Solutions to Problem Set 5: Volatility Modeling

Jura Ivanković

MIT Financial Mathematics course website: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/`

Problem sets: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/`

Problem set 5: `https://ocw.mit.edu/courses/mathematics/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/assignments/MIT18_S096F13_pset5.pdf`

1. (a)

$$\begin{aligned}
\ln f_{Y_1,Y_2,\ldots,Y_n}(y_1,y_2,\ldots,y_n;\mu,\sigma,h) &= \ln \prod_{i=1}^{n} f_{Y_i}(y_i;\mu,\sigma,h) \\
&= \sum_{i=1}^{n} \ln \frac{e^{\frac{-(y_i-\mu h)^2}{2\sigma^2 h}}}{\sigma\sqrt{h}\sqrt{2\pi}} \\
&= \sum_{i=1}^{n} \left(\frac{-(y_i-\mu h)^2}{2\sigma^2 h} - \ln\sigma - \ln\sqrt{h2\pi} \right) \\
&= \frac{-\sum_{i=1}^{n}(y_i-\mu h)^2}{2\sigma^2 h} - \frac{n\ln\sigma^2}{2} - n\ln\sqrt{h2\pi}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial\mu}(y_1,y_2,\ldots,y_n;\mu,\sigma,h) &= \frac{-\sum_{i=1}^{n}\frac{\partial(y_i-\mu h)^2}{\partial\mu}}{2\sigma^2 h} \\
&= \frac{-\sum_{i=1}^{n}2(y_i-\mu h)(-h)}{2\sigma^2 h} \\
&= \frac{\sum_{i=1}^{n}(y_i-\mu h)}{\sigma^2} \\
&= \frac{\sum_{i=1}^{n}y_i-\mu T}{\sigma^2}
\end{aligned}$$

$$\frac{\partial^2 \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial\mu^2}(y_1,y_2,\ldots,y_n;\mu,\sigma,h) = \frac{-T}{\sigma^2} < 0$$

$$\frac{\partial \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial\mu}(y_1,y_2,\ldots,y_n;\mu,\sigma,h) = 0 \implies \mu = \frac{\sum_{i=1}^{n}y_i}{T}$$

$$\therefore \hat{\mu} = \frac{\sum_{i=1}^n y_i}{T} \quad \left(\hat{\mu} = \frac{\sum_{i=1}^n y_i}{n} \text{ if } h = 1\right)$$

$$\begin{aligned}
\frac{\partial \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial \sigma^2}(y_1, y_2, \ldots, y_n; \mu, \sigma, h) &= \frac{-\sum_{i=1}^n (y_i - \mu h)^2 \frac{\partial \sigma^{-2}}{\partial \sigma^2}}{2h} - \frac{n \frac{\partial \ln \sigma^2}{\partial \sigma^2}}{2} \\
&= \frac{\sum_{i=1}^n (y_i - \mu h)^2}{2\sigma^4 h} - \frac{n}{2\sigma^2} \\
&= \frac{\sum_{i=1}^n (y_i - \mu h)^2 - \sigma^2 T}{2\sigma^4 h}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ln f_{Y_1,Y_2,\ldots,Y_n}}{\left(\partial \sigma^2\right)^2}(y_1, y_2, \ldots, y_n; \mu, \sigma, h) &= \frac{\sum_{i=1}^n (y_i - \mu h)^2 \frac{\partial \sigma^{-4}}{\partial \sigma^2}}{2h} - \frac{n \frac{\partial \sigma^{-2}}{\partial \sigma^2}}{2} \\
&= \frac{-\sum_{i=1}^n (y_i - \mu h)^2}{\sigma^6 h} + \frac{n}{2\sigma^4} \\
&= \frac{\sigma^2 T - 2\sum_{i=1}^n (y_i - \mu h)^2}{2\sigma^6 h}
\end{aligned}$$

$$\frac{\partial \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial \sigma^2}(y_1, y_2, \ldots, y_n; \mu, \sigma, h) = 0 \implies \sigma^2 = \frac{\sum_{i=1}^n (y_i - \mu h)^2}{T}$$

$$\begin{aligned}
\frac{\partial^2 \ln f_{Y_1,\ldots,Y_n}}{\left(\partial \sigma^2\right)^2}\left(y_1, \ldots, y_n; \mu, \sqrt{\frac{\sum_{i=1}^n (y_i - \mu h)^2}{T}}, h\right) &= \frac{\sqrt{\frac{\sum_{i=1}^n (y_i - \mu h)^2}{T}}^2 T - 2\sum_{i=1}^n (y_i - \mu h)^2}{2\sqrt{\frac{\sum_{i=1}^n (y_i - \mu h)^2}{T}}^6 h} \\
&= \frac{-\sum_{i=1}^n (y_i - \mu h)^2}{2\sqrt{\frac{\sum_{i=1}^n (y_i - \mu h)^2}{T}}^6 h} \\
&< 0
\end{aligned}$$

$$\therefore \hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \hat{\mu} h)^2}{T} \quad \left(\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \hat{\mu})^2}{n} \text{ if } h = 1\right)$$

(b)

$$y_1, y_2, \ldots, y_n \overset{i.i.d.}{\sim} \mathcal{N}\left(\mu h, \sigma^2 h\right)$$

$$\sum_{i=1}^n y_i \sim \mathcal{N}\left(T\mu, T\sigma^2\right)$$

$$\therefore \hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right) \quad \left(\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \text{ if } h = 1\right)$$

(c) Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T$.

$$\begin{pmatrix} \frac{1}{T} & \frac{1}{T} & \cdots & \frac{1}{T} \\ \frac{n-1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \frac{-1}{n} & \frac{n-1}{n} & \cdots & \frac{-1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{n} & \frac{-1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix} \mathbf{y} = \begin{pmatrix} \hat{\mu} \\ \mathbf{y} - \hat{\mu} h \mathbf{1}_n \end{pmatrix} \sim \mathcal{N}_{n+1}$$

For any $1 \le i \le n$,

$$\begin{aligned}
Cov(y_i - \hat{\mu}h, \hat{\mu}) &= E\left[y_i\hat{\mu} - \hat{\mu}^2 h\right] - E[y_i - \hat{\mu}h]E[\hat{\mu}] \\
&= \frac{E\left[y_i \sum_{j=1}^n y_j\right]}{T} - hE\left[\hat{\mu}^2\right] - 0E[\hat{\mu}] \\
&= \frac{(n-1)\mu^2 h^2 + E\left[y_i^2\right]}{T} - h\left(Var\left(\hat{\mu}\right) + \mu^2\right) \\
&= \frac{n\mu^2 h^2 + Var(y_i)}{T} - h\left(\frac{\sigma^2}{T} + \mu^2\right) \\
&= \mu^2 h + \frac{\sigma^2 h}{T} - \frac{h\sigma^2}{T} - h\mu^2 \\
&= 0
\end{aligned}$$

Since $\left(\hat{\mu} \quad \mathbf{y}^T - \hat{\mu}h\mathbf{1}_n^T\right)^T$ has a multivariate normal distribution and $Cov(y_i - \hat{\mu}h, \hat{\mu}) = 0$ for all $1 \le i \le n$, $\hat{\mu}$ and $\mathbf{y} - \hat{\mu}h\mathbf{1}_n$ are independent.

$$\begin{aligned}
\frac{\sum_{i=1}^n (y_i - \mu h)^2}{\sigma^2 h} &= \frac{\sum_{i=1}^n (y_i - \hat{\mu}h + \hat{\mu}h - \mu h)^2}{\sigma^2 h} \\
&= \frac{\sum_{i=1}^n \left((y_i - \hat{\mu}h)^2 + 2(y_i - \hat{\mu}h)(\hat{\mu}h - \mu h) + (\hat{\mu}h - \mu h)^2\right)}{\sigma^2 h} \\
&= \frac{\sum_{i=1}^n (y_i - \hat{\mu}h)^2 + n(\hat{\mu}h - \mu h)^2}{\sigma^2 h} \qquad \left(\because \sum_{i=1}^n (y_i - \hat{\mu}h) = 0\right) \\
&= \frac{n\hat{\sigma}^2 + T(\hat{\mu} - \mu)^2}{\sigma^2}
\end{aligned}$$

Since $n\hat{\sigma}^2/\sigma^2$ and $T(\hat{\mu} - \mu)^2/\sigma^2$ are functions only of $\mathbf{y} - \hat{\mu}h\mathbf{1}_n$ and $\hat{\mu}$ respectively and $\mathbf{y} - \hat{\mu}h\mathbf{1}_n$ and $\hat{\mu}$ are independent, $n\hat{\sigma}^2/\sigma^2$ and $T(\hat{\mu} - \mu)^2/\sigma^2$ are independent.

$$y_i \overset{i.i.d.}{\sim} \mathcal{N}\left(\mu h, \sigma^2 h\right) \implies \frac{y_i - \mu h}{\sigma\sqrt{h}} \overset{i.i.d.}{\sim} \mathcal{N}(0,1) \implies \frac{\sum_{i=1}^n (y_i - \mu h)^2}{\sigma^2 h} \sim \chi_n^2$$

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right) \implies \frac{\sqrt{T}(\hat{\mu} - \mu)}{\sigma} \sim \mathcal{N}(0,1) \implies \frac{T(\hat{\mu} - \mu)^2}{\sigma^2} \sim \chi_1^2$$

Since $n\hat{\sigma}^2/\sigma^2$ and $T(\hat{\mu} - \mu)^2/\sigma^2$ are independent, $n\hat{\sigma}^2/\sigma^2 + \chi_1^2 = \chi_n^2$.

$$\therefore \quad \hat{\sigma}^2 = \frac{\sigma^2 \chi_{n-1}^2}{n}, \quad E\left[\hat{\sigma}^2\right] = \frac{(n-1)\sigma^2}{n}, \quad Var\left(\hat{\sigma}^2\right) = \frac{2(n-1)\sigma^4}{n^2}$$

(d)

$$\hat{\mu}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right) \qquad \forall n \in \mathbb{Z}^+$$

$$\therefore \hat{\mu}_n \xrightarrow{p} \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right)$$

$$\lim_{n\to\infty} E\left[\hat{\sigma}_n^2\right] = \lim_{n\to\infty} \frac{(n-1)\sigma^2}{n} = \sigma^2$$

$$\lim_{n\to\infty} Var\left(\hat{\sigma}_n^2\right) = \lim_{n\to\infty} \frac{2(n-1)\sigma^4}{n^2} = 0$$

$$\therefore \hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$$

(e) $\hat{\mu}_n$ is not weakly consistent. $\hat{\sigma}_n^2$ is weakly consistent.

2. (a)

$$\begin{aligned}
\ln f_{Y_1,Y_2,\ldots,Y_n}(y_1,y_2,\ldots,y_n;\mu,\sigma,h_1,h_2,\ldots,h_n) &= \ln \prod_{i=1}^{n} f_{Y_i}(y_i;\mu,\sigma,h_i) \\
&= \sum_{i=1}^{n} \ln \frac{e^{\frac{-(y_i-\mu h_i)^2}{2\sigma^2 h_i}}}{\sigma\sqrt{h_i}\sqrt{2\pi}} \\
&= \sum_{i=1}^{n} \left(\frac{-(y_i-\mu h_i)^2}{2\sigma^2 h_i} - \ln\sigma - \ln\sqrt{h_i 2\pi} \right) \\
&= -\sum_{i=1}^{n} \frac{(y_i-\mu h_i)^2}{2\sigma^2 h_i} - \frac{n\ln\sigma^2}{2} - \sum_{i=1}^{n} \ln\sqrt{h_i 2\pi}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial\mu}(y_1,y_2,\ldots,y_n;\mu,\sigma,h_1,h_2,\ldots,h_n) &= -\sum_{i=1}^{n} \frac{\frac{\partial(y_i-\mu h_i)^2}{\partial\mu}}{2\sigma^2 h_i} \\
&= -\sum_{i=1}^{n} \frac{2(y_i-\mu h_i)(-h_i)}{2\sigma^2 h_i} \\
&= \sum_{i=1}^{n} \frac{y_i-\mu h_i}{\sigma^2} \\
&= \frac{\sum_{i=1}^{n} y_i - \mu T}{\sigma^2}
\end{aligned}$$

$$\frac{\partial^2 \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial\mu^2}(y_1,y_2,\ldots,y_n;\mu,\sigma,h_1,h_2,\ldots,h_n) = \frac{-T}{\sigma^2} < 0$$

$$\frac{\partial \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial\mu}(y_1,y_2,\ldots,y_n;\mu,\sigma,h_1,h_2,\ldots,h_n) = 0 \implies \mu = \frac{\sum_{i=1}^{n} y_i}{T}$$

$$\therefore \hat{\mu} = \frac{\sum_{i=1}^{n} y_i}{T}$$

$Y_1, Y_2, \ldots, Y_n$ are mutually independent and $Y_i \sim \mathcal{N}\left(\mu h_i, \sigma^2 h_i\right)$ for all $1 \le i \le n$. Hence,

$$\sum_{i=1}^{n} Y_i \sim \mathcal{N}\left(\mu T, \sigma^2 T\right)$$

$$\therefore \hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right)$$

(b)

$$\begin{aligned}
\frac{\partial \ln f_{Y_1,Y_2\ldots,Y_n}}{\partial \sigma^2}(y_1, y_2 \ldots, y_n; \mu, \sigma, h_1, h_2, \ldots, h_n) &= -\sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2 \frac{\partial \sigma^{-2}}{\partial \sigma^2}}{2h_i} - \frac{n \frac{\partial \ln \sigma^2}{\partial \sigma^2}}{2} \\
&= \sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{2\sigma^4 h_i} - \frac{n}{2\sigma^2} \\
&= \frac{\sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{h_i} - n\sigma^2}{2\sigma^4}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ln f_{Y_1,Y_2\ldots,Y_n}}{\left(\partial \sigma^2\right)^2}(y_1, y_2 \ldots, y_n; \mu, \sigma, h_1, h_2, \ldots, h_n) &= \sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2 \frac{\partial \sigma^{-4}}{\partial \sigma^2}}{2h_i} - \frac{n \frac{\partial \sigma^{-2}}{\partial \sigma^2}}{2} \\
&= -\sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{\sigma^6 h_i} + \frac{n}{2\sigma^4} \\
&= \frac{n\sigma^2 - 2\sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{h_i}}{2\sigma^6}
\end{aligned}$$

$$\frac{\partial \ln f_{Y_1,Y_2,\ldots,Y_n}}{\partial \sigma^2}(y_1, y_2, \ldots, y_n; \mu, \sigma, h_1, h_2, \ldots, h_n) = 0 \implies \sigma^2 = \sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{n h_i}$$

$$\begin{aligned}
\frac{\partial^2 \ln f_{Y_1,Y_2,\ldots,Y_n}}{\left(\partial \sigma^2\right)^2}\left(y_1, y_2, \ldots, y_n; \mu, \sqrt{\sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{n h_i}}, h_1, h_2, \ldots, h_n\right) &= \frac{-\sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{h_i}}{2\sigma^6} \\
&< 0
\end{aligned}$$

$$\therefore \hat{\sigma}^2 = \sum_{i=1}^{n} \frac{(y_i - \hat{\mu} h_i)^2}{n h_i}$$

Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T$ and $\mathbf{h} = \begin{pmatrix} h_1 & h_2 & \cdots & h_n \end{pmatrix}^T$.

$$\begin{pmatrix} \frac{1}{T} & \frac{1}{T} & \cdots & \frac{1}{T} \\ \frac{n-1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \frac{-1}{n} & \frac{n-1}{n} & \cdots & \frac{-1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{n} & \frac{-1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix} \mathbf{y} = \begin{pmatrix} \hat{\mu} \\ \mathbf{y} - \hat{\mu}\mathbf{h} \end{pmatrix} \sim \mathcal{N}_{n+1}$$

For any $1 \leq i \leq n$,

$$Cov(y_i - \hat{\mu} h_i, \hat{\mu}) = E\left[y_i \hat{\mu} - \hat{\mu}^2 h_i\right] - E[y_i - \hat{\mu} h_i]E[\hat{\mu}]$$

$$
\begin{aligned}
&= \frac{E\left[y_i \sum_{j=1}^{n} y_j\right]}{T} - h_i E\left[\hat{\mu}^2\right] - 0E[\hat{\mu}] \\
&= \frac{(n-1)\mu^2 h_i^2 + E\left[y_i^2\right]}{T} - h_i \left(Var\left(\hat{\mu}\right) + \mu^2\right) \\
&= \frac{n\mu^2 h_i^2 + Var(y_i)}{T} - h_i \left(\frac{\sigma^2}{T} + \mu^2\right) \\
&= \mu^2 h_i + \frac{\sigma^2 h_i}{T} - \frac{h_i \sigma^2}{T} - h_i \mu^2 \\
&= 0
\end{aligned}
$$

Since $\left(\hat{\mu} \quad \mathbf{y}^T - \hat{\mu}\mathbf{h}^T\right)^T$ has a multivariate normal distribution and $Cov(y_i - \hat{\mu}h, \hat{\mu}) = 0$ for all $1 \leq i \leq n$, $\hat{\mu}$ and $\mathbf{y} - \hat{\mu}\mathbf{h}$ are independent.

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{\sigma^2 h_i} &= \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}h_i + \hat{\mu}h_i - \mu h_i)^2}{\sigma^2 h_i} \\
&= \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}h_i)^2 + 2(y_i - \hat{\mu}h_i)(\hat{\mu}h_i - \mu h_i) + (\hat{\mu}h_i - \mu h_i)^2}{\sigma^2 h_i} \\
&= \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}h_i)^2 + h_i^2(\hat{\mu} - \mu)^2}{\sigma^2 h_i} \qquad \left(\because \sum_{i=1}^{n}(y_i - \hat{\mu}h_i) = 0\right) \\
&= \frac{n\hat{\sigma}^2 + T(\hat{\mu} - \mu)^2}{\sigma^2}
\end{aligned}
$$

Since $n\hat{\sigma}^2/\sigma^2$ and $T(\hat{\mu} - \mu)^2/\sigma^2$ are functions only of $\mathbf{y} - \hat{\mu}\mathbf{h}$ and $\hat{\mu}$ respectively and the latter are independent, $n\hat{\sigma}^2/\sigma^2$ and $T(\hat{\mu}-\mu)^2/\sigma^2$ are independent.

$$
y_i \sim \mathcal{N}\left(\mu h_i, \sigma^2 h_i\right) \forall 1 \leq i \leq n, y_1, y_2, \ldots, y_n \text{ mutually independent} \implies \sum_{i=1}^{n} \frac{(y_i - \mu h_i)^2}{\sigma^2 h_i} \sim \chi_n^2
$$

$$
\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{T}\right) \implies \frac{T(\hat{\mu} - \mu)^2}{\sigma^2} \sim \chi_1^2
$$

Since $n\hat{\sigma}^2/\sigma^2$ and $T(\hat{\mu} - \mu)^2/\sigma^2$ are independent, $n\hat{\sigma}^2/\sigma^2 + \chi_1^2 = \chi_n^2$.

$$
\therefore \quad \hat{\sigma}^2 = \frac{\sigma^2 \chi_{n-1}^2}{n}, \quad E\left[\hat{\sigma}^2\right] = \frac{(n-1)\sigma^2}{n}, \quad Var\left(\hat{\sigma}^2\right) = \frac{2(n-1)\sigma^4}{n^2}
$$

(c) The ML estimator of σ^2, $\hat{\sigma}^2 = \sum_{i=1}^{n}(y_i - \hat{\mu}h_i)^2/nh_i$, depends on the terms $h_1, h_2, \ldots, h_n$, while $Var\left(\hat{\sigma}^2\right) = 2(n-1)\sigma^4/n^2$ does not.
The ML estimator of μ, $\hat{\mu} = \sum_{i=1}^{n} y_i/T$, is distributed as $\mathcal{N}\left(\mu, \sigma^2/T\right)$, and therefore does not depend on the terms $h_1, h_2, \ldots, h_n$.

3. (a)

$$
E\left[\epsilon_t^2\right] = E\left[Z_t^2 \sigma_t^2\right]
$$

$$
\begin{aligned}
&= E\left[Z_t^2\right] E\left[\sigma_t^2\right] \\
&= 1E\left[\alpha_0 + \alpha_1 \epsilon_{t-1}^2\right] \\
&= \alpha_0 + \alpha_1 E\left[\epsilon_{t-1}^2\right] \\
&= \lim_{n\to\infty} \left(\alpha_0 \sum_{i=0}^{n-1} \alpha_1^i + \alpha_1^n E\left[\epsilon_{t-n}^2\right] \right) \\
&= \alpha_0 \sum_{i=0}^{\infty} \alpha_1^i \quad (\text{for } |\alpha_1| < 1) \\
&= \frac{\alpha_0}{1-\alpha_1}
\end{aligned}
$$

(b)

$$
E\left[\epsilon_t^3\right] = E\left[Z_t^3 \sigma_t^3\right] = E\left[Z_t^3\right] E\left[\sigma_t^3\right] = 0E\left[\sigma_t^3\right] = 0
$$

(c)

$$
\begin{aligned}
E\left[\epsilon_t^4\right] &= E\left[Z_t^4 \sigma_t^4\right] \\
&= E\left[Z_t^4\right] E\left[\left(\sigma_t^2\right)^2\right] \\
&= \kappa E\left[\left(\alpha_0 + \alpha_1 \epsilon_{t-1}^2\right)^2\right] \\
&= \kappa E\left[\alpha_0^2 + 2\alpha_0\alpha_1 \epsilon_{t-1}^2 + \alpha_1^2 \epsilon_{t-1}^4\right] \\
&= \kappa \left(\alpha_0^2 + 2\alpha_0\alpha_1 E\left[\epsilon_{t-1}^2\right] + \alpha_1^2 E\left[\epsilon_{t-1}^4\right]\right) \\
&= \kappa \left(\alpha_0^2 + \frac{2\alpha_0^2\alpha_1}{1-\alpha_1} + \alpha_1^2 E\left[\epsilon_{t-1}^4\right]\right) \quad (\text{for } |\alpha_1| < 1) \\
&= \frac{\kappa\alpha_0^2(1+\alpha_1)}{1-\alpha_1} + \kappa\alpha_1^2 E\left[\epsilon_{t-1}^4\right] \\
&= \lim_{n\to\infty} \left(\frac{\kappa\alpha_0^2(1+\alpha_1)}{1-\alpha_1} \sum_{i=0}^{n-1} \left(\kappa\alpha_1^2\right)^i + \left(\kappa\alpha_1^2\right)^n E\left[\epsilon_{t-n}^4\right] \right) \\
&= \frac{\kappa\alpha_0^2(1+\alpha_1)}{1-\alpha_1} \sum_{i=0}^{\infty} \left(\kappa\alpha_1^2\right)^i \quad \left(\text{for } |\alpha_1| < \frac{1}{\sqrt{\kappa}}\right) \\
&= \frac{\kappa\alpha_0^2(1+\alpha_1)}{\left(1-\alpha_1\right)\left(1-\kappa\alpha_1^2\right)}
\end{aligned}
$$

(d) The constraint $|\alpha_1| < 1/\sqrt{\kappa}$ is needed to maintain 4th-order stationarity. No constraint is needed on α_0.

(e) If Z_t is Gaussian/normal, then $E\left[Z_t^4\right] = \kappa = 3$. Then,

$$
\kappa_\epsilon = \frac{E\left[\epsilon_t^4\right]}{\left(E\left[\epsilon_t^2\right]\right)^2} = \frac{\frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)\left(1-3\alpha_1^2\right)}}{\left(\frac{\alpha_0}{1-\alpha_1}\right)^2} = \frac{3(1+\alpha_1)(1-\alpha_1)}{1-3\alpha_1^2} = \frac{3\left(1-\alpha_1^2\right)}{1-3\alpha_1^2} > 3 \quad \left(\because |\alpha_1| < \frac{1}{\sqrt{3}}\right)
$$

$\therefore$ If Z_t is Gaussian/normal, then the unconditional distribution of ϵ_t has a higher kurtosis and heavier tails than a Gaussian distribution.

4. (a) Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T$.

$$\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{n-1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \frac{-1}{n} & \frac{n-1}{n} & \cdots & \frac{-1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{n} & \frac{-1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix} \mathbf{y} = \begin{pmatrix} \hat{\mu} \\ \mathbf{y} - \hat{\mu}\mathbf{1}_n \end{pmatrix} \sim \mathcal{N}_{n+1}$$

For any $1 \leq i \leq n$,

$$\begin{aligned} Cov(y_i - \hat{\mu}, \hat{\mu}) &= E\left[y_i\hat{\mu} - \hat{\mu}^2\right] - E[y_i - \hat{\mu}]E[\hat{\mu}] \\ &= \frac{E\left[y_i \sum_{j=1}^n y_j\right]}{n} - E\left[\hat{\mu}^2\right] - 0E[\hat{\mu}] \\ &= \frac{(n-1)\mu^2 + E\left[y_i^2\right]}{n} - \left(Var\left(\hat{\mu}\right) + \mu^2\right) \\ &= \frac{n\mu^2 + Var(y_i)}{n} - \frac{\sigma^2}{n} - \mu^2 \\ &= 0 \end{aligned}$$

Since $\begin{pmatrix} \hat{\mu} & \mathbf{y}^T - \hat{\mu}\mathbf{1}_n^T \end{pmatrix}^T$ has a multivariate normal distribution and $Cov(y_i - \hat{\mu}, \hat{\mu}) = 0$ for all $1 \leq i \leq n$, $\hat{\mu}$ and $\mathbf{y} - \hat{\mu}\mathbf{1}_n$ are independent.

$$\begin{aligned} \frac{\sum_{i=1}^n (y_i - \mu)^2}{\sigma^2} &= \frac{\sum_{i=1}^n (y_i - \hat{\mu} + \hat{\mu} - \mu)^2}{\sigma^2} \\ &= \frac{\sum_{i=1}^n \left((y_i - \hat{\mu})^2 + 2(y_i - \hat{\mu})(\hat{\mu} - \mu) + (\hat{\mu} - \mu)^2\right)}{\sigma^2} \\ &= \frac{n\left(\hat{\sigma}^2 + (\hat{\mu} - \mu)^2\right)}{\sigma^2} \quad \left(\because \sum_{i=1}^n (y_i - \hat{\mu}) = 0\right) \end{aligned}$$

Since $n\hat{\sigma}^2/\sigma^2$ and $n(\hat{\mu} - \mu)^2/\sigma^2$ are functions only of $\mathbf{y} - \hat{\mu}\mathbf{1}_n$ and $\hat{\mu}$ respectively and the latter are independent, $n\hat{\sigma}^2/\sigma^2$ and $n(\hat{\mu}-\mu)^2/\sigma^2$ are independent.

$$y_i \overset{i.i.d.}{\sim} \mathcal{N}\left(\mu, \sigma^2\right) \implies \frac{y_i - \mu}{\sigma} \overset{i.i.d.}{\sim} \mathcal{N}(0,1) \implies \frac{\sum_{i=1}^n (y_i - \mu)^2}{\sigma^2} \sim \chi_n^2$$

$$\hat{\mu} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \implies \frac{\sqrt{n}(\hat{\mu} - \mu)}{\sigma} \sim \mathcal{N}(0,1) \implies \frac{n(\hat{\mu} - \mu)^2}{\sigma^2} \sim \chi_1^2$$

Since $n\hat{\sigma}^2/\sigma^2$ and $n(\hat{\mu} - \mu)^2/\sigma^2$ are independent, $n\hat{\sigma}^2/\sigma^2 + \chi_1^2 = \chi_n^2$.

$$\therefore \quad \hat{\sigma}^2 = \frac{\sigma^2\chi_{n-1}^2}{n}$$

(b)

$$\hat{\sigma}^2 \frac{n-1}{q_{0.975}} = 6.677100 \times 10^{-4} \times 0.8460245 = 5.648990 \times 10^{-4}$$

$$\hat{\sigma}^2 \frac{n-1}{q_{0.025}} = 6.677100 \times 10^{-4} \times 1.200442 = 8.015471 \times 10^{-4}$$

The 95% confidence interval is $(5.648990 \times 10^{-4}, 8.015471 \times 10^{-4})$.

$$\sqrt{253}\sqrt{5.648990 \times 10^{-4}} = 0.378047$$

$$\sqrt{253}\sqrt{8.015471 \times 10^{-4}} = 0.450324$$

In terms of annualized volatility, the interval is $(0.378047, 0.450324)$. The sample annual volatility does not fall in this confidence interval for any other year.

(c)

$$S_0 = \frac{\hat{\sigma}^2_{2008}}{\hat{\sigma}^2_{2007}} = \frac{6.677100 \times 10^{-4}}{1.018599 \times 10^{-4}} = 6.555180$$

```
> 2*(1-pf((6.677100*(10**(-4)))/(1.018599*(10**(-4))), df1=252, df2=250))
[1] 0
```

The α-level is very close to 0, too small to be calculated.

$$S_0 = \frac{\hat{\sigma}^2_{2008}}{\hat{\sigma}^2_{2006}} = \frac{6.677100 \times 10^{-4}}{3.981351 \times 10^{-5}} = 16.77094$$

```
> 2*(1-pf((6.677100*(10**(-4)))/(3.981351*(10**(-5))), df1=252, df2=250))
[1] 0
```

The α-level is very close to 0, too small to be calculated.