Math 371 Review Sheet for Final Exam Winter 2021

1. True or False? Give a reason to justify your answer.

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a) $D_+D_-y_i = D_-D_+y_i$

TRUE

$D_+y_i = \frac{y_{i+1}-y_i}{h}$, $D_-y_i = \frac{y_i-y_{i-1}}{h}$

$D_+D_-y_i = D_+\left(\frac{y_i-y_{i-1}}{h}\right) = \frac{D_+y_i-D_+y_{i-1}}{h} = \frac{1}{h}\left(\frac{y_{i+1}-y_i}{h} - \frac{y_i-y_{i-1}}{h}\right) = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$

$D_-D_+y_i = D_-\left(\frac{y_{i+1}-y_i}{h}\right) = \frac{D_-y_{i+1}-D_-y_i}{h} = \frac{1}{h}\left(\frac{y_{i+1}-y_i}{h} - \frac{y_i-y_{i-1}}{h}\right) = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$

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b) $D_0f(x)$ is a 2nd order accurate approximation for the 2nd derivative $f''(x)$.

FALSE $D_0f(x)$ is a 2nd order accurate approximation for the 1st derivative $f'(x)$. This can be shown by Taylor expansion.

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c) If $Ax = 0$, then $A = 0$ or $x = 0$.

FALSE It is easy to find a counterexample. A modified true statement is: if $Ax = 0$ and A is invertible, then $x = 0$.

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d) If A is invertible, then $||A||^{-1} = ||A^{-1}||$.

FALSE For example consider $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$, so $||A|| = 3 \Rightarrow ||A||^{-1} = \frac{1}{3}$, but $||A^{-1}|| = 1$.

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e) $\rho(B) \le ||B||$ for any matrix B

TRUE If $Bx = \lambda x$, then $|\lambda| \cdot ||x|| = ||\lambda x|| = ||Bx|| \le ||B|| \cdot ||x||$, and it follows that $|\lambda| \le ||B||$, and since this holds for any eigenvalue, it follows that $\rho(B) = \max\{|\lambda|\} \le ||B||$.

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f) The spectral radius of a matrix satisfies the properties required to be a matrix norm.

FALSE Consider for example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, satisfying $\rho(A) = 0$ (can you justify this?), but $A \ne 0$. This violates the first property of a matrix norm, $||A|| = 0$ if and only if $A = 0$.

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g) In solving an $n \times n$ system of linear equations by Gaussian elimination, if n increases by a factor of 3, then the operation count increases by a factor of approximately 9.

FALSE The operation count for Gaussian elimination is $O(n^3)$, therefore the operation count increases by a factor of approximately $3^3 = 27$.

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h) In computing the solution of a linear system $Ax = b$, if the residual norm $||r||$ is small, then the error norm $||e||$ is also small.

FALSE An ill-conditioned system will not have this property according to the result derived in class, $\frac{||e||}{||x||} \le \kappa(A)\frac{||r||}{||b||}$, where $\kappa(A) = ||A|| \cdot ||A^{-1}||$ is the condition number of the matrix.

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i) If $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, then Jacobi's method applied to solve $Ax = b$ converges.

TRUE We have $B_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \Rightarrow ||B_J|| = 1$, but this is inconclusive, so we must consider the spectral radius, $\det(B_J - \lambda I) = -\lambda^3 + \frac{1}{2}\lambda = 0 \Rightarrow \lambda = 0\,,\, \pm\frac{1}{\sqrt{2}} \Rightarrow \rho(B_J) = \frac{1}{\sqrt{2}} = 0.7071 < 1$, and so the method converges.

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j) In solving the linear system $Ax = b$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, one step of Gauss-Seidel reduces the norm of the error as much as two steps of Jacobi.

TRUE We considered this example in class and the numerical results indicate this.

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k) In solving a linear system $Ax = b$ by an iterative method $x_{k+1} = Bx_k + c$, if $||B|| < 1$, then $\lim_{k\to\infty} x_k = x$ for any initial guess x_0.

TRUE This is the convergence theorem for iterative methods derived in class.

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l) In solving a linear system $Ax = b$ by an iterative method such as Jacobi or Gauss-Seidel, if the matrix A has dimension $n \times n$, then the exact solution is obtained after n iterations.

FALSE An iterative method gives better accuracy with increasing number of iterations k, but it will not give the exact solution for some specific value of k (except in some special cases).

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m) Consider two iterative methods for solving $Ax = b$. If the two iteration matrices B_1, B_2 satisfy $||B_1|| = ||B_2||$, then the two iterative methods converge at the same rate.

FALSE The convergence rate of an iterative method is determined by the spectral radius of the iteration matrix, $\rho(B)$, not by the matrix norm of the iteration matrix $||B||$.

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n) Suppose a two-dimensional boundary value problem is solved using a finite-difference scheme and the resulting linear system is solved by Jacobi's method with stopping criterion $||r_k|| \leq 10^{-2}$. If the mesh size h is decreased, then the number of iterations needed to satisfy the stopping criterion is also decreased.

FALSE The spectral radius of the iteration matrix $\rho(B_J) = \cos \pi h$ has the property that it approaches 1 as the mesh size h goes to zero. Hence more iterations are required if h is decreased.

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o) Jacobi and Gauss-Seidel converge linearly, but optimal SOR converges quadratically.

FALSE All three methods converge linearly; the convergence factor is the spectral radius of the iteration matrix.

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p) A matrix A is positive definite if there exists at least one vector $x \neq 0$ such that $x^T Ax > 0$.

FALSE By definition, a matrix A is positive definite if $x^T Ax > 0$ for all $x \neq 0$.

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q) If $\lambda = 0$ is an eigenvalue of A, then A is not invertible.

TRUE If $\lambda = 0$ is an eigenvalue of A, then there exists an eigenvector, $x \neq 0$, corresponding to the eigenvalue $\lambda = 0$, and hence $Ax = \lambda x = 0 \cdot x = 0$. If A were invertible, then we would have $x = Ix = A^{-1}Ax = A^{-1}0 = 0$, but this contradicts the fact that $x \neq 0$. Hence A cannot be invertible.

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r) If A is symmetric and positive definite, then A is invertible.

TRUE Let $x \neq 0$ be any nonzero vector. If $Ax = 0$, then $x^T Ax = 0$, which contradicts the assumption that A is positive definite. Hence $Ax \neq 0$. This ensures that A is invertible.

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s) The inverse power method is used to find the inverse of a matrix.

FALSE The inverse power methods is used to find the smallest eigenvalue λ_n of matrix.

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t) Wilkinson's example shows that the coefficients of a polynomial can depend sensitively on the roots.

FALSE Wilkinson's example shows that the roots of the characteristic polynomial are sensitive to perturbations in the coefficients, and hence solving $f_A(\lambda) = 0$ is not a practical method for computing e-values (in general).

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u) When the power method is applied to find the largest eigenvalue and corresponding eigenvector of a matrix, the vectors $v^{(k)}$ are normalized at each step in order to accelerate convergence of the method.

FALSE The vectors $v^{(k)}$ are normalized at each step to avoid overflow/underflow.

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v) If $p_n(x)$ is the interpolating polynomial of degree n for a given function $f(x)$ at points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$, then $\lim\limits_{n\to\infty} p_n(x) = f(x)$ for all x in the interval $[a, b]$.

FALSE As we saw in class, the interpolating polynomial based on uniform points has wild oscillations near the endpoints of the interval as the degree n increases.

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w) Polynomial interpolation at the Chebyshev points on the interval $a \leq x \leq b$ gives a good approximation near the endpoints of the interval and a bad approximation near the center of the interval.

FALSE As we saw in class and on homework, polynomial interpolation at the Chebyshev points gives a good approximation over the entire interval.

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x) Suppose $f(x)$ is approximated by a cubic spline interpolant $s(x)$ on the interval $a \leq x \leq b$ with interpolation points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$. Then if n is doubled, the error defined by $\max\limits_{a\leq x\leq b} |f(x) - s(x)|$ is reduced by a factor of approximately 1/16.

TRUE It was stated in class that cubic spline interpolation on uniform points is 4th order accurate and $(\frac{1}{2})^4 = \frac{1}{16}$.

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2. State one advantage of . . .
a) . . . Newton's method over the bisection method.

Newton's method converges quadratically; the bisection method converges linearly.

b) . . . Gaussian elimination with pivoting over Gaussian elimination without pivoting.

Pivoting prevents breakdown if a zero pivot arises.

c) . . . optimal SOR over Gauss-Seidel.

Optimal SOR converges faster than Gauss-Seidel.

d) . . . Chebyshev points over uniform points.

In polynomial interpolation, Chebyshev points control the error over the entire interval, while uniform points control the error only near the middle of the interval.

e) . . . cubic spline interpolation over Taylor approximation.

Cubic spline interpolation controls the error over the entire interval, while Taylor approximation controls the error only near the expansion point.

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3. Consider the following approximation for the first derivative, $f'(x) \approx \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$.

a) Apply the method to compute $f'(1)$, for $f(x) = e^x$, with step size $h = 1, \frac{1}{2}, \frac{1}{4}$.

h	approximation	error	error/h	error/h^2
1	0.6579	2.0604	2.0604	2.0604
$\frac{1}{2}$	2.3829	0.3354	0.6709	1.3417
$\frac{1}{4}$	2.6497	0.0686	0.2744	1.0977

The last column converges to a non-zero constant, so the method is 2nd order accurate.

b) The error has the form: error $= cf^{(m)}(x)h^n + \cdots$. Find the constants c, m, n by Taylor expansion. Are the results of parts (a) and (b) consistent? Explain.

$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \frac{f'''(x)h^3}{6} + O(h^4)$

$f(x+2h) = f(x) + f'(x)2h + \frac{4f''(x)h^2}{2} + \frac{8f'''(x)h^3}{6} + O(h^4)$

$\Rightarrow Df(x) = \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h} = f'(x) - \frac{1}{3}f'''(x)h^2 + O(h^3)$

Therefore, $c = -\frac{1}{3}$, $m = 3$, $n = 2$. The last column converges to $\frac{1}{3}e = 0.9061$.

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4. a) $f(0) = -5$, $f(4) = 9$, $f(0)f(4) < 0$, thus $[0, 4]$ is a suitable starting intervals.

b)

i	x_i	$f(x_i)$	a_i	b_i
0	2	-1	0	4
1	3	4	2	4
2	2.5	1.25	2	3

c) $\frac{1}{2}^n|b-a| < 10^{-4} \Rightarrow \frac{1}{2}^n < \frac{1}{4} \cdot 10^{-4} \Rightarrow 2^n > 4 \cdot 10^4 \Rightarrow n = 16$

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5. Fixed point: $x = x^2 - \frac{1}{2}x + \frac{1}{2} \Rightarrow 2x^2 - 3x + 1 = 0 \Rightarrow x_1 = \frac{1}{2},\ x_2 = 1.$
$g'(x) = 2x - \frac{1}{2}x$, $g'(\frac{1}{2}) = \frac{3}{4} < 1$, $g'(1) = \frac{3}{2} > 1$, therefore the iteration converges for starting value sufficiently close to the fixed point $\frac{1}{2}$.

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6. The correct form is below.

$$\begin{aligned} a_{11}x_1^{(k+1)} &= a_{11}x_1^{(k)} - \omega(a_{11}x_1^{(k)} + a_{12}x_2^{(k)} - b_1), \\ a_{22}x_2^{(k+1)} &= a_{22}x_2^{(k)} - \omega(a_{21}x_1^{(k+1)} + a_{22}x_2^{(k)} - b_2). \end{aligned}$$

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7. a) Jacobi

$2x_1^{(k+1)} = x_2^{(k)} + 1$

$2x_2^{(k+1)} = x_1^{(k)} + x_3^{(k)}$

$2x_3^{(k+1)} = x_2^{(k)} + 1$

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ e_0 = x - x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \| e_0 \| = 1 \ ; \quad x_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ e_1 = x - x_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}, \| e_1 \| = 1$$

b) Gauss-Seidel

$2x_1^{(k+1)} = x_2^{(k)} + 1$

$2x_2^{(k+1)} = x_1^{(k+1)} + x_3^{(k)}$

$2x_3^{(k+1)} = x_2^{(k+1)} + 1$

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},\ e_0 = x - x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \| e_0 \| = 1 \ ; \quad x_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{5}{8} \end{pmatrix},\ e_1 = x - x_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ \frac{3}{8} \end{pmatrix}, \| e_1 \| = \frac{3}{4}$$

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8. For A_1.

$$B_J^{(1)} = -D^{-1}(L+U) = -\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \| B_J^{(1)} \| = \frac{1}{2} < 1$$

$$B_{GS}^{(1)} = -(D+L)^{-1}U = -\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}, \| B_{GS}^{(1)} \| = \frac{1}{2} < 1$$

For A_1, both methods converge.

For A_2.

$$B_J^{(2)} = -D^{-1}(L+U) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \| B_J^{(2)} \| = 2 > 1$$

$$B_{GS}^{(2)} = -(D+L)^{-1}U = -\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 2 \\ 0 & -4 \end{pmatrix}, \| B_{GS}^{(2)} \| = 4 > 1$$

Thus we need to consider the spectral radius.

Jacobi : eigenvalues for $B_J^{(2)}$, $\lambda_1 = \lambda_2 = 2 \Rightarrow \rho(B_J^{(2)}) = 2 > 1$

Gauss-Seidel : eigenvalues for $B_{GS}^{(2)}$, $\lambda_1 = 0, \lambda_2 = 4 \Rightarrow \rho(B_J^{(2)}) = 4 > 1$

Hence both methods will not converge.

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9. a) Let $x = (1\ 1\ 1)$, then $xAx^T = 0$, so it is not positive definite.

b) The eigenvalues of A are all positive, so it is positive definite.

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10. a) $x = (1\ 1\ 1)^T$

b) $B_J = -D^{-1}(L+U) = -\begin{pmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$, $\| B_J \| = \frac{1}{2} < 1$: convergent

c) $B_{GS} = -(D+L)^{-1}U = -\begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{16} & \frac{1}{4} \\ 0 & \frac{1}{64} & -\frac{1}{16} \end{pmatrix}$, $\| B_{GS} \| = \frac{5}{16} < 1$: convergent

d) $\omega^* = \dfrac{2}{1+\sqrt{1-\rho(B_J)^2}}$, $\rho(B_J) = \dfrac{1}{2\sqrt{2}} \Rightarrow \omega^* = 1.0334$

e) 5.395348837209301

f) 5.090909090909090

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11. a) True. Proof. Suppose A is not invertible, we want to show contradiction. If A is not invertible, then there exists $x \neq 0$ such that $Ax = 0$, and moreover, we have $x^T Ax = 0$, which violates the positive definite assumption of A.

b) True. Proof. Choose $x_i = e_i$, where e_i is the unit vector, i.e., the ith component of e_i is 1, and rest components are 0. Then $e_i^T Ae_i = a_{ii} > 0$.

c) True. Proof. Suppose λ is a eigenvalue of A and $v \neq 0$ is the corresponding eigenvector. Since A is positive definite, $v^T Av = \lambda v^T v > 0$, note that $v^T v > 0$, we have $\lambda > 0$.

d) True. Proof. $(A^T A)^T = A^T (A^T)^T = A^T A$, so $A^T A$ is symmetric. For any $x \neq 0$, $x^T (A^T A)x = (Ax)^T Ax > 0$, so $A^T A$ is positive definite.

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12. Here $n = 3$, thus $h = \frac{1}{3+1} = \frac{1}{4}$.

$$-\left(\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}\right) + w_i = x_i, \quad \text{for } i = 1, 2, 3, \quad w_0 = 1,\ w_4 = 0.$$

In matrix form the system is

$$16 \begin{pmatrix} 2+\frac{1}{16} & -1 & 0 \\ -1 & 2+\frac{1}{16} & -1 \\ 0 & -1 & 2+\frac{1}{16} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+16 \\ \frac{2}{4} \\ \frac{3}{4} \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} 33 & -16 & 0 \\ -16 & 33 & -16 \\ 0 & -16 & 33 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \frac{65}{4} \\ \frac{2}{4} \\ \frac{3}{4} \end{pmatrix}.$$

13.

$$-\frac{4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1}}{h^2} = f_{i,j}, \quad \text{for } i = 1, 2, 3, j = 1, 2, 3.$$

$$A_h = \frac{1}{h^2} \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 4 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 4 \end{pmatrix}$$

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14. The eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = 1$, and the corresponding eigenvectors are $q_1 = \frac{1}{\sqrt{2}}(1,\ 1)^T$, $q_2 = \frac{1}{\sqrt{2}}(1,\ 1)^T$. Then the solution is $x = \lambda_1^{-1}(q_1^T b)q_1 + \lambda_2^{-1}(q_2^T b)q_2 = (3,\ -2)^T$.

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15. a) eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 3 \Rightarrow \rho(A) = 3$, $\parallel A \parallel = 3$

b) eigenvalues of A are $\lambda_1 = \lambda_2 = 2 \Rightarrow \rho(A) = 2$, $\parallel A \parallel = 3$

c) eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -3 \Rightarrow \rho(A) = 3$, $\parallel A \parallel = 3$

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16. Let λ_1, λ_2 be the eigenvalues of A, with $\lambda_1 > \lambda_2$, and let v_1, v_2 be the corresponding eigenvectors. Since the matrix is real and symmetric, the eigenvectors form an orthonormal basis (they are automatically orthogonal and we assume they've been normalized). Any vector can be expanded as a linear combination of the eigenvectors, $x = a_1v_1 + a_2v_2$, and the Rayleigh quotient can be evaluated as follows.

$$x^Tx = (a_1v_1 + a_2v_2)^T(a_1v_1 + a_2v_2) = a_1^2v_1^Tv_1 + a_1a_2v_1^Tv_2 + a_1a_2v_2^Tv_1 + a_2^2v_2^Tv_2 = a_1^2 + a_2^2$$

$$x^TAx = (a_1v_1 + a_2v_2)^TA(a_1v_1 + a_2v_2) = \cdots = \lambda_1a_1^2 + \lambda_2a_2^2$$

$$\frac{x^TAx}{x^Tx} = \frac{\lambda_1a_1^2 + \lambda_2a_2^2}{a_1^2 + a_2^2}$$

$$\max_{x\neq0} \frac{x^TAx}{x^Tx} = \max_{x\neq0} \frac{\lambda_1a_1^2 + \lambda_2a_2^2}{a_1^2 + a_2^2} \leq \max_{x\neq0} \frac{\lambda_1a_1^2 + \lambda_1a_2^2}{a_1^2 + a_2^2} = \lambda_1$$

$$\min_{x\neq0} \frac{x^TAx}{x^Tx} = \min_{x\neq0} \frac{\lambda_1a_1^2 + \lambda_2a_2^2}{a_1^2 + a_2^2} \geq \min_{x\neq0} \frac{\lambda_2a_1^2 + \lambda_2a_2^2}{a_1^2 + a_2^2} = \lambda_2$$

so $\max_{x\neq0} \frac{x^TAx}{x^Tx} = \lambda_1 = 3$, $\min_{x\neq0} \frac{x^TAx}{x^Tx} = \lambda_2 = 1$

17.

$$\begin{array}{cccc}
x & y & & \\
2 & -1 & & \\
4 & 4 & f[x_0,x_1] = \frac{4+1}{4-2} = 2.5 & \\
5 & 8 & f[x_1,x_2] = \frac{8-4}{5-4} = 4 & f[x_0,x_1,x_2] = \frac{4-2.5}{5-2} = 0.5
\end{array}$$

$\Rightarrow P_2(x) = -1 + 2.5(x-2) + 0.5(x-2)(x-4) \Rightarrow P_2(3) = 1$

18.

$$\begin{array}{cccccc}
x & f(x) & & & \\
1 & 1 & & & \\
2 & \frac{1}{2} & -\frac{1}{2} & & \\
3 & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \\
4 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{24} & -\frac{1}{24}
\end{array}$$

$\Rightarrow P_3(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{6}(x-1)(x-2) - \frac{1}{24}(x-1)(x-2)(x-3) \Rightarrow P_3(4) = \frac{1}{4}$

19.

$$\begin{array}{ccccc}
t & T(t) & & & \\
8 & 30 & & & \\
10 & 40 & 5 & & \\
14 & 50 & 2.5 & -\frac{5}{12} & \\
16 & 60 & 5 & \frac{5}{12} & \frac{5}{48}
\end{array}$$

$\Rightarrow P_3(x) = 30 + 5(t-8) - \frac{5}{12}(t-8)(t-10) + \frac{5}{48}(t-8)(t-10)(t-14) \Rightarrow P_3(12) = 45$

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20. Similar to 17, 18, 19. $P_5(240) = 22.3099$.

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21. Need to check the following conditions.
(1) cubic polynominal on each subinterval
(2) $s''(a) = s''(b) = 0$
(3) $s_i(x_{i+1}) = s_{i+1}(x_i)$, $s'_i(x_{i+1}) = s'_{i+1}(x_i)$, $s''_i(x_{i+1}) = s''_{i+1}(x_i)$.

a) $s_0(x) = 0$, $0 \le x \le 1$; $s_1(x) = x^3 - 3x^2 + 3x - 1$, $1 \le x \le 2$
$s_0(x) = s'_0(x) = s''_0(x) = 0$, $s'_1(x) = 3(x-1)^2$, $s''_2(x) = 6(x-1)$
Since $s''(2) = s''_2(2) = 6 \ne 0$, it is not cubic spline.

b) $s_0(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$, $-1 \le x \le 0$; $s_1(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$, $0 \le x \le 1$
$s'_0(x) = -\frac{3}{2}x^2 - 3x$, $s''_0(x) = -3x - 3$; $s'_1(x) = \frac{3}{2}x^2 - 3x$, $s''_1(x) = 3x - 3$
Then $s''_0(-1) = 0$, $s''_1(1) = 0$, satisfy condition (2). $s_0(0) = 1 = s_1(0)$, $s'_0(0) = 0 = s'_1(0)$, $s''_0(0) = -3 = s''_1(0)$, satisfy condition (3). So it is a cubic spline.

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22. Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $h = 1$. $f_0 = f(x_0) = 0$, $f_1 = f(x_1) = 1$, $f_2 = f(x_2) = 0$. Now go through the four steps in the notes.
(1) second derivative

$$\begin{aligned} s''_0(x) &= a_0\frac{x_1 - x}{h} + a_1\frac{x - x_0}{h} = a_0(x_1 - x) + a_1(x - x_0), \\ s''_1(x) &= a_1\frac{x_2 - x}{h} + a_2\frac{x - x_1}{h} = a_1(x_2 - x) + a_2(x - x_1), \end{aligned}$$

(2) interpolation

$$\begin{aligned} s_0(x) &= a_0\frac{(x_1 - x)^3}{6} + a_1\frac{(x - x_0)^3}{6} + (f_0 - \frac{a_0}{6})(x_1 - x) + (f_1 - \frac{a_1}{6})(x - x_0) \\ s_1(x) &= a_1\frac{(x_2 - x)^3}{6} + a_2\frac{(x - x_1)^3}{6} + (f_1 - \frac{a_1}{6})(x_2 - x) + (f_2 - \frac{a_2}{6})(x - x_1) \end{aligned}$$

(3) first derivative $a_0 + 4a_1 + a_2 = 6(f_0 - 2f_1 + f_2) = -12$
(4) apply BC $s''_0(x_0) = a_0 = 0$, $s''_1(x_1) = a_2 = 0$
Therefore, $s_0(x) = -\frac{1}{2}x^3 + \frac{3}{2}x$, $s_1(x) = -\frac{1}{2}(1-x)^3 + \frac{3}{2}(2-x)$.

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23. Similar to 25.

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24. a) $\int_0^{2h} f(x)dx = c_0 f(0) + c_1 f(h) + c_2 f(2h)$.

$$\begin{aligned} f(x) = 1, &\quad \int_0^{2h} 1dx = 2h = c_0 + c_1 + c_2 \\ f(x) = x, &\quad \int_0^{2h} 2xdx = 2h^2 = c_1 h + c_2 2h \\ f(x) = x^2, &\quad \int_0^{2h} x^2 dx = \frac{8}{3}h^3 = c_1 h^2 + c_2(2h)^2 \end{aligned}$$

Thus $c_0 = \frac{1}{3}h$, $c_1 = \frac{4}{3}h$, $c_2 = \frac{1}{3}h$, and therefore $\int_0^{2h} f(x)dx = \frac{2h}{6}\left[f(0) + 4f(h) + f(2h)\right]$.
b) For x^3, $LHS = \int_0^{2h} x^3 dx = 4h^4$, $RHS = 4h^4 \Rightarrow LHS = RHS$.
c) For x^4, $LHS = \int_0^{2h} x^4 dx = \frac{32}{5}h^5$, $RHS = \frac{20}{3}h^5 \Rightarrow LHS \ne RHS$.

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25. a)

h	$R_0(h)$	$R_1(h)$	$R_2(h)$	$R_3(h)$
1	0.183939720585721			
$\frac{1}{2}$	0.286670056060712	0.320913501219042		
$\frac{1}{4}$	0.308882624093246	0.316286813437425	0.315978367585317	
$\frac{1}{8}$	0.314275892570701	0.316073648729853	0.316059437749348	0.316060724577349

b) For $R_3\left(\frac{1}{8}\right)$, the error is $O(h^8) = O\left((\frac{1}{8})^8\right)$. Using trapezoidal rule with mesh size h, we should have approximately $h^2 < (\frac{1}{8})^8 \Rightarrow h \approx \frac{1}{2^{12}}$.

Note that the exact value of the integral can be computed using calculus.

$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}|_0^1 = \frac{1}{2}(1-e^{-1}) = 0.316060279414279$

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26. Orthogonal pairs are $(1,x)$, $(1, \sin\pi x)$, $(1,\sin 2\pi x)$, (x,x^2), $(x,\cos\pi x)$, $(x, \sin^2\pi x)$, $(x^2, \sin\pi x)$, $(x^2, \sin 2\pi x)$, $(\sin\pi x, \cos\pi x)$, $(\sin\pi x, \sin^2\pi x)$, $(\cos\pi x, \sin 2\pi x)$, $(\sin 2\pi x, \sin^2\pi x)$.

27.

$$\begin{aligned} P_4(x) &= x^4 - \frac{\langle x^4, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle}P_0(x) - \frac{\langle x^4, P_1(x)\rangle}{\langle P_1(x), P_1(x)\rangle}P_1(x) - \frac{\langle x^4, P_2(x)\rangle}{\langle P_2(x), P_2(x)\rangle}P_2(x) - \frac{\langle x^4, P_3(x)\rangle}{\langle P_3(x), P_3(x)\rangle}P_3(x) \\ &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}$$

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28. a), b), c) similar to 25.

e) substituting $t = 2x - 1 \Rightarrow$

$$\begin{aligned} \int_0^1 f(x)dx = \frac{1}{2}\int_{-1}^1 f\left(\frac{t+1}{2}\right)dt = \frac{1}{2}\int_{-1}^1 e^{-\frac{t+1}{2}}\sin\pi\frac{t+1}{2}dt \\ = \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right) = 0.395269764820483 \end{aligned}$$

f)
$$\begin{aligned} \int_0^1 e^{-x}\sin\pi x dx = -\int_0^1 \sin\pi x de^{-x} = \pi\int_0^1 e^{-x}\cos\pi x dx = -\pi\int_0^1 \cos\pi x de^{-x} \\ -\pi\cos\pi x e^{-x}\Big|_0^1 - \pi^2\int_0^1 e^{-x}\sin\pi x dx = \pi + \pi e^{-1} - \pi^2\int_0^1 e^{-x}\sin\pi x dx \\ \Rightarrow \int_0^1 e^{-x}\sin\pi x dx = \frac{\pi(1+e^{-1})}{1+\pi^2} = 0.395352015106459 \end{aligned}$$
