Solutions to Problem Set 1: Linear Algebra

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MIT Financial Mathematics course website: https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/

Problem sets: https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/

Problem set 1: https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset1.pdf

Part A

1. (a) True (b) True (c) False (d) True (e) False (f) F (g) False

2. (a)

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\therefore \text{The rank of } \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \text{ is 3.}$$

(b)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 & -2 & 0 \end{pmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -2 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2 \to R_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_4-3R_3\to R_4}\begin{pmatrix}1&1&0&0&0&0\\0&0&4&0&-2&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}$$

$$\therefore \text{The rank of }\begin{pmatrix}1&1&0&0&0&0\\1&1&0&3&0&0\\0&0&2&1&-1&0\\0&0&4&0&-2&0\end{pmatrix}\text{ is 3.}$$

3. (a)

$$\begin{vmatrix}1&2\\4&-1\end{vmatrix}=1\times(-1)-2\times 4=-9$$

$$\begin{pmatrix}1&2\\4&-1\end{pmatrix}^{-1}=\frac{1}{-9}\begin{pmatrix}-1&-2\\-4&1\end{pmatrix}=\begin{pmatrix}\frac{1}{9}&\frac{2}{9}\\\frac{4}{9}&-\frac{1}{9}\end{pmatrix}$$

(b)

$$\begin{aligned}\begin{vmatrix}-1&-2&3\\1&2&0\\4&6&3\end{vmatrix}&=-1\begin{vmatrix}2&0\\6&3\end{vmatrix}-(-2)\begin{vmatrix}1&0\\4&3\end{vmatrix}+3\begin{vmatrix}1&2\\4&6\end{vmatrix}\\&=-(2\times3-0\times6)+2(1\times3-0\times4)+3(1\times6-2\times4)\\&=-6+2(3)+3(-2)\\&=-6\end{aligned}$$

$$\left(\begin{array}{ccc|ccc}-1&-2&3&1&0&0\\1&2&0&0&1&0\\4&6&3&0&0&1\end{array}\right)\xrightarrow[R_3+4R_1\to R_3]{R_2+R_1\to R_2}\left(\begin{array}{ccc|ccc}-1&-2&3&1&0&0\\0&0&3&1&1&0\\0&-2&15&4&0&1\end{array}\right)$$

$$\xrightarrow[R_2\leftrightarrow R_3]{R_1-R_3\to R_1}\left(\begin{array}{ccc|ccc}-1&0&-12&-3&0&-1\\0&-2&15&4&0&1\\0&0&3&1&1&0\end{array}\right)$$

$$\xrightarrow[R_2-5R_3\to R_2]{R_1+4R_3\to R_1}\left(\begin{array}{ccc|ccc}-1&0&0&1&4&-1\\0&-2&0&-1&-5&1\\0&0&3&1&1&0\end{array}\right)$$

$$\xrightarrow[\frac{1}{3}R_3\to R_3]{-R_1\to R_1,\ \frac{-1}{2}R_2\to R_2}\left(\begin{array}{ccc|ccc}1&0&0&-1&-4&1\\0&1&0&\frac{1}{2}&\frac{5}{2}&-\frac{1}{2}\\0&0&1&\frac{1}{3}&\frac{1}{3}&0\end{array}\right)$$

$$\therefore\begin{pmatrix}-1&-2&3\\1&2&0\\4&6&3\end{pmatrix}^{-1}=\begin{pmatrix}-1&-4&1\\\frac{1}{2}&\frac{5}{2}&-\frac{1}{2}\\\frac{1}{3}&\frac{1}{3}&0\end{pmatrix}$$

4.

$$\left|\begin{pmatrix}-3&3&2\\1&-1&-2\\-1&-3&0\end{pmatrix}-\lambda I\right|=\begin{vmatrix}-3-\lambda&3&2\\1&-1-\lambda&-2\\-1&-3&-\lambda\end{vmatrix}$$

$$= (-3-\lambda)\begin{vmatrix}-1-\lambda & -2\\ -3 & -\lambda\end{vmatrix} - 3\begin{vmatrix}1 & -2\\ -1 & -\lambda\end{vmatrix} + 2\begin{vmatrix}1 & -1-\lambda\\ -1 & -3\end{vmatrix}$$
$$= (-\lambda-3)(\lambda^2+\lambda-6) - 3(-\lambda-2) + 2(-3-(\lambda+1))$$
$$= -\lambda^3 - 4\lambda^2 + 3\lambda + 18 + 3\lambda + 6 - 6 - 2\lambda - 2$$
$$= -\lambda^3 - 4\lambda^2 + 4\lambda + 16$$

$\therefore$ The characteristic polynomial of $\begin{pmatrix}-3 & 3 & 2\\ 1 & -1 & -2\\ -1 & -3 & 0\end{pmatrix}$ is $-\lambda^3-4\lambda^2+4\lambda+16$.

$$-\lambda^3 - 4\lambda^2 + 4\lambda + 16 = (\lambda+4)(-\lambda^2+4) = (\lambda+4)(\lambda+2)(-\lambda+2)$$

$\therefore$ The eigenvalues of $\begin{pmatrix}-3 & 3 & 2\\ 1 & -1 & -2\\ -1 & -3 & 0\end{pmatrix}$ are -4, -2 and 2.

Case $\lambda = -4$:

$$\begin{pmatrix}-3-(-4) & 3 & 2\\ 1 & -1-(-4) & -2\\ -1 & -3 & -(-4)\end{pmatrix} = \begin{pmatrix}1 & 3 & 2\\ 1 & 3 & -2\\ -1 & -3 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & 3 & 2\\ 1 & 3 & -2\\ -1 & -3 & 4\end{pmatrix} \xrightarrow[R_1+R_3\to R_3]{R_2-R_1\to R_2} \begin{pmatrix}1 & 3 & 2\\ 0 & 0 & -4\\ 0 & 0 & 6\end{pmatrix} \xrightarrow[\frac{3}{2}R_2+R_3\to R_3]{R_1+\frac{1}{2}R_2\to R_1} \begin{pmatrix}1 & 3 & 0\\ 0 & 0 & -4\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{aligned} x+3y&=0\\ -4z&=0\end{aligned} \implies \begin{aligned} y&=\tfrac{-1}{3}x\\ z&=0\end{aligned}$$

$\therefore$ The eigenvector of $\begin{pmatrix}-3 & 3 & 2\\ 1 & -1 & -2\\ -1 & -3 & 0\end{pmatrix}$ with eigenvalue -4 is $\begin{pmatrix}3\\ -1\\ 0\end{pmatrix}$.

Case $\lambda = -2$:

$$\begin{pmatrix}-3-(-2) & 3 & 2\\ 1 & -1-(-2) & -2\\ -1 & -3 & -(-2)\end{pmatrix} = \begin{pmatrix}-1 & 3 & 2\\ 1 & 1 & -2\\ -1 & -3 & 2\end{pmatrix}$$

$$\begin{pmatrix}-1 & 3 & 2\\ 1 & 1 & -2\\ -1 & -3 & 2\end{pmatrix} \xrightarrow[R_3-R_1\to R_3]{R_1+R_2\to R_2} \begin{pmatrix}-1 & 3 & 2\\ 0 & 4 & 0\\ 0 & -6 & 0\end{pmatrix} \xrightarrow[\frac{3}{2}R_2+R_3\to R_3]{R_1-\frac{3}{4}R_2\to R_1} \begin{pmatrix}-1 & 0 & 2\\ 0 & 4 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\begin{aligned} -x+2z&=0\\ 4y&=0\end{aligned} \implies \begin{aligned} y&=0\\ z&=\tfrac{1}{2}x\end{aligned}$$

$\therefore$ The eigenvector of $\begin{pmatrix}-3 & 3 & 2\\ 1 & -1 & -2\\ -1 & -3 & 0\end{pmatrix}$ with eigenvalue -2 is $\begin{pmatrix}2\\ 0\\ 1\end{pmatrix}$.

Case $\lambda = 2$:

$$\begin{pmatrix} -3-2 & 3 & 2 \\ 1 & -1-2 & -2 \\ -1 & -3 & -2 \end{pmatrix} = \begin{pmatrix} -5 & 3 & 2 \\ 1 & -3 & -2 \\ -1 & -3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 3 & 2 \\ 1 & -3 & -2 \\ -1 & -3 & -2 \end{pmatrix} \xrightarrow[R_3 - R_2 \to R_3]{R_1 + R_2 \to R_1} \begin{pmatrix} -4 & 0 & 0 \\ 1 & -3 & -2 \\ -2 & 0 & 0 \end{pmatrix} \xrightarrow[R_2 + \frac{1}{2}R_3 \to R_2]{R_1 - 2R_3 \to R_1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & -2 \\ -2 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} -3y - 2z &= 0 \\ -2x &= 0 \end{aligned} \implies \begin{aligned} x &= 0 \\ y &= \tfrac{-2}{3} z \end{aligned}$$

$\therefore$ The eigenvector of $\begin{pmatrix} -3 & 3 & 2 \\ 1 & -1 & -2 \\ -1 & -3 & 0 \end{pmatrix}$ with eigenvalue 2 is $\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$.

5. (a)

$$e_1 = \frac{1}{\sqrt{1^2 + 0^2 + 1^2 + 0^2 + 1^2}} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\begin{aligned} u_2 &= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \end{pmatrix} - \left(\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \right) \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & 1 & \frac{1}{3} & 0 & \frac{-2}{3} \end{pmatrix} \end{aligned}$$

$$e_2 = \frac{u_2}{\sqrt{\frac{1}{9} + 1 + \frac{1}{9} + \frac{4}{9}}} = \frac{u_2}{\sqrt{\frac{15}{9}}} = \frac{3}{\sqrt{15}} \begin{pmatrix} \frac{1}{3} & 1 & \frac{1}{3} & 0 & \frac{-2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{15}} & \sqrt{\frac{3}{5}} & \frac{1}{\sqrt{15}} & 0 & \frac{-2}{\sqrt{15}} \end{pmatrix}$$

$$\begin{aligned} u_3 &= \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix} - \left(\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix} \cdot e_1 \right) e_1 - \left(\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix} \cdot e_2 \right) e_2 \\ &= \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} - \frac{-1}{\sqrt{15}} \begin{pmatrix} \frac{1}{\sqrt{15}} & \sqrt{\frac{3}{5}} & \frac{1}{\sqrt{15}} & 0 & \frac{-2}{\sqrt{15}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{2}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{15} & \frac{1}{5} & \frac{1}{15} & 0 & \frac{2}{15} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3}{5} & \frac{1}{5} & \frac{2}{5} & 1 & \frac{1}{5} \end{pmatrix} \end{aligned}$$

$$e_3 = \frac{u_3}{\sqrt{\frac{9}{25} + \frac{1}{25} + \frac{4}{25} + 1 + \frac{1}{25}}} = \frac{u_3}{\sqrt{\frac{40}{25}}} = \frac{5}{2\sqrt{10}} u_3 = \begin{pmatrix} \frac{-3}{2\sqrt{10}} & \frac{1}{2\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{\sqrt{5}}{2\sqrt{2}} & \frac{1}{2\sqrt{10}} \end{pmatrix}$$

$\therefore \left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{15}} & \sqrt{\frac{3}{5}} & \frac{1}{\sqrt{15}} & 0 & \frac{-2}{\sqrt{15}} \end{pmatrix}, \begin{pmatrix} \frac{-3}{2\sqrt{10}} & \frac{1}{2\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{\sqrt{5}}{2\sqrt{2}} & \frac{1}{2\sqrt{10}} \end{pmatrix} \right\}$
is an orthonormal basis of the subspace of $\mathbb{R}^5$ spanned by v_1, v_2 and v_3.

(b) We must find two linearly independent vectors that are not in the space spanned by v_1, v_2 and v_3. Using row reduction,

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

we can see that $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ are two such vectors. We will now apply the Gram-Schmidt process on them.
Let $v_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ and $v_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

$$\begin{aligned} u_4 &= v_4 - (v_4 \cdot e_1)e_1 - (v_4 \cdot e_2)e_2 - (v_4 \cdot e_3)e_3 \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} - 0e_1 - 0e_2 - \frac{\sqrt{5}}{2\sqrt{2}} \begin{pmatrix} \frac{-3}{2\sqrt{10}} & \frac{1}{2\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{\sqrt{5}}{2\sqrt{2}} & \frac{1}{2\sqrt{10}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \frac{-3}{8} & \frac{1}{8} & \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{8} & \frac{-1}{8} & \frac{-1}{4} & \frac{3}{8} & \frac{-1}{8} \end{pmatrix} \end{aligned}$$

$$e_4 = \frac{u_4}{\sqrt{\frac{9}{64} + \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{64}}} = \frac{u_4}{\sqrt{\frac{24}{64}}} = \frac{8}{2\sqrt{6}} u_4 = \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2\sqrt{6}} \end{pmatrix}$$

$$\begin{aligned} u_5 &= v_5 - (v_5 \cdot e_1)e_1 - (v_5 \cdot e_2)e_2 - (v_5 \cdot e_3)e_3 - (v_5 \cdot e_4)e_4 \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T - \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}^T - \frac{-2}{\sqrt{15}} \begin{pmatrix} \frac{1}{\sqrt{15}} \\ \sqrt{\frac{3}{5}} \\ \frac{1}{\sqrt{15}} \\ 0 \\ \frac{-2}{\sqrt{15}} \end{pmatrix}^T - \frac{1}{2\sqrt{10}} \begin{pmatrix} \frac{-3}{2\sqrt{10}} \\ \frac{1}{2\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{\sqrt{5}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{10}} \end{pmatrix}^T - \frac{-1}{2\sqrt{6}} \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{6}} \end{pmatrix}^T \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T - \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}^T + \begin{pmatrix} \frac{2}{15} \\ \frac{2}{5} \\ \frac{2}{15} \\ 0 \\ \frac{-4}{15} \end{pmatrix}^T - \begin{pmatrix} \frac{-3}{40} \\ \frac{1}{40} \\ \frac{1}{20} \\ \frac{1}{8} \\ \frac{1}{40} \end{pmatrix}^T + \begin{pmatrix} \frac{1}{8} \\ \frac{-1}{24} \\ \frac{-1}{12} \\ \frac{1}{8} \\ \frac{-1}{24} \end{pmatrix}^T \\ &= \begin{pmatrix} 0 & \frac{1}{3} & \frac{-1}{3} & 0 & \frac{1}{3} \end{pmatrix} \end{aligned}$$

$$e_5 = \frac{u_5}{\sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}}} = \frac{u_5}{\sqrt{\frac{1}{3}}} = \sqrt{3} u_5 = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$\therefore \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2\sqrt{6}} \end{pmatrix}$ and $\begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$ complete the basis found in (a) into an orthonormal basis of $\mathbb{R}^5$.

(c) $\begin{pmatrix} e_1^T & e_2^T & e_3^T & e_4^T & e_5^T \end{pmatrix}^T$ transforms the standard basis of $\mathbb{R}^5$ into the basis found in (b), so $\begin{pmatrix} e_1^T & e_2^T & e_3^T & e_4^T & e_5^T \end{pmatrix}^{T^{-1}}$ transforms the basis found in (b) into the standard basis. As e_1, e_2, e_3, e_4 and e_5 are orthonormal, $\begin{pmatrix} e_1^T & e_2^T & e_3^T & e_4^T & e_5^T \end{pmatrix}^{T^{-1}} = \begin{pmatrix} e_1^T & e_2^T & e_3^T & e_4^T & e_5^T \end{pmatrix}$.

$$\therefore U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{15}} & \frac{-3}{2\sqrt{10}} & \frac{\sqrt{3}}{2\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{3}{5}} & \frac{1}{2\sqrt{10}} & \frac{-1}{2\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{15}} & \frac{1}{\sqrt{10}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{5}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{15}} & \frac{1}{2\sqrt{10}} & \frac{-1}{2\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

6. (a) First, let us find Σ:

$$A^T A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

$$\begin{aligned} |A^T A - \lambda I| &= \begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} \\ &= (10-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} + 0\begin{vmatrix} 0 & 4 \\ 2 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & 10-\lambda \\ 2 & 4 \end{vmatrix} \\ &= (10-\lambda)(\lambda^2 - 12\lambda + 20 - 16) + 2(-(10-\lambda)2) \\ &= (10-\lambda)(\lambda^2 - 12\lambda + 4) - 4(10-\lambda) \\ &= (10-\lambda)(\lambda^2 - 12\lambda + 4 - 4) \\ &= (10-\lambda)\lambda(\lambda - 12) \end{aligned}$$

$$\Sigma = \begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}$$

Now, let us find V^T.
Case $\lambda = 12$:

$$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{pmatrix} \xrightarrow{R_1 + R_3 \to R_3} \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{pmatrix} \xrightarrow{2R_2 + R_3 \to R_3} \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} -2x + 2z &= 0 \\ -2y + 4z &= 0 \end{aligned} \implies \begin{aligned} y &= 2z \\ z &= x \end{aligned}$$

The unit eigenvector with eigenvalue 12 is $\frac{1}{\sqrt{1+4+1}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

Case $\lambda = 10$:

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{pmatrix} \xrightarrow[R_3+4R_1\to R_3]{R_2-2R_1\to R_2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 4 & 0 \end{pmatrix} \xrightarrow[R_3\to\frac{1}{2}R_3]{R_1\to\frac{1}{2}R_2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

$$\begin{aligned} x + 2y &= 0 \\ z &= 0 \end{aligned} \implies \begin{aligned} y &= \tfrac{-1}{2}x \\ z &= 0 \end{aligned}$$

The unit eigenvector with eigenvalue 10 is $\frac{1}{\sqrt{4+1}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{pmatrix}$.

To find the third column of V, we calculate the cross product of the two eigenvectors:

$$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \end{pmatrix}$$

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{pmatrix}$$

Now, let us find U:

$$U = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{6}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \sqrt{6} & \sqrt{5} \\ \sqrt{6} & -\sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{pmatrix}$$

(b) First, let us find Σ:

$$A^TA = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$|A^TA - \lambda I| = \begin{vmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 - 25 = \lambda(\lambda - 10)$$

$$\Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}$$

Now, let us find V^T.

Case $\lambda = 10$:

$$\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \xrightarrow{R_1+R_2\to R_2} \begin{pmatrix} -5 & 5 \\ 0 & 0 \end{pmatrix} \xrightarrow{\frac{-1}{5}R_1\to R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$x - y = 0 \implies x = y$$

The unit eigenvector with eigenvalue 10 is $\frac{1}{\sqrt{1+1}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$.

To find the second column of V, we take an orthogonal unit vector:

$$V^T = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$$

Now, let us find U:

$$\frac{1}{\sqrt{10}}\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix}\sqrt{2}\\ 2\sqrt{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix}$$

The second column of U is a unit vector orthogonal to $\begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix}$:

$$U = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{pmatrix}$$

$$\therefore \begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}\sqrt{10} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$$

(c) First, let us find Σ:

$$A^TA = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & 0 & 2\\ 0 & 1 & 0\\ 2 & 0 & 2\end{pmatrix}$$

$$\begin{aligned}
|A^TA - \lambda I| &= \begin{vmatrix}2-\lambda & 0 & 2\\ 0 & 1-\lambda & 0\\ 2 & 0 & 2-\lambda\end{vmatrix}\\
&= (2-\lambda)\begin{vmatrix}1-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix} - 0\begin{vmatrix}0 & 0\\ 2 & 2-\lambda\end{vmatrix} + 2\begin{vmatrix}0 & 1-\lambda\\ 2 & 0\end{vmatrix}\\
&= (2-\lambda)(1-\lambda)(2-\lambda) + 2(-(1-\lambda)2)\\
&= (2-\lambda)^2(1-\lambda) - 4(1-\lambda)\\
&= (1-\lambda)(\lambda^2 - 4\lambda + 4 - 4)\\
&= (1-\lambda)\lambda(\lambda - 4)
\end{aligned}$$

$$\Sigma = \begin{pmatrix}\sqrt{4} & 0 & 0\\ 0 & \sqrt{1} & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Now, let us find V^T.

Case $\lambda = 4$:

$$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -2 \end{pmatrix} \xrightarrow{R_1+R_3\to R_3} \begin{pmatrix} -2 & 0 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\frac{-1}{3}R_2\to R_2]{\frac{-1}{2}R_1\to R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x - z &= 0 \\ y &= 0 \end{aligned} \implies \begin{aligned} y &= 0 \\ z &= x \end{aligned}$$

The unit eigenvector with eigenvalue 4 is $\frac{1}{\sqrt{1+1}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Case $\lambda = 1$:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-2R_1\to R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{R_1+\frac{2}{3}R_3\to R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{\frac{-1}{3}R_3\to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x = 0$$

$$z = 0$$

The unit eigenvector with eigenvalue 1 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

To find the third column of V, we calculate the cross product of the two eigenvectors:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now, let us find U:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

As $\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ is orthogonal to both $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Part B

1. (a) True.

   Since Σ is diagonal, the number of non-zero diagonal entries of Σ equals the rank of Σ. Since V^T is orthonormal, V^T has full rank, and since V^T has full rank, the rank of Σ equals the rank of ΣV^T. Since U is orthonormal, U has full rank, and since U has full rank, the rank of ΣV^T equals the rank of $U\Sigma V^T$, which is the rank of A.

   $\therefore$ The number of non-zero diagonal entries of Σ equals the rank of A.

   (b) True.

   The diagonal entries of Σ are the singular values of A. If A is an $m \times n$ matrix, the largest singular value of A equals the maximum value of $\frac{\|Av\|}{\|v\|}$ for any v in $\mathbb{R}^n$. Denote one such v by v_1. The next largest singular value of A is the maximum value of $\frac{\|Av\|}{\|v\|}$ for any v in the subspace of $\mathbb{R}^n$ orthogonal to v_1. Denote one such v by v_2. The next largest singular value of A is the maximum value of $\frac{\|Av\|}{\|v\|}$ for any v in the subspace of $\mathbb{R}^n$ orthogonal to v_1 and v_2. Continuing this process gives us the singular values of A in descending order, which are the diagonal entries of Σ.

   $\therefore$ Σ is uniquely determined up to permuting the rows and columns.

   (c) False.

   Let A be any $n \times n$ matrix such that $A^T \neq A$ and suppose that $U = V$. Since $A = U\Sigma U^T$ and U is orthonormal, U is $n \times n$. Then, Σ is $n \times n$, and as Σ is diagonal, $\Sigma^T = \Sigma$. Then,

   $$A^T = (U\Sigma U^T)^T = U^{T^T}(U\Sigma)^T = U\Sigma^T U^T = U\Sigma U^T = A$$

   This is a contradiction.

   $\therefore$ If A is $n \times n$, U and V cannot always be chosen so that $U = V$.

   (d) True.

   Assume that A is $n \times n$ symmetric. Then A has n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $v_1, v_2, \ldots, v_n$. For any λ_i and λ_j such that $\lambda_i = \lambda_j$, $A(av_i + bv_j) = \lambda_i(av_i + bv_j)$ for any real numbers a and b. Hence, for any eigenvalue λ_i, the span of the eigenvectors with eigenvalue λ_i has an orthonormal basis. Now, consider any v_i and v_j such that $\lambda_i \neq \lambda_j$:

   $$\lambda_i(v_i^T v_j) = (Av_i)^T v_j = v_i^T A^T v_j = v_i^T A v_j = \lambda_j(v_i^T v_j) \implies v_i^T v_j = 0$$

   Hence, $v_1, v_2, \ldots, v_n$ can be chosen to be orthonormal. Let $v_1, v_2, \ldots, v_n$ be orthonormal, let the ith column of U be v_i, and let the ith diagonal entry of Σ be λ_i. Then for any v_i,

   $$U\Sigma U^T v_i = U\Sigma e_i = U(\lambda_i e_i) = \lambda_i v_i = Av_i$$

Since $v_1, v_2, \ldots, v_n$ span $\mathbb{R}^n$, $U\Sigma U^T v = Av$ for any v in $\mathbb{R}^n$.

$\therefore$ If A is symmetric, U and V can be chosen so that $U = V$.

2. Let $A = U\Sigma V^T$ such that U and V are orthonormal and Σ is diagonal. Then the non-zero values of Σ are the square roots of the non-zero eigenvalues of $A^T A$. Also, $A^T = (U\Sigma V^T)^T = V^{T^T}(U\Sigma)^T = V\Sigma^T U^T$, so the non-zero values of Σ^T, which are the non-zero values of Σ, are the square roots of the non-zero eigenvalues of $A^{T^T} A^T = AA^T$.

   $\therefore$ AA^T and $A^T A$ have the same set of non-zero eigenvalues.

3. Assume that A is a symmetric positive semi-definite matrix. From 1. (d), it follows that $A = UDU^T$ for some U and D such that U is orthonormal, D is diagonal and the ith column of U is the eigenvector of A with eigenvalue equal to the ith diagonal entry of D. It remains to show that all entries of D are non-negative. Let u_i denote the ith column of U. Then, u_i^T is the ith row of U^T and Au_i is the ith column of AU. Then, $u_i^T A u_i$ is the ith diagonal entry of $U^T AU = D$ and $u_i^T A u_i \geq 0$, so all diagonal entries of D are non-negative.

   $\therefore$ For any symmetric positive semi-definite U such that $D = U^T AU$ is a diagonal matrix with all non-negative entries.

4. (a) Assume that A is diagonal and let $v_i = \begin{pmatrix} v_{i,1} & v_{i,2} & \cdots & v_{i,m} \end{pmatrix}^T$ and $w = \begin{pmatrix} w_1 & w_2 & \cdots & w_m \end{pmatrix}^T$. Then, the ith diagonal entry of A is $v_{i,i}$. Either $m \leq n$ or $m > n$. If $m \leq n$, then:

$$L = \|A\vec{x} - w\| = \left\| \begin{pmatrix} v_{1,1}x_1 \\ v_{2,2}x_2 \\ \vdots \\ v_{m,m}x_m \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \right\| = \sqrt{\sum_{i=1}^{m} (v_{i,i}x_i - w_i)^2}$$

   If $m > n$, then:

$$L = \|A\vec{x} - w\| = \left\| \begin{pmatrix} v_{1,1}x_1 \\ v_{2,2}x_2 \\ \vdots \\ v_{n,n}x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ w_{n+1} \\ \vdots \\ w_m \end{pmatrix} \right\| = \sqrt{\sum_{i=1}^{n} (v_{i,i}x_i - w_i)^2 + \sum_{i=n+1}^{m} w_i^2}$$

   Clearly, L is minimised if $x_i = \frac{w_i}{v_{i,i}}$ for all i such that $v_{i,i} \neq 0$ and $1 \leq i \leq \min(m, n)$.

   $\therefore$ The problem can be solved directly is A is a diagonal matrix.

(b) When A is a diagonal matrix, $V^T\vec{x}$ and U^Tw are permutations of the rows of $\vec{x}$ and w respectively, Σ has the same diagonal entries as A (possibly in a different order) and $U(\Sigma V^T\vec{x}-U^Tw)$ is a permutation of the rows of $\Sigma V^T\vec{x}-U^Tw$. Then, $\|A\vec{x}-w\|$ is minimised when the ith entry of $V^T\vec{x}$ is set to equal the ith entry of U^Tw divided by the ith diagonal entry of Σ.

(c) Let

$$A = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix}, \vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}, w = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Then

$$\sum_{i=1}^{n}(ax_i+b-y_i)^2 = \left\| \begin{pmatrix} ax_1+b \\ ax_2+b \\ \vdots \\ ax_n+b \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - w \right\|^2 = \|A\vec{x}-w\|^2$$

Since $\|A\vec{x}-w\| > 0$, $\sum_{i=1}^{n}(ax_i+b-y_i)^2$ is minimised when $\|A\vec{x}-w\|$ is minimised.

$\therefore$ This is a special case of the problem given above.

(d) Let

$$A = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{pmatrix}, \vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, w = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Then

$$\begin{aligned} \sum_{i=1}^{n}(ax_i^2+bx_i+c-y_i)^2 &= \left\| \begin{pmatrix} ax_1^2+bx_1+c \\ ax_2^2+bx_2+c \\ \vdots \\ ax_n^2+bx_n+c \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\|^2 \\ &= \left\| \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} - w \right\|^2 \\ &= \|A\vec{x}-w\|^2 \end{aligned}$$

Since $\|A\vec{x}-w\| > 0$, $\sum_{i=1}^{n}(ax_i+bx_i+c-y_i)^2$ is minimised when $\|A\vec{x}-w\|$ is minimised.

$\therefore$ This is a special case of the problem given above.