chapter 5 : polynomial approximation and interpolation

5.1 introduction

problem : Given a function $f(x)$, find a polynomial approximation $p_n(x)$.

application : $\int_a^b f(x)dx \rightarrow \int_a^b p_n(x)dx$, ...

one solution : The Taylor polynomial of degree n about a point $x = a$ is

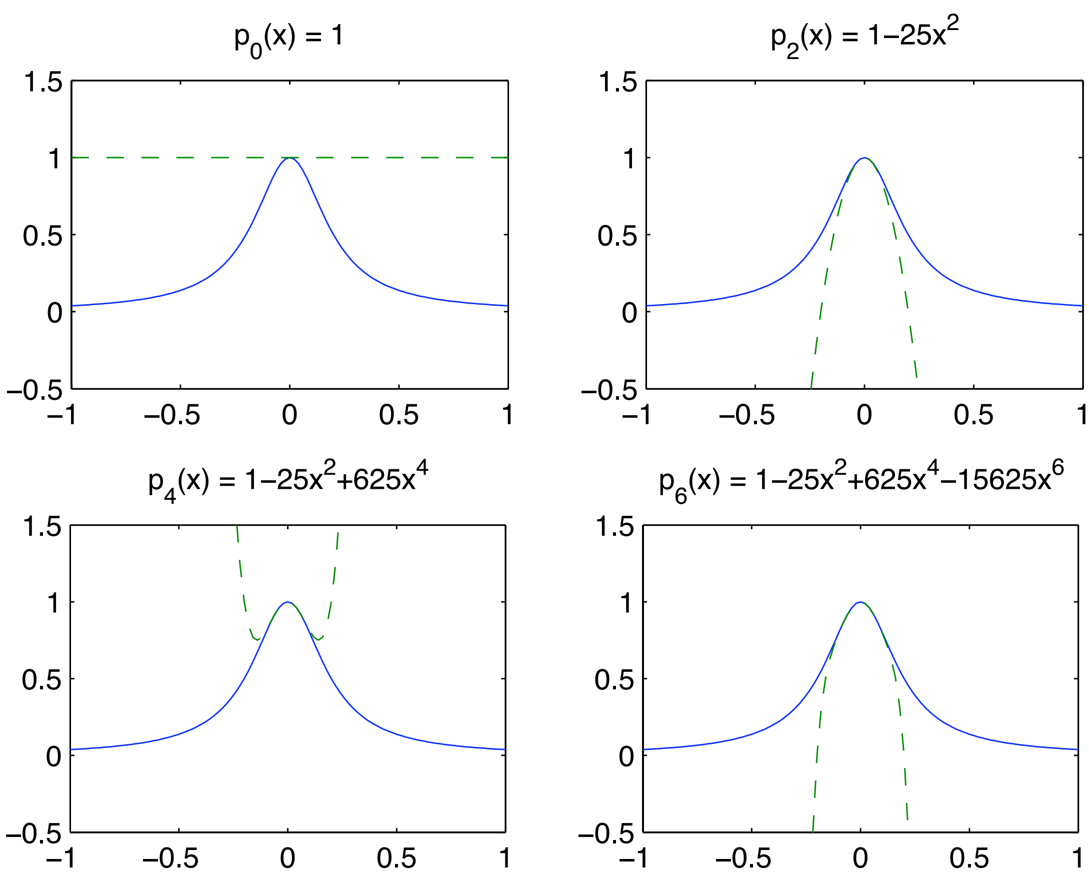
$$p_n(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \cdots + \tfrac{1}{n!}f^{(n)}(a)(x-a)^n.$$

ex : $f(x) = \dfrac{1}{1+25x^2}$, $a = 0$, $p_n(x) = ?$

In this case we can find $p_n(x)$ without computing $f(a), f'(a), \ldots, f^{(n)}(a)$.

recall the geometric series : $\dfrac{1}{1-r} = 1 + r + r^2 + \cdots$, converges for $|r| < 1$

$$\frac{1}{1+25x^2} = \frac{1}{1-(-25x^2)} = 1 + (-25x^2) + (-25x^2)^2 + \cdots \text{, converges for } |x| \leq \tfrac{1}{5}$$



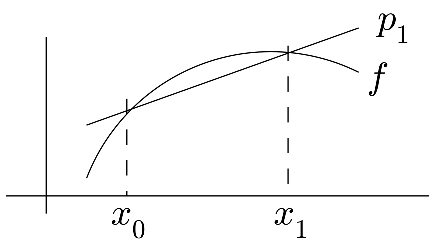
The Taylor polynomial $p_n(x)$ is a good approximation to $f(x)$ when x is close to a, but in general we need to consider other methods of approximation.

5.2 polynomial interpolation

thm: Assume $f(x)$ is given and let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points. Then there exists a unique polynomial $p_n(x)$ of degree $\leq n$ which interpolates $f(x)$ at the given points, i.e. such that $p_n(x_i) = f(x_i)$ for $i = 0 : n$.

pf : omit

ex : $n = 1 \Rightarrow x_0 , x_1$



$$p_1(x) = f(x_0) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_0)$$

check : $\begin{cases} \deg p_1 \leq 1 , \\ p_1(x_0) = f(x_0) , \ p_1(x_1) = f(x_1) \end{cases}$ ok

questions

1. What is the form of $p_n(x)$ for $n \geq 2$?
2. What is the best choice of the interpolation points $x_0, \ldots, x_n$?

note : The interpolating polynomial $p_n(x)$ can be written in different forms.

standard form

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Newton's form

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1})$$

note

1. The example above with $n = 1$ used Newton's form for $p_1(x)$.
2. The coefficients in each form are different; how can they be computed?

thm: The coefficients in Newton's form of $p_n(x)$ can be computed as follows.

$a_0 = f(x_0) = f[x_0]$

$$a_1 = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1]$$: 1st divided difference

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]$$: 2nd divided difference

...

$$a_n = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} = f[x_0, \ldots, x_n]$$: nth divided difference

<u>pf</u> : skip

$n = 0$: ok because $p_0(x) = a_0$, $p_0(x_0) = f(x_0) \Rightarrow a_0 = f(x_0)$

$n = 1$: ok because $p_1(x) = a_0 + a_1(x - x_0)$ and we showed that $a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$n = 2$: need to work

$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

define $g(x) = \left(\dfrac{x - x_0}{x_2 - x_0}\right) q_1(x) + \left(\dfrac{x_2 - x}{x_2 - x_0}\right) p_1(x)$

where

$p_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$: interpolates $f(x)$ at x_0 , x_1

$q_1(x) = f[x_1] + f[x_1, x_2](x - x_1)$: interpolates $f(x)$ at x_1 , x_2

then $g(x)$ has the following properties

$\deg g \le 2$

$g(x_0) = p_1(x_0) = f(x_0)$

$g(x_1) = \left(\dfrac{x_1 - x_0}{x_2 - x_0}\right) q_1(x_1) + \left(\dfrac{x_2 - x_1}{x_2 - x_0}\right) p_1(x_1) = \cdots = f(x_1)$

$g(x_2) = q_1(x_2) = f(x_2)$

then $g(x) = p_2(x)$ for all x (by uniqueness theorem on polynomial interpolation)

note : the coefficient of x^2 in $g(x)$ is $\dfrac{f[x_1, x_2]}{x_2 - x_0} - \dfrac{f[x_0, x_1]}{x_2 - x_0}$ 18 Thu 3/25

the coefficient of x^2 in $p_2(x)$ is a_2

$\Rightarrow a_2 = f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2]}{x_2 - x_0} - \dfrac{f[x_0, x_1]}{x_2 - x_0}$ as required

$n \ge 3$: follows the same way <u>ok</u>

ex : $n = 2 \;\Rightarrow\; x_0, x_1, x_2$

$$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

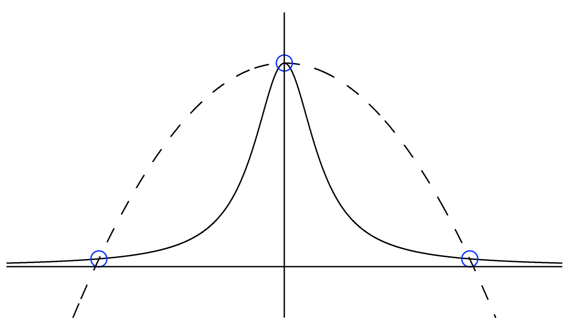
divided difference table

$$\begin{array}{lllll}
x_0 \;—\; f[x_0]^* & & & & \\
 & \diagdown & & & \\
 & & f[x_0,x_1]^* & & \\
 & \diagup & & \diagdown & \\
x_1 \;—\; f[x_1] & & & & f[x_0,x_1,x_2]^* \\
 & \diagdown & & \diagup & \\
 & & f[x_1,x_2] & & \\
 & \diagup & & & \\
x_2 \;—\; f[x_2] & & & &
\end{array}$$

The starred values are the coefficients in Newton's form of $p_2(x)$.

ex

$$f(x) = \frac{1}{1 + 25x^2} \;,\; x_0 = -1 \;,\; x_1 = 0 \;,\; x_2 = 1 \;\Rightarrow\; p_2(x) = \,?$$



$$\begin{array}{ccccccc}
x_i & & f(x_i) & & & & \\
-1 & — & \frac{1}{26} & & & & \\
 & & & \diagdown & & & \\
 & & & & \frac{25}{26} & & \\
 & & & \diagup & & \diagdown & \\
0 & — & 1 & & & & -\frac{25}{26} \\
 & & & \diagdown & & \diagup & \\
 & & & & -\frac{25}{26} & & \\
 & & & \diagup & & & \\
1 & — & \frac{1}{26} & & & &
\end{array}$$

$$p_2(x) \;=\; \tfrac{1}{26} + \tfrac{25}{26}(x - (-1)) - \tfrac{25}{26}(x - (-1))(x - 0) \;:\; \text{Newton's form}$$

$$= \tfrac{1}{26} + \tfrac{25}{26}(x + 1) - \tfrac{25}{26}(x + 1)x$$

$$= 1 - \tfrac{25}{26}x^2 \;:\; \text{standard form}$$

check : $p_2(-1) = \frac{1}{26}$, $p_2(0) = 1$, $p_2(1) = \frac{1}{26}$ <u>ok</u>

$$\int_{-1}^{1} f(x)dx = 2\int_0^1 \frac{dx}{1 + 25x^2} = \cdots = 2 \cdot \tfrac{1}{5}\tan^{-1}5x\Big|_0^1 = \tfrac{2}{5}\tan^{-1}5 = 0.5494$$

$$\int_{-1}^{1} p_2(x)dx = 2\int_0^1 (1 - \tfrac{25}{26}x^2)dx = 2(x - \tfrac{25}{26}\cdot\tfrac{1}{3}x^3)\Big|_0^1 = 2(1 - \tfrac{25}{78}) = \tfrac{106}{78} = 1.3590$$

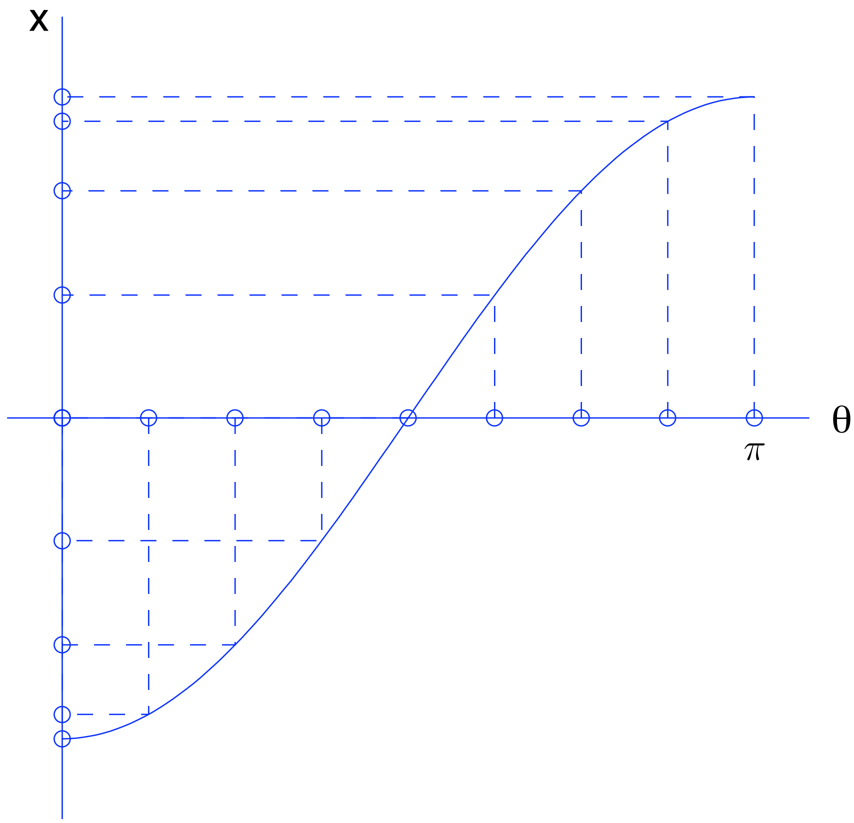
Hence $p_2(x)$ is a poor approximation to $f(x)$. Can we do better?

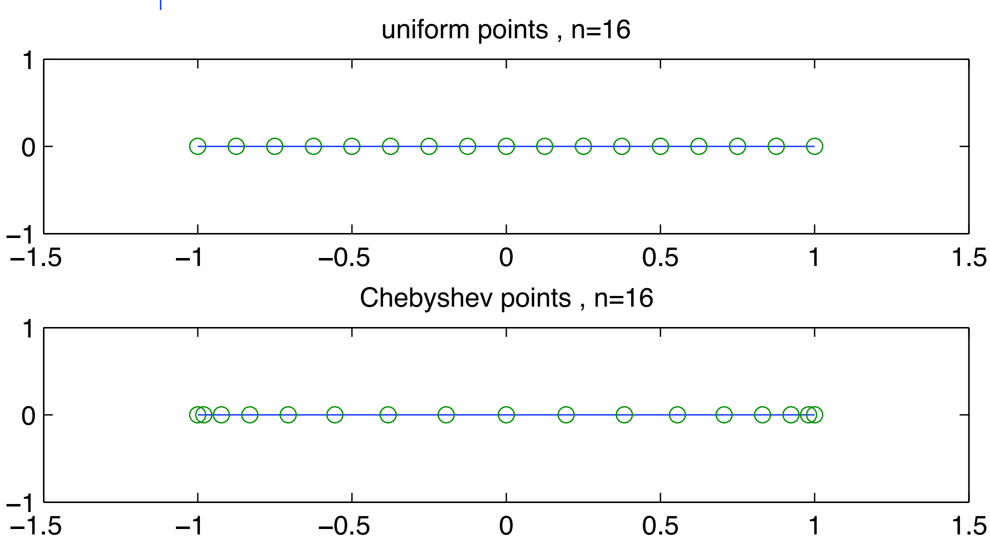
<u>5.3 optimal interpolation points</u>

Given $f(x)$ for $-1 \le x \le 1$, how should the interpolation points $x_0, \ldots, x_n$ be chosen? Consider two options.

1. uniform points : $x_i = -1 + ih \ ,\ h = \dfrac{2}{n} \ ,\ i = 0:n$

2. Chebyshev points : $x_i = -\cos\theta_i \ ,\ \theta_i = ih \ ,\ h = \dfrac{\pi}{n} \ ,\ i = 0:n$



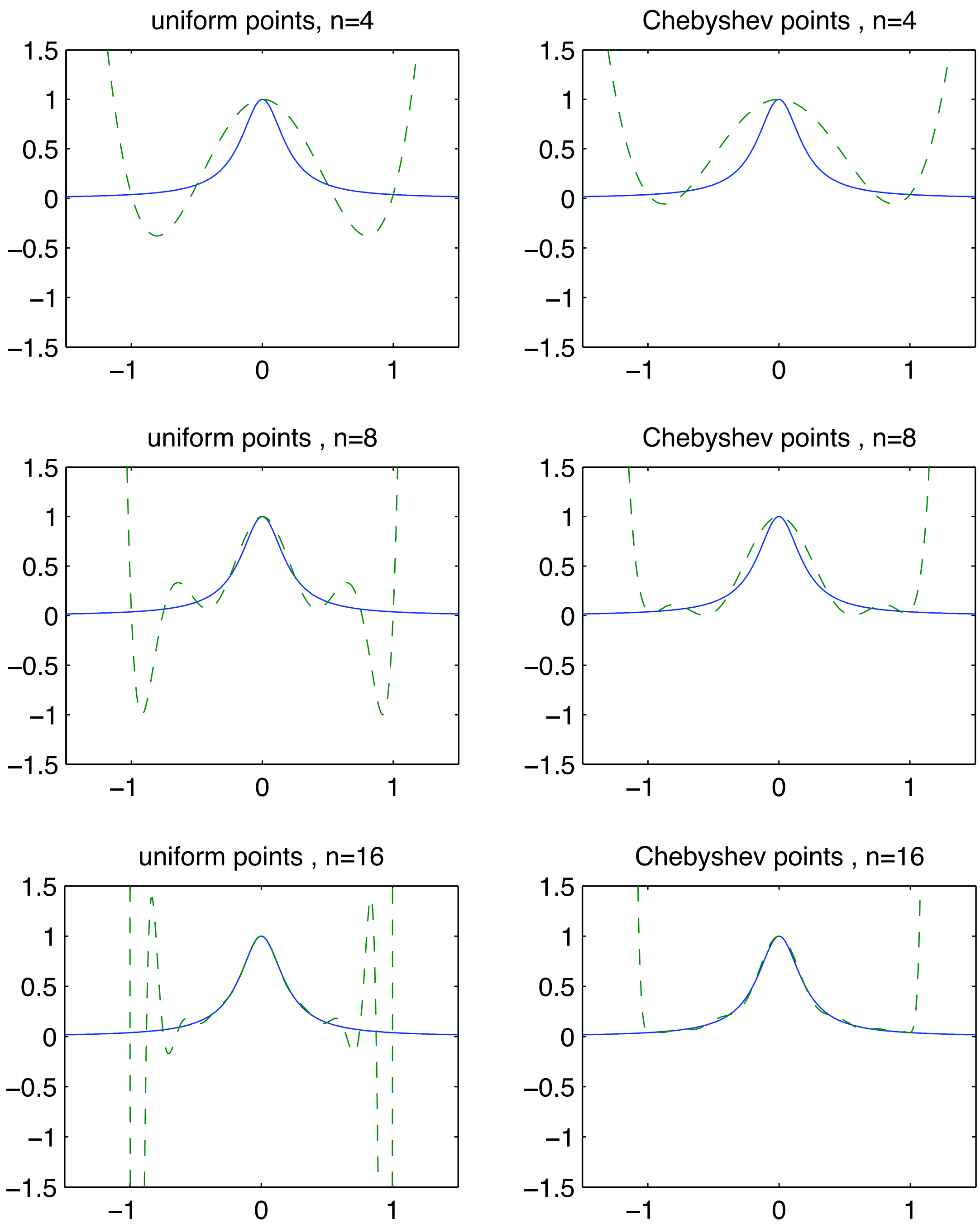


<u>note</u> : The Chebyshev points are clustered near the endpoints of the interval.

$\underline{\text{ex}}$: $f(x) = \dfrac{1}{1+25x^2}$, $-1 \le x \le 1$

solid line : $f(x)$, given function

dashed line : $p_n(x)$, interpolating polynomial



1. Interpolation at the uniform points gives a good approximation near the center of the interval, but it gives a bad approximation near the endpoints.

2. Interpolation at the Chebyshev points gives a good approximation on the entire interval.

<u>5.4 spline interpolation</u>

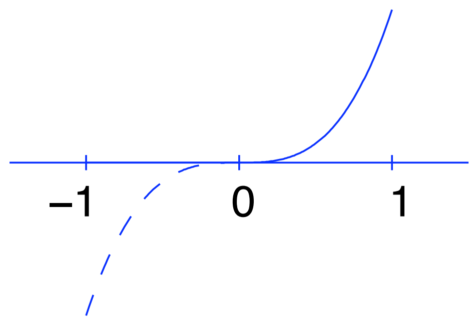
Let $x_0 < x_1 < \cdots < x_{n-1} < x_n$. A <u>cubic spline</u> is a function $s(x)$ satisfying the following conditions.

1. $s(x)$ is a cubic polynomial on each interval $x_i \le x \le x_{i+1}$.

2. $s(x)\,,\ s'(x)\,,\ s''(x)$ are continuous at the interior points $x_1, \ldots, x_{n-1}$

<u>ex</u> : $x_0 = -1\ ,\ x_1 = 0\ ,\ x_2 = 1$

$$s(x) = \begin{cases} 0 & , \ -1 \le x \le 0 \\ x^3 & , \quad 0 \le x \le 1 \end{cases}$$

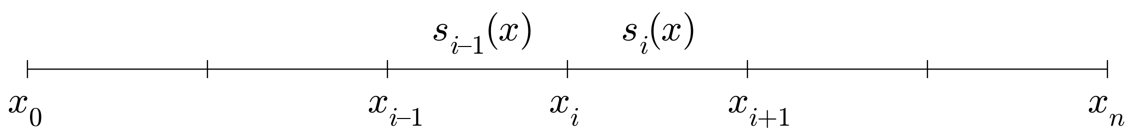


$$s'(x) = \begin{cases} 0 & , \ -1 \le x \le 0 \\ 3x^2 & , \quad 0 \le x \le 1 \end{cases}$$

$$s''(x) = \begin{cases} 0 & , \ -1 \le x \le 0 \\ 6x & , \quad 0 \le x \le 1 \end{cases}$$

check : $s(x)$ satisfies the conditions required to be a cubic spline

<u>problem</u> : Given $f(x)$ and $x_0 < x_1 < \cdots < x_{n-1} < x_n$, find the cubic spline $s(x)$ that interpolates $f(x)$ at the given points, i.e. $s(x_i) = f(x_i)\,,\ i = 0:n$. 19 Tue 3/30



$x_i \le x \le x_{i+1} \ \Rightarrow\ s(x) = s_i(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3\ ,\ i = 0 : n-1$

$n+1$ points $\Rightarrow$ n intervals $\Rightarrow$ $4n$ unknown coefficients

interpolation conditions $\Rightarrow$ $2n$ equations

continuity of $s'(x)\,,\ s''(x)$ at interior points $\Rightarrow$ $2(n-1)$ equations

Hence we can choose 2 more conditions; a popular choice is $s''(x_0) = s''(x_n) = 0$, which gives the <u>natural cubic spline interpolant</u>.

<u>how to find $s(x)$</u>

<u>ex</u> : $-1 \le x \le 1\ ,\ x_i = -1 + ih\ ,\ h = \frac{2}{n}\ ,\ i = 0, \ldots, n$: uniform points

<u>step 1</u> : 2nd derivative conditions

$s_i''(x)$ is a linear polynomial

$$\Rightarrow\ s_i''(x) = a_i\left(\frac{x_{i+1} - x}{h}\right) + a_{i+1}\left(\frac{x - x_i}{h}\right) \quad , \quad a_i\,,\ a_{i+1} : \text{to be determined}$$

$$\Rightarrow\ s_i''(x_i) = a_i\ ,\ s_i''(x_{i+1}) = a_{i+1}$$

$$\Rightarrow\ s_{i-1}''(x_i) = a_i = s_i''(x_i)\ \Rightarrow\ s''(x) \text{ is continuous at the interior points}$$

<u>step 2</u> : interpolation

integrate twice

$$s_i(x) \;=\; \frac{a_i(x_{i+1}-x)^3}{6h} \;+\; \frac{a_{i+1}(x-x_i)^3}{6h} \;+\; b_i\left(\frac{x_{i+1}-x}{h}\right) + \; c_i\left(\frac{x-x_i}{h}\right)$$

$$s_i(x_i) = \frac{a_i h^2}{6} + b_i = f_i \;\Rightarrow\; b_i = f_i - \frac{a_i h^2}{6}$$

$$s_i(x_{i+1}) = \frac{a_{i+1} h^2}{6} + c_i = f_{i+1} \;\Rightarrow\; c_i = f_{i+1} - \frac{a_{i+1} h^2}{6}$$

<u>step 3</u> : 1st derivative conditions

$$s_i'(x) \;=\; -\frac{a_i(x_{i+1}-x)^2}{2h} \;+\; \frac{a_{i+1}(x-x_i)^2}{2h} + \left(f_i - \frac{a_i h^2}{6}\right)\cdot\frac{-1}{h} + \left(f_{i+1} - \frac{a_{i+1}h^2}{6}\right)\cdot\frac{1}{h}$$

$$s_i'(x_i) \;=\; -\frac{a_i h}{2} \;-\; \frac{f_i}{h} \;+\; \frac{a_i h}{6} \;+\; \frac{f_{i+1}}{h} \;-\; \frac{a_{i+1} h}{6}$$

$$s_i'(x_{i+1}) \;=\; \frac{a_{i+1} h}{2} \;-\; \frac{f_i}{h} \;+\; \frac{a_i h}{6} \;+\; \frac{f_{i+1}}{h} \;-\; \frac{a_{i+1} h}{6}$$

we require $s_{i-1}'(x_i) = s_i'(x_i)$

$$\Rightarrow\; \frac{a_i h}{2} - \frac{f_{i-1}}{h} + \frac{a_{i-1} h}{6} + \frac{f_i}{h} - \frac{a_i h}{6} \;=\; -\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1} h}{6}$$

$$\Rightarrow\; \frac{a_{i-1} h}{6} \;+\; a_i\left(\frac{h}{2} - \frac{h}{6} + \frac{h}{2} - \frac{h}{6}\right) + \frac{a_{i+1} h}{6} \;=\; \frac{f_{i-1} - 2f_i + f_{i+1}}{h}$$

$$\Rightarrow\; a_{i-1} \;+\; 4a_i + a_{i+1} \;=\; \tfrac{6}{h^2}\left(f_{i-1} - 2f_i + f_{i+1}\right)\,,\;\; i = 1 : n-1$$

<u>step 4</u> : apply BC

$$s_0''(x_0) = a_0 = 0\;,\;\; s_{n-1}''(x_n) = a_n = 0$$

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} f_0 - 2f_1 + f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix}$$

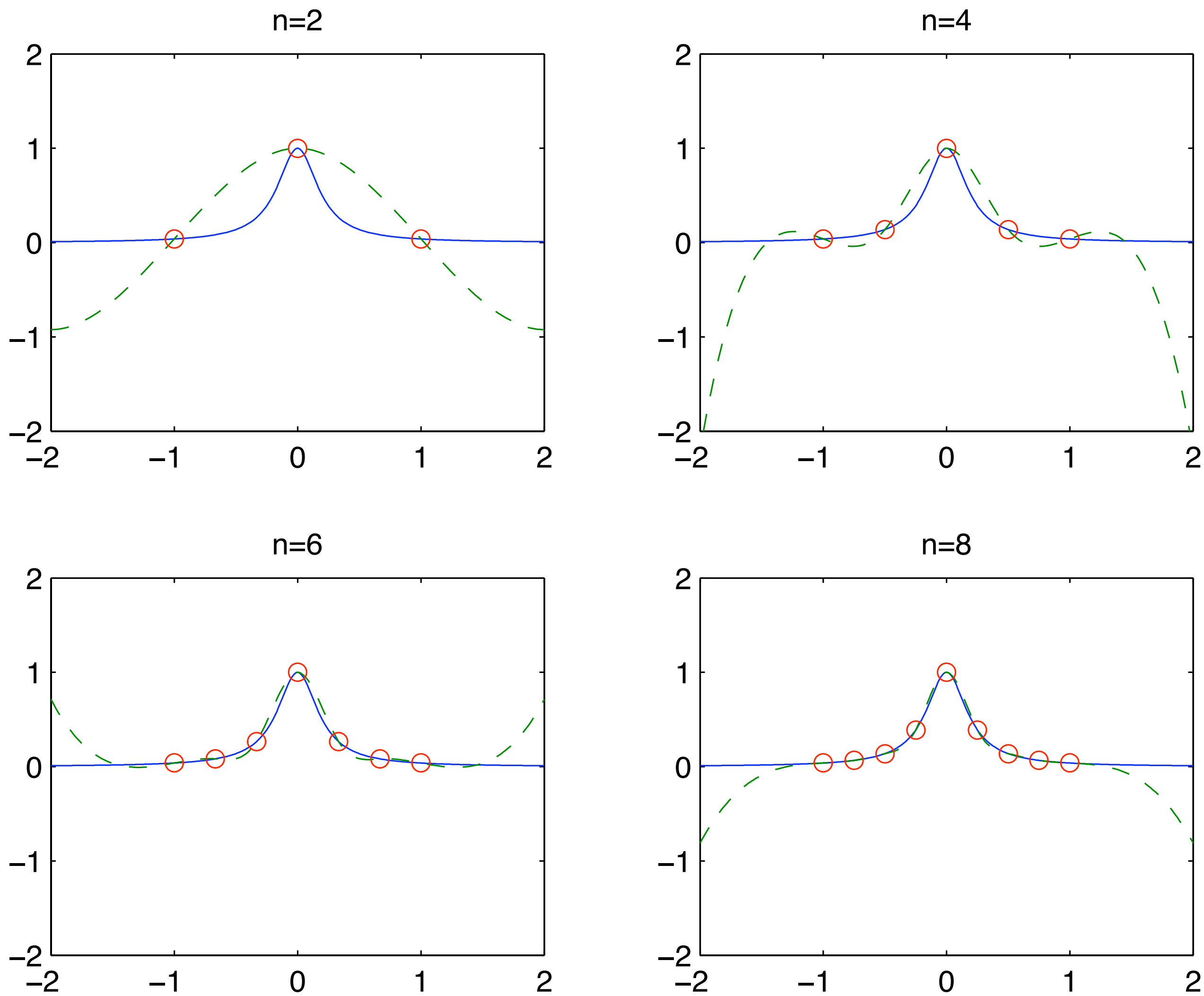
A : symmetric , tridiagonal , positive definite

<u>ex</u> : natural cubic spline interpolation

$$f(x) = \frac{1}{1+25x^2} \ , \quad -1 \le x \le 1 \ , \ x_i = -1 + ih \ , \ h = \frac{2}{n} \ , \ i = 0 : n$$

solid line : $f(x)$, given function

dashed line : $s(x)$, natural cubic spline interpolant



1. error bound : $|f(x) - s(x)| \le \frac{5}{384} \max_{a \le x \le b} |f^{(4)}(x)| \, h^4$: 4th order accurate

2. The natural cubic spline interpolant has inflection points at the endpoints of the interval, due to the boundary conditions $s''(x_0) = s''(x_n) = 0$; there are also inflection points in the interior of the interval which are not present in the original $f(x)$, and these are problematic in some applications.