Solutions to Problem Set 4: Time Series Analysis

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MIT Financial Mathematics course website: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/`

Problem sets: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/`

Problem set 3: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset4.pdf`

1. (a)

$$\mu = E[X_t] = \phi_0 + \phi_1 E[X_{t-1}] + \phi_2 E[X_{t-2}] + E[\eta_t] = \phi_0 + (\phi_1 + \phi_2)\mu = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

(b)

$$\begin{aligned}
\gamma(k) &= Cov[X_t, X_{t-k}] \\
&= Cov[\phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \eta_t, X_{t-k}] \\
&= \phi_1 Cov[X_{t-1}, X_{t-k}] + \phi_2 Cov[X_{t-2}, X_{t-k}] + Cov[\eta_t, X_{t-k}] \\
&= \phi_1 Cov[X_t, X_{t-(k-1)}] + \phi_2 Cov[X_t, X_{t-(k-2)}] \\
&= \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2)
\end{aligned}$$

(c)

$$\rho_k = \frac{\gamma(k)}{\gamma(0)} = \frac{\phi_1 \gamma(k-1) + \phi_2 \gamma(k-2)}{\gamma(0)} = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$$

(d)

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \frac{1}{1-\rho_1^2} \begin{pmatrix} 1 & -\rho_1 \\ -\rho_1 & 1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \frac{1}{1-\rho_1^2} \begin{pmatrix} \rho_1 - \rho_1\rho_2 \\ -\rho_1^2 + \rho_2 \end{pmatrix} = \begin{pmatrix} \frac{\rho_1(1-\rho_2)}{1-\rho_1^2} \\ \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} \end{pmatrix}$$

(e)

$$\rho_1 = \phi_1 + \phi_2 \rho_1 = \frac{\phi_1}{1 - \phi_2}$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2$$

(f) We will prove the following formula for $k > 2$ and $\phi_1^2 + 4\phi_2 \neq 0$ by induction.

$$\rho_k = \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)} + 1\right)\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)^k - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)} - 1\right)\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)^k}{2^{k+1}}$$

Base case $k = 0$:

$$\rho_0 = \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)} + 1\right)\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)^0 - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)} - 1\right)\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)^0}{2^{0+1}}$$

$$
\begin{aligned}
&= \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right) - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)}{2} \\
&= 1
\end{aligned}
$$

Base case $k = 1$:

$$
\begin{aligned}
\rho_1 &= \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right)\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^1 - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^1}{2^{1+1}} \\
&= \frac{\frac{\phi_1^2(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)} - \frac{\phi_1(1+\phi_2)}{\phi_2-1} + \phi_1 - \sqrt{\phi_1^2+4\phi_2} - \frac{\phi_1^2(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)} - \frac{\phi_1(1+\phi_2)}{\phi_2-1} + \phi_1 + \sqrt{\phi_1^2+4\phi_2}}{4} \\
&= \frac{\phi_1 - \frac{\phi_1(1+\phi_2)}{\phi_2-1}}{2} \\
&= \frac{\phi_1}{1-\phi_2}
\end{aligned}
$$

Inductive step: let $k \geq 2$ and assume that the formula holds for ρ_{k-2}, ρ_{k-1}. Then,

$$
\begin{aligned}
\rho_k &= \phi_1\rho_{k-1} + \phi_2\rho_{k-2} \\
&= \phi_1 \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right)\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^{k-1} - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^{k-1}}{2^k} \\
&\quad + \phi_2 \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right)\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^{k-2} - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^{k-2}}{2^{k-1}} \\
&= \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right)\left(2\phi_1\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^{k-1} + 4\phi_2\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^{k-2}\right)}{2^{k+1}} \\
&\quad - \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)\left(2\phi_1\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^{k-1} + 4\phi_2\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^{k-2}\right)}{2^{k+1}} \\
&= \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right)\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^{k-2}\left(\phi_1^2 - 2\phi_1\sqrt{\phi_1^2+4\phi_2} + \phi_1^2 + 4\phi_2\right)}{2^{k+1}} \\
&\quad - \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^{k-2}\left(\phi_1^2 + 2\phi_1\sqrt{\phi_1^2+4\phi_2} + \phi_1^2 + 4\phi_2\right)}{2^{k+1}} \\
&= \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right)\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^{k} - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^{k}}{2^{k+1}}
\end{aligned}
$$

$\therefore$ For all $k > 2$ and $\phi_1^2 + 4\phi_2 \neq 0$,

$$
\rho_k = \frac{\left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}+1\right)\left(\phi_1 - \sqrt{\phi_1^2+4\phi_2}\right)^{k} - \left(\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(\phi_2-1)}-1\right)\left(\phi_1 + \sqrt{\phi_1^2+4\phi_2}\right)^{k}}{2^{k+1}}
$$

2. (a)

$$
\mu = E[X_t] = \phi_0 + \sum_{i=1}^{p} \phi_i E[X_{t-i}] + E[\eta_t] = \phi_0 + \sum_{i=1}^{p} \phi_i \mu = \frac{\phi_0}{1 - \sum_{i=1}^{p} \phi_i}
$$

(b)

$$
\begin{aligned}
\phi(L)\gamma(t) &= \gamma(t) - \sum_{i=1}^{p} \phi_i \gamma(t-i) \\
&= Cov\left(\phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i} + \eta_t, X_0\right) - \sum_{i=1}^{p} \phi_i \gamma(t-i) \\
&= \sum_{i=1}^{p} \phi_i \gamma(t-i) + Cov(\eta_t, X_0) - \sum_{i=1}^{p} \phi_i \gamma(t-i) \\
&= 0
\end{aligned}
$$

(c)

$$
\phi(L)\rho(t) = \frac{\phi(L)\gamma(t)}{\gamma(0)} = 0
$$

(d)

$$
\begin{aligned}
\hat{x}_{t^*}(h) &= g_{t^*}(h) \\
&= E[X_{t^*+h} \mid (X_{t^*}, \ldots, X_1, X_0) = (x_{t^*}, \ldots, x_1, x_0)] \\
&= E\left[\phi_0 + \sum_{i=1}^{p} \phi_i X_{t^*+h-i} + \eta_t \,\middle|\, (X_{t^*}, \ldots, X_1, X_0) = (x_{t^*}, \ldots, x_1, x_0)\right] \\
&= E\left[\phi_0 + \sum_{i=1}^{h-1} \phi_i X_{t^*+h-i} + \sum_{i=h}^{p} \phi_i X_{t^*+h-i} + \eta_t \,\middle|\, (X_{t^*}, \ldots, X_1, X_0) = (x_{t^*}, \ldots, x_1, x_0)\right] \\
&= \phi_0 + \sum_{i=1}^{h-1} \phi_i g_{t^*}(h-i) + \sum_{i=h}^{p} \phi_i \hat{x}_{t^*+h-i} \\
&= \phi_0 + \sum_{i=1}^{h-1} \phi_i \hat{x}_{t^*}(h-i) + \sum_{i=h}^{p} \phi_i \hat{x}_{t^*}(h-i) \\
&= \phi_0 + \sum_{i=1}^{p} \phi_i \hat{x}_{t^*}(h-i)
\end{aligned}
$$

(e)

$$
\phi(L)g_{t^*}(t) = \left(I - \sum_{i=1}^{p} \phi_i L^i\right) \hat{x}_{t^*}(t) = \hat{x}_{t^*}(t) - \sum_{i=1}^{p} \phi_i \hat{x}_{t^*}(t-i) = \sum_{i=1}^{p} \phi_i \hat{x}_{t^*}(t-i) - \sum_{i=1}^{p} \phi_i \hat{x}_{t^*}(t-i) = 0
$$

3. (a)

$$
\phi(L)g(t) = g(t) - \sum_{i=1}^{p} \phi_i g(t-i) = Ce^{\lambda t} - \sum_{i=1}^{p} \phi_i Ce^{\lambda(t-i)} = Ce^{\lambda t}\left(1 - \sum_{i=1}^{p} \phi_i \left(e^{-\lambda}\right)^i\right) = Ce^{\lambda t}\phi\left(e^{-\lambda}\right) = 0
$$

(b)

$$
\phi(L)g(t) = g(t) - \sum_{i=1}^{p} \phi_i g(t-i) = CG^t - \sum_{i=1}^{p} \phi_i CG^{t-i} = CG^t\left(1 - \sum_{i=1}^{p} \phi_i \left(G^{-1}\right)^i\right) = CG^t\phi\left(G^{-1}\right) = 0
$$

(c)

$$\phi(L)g(t)=\sum_{j=1}^{p}C_jG_j^t-\sum_{i=1}^{p}\phi_i\sum_{j=1}^{p}C_jG_j^{t-i}=\sum_{j=1}^{p}C_jG_j^t\left(1-\sum_{i=1}^{p}\phi_i\left(G_j^{-1}\right)^i\right)=\sum_{j=1}^{p}C_jG_j^t\phi\left(G_j^{-1}\right)=0$$

(d) Let all roots of $\phi(z)$ be outside the complex unit circle and $g(t)=CG^t$ solve $\phi(L)g(t)=0$. Then,

$$CG^t\phi\left(G^{-1}\right)=0$$
$$\phi\left(G^{-1}\right)=0$$
$$\left|G^{-1}\right|>1$$
$$|G|<1$$
$$\therefore |g(t)|=\left|CG^t\right|=|C||G|^t\leq|C|$$

4. (a)

$$z_1,z_2=\frac{\phi_1\pm\sqrt{\phi_1^2+4\phi_2}}{-2\phi_2}$$
$$\phi_1^2+4\phi_2>0\implies z_1\neq z_2, z_1,z_2\in\mathbb{R}$$
$$\phi_1^2+4\phi_2=0\implies z_1=z_2\in\mathbb{R}$$
$$\phi_1^2+4\phi_2<0\implies z_1=z_2^*, z_1,z_2\in\mathbb{C}\setminus\mathbb{R}$$

(b)

$$\begin{aligned}\rho_k-\phi_1\rho_{k-1}-\phi_2\rho_{k-2}&=C_1G_1^k+C_2G_2^k-\phi_1\left(C_1G_1^{k-1}+C_2G_2^{k-1}\right)-\phi_2\left(C_1G_1^{k-2}+C_2G_2^{k-2}\right)\\&=C_1G_1^k\left(1-\phi_1z_1-\phi_2z_1^2\right)+C_2G_2^k\left(1-\phi_1z_2-\phi_2z_2^2\right)\\&=0\end{aligned}$$

$$G_1,G_2=\frac{-2\phi_2}{\phi_1\pm\sqrt{\phi_1^2+4\phi_2}}=\frac{-2\phi_2\left(\phi_1\mp\sqrt{\phi_1^2+4\phi_2}\right)}{\phi_1^2-(\phi_1^2+4\phi_2)}=\frac{-2\phi_2\left(\phi_1\mp\sqrt{\phi_1^2+4\phi_2}\right)}{-4\phi_2}=\frac{\phi_1\mp\sqrt{\phi_1^2+4\phi_2}}{2}$$
$$\rho_0=1=C_1+C_2$$
$$C_2=1-C_1$$
$$\rho_1=\frac{\phi_1}{1-\phi_2}=\frac{C_1\left(\phi_1-\sqrt{\phi_1^2+4\phi_2}\right)+(1-C_1)\left(\phi_1+\sqrt{\phi_1^2+4\phi_2}\right)}{2}=\frac{\phi_1+(1-2C_1)\sqrt{\phi_1^2+4\phi_2}}{2}$$
$$\therefore C_1=\frac{\frac{\frac{2\phi_1}{1-\phi_2}-\phi_1}{\sqrt{\phi_1^2+4\phi_2}}-1}{-2}=\frac{\frac{\phi_1(1+\phi_2)}{\sqrt{\phi_1^2+4\phi_2}(1-\phi_2)}-1}{-2}=\frac{\phi_1(1+\phi_2)}{2(\phi_2-1)\sqrt{\phi_1^2+4\phi_2}}+\frac{1}{2}$$
$$\therefore C_2=1-\left(\frac{\phi_1(1+\phi_2)}{2(\phi_2-1)\sqrt{\phi_1^2+4\phi_2}}+\frac{1}{2}\right)=\frac{\phi_1(1+\phi_2)}{2(1-\phi_2)\sqrt{\phi_1^2+4\phi_2}}+\frac{1}{2}$$

$$\frac{G_1\left(1-G_2^2\right)}{(G_1-G_2)(1+G_1G_2)}=\frac{\frac{\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2}\left(1-\frac{\left(\phi_1+\sqrt{\phi_1^2+4\phi_2}\right)^2}{4}\right)}{\frac{\phi_1-\sqrt{\phi_1^2+4\phi_2}-\left(\phi_1+\sqrt{\phi_1^2+4\phi_2}\right)}{2}\left(1+\frac{\phi_1^2-\left(\phi_1^2+4\phi_2\right)}{4}\right)}$$

$$
\begin{aligned}
&= \frac{\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)\left(4 - \left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)^2\right)}{8\sqrt{\phi_1^2 + 4\phi_2}\,(\phi_2 - 1)} \\
&= \frac{4\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right) + 4\phi_2\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)}{8\,(\phi_2 - 1)\sqrt{\phi_1^2 + 4\phi_2}} \\
&= \frac{\phi_1(1 + \phi_2) + (\phi_2 - 1)\sqrt{\phi_1^2 + 4\phi_2}}{2\,(\phi_2 - 1)\sqrt{\phi_1^2 + 4\phi_2}} \\
&= C_1
\end{aligned}
$$

$$
\begin{aligned}
\frac{-G_2\left(1 - G_1^2\right)}{(G_1 - G_2)(1 + G_1 G_2)} &= \frac{-\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} + \phi_2\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)\right)}{2\,(\phi_2 - 1)\sqrt{\phi_1^2 + 4\phi_2}} \\
&= \frac{\phi_1(1 + \phi_2) + (1 - \phi_2)\sqrt{\phi_1^2 + 4\phi_2}}{2\,(1 - \phi_2)\sqrt{\phi_1^2 + 4\phi_2}} \\
&= C_2
\end{aligned}
$$

$$
\therefore \rho_k = C_1 G_1^k + C_2 G_2^k = \frac{G_1\left(1 - G_2^2\right)G_1^k - G_2\left(1 - G_1^2\right)G_2^k}{(G_1 - G_2)(1 + G_1 G_2)}
$$

(c) Let γ_k be the autocovariance function of an $AR(2)$ process with a characteristic equation with distinct roots. Then, for all $k \geq 0$,

$$
|\gamma_k| = |\rho_k||\gamma_0| \leq \left(|C_1||G_1|^k + |C_2||G_2|^k\right)|\gamma_0| \leq (|C_1| + |C_2|)|\gamma_0| \max\{|G_1|, |G_2|\}^k
$$

$\therefore$ Since $0 < \max\{|G_1|, |G_2|\} < 1$, $|\gamma_k|$ decreases exponentially.

(d)

$$
G_1 - G_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} - \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} = -\sqrt{\phi_1^2 + 4\phi_2} < 0
$$

$$
1 - G_2^2, 1 - G_1^2, 1 + G_1 G_2 > 0 \quad (\because |G_1|, |G_2| < 1)
$$

$$
\frac{C_1}{|C_1|} = \frac{-G_1}{|G_1|} \wedge \frac{C_2}{|C_2|} = \frac{G_2}{|G_2|}
$$

$$
C_1 > 0 \vee C_2 > 0 \quad (\because \rho_0 = C_1 + C_2 = 1)
$$

For even $k \geq 0$,

$$
C_1, C_2 > 0 \implies \rho_k = C_1 G_1^k + C_2 G_2^k > 0
$$

$$
C_1 < 0 < C_2 \implies 0 < G_1 < G_2 \implies \rho_k = G_2^k\left(C_1\left(\frac{G_1}{G_2}\right)^k + C_2\right) > 0
$$

$$
C_2 < 0 < C_1 \implies G_1 < G_2 < 0 \implies \rho_k = G_1^k\left(C_1 + C_2\left(\frac{G_2}{G_1}\right)^k\right) > 0
$$

$$
G_1^2, G_2^2 = \left(\frac{\phi_1 \mp \sqrt{\phi_1^2 + 4\phi_2}}{2}\right)^2 = \frac{\phi_1^2 \mp 2\phi_1\sqrt{\phi_1^2 + 4\phi_2} + \phi_1^2 + 4\phi_2}{4} = \frac{\phi_1\left(\phi_1 \mp \sqrt{\phi_1^2 + 4\phi_2}\right)}{2} + \phi_2
$$

For odd $k \geq 0$,

$$
\phi_1 > 0 \implies G_1^2 < G_2^2
$$

$$
\begin{aligned}
&\implies 0 < \left(1-G_2^2\right) < \left(1-G_1^2\right) \wedge 0 < G_1^{k+1} < G_2^{k+1} \\
&\implies G_1\left(1-G_2^2\right)G_1^k - G_2\left(1-G_1^2\right)G_2^k < 0 \\
&\implies \rho_k > 0
\end{aligned}
$$

$$
\phi_1 = 0 \implies G_1^2 = G_2^2 \implies G_1\left(1-G_2^2\right)G_1^k - G_2\left(1-G_1^2\right)G_2^k = 0 \implies \rho_k = 0
$$

$$
\begin{aligned}
\phi_1 < 0 &\implies G_1^2 > G_2^2 \\
&\implies 0 < \left(1-G_1^2\right) < \left(1-G_2^2\right), 0 < G_2^{k+1} < G_1^{k+1} \\
&\implies G_1\left(1-G_2^2\right)G_1^k - G_2\left(1-G_1^2\right)G_2^k > 0 \\
&\implies \rho_k < 0
\end{aligned}
$$

$\therefore$ If $\phi_1 > 0$, then $\rho_k > 0$ for all $k \geq 0$; if $\phi_1 = 0$, then $\rho_k > 0$ for even $k \geq 0$ and $\rho_k = 0$ for odd $k \geq 0$; if $\phi_1 < 0$, then $\rho_k > 0$ for even $k \geq 0$ and $\rho_k < 0$ for odd $k \geq 0$.

(e)

$$
d = \frac{d\cos(2\pi f_0)2\sqrt{-\phi_2}}{|\phi_1|} = \frac{d\left(e^{i2\pi f_0} + e^{-i2\pi f_0}\right)2|d|}{2|\phi_1|} = \frac{(G_1+G_2)|d|}{|\phi_1|} = \frac{\phi_1|d|}{|\phi_1|} = \operatorname{sgn}(\phi_1)|d|
$$

$$
\begin{aligned}
\therefore \rho_k &= \frac{de^{i2\pi f_0}\left(1-\left(de^{-i2\pi f_0}\right)^2\right)\left(de^{i2\pi f_0}\right)^k - de^{-i2\pi f_0}\left(1-\left(de^{i2\pi f_0}\right)^2\right)\left(de^{-i2\pi f_0}\right)^k}{\left(de^{i2\pi f_0} - de^{-i2\pi f_0}\right)\left(1+de^{i2\pi f_0}de^{-i2\pi f_0}\right)} \\
&= \frac{d^{k+1}e^{i2(k+1)\pi f_0} - d^{k+3}e^{i2(k-1)\pi f_0} - d^{k+1}e^{-i2(k+1)\pi f_0} + d^{k+3}e^{-i2(k-1)\pi f_0}}{d\left(e^{i2\pi f_0} - e^{-i2\pi f_0}\right)\left(1+d^2\right)} \\
&= \frac{d^k\left(2i\sin(2(k+1)\pi f_0) - d^2 2i\sin(2(k-1)\pi f_0)\right)}{2i\sin(2\pi f_0)\left(1+d^2\right)} \\
&= \frac{d^k\left(\sin(2k\pi f_0)\cos(2\pi f_0) + \sin(2\pi f_0)\cos(2k\pi f_0) - d^2\sin(2k\pi f_0)\cos(2\pi f_0) + d^2\sin(2\pi f_0)\cos(2k\pi f_0)\right)}{\sin(2\pi f_0)\left(1+d^2\right)} \\
&= [\operatorname{sgn}(x)]^k|d|^k\left(\frac{\left(1-d^2\right)\sin(2k\pi f_0)}{\tan(2\pi f_0)\left(1+d^2\right)} + \cos(2k\pi f_0)\right) \\
&= [\operatorname{sgn}(x)]^k|d|^k\left(\frac{\sin(2k\pi f_0)}{\tan(F)} + \cos(2k\pi f_0)\right) \\
&= [\operatorname{sgn}(x)]^k|d|^k\left(\frac{\sin(2k\pi f_0)\cos(F) + \cos(2k\pi f_0)\sin(F)}{\sin(F)}\right) \\
&= [\operatorname{sgn}(x)]^k|d|^k\frac{\sin(2k\pi f_0 + F)}{\sin(F)}
\end{aligned}
$$

5. (a)

$$
\begin{aligned}
\rho_k &= \frac{E[X_tX_{t-k}] - E[X_t]E[X_{t-k}]}{E\left[X_t^2\right] - E[X_t]^2} \\
&= \frac{E[(\eta_t + \theta_1\eta_{t-1})(\eta_{t-k} + \theta_1\eta_{t-k-1})]}{E\left[(\eta_t + \theta_1\eta_{t-1})^2\right]} \\
&= \frac{E[\eta_t\eta_{t-k}] + \theta_1E[\eta_t\eta_{t-k-1}] + \theta_1E[\eta_{t-1}\eta_{t-k}] + \theta_1^2E[\eta_{t-1}\eta_{t-k-1}]}{E\left[\eta_t^2\right] + 2\theta_1E[\eta_t\eta_{t-1}] + \theta_1^2E\left[\eta_{t-1}^2\right]} \\
&= \frac{\theta_1E[\eta_{t-1}\eta_{t-k}]}{\left(1+\theta_1^2\right)\sigma^2}
\end{aligned}
$$

$$= \begin{cases} \frac{\theta_1}{1+\theta_1^2} & k = 1 \\ 0 & k > 1 \end{cases}$$

(b) If $\theta_1 = 0.5$, then $\rho_1 = \frac{0.5}{1+0.5^2} = 0.4$. If $\theta_1 = 2$, then $\rho_1 = \frac{2}{1+2^2} = 0.4$.

(c)

$$\rho_1\theta_1^2 - \theta_1 + \rho_1 = 0$$

$\therefore \theta_1 = \frac{1 \pm \sqrt{1-4\rho_1^2}}{2\rho_1}$ and the solution for θ_1 in terms of ρ_1 is not unique.

(d)

$$(1+\theta_1 L)\sum_{i=0}^{\infty}(-\theta_1)^i L^i = \sum_{i=0}^{\infty}(-\theta_1)^i L^i - \sum_{i=1}^{\infty}(-\theta_1)^i L^i = 1 + \sum_{i=1}^{\infty}(-\theta_1)^i L^i - \sum_{i=1}^{\infty}(-\theta_1)^i L^i$$

Hence, $(1+\theta_1 L)^{-1} = \sum_{i=0}^{\infty}(-\theta_1)^i L^i$ if $|\theta_1| < 1$ and is undefined otherwise. If $|\theta_1| < 1$, then

$$\sum_{i=0}^{\infty}(-\theta_1)^i L^i X_t = \eta_t$$

$$X_t = \eta_t - \sum_{i=1}^{\infty}(-\theta_1)^i X_{t-i}$$

$\therefore \{X_t\}$ is invertible only if $\theta_1 = 0.5$, in which case $X_t = \eta_t - \sum_{i=1}^{\infty}(-0.5)^i X_{t-i}$.

6. (a)

$$\mu = E[X_t] = \phi_0 + \phi_1 E[X_{t-1}] + E[\eta_t] + \theta_1 E[\eta_{t-1}] = \phi_0 + \phi_1\mu = \frac{\phi_0}{1-\phi_1}$$

(b)

$$\begin{aligned}
\sigma_X^2 &= Var(X_t) \\
&= \gamma(0) \\
&= Var(\phi_0 + \phi_1 X_{t-1} + \eta_t + \theta_1\eta_{t-1}) \\
&= \phi_1^2\sigma_X^2 + \sigma^2 + \theta_1^2\sigma^2 + 2(\phi_1 Cov(X_{t-1},\eta_t) + \phi_1\theta_1 Cov(X_{t-1},\eta_{t-1}) + \theta_1 Cov(\eta_t,\eta_{t-1})) \\
&= \frac{\sigma^2 + \theta_1^2\sigma^2 + 2\phi_1\theta_1\sigma^2}{1-\phi_1^2} \\
&= \frac{\sigma^2\left(1+\theta_1^2+2\phi_1\theta_1\right)}{1-\phi_1^2} \\
&= \frac{\sigma^2\left(1-\phi_1^2+\theta_1^2+2\phi_1\theta_1+\phi_1^2\right)}{1-\phi_1^2} \\
&= \sigma^2\left(1+\frac{(\theta_1+\phi_1)^2}{1-\phi_1^2}\right)
\end{aligned}$$

(c)

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{Cov(\phi_0 + \phi_1 X_{t-1} + \eta_t + \theta_1\eta_{t-1}, X_{t-k})}{\gamma_0} = \phi_1\rho_{k-1} + \frac{\theta_1 Cov(\eta_{t-1}, X_{t-k})}{\gamma_0} = \begin{cases} \phi_1 + \frac{\theta_1\sigma^2}{\gamma_0} & k = 1 \\ \phi_1\rho_{k-1} & k > 1 \end{cases}$$

(d) If $0 < \phi_1 < 1$ and $-1 < \theta_1 < -\phi_1$, then

$$\theta_1 + \phi_1 < 0, \quad 1 + \phi_1\theta_1 > 0, \quad 1 + \theta_1^2 + 2\phi_1\theta_1 > 1 + 2\theta_1 + \theta_1^2 = (1+\theta_1)^2 > 0$$

$$\rho_1 = \phi_1 + \frac{\theta_1\left(1-\phi_1^2\right)}{1+\theta_1^2+2\phi_1\theta_1} = \frac{\phi_1+\phi_1\theta_1^2+2\phi_1^2\theta_1+\theta_1-\theta_1\phi_1^2}{1+\theta_1^2+2\phi_1\theta_1} = \frac{(1+\phi_1\theta_1)(\theta_1+\phi_1)}{1+\theta_1^2+2\phi_1\theta_1} < 0$$

Under these conditions, the ACF is always negative and declines exponentially in magnitude from the second time-step on, which is not possible with an $AR(1)$ model. In a time series following such an $ARMA(1,1)$ process, points would be strongly negatively correlated with nearby points and more weakly negatively correlated with points further away, so there would tend to be sharp oscillation.