Solutions to Problem Set 6: Time Series II and Portfolio Theory

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MIT Financial Mathematics course website: https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/

Problem sets: https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/

Problem set 6: https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset6.pdf

1. (a)

$$
\begin{aligned}
\mu &= \begin{pmatrix} E[X_{1,t}] \\ E[X_{2,t}] \end{pmatrix} \\
&= \begin{pmatrix} 0.3 + 0.8E[X_{1,t-1}] + E[\epsilon_{1,t}] \\ 0.2 + 0.6E[X_{1,t-1}] + 0.4E[X_{2,t-1}] + E[\epsilon_{2,t}] \end{pmatrix} \\
&= \begin{pmatrix} 0.3 + 0.8E[X_{1,t}] \\ 0.2 + 0.6E[X_{1,t}] + 0.4E[X_{2,t}] \end{pmatrix} \quad (\because X_t \text{ is covariance stationary}) \\
&= \begin{pmatrix} \frac{0.3}{1-0.8} \\ 0.2 + 0.6E[X_{1,t}] + 0.4E[X_{2,t}] \end{pmatrix} \\
&= \begin{pmatrix} 1.5 \\ 0.2 + 0.6 \times 1.5 + 0.4E[X_{2,t}] \end{pmatrix} \\
&= \begin{pmatrix} 1.5 \\ 1.1 + 0.4E[X_{2,t}] \end{pmatrix} \\
&= \begin{pmatrix} 1.5 \\ \frac{1.1}{1-0.4} \end{pmatrix} \\
&= \begin{pmatrix} 1.5 \\ \frac{11}{6} \end{pmatrix}
\end{aligned}
$$

(b)

$$
\begin{aligned}
Var(X_{1,t}) &= E\left[(0.3 + 0.8X_{1,t-1} + \epsilon_{1,t})^2\right] - E^2[X_{1,t}] \\
&= 0.09 + 2 \times 0.24E[X_{1,t-1}] + 0.64E\left[X_{1,t-1}^2\right] + E\left[\epsilon_{1,t}^2\right] - E^2[X_{1,t}]
\end{aligned}
$$

$$
\begin{aligned}
&= 1.56 + 0.64E\left[X_{1,t-1}^2\right] \\
&= 1.56 + 0.64\left(Var(X_{1,t-1}) + E^2[X_{1,t-1}]\right) \\
&= 3 + 0.64Var(X_{1,t}) \quad (\because X_t \text{ is covariance stationary}) \\
&= \frac{3}{1-0.64} \\
&= \frac{25}{3}
\end{aligned}
$$

$$
\begin{aligned}
Cov(X_{1,t}X_{2,t}) &= E[(0.3 + 0.8X_{1,t-1} + \epsilon_{1,t})(0.2 + 0.6X_{1,t-1} + 0.4X_{2,t-1} + \epsilon_{2,t})] \\
&\quad - E[X_{1,t}]E[X_{2,t}] \\
&= 0.06 + 0.18E[X_{1,t-1}] + 0.12E[X_{2,t-1}] \\
&\quad + 0.16E[X_{1,t-1}] + 0.48E\left[X_{1,t-1}^2\right] + 0.32E[X_{1,t-1}X_{2,t-1}] \\
&\quad + E[\epsilon_{1,t-1}\epsilon_{2,t-1}] \\
&\quad - E[X_{1,t}]E[X_{2,t}] \\
&= -2.96 + 0.48E\left[X_{1,t-1}^2\right] + 0.32E[X_{1,t-1}X_{2,t-1}] \\
&= -2.96 \\
&\quad + 0.48\left(Var(X_{1,t-1}) + E^2[X_{1,t-1}]\right) \\
&\quad + 0.32(Cov(X_{1,t-1}X_{2,t-1}) + E[X_{1,t-1}]E[X_{2,t-1}]) \\
&= 3 + 0.32Cov(X_{1,t}X_{2,t}) \quad (\because X_t \text{ is covariance stationary}) \\
&= \frac{3}{1-0.32} \\
&= \frac{75}{17}
\end{aligned}
$$

$$
\begin{aligned}
Var(X_{2,t}) &= E\left[(0.2 + 0.6X_{1,t-1} + 0.4X_{2,t-1} + \epsilon_{2,t})^2\right] - E^2[X_{2,t}] \\
&= 0.04 + 2 \times 0.12E[X_{1,t-1}] + 2 \times 0.08E[X_{2,t-1}] \\
&\quad + 0.36E\left[X_{1,t-1}^2\right] + 2 \times 0.24E[X_{1,t-1}X_{2,t-1}] \\
&\quad + 0.16E\left[X_{2,t-1}^2\right] \\
&\quad + E\left[\epsilon_{2,t}^2\right] \\
&\quad - E^2[X_{2,t}] \\
&= \frac{299}{900} + 0.36E\left[X_{1,t-1}^2\right] + 0.48E[X_{1,t-1}X_{2,t-1}] + 0.16E\left[X_{2,t-1}^2\right] \\
&= \frac{299}{900} \\
&\quad + 0.36\left(Var(X_{1,t-1}) + E^2[X_{1,t-1}]\right) \\
&\quad + 0.48\left(Cov(X_{1,t-1}X_{2,t-1}) + E[X_{1,t-1}]E[X_{2,t-1}]\right) \\
&\quad + 0.16\left(Var(X_{2,t-1}) + E^2[X_{2,t-1}]\right) \\
&= \frac{138}{17} + 0.16Var(X_{2,t}) \quad (\because X_t \text{ is covariance stationary})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{138}{17(1-0.16)} \\
&= \frac{1150}{119}
\end{aligned}
$$

$$
\therefore \Gamma(0) = \begin{pmatrix} Var[X_{1,t}] & Cov[X_{1,t}, X_{2,t}] \\ Cov[X_{2,t}, X_{1,t}] & Var[X_{2,t}] \end{pmatrix} = \begin{pmatrix} \frac{25}{3} & \frac{75}{17} \\ \frac{75}{17} & \frac{1150}{119} \end{pmatrix}
$$

(c)

$$
\begin{aligned}
Cov(X_{1,t}, X_{1,t-1}) &= E[(0.3 + 0.8X_{1,t-1} + \epsilon_{1,t})X_{1,t-1}] - E[X_{1,t}]E[X_{1,t-1}] \\
&= -1.8 + 0.8E\left[X_{1,t-1}^2\right] \\
&= -1.8 + 0.8\left(Var(X_{1,t-1}) + E^2[X_{1,t-1}]\right) \\
&= 0.8Var(X_{1,t-1}) \\
&= \frac{20}{3}
\end{aligned}
$$

$$
\begin{aligned}
Cov(X_{1,t}, X_{2,t-1}) &= E[(0.3 + 0.8X_{1,t-1} + \epsilon_{1,t})X_{2,t-1}] - E[X_{1,t}]E[X_{2,t-1}] \\
&= -2.2 + 0.8E[X_{1,t-1}X_{2,t-1}] \\
&= -2.2 + 0.8(Cov(X_{1,t-1}X_{2,t-1}) + E[X_{1,t-1}]E[X_{2,t-1}]) \\
&= 0.8Cov(X_{1,t-1}X_{2,t-1}) \\
&= \frac{60}{17}
\end{aligned}
$$

$$
\begin{aligned}
Cov(X_{2,t}, X_{1,t-1}) &= E[(0.2 + 0.6X_{1,t-1} + 0.4X_{2,t-1} + \epsilon_{2,t})X_{1,t-1}] - E[X_{2,t}]E[X_{1,t-1}] \\
&= -2.45 + 0.6E\left[X_{1,t-1}^2\right] + 0.4E[X_{2,t-1}X_{1,t-1}] \\
&= -2.45 \\
&\quad + 0.6\left(Var(X_{1,t-1}) + E^2[X_{1,t-1}]\right) \\
&\quad + 0.4(Cov(X_{2,t-1}, X_{1,t-1}) + E[X_{2,t-1}]E[X_{1,t-1}]) \\
&= 0.6Var(X_{1,t-1}) + 0.4Cov(X_{2,t-1}, X_{1,t-1}) \\
&= \frac{115}{17}
\end{aligned}
$$

$$
\begin{aligned}
Cov(X_{2,t}, X_{2,t-1}) &= E[(0.2 + 0.6X_{1,t-1} + 0.4X_{2,t-1} + \epsilon_{2,t})X_{2,t-1}] - E[X_{2,t}]E[X_{2,t-1}] \\
&= \frac{-539}{180} + 0.6E[X_{1,t-1}X_{2,t-1}] + 0.4E\left[X_{2,t-1}^2\right] \\
&= \frac{-539}{180} \\
&\quad + 0.6(Cov(X_{1,t-1}, X_{2,t-1}) + E[X_{1,t-1}]E[X_{2,t-1}]) \\
&\quad + 0.4\left(Var(X_{2,t-1}) + E^2[X_{2,t-1}]\right) \\
&= 0.6Cov(X_{1,t-1}, X_{2,t-1}) + 0.4Var(X_{2,t-1})
\end{aligned}
$$

$$= \frac{775}{119}$$

$$\therefore \Gamma(1) = \begin{pmatrix} Cov(X_{1,t}, X_{1,t-1}) & Cov(X_{1,t}, X_{2,t-1}) \\ Cov(X_{2,t}, X_{1,t-1}) & Cov(X_{2,t}, X_{2,t-1}) \end{pmatrix} = \begin{pmatrix} \frac{20}{3} & \frac{60}{17} \\ \frac{115}{17} & \frac{775}{119} \end{pmatrix}$$

(d) For $h \geq 1$,

$$\begin{aligned} Cov(X_{1,t}, X_{1,t-h}) &= E[(0.3 + 0.8X_{1,t-1} + \epsilon_{1,t})X_{1,t-h}] - E[X_{1,t}]E[X_{1,t-h}] \\ &= -1.8 + 0.8E[X_{1,t-1}X_{1,t-h}] \\ &= -1.8 + 0.8(Cov(X_{1,t-1}, X_{1,t-h}) + E[X_{1,t-1}]E[X_{1,t-h}]) \\ &= 0.8Cov(X_{1,t}, X_{1,t-(h-1)}) \quad (\because X_t \text{ is covariance stationary}) \\ &= 0.8^h Var(X_{1,t}) \\ &= \frac{0.8^h 25}{3} \end{aligned}$$

$$\begin{aligned} Cov(X_{1,t}, X_{2,t-h}) &= E[(0.3 + 0.8X_{1,t-1} + \epsilon_{1,t})X_{2,t-h}] - E[X_{1,t}]E[X_{2,t-h}] \\ &= -2.2 + 0.8E[X_{1,t-1}X_{2,t-h}] \\ &= -2.2 + 0.8(Cov(X_{1,t-1}X_{2,t-h}) + E[X_{1,t-1}]E[X_{2,t-h}]) \\ &= 0.8Cov(X_{1,t}X_{2,t-(h-1)}) \quad (\because X_t \text{ is covariance stationary}) \\ &= 0.8^h Cov(X_{1,t}X_{2,t}) \\ &= \frac{0.8^h 75}{17} \end{aligned}$$

$$\begin{aligned} Cov(X_{2,t}, X_{1,t-h}) &= E[(0.2 + 0.6X_{1,t-1} + 0.4X_{2,t-1} + \epsilon_{2,t})X_{1,t-h}] - E[X_{2,t}]E[X_{1,t-h}] \\ &= -2.45 + 0.6E[X_{1,t-1}X_{1,t-h}] + 0.4E[X_{2,t-1}X_{1,t-h}] \\ &= -2.45 \\ &\quad + 0.6(Cov(X_{1,t-1}, X_{1,t-h}) + E[X_{1,t-1}]E[X_{1,t-h}]) \\ &\quad + 0.4(Cov(X_{2,t-1}, X_{1,t-h}) + E[X_{2,t-1}]E[X_{1,t-h}]) \\ &= 0.6Cov(X_{1,t}, X_{1,t-(h-1)}) + 0.4Cov(X_{2,t}, X_{1,t-(h-1)}) \quad (\because X_t \text{ is WSS}) \\ &= 0.8^{h-1}5 + 0.4Cov(X_{2,t}, X_{1,t-(h-1)}) \\ &= 0.8^{h-1}5 \sum_{i=0}^{h-1} 0.4^i + 0.4^h Cov(X_{2,t}, X_{1,t}) \\ &= 0.8^{h-1}5 \frac{1 - 0.4^h}{1 - 0.4} + 0.4^h \frac{75}{17} \\ &= \frac{0.8^{h-1}25\left(1 - 0.4^h\right)}{3} + \frac{0.4^h 75}{17} \end{aligned}$$

$$Cov(X_{2,t}, X_{2,t-h}) = E[(0.2 + 0.6X_{1,t-1} + 0.4X_{2,t-1} + \epsilon_{2,t})X_{2,t-h}] - E[X_{2,t}]E[X_{2,t-h}]$$

$$
\begin{aligned}
&= \frac{-539}{180} + 0.6E[X_{1,t-1}X_{2,t-h}] + 0.4E[X_{2,t-1}X_{2,t-h}] \\
&= \frac{-539}{180} \\
&\quad + 0.6(Cov(X_{1,t-1}, X_{2,t-h}) + E[X_{1,t-1}]E[X_{2,t-h}]) \\
&\quad + 0.4(Cov(X_{2,t-1}, X_{2,t-h}) + E[X_{2,t-1}]E[X_{2,t-h}]) \\
&= 0.6Cov(X_{1,t}, X_{2,t-(h-1)}) + 0.4Cov(X_{2,t}, X_{2,t-(h-1)}) \quad (\because X_t \text{ is WSS}) \\
&= \frac{0.8^{h-1}45}{17} + 0.4Cov(X_{2,t}, X_{2,t-(h-1)}) \\
&= \frac{0.8^{h-1}45}{17} \sum_{i=0}^{h-1} 0.4^i + 0.4^h Cov(X_{2,t}, X_{2,t}) \\
&= \frac{0.8^{h-1}45}{17} \left(\frac{1-0.4^h}{1-0.4} \right) + 0.4^h \frac{1150}{119} \\
&= \frac{0.8^{h-1}75\left(1-0.4^h\right)}{17} + \frac{0.4^h 1150}{119}
\end{aligned}
$$

$$
\therefore \Gamma(h) = \begin{pmatrix} \frac{0.8^h 25}{3} & \frac{0.8^h 75}{17} \\ \frac{0.8^{h-1}25\left(1-0.4^h\right)}{3} + \frac{0.4^h 75}{17} & \frac{0.8^{h-1}75\left(1-0.4^h\right)}{17} + \frac{0.4^h 1150}{119} \end{pmatrix} \quad \forall h \in \mathbb{Z}^+
$$

2. (a)

$$
\begin{aligned}
x_t &= w_t - v_t \\
&= \left(5(1-0.5L)^{-1} - 4(1-0.4L)^{-1}\right)\epsilon_t \\
&= \left(5\sum_{i=0}^{\infty} 0.5^i L^i - 4\sum_{i=0}^{\infty} 0.4^i L^i\right)\epsilon_t \\
&= \epsilon_t + \sum_{i=1}^{\infty}\left(5 \times 0.5^i - 4 \times 0.4^i\right)\epsilon_{t-i}
\end{aligned}
$$

$$
\therefore \theta_i = 5 \times 0.5^i - 4 \times 0.4^i
$$

(b)

$$
\begin{aligned}
x_t &= \left(5(1-0.5L)^{-1} - 4(1-0.4L)^{-1}\right)\epsilon_t \\
(1-0.5L)(1-0.4L)x_t &= (5(1-0.4L) - 4(1-0.5L))\epsilon_t \\
\left(1-0.9L+0.2L^2\right)x_t &= \epsilon_t \\
x_t &= \epsilon_t + 0.9x_{t-1} - 0.2x_{t-2}
\end{aligned}
$$

$\therefore \{x_t\}$ is an $AR(2)$ process.

(c) $\phi_1 = 0.9$ and $\phi_2 = -0.2$.

(d) Let $X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$ be a stationary $AR(2)$ process and z_1, z_2 the roots of $1 - \phi_1 z - \phi_2 z^2$. Because X_t is stationary,

$|z_1|, |z_2| > 1$.

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t = \frac{\phi_0 + \epsilon_t}{1 - \phi_1 L - \phi_2 L^2} = \frac{\phi_0}{1 - \phi_1 - \phi_2} + \frac{\epsilon_t}{\left(1 - z_1^{-1} L\right)\left(1 - z_2^{-1} L\right)}$$

Either $z_1 \neq z_2$ and $z_1, z_2 \in \mathbb{R}$, $z_1 = z_2 \in \mathbb{R}$ or $z_1 = \overline{z_2}$ and $z_1, z_2 \notin \mathbb{R}$.
Case $z_1 \neq z_2$ and $z_1, z_2 \in \mathbb{R}$:

$$\begin{aligned}
X_t &= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \frac{\epsilon_t}{\left(1 - z_1^{-1} L\right)\left(1 - z_2^{-1} L\right)} \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \frac{\left(z_1^{-1} - z_2^{-1}\right)\epsilon_t}{\left(1 - z_1^{-1} L\right)\left(1 - z_2^{-1} L\right)\left(z_1^{-1} - z_2^{-1}\right)} \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \frac{\left(z_1^{-1} - z_2^{-1} + z_1^{-1} z_2^{-1} L - z_1^{-1} z_2^{-1} L\right)\epsilon_t}{\left(1 - z_1^{-1} L\right)\left(1 - z_2^{-1} L\right)\left(z_1^{-1} - z_2^{-1}\right)} \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \frac{\left(z_1^{-1}\left(1 - z_2^{-1} L\right) - z_2^{-1}\left(1 - z_1^{-1} L\right)\right)\epsilon_t}{\left(1 - z_1^{-1} L\right)\left(1 - z_2^{-1} L\right)\left(z_1^{-1} - z_2^{-1}\right)} \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \left(\frac{z_2}{\left(1 - z_1^{-1} L\right)\left(z_2 - z_1\right)} - \frac{z_1}{\left(1 - z_2^{-1} L\right)\left(z_2 - z_1\right)}\right)\epsilon_t \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \frac{\left(z_2 \sum_{i=0}^{\infty} z_1^{-i} L^i - z_1 \sum_{i=0}^{\infty} z_2^{-i} L^i\right)\epsilon_t}{z_2 - z_1} \qquad \left(\because \left|z_1^{-1}\right|, \left|z_2^{-1}\right| < 1\right) \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \epsilon_t + \sum_{i=1}^{\infty} \frac{z_2 \epsilon_{t-i}}{(z_2 - z_1) z_1^i} - \sum_{i=1}^{\infty} \frac{z_1 \epsilon_{t-i}}{(z_2 - z_1) z_2^i}
\end{aligned}$$

Case $z_1 = z_2 \in \mathbb{R}$:

$$\begin{aligned}
X_t &= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \frac{\epsilon_t}{\left(1 - z_1^{-1} L\right)^2} \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \left(\sum_{i=0}^{\infty} z_1^{-i} L^i\right)^2 \epsilon_t \qquad \left(\because \left|z_1^{-1}\right| < 1\right) \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \sum_{i=0}^{\infty} \sum_{j=0}^{i} z_1^{-j} z_1^{-i+j} L^j L^{i-j} \epsilon_t \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \sum_{i=0}^{\infty} (i+1) z_1^{-i} L^i \epsilon_t \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \epsilon_t + \sum_{i=1}^{\infty} \frac{(i+1)\epsilon_{t-i}}{z_1^i} \\
&= \frac{\phi_0}{1 - \phi_1 - \phi_2} + \epsilon_t + \sum_{i=1}^{\infty} \frac{(i+2)\epsilon_{t-i}}{z_1^i} - \sum_{i=1}^{\infty} \frac{\epsilon_{t-i}}{z_1^i}
\end{aligned}$$

Case $z_1 = \overline{z_2}$ and $z_1, z_2 \notin \mathbb{R}$:

$$\begin{aligned}
X_t &= \frac{\phi_0}{1-\phi_1-\phi_2} + \frac{\epsilon_t}{\left(1-z_1^{-1}L\right)\left(1-\overline{z_1}^{-1}L\right)} \\
&= \frac{\phi_0}{1-\phi_1-\phi_2} + \left(\sum_{j=0}^{\infty} z_1^{-j}L^j\right)\left(\sum_{j=0}^{\infty} \overline{z_1}^{-j}L^j\right)\epsilon_t \quad \left(\because \left|z_1^{-1}\right|, \left|\overline{z_1}^{-1}\right| < 1\right) \\
&= \frac{\phi_0}{1-\phi_1-\phi_2} + \sum_{j=0}^{\infty}\sum_{k=0}^{j} z_1^{-k}\overline{z_1}^{k-j}L^j\epsilon_t \\
&= \frac{\phi_0}{1-\phi_1-\phi_2} + \sum_{j=0}^{\infty} \frac{\overline{z_1}^{-j}\left(1-z_1^{-j-1}\overline{z_1}^{j+1}\right)\epsilon_{t-j}}{1-z_1^{-1}\overline{z_1}} \\
&= \frac{\phi_0}{1-\phi_1-\phi_2} + \sum_{j=0}^{\infty} \frac{\left(\overline{z_1^{-j-1}} - z_1^{-j-1}\right)\epsilon_{t-j}}{\overline{z_1^{-1}} - z_1^{-1}} \\
&= \frac{\phi_0}{1-\phi_1-\phi_2} + \sum_{j=0}^{\infty} \frac{-2\mathrm{Im}\left(z_1^{-j-1}\right)i\epsilon_{t-j}}{-2\mathrm{Im}\left(z_1^{-1}\right)i} \\
&= \frac{\phi_0}{1-\phi_1-\phi_2} + \epsilon_t + \sum_{j=1}^{\infty} \frac{\mathrm{Im}\left(z_1^{-j-1}\right)\epsilon_{t-j}}{\mathrm{Im}\left(z_1^{-1}\right)} \\
&= \frac{\phi_0}{1-\phi_1-\phi_2} + \epsilon_t + \sum_{j=1}^{\infty} \frac{2\mathrm{Im}\left(z_1^{-j-1}\right)\epsilon_{t-j}}{\mathrm{Im}\left(z_1^{-1}\right)} - \sum_{j=1}^{\infty} \frac{\mathrm{Im}\left(z_1^{-j-1}\right)\epsilon_{t-j}}{\mathrm{Im}\left(z_1^{-1}\right)}
\end{aligned}$$

$\therefore$ Any stationary $AR(2)$ process can be expressed as the difference of two moving average processes on the same innovation process $\{\epsilon_t\}$.

3. (a)

$$\begin{aligned}
Var(R_w) &= Var(w_1R_1 + w_2R_2) \\
&= w_1^2Var(R_1) + 2w_1w_2Cov(R_1,R_2) + w_2^2Var(R_2) \\
&= w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2 \\
&\leq \left(w_1^2 + 2w_1w_2 + w_2^2\right)\max\left(\sigma_1^2,\sigma_2^2\right) \quad \left(\because \rho\sigma_1\sigma_2 \leq \max\left(\sigma_1^2,\sigma_2^2\right), w_1, w_2 \geq 0\right) \\
&= (w_1+w_2)^2\max\left(\sigma_1^2,\sigma_2^2\right) \\
&= \max\left(\sigma_1^2,\sigma_2^2\right)
\end{aligned}$$

$\therefore Var(R_w) \leq \max\left(\sigma_1^2,\sigma_2^2\right)$ for all portfolios w.

(b)

$$Var(R_w) = w_1^2\sigma_1^2 + 2w_1w_20\sigma_1^2 + w_2^2\sigma_1^2 = \left(w_1^2 + (1-w_1)^2\right)\sigma_1^2 = \left(1-2w_1+2w_1^2\right)\sigma_1^2$$

The minimum of $\left(1-2w_1+2w_1^2\right)\sigma_1^2$ is at $w_1 = \frac{-(-2)}{2\times 2} = \frac{1}{2}$.

$$\therefore w^* = \begin{pmatrix} \frac{1}{2} \\ 1-\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\therefore Var(R_{w^*}) = \left(1 - 2\times\frac{1}{2} + 2\left(\frac{1}{2}\right)^2\right)\sigma_1^2 = \frac{\sigma_1^2}{2}$$

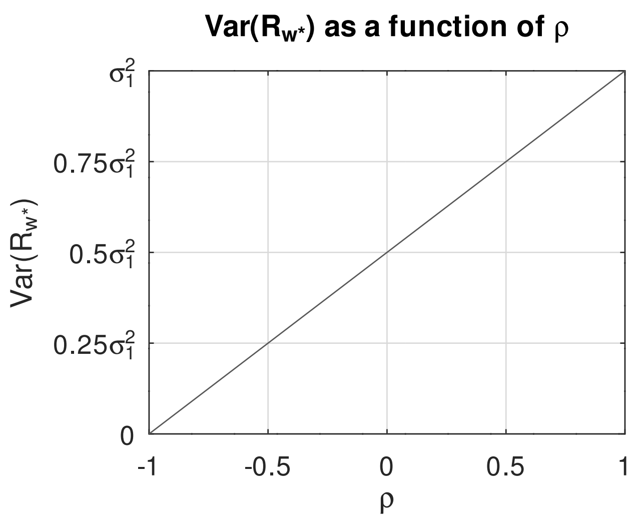
(c)

$$\begin{aligned} Var(R_w) &= w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1^2 + w_2^2\sigma_1^2 \\ &= \left(w_1^2 + 2w_1(1-w_1)\rho + (1-w_1)^2\right)\sigma_1^2 \\ &= \left(2(1-\rho)w_1^2 + 2(\rho-1)w_1 + 1\right)\sigma_1^2 \end{aligned}$$

The minimum of $Var(R_w)$ is at $w_1 = \frac{-2(\rho-1)}{2\times 2(1-\rho)} = \frac{1}{2}$.

$$\therefore w^* = \begin{pmatrix} \frac{1}{2} \\ 1-\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\therefore Var(R_{w^*}) = \left(2(1-\rho)\left(\frac{1}{2}\right)^2 + 2(\rho-1)\frac{1}{2} + 1\right)\sigma_1^2 = \frac{(\rho+1)\sigma_1^2}{2}$$



4. (a)

$$\begin{aligned} Var(R_w) &= Var\left(\sum_{i=0}^{m} w_iR_i\right) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_iw_jCov(R_i, R_j) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i w_j \rho_{i,j}\sigma_i\sigma_j \\
&\leq \sum_{i=1}^{m}\sum_{j=1}^{m} w_i w_j \max\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\right) \quad \left(\because \rho_{i,j}\sigma_i\sigma_j \leq \max\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\right) \forall 1 \leq i, j \leq m\right) \\
&= \sum_{i=1}^{m} w_i \max\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\right) \\
&= \max\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\right)
\end{aligned}
$$

$\therefore Var(R_w) \leq \max\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\right)$ for all portfolios w.

(b)

$$
Var(R_w) = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i w_j \rho_{i,j}\sigma_i\sigma_j = \sum_{i=1}^{m} w_i^2\sigma_1^2
$$

Let $L(w_1, w_2, \ldots, w_m, \lambda) = \sum_{i=1}^{m} w_i^2\sigma_1^2 - \lambda\left(\sum_{i=1}^{m} w_i - 1\right)$.

$$
\begin{aligned}
\nabla L(w_1, w_2, \ldots, w_m, \lambda) = \begin{pmatrix} 2w_1\sigma_1^2 - \lambda \\ 2w_2\sigma_1^2 - \lambda \\ \vdots \\ 2w_m\sigma_1^2 - \lambda \\ 1 - \sum_{i=1}^{m} w_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} &\implies w = \begin{pmatrix} \frac{\lambda}{2\sigma_1^2} \\ \frac{\lambda}{2\sigma_1^2} \\ \vdots \\ \frac{\lambda}{2\sigma_1^2} \end{pmatrix} \\
&\implies 1 - \sum_{i=1}^{m} \frac{\lambda}{2\sigma_1^2} = 0 \\
&\implies \lambda = \frac{2\sigma_1^2}{m} \\
&\implies w = \begin{pmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix}
\end{aligned}
$$

Since $\sigma_1^2 \geq 0$, the minimum of $Var(R_w)$ is at $w = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \end{pmatrix}^T$.

$$
\therefore w^* = \begin{pmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix}
$$

$$
\therefore Var(R_{w^*}) = \sum_{i=1}^{m} \left(\frac{1}{m}\right)^2 \sigma_1^2 = \frac{\sigma_1^2}{m}
$$

$$
\therefore \lim_{m\to\infty} Var(R_{w^*}) = \lim_{m\to\infty} \frac{\sigma_1^2}{m} = 0
$$

(c)

$$\mathbf{1}_m^T \Sigma \mathbf{1}_m = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \rho\sigma_1^2 & \cdots & \rho\sigma_1^2 \\ \rho\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho\sigma_1^2 & \rho\sigma_1^2 & \cdots & \sigma_1^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} (\rho(m-1)+1)\sigma_1^2 \\ (\rho(m-1)+1)\sigma_1^2 \\ \vdots \\ (\rho(m-1)+1)\sigma_1^2 \end{pmatrix}$$

$$= m(\rho(m-1)+1)\sigma_1^2$$

Since Σ is positive semi-definite,

$$\begin{aligned} m(\rho(m-1)+1)\sigma_1^2 &\geq 0 \\ m(\rho(m-1)+1) &\geq 0 \\ \rho(m-1)+1 &\geq 0 \\ \rho(m-1) &\geq -1 \end{aligned}$$

$$\therefore \rho \geq \frac{-1}{m-1}$$

$$Var(R_w) = \sum_{i=1}^m \sum_{j=1}^m w_i w_j \rho_{i,j} \sigma_i \sigma_j = \left(\sum_{i=1}^m w_i^2 (1-\rho) + \sum_{i=1}^m \sum_{j=1}^m w_i w_j \rho \right) \sigma_1^2 = \left(\sum_{i=1}^m w_i^2 (1-\rho) + \rho \right) \sigma_1^2$$

Let $L(w_1, w_2, \ldots, w_m, \lambda) = \left(\sum_{i=1}^m w_i^2 (1-\rho) + \rho \right) \sigma_1^2 - \lambda \left(\sum_{i=1}^m w_i - 1 \right)$.

$$\nabla L(w_1, w_2, \ldots, w_m, \lambda) = \begin{pmatrix} 2w_1(1-\rho)\sigma_1^2 - \lambda \\ 2w_2(1-\rho)\sigma_1^2 - \lambda \\ \vdots \\ 2w_m\sigma_1^2 - \lambda \\ 1 - \sum_{i=1}^m w_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \implies w = \begin{pmatrix} \frac{\lambda}{2(1-\rho)\sigma_1^2} \\ \frac{\lambda}{2(1-\rho)\sigma_1^2} \\ \vdots \\ \frac{\lambda}{2(1-\rho)\sigma_1^2} \end{pmatrix}$$

$$\implies 1 - \sum_{i=1}^m \frac{\lambda}{2(1-\rho)\sigma_1^2} = 0$$

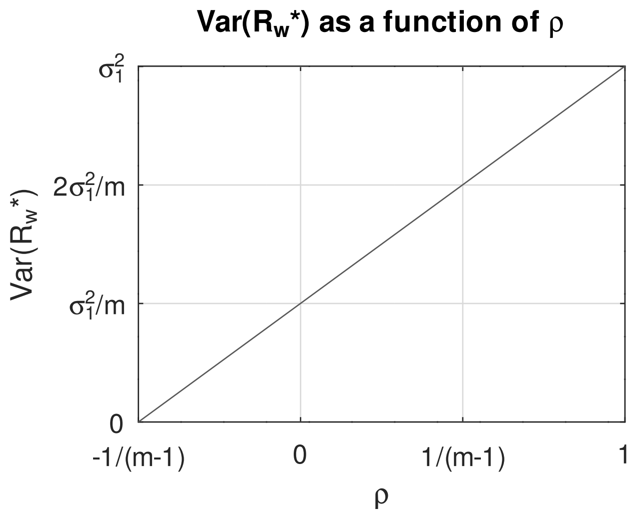
$$\implies \lambda = \frac{2(1-\rho)\sigma_1^2}{m}$$

$$\implies w = \begin{pmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix}$$

Since $(1-\rho)\sigma_1^2 \geq 0$, the minimum of $Var(R_w)$ is at $w = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \end{pmatrix}^T$.

$$\therefore w^* = \begin{pmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix}$$

$$\therefore Var(R_{w^*}) = \left(\sum_{i=1}^{m} \left(\frac{1}{m}\right)^2 (1-\rho) + \rho \right) \sigma_1^2 = \left(\frac{1-\rho}{m} + \rho \right) \sigma_1^2 = \frac{((m-1)\rho + 1)\sigma_1^2}{m}$$



$$\therefore \lim_{m\to\infty} Var(R_{w^*}) = \left(\lim_{m\to\infty} \frac{1-\rho}{m} + \rho \right) \sigma_1^2 = \rho\sigma_1^2$$

$\therefore$ For any $0 < \rho \leq 1$, adding equi-correlated assets with covariance $\rho\sigma_1^2$ causes portfolio variance to decrease closer and closer to $\rho\sigma_1^2$. For any $-1 \leq \rho < 0$, adding equi-correlated assets with covariance $\rho\sigma_1^2$ also causes portfolio variance to decrease with an upper limit of $\frac{\rho-1}{\rho}$ on the number of assets, at which point $0 \leq \rho\sigma_1^2 < \rho^2\sigma_1^2$.

5. (a)

$$Var(R_w) = w_1^2\sigma_1^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2 + w_2^2\sigma_2^2 = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 = (\sigma_1^2+\sigma_2^2)w_1^2 - 2\sigma_2^2 w_1 + \sigma_2^2$$

The minimum of $Var(R_w)$ is at $w_1 = \frac{-(-2\sigma_2^2)}{2(\sigma_1^2+\sigma_2^2)} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.

$$\therefore w^* = \begin{pmatrix} \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \\ 1 - \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \end{pmatrix} = \begin{pmatrix} \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \\ \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} \end{pmatrix}$$

$$\therefore Var(R_{w^*}) = (\sigma_1^2+\sigma_2^2)\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 - 2\sigma_2^2\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} + \sigma_2^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

(b)

$$Var(R_w) = \sum_{i=1}^m \sum_{j=1}^m w_i w_j \rho_{i,j} \sigma_i \sigma_j = \sum_{i=1}^m w_i^2 \sigma_i^2$$

Let $L(w_1, w_2, \ldots, w_m, \lambda) = \sum_{i=1}^m w_i^2 \sigma_i^2 - \lambda \left(\sum_{i=1}^m w_i - 1\right)$.

$$\nabla L(w_1, w_2, \ldots, w_m, \lambda) = \begin{pmatrix} 2w_1\sigma_1^2 - \lambda \\ 2w_2\sigma_2^2 - \lambda \\ \vdots \\ 2w_m\sigma_m^2 - \lambda \\ 1 - \sum_{i=1}^m w_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \implies w = \begin{pmatrix} \frac{\lambda}{2\sigma_1^2} \\ \frac{\lambda}{2\sigma_2^2} \\ \vdots \\ \frac{\lambda}{2\sigma_m^2} \end{pmatrix}$$

$$\implies 1 - \sum_{i=1}^m \frac{\lambda}{2\sigma_i^2} = 0$$

$$\implies \lambda = \frac{1}{\sum_{i=1}^m \frac{1}{2\sigma_i^2}} = \frac{2}{\sum_{i=1}^m \frac{1}{\sigma_i^2}}$$

$$\implies w = \begin{pmatrix} \frac{1}{\sigma_1^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} \\ \frac{1}{\sigma_2^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} \\ \vdots \\ \frac{1}{\sigma_m^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} \end{pmatrix}$$

The minimum of $\sum_{i=1}^m w_i^2 \sigma_i^2$ is at $w = \begin{pmatrix} \frac{1}{\sigma_1^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} & \frac{1}{\sigma_2^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} & \cdots & \frac{1}{\sigma_m^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} \end{pmatrix}^T$.

$$\therefore w^* = \begin{pmatrix} \frac{1}{\sigma_1^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} \\ \frac{1}{\sigma_2^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} \\ \vdots \\ \frac{1}{\sigma_m^2 \sum_{i=1}^m \frac{1}{\sigma_i^2}} \end{pmatrix}$$

$$\therefore Var(R_{w^*}) = \sum_{i=1}^m \left(\frac{1}{\sigma_i^2 \sum_{j=1}^m \frac{1}{\sigma_j^2}} \right)^2 \sigma_i^2 = \frac{1}{\sum_{j=1}^m \frac{1}{\sigma_j^2}}$$

$$\therefore Var(R_{w^*}) = \frac{\frac{1}{\frac{1}{m}\sum_{j=1}^m \sigma_j^{-2}}}{m} = \frac{\tilde{\sigma}^2}{m}$$

(c) Let $L(w, \lambda) = \frac{1}{2} w^T \Sigma w - \lambda \left(w^T \mathbf{1}_m - 1\right)$.

$$\nabla L(w, \lambda) = \begin{pmatrix} \Sigma w - \lambda \mathbf{1}_m \\ 1 - w^T \mathbf{1}_m \end{pmatrix} = \mathbf{0}_{m+1} \implies w = \lambda \Sigma^{-1} \mathbf{1}_m$$

$$\implies 1 - \left(\lambda \Sigma^{-1} \mathbf{1}_m\right)^T \mathbf{1}_m = 1 - \lambda \mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m = 0$$

$$\implies \lambda = \frac{1}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}$$

$$\implies w = \frac{\Sigma^{-1} \mathbf{1}_m}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}$$

$$Var\left(w^T R\right) = w^T \Sigma w > 0 \quad (\because \Sigma \text{ is positive definite})$$

$$\therefore w^* = \frac{\Sigma^{-1} \mathbf{1}_m}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}$$

$$\therefore Var(R_{w^*}) = \left(\frac{\Sigma^{-1} \mathbf{1}_m}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}\right)^T \Sigma \left(\frac{\Sigma^{-1} \mathbf{1}_m}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}\right) = \frac{\mathbf{1}_m^T \Sigma^{-1} \Sigma \Sigma^{-1} \mathbf{1}_m}{\left(\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m\right)^2} = \frac{1}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}$$