Functional Analysis

UCL MATH0018 Problem Classes

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Term 2, 2023/24

Contents

1 Week 1 - No problem class for week 1

2 Week 2 - The ℓ^p sequence spaces

Exercise 2.1. Prove that the following table of properties hold. Let $p \in [1, \infty)$.

Space	Banach	Separable
$(\ell^\infty, \|\cdot\|_\infty)$	Yes	No
$(\ell^p, \|\cdot\|_p)$	Yes	Yes
$(\mathbf{c}, \|\cdot\|_\infty)$	Yes	Yes
$(\mathbf{c}_0, \|\cdot\|_\infty)$	Yes	Yes
$(\mathbf{c}_{00}, \|\cdot\|_p)$	No	Yes
$(\mathbf{c}_{00}, \|\cdot\|_\infty)$	No	Yes

Remark. Recall that

▷
$$\ell^\infty = \left\{(x_n)_{n=1}^\infty : \sup_{n\in\mathbb{N}} |x_n| < \infty\right\}$$
ℓ^∞ is the space (set) of bounded sequences;

▷
$$\mathbf{c} = \left\{(x_n)_{n=1}^\infty : \lim_{n\to\infty} x_n \text{ exists (i.e. } (x_n)_n^\infty \text{ converges)}\right\} \subset \ell^\infty$$
$\mathbf{c}$ is the subspace (set) of convergent sequences;

▷
$$\mathbf{c}_0 = \left\{(x_n)_{n=1}^\infty : \lim_{n\to\infty} x_n = 0 \ \text{ (i.e. } x_n \to 0)\right\} \subset \mathbf{c}$$
$\mathbf{c}_0$ is the subspace (set) of sequences converging to 0;

▷
$$\mathbf{c}_{00} = \left\{(x_n)_{n=1}^\infty : \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, x_n = 0\right\} \subset \ell^p$$
$\mathbf{c}_{00}$ is the subspace (set) of finite sequences.

Solution 2.1. Let us show the above properties one by one:
► $(\ell^\infty, \|\cdot\|_\infty)$ is complete.
Let $(x^{(n)})_n \in \ell^\infty$ be a Cauchy sequence. For each n, let us denote $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots)$.
Claim: $\forall k \in \mathbb{N}$, $\left(x_k^{(n)}\right)_n \subset \mathbb{F}$ is a Cauchy sequence.

Proof of the claim: Let $\epsilon > 0$. $\left(x^{(n)}\right)_n$ is Cauchy $\iff$ $(\exists K)(\forall m, n \geq K), \|x^{(m)} - x^{(n)}\|_\infty < \epsilon$. By the definition of $\|\cdot\|_\infty$,
$$\|x^{(m)} - x^{(n)}\|_\infty = \sup_{k\in\mathbb{N}} |x_k^{(m)} - x_k^{(n)}| < \epsilon \implies \forall k \in \mathbb{N}, |x_k^{(m)} - x_k^{(n)}| \leq \|x^{(m)} - x^{(n)}\|_\infty < \epsilon. \tag{1}$$
Thus, the same K works for $\left(x_k^{(n)}\right)_n$ and hence it is Cauchy. √

Since $\mathbb{F}$ is complete, we have $\forall k \in \mathbb{N}$, $\left(x_k^{(n)}\right)_n \to x_k$ as $n \to \infty$ for some $x_k \in \mathbb{F}$. Denote $x = (x_1, x_2, \ldots, x_m, \ldots)$. It remains to show that $x \in \ell^\infty$ and $x^{(n)} \to x$ in ℓ^∞ as $n \to \infty$.

$x \in \ell^\infty$: Let $\epsilon > 0$. By (1), we have that $(\exists K)(\forall k \in \mathbb{N})(\forall m, n \geq K), |x_k^{(m)} - x_k^{(n)}| < \epsilon$. Taking $m \to \infty$, we have
$$(\exists K)(\forall k \in \mathbb{N})(\forall n \geq K), |x_k - x_k^{(n)}| \leq \epsilon. \tag{2}$$
In particular, $\forall k \in \mathbb{N}$,
$$|x_k| = |x_k - x_k^{(n)} + x_k^{(n)}| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}| \leq \epsilon + |x_k^{(n)}|.$$
Taking the $\sup_{k\in\mathbb{N}}$ on both sides, since $x^{(n)} \in \ell^\infty$, we have
$$\|x\|_\infty \leq \epsilon + \|x^{(n)}\|_\infty < \infty \implies x \in \ell^\infty.$$
$x^{(n)} \to x$ in ℓ^∞: By (2), we have $(\exists K)(\forall n \geq K), \|x^{(n)} - x\|_\infty \leq \epsilon \implies x^{(n)} \to x$.

► $(\ell^\infty, \|\cdot\|_\infty)$ is not separable.
Let us first show that $\exists S \subset \ell^\infty$ s.t.

- S is uncountable;
- $(\exists C > 0)(\forall x, y \in S, x \neq y), \|x - y\|_\infty = C$ i.e. the distance between a distinct pair of points in S is the same and positive.

For any $G \subset \mathbb{N}$, let $e^G \in \ell^\infty$ be the sequence such that

$$e_n^G = \begin{cases} 1 & \text{if } n \in G, \\ 0 & \text{otherwise.} \end{cases}$$

So e.g. $e^{\{2,4,5\}} = (0, 1, 0, 1, 1, 0, 0, \ldots)$. Let $S = \left\{e^G : G \subset \mathbb{N}\right\}$.
$\underline{S \text{ is an uncountable set}}$ by the diagonal argument. $\underline{\forall G \neq G', \left\|e^G - e^{G'}\right\|_\infty = 1}$ simply by the definition of $\|\cdot\|_\infty$. Hence there are uncountably many non-intersecting open balls of radius $1/2$ if we center them at e^G. Here disjointness holds because $\left\|e^G - e^{G'}\right\|_\infty = 1 \geq 1/2 + 1/2$ for $G \neq G'$.
Suppose D is a countable set in ℓ^∞. It cannot have points in all the balls above so some balls will have no points from D in them. The centre of any such ball is not a limit point of D. Or, suppose that D is dense in ℓ^∞. Then $\forall x \in \ell^\infty \setminus D, x$ is a limit point of D. Thus, $\forall s \in S, D \cap B(s, \frac{1}{2}) \neq \emptyset$. Since there are uncountable many such balls, D cannot be countable. Hence ℓ^∞ is not separable.

► $(\ell^p, \|\cdot\|_p)$ is complete.

3 Week 3 - Precompactness and linear bounded operators

Exercise 3.1. Are the following subsets of $C[0,1]$ precompact?

$$A = \left\{ \frac{\sin(nx)}{n} \right\}_{n \in \mathbb{N}} \qquad B = \{x^n\}_n$$

Solution 3.1.

Exercise 3.2. Let $\varphi : C[a,b] \to \mathbb{R}$ be given by

$$\varphi(f) = \int_a^b m(x) f(x) dx$$

where $m \in C[a,b]$ and $m > 0$. Show that φ is continuous and linear. Find $\|\varphi\|$ in terms of m.

Solution 3.2.

4 Week 4 - Riesz's ℓ^p representation theorem and compact operators

Exercise 4.1. Let $(\varphi_n)_n$ be a sequence of linear functionals on ℓ^p for $1 \leq p < +\infty$ given by

$$\varphi_n(x) = \sum_{k=1}^{n} x_k.$$

1. Prove that $\varphi_n \in (\ell^p)^*$ and find $\|\varphi_n\|$.
2. Is there a $\varphi \in (\ell^p)^*$ s.t. $\varphi_n \to \varphi$?
3. Answer the above 2 questions for $p = +\infty$.

Solution 4.1. 1. Note that for each $n \in \mathbb{N}$,

$$\varphi_n(x) = \sum_{k=1}^{\infty} x_k y_k^{(n)}$$

where $y^{(n)} = (\underbrace{1, 1, \dots, 1}_{n\ 1's}, 0, 0, \dots) \in \ell^q$. By Riesz's ℓ^p representation theorem, we have $\varphi_n \in (\ell^p)^*$ and

$$\|\varphi_n\| = \|y^{(n)}\|_q = \left(\sum_{k=1}^{\infty} \left(y_k^{(n)}\right)^q\right)^{\frac{1}{q}} = n^{\frac{1}{q}}.$$

2. No. Suppose there is a $\varphi \in (\ell^p)^*$ s.t. $\varphi_n \to \varphi$. Then by the inverse triangle inequality,

$$0 \leftarrow \|\varphi_n - \varphi\| \geq |\|\varphi_n\| - \|\varphi\|| \to +\infty.$$

Contradiction.
3.

Exercise 4.2. Let $X = L^2[0, 1]$, $Y = (C[0, 1], \|\cdot\|_\infty)$, and let $k \in C([0, 1] \times [0, 1])$. Let K be defined for $f \in X$ as

$$[Kf][x] = \int_0^1 k(x, y) f(y) dy.$$

1. Prove that $Kf \in C[0, 1]$.
2. Prove that $K \in \mathcal{L}(X, Y)$.
3. Show that K is a compact operator.

Solution 4.2.

5 Week 5 - Hilbert spaces and orthogonal complements

Exercise 5.1. Let $H = L^2[0,1]$ and $M = \left\{ f \in H : \int_0^1 f(x)dx = 0 \right\}$ be a subspace of H.

1. Prove that M is closed;
2. Find $M^\perp$;
3. Find the orthogonal projection P_M.

Solution 5.1.

Exercise 5.2. Let $H = L^2[-1,1]$ and $M = \{f \in H : f(-x) = -f(x)\}$ be a subspace of H (i.e. the subspace of all odd functions in H).

1. Prove that M is closed;
2. Find $M^\perp$;
3. Find the orthogonal projection P_M.

Solution 5.2.

6 Week 6 - Adjoint operators

Exercise 6.1. Let $H = L^2([0,1],\mathbb{C})$ and $A : H \to H$ be given by $\forall t \in [0,1]$ and $m \in C[0,1]$,

$$(Af)(t) = m(t)f(t).$$

Prove that A is bounded and find A^*.

Solution 6.1.

Exercise 6.2. Let $H = L^2([0,1],\mathbb{C})$ and $A : H \to H$ be given by $\forall t \in [0,1]$ and $k \in L^2([0,1] \times [0,1])$,

$$(Af)(t) = \int_0^1 k(t,\tau)f(\tau)d\tau.$$

Prove that A is bounded and find A^*.

Solution 6.2.

7 Week 7 - Open mapping theorem and closed graph theorem

Exercise 7.1. Let $X = (C[0,1], \|\cdot\|_\infty)$ and $Y = (C[0,1], \|\cdot\|_1)$. Show that

1. the operator $\mathrm{Id}_{X,Y}$ is bounded;
2. the operator $\mathrm{Id}_{Y,X}$ is not bounded;
3. the operator $\mathrm{Id}_{X,Y}$ is not open;
4. Y is not complete.

Solution 7.1.

Exercise 7.2. Let $a = (a_n)$ be a sequence. Suppose that $\forall b = (b_n) \subset \mathbf{c}_0$, $\sum_{n=1}^\infty a_n b_n$ exists. Show that $a \in \ell^1$.

Solution 7.2.

Exercise 7.3. Let X be Banach and $Y, Z \subset X$ be closed subspaces such that $\forall x \in X\ \exists! y \in Y, z \in Z, x = y + z$. Define $\prod : X \to Y$ as $\prod(x) = y$. Show that $\prod$ is bounded.

Solution 7.3.

Reacll that

Theorem (Open mapping theorem). *Suppose that*

- *X, Y - Banach*
- *$A \in \mathcal{L}(X, Y)$ - surjective*

Then A is open.

Theorem (Closed graph theorem). *Suppose that*

1. *X, Y - Banach*
2. *$A : X \to Y$ - linear operator*
3. *Γ_A - closed*

Then $A \in \mathcal{L}(X, Y)$.

8 Week 8 - Weak convergence and weak-$*$ convergence

Exercise 8.1. Suppose that $f_n \rightharpoonup f$ in $C[0,1]$. Show that $\forall x \in [0,1]$, $f_n(x) \to f(x)$.

Solution 8.1. $f_n \rightharpoonup f \iff \forall \varphi \in (C[0,1])^*, \varphi(f_n) \to \varphi(f)$. Let us choose a suitable functional in $(C[0,1])^*$ to achieve $f_n(x) \to f(x)$. Let $x \in [0,1]$ and define $\varphi_x \in (C[0,1])^*$ as $\forall f \in C[0,1], \varphi_x : f \mapsto f(x)$. Now, let us check that φ_x is indeed linear and continuous.
φ_x is linear: Let $f, g \in C[0,1]$ and $\alpha \in \mathbb{F}$. Then

$$\varphi_x(f + \alpha g) = (f + \alpha g)(x) = f(x) + \alpha g(x) = \varphi_x(f) + \alpha \varphi_x(g).$$

φ_x is continuous: $\forall f \in C[0,1]$,

$$|\varphi_x(f)| = |f(x)| \leq \max_{x \in [0,1]} |f(x)| = \|f\|_\infty.$$

Then $f_n \rightharpoonup f \implies \varphi_x(f_n) \to \varphi_x(f) \implies f_n(x) \to f(x)$.

Exercise 8.2. Let $\varphi_n \in (\ell^\infty)^*$ given by $\varphi_n(x) = x_n$. Show that φ_n does not converge weakly-$*$.

Solution 8.2. $\varphi_n \overset{*}{\rightharpoonup} \varphi$ for some $\varphi \iff \forall x \in \ell^\infty, \varphi_n(x) \to \varphi(x)$. It suffices to find an $x \in \ell^\infty$ s.t. $\varphi_n(x)$ diverges. Choose $x = (-1, 1, -1, 1 \dots)$ where $x_k = (-1)^k$. Then $\varphi_n(x) = x_n = (-1)^n$ and diverges.

Exercise 8.3. Let $\varphi_n \in (\ell^1)^*$ given by $\varphi_n(x) = \sum_{k=1}^n x_k$. Show that $\varphi_n \overset{*}{\rightharpoonup} \varphi$ with $\varphi(x) = \sum_{k=1}^\infty x_k$ but φ_n does not converge weakly.

Remark. Note that the weak convergence and weak-$*$ convergence does not coincide in $(\ell^1)^*$ since ℓ^1 is not reflexive.

Solution 8.3. $\varphi_n \overset{*}{\rightharpoonup} \varphi \iff \forall x \in \ell^1, \varphi_n(x) \to \varphi(x)$. For $x \in \ell^1$, we have

$$\sum_{k=1}^\infty |x_k| < +\infty \implies \sum_{k=1}^\infty x_k < +\infty \implies \sum_{k=1}^n x_k \overset{n\to\infty}{\to} \sum_{k=1}^\infty x_k \implies \varphi_n \overset{*}{\rightharpoonup} \varphi.$$

Let us show φ_n does not converge weakly. By Riesz's ℓ^p representation theorem, $\ell^\infty \cong (\ell^1)^*$. Thus,

$$\varphi_n \mapsto y^{(n)} = (\underbrace{1, 1, \dots, 1}_{n\ 1's}, 0, 0, \dots) \in \ell^\infty,$$

$$\varphi \mapsto y = (1, 1, \dots, 1, \dots) \in \ell^\infty.$$

Note that if φ_n converges weakly, then $\varphi_n \not\rightharpoonup$something not φ. This is because we have $\varphi_n \overset{*}{\rightharpoonup} \varphi$. Now, it suffices to show that $\varphi_n \not\rightharpoonup \varphi \iff \exists f \in (\ell^1)^{**}$ s.t. $f(\varphi_n) \not\to f(\varphi) \overset{\text{Riesz's}}{\iff} \Psi \in (\ell^\infty)^*$ s.t. $\Psi(y^{(n)}) \not\to \Psi(y)$ i.e. $\boxed{\text{WTS}}$ $\exists$ such a Ψ.
Idea: Let first define such a functional on $\mathbf{c}^*$ and then extend it to $(\ell^\infty)^*$ using Hahn-Banach theorem.
Define $\Psi : \mathbf{c} \to \mathbb{F}$ as

$$\Psi(y) = \lim_{n\to\infty} y_n.$$

Let us show that $\Psi_{\mathbf{c}} \in \mathbf{c}^*$.
$\Psi_{\mathbf{c}}$ is linear: Let $x, y \in \mathbf{c}$ and $\alpha \in \mathbb{F}$.

$$\Psi_{\mathbf{c}}(x + \alpha y) = \lim_{n\to\infty} (x + \alpha y)_n = \lim_{n\to\infty} (x_n + \alpha y_n) \overset{\text{AOL}}{=} \lim_{n\to\infty} x_n + \alpha \lim_{n\to\infty} y_n = \Psi_{\mathbf{c}}(x) + \alpha \Psi_{\mathbf{c}}(y).$$

$\Psi_{\mathbf{c}}$ is continuous: Let $y \in \mathbf{c}$,

$$|\Psi_{\mathbf{c}}(y)| = |\lim_{n\to\infty} y_n| \leq \|y\|_\infty.$$

By Hahn-Banach theorem, we can extend $\Psi_{\mathbf{c}} \in \mathbf{c}^*$ to $\Psi \in (\ell^\infty)^*$. Observe that $\Psi(y_n) = 0$ and $\Psi(y) = 1$. Hence $\Psi(y_n) \not\to \Psi(y)$, which is exactly what $\boxed{\text{WTS}}$.

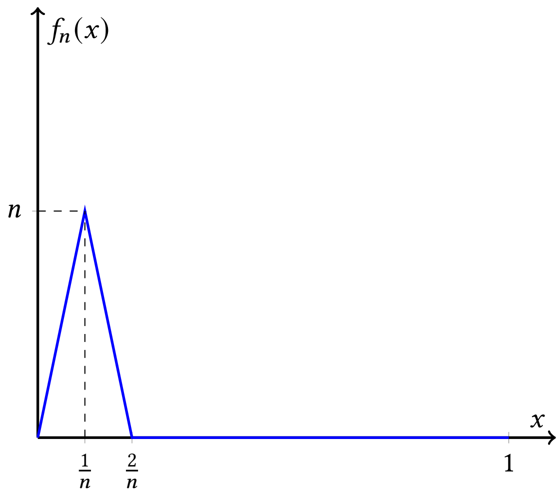
Exercise 8.4. Given an example of a sequence $(f_n)_n \subset C[0,1]$ and $f \in C[0,1]$ such that $\forall x \in [0,1], f_n(x) \to f(x)$ (i.e. $f_n \xrightharpoonup{*} f$) but $f_n \not\rightharpoonup f$.

Remark. Note that the weak convergence and weak-$*$ convergence does not coincide in $C[0,1]$ since $C[0,1]$ is <u>not</u> reflexive (see later in week 9 problem class).

Solution 8.4. $f_n \not\rightharpoonup f \iff \exists \varphi \in (C[0,1])^*$ s.t. $\varphi(f_n) \not\to \varphi(f)$. $\boxed{\text{WTS}}$ $\exists$ such a φ, a sequence $(f_n)_n$, and f. Define $\varphi : C[0,1] \to \mathbb{F}$ as

$$\varphi(f) = \int_0^1 f(x)dx.$$

Indeed, $\varphi \in (C[0,1])^*$ as <u>linearity</u> comes from the linearity of integrals and <u>continuity</u> comes from an obvious observation that $\|\varphi\| = 1$.
Let f_n be as follows:



Then $\varphi(f_n) = \int_0^1 f_n(x)dx = \frac{2}{n} \cdot n \cdot \frac{1}{2} = 1$ but $f_n \to f = 0$ pointwise (standard exercise in Analysis 4) and $\varphi(f) = \varphi(0) \overset{\text{linear}}{=} 0$. Hence we have $\varphi(f_n) \not\to \varphi(f)$, which is exactly $\boxed{\text{WTS}}$.

9 Week 9 - Reflexivity

Exercise 9.1. Prove that $\mathbf{c}_0$ is not reflexive.

Solution 9.1. Recall that a normed space X is reflexive $\iff$ the canonical embedding $J : X \to X^{**}$ is an isomorphisim $\implies X \cong X^{**}$. Thus, if we can show that $\mathbf{c}_0 \not\cong (\mathbf{c}_0)^{**}$, then we know $\mathbf{c}_0$ is not reflexive. $\boxed{\text{WTS}}$ $\mathbf{c}_0 \not\cong (\mathbf{c}_0)^{**}$. By Riesz's ℓ^p representation theorem, $(\mathbf{c}_0)^* \cong \ell^1$ and $\ell^1 \cong \ell^\infty \implies (\mathbf{c}_0)^{**} \cong \ell^\infty$. However, in week 2 problem class, we have proved that $\mathbf{c}_0 \subsetneq \ell^\infty$. Thus, $\mathbf{c}_0 \not\cong (\mathbf{c}_0)^{**}$, which is exactly $\boxed{\text{WTS}}$.

Exercise 9.2. Prove that $C[0,1]$ is not reflexive.

Solution 9.2. We know that $C[0,1]$ is separable and Banach. Recall that

$$X - \text{ Banach, separable, reflexive} \iff X^* - \text{ Banach, separable, reflexive}$$

It suffices to show that $(C[0,1])^*$ is <u>not</u> separable. Let us mimic the proof of the non-separability of ℓ^∞.
$\boxed{\text{WTS}}$ $(C[0,1])^*$ is <u>not</u> separable.
Let us first show that $\exists S \subset (C[0,1])^*$ s.t.

- S is uncountable;
- $(\exists C > 0)(\forall x, y \in S, x \neq y), \|x - y\|_\infty = C$ i.e. the distance between a distinct pair of points in S is the same and positive.

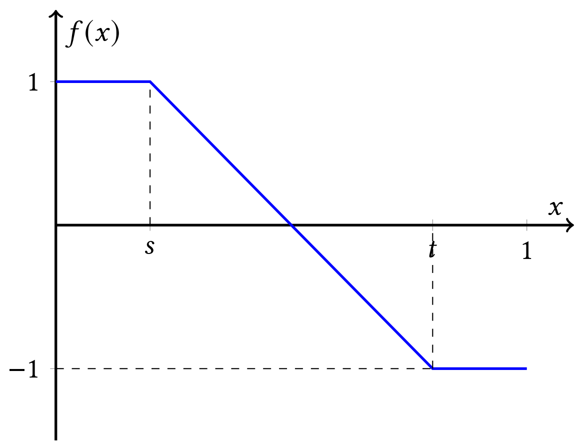
Let $t \in [0,1]$. Define $\varphi_t \in (C[0,1])^*$ as $\forall f \in C[0,1]$,

$$\varphi_t(f) = f(t).$$

Let $S := \{\varphi_t : t \in [0,1]\}$.
<u>S is uncountable:</u> By the construction of S, $|S| = |[0,1]|$. Since $[0,1]$ is uncountable, we have S is uncountable.
<u>'$\|x - y\|_\infty = C$':</u> Let $\varphi_s, \varphi_t \in S$ and $\varphi_s \neq \varphi_t$. Then $\forall f \in C[0,1]$,

$$|(\varphi_s - \varphi_t)(f)| = |\varphi_t(f) - \varphi_s(f)| = |f(t) - f(s)| \leq 2\|f\|_\infty \implies \|\varphi_s - \varphi_t\| \leq 2.$$

To see that $\|\varphi_s - \varphi_t\| = 2$, note that we can take an f as follows:



Then $|(\varphi_t - \varphi_s)(f)| = |f(t) - f(s)| = 2$ and $\|f\|_\infty = 1$. Thus, $\|\varphi_s - \varphi_t\| = 2$. Then the rest is exactly the same as the procedure for ℓ^∞ where we can show that any dense subset in $(C[0,1])^*$ cannot be countable. Hence $(C[0,1])^*$ cannot contain a countable dense subset, which is exactly $\boxed{\text{WTS}}$.

Exercise 9.3. Prove that $\mathbf{c}$ is not reflexive.

10 Week 10 - Q&A Session