Solutions to Problem Set 8: Stochastic Calculus

Jura Ivanković

MIT Financial Mathematics course website: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/`

Problem sets: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/`

Problem set 8: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset8.pdf`

Part A

1. (a) Let X_t be a simple random walk.

$$\mathbb{E}[X_{t+1}] = \frac{1}{2}(\mathbb{E}[X_t] - 1) + \frac{1}{2}(\mathbb{E}[X_t] + 1) = \mathbb{E}[X_t]$$

$\therefore$ A simple random walk is a martingale.

(b)

$$\mathbb{E}[X_{t+1} \mid X_t = 0] = \mathbb{E}[X_{t+1} \mid S_t = 0] = \frac{1}{2}|0-1| + \frac{1}{2}|0+1| = 1$$

$\therefore X_t$ is not a martingale.

(c)

$$\mathbb{E}[X_{t+1}] = \mathbb{E}[X_t] + \mathbb{E}[Y_t] - \frac{1}{\lambda} = \mathbb{E}[X_t] + \frac{1}{\lambda} - \frac{1}{\lambda} = \mathbb{E}[X_t]$$

$\therefore X_t$ is a martingale.

(d)

$$\mathbb{E}[X_{t+1}] = \mathbb{E}[X_t] + \mathbb{E}[Z_t]\mathbb{E}[Z_{t-1}]\mathbb{E}[Z_0] > \mathbb{E}[X_t]$$

$\therefore X_t$ is not a martingale.

(e) Let $s < t$.

$$\mathbb{E}[X_t] = \mathbb{E}\left[B_t^{(1)} B_t^{(2)}\right] = \mathbb{E}\left[B_t^{(1)}\right]\mathbb{E}\left[B_t^{(2)}\right] = \mathbb{E}\left[B_s^{(1)}\right]\mathbb{E}\left[B_s^{(2)}\right] = \mathbb{E}\left[B_s^{(1)} B_s^{(2)}\right] = \mathbb{E}[X_s]$$

$\therefore X_t$ is a martingale.

2. (a) $\mathbb{E}[B(t) \mid B(s)] = B(s)$ and $\mathbb{V}[B(t) \mid B(s)] = t - s$.

(b)

$$\mathbb{E}[X(t) \mid X(s)] = \mathbb{E}[B(t) + \mu t \mid B(s)] = B(s) + \mu t = X(s) + \mu(t - s)$$

$$\mathbb{V}[X(t) \mid X(s)] = \mathbb{V}[B(t) + \mu t \mid B(s)] = \mathbb{V}[B(t) \mid B(s)] = t - s$$

(c)

$$\begin{aligned} B(t) - B(s) &\sim \mathcal{N}(0, t - s) \\ \sigma(B(t) - B(s)) &\sim \mathcal{N}\left(0, \sigma^2(t - s)\right) \\ e^{\sigma(B(t) - B(s))} &\sim \text{Lognormal}\left(0, \sigma^2(t - s)\right) \end{aligned}$$

$$\therefore \mathbb{E}\left[e^{\sigma(B(t) - B(s))}\right] = e^{\frac{\sigma^2(t-s)}{2}}$$

3. (a) Let $t = \frac{T}{3}$. Then $Y_t = X_{\frac{2T}{3}} = X_{2t}$. $\therefore Y_t$ is not adapted to X_t.

(b) Y_t depends on X_s such that $t < s \leq 2t$. $\therefore Y_t$ is not adapted to X_t.

(c) Y_t depends only on X_i such that $0 \leq i \leq t$. $\therefore Y_t$ is adapted to X_t.

4. (a) $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = 3x^2$, $\frac{\partial^2 f}{\partial x^2} = 6x$

$$\therefore df(t, B_t) = \left(0 + \frac{6B_t}{2}\right) dt + 3B_t^2 dB_t = 3B_t dt + 3B_t^2 dB_t$$

(b) $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = \cos x$, $\frac{\partial^2 f}{\partial x^2} = -\sin x$

$$\therefore df(t, B_t) = \left(0 + \frac{-\sin B_t}{2}\right) dt + \cos B_t dB_t = \frac{-\sin B_t}{2} dt + \cos B_t dB_t$$

(c)

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\sin\left(t^3 + x^2\right) 3t^2 = -3t^2 \sin\left(t^3 + x^2\right) \\ \frac{\partial f}{\partial x} &= -\sin\left(t^3 + x^2\right) 2x = -2x \sin\left(t^3 + x^2\right) \\ \frac{\partial^2 f}{\partial x^2} &= -2\left(\sin\left(t^3 + x^2\right) + x\cos\left(t^3 + x^2\right) 2x\right) = -2\left(\sin\left(t^3 + x^2\right) + 2x^2 \cos\left(t^3 + x^2\right)\right) \end{aligned}$$

$$\begin{aligned} &\therefore df(t, B_t) \\ &= \left(-3t^2 \sin\left(t^3 + B_t^2\right) + \frac{-2\left(\sin\left(t^3 + B_t^2\right) + 2B_t^2 \cos\left(t^3 + B_t^2\right)\right)}{2}\right) dt \\ &\quad + \left(-2B_t \sin\left(t^3 + B_t^2\right)\right) dB_t \\ &= -\left(\left(3t^2 + 1\right) \sin\left(t^3 + B_t^2\right) + 2B_t^2 \cos\left(t^3 + B_t^2\right)\right) dt - 2B_t \sin\left(t^3 + B_t^2\right) dB_t \end{aligned}$$

(d) $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = 2xe^{x^2}$, $\frac{\partial^2 f}{\partial x^2} = 2(1+2x)e^{x^2}$

$$df(t, B_t) = \left(0 + \frac{2(1+2B_t)e^{B_t^2}}{2}\right) dt + 2B_t e^{B_t^2} dB_t = \left(1 + 2B_t^2\right) e^{B_t^2} dt + 2B_t e^{B_t^2} dB_t$$

(e) $df(t, B_t) = B_t^2 dB_t$

(f) $df(t, B_t) = B_t dt$

(g) $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial^2 f}{\partial x^2} = 2$

$$df(t, X_t) = \left(0 + \mu_t 2X_t + \frac{2\sigma_t^2}{2}\right) dt + 2X_t \sigma_t dB_t = \left(2X_t \mu_t + \sigma_t^2\right) dt + 2X_t \sigma_t dB_t$$

5. Let $\mathbf{P}, \mathbf{Q}$ be the probability distributions of $B(t), B(t)^2$, respectively, let $T > 0$ and let X be the set of paths such that $B(T) < 0$. Then, $\mathbf{P}(X) > 0$ and $\mathbf{Q}(X) = 0$. $\therefore B(t), B(t)^2$ are not equivalent probability distributions.

Part B

1. (a)

$$f_{Y(t)}(x_1, x_2) = f_{X_1(t)}(x_1) f_{X_2(t)}(x_2) = \frac{e^{\frac{-x_1^2}{2t}}}{\sqrt{2\pi t}} \frac{e^{\frac{-x_2^2}{2t}}}{\sqrt{2\pi t}} = \frac{e^{\frac{-\left(x_1^2+x_2^2\right)}{2t}}}{2\pi t}$$

(b)

$$\begin{aligned}
\mathbf{P}(Y(t) \in D_\rho) &= \iint\limits_{D_\rho} \frac{e^{\frac{-\left(x_1^2+x_2^2\right)}{2t}}}{2\pi t} dx_1 dx_2 \\
&= \int_0^{2\pi} \int_0^{\rho} \frac{e^{\frac{-r^2}{2t}}}{2\pi t} r dr d\theta \\
&= \int_0^{\rho} \frac{e^{\frac{-r^2}{2t}}}{t} r dr \\
&= -e^{\frac{-r^2}{2t}} \Big|_{r=0}^{\rho} \\
&= 1 - e^{\frac{-\rho^2}{2t}}
\end{aligned}$$

2. Let $f(t, x) = tx^2$.

$$\frac{\partial f}{\partial t} = x^2 \quad \frac{\partial f}{\partial x} = 2tx \quad \frac{\partial^2 f}{\partial x^2} = 2t$$

$$df(t, B_t) = \left(B_t^2 + \frac{2t}{2}\right) dt + 2tB_t dB_t = \left(B_t^2 + t\right) dt + 2tB_t dB_t$$

$$f(t, B_t) = \int \left(B_t^2 + t\right) dt + \int 2tB_t dB_t$$

$$\begin{aligned}
tB_t^2 &= tB_t^2 - 0B_0^2 \\
&= f(t, B_t) - f(0, B_0) \\
&= \int_0^t \left(B_s^2 + s\right) ds + \int_0^t 2sB_s dB_s \\
&= \int_0^t B_s^2 ds + \frac{t^2}{2} + \int_0^t 2sB_s dB_s
\end{aligned}$$

$$\begin{aligned}
\therefore \mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\int_0^t B_s^2 ds\right] \\
&= \mathbb{E}\left[tB_t^2 - \frac{t^2}{2} - \int_0^t 2sB_s dB_s\right] \\
&= t\mathbb{E}\left[B_t^2\right] - \frac{t^2}{2} - \mathbb{E}\left[\int_0^t 2sB_s dB_s\right] \\
&= t \times t - \frac{t^2}{2} - 0 \\
&= \frac{t^2}{2}
\end{aligned}$$

3. (a) Let $f(t, x) = h(t)x$.

$$\frac{\partial f}{\partial t} = h'(t)x \quad \frac{\partial f}{\partial x} = h(t) \quad \frac{\partial^2 f}{\partial x^2} = 0$$

$$df(t, B_t) = \left(h'(t)B_t + \frac{0}{2}\right) dt + h(t)dB_t = h'(t)B_t dt + h(t)dB_t$$

$$\begin{aligned}
f(t, B_t) &= \int h'(t)B_t dt + \int h(t)dB_t \\
f(t, B_t) - f(0, B_0) &= \int_0^t h'(s)B_s ds + \int_0^t h(s)dB_s \\
\therefore \int_0^t h(s)dB_s &= f(t, B_t) - f(0, B_0) - \int_0^t h'(s)B_s ds \\
&= h(t)B_t - h(0)B_0 - \int_0^t h'(s)B_s ds \\
&= h(t)B_t - \int_0^t h'(s)B_s ds
\end{aligned}$$

(b)

$$\int_0^T s dB_s = TB_T - \int_0^T B_s ds$$

$$\int_0^T B_s ds = TB_T - \int_0^T s dB_s$$

$\left(TB_T, -\int_0^T s dB_s\right)$ has a joint normal distribution.

$\therefore \int_0^T B_s ds$ is normally distributed.