Numerical Linear Algebra

Sheet 1 — MT24

Norms and SVD, up to lecture 4

Questions are split into three sections: Section A (basic, not marked, solutions provided): 1–3. Section B (will be marked): 4–8. Section C (new, solutions provided): 9.

1. Show that $\|x\|_\infty = \max_i |x_i|$ satisfies the axioms for a vector norm.

2. Show that if $\|x\|$ is a vector norm then $\sup_x \frac{\|A\,x\|}{\|x\|}$ satisfies the axioms for a matrix norm. Further show that
$$\|A\,B\| \le \|A\|\,\|B\|.$$

3. By considering the individual columns a_j of A and b_j of $B = Q\,A$, show that
$$\|Q\,A\|_{\mathrm{F}} = \|A\|_{\mathrm{F}}$$
if Q is an orthogonal matrix.

4. From the definition of the vector 1–norm show that
$$\|A\|_1 = \max_j \sum_i |a_{ij}|.$$

5. Full SVD. Prove the existence of $A = U \begin{bmatrix} \Sigma \\ 0_{(m-n)\times n} \end{bmatrix} V^*$, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices i.e., $U^*U = I_m$ and $V^*V = I_n$, and $\Sigma \in \mathbb{R}^{n\times n}$ is diagonal.

6. What is the SVD of a normal matrix A, with respect to the eigenvalues and eigenvectors? What if A is (real) symmetric? And unitary?

7. If $A \in \mathbb{R}^{n\times n}$ is nonsingular, what is the SVD of A^{-1} in terms of that of A?

8. Let B be a square $n \times n$ matrix. Bound the ith singular values of AB using $\sigma_i(A)$ and $\sigma_i(B)$: Specifically, prove that for each i,
$$\sigma_i(A)\sigma_n(B) \le \sigma_i(AB) \le \sigma_i(A)\sigma_1(B).$$

9. (optional; harder) Let $A \in \mathbb{R}^{m\times n}, m \ge n$ and $\sigma_1(A) \ge \sigma_2(A) \ge \cdots \ge \sigma_n(A) \ge 0$ be its singular values. Prove that for $k = 1, 2, \ldots, n$,
$$\sum_{i=1}^{k} \sigma_i(A) = \max_{Q^TQ=I_k, W^TW=I_k} \mathrm{trace}(Q^TAW).$$
($Q \in \mathbb{R}^{m\times k}, W \in \mathbb{R}^{n\times k}$ are orthonormal. Recall for an $k \times k$ matrix B, $\mathrm{trace}(B) = \sum_{i=1}^{k} B_{ii}$; a useful property is $\mathrm{trace}(CD) = \mathrm{trace}(DC)$ as long as CD is square.)

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