Math 371 Review Sheet Solutions for Midterm Exam Winter 2021

1. True or False? Give a reason to justify your answer.

a) TRUE $(10101.01)_2 = 2^4 + 2^2 + 1 + \frac{1}{4} = 16 + 4 + 1 + 0.25 = (21.25)_{10}$

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b) TRUE

$D_+D_-f(x) = D_+\left(D_-f(x)\right) = D_+\left(\frac{f(x)-f(x-h)}{h}\right) = \frac{1}{h}\left(D_+f(x) - D_+f(x-h)\right)$

$= \frac{1}{h}\left(\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}\right) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$

$D_-D_+f(x) = D_-\left(D_+f(x)\right) = D_-\left(\frac{f(x+h)-f(x)}{h}\right) = \frac{1}{h}\left(D_-f(x+h) - D_-f(x)\right)$

$= \frac{1}{h}\left(\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}\right) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ hence the 2 expressions are equal

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c) FALSE

$D_+D_+f(x) = D_+\left(D_+f(x)\right) = D_+\left(\frac{f(x+h)-f(x)}{h}\right) = \frac{1}{h}\left(D_+f(x+h) - D_+f(x)\right)$

$= \frac{1}{h}\left(\frac{f(x+2h)-f(x+h)}{h} - \frac{f(x+h)-f(x)}{h}\right) = \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$

$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + O(h^3)$

$f(x+2h) = f(x) + f'(x)(2h) + \frac{1}{2}f''(x)(2h)^2 + O((2h)^3) = f(x) + 2f'(x)h + 2f''(x)h^2 + O(h^3)$

$f(x+2h) - 2f(x+h) + f(x) = f''(x)h^2 + O(h^3) \Rightarrow D_+D_+f(x) = f''(x) + O(h)$

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d) FALSE

The correct statement is "When the derivative $f'(x)$ is approximated by the forward difference approximation $D_+f(x)$ with step size h in finite precision arithmetic, for large h the TRUNCATION error dominates the ROUNDOFF error, but for small h the ROUNDOFF error dominates the TRUNCATION error."

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e) TRUE

$D_0f(x) = \frac{f(x+h)-f(x-h)}{2h}$

$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f^{(4)}(x)h^4 + O(h^5)$

$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f^{(4)}(x)h^4 + O(h^5)$

$D_0f(x) = f'(x) + \frac{1}{6}f'''(x)h^2 + O(h^4)$ hence the discretization error is $O(h^2)$

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f) FALSE The approximation is exact if and only if the discretization error is zero. Hence if $f(x)$ is a polynomial of degree less than or equal to three, then $f(x) = ax^3 + bx^2 + cx + d$ and $f'''(x) = 6a \neq 0$ in general, so the discretization error is nonzero in general and the approximation is not exact in that case. However, this also shows that the approximation is exact if $f(x)$ is a polynomial of degree less than or equal to two.

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g) TRUE The statement as written is true because a theorem in class stated that fixed-point iteration converges whenever x_0 is sufficiently close to the root r and $|g'(r)| < 1$.

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h) FALSE If A is invertible, then $Ax = 0$ has the unique solution $x = 0$.

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i) FALSE In solving a linear system of equations with three equations and three unknowns by Gaussian elimination, in step 1 variable x_1 is eliminated from equations 2 and 3.

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j) FALSE The operation count for Gaussian elimination is $O(n^3)$. Hence if the dimension n is doubled, then the operation count is increased by approximately a factor of eight.

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k) FALSE Pivoting is recommended for stability, not to reduce the operation count.

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l) TRUE $||Ax|| \leq ||A|| \cdot ||x||$ - this is one of the properties satisfied by the matrix norm

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m) FALSE Gaussian elimination is an unstable method for solving for solving $Ax = b$ because it can replace an WELL-conditioned matrix A by an ILL-conditioned matrix U.

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n) FALSE In solving a linear system $Ax = b$ by a numerical method, if the residual is small, then the error is not guaranteed to be small and we saw an example of this in class.

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o) TRUE A property of a 2nd order acurate method is that if the mesh size h is reduced by one half, then the norm of the error is reduced by approximately one fourth.

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p) TRUE This was proven in class.

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2. Matlab gives $\sqrt{5} = 2.236067977499790$. Express $\sqrt{5}$ in normalized floating point form, $\pm(0.d_1 \dots d_n)_\beta \cdot \beta^e$, with $d_1 \neq 0$, on a computer with $\beta = 2, n = 4, M = 3$ and then express the result in decimal form.

SOLUTION

$\sqrt{5} = 2.236067977499790 = 2 + 0.125 + \cdots = 1 \cdot 2^1 + 0 \cdot 2^0 + 0 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} + \cdots$

$\text{fl}(\sqrt{5}) = (0.1000)_2 \cdot 2^2 = 2$

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3. Let $f(x) = \sqrt{1+x} - \sqrt{1-x}$ and $g(x) = 2x/(\sqrt{1+x} + \sqrt{1-x}\,)$. Show that $f(x) = g(x)$ for all x such that $|x| \leq 1$. If you are using finite precision arithmetic, which expression is better to use when $x \approx 0$? Explain.

SOLUTION

$$f(x) = \sqrt{1+x} - \sqrt{1-x} = \left(\sqrt{1+x} - \sqrt{1-x}\,\right) \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+1} + \sqrt{1-x}} = \frac{(1+x) - (1-x)}{\sqrt{1+x} + \sqrt{1-x}}$$
$$= \frac{2x}{\sqrt{1+x} + \sqrt{1-x}} = g(x)$$

The reason for requiring $|x| \leq 1$ is to ensure that $\sqrt{1+x}$ and $\sqrt{1-x}$ are real numbers.
It is better to use $g(x)$ when $x \approx 0$ to avoid cancellation of digits in $f(x)$ when $\sqrt{1+x} \approx \sqrt{1-x}$.

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4. Consider the finite-difference approximation $f'(x) \approx \dfrac{af(x+h) + bf(x) + cf(x-h)}{h}$, where a, b, c are constants. The forward approximation D_+f has $(a, b, c) = (1, -1, 0)$ and is 1st order accurate. The central approximation $D_0 f$ has $(a, b, c) = (\frac{1}{2}, 0, -\frac{1}{2})$ and is 2nd order accurate. Are there any values of (a, b, c) that yield 3rd order accuracy?

SOLUTION

3rd order accuracy means $\dfrac{af(x+h) + bf(x) + cf(x-h)}{h} = f'(x) + O(h^3)$

$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + O(h^4)$

$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + O(h^4)$

$\dfrac{af(x+h) + bf(x) + cf(x-h)}{h} =$

$$= \frac{(a+b+c)f(x) + (a-c)f'(x)h + (a+c)\frac{1}{2}f''(x)h^2 + (a-c)f'''(x)\frac{1}{6}h^3 + O(h^4)}{h}$$

$$= f'(x) + O(h^3) \;\Rightarrow\; a+b+c=0\,,\; a-c=1\,,\; a+c=0\,,\; a-c=0$$

We see that these equations cannot be satisfied and hence there are no values of (a, b, c) that yield 3rd order accuracy.

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5. Below is the algorithm for the bisection method. Find and correct any errors.

   <u>bisection method</u> (assume $f(a)\cdot f(b) < 0$)
   1. $n = 0$, $a_0 = a$, $b_0 = b$
   2. $x_n = \frac{1}{2}(a_n - b_n)$
   3. if $f(x_n)\cdot f(a_n) < 0$, then $a_{n+1} = a_n$, $b_{n+1} = x_n$
   4. else $a_{n+1} = x_n$, $b_{n+1} = b_n$
   5. set $n = n+1$ and go to line 1

SOLUTION

There are two bugs; line 2 should be $x_n = \frac{1}{2}(a_n + b_n)$, line 5 should say "go to line 2".

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6. Consider solving $f(x) = 0$. (a) State one advantage of Newton's method over the bisection method; (b) State one advantage of the bisection method over Newton's method.

SOLUTION

a) Newton's method converges quadratically, while the bisection method converges linearly.

b) The bisection method requires evaluation of $f(x)$ in each step and convergence is guaranteed when $f(a)\cdot f(b) < 0$. On the other hand, Newton's method requires evaluation of $f(x)$ and $f'(x)$ in each step and the initial guess x_0 must be sufficiently close to the root in order for the method to converge. Hence the bisection method is less costly and less sensitive to the initial guess than Newton's method.

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7. Show that $f(x) = x^2 - 3x + 2 = 0$ is equivalent to $x = g(x) = \frac{1}{3}x^2 + \frac{2}{3}$. Suppose fixed-point iteration $x_{n+1} = g(x_n)$ is applied with initial guess $x_0 = 0$. Find $\lim_{n\to\infty} x_n$. Justify your answer.

SOLUTION : A fixed point of $g(x)$ is defined by $x = g(x)$.

$x = \frac{1}{3}x^2 + \frac{2}{3} \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) = 0 \Rightarrow x = 1\,,\, 2$: two fixed points

$g'(x) = \frac{2}{3}x \Rightarrow g'(1) = \frac{2}{3}\,,\; g'(\frac{4}{3}) \Rightarrow |g'(1)| < 1\,,\; |g'(2)| > 1$

The iteration converges to $x = 1$.

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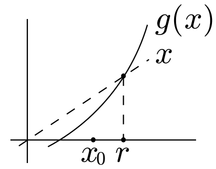
8. Consider fixed-point iteration $x_{n+1} = g(x_n)$. The figure shows the function $y = g(x)$, the line $y = x$, the fixed point r, and the initial guess x_0. Does the sequence x_n converge to r in this case? Explain.



SOLUTION

The sequence x_n diverges in this case because it is evident from the figure that $|g'(r)| > 1$.

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9. The screened Coulomb potential is defined by $\phi(x) = \frac{e^{-\kappa x}}{4\pi\epsilon x}$, where x is the distance from a charged particle to a point in space, ϵ is the dielectric constant, and κ controls the screening effect. Let $\epsilon = 2, \kappa = \frac{1}{2}$. Apply Newton's method to find the value of x for which $\phi(x) = 0.005$. Let $x_0 = 2$ be the starting value and take two steps, x_1, x_2. How many digits in x_1 are correct?

SOLUTION

$\phi(x) = \dfrac{e^{-\kappa x}}{4\pi\epsilon x} = 0.005 \Rightarrow f(x) = \dfrac{e^{-\kappa x}}{4\pi\epsilon x} - 0.005 \Rightarrow f'(x) = \dfrac{1}{4\pi\epsilon}\dfrac{x\cdot(-\kappa)e^{-\kappa x} - e^{-\kappa x}}{x^2}$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \Rightarrow$ after some algebra $\Rightarrow x_{n+1} = x_n + \dfrac{x_n - 0.005\cdot 4\pi\epsilon x_n^2 e^{\kappa x_n}}{1+\kappa x_n}$

n	x_n	
0	2	1 correct digit
1	2.3168 2126 2186 115	2 correct digits
2	2.3949 1766 0163 369	3 correct digits
3	2.3985 4982 1389 847	5 correct digits
4	2.3985 5714 4463 364	11 correct digits
5	2.3985 5714 4493 033	

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10. Consider the nonlinear system, $f(x,y) = (x-1)^2 + y^2 - 4 = 0, g(x,y) = xy - 1 = 0$, the solution of which is the intersection of a circle and a hyperbola. Find an approximate solution using Newton's method for systems. Take one step starting from $(x_0, y_0) = (3, 0)$.

SOLUTION

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\Bigg|_{(x_n,y_n)} \cdot \begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix} = \begin{pmatrix} -f(x_n,y_n) \\ -g(x_n,y_n) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2(x-1) & 2y \\ y & x \end{pmatrix}\Bigg|_{(3,0)} \cdot \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow 4(x_1 - x_0) = 0 \Rightarrow x_1 = x_0 = 3\,,\ 3(y_1 - y_0) = 1 \Rightarrow y_1 = \frac{1}{3} + y_0 = \frac{1}{3} \Rightarrow (x_1, y_1) = (3, \frac{1}{3})$

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11. Solve $2x_1 - x_2 + x_3 = -1, 4x_1 + 2x_2 + x_3 = 4, 6x_1 - 4x_2 + 2x_3 = -2$ by Gaussian elimination.

SOLUTION

$$\left(\begin{array}{rrr|r} 2 & -1 & 1 & -1 \\ 4 & 2 & 1 & 4 \\ 6 & -4 & 2 & -2 \end{array}\right) \to \left(\begin{array}{rrr|r} 2 & -1 & 1 & -1 \\ 0 & 4 & -1 & 6 \\ 0 & -1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 2 & -1 & 1 & -1 \\ 0 & 4 & -1 & 6 \\ 0 & 0 & -\frac{5}{4} & \frac{5}{2} \end{array}\right) \Rightarrow \begin{cases} x_1 = \frac{-1-((-1)\cdot 1 + 1\cdot(-2))}{2} = 1 \\ x_2 = \frac{6-(-1)(-2)}{4} = 1 \\ x_3 = \frac{\frac{5}{2}}{-\frac{5}{4}} = -2 \end{cases}$$

$m_{21} = \frac{4}{2} = 2 \qquad m_{32} = \frac{-1}{4} = -\frac{1}{4}$

$m_{31} = \frac{6}{2} = 3$

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12. Solve $Ax = b$ by Gaussian elimination with partial pivoting.

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 0 & 4 & -15 \\ 10 & 0 & 15 \\ 1 & -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} -12 \\ 100 \\ 0 \end{pmatrix}$

SOLUTION

part (a)

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{cases} x_1 = \frac{1-(1\cdot(-1)+1\cdot 1)}{1} = 1 \\ x_2 = \frac{0-1\cdot 1}{1} = -1 \\ x_3 = \frac{1}{1} = 1 \end{cases}$$

$m_{21} = \frac{1}{1} = 1 \qquad a_{22}^{(2)} = 0 \Rightarrow$ pivot

$m_{31} = \frac{1}{1} = 1$

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part (b)

$$\begin{pmatrix} 0 & 4 & -15 & \vdots & -12 \\ 10 & 0 & 15 & \vdots & 100 \\ 1 & -1 & -1 & \vdots & 0 \end{pmatrix} \to \begin{pmatrix} 10 & 0 & 15 & \vdots & 100 \\ 0 & 4 & -15 & \vdots & -12 \\ 1 & -1 & -1 & \vdots & 0 \end{pmatrix} \to \begin{pmatrix} 10 & 0 & 15 & \vdots & 100 \\ 0 & 4 & -15 & \vdots & -12 \\ 0 & -1 & -\frac{5}{2} & \vdots & -10 \end{pmatrix} \to$$

$a_{11}^{(1)} = 0 \Rightarrow$ pivot $\qquad m_{21} = \frac{0}{10} = 0 \qquad m_{32} = \frac{-1}{4} = -\frac{1}{4}$

$m_{31} = \frac{1}{10} = \frac{1}{10}$

$$\begin{pmatrix} 10 & 0 & 15 & \vdots & 100 \\ 0 & 4 & -15 & \vdots & -12 \\ 0 & 0 & -\frac{25}{4} & \vdots & -13 \end{pmatrix} \Rightarrow \begin{cases} x_1 = \frac{100-(0\cdot\frac{24}{5}+15\cdot\frac{52}{25})}{10} = \frac{172}{25} = 6.88 \\ x_2 = \frac{-12-(-15)\cdot\frac{52}{25}}{4} = \frac{24}{5} = 4.8 \\ x_3 = \frac{-13}{\frac{-25}{4}} = \frac{52}{25} = 2.08 \end{cases}$$

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13. Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find a vector x such that $\dfrac{||Ax||}{||x||} = ||A||$.

SOLUTION

$||A|| = \max\{2+1, 1+2+1, 1+2\} = 4$, $Ax = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix}$

It is convenient to choose x such that $||x|| = 1$, e.g. if we choose $x = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$, then we have $Ax = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -3 \end{pmatrix}$, so $||Ax|| = 4$, and $\dfrac{||Ax||}{||x||} = 4 = ||A||$ as required.

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14. Fill in the blanks. In solving a linear system $Ax = b$, the ________________ of the matrix A controls the relative error in the solution x due to _____________ in the right hand side b.

SOLUTION

In solving a linear system $Ax = b$, the <u>condition number</u> of the matrix A controls the relative error in the solution x due to <u>changes</u> in the right hand side b.

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15. Suppose $Ax = b$ and $A\tilde{x} = \tilde{b}$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $||b - \tilde{b}|| \le 10^{-2}$. Find the maximum value that $||x - \tilde{x}||$ can attain.

SOLUTION

In class we showed that $\dfrac{||x - \tilde{x}||}{||x||} \le \kappa(A) \dfrac{||b - \tilde{b}||}{||b||}$, where $\kappa(A) = ||A|| \cdot ||A^{-1}||$.

The exact solution is $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $||x|| = 1$. Also, $||b|| = 1$.

We have $||A|| = 3, A^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, so $||A^{-1}|| = 1, \kappa(A) = 3$.

Then $||x - \tilde{x}|| \le ||x|| \cdot \kappa(A) \dfrac{||b - \tilde{b}||}{||b||} = 1 \cdot 3 \cdot \dfrac{10^{-2}}{1} = 0.03.$

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16. Consider the linear system $2x_1 - x_2 = 1, -x_1 + 2x_2 - x_3 = 0, -x_2 + 2x_3 = 1$, with solution $x_1 = x_2 = x_3 = 1$. a) Write Jacobi's method in component form. Take two steps starting from the zero vector. Compute the error norms $||e_k||, k = 0 : 2$. b) Repeat for Gauss-Seidel.

SOLUTION

Note that the exact solution is $x_1 = x_2 = x_3 = 1$.

a) Jacobi's method

$$\begin{aligned} 2x_1 - x_2 \qquad &= 1 \Rightarrow 2x_1^{(k+1)} = 1 + x_2^{(k)} \\ -x_1 + 2x_2 - x_3 &= 0 \Rightarrow 2x_2^{(k+1)} = x_1^{(k)} + x_3^{(k)} \\ -x_2 + 2x_3 &= 1 \Rightarrow 2x_3^{(k+1)} = 1 + x_2^{(k)} \end{aligned}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\|\|e_k\|\|$
0	0	0	0	1
1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

b) Gauss-Seidel method

$$\begin{aligned} 2x_1 - x_2 \qquad &= 1 \Rightarrow 2x_1^{(k+1)} = 1 + x_2^{(k)} \\ -x_1 + 2x_2 - x_3 &= 0 \Rightarrow 2x_2^{(k+1)} = x_1^{(k+1)} + x_3^{(k)} \\ -x_2 + 2x_3 &= 1 \Rightarrow 2x_3^{(k+1)} = 1 + x_2^{(k+1)} \end{aligned}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\|\|e_k\|\|$
0	0	0	0	1
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{3}{4}$
2	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{13}{16}$	$\frac{3}{8}$

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17. Let $A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. a) For which of these does Jacobi's method converge? b) For which of these does Gauss-Seidel converge?

SOLUTION

a) Jacobi's method

$A_1 \,:\, B_J = -D^{-1}(L+U) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \Rightarrow ||B_J|| = \frac{1}{2}$

$A_2 \,:\, B_J = -D^{-1}(L+U) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \Rightarrow ||B_J|| = 2$

Jacobi's method converges for A_1.

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b) Gauss-Seidel

$A_1 \,:\, B_{GS} = -(D+L)^{-1}U = -\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \Rightarrow$
$||B_{GS}|| = \frac{1}{2}$

$A_2 \,:\, B_{GS} = -(D+L)^{-1}U = -\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 4 \end{pmatrix} \Rightarrow$
$||B_{GS}|| = 4$

Gauss-Seidel converges for A_1.
