23
Tue
4/13

chapter 7 : time-dependent differential equations

ordinary differential equations

Let $x(t)$ be the position of a particle moving on the $x$-axis at time $t$.

1st order ODE

$\dfrac{dx}{dt} = f(x)$ : velocity is a function of position

$x(0)$ : initial position

The problem is to find the position $x(t)$ for $t > 0$.

ex

1. $\dfrac{dx}{dt} = x \ , \ x(0) = 1 \ \Rightarrow \ x(t) = e^t$

2. $\dfrac{dx}{dt} = x^2 \ , \ x(0) = 1 \ \Rightarrow \ x(t) = \dfrac{1}{1-t}$

3. $\dfrac{dx}{dt} = \sin x \ , \ x(0) = 1 \ \Rightarrow \ x(t) = \ ?$

The simplest numerical method is Euler's method.

choose $\Delta t$ : time step

define $w_n$ : numerical solution at time $t_n = n\Delta t$

$$\frac{w_{n+1} - w_n}{\Delta t} = f(w_n) \ \Rightarrow \ w_{n+1} = w_n + \Delta t f(w_n)$$

given $w_0$, compute $w_1, w_2, \ldots$

questions : accuracy , stability , efficiency

2nd order ODE

$\dfrac{d^2x}{dt^2} = f(x)$ : acceleration is a function of position (Newton's equation)

$x(0)$ , $x'(0)$ : initial position , velocity

$$\frac{w_{n+1} - 2w_n + w_{n-1}}{(\Delta t)^2} = f(w_n) \ \Rightarrow \ w_{n+1} = 2w_n - w_{n-1} + (\Delta t)^2 f(w_n)$$

given $w_0$ and $w_1$, compute $w_2, w_3, \ldots$

<u>partial differential equations</u>

<u>ex</u> : heat equation

$u(x,t)$ : temperature of a metal rod at position $x$ and time $t$

$\dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2}$ , $\kappa$ : coefficient of thermal expansion

initial condition : $u(x,0) = f(x)$ , boundary conditions : $u(0,t) = u(1,t) = 0$

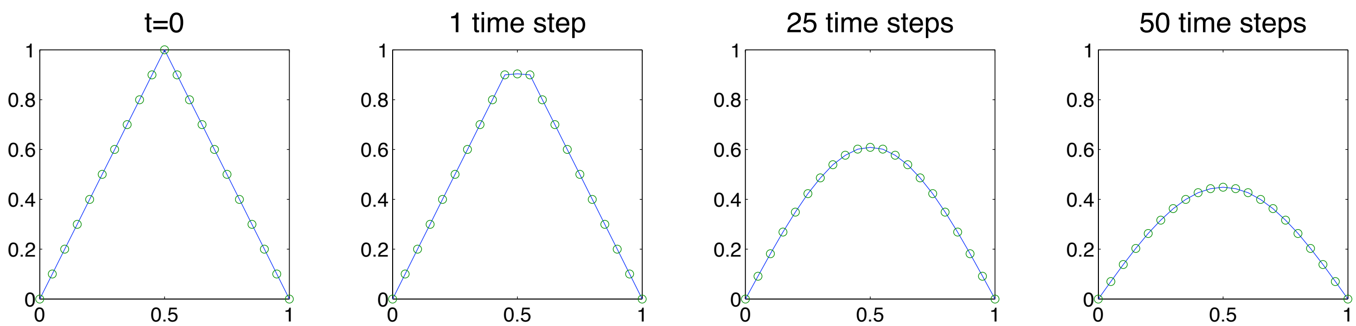
The simplest numerical method is a <u>finite-difference scheme</u>.

choose $\Delta x$ : space step , $\Delta t$ : time step

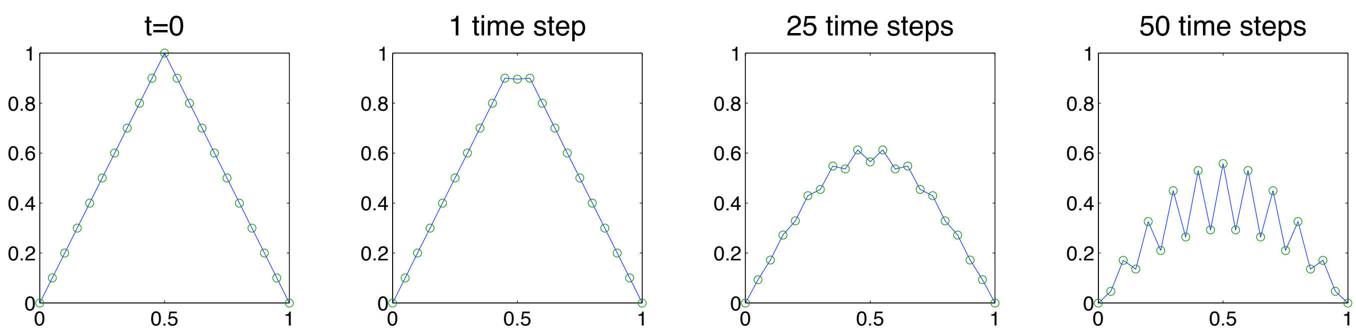
define $w_j^n$ : numerical solution at position $x_j = j\Delta x$ and time $t_n = n\Delta t$

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} = \kappa \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{(\Delta x)^2} \Rightarrow w_j^{n+1} = w_j^n + \frac{\kappa \Delta t}{(\Delta x)^2}(w_{j+1}^n - 2w_j^n + w_{j-1}^n)$$

<u>case 1</u> : $\kappa = 1$ , $\Delta x = 0.05$ , $\Delta t = 0.0012$




<u>case 2</u> : $\kappa = 1$ , $\Delta x = 0.05$ , $\Delta t = 0.0013$




explanation : the method is stable $\Leftrightarrow \dfrac{\kappa \Delta t}{(\Delta x)^2} \le \dfrac{1}{2}$

24
Thu
4/15

<u>Lax equivalence theorem</u> (Peter Lax)

Given a <u>well-posed</u> initial/boundary value problem and a <u>consistent</u> finite-difference scheme, <u>stability</u> is necessary and sufficient for <u>convergence</u>.

Fourier analysis

differential equation

$u(x,t) = e^{\omega t + ikx}$ : Fourier mode , $\omega$ : growth rate , $k$ : wavenumber

$\Rightarrow \omega = -k^2 \Rightarrow u(x,t) = e^{-k^2 t + ikx}$

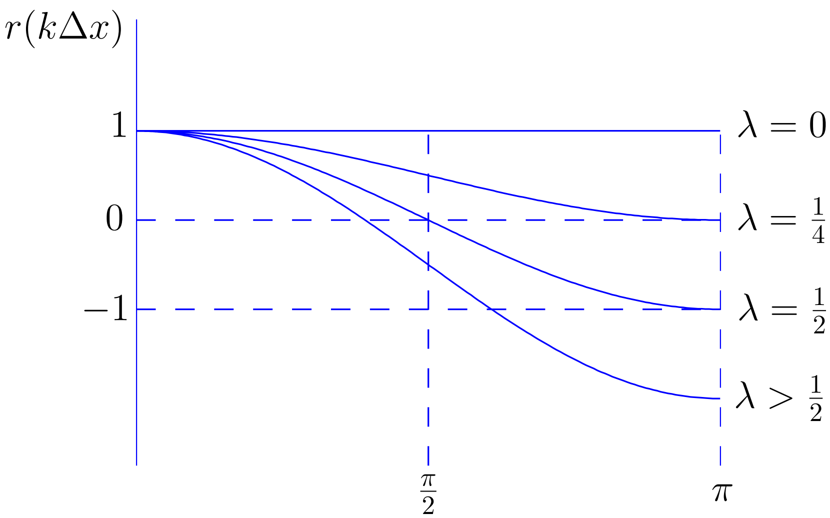
$\Rightarrow$ all modes with $k \neq 0$ decay monotonically in time

finite-difference scheme

$w_j^n = r^n e^{ikj\Delta x}$ : discrete Fourier mode , $r$ : amplification factor

$$r^{n+1} e^{ikj\Delta x} = r^n e^{ikj\Delta x} + \lambda \left( r^n e^{ik(j+1)\Delta x} - 2r^n e^{ikj\Delta x} + r^n e^{ik(j-1)\Delta x} \right) \ , \ \lambda = \frac{\kappa \Delta t}{(\Delta x)^2}$$

$$\Rightarrow r = 1 + \lambda \left( e^{ik\Delta x} - 2 + e^{-ik\Delta x} \right) = 1 + 2\lambda(\cos k\Delta x - 1) = 1 - 4\lambda \sin^2 \tfrac{1}{2} k\Delta x$$




note

1. $|r(k\Delta x)| \leq 1$ for all $k\Delta x \Leftrightarrow \lambda \leq \frac{1}{2}$

2. $|w_j^n| = |r|^n$

$0 \leq \lambda \leq \frac{1}{4}$ : all modes decay monotonically in time

$\frac{1}{4} \leq \lambda \leq \frac{1}{2}$ : $\begin{cases} \text{long waves decay monotonically in time} \\ \text{short waves oscillate in sign, amplitude decays} \end{cases}$

$\lambda > \frac{1}{2}$ : $\begin{cases} \text{long waves decay monotonically in time} \\ \text{intermediate waves oscillate in sign, amplitude decays} \\ \text{short waves oscillate in sign, amplitude grows} \end{cases}$

This is in sharp contrast with the PDE.

<u>ex</u> : wave equation

$u(x,t)$ : displacement of a string at position $x$ and time $t$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$c$ : wave speed

$$\frac{w_j^{n+1} - 2w_j^n + w_j^{n-1}}{(\Delta t)^2} = c^2 \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{(\Delta x)^2}$$

$$\Rightarrow \; w_j^{n+1} = 2w_j^n - w_j^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 (w_{j+1}^n - 2w_j^n + w_{j-1}^n) \quad , \quad \text{stable} \Leftrightarrow \left|\frac{c\Delta t}{\Delta x}\right| \leq 1$$