Step-Level Reachability of the U-Layer Corners under a Roux Block Stabilizer

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Abstract

We formalize a step-level notion of reachability for the CMLL stage in the Roux method: throughout the entire sequence, two $1 \times 2 \times 3$ blocks (left and right, spanning the D and middle layers) must remain fixed in both position and orientation. This constraint defines a subgroup H of the Rubik's Cube group. We study the induced action of H on the four U-layer corner cubies and define a natural "corner projection" homomorphism Φ onto a semidirect product $S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$. We provide a group-theoretic surjectivity criterion for $\Phi(\langle S \rangle)$ and show that, under mild explicit hypotheses on two block-preserving macro moves, the set of step-level reachable U-layer corner states has cardinality $|S_4| \cdot 3^3 = 648$.

1 Introduction

In the CMLL stage of the Roux method, it is common to distinguish two notions of reachability.

- *Endpoint-only (global) reachability:* one only requires the existence of a cube state in which the Roux blocks are solved and the U-layer corners realize a desired configuration; intermediate states are unconstrained.
- *Step-level reachability:* one starts from a state with solved Roux blocks and applies a sequence of allowed steps, each step preserving the blocks throughout, ending at the desired U-layer corner configuration.

The second notion is the one consistent with the operational meaning of a "CMLL step." This note develops a concise group-theoretic framework for step-level reachability and reduces the problem to a small set of explicit macro moves whose verification may be carried out by hand tracking or by a short computation in the standard cubie model.

2 Cubie model and the Roux block stabilizer

2.1 Cubie model: permutation and orientation data

Conventions on evaluation and indexing

We interpret move words in Singmaster notation as executed *left-to-right*: a word XY means "perform X and then Y." All induced data below use the same convention. For the four U-layer corner positions (T_0, T_1, T_2, T_3), we write $\tau_h \in S_4$ for the induced permutation, defined by

$$\tau_h(i) = j \quad \Longleftrightarrow \quad \text{the corner cubie initially at position } T_i \text{ moves to } T_j \text{ after applying } h.$$

We compose permutations in the same left-to-right order: $(\tau\sigma)(i) := \sigma(\tau(i))$. For twists, we record $t(h) = (t_0, t_1, t_2, t_3) \in (\mathbb{Z}/3\mathbb{Z})^4$ where $t_i(h)$ is the twist increment (mod 3) of the corner cubie initially at T_i. With this convention, under concatenation one has

$$t(hk) = t(h) + \tau_h \cdot t(k), \qquad (\tau \cdot x)_i := x_{\tau(i)}.$$

Remark 1 (Sanity check for conventions). Fix $h, k \in G$. For a corner cubie initially at T_i, applying h sends it to $T_{\tau_h(i)}$ and contributes twist $t_i(h)$. Applying k next contributes twist $t_{\tau_h(i)}(k)$ to that same cubie. Hence the total twist increment is $t_i(h) + t_{\tau_h(i)}(k)$, which is exactly the ith coordinate of $t(h) + \tau_h \cdot t(k)$.

We work in the standard cubie model of the 3×3 Rubik's Cube group G. Let C be the set of the 8 *corner positions* and E the set of the 12 *edge positions*. For each $g \in G$ one may associate:

- a corner-position permutation $\pi_C(g) \in \mathrm{Sym}(C)$ and a corner-orientation increment function $\omega_C(g) : C \to \mathbb{Z}/3\mathbb{Z}$;
- an edge-position permutation $\pi_E(g) \in \mathrm{Sym}(E)$ and an edge-orientation increment function $\omega_E(g) : E \to \mathbb{Z}/2\mathbb{Z}$.

Intuitively, $\pi_C(g)$ records where each corner position is sent, while $\omega_C(g)(p)$ records the twist increment (mod 3) imparted to the corner cubie occupying position p under g; similarly for edges with a flip increment mod 2.
For our purposes it suffices to use the following shorthand: a set of cubies is *fixed pointwise and orientation-wise* if both its position permutation and its orientation increment are trivial on that set.

2.2 The Roux blocks

Fix the two $1 \times 2 \times 3$ Roux blocks as the left and right blocks occupying the D-layer and the middle layer. In cubie notation, we take:

- Block corner cubies:
$$B_C := \{\mathrm{DFL}, \mathrm{DBL}, \mathrm{DFR}, \mathrm{DBR}\}.$$
- Block edge cubies:
$$B_E := \{\mathrm{DL}, \mathrm{FL}, \mathrm{BL}, \mathrm{DR}, \mathrm{FR}, \mathrm{BR}\}.$$

Write $B := (B_C, B_E)$.

2.3 Step-level subgroup

Definition 1 (Roux block stabilizer)**.** Define the *block stabilizer* subgroup

$$H := \mathrm{Stab}_G(B)$$

to be the set of all $g \in G$ that fix every cubie in $B_C \cup B_E$ *pointwise and orientation-wise*, i.e.

$$\pi_C(g)(c) = c,\ \omega_C(g)(c) = 0\ \ \forall c \in B_C, \qquad \pi_E(g)(e) = e,\ \omega_E(g)(e) = 0\ \ \forall e \in B_E.$$

Lemma 1. *H is a subgroup of G.*

Proof. The identity fixes all cubies and orientations. If $g, h \in H$, then for each specified cubie the composition gh fixes its position and adds orientation changes $0 + 0 = 0$, hence $gh \in H$. Similarly g^{-1} fixes the same cubies with inverse orientation change 0, hence $g^{-1} \in H$. $\square$

Definition 2 (Allowed steps and step-level reachability)**.** Let $S \subseteq H$ be a finite set of *allowed steps* and let $\langle S \rangle \leq H$ be the subgroup they generate. A U-layer corner configuration is *step-level reachable (from a fixed reference state)* if it is realized by the action of some $h \in \langle S \rangle$.

3 U-layer corner state space and the corner projection

3.1 U-layer corner labels and cycle notation

Let $\mathcal{T} = \{T_0, T_1, T_2, T_3\}$ be the set of U-layer corner *positions*, and fix the concrete labeling

$$(T_0, T_1, T_2, T_3) = (\mathrm{UFR}, \mathrm{UBR}, \mathrm{UBL}, \mathrm{UFL}).$$

Thus S_4 will act on $\mathcal{T}$ by permuting these four positions.

Definition 3 (k-cycles)**.** A *k-cycle* in S_4 is a permutation of the form $(i_1\, i_2\, \ldots\, i_k)$ that sends $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_k \mapsto i_1$ and fixes all other elements. In particular, a 3-cycle has the form (abc) and a 4-cycle has the form $(abcd)$.

3.2 Corner coordinates

For $h \in H$, the induced action on the four U-layer corners determines:

- a permutation $\tau_h \in S_4$ of the four positions;
- a twist vector $t(h) = (t_0, t_1, t_2, t_3) \in (\mathbb{Z}/3\mathbb{Z})^4$ (twist increments of the corner cubies initially at T_0, T_1, T_2, T_3).

In the cubie model, corner twists satisfy a global sum constraint on all eight corners. Since elements of H fix the four block corners in B_C orientation-wise, those four twists are always 0, so the constraint reduces to the four U-layer corners:

$$t_0 + t_1 + t_2 + t_3 = 0 \quad \text{in } \mathbb{Z}/3\mathbb{Z}.$$

Thus the natural twist space is

$$(\mathbb{Z}/3\mathbb{Z})^4_{\sum=0} := \{(t_0, t_1, t_2, t_3) \in (\mathbb{Z}/3\mathbb{Z})^4 : \sum_i t_i = 0\}.$$

We parameterize it by three coordinates via the bijection

$$\Theta : (\mathbb{Z}/3\mathbb{Z})^4_{\sum=0} \to (\mathbb{Z}/3\mathbb{Z})^3, \qquad \Theta(t_0, t_1, t_2, t_3) = (t_0, t_1, t_2),$$

with $t_3 = -(t_0 + t_1 + t_2)$ in $\mathbb{Z}/3\mathbb{Z}$.

3.3 The semidirect product structure

Permuting corner positions reindexes twist coordinates, so the correct target group is a semidirect product. Define an embedding

$$\iota : (\mathbb{Z}/3\mathbb{Z})^3 \to (\mathbb{Z}/3\mathbb{Z})^4_{\sum=0}, \qquad \iota(x_0, x_1, x_2) = (x_0, x_1, x_2, -x_0 - x_1 - x_2).$$

Let S_4 act on $(\mathbb{Z}/3\mathbb{Z})^4_{\sum=0}$ by permuting the four coordinates according to the labeling $(0, 1, 2, 3)$; denote this action by $\tau \cdot (\cdot)$. Explicitly, for $x = (x_0, x_1, x_2, x_3)$ we set $(\tau \cdot x)_i := x_{\tau(i)}$, consistent with the left-to-right composition convention above. Pull it back to $(\mathbb{Z}/3\mathbb{Z})^3$ by

$$\tau \star u \ := \ \Theta\big(\tau \cdot \iota(u)\big) \in (\mathbb{Z}/3\mathbb{Z})^3.$$

Define the group law on $S_4 \times (\mathbb{Z}/3\mathbb{Z})^3$ by

$$(\tau, u) \cdot (\sigma, v) \ := \ (\tau\sigma, \ u + \tau \star v).$$

This is the semidirect product group $S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$.

3.4 Corner projection

Definition 4 (Corner projection)**.** Define

$$\Phi : H \to S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3, \qquad \Phi(h) := \big(\tau_h, \ \Theta(t(h))\big).$$

Proposition 1. *Φ is a group homomorphism.*

Proof. The permutation parts compose: $\tau_{h_1 h_2} = \tau_{h_1}\tau_{h_2}$. For twists, the twist contributed by h_2 is reindexed by τ_{h_1} under composition, yielding (in four coordinates)

$$t(h_1 h_2) = t(h_1) + \tau_{h_1} \cdot t(h_2).$$

Applying Θ and using the definition of $\star$ gives

$$\Theta(t(h_1 h_2)) = \Theta(t(h_1)) + \tau_{h_1} \star \Theta(t(h_2)),$$

which matches the semidirect product law. Hence $\Phi(h_1 h_2) = \Phi(h_1) \cdot \Phi(h_2)$. □

4 A surjectivity criterion for the image

Fix $S \subseteq H$ and let

$$K := \Phi(\langle S \rangle) \ \leq \ S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3.$$

Our step-level reachability goal is to prove $K = S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$.
Let $p : S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3 \to S_4$ be the projection $p(\tau, u) = \tau$. Its kernel is

$$N := \ker(p) = \{(\mathrm{id}, u) : u \in (\mathbb{Z}/3\mathbb{Z})^3\} \ \cong \ (\mathbb{Z}/3\mathbb{Z})^3,$$

the *twist-only* subgroup.

Lemma 2 (Conjugation formula)**.** *For all $\tau \in S_4$ and $u \in (\mathbb{Z}/3\mathbb{Z})^3$,*

$$(\tau, 0)\,(\mathrm{id}, u)\,(\tau, 0)^{-1} = (\mathrm{id}, \tau \star u).$$

More generally, conjugation by (τ, a) has the same effect on N.

Proof. Using the multiplication rule $(\tau, a)(\sigma, b) = (\tau\sigma,\ a + \tau \star b)$ and the inverse $(\tau, a)^{-1} = \big(\tau^{-1},\ -\tau^{-1} \star a\big)$, we compute

$$(\tau, 0)(\mathrm{id}, u) = (\tau, \tau \star u), \qquad (\tau, \tau \star u)(\tau, 0)^{-1} = (\tau, \tau \star u)(\tau^{-1}, 0) = (\mathrm{id}, \tau \star u).$$

For the general case, the a-terms cancel:

$$(\tau, a)(\mathrm{id}, u)(\tau, a)^{-1} = (\tau, a)(\mathrm{id}, u)\big(\tau^{-1}, -\tau^{-1} \star a\big) = (\mathrm{id}, \tau \star u).$$

□

Proposition 2 (Semidirect-product criterion)**.** *Let $K \leq S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$, and let $p : S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3 \to S_4$ be the projection $p(\tau, u) = \tau$. Assume that:*

1. *$p(K) = S_4$;*
2. *K contains (id, e) for some $e \neq 0$ such that the S_4-orbit $\{\tau \star e : \tau \in S_4\}$ generates $(\mathbb{Z}/3\mathbb{Z})^3$.*

Then $K = S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$.

Proof. Let

$$N := \ker(p) = \{(\mathrm{id}, u) : u \in (\mathbb{Z}/3\mathbb{Z})^3\} \ \cong\ (\mathbb{Z}/3\mathbb{Z})^3$$

be the twist-only subgroup. By assumption (2), $(\mathrm{id}, e) \in K \cap N$. Since $p(K) = S_4$, for each $\tau \in S_4$ we may choose some lift $(\tau, a_\tau) \in K$. By Lemma 2, conjugation by (τ, a_τ) acts on N as $u \mapsto \tau \star u$, hence

$$(\mathrm{id}, \tau \star e) = (\tau, a_\tau)\,(\mathrm{id}, e)\,(\tau, a_\tau)^{-1} \ \in\ K \cap N.$$

Therefore $K \cap N$ contains the subgroup generated by $\{\tau \star e : \tau \in S_4\}$, which equals all of $(\mathbb{Z}/3\mathbb{Z})^3$ by assumption (2). Hence $K \cap N = N$.
Now let $(\tau, u) \in S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$ be arbitrary. Choose a lift $(\tau, a) \in K$ (possible since $p(K) = S_4$). Because $\star$ is induced by permuting coordinates, $\tau \star (\cdot)$ is an automorphism of $(\mathbb{Z}/3\mathbb{Z})^3$ with inverse $\tau^{-1} \star (\cdot)$. Set

$$v := \tau^{-1} \star (u - a) \in (\mathbb{Z}/3\mathbb{Z})^3.$$

Since $K \cap N = N$, we have $(\mathrm{id}, v) \in K$, and therefore

$$(\tau, a)\,(\mathrm{id}, v) = (\tau,\ a + \tau \star v) = (\tau,\ a + (u - a)) = (\tau, u) \in K.$$

Thus K is the whole semidirect product. □

Lemma 3 (Orbit generation for a difference vector)**.** *Let $e = (2, 0, 1) \in (\mathbb{Z}/3\mathbb{Z})^3$. Then the S_4-orbit of e under $\star$ generates $(\mathbb{Z}/3\mathbb{Z})^3$.*

Proof. View e in four coordinates via ι:

$$\iota(e) = (2, 0, 1, 0) \in (\mathbb{Z}/3\mathbb{Z})^4_{\sum=0}.$$

Permuting the four coordinates by elements of S_4 yields all vectors obtained by relocating the two nonzero entries. In particular, there exist $\tau_0, \tau_1, \tau_2 \in S_4$ sending the nonzero entry 1 to the fourth coordinate while keeping the other nonzero entry among the first three coordinates; explicitly, the orbit contains vectors of the form

$$(2, 0, 0, 1), \quad (0, 2, 0, 1), \quad (0, 0, 2, 1) \in (\mathbb{Z}/3\mathbb{Z})^4_{\sum=0}.$$

Applying Θ gives

$$(2, 0, 0), \quad (0, 2, 0), \quad (0, 0, 2) \in (\mathbb{Z}/3\mathbb{Z})^3.$$

Since 2 is a unit in $\mathbb{Z}/3\mathbb{Z}$, these three vectors generate the standard basis of $(\mathbb{Z}/3\mathbb{Z})^3$, hence generate all of $(\mathbb{Z}/3\mathbb{Z})^3$. Therefore the S_4-orbit of e generates $(\mathbb{Z}/3\mathbb{Z})^3$. □

5 Choice of allowed steps, explicit macro moves, and the main theorem

We adopt the following choice of allowed steps:

$$S = \{U,\ A,\ T\} \subseteq H,$$

where U is the U-face quarter turn, and A, T are block-preserving macro moves.

5.1 Move-word conventions

We use standard Singmaster notation for face turns (U, D, L, R, F, B) and their inverses $(U', D', \dots)$, with X^2 denoting a half-turn. A word such as XY denotes performing X followed by Y (left-to-right execution), consistent with the conventions fixed in Section 2. The inverse word w^{-1} is obtained by reversing the order and inverting each face turn.

5.2 Explicit macro moves

Definition 5 (Macro moves used in Scheme S)**.** Define the following words in face turns:

$$A := R^2B^2RFR'B^2RF'R, \qquad f := R'DRD'R'DR,$$

$$W := f\,U\,f^{-1}\,U', \qquad T := W^{-1}\,A\,W\,A^{-1}.$$

Lemma 4. *$U \in H$ and $\Phi(U) = (\tau_U, 0)$ where τ_U is a 4-cycle (hence an odd permutation). More precisely, with $(T_0, T_1, T_2, T_3) = (\mathrm{UFR}, \mathrm{UBR}, \mathrm{UBL}, \mathrm{UFL})$,*

$$\tau_U = (0123)\,.$$

Proof. U acts only on the U layer, and does not move or reorient any cubie in the two Roux blocks (which lie in the D and middle layers), hence $U \in H$. On the four U-layer corners it induces the 4-cycle (0123), and in the standard cubie convention the U-turn contributes zero corner twist. Therefore $\Phi(U) = (\tau_U, 0)$. □

Lemma 5. *The macro move A of Definition 5 lies in H and satisfies*

$$\Phi(A) = \big((012)\,,\,0\big).$$

Proof. This is a finite verification in the standard cubie model (see Appendix A–C for a tracking checklist and a reproducible record). One checks that A fixes each cubie in $B_C \cup B_E$ pointwise and orientation-wise, hence $A \in H$. Restricting to the four U-layer corner positions $(T_0, T_1, T_2, T_3) = (\mathrm{UFR}, \mathrm{UBR}, \mathrm{UBL}, \mathrm{UFL})$, one verifies that A induces the 3-cycle (012) and produces zero corner twist increments on all four U-corners. Thus $\Phi(A) = \big((012)\,,0\big)$. □

Lemma 6. *Let $a = (012)$ and $b = (0123)$ in S_4. Then $\langle a, b\rangle = S_4$.*

Proof. Conjugating a by powers of b yields

$$bab^{-1} = (123)\,, \qquad b^2ab^{-2} = (023)\,, \qquad b^3ab^{-3} = (013)\,.$$

Hence the subgroup $G := \langle a, b\rangle$ contains the 3-cycles (012) and (013), so the even subgroup $G \cap A_4$ acts transitively on $\{0, 1, 2, 3\}$. Indeed, the orbit of 0 under $\langle (012)\,, (013)\rangle$ contains 1 (via (013)), 2 (via (012)), and 3 (via $(013)^2$). Moreover, the stabilizer of 0 in $G \cap A_4$ contains (123) (of order 3), so by orbit–stabilizer

$$|G \cap A_4| \ \geq \ 4 \cdot 3 \ = \ 12.$$

Since $G \cap A_4 \leq A_4$ and $|A_4| = 12$, we conclude $G \cap A_4 = A_4$. Finally, b is an odd permutation, so G is not contained in A_4; therefore $G = S_4$. □

Lemma 7. *The macro move T of Definition 5 lies in H and satisfies*

$$\Phi(T) = \big(\mathrm{id},\ (2,0,1)\big).$$

Equivalently, in four twist coordinates one has $t(T) = (2,0,1,0) \in (\mathbb{Z}/3\mathbb{Z})^4_{\sum=0}$.

Proof. Again this is a finite verification in the standard cubie model (see Appendix A–C for a tracking checklist and a reproducible record). One checks that T fixes each cubie in $B_C \cup B_E$ pointwise and orientation-wise, hence $T \in H$. On the four U-layer corners it induces the identity permutation, and its corner twist increments (in the order UFR, UBR, UBL, UFL) are $(2,0,1,0)$. Applying Θ yields $\Theta(t(T)) = (2,0,1)$, hence $\Phi(T) = \big(\mathrm{id}, (2,0,1)\big)$. □

Remark 2 (On verification and independent checking). Lemmas 5 and 7 require only a finite computation of the induced action on specified cubies. They may be verified by hand tracking (e.g. via a tracking table) or by a short script implementing the standard cubie model (for example, `code/full_cubie_tracking.py` in the accompanying repository). The main argument of this paper does not rely on any computer assistance beyond this explicit verification.

Theorem 1 (Step-level reachability of U-layer corners)**.** *With $S = \{U, A, T\}$ as above,*

$$\Phi(\langle U, A, T\rangle) \ = \ S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3.$$

In particular, the number of step-level reachable U-layer corner states equals

$$|S_4| \cdot |(\mathbb{Z}/3\mathbb{Z})^3| = 24 \cdot 27 = 648.$$

Proof. Let $K := \Phi(\langle U, A, T\rangle) \leq S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$. By Lemmas 4 and 5, the projection $p(K) \leq S_4$ contains $\tau_U = (0123)$ and $\tau_A = (012)$. By Lemma 6 we have $p(K) = S_4$.
By Lemma 7, K contains (id, e) with $e = (2,0,1)$. Lemma 3 shows that the S_4-orbit $\{\tau \star e : \tau \in S_4\}$ generates $(\mathbb{Z}/3\mathbb{Z})^3$. Therefore Proposition 2 implies $K = S_4 \ltimes (\mathbb{Z}/3\mathbb{Z})^3$. □

6 Appendix (optional): Verification record

A. Tracking checklist

To verify $A, T \in H$ and compute $\Phi(A), \Phi(T)$, it suffices to record:

- **Block fixation:** for each $c \in B_C$ and $e \in B_E$, confirm $\pi_C(\cdot)(c) = c$, $\omega_C(\cdot)(c) = 0$, and $\pi_E(\cdot)(e) = e$, $\omega_E(\cdot)(e) = 0$.
- **Top-corner permutation:** track the images of T_0, T_1, T_2, T_3.
- **Top-corner twists:** track twist increments of the four U-corners and convert to three coordinates via Θ.

B. Example computed outcomes (one-line record)

For the words in Definition 5, the induced actions on U-layer corners are:

$$\Phi(A) = \big((012)\,,0\big), \qquad \Phi(U) = \big((0123)\,,0\big), \qquad \Phi(W) = \big(\mathrm{id}, (1,0,0)\big), \qquad \Phi(T) = \big(\mathrm{id}, (2,0,1)\big),$$

and A, U, W, T preserve $B_C \cup B_E$ pointwise and orientation-wise.

C. Reproducible verification (script)

As discussed in Remark 2, we provide a reference implementation in the repository (file `code/full_cubie_tracking`

Running the following command from the repository root reproduces the one-line outcomes recorded in Appendix B:

```
python3 code/full_cubie_tracking.py
```

The SHA-256 hash of `code/full_cubie_tracking.py` at the time of writing is:

```
c4757602ce0c19435443462df1a7ab1896c38a2d203eaab780f8ae6d5400f702
```

Note. The script is provided purely as a convenience for checking these explicit finite computations; the group-theoretic proof of Theorem 1 does not rely on any computer assistance beyond this verification.

Acknowledgement. This note emphasizes the structural separation between (i) a group-theoretic surjectivity argument and (ii) the explicit construction/verification of a small set of macro moves.