06051540-MATH70076-assessment-1

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Question 1

For $\xi \neq 0$:

We know from the cumulative distribution function that:

$$F(x;\sigma,\xi,u) = 1 - \left(1 + \frac{\xi(x-u)}{\sigma}\right)_+^{-1/\xi} \tag{1}$$

Rewriting to make x the subject:

$$\left(1 + \frac{\xi(x-u)}{\sigma}\right)_+^{-1/\xi} = (1-F) \tag{2}$$

$$1 = (1-F)\left(1 + \frac{\xi(x-u)}{\sigma}\right)_+^{1/\xi} \tag{3}$$

$$(1-F)^{-1} = \left(1 + \frac{\xi(x-u)}{\sigma}\right)_+^{1/\xi} \tag{4}$$

$$\tag{5}$$

For $\xi > 0$, we see that $\left(1 + \frac{\xi(x-u)}{\sigma}\right)^{1/\xi} > 0$, so we get:

$$(1-F)^{-\xi} = 1 + \frac{\xi(x-u)}{\sigma} \tag{6}$$

$$(1-F)^{-\xi} - 1 = \frac{\xi(x-u)}{\sigma} \tag{7}$$

$$\left((1-F)^{-\xi} - 1\right) \times \frac{\sigma}{\xi} = x - u \tag{8}$$

$$u + \left((1-F)^{-\xi} - 1\right) \times \frac{\sigma}{\xi} = x \tag{9}$$

$$\tag{10}$$

So $F_X^{-1} = u + \left((1-x)^{-\xi} - 1\right) \times \frac{\sigma}{\xi}$ But given our inputs of $F_X^{-1}(x)$ vary between 0 and 1, we can write that:

$$F_X^{-1}(x) = u + \left((x)^{-\xi} - 1\right) \times \frac{\sigma}{\xi} \tag{11}$$

For $\xi < 0$, we know that we have quickly decaying tails with finite upper endpoint. With $x > u$, this finite endpoint is met when

$$\begin{align} 1 + \frac{\xi(x-u)}{\sigma} &= 0 \tag{12}\\ \frac{\xi(x-u)}{\sigma} &= -1 \tag{13}\\ x - u &= -\frac{\sigma}{\xi} \tag{14}\\ x &= u - \frac{\sigma}{\xi} \tag{15}\\ & \tag{16} \end{align}$$

So for $x > u - \frac{\sigma}{\xi}$, produced by the inverse function above we discard the values of x.

For $\xi = 0$:

$$\begin{align} F = 1 - exp\left(-\frac{x-u}{\sigma}\right) \tag{17}\\ exp\left(-\frac{x-u}{\sigma}\right) &= 1 - F \tag{18}\\ -\frac{x-u}{\sigma} &= ln(1-F) \tag{19}\\ -x + u &= \sigma ln(1-F) \tag{20}\\ u - \sigma ln(1-F) = x \tag{21}\\ & \tag{22} \end{align}$$

So $F_X^{-1} = u - \sigma ln(1-x)$

But given our inputs of $F_X^{-1}(x)$ vary between 0 and 1, we can write that:

$$F_X^{-1}(x) = u - \sigma ln(x) \tag{23}$$

Question 2a

Defining the quantile function, and using the basis of the cdf from question 1, we get that:

```r
qgpd <- function (p, sigma=1, xi=0, u=0){
  if (p<0||p>1){
    return(warning("NaNs produced - p must be between 0 and 1"))
  }
  else if (sigma<=0){
    return(warning("NaNs produced - sigma must be greater than 0"))
  }
  else if (xi != 0){
    return(u + ((1-p)^(-xi)-1) * sigma / xi)
  } else {
    return(u - sigma * log(1-p))
  }
}
```

By default, the expected inputs for the function are sigma=1, xi=0, u=0. The code also prevents inputs where p values are less than 0, where p values exceed 1, or where sigma is less than or equal to 0.

The expected output is a real number greater than u that is unbounded if xi is greater than or equal to 0. The expected output is less than u - sigma/xi if xi is less than 0.

Regarding the behaviours of the quantile function: The larger the value of xi, the slower the tail decays. In this for xi >= 0 the functions output approaches infinity as p -> 1. For xi < 0, the functions output has a maximum at u - sigma/xi (when p -> 1). For larger values of sigma, the slower the tail decays.

Question 2b

```r
qgpd(0.5,2,-0.4,1.5)
```

```
[1] 2.710709
```

```r
qgpd(0.75,2,-0.4,1.5)
```

```
[1] 3.628254
```

```r
qgpd(0.99,2,-0.4,1.5)
```

```
[1] 5.707553
```

Question 3

Graphing our actual vs expected, we see below: