Solutions to Problem Set 9: Stochastic Differential Equations

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MIT Financial Mathematics course website: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/`

Problem sets: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/`

Problem set 9: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset9.pdf`

1. (a)

$$\frac{\partial X_t}{\partial t} = 0, \quad \frac{\partial X_t}{\partial B_t} = e^{B_t} = X_t, \quad \frac{\partial^2 X_t}{\partial B_t^2} = e^{B_t} = X_t$$

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 X_t}{\partial B_t^2}\right)dt + \frac{\partial X_t}{\partial B_t}dB_t = \left(0 + \frac{1}{2}X_t\right)dt + X_t dB_t = \frac{1}{2}X_t dt + X_t dB_t$$

(b)

$$\frac{\partial X_t}{\partial t} = \frac{-B_t}{(1+t)^2} = -\frac{1}{1+t}X_t, \quad \frac{\partial X_t}{\partial B_t} = \frac{1}{1+t}, \quad \frac{\partial^2 X_t}{\partial B_t^2} = 0$$

$$\begin{aligned} dX_t &= \left(\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 X_t}{\partial B_t^2}\right)dt + \frac{\partial X_t}{\partial B_t}dB_t \\ &= \left(-\frac{1}{1+t}X_t + \frac{1}{2}\times 0\right)dt + \frac{1}{1+t}dB_t \\ &= -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t \end{aligned}$$

(c)

$$\frac{\partial X_t}{\partial t} = 0, \quad \frac{\partial X_t}{\partial B_t} = \cos B_t = \sqrt{1-\sin^2 B_t} = \sqrt{1-X_t^2}, \quad \frac{\partial^2 X_t}{\partial B_t^2} = -\sin B_t = -X_t$$

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 X_t}{\partial B_t^2}\right)dt + \frac{\partial X_t}{\partial B_t}dB_t$$

$$
\begin{aligned}
&= \left(0 + \frac{1}{2}(-X_t)\right) dt + \sqrt{1 - X_t^2} dB_t \\
&= -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dB_t
\end{aligned}
$$

2.
$$
\begin{aligned}
\frac{\partial X_t}{\partial t} &= 0 \\
\frac{\partial X_t}{\partial B_t} &= 3\left(a^{\frac{1}{3}} + \frac{1}{3}B_t\right)^2 \frac{1}{3} = \left(a^{\frac{1}{3}} + \frac{1}{3}B_t\right)^2 = X_t^{\frac{2}{3}} \\
\frac{\partial^2 X_t}{\partial B_t^2} &= 2\left(a^{\frac{1}{3}} + \frac{1}{3}B_t\right)\frac{1}{3} = \frac{2}{3}\left(a^{\frac{1}{3}} + \frac{1}{3}B_t\right) = \frac{2}{3}X_t^{\frac{1}{3}}
\end{aligned}
$$

$$
dX_t = \left(\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 X_t}{\partial B_t^2}\right) dt + \frac{\partial X_t}{\partial B_t} dB_t = \left(0 + \frac{1}{2} \times \frac{2}{3} X_t^{\frac{1}{3}}\right) dt + X_t^{\frac{2}{3}} dB_t = \frac{1}{3} X_t^{\frac{1}{3}} dt + X_t^{\frac{2}{3}} dB_t
$$

$$
X_0 = \left(a^{\frac{1}{3}} + \frac{1}{3}B_0\right)^3 = \left(a^{\frac{1}{3}} + \frac{1}{3} \times 0\right)^3 = a
$$

3.

$$
\begin{aligned}
dR(t) &= -\beta e^{-\beta t} R(0) dt - \beta \frac{\alpha}{\beta} e^{-\beta t} dt - \beta \sigma e^{-\beta t} \left(\int_0^t e^{\beta s} dB_s\right) dt + \sigma e^{-\beta t} e^{\beta t} dB_t \\
&= \left(-\beta e^{-\beta t} R(0) - \beta \frac{\alpha}{\beta}\left(-e^{-\beta t}\right) - \beta \sigma e^{-\beta t} \left(\int_0^t e^{\beta s} dB_s\right)\right) dt + \sigma dB_t \\
&= \left(\alpha - \beta e^{-\beta t} R(0) - \beta \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) - \beta \sigma e^{-\beta t} \left(\int_0^t e^{\beta s} dB_s\right)\right) dt + \sigma dB_t \\
&= \left(\alpha - \beta\left(e^{-\beta t} R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \left(\int_0^t e^{\beta s} dB_s\right)\right)\right) dt + \sigma dB_t \\
&= (\alpha - \beta R(t)) dt + \sigma dB_t
\end{aligned}
$$