

Solutions to Problem Set 9: Stochastic Differential Equations

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MIT Financial Mathematics course website: <https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/>

Problem sets: <https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/>

Problem set 9: https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset9.pdf

1. (a)

$$\frac{\partial X_t}{\partial t} = 0, \quad \frac{\partial X_t}{\partial B_t} = e^{B_t} = X_t, \quad \frac{\partial^2 X_t}{\partial B_t^2} = e^{B_t} = X_t$$

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \frac{1}{2} \frac{\partial^2 X_t}{\partial B_t^2} \right) dt + \frac{\partial X_t}{\partial B_t} dB_t = \left(0 + \frac{1}{2} X_t \right) dt + X_t dB_t = \frac{1}{2} X_t dt + X_t dB_t$$

(b)

$$\frac{\partial X_t}{\partial t} = \frac{-B_t}{(1+t)^2} = -\frac{1}{1+t} X_t, \quad \frac{\partial X_t}{\partial B_t} = \frac{1}{1+t}, \quad \frac{\partial^2 X_t}{\partial B_t^2} = 0$$

$$\begin{aligned} dX_t &= \left(\frac{\partial X_t}{\partial t} + \frac{1}{2} \frac{\partial^2 X_t}{\partial B_t^2} \right) dt + \frac{\partial X_t}{\partial B_t} dB_t \\ &= \left(-\frac{1}{1+t} X_t + \frac{1}{2} \times 0 \right) dt + \frac{1}{1+t} dB_t \\ &= -\frac{1}{1+t} X_t dt + \frac{1}{1+t} dB_t \end{aligned}$$

(c)

$$\frac{\partial X_t}{\partial t} = 0, \quad \frac{\partial X_t}{\partial B_t} = \cos B_t = \sqrt{1 - \sin^2 B_t} = \sqrt{1 - X_t^2}, \quad \frac{\partial^2 X_t}{\partial B_t^2} = -\sin B_t = -X_t$$

$$dX_t = \left(\frac{\partial X_t}{\partial t} + \frac{1}{2} \frac{\partial^2 X_t}{\partial B_t^2} \right) dt + \frac{\partial X_t}{\partial B_t} dB_t$$

$$\begin{aligned}
&= \left(0 + \frac{1}{2}(-X_t)\right) dt + \sqrt{1 - X_t^2} dB_t \\
&= \frac{-X_t}{2} dt + \sqrt{1 - X_t^2} dB_t
\end{aligned}$$

2.

$$\begin{aligned}
\frac{\partial X_t}{\partial t} &= 0 \\
\frac{\partial X_t}{\partial B_t} &= 3 \left(a^{\frac{1}{3}} + \frac{1}{3} B_t\right)^2 \frac{1}{3} = \left(a^{\frac{1}{3}} + \frac{1}{3} B_t\right)^2 = X_t^{\frac{2}{3}} \\
\frac{\partial^2 X_t}{\partial B_t^2} &= 2 \left(a^{\frac{1}{3}} + \frac{1}{3} B_t\right) \frac{1}{3} = \frac{2}{3} \left(a^{\frac{1}{3}} + \frac{1}{3} B_t\right) = \frac{2}{3} X_t^{\frac{1}{3}} \\
dX_t &= \left(\frac{\partial X_t}{\partial t} + \frac{1}{2} \frac{\partial^2 X_t}{\partial B_t^2}\right) dt + \frac{\partial X_t}{\partial B_t} dB_t = \left(0 + \frac{1}{2} \times \frac{2}{3} X_t^{\frac{1}{3}}\right) dt + X_t^{\frac{2}{3}} dB_t = \frac{1}{3} X_t^{\frac{1}{3}} dt + X_t^{\frac{2}{3}} dB_t \\
X_0 &= \left(a^{\frac{1}{3}} + \frac{1}{3} B_0\right)^3 = \left(a^{\frac{1}{3}} + \frac{1}{3} \times 0\right)^3 = a
\end{aligned}$$

3.

$$\begin{aligned}
R(t) - e^{-\beta t} R(0) &= \frac{\alpha}{\beta} (1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dB_s \\
e^{\beta t} R(t) - R(0) &= \frac{\alpha}{\beta} (e^{\beta t} - 1) + \sigma \int_0^t e^{\beta s} dB_s \\
\int_0^t d(e^{\beta s} R(s)) &= \alpha \int_0^t e^{\beta s} ds + \sigma \int_0^t e^{\beta s} dB_s \\
d(e^{\beta t} R(t)) &= \alpha e^{\beta t} dt + \sigma e^{\beta t} dB_t \\
\beta e^{\beta t} R(t) + e^{\beta t} dR(t) &= \alpha e^{\beta t} dt + \sigma e^{\beta t} dB_t \\
\beta R(t) dt + dR(t) &= \alpha dt + \sigma dB_t \\
dR(t) &= (\alpha - \beta R(t)) dt + \sigma dB_t
\end{aligned}$$