Solutions to Problem Set 2: Probability Theory and Stochastic Process

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MIT Financial Mathematics course website: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/`

Problem sets: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/`

Problem set 2: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset2.pdf`

Part A

1. (a)

$$\begin{aligned}
M_X(t) &= \mathbb{E}\left[e^{tX}\right] \\
&= \int_0^\infty e^{tx}\lambda e^{-\lambda x} dx \\
&= \lambda \int_0^\infty e^{(t-\lambda)x} dx \\
&= \frac{\lambda}{t-\lambda}\, e^{(t-\lambda)x}\Big|_{x=0}^{\infty} \\
&= \frac{\lambda}{t-\lambda}\left(\lim_{x\to\infty} e^{(t-\lambda)x} - 1\right) \\
&= \frac{\lambda}{\lambda-t}\left(1 - \lim_{x\to\infty} e^{(t-\lambda)x}\right) \\
&= \frac{\lambda}{\lambda-t} \quad \text{for } t < \lambda
\end{aligned}$$

(b)

$$M_X'(t) = \lambda\left((\lambda-t)^{-1}\right)' = \lambda\left(-(\lambda-t)^{-2}\right)(-1) = \frac{\lambda}{(\lambda-t)^2}$$

$$M_X''(t) = \lambda\left((\lambda-t)^{-2}\right)' = \lambda\left(-(\lambda-t)^{-2}\right)(-1) = \frac{2\lambda}{(\lambda-t)^3}$$

$$\mathbb{E}[X] = M_X'(0) = \frac{\lambda}{(\lambda - 0)^2} = \frac{1}{\lambda}$$

$$V(X) = M_X''(0) - (M_X'(0))^2 = \frac{2\lambda}{(\lambda - 0)^3} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

(c)

$$\mathbf{P}(X > t) = \int_t^\infty \lambda e^{-\lambda x} dx = \lambda \left. \frac{e^{-\lambda x}}{-\lambda} \right|_{x=t}^{\infty} = e^{-\lambda t} - \lim_{x \to \infty} e^{-\lambda x} = e^{-\lambda t} \quad (\because \lambda > 0)$$

(d) Let $t, s > 0$. Then,

$$\begin{aligned} \mathbf{P}(X > s + t \mid X > s) &= \frac{\mathbf{P}(X > s + t, X > s)}{\mathbf{P}(X > s)} \\ &= \frac{\mathbf{P}(X > s + t)}{\mathbf{P}(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= \mathbf{P}(X > t) \end{aligned}$$

$\therefore \mathbf{P}(X > s + t \mid X > s) = \mathbf{P}(X > t) \quad \forall t, s > 0$

(e)

$$\begin{aligned} \mathbf{P}(\min\{X_1, X_2, \ldots, X_n\} > t) &= \mathbf{P}(X_1 > t, X_2 > t, \ldots, X_n > t) \\ &= \mathbf{P}(X_1 > t) \times \mathbf{P}(X_2 > t) \times \cdots \times \mathbf{P}(X_n > t) \\ &= \left(e^{-\lambda t}\right)^n \\ &= e^{-(n\lambda)t} \end{aligned}$$

$\therefore \min\{X_1, X_2, \ldots, X_n\} \sim \mathrm{Exp}(n\lambda)$

(f) Consider the point in time when the first of the other three customers leaves and I begin being served. By the memoryless property, the probability that I will leave more than t time units after that point in time equals the probability that either of the other two customers will leave more than t time units after that point in time, for any t. $\therefore$ By symmetry, the probability that I will be the last to leave among the four customers is $^1/_3$.

2. (a)

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda\left(e^t - 1\right)}$$

(b)

$$M_X'(t) = \left(e^{\lambda\left(e^t-1\right)}\right)' = \lambda e^{\lambda\left(e^t-1\right)}\left(e^t-1\right)' = \lambda e^{\lambda\left(e^t-1\right)}e^t = \lambda e^{\lambda\left(e^t-1\right)+t}$$

$$M_X''(t) = \left(\lambda e^{\lambda\left(e^t-1\right)+t}\right)' = \lambda e^{-\lambda}\left(e^{\lambda e^t+t}\right)' = \lambda e^{-\lambda}e^{\lambda e^t+t}\left(\lambda e^t+t\right)' = \lambda e^{-\lambda}e^{\lambda e^t+t}\left(\lambda e^t+1\right)$$

$$M_X''(0) = \lambda e^{-\lambda}e^{\lambda e^0+0}\left(\lambda e^0+1\right) = \lambda e^{-\lambda}e^{\lambda}(\lambda+1) = \lambda(\lambda+1)$$

$$\mathbb{E}[X] = M_X'(0) = \lambda e^{\lambda\left(e^0-1\right)+0} = \lambda$$

$$V(X) = M_X''(0) - (M_X'(0))^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

(c)

$$\begin{aligned}\mathbb{E}\left[e^{t(X_1+X_2\cdots+X_n)}\right] &= \mathbb{E}\left[e^{tX_1}\times e^{tX_2}\times\cdots\times e^{tX_n}\right]\\ &= \mathbb{E}\left[e^{tX_1}\right]\times\mathbb{E}\left[e^{tX_2}\right]\times\cdots\times\mathbb{E}\left[e^{tX_n}\right] \quad (\because X_1, X_2, \ldots, X_n \text{ are i.i.d.})\\ &= e^{\lambda_1\left(e^t-1\right)}\times e^{\lambda_2\left(e^t-1\right)}\times\cdots\times e^{\lambda_n\left(e^t-1\right)}\\ &= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)\left(e^t-1\right)}\end{aligned}$$

$\therefore X_1 + X_2 + \cdots + X_n \sim \text{Poisson}(\lambda_1+\lambda_2+\cdots+\lambda_n)$

3. (a) Let $X_0, X_1, X_2, \ldots$ be a simple random walk, $t \geq 1$ and $x_{t-1}, x_{t-2}, \ldots, x_0$ such that $p_{X_{t-1},X_{t-2},\ldots,X_0}(x_{t-1}, x_{t-2}, \ldots, x_0) > 0$. Then,

$$p_{X_t|X_{t-1},X_{t-2},\ldots,X_0}(x_t \mid x_{t-1}, x_{t-2}, \ldots, x_0) = \begin{cases}\frac{1}{2} & x_t = x_{t-1}-1\\ \frac{1}{2} & x_t = x_{t-1}+1\\ 0 & \text{otherwise}\end{cases}$$

X_t depends only on X_{t-1}. $\therefore$ A simple random walk is a Markov process.

(b) Let $t \geq 1$ and $s_{t-1}, \ldots, s_0$ be such that $p_{S_{t-1},\ldots,S_0}(s_{t-1}, \ldots, s_0) > 0$. Note that

$$X_t = |S_{t-1} \pm 1| = \begin{cases}|S_{t-1}| \mp 1 = X_{t-1} \mp 1 & S_{t-1} \leq -1\\ 1 & S_{t-1} = 0\\ |S_{t-1}| \pm 1 = X_{t-1} \pm 1 & S_{t-1} \geq 1\end{cases}$$

Hence,

$$p_{X_t|X_{t-1},\ldots,X_0}(|s_t| \mid |s_{t-1}|, |s_{t-2}|, \ldots, |s_0|) = \begin{cases}1 & |s_t| = 1, |s_{t-1}| = 0\\ \frac{1}{2} & |s_t| = |s_{t-1}|-1, |s_{t-1}| \geq 1\\ \frac{1}{2} & |s_t| = |s_{t-1}|+1, |s_{t-1}| \geq 1\\ 0 & \text{otherwise}\end{cases}$$

X_t depends only on X_{t-1}. $\therefore X_0, X_1, X_2, \ldots$ is a Markov process.

(c) Let $t \geq 1$. Then,

$$\begin{aligned} f_{X_t \mid X_{t-1},\ldots,X_0}(x_t \mid x_{t-1},\ldots,x_0) &= f_{X_{t-1}+Y_{t-1}-\frac{1}{\lambda} \mid X_{t-1},\ldots,X_0}(x_t \mid x_{t-1},\ldots,x_0) \\ &= f_{X_{t-1},Y_{t-1}-\frac{1}{\lambda} \mid X_{t-1},\ldots,X_0}(x_{t-1}, x_t - x_{t-1} \mid x_{t-1},\ldots,x_0) \\ &= f_{Y_{t-1}-\frac{1}{\lambda} \mid X_{t-1},\ldots,X_0}(x_t - x_{t-1} \mid x_{t-1},\ldots,x_0) \\ &= f_{Y_{t-1}-\frac{1}{\lambda}}(x_t - x_{t-1}) \end{aligned}$$

X_t only depends on X_{t-1}. $\therefore X_0, X_1, X_2, \ldots$ is a Markov process.

(d)

$$X_1 = X_0 + Z_0 = Z_0$$
$$X_2 = X_1 + Z_1 Z_0 \implies Z_1 Z_0 = X_2 - X_1$$
$$X_3 = X_2 + Z_2 Z_1 Z_0 = X_2 + Z_2(X_2 - X_1) = (Z_2 + 1)X_2 - Z_2 X_1$$

X_3 depends on X_1 even after taking X_2 into account. $\therefore X_0, X_1, X_2, \ldots$ is not a Markov process.

(e)

$$W_0 = X_0 + W_0 = X_1$$
$$W_1 = X_2 - X_1 - W_0 = X_2 - 2X_1$$
$$X_3 = X_2 + W_2 + (X_2 - 2X_1) + X_1 = W_2 + 2X_2 - X_1$$

X_3 depends on X_1 even after taking X_2 into account. $\therefore X_0, X_1, X_2, \ldots$ is not a Markov process.

4. (a) Let I_2 be the transition matrix of a Markov chain. Then, for any $0 \leq p \leq 1$, $\begin{pmatrix} p & 1-p \end{pmatrix}^T$ is a stationary distribution of that Markov chain.

(b)

$$\begin{aligned} \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{m1} \\ p_{12} & p_{22} & \cdots & p_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1m} & p_{2m} & \cdots & p_{mm} \end{pmatrix} \begin{pmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix} &= \begin{pmatrix} \sum_{i=1}^m p_{i1} \frac{1}{m} \\ \sum_{i=1}^m p_{i2} \frac{1}{m} \\ \vdots \\ \sum_{i=1}^m p_{im} \frac{1}{m} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix} \quad \left(\because \sum_{i=1}^m p_{ij} = 1 \quad j = 1, 2, \ldots, m \right) \end{aligned}$$

$\therefore \pi_j = \frac{1}{m}$ gives the stationary distribution of a doubly stochastic process over a state space S of size m.

Part B

1. (a)

$$\mathbf{P}(\tau = 0) = \mathbf{P}(\max\{t \geq 0 : X_1 + X_2 + \cdots + X_t \leq 1\} = 0) = \mathbf{P}(X_1 > 1) = e^{-\lambda}$$

   (b)

$$\begin{aligned}
\mathbf{P}(\tau = n) &= \mathbf{P}(S_n \leq 1, S_{n+1} > 1) \\
&= \mathbf{P}(S_n \leq 1) - \mathbf{P}(S_n \leq 1, S_{n+1} \leq 1) \\
&= \mathbf{P}(S_n \leq 1) - \mathbf{P}(S_{n+1} \leq 1) \\
&= F_{S_n}(1) - F_{S_{n+1}}(1) \\
&= \left(1 - \sum_{k=0}^{n-1} \frac{1}{k!} e^{-\lambda} \lambda^k\right) - \left(1 - \sum_{k=0}^{n} \frac{1}{k!} e^{-\lambda} \lambda^k\right) \\
&= \frac{1}{n!} e^{-\lambda} \lambda^n
\end{aligned}$$

   $\frac{1}{n!} e^{-\lambda} \lambda^n$ is the probability mass function of a Poisson distribution with parameter λ.
   $\therefore \tau \sim \text{Poisson}(\lambda)$

2. (a) $\sigma > 0$ and $e^y > 0$ for all $-\infty < y < \infty$, so

$$\phi(x) = \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} > 0$$

   for all $-\infty < x < \infty$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)dx &= \sqrt{\left(\int_{-\infty}^{\infty} \phi(x)dx\right)^2} \quad (\because \phi(x) > 0 \text{ for all } -\infty < x < \infty) \\
&= \sqrt{\left(\int_{-\infty}^{\infty} \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx\right)^2} \\
&= \sqrt{\left(\int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{\pi}} du\right)^2} \quad \left(\text{substitute } u = \frac{x-\mu}{\sqrt{2}\sigma}, \frac{du}{dx} = \frac{1}{\sqrt{2}\sigma}, \sqrt{2}\sigma du = dx\right) \\
&= \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-u^2-v^2}}{\pi} dudv} \\
&= \sqrt{\int_{0}^{2\pi} \int_{0}^{\infty} \frac{e^{-r^2\sin^2\theta - r^2\cos^2\theta}}{\pi} rdrd\theta} \quad (\text{substitute } u = r\cos(\theta), v = r\sin(\theta)) \\
&= \sqrt{\int_{0}^{\infty} 2e^{-r^2} rdrd\theta}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\int_0^\infty e^{-s} ds} \quad \left(\text{substitute } s = r^2, \frac{ds}{dr} = 2r, ds = 2rdr\right) \\
&= \sqrt{-e^{-s}|_{s=0}^\infty} \\
&= \sqrt{1 - \lim_{s\to\infty}(e^{-s})} \\
&= 1
\end{aligned}
$$

$\therefore \phi(x)$ is a probability density function.

(b) Let $Y \sim \text{Lognormal}(\mu, \sigma)$. Then, $\ln(Y) \sim \mathcal{N}(\mu, \sigma)$. Let $X = \ln(Y)$. Then,

$$
\begin{aligned}
\mathbb{E}\left[Y^n\right] &= \mathbb{E}\left[e^{nX}\right] \\
&= \int_{-\infty}^{\infty} e^{nx} \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx \\
&= \int_{-\infty}^{\infty} e^{n\mu} e^{n(x-\mu)} \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx \\
&= e^{n\mu} \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-\mu)^2+2\sigma^2 n(x-\mu)}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx \\
&= e^{n\mu} \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-\mu)^2+2\sigma^2 n(x-\mu)-\sigma^4 n^2+\sigma^4 n^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx \\
&= e^{n\mu} \int_{-\infty}^{\infty} \frac{e^{\frac{-\left(x-\mu-\sigma^2 n\right)^2+\sigma^4 n^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx \\
&= e^{n\mu+\frac{\sigma^2 n^2}{2}} \int_{-\infty}^{\infty} \frac{e^{\frac{-\left(x-\mu-\sigma^2 n\right)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx \\
&= e^{n\mu+\frac{\sigma^2 n^2}{2}}
\end{aligned}
$$

$$
\mathbb{E}[Y] = e^{\mu+\frac{\sigma^2}{2}}
$$

$$
V(Y) = \mathbb{E}\left[Y^2\right] - \mathbb{E}^2[Y] = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)
$$

3. Log-normal:

Let

$$
\theta = \begin{pmatrix}\mu \\ \sigma\end{pmatrix} \quad h(x) = \frac{1}{x} \quad c(\theta) = \frac{e^{\frac{-\mu^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \quad k = 2
$$

$$
w_1(\theta) = \frac{-1}{2\sigma^2} \quad t_1(x) = \ln^2 x \quad w_2(\theta) = \frac{\mu}{\sigma^2} \quad t_2(x) = \ln x
$$

Then,

$$f(x \mid \theta) = \frac{1}{x}\frac{e^{\frac{-\mu^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}\ln^2 x + \frac{\mu}{\sigma^2}\ln x} = \frac{e^{\frac{-\ln^2 x + 2\mu \ln x - \mu^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} = \frac{e^{\frac{-(\ln x - \mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}}$$

$\therefore$ The log-normal distribution is an exponential family.

Poisson:

Let

$$\theta = \lambda \quad h(x) = \frac{1}{x!} \quad c(\theta) = e^{-\lambda} \quad k = 1 \quad w_1(\theta) = \ln\lambda \quad t_1(x) = x$$

Then,

$$f(x \mid \theta) = \frac{1}{x!}e^{-\lambda}e^{x\ln\lambda} = \frac{\lambda^x e^{-\lambda}}{x!}$$

$\therefore$ The Poisson distribution is an exponential family.

Exponential:

Let

$$\theta = \lambda \quad h(x) = 1 \quad c(\theta) = \lambda \quad k = 1 \quad w_1(\theta) = -\lambda \quad t_1(x) = x$$

Then,

$$f(x \mid \theta) = \lambda e^{-\lambda x}$$

$\therefore$ The exponential distribution is an exponential family.

4. (a) Let S_T be the price of the stock after T years. Then, for $T \geq 1$,

$$\frac{S_T}{S_{T-1}} \sim \text{Lognormal}(\ln(1.1), \ln(1.2))$$

$$\ln\left(\frac{S_T}{S_{T-1}}\right) \sim \mathcal{N}(\ln(1.1), \ln(1.2))$$

$$\ln\left(\frac{S_T}{S_0}\right) = \ln\left(\frac{S_1}{S_0}\frac{S_2}{S_1}\cdots\frac{S_T}{S_{T-1}}\right) = \sum_{i=1}^{T}\ln\left(\frac{S_i}{S_{i-1}}\right) \sim \mathcal{N}(T\ln(1.1), \sqrt{T}\ln(1.2))$$

Let $Z \sim \mathcal{N}(0,1)$. Then,

$$S_T = S_0 e^{\ln\left(\frac{S_T}{S_0}\right)} = S_0 e^{T\ln(1.1) + \sqrt{T}\ln(1.2)Z} = S_0 1.1^T 1.2^{\sqrt{T}Z}$$

$\therefore$ The fraction of wealth lost with 0.1% chance when invested over T years is

$$h(T) = \frac{S_0 - S_0 1.1^T 1.2^{\sqrt{T}F_Z^{-1}(0.001)}}{S_0} = 1 - 1.1^T 1.2^{\sqrt{T}F_Z^{-1}(0.001)}$$

(b)

$$\begin{aligned}
&h(T^*) = \max_{T\geq 0}\{h(T)\} \\
\iff &1.1^{T^*}1.2^{\sqrt{T^*}F_Z^{-1}(0.001)} = \min_{T\geq 0}\left\{1.1^T1.2^{\sqrt{T}F_Z^{-1}(0.001)}\right\} \\
\iff &T^*\ln(1.1) + \sqrt{T^*}\ln(1.2)F_Z^{-1}(0.001) = \min_{T\geq 0}\left\{T\ln(1.1) + \sqrt{T}\ln(1.2)F_Z^{-1}(0.001)\right\} \\
\iff &T^* = \left(\frac{-\ln(1.2)F_Z^{-1}(0.001)}{2\ln(1.1)}\right)^2 \simeq 8.736
\end{aligned}$$

$\therefore \left(\frac{-\ln(1.2)F_Z^{-1}(0.001)}{2\ln(1.1)}\right)^2 \simeq 8.736$ maximises the value of $h(T)$. $h(T)$ is not an increasing function of time.

(c) The magnitude of wealth lost does not increase with time, but reaches a maximum and decreases thereafter. Therefore, time diminishes risk even if risk is defined as magnitude of wealth lost.

5. (a) The transition matrix is

$$\begin{pmatrix}
1-b & 1-b & 0 & \cdots & 0 \\
b & 0 & 1-b & \ddots & \vdots \\
0 & b & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & 1-b \\
0 & \cdots & 0 & b & b
\end{pmatrix}$$

(b) We want to find $\begin{pmatrix}\pi_1 & \pi_2 & \cdots & \pi_m\end{pmatrix}^T$ such that $\sum_{i=1}^m \pi_i = 1$ and

$$\begin{pmatrix}
1-b & 1-b & 0 & \cdots & 0 \\
b & 0 & 1-b & \ddots & \vdots \\
0 & b & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & 1-b \\
0 & \cdots & 0 & b & b
\end{pmatrix}
\begin{pmatrix}\pi_1 \\ \pi_2 \\ \vdots \\ \pi_m\end{pmatrix} = \begin{pmatrix}\pi_1 \\ \pi_2 \\ \vdots \\ \pi_m\end{pmatrix}$$

We will show by induction that $\pi_i = \left(\frac{b}{1-b}\right)^{i-1}\pi_1$, for $1 \leq i \leq m$.

Base case:

$$\begin{aligned}
(1-b)\pi_1 + (1-b)\pi_2 &= \pi_1 \\
(1-b)\pi_2 &= b\pi_1 \\
\pi_2 &= \frac{b}{1-b}\pi_1
\end{aligned}$$

Inductive step:

Let $i \geq 2$ and assume that $\pi_i = \left(\frac{b}{1-b}\right)^{i-1} \pi_1$ for $j \leq i$. Then,

$$\begin{aligned}
b\pi_{i-1} + (1-b)\pi_{i+1} &= \pi_i \\
b\left(\frac{b}{1-b}\right)^{i-2}\pi_1 + (1-b)\pi_{i+1} &= \left(\frac{b}{1-b}\right)^{i-1}\pi_1 \\
\frac{b^{i-1}(1-b)}{(1-b)^{i-1}}\pi_1 + (1-b)\pi_{i+1} &= \frac{b^{i-1}}{(1-b)^{i-1}}\pi_1 \\
(1-b)\pi_{i+1} &= \frac{b^{i-1}b}{(1-b)^{i-1}}\pi_1 \\
\pi_{i+1} &= \frac{b^{(i+1)-1}}{(1-b)^{(i+1)-1}}\pi_1
\end{aligned}$$

Hence, $\pi_i = \left(\frac{b}{1-b}\right)^{i-1} \pi_1$ for i such that $1 \leq i \leq m$.

$$\begin{aligned}
\pi_1 + \left(\frac{b}{1-b}\right)\pi_1 + \cdots + \left(\frac{b}{1-b}\right)^{m-1}\pi_1 = 1 \implies \pi_1 \left(\frac{1-\left(\frac{b}{1-b}\right)^m}{1-\left(\frac{b}{1-b}\right)}\right) &= 1 \\
\implies \pi_1 &= \frac{1-\left(\frac{b}{1-b}\right)}{1-\left(\frac{b}{1-b}\right)^m}
\end{aligned}$$

$\therefore$ The stationary distribution is

$$\frac{1-\left(\frac{b}{1-b}\right)}{1-\left(\frac{b}{1-b}\right)^m} \begin{pmatrix} 1 \\ \left(\frac{b}{1-b}\right) \\ \vdots \\ \left(\frac{b}{1-b}\right)^{m-1} \end{pmatrix}$$