = \operatorname{sgn}(\tau)$.

Proof. If $\operatorname{sgn}(\tau) = +1$, take $\pi = \mathrm{id}$. If $\operatorname{sgn}(\tau) = -1$, take π to be any transposition in S_6, e.g. $(1\ 2)$ in a fixed labeling of the 6 elements. A transposition is an odd permutation, hence has sign -1. Therefore $\operatorname{sgn}(\pi) = \operatorname{sgn}(\tau)$ in all cases. □

5.4 Assembling a reachable full state

Proof of Theorem 2. Fix an arbitrary $\tau \in S_4$ and an orientation vector (t_1, t_2, t_3, t_4) satisfying $\sum_{i=1}^{4} t_i \equiv 0 \pmod 3$. Define:

- $\sigma_c \in S_8$ by setting $\sigma_c|_{\{1,2,3,4\}} = \tau$ and $\sigma_c(i) = i$ for $i = 5, 6, 7, 8$;

- $t_c = (t_1, \dots, t_8)$ by setting $t_5 = \cdots = t_8 = 0$;
- choose $\pi \in S_6$ as in Lemma 3;
- define $\sigma_e \in S_{12}$ by $\sigma_e(k) = k$ for $k \in B$ and $\sigma_e|_R = \pi$ (after identifying R with a 6-element set);
- choose $t_e = (f_1, \dots, f_{12})$ as in Lemma 2.

Then:

1. By Lemma 1, $\sum_{i=1}^{8} t_i \equiv 0 \pmod 3$.
2. By Lemma 2, $\sum_{j=1}^{12} f_j \equiv 0 \pmod 2$.
3. For parity, since σ_c fixes corners 5–8, we have $\mathrm{sgn}(\sigma_c) = \mathrm{sgn}(\tau)$. Since σ_e fixes all edges in B, we have $\mathrm{sgn}(\sigma_e) = \mathrm{sgn}(\pi)$. By Lemma 3,
$$\mathrm{sgn}(\sigma_c) = \mathrm{sgn}(\tau) = \mathrm{sgn}(\pi) = \mathrm{sgn}(\sigma_e).$$

Thus all conditions of Theorem 1 hold, so the constructed state $(\sigma_c, t_c, \sigma_e, t_e)$ is globally reachable from the solved cube. By construction it satisfies the fixed bottom corners and the fixed edge set B, and its top-corner component is $(\tau, (t_1, t_2, t_3, t_4))$. This proves admissibility of every such top-corner configuration.
For the count: τ can be chosen arbitrarily in S_4, giving $4! = 24$ choices. For orientations, the constraint $\sum_{i=1}^{4} t_i \equiv 0 \pmod 3$ leaves 3^3 choices (choose three entries freely and determine the fourth). Hence the total is $4! \cdot 3^3 = 648$. □

6 Remarks and Variants

Remark 4. If instead *all* 12 edges are required solved, then $\sigma_e = \mathrm{id}$ forces $\mathrm{sgn}(\sigma_c) = +1$. With bottom corners fixed, the top-corner permutation must lie in A_4 (size 12), yielding $12 \cdot 3^3 = 324$ reachable top-corner states.

7 Orbit-counting setup: the state space and symmetries

We consider four *positions* arranged cyclically, corresponding to the four corner positions in the top layer. For definiteness, label the positions by

$$P := \{0, 1, 2, 3\},$$

with the cyclic order $0 \to 1 \to 2 \to 3 \to 0$.

Definition 2 (Permutation data)**.** Let $\mathfrak{S}(P) \cong S_4$ be the symmetric group on P. We represent a corner permutation state by a bijection

$$\tau : P \to P,$$

interpreted as: *the corner cubie labeled $x \in P$ occupies position $\tau^{-1}(x)$*, equivalently *the cubie occupying position p is $\tau(p)$*. Concretely, τ is a 4-tuple $(\tau(0), \tau(1), \tau(2), \tau(3))$ with distinct entries in $\{0, 1, 2, 3\}$.

Definition 3 (Orientation data)**.** Let $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$. An orientation state is a 4-tuple

$$t = (t_0, t_1, t_2, t_3) \in \mathbb{Z}_3^4$$

subject to the corner twist constraint

$$t_0 + t_1 + t_2 + t_3 \equiv 0 \pmod 3.$$

Definition 4 (Corner state space)**.** Define

$$X := \{\, (\tau, t) \mid \tau \in \mathfrak{S}(P),\ t \in \mathbb{Z}_3^4,\ t_0 + t_1 + t_2 + t_3 \equiv 0 \pmod 3 \,\}.$$

Proposition 1 (Cardinality of X)**.** *We have* $|X| = 4! \cdot 3^3 = 648$.

Proof. There are $4! = 24$ choices for τ. For t, choose $(t_0, t_1, t_2) \in (\mathbb{Z}/3\mathbb{Z})^3$ arbitrarily (giving 3^3 choices), and then t_3 is uniquely determined by the constraint:

$$t_3 \equiv -(t_0 + t_1 + t_2) \pmod 3.$$

Hence $|X| = 24 \cdot 27 = 648$. □

8 The symmetry group acting on X

We define two types of transformations of X:

1. position symmetries of the 4-cycle (a dihedral action on positions),
2. an involution that inverts all corner twists.

Together these generate a group of order 16.

8.1 Dihedral symmetries on positions

Let D_4 be the dihedral group of the square, realized here as the group of symmetries of the cyclically ordered set P.

Definition 5 (The group D_4)**.** Let r be the rotation $r(p) = p+1 \pmod 4$, and let s be the reflection $s(0) = 0,\ s(1) = 3,\ s(2) = 2,\ s(3) = 1$. Let

$$D_4 := \langle r, s \mid r^4 = e,\ s^2 = e,\ srs = r^{-1} \rangle.$$

We view each $g \in D_4$ as a permutation of the position set P.

Definition 6 (Action of D_4 on X)**.** For $g \in D_4$ and $(\tau, t) \in X$, define

$$g \cdot (\tau, t) := (\tau \circ g^{-1},\ t \circ g^{-1}),$$

where $(\tau \circ g^{-1})(p) = \tau(g^{-1}(p))$ and similarly $(t \circ g^{-1})(p) = t_{g^{-1}(p)}$.

Remark 5. This is the standard *relabeling of positions* action: applying g rotates or reflects the top layer positions, so the cubie and twist values attached to a position p are transported from the previous position $g^{-1}(p)$.

8.2 Twist inversion

Definition 7 (Twist inversion)**.** Define $\iota : X \to X$ by

$$\iota(\tau, t) := (\tau, -t), \qquad \text{where } (-t)_p := -t_p \in \mathbb{Z}_3.$$

Lemma 4. *ι is well-defined on X and satisfies $\iota^2 = \mathrm{id}_X$.*

Proof. If $\sum_{p \in P} t_p \equiv 0 \pmod 3$, then

$$\sum_{p \in P} (-t_p) \equiv -\sum_{p \in P} t_p \equiv 0 \pmod 3,$$

so $-t$ also satisfies the constraint and $\iota(\tau, t) \in X$. Also, $-(-t) = t$, hence $\iota^2 = \mathrm{id}_X$. $\square$

Lemma 5. *For every $g \in D_4$ we have $\iota(g \cdot x) = g \cdot \iota(x)$ for all $x \in X$.*

Proof. Let $x = (\tau, t)$. Then

$$\iota(g \cdot (\tau, t)) = \iota(\tau \circ g^{-1},\, t \circ g^{-1}) = (\tau \circ g^{-1},\, -(t \circ g^{-1})).$$

On the other hand

$$g \cdot \iota(\tau, t) = g \cdot (\tau, -t) = (\tau \circ g^{-1},\, (-t) \circ g^{-1}) = (\tau \circ g^{-1},\, -(t \circ g^{-1})).$$

Thus they agree. $\square$

8.3 The full action group

Definition 8 (The action group K)**.** Let $C_2 = \langle \iota \rangle$ be the order-2 group generated by twist inversion. By Lemma 5, the actions of D_4 and C_2 commute, so we obtain a well-defined action of the direct product

$$K := D_4 \times C_2$$

on X, given by

$$(g, \epsilon) \cdot (\tau, t) := g \cdot (\tau, \epsilon t), \quad \text{where } \epsilon \in \{+1, -1\} \text{ and } (-1)t := -t.$$

Proposition 2. *$|K| = 16$.*

Proof. $|D_4| = 8$ and $|C_2| = 2$, so $|K| = |D_4||C_2| = 16$. $\square$

9 Orbit count via Burnside's lemma

Definition 9 (Equivalence classes)**.** Define an equivalence relation on X by

$$x \sim y \iff \exists k \in K \text{ such that } y = k \cdot x.$$

The equivalence classes are precisely the K-orbits in X.

Theorem 3 (Orbit count)**.** *The action of K on X has exactly 42 orbits.*

The proof uses the classical orbit-counting tool.

Theorem 4 (Burnside's lemma)**.** *Let a finite group K act on a finite set X. Then the number of orbits is*

$$|X/K| = \frac{1}{|K|} \sum_{k \in K} |\mathrm{Fix}(k)|,$$

where $\mathrm{Fix}(k) = \{x \in X : k \cdot x = x\}$.

9.1 Fixed-point counts

We now compute $|\mathrm{Fix}(k)|$ for each $k \in K$.

Lemma 6. *For the identity element $e \in K$, we have $|\mathrm{Fix}(e)| = |X| = 648$.*

Proof. Trivial: $e \cdot x = x$ for all $x \in X$. $\square$

Lemma 7. *For the element $\iota \in K$ (i.e. $(e, -1)$ in $D_4 \times C_2$), we have $|\mathrm{Fix}(\iota)| = 24$.*

Proof. A state $(\tau, t) \in X$ is fixed by ι iff

$$(\tau, t) = \iota(\tau, t) = (\tau, -t),$$

which holds iff $t = -t$ in $(\mathbb{Z}/3\mathbb{Z})^4$. In $\mathbb{Z}/3\mathbb{Z}$, the equation $u = -u$ implies $2u \equiv 0 \pmod 3$, hence $u \equiv 0 \pmod 3$. Therefore $t = (0, 0, 0, 0)$ is the only orientation vector fixed by ι.
With t forced to be the zero vector, τ may be chosen arbitrarily in S_4, giving $4! = 24$ fixed states. $\square$

Lemma 8. *Let $g \in D_4$ be nontrivial ($g \neq e$). Then*

$$|\mathrm{Fix}((g, +1))| = 0.$$

Proof. Suppose $(\tau, t) \in X$ is fixed by $(g, +1)$, i.e. by the position relabeling action of g. By Definition 6, the condition $(g, +1) \cdot (\tau, t) = (\tau, t)$ implies in particular

$$\tau \circ g^{-1} = \tau.$$

Equivalently, $\tau(g^{-1}(p)) = \tau(p)$ for all $p \in P$.
Since $g \neq e$ as a permutation of P, there exists some $p \in P$ with $g^{-1}(p) \neq p$. Applying the equality above at this p yields

$$\tau(g^{-1}(p)) = \tau(p) \quad \text{with} \quad g^{-1}(p) \neq p.$$

But τ is a bijection $P \to P$ (Definition 2), so it cannot take the same value at two distinct inputs. This is a contradiction. Hence no such (τ, t) exists, and the fixed-point set is empty. $\square$

Lemma 9. *Let $g \in D_4$ be nontrivial ($g \neq e$). Then*

$$|\mathrm{Fix}((g, -1))| = 0.$$

Proof. If (τ, t) is fixed by $(g, -1)$, then by Definition 8 and Definition 6 we have

$$(\tau, t) = (g, -1) \cdot (\tau, t) = g \cdot (\tau, -t) = (\tau \circ g^{-1},\ (-t) \circ g^{-1}).$$

In particular, $\tau = \tau \circ g^{-1}$. The same bijectivity argument as in Lemma 8 yields a contradiction whenever $g \neq e$. Thus the fixed-point set is empty. $\square$

9.2 Conclusion of the orbit count

Proof of Theorem 3. By Proposition 2, $|K| = 16$. By Lemmas 6, 7, 8, and 9, the fixed-point counts satisfy:

$$|\mathrm{Fix}(k)| = \begin{cases} 648, & k = e, \\ 24, & k = \iota, \\ 0, & \text{for all other } k \in K. \end{cases}$$

Therefore Burnside's lemma (Theorem 4) gives

$$|X/K| = \frac{1}{16}\Big(648 + 24\Big) = \frac{672}{16} = 42.$$

Hence the action has exactly 42 orbits. $\square$

10 Interpretation (optional)

Remark 6. The equivalence relation generated by the D_4 action identifies states that differ only by relabeling of the four top-layer corner *positions* via rotations and reflections of the layer. The twist-inversion involution ι identifies states differing by reversing the direction of each corner twist simultaneously. Theorem 3 states that, under these identifications, the 648 admissible states fall into precisely 42 equivalence classes.