Solutions to Problem Set 9: Stochastic Differential Equations

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MIT Financial Mathematics course website: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/`

Problem sets: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/`

Problem set 9: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset9.pdf`

1. (a) Let $f(t,x) = e^x$.

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = e^x = f(t,x), \quad \frac{\partial^2 f}{\partial x^2} = e^x = f(t,x)$$

$$df(t,B_t) = \left(0 + \frac{f(t,B_t)}{2}\right)dt + f(t,B_t)dB_t = \frac{f(t,B_t)}{2}dt + f(t,B_t)dB_t$$

$\therefore X_t = e^{B_t}$ solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$.

(b) Let $f(t,x) = \frac{x}{1+t}$.

$$\frac{\partial f}{\partial t} = \frac{-x}{(1+t)^2} = \frac{-f(t,x)}{1+t}, \quad \frac{\partial f}{\partial x} = \frac{1}{1+t}, \quad \frac{\partial^2 f}{\partial x^2} = 0$$

$$df(t,B_t) = \left(\frac{-f(t,B_t)}{1+t} + \frac{0}{2}\right)dt + \frac{1}{1+t}dB_t = \frac{-f(t,B_t)}{1+t}dt + \frac{1}{1+t}dB_t$$

$\therefore X_t = \frac{B_t}{1+t}$ solves $dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t$.

(c) Let $f(t,x) = \sin x$.

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-f^2(t,x)}, \quad \frac{\partial^2 f}{\partial x^2} = -\sin x = -f(t,x)$$

$$df(t,B_t) = \left(0 + \frac{-f(t,B_t)}{2}\right)dt + \sqrt{1-f^2(t,B_t)}dB_t = \frac{-f(t,B_t)}{2}dt + \sqrt{1-f^2(t,B_t)}dB_t$$

$\therefore X_t = \sin B_t$ solves $dX_t = -\frac{1}{2}X_t dt + \sqrt{1-X_t^2}dB_t$.

2. Let $f(t,x) = \left(\sqrt[3]{a} + \frac{x}{3}\right)^3$.

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial x} = 3\left(\sqrt[3]{a} + \frac{x}{3}\right)^2 \frac{1}{3} = \left(\sqrt[3]{a} + \frac{x}{3}\right)^2 = f^{\frac{2}{3}}(t,x)$$

$$\frac{\partial^2 f}{\partial x^2} = 2\left(\sqrt[3]{a} + \frac{x}{3}\right)\frac{1}{3} = \frac{2\left(\sqrt[3]{a} + \frac{x}{3}\right)}{3} = \frac{2\sqrt[3]{f(t,x)}}{3}$$

$$df(t,B_t) = \left(0 + \frac{\frac{2\sqrt[3]{f(t,B_t)}}{3}}{2}\right)dt + f^{\frac{2}{3}}(t,B_t)dB_t = \frac{\sqrt[3]{f(t,B_t)}}{3}dt + f^{\frac{2}{3}}(t,B_t)dB_t$$

$$f(0,B_0) = \left(\sqrt[3]{a} + \frac{B_0}{3}\right)^3 = \left(\sqrt[3]{a} + \frac{0}{3}\right)^3 = a$$

$\therefore X_t = \left(\sqrt[3]{a} + \frac{B_t}{3}\right)^3$ solves $dX_t = \frac{1}{3}X_t^{\frac{1}{3}}dt + X_t^{\frac{2}{3}}dB_t$ subject to $X_0 = a$.

3. Let $R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB_s$.

$$\begin{aligned}
dR(t) &= d\left(e^{-\beta t}\left(R(0) - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta} + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB_s\right) \\
&= d\left(e^{-\beta t}\left(R(0) - \frac{\alpha}{\beta}\right) + \frac{\alpha}{\beta}\right) + \sigma d\left(e^{-\beta t}\int_0^t e^{\beta s}dB_s\right) \\
&= -\beta e^{-\beta t}\left(R(0) - \frac{\alpha}{\beta}\right)dt + \sigma d\left(e^{-\beta t}\right)\int_0^t e^{\beta s}dB_s + \sigma e^{-\beta t}d\left(\int_0^t e^{\beta s}dB_s\right) \\
&= -\beta\left(e^{-\beta t}R(0) - \frac{\alpha}{\beta}e^{-\beta t}\right)dt + \sigma\left(-\beta e^{-\beta t}dt\right)\int_0^t e^{\beta s}dB_s + \sigma e^{-\beta t}e^{\beta t}dB_t \\
&= -\beta\left(e^{-\beta t}R(0) - \frac{\alpha}{\beta}e^{-\beta t} + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB_s\right)dt + \sigma dB_t \\
&= \left(\alpha - \beta\left(e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB_s\right)\right)dt + \sigma dB_t \\
&= (\alpha - \beta R(t))dt + \sigma dB_t
\end{aligned}$$

$\therefore R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB_s$ solves the SDE $dR(t) = (\alpha - \beta R(t))dt + \sigma dB_t$.