1 Possibly an error in HK model specification

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1.0.1 Final good production

HK assumes that there is only one final good Y produced by a representative firm in a perfectly competitive market, which combines the output Y_s of S manufacturing industries using a Cobb-Douglas production function, that is:

$$Y = \prod_{s=1}^{S} Y_s^{\theta_s} \tag{1}$$

where $\sum_{s=1}^{S} \theta_s = 1$. Therefore, profit maximization of the final good producer can be written as follows (in a perfectly competitive market, profits are zero):

$$\max_{Y_s} \sum_{s=1}^{S} P_s Y_s - PY$$

$$\text{subject to} \quad Y = \prod_{s=1}^{S} Y_s^{\theta_s}$$

$$\text{FOC:} \quad P_s - P\theta_s \frac{Y}{Y_s} = 0 \longrightarrow P_s Y_s = \theta_s PY \tag{2}$$

Equation 2 (we refer to the last equation in the row) gives us the demand function for the final good. Now let's derive the price index for the final good, which we will be using later. For this, we will insert the demand function (2) into the production function (1), that is:

$$Y = \prod_{s=1}^{S} Y_s^{\theta_s} = \prod_{s=1}^{S} \Big(\frac{\theta_s PY}{P_s}\Big)^{\theta_s} = (PY)^{\sum_{s=1}^{S} \theta_s} \prod_{s=1}^{S} \Big(\frac{\theta_s}{P_s}\Big)^{\theta_s}$$

we know that $\sum_{s=1}^{S} \theta_s = 1$, thus:

$$Y = PY \prod_{s=1}^{S} \Big(\frac{\theta_s}{P_s}\Big)^{\theta_s}$$

Canceling out the Ys, we can write the price index for the final good as follows:

$$P = \prod_{s=1}^{S} \Big(\frac{P_s}{\theta_s}\Big)^{\theta_s} \tag{3}$$

1.0.2 Industry level production

Now, having found the main equations for the final good, we go down one step lower from the industrial hierarchy. HK assumes that there are S number of industries within a sector, and industry level production function Y_s takes a CES form, that is:

$$Y_s = \Big(\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}}\Big)^{\frac{\sigma}{\sigma-1}} \tag{4}$$

where σ is the constant elasticity of substitution of M_s differentiated product ouputs Y_{si}. As we did earlier, here we also assume that industries exist in a perfectly competitive market with zero profits. Thus, profit maximization problem of an industry can be written as such:

$$\max_{Y_{si}} \sum_{i=1}^{M_s} P_{si}Y_{si} - P_sY_s$$

$$\text{subject to} \quad Y_s = \Big(\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}}\Big)^{\frac{\sigma}{\sigma-1}}$$

$$\begin{aligned}
\text{FOC:} \quad & P_{si} - \frac{\sigma}{\sigma-1}P_sY_s^{\frac{1}{\sigma}}\frac{\sigma-1}{\sigma}Y_{si}^{-\frac{1}{\sigma}} = 0 \\
& P_{si} = P_sY_s^{\frac{1}{\sigma}}Y_{si}^{-\frac{1}{\sigma}} \longrightarrow P_{si}^{\sigma} = P_s^{\sigma}Y_sY_{si}^{-1} \\
& P_s^{\sigma}Y_s = P_{si}^{\sigma}Y_{si} \qquad (5)
\end{aligned}$$

Equation 5 is a demand function for an industry output. Now, lets derive the price index for the industry output, which we will be using later. For this, we will insert the industry demand function (5) into industry production function (4), that is:

$$Y_s = \Big(\sum_{i=1}^{M_s} Y_{si}^{\frac{\sigma-1}{\sigma}}\Big)^{\frac{\sigma}{\sigma-1}} = \Big(\sum_{i=1}^{M_s}\Big(P_s^{\sigma}Y_sP_{si}^{-\sigma}\Big)^{\frac{\sigma-1}{\sigma}}\Big)^{\frac{\sigma}{\sigma-1}} = Y_sP_s^{\sigma}\Big(\sum_{i=1}^{M_s} P_{si}^{1-\sigma}\Big)^{\frac{\sigma}{\sigma-1}}$$

$$P_s = \Big(\sum_{i=1}^{M_s} P_{si}^{1-\sigma}\Big)^{\frac{1}{1-\sigma}} \qquad (6)$$

Equation 6 gives us the price index for industry output.

1.0.3 Firm level production

Now, as we have derived the main equations for the industry level, we go one step down to the firm level. Specifically, HK assumes that the firm-level production takes a Cobb-Douglas form, that is:

$$Y_{si} = A_{si}K_{si}^{\alpha_s}L_{si}^{1-\alpha_s} \qquad (7)$$

where A_{si} is a firm's total factor productivity, K_{si} and L_{si} are firm i's capital and labor respectively. Note, the shares of capital and labor in production, which are α_s and $1-\alpha_s$ are assumed to be constant within an industry, but varying across industries, which is why i subscript is being left out.

At the firm level, firms operate in a monopolistic competitive market, and they face two firm-specific distortions: mark-up on the cost of capital $\tau_{K_{si}}$ and tax on the price of good $\tau_{Y_{si}}$. HK explains that $\tau_{Y_{si}}$ exists because some firms may be subsidized and/or the others may face government restrictions. And, $\tau_{K_{si}}$ captures, among others, differences in the firms' ability to access financial credit. Thus, profit function of a firm can be written as such:

$$\pi_{si} = (1-\tau_{Y_{si}})P_{si}Y_{si} - wL_{si} - (1-\tau_{K_{si}})RK_{si}$$

Using $P_{si} = P_sY_s^{\frac{1}{\sigma}}Y_{si}^{-\frac{1}{\sigma}}$ from equation 5, we get:

$$\pi_{si} = (1-\tau_{Y_{si}})(P_sY_s^{\frac{1}{\sigma}}Y_{si}^{-\frac{1}{\sigma}})Y_{si} - wL_{si} - (1-\tau_{K_{si}})RK_{si}$$

Using equation 7, profit maximization problem of a firm becomes:

$$\max_{K_{si},L_{si}} \pi_{si} = (1-\tau_{Y_{si}})P_sY_s^{\frac{1}{\sigma}}(A_{si}K_{si}^{\alpha_s}L_{si}^{1-\alpha_s})^{\frac{\sigma-1}{\sigma}} - wL_{si} - (1-\tau_{K_{si}})RK_{si} \qquad (8)$$

First, FOC with respect to K_{si}:

$$(1-\tau_{Y_{si}})P_sY_s^{\frac{1}{\sigma}}\frac{\sigma-1}{\sigma}Y_{si}^{-\frac{1}{\sigma}}A_{si}\alpha_s\Big(\frac{K_{si}}{L_{si}}\Big)^{\alpha_s-1} = (1+\tau_{K_{si}})R \qquad (9)$$

Again, using $P_{si} = P_s Y_s^{\frac{1}{\sigma}} Y_{si}^{-\frac{1}{\sigma}}$ from equation 5, we get:

$$(1-\tau_{Y_{si}})P_{si}\frac{\sigma-1}{\sigma}A_{si}\alpha_s\Big(\frac{K_{si}}{L_{si}}\Big)^{\alpha_s-1} = (1+\tau_{K_{si}})R \tag{10}$$

or

$$(1-\tau_{Y_{si}})P_{si}\frac{\sigma-1}{\sigma}\alpha_s\frac{Y_{si}}{K_{si}} = (1+\tau_{K_{si}})R \tag{11}$$

Now, FOC with respect to L_{si} is:

$$(1-\tau_{Y_{si}})P_s Y_s^{\frac{1}{\sigma}}\frac{\sigma-1}{\sigma}Y_{si}^{-\frac{1}{\sigma}}A_{si}(1-\alpha_s)\Big(\frac{K_{si}}{L_{si}}\Big)^{\alpha_s} = w \tag{12}$$

Following the same steps as above, we get:

$$(1-\tau_{Y_{si}})P_{si}\frac{\sigma-1}{\sigma}(1-\alpha_s)\frac{Y_{si}}{L_{si}} = w \tag{13}$$

Dividing equation 13 by equation 11, we will get the following capital-labor ratio:

$$\frac{K_{si}}{L_{si}} = \frac{\alpha_s}{1-\alpha_s}\frac{w}{(1+\tau_{K_{si}})R} \tag{14}$$

Plugging equation 14 to equation 10, we can find P_{si}, that is:

$$P_{si} = \frac{\sigma}{\sigma-1}\Big(\frac{R}{\alpha_s}\Big)^{\alpha_s}\Big(\frac{w}{1-\alpha_s}\Big)^{1-\alpha_s}\frac{(1+\tau_{K_{si}})^{\alpha_s}}{A_{si}(1-\tau_{Y_{si}})} \tag{15}$$

Note, since the market is monopolistic competition, profit maximization yields the standard condition that the firm's output price is a fixed markup over its marginal cost, where marginal cost is given by: $MC_{si} = \Big(\frac{R}{\alpha_s}\Big)^{\alpha_s}\Big(\frac{w}{1-\alpha_s}\Big)^{1-\alpha_s}\frac{(1+\tau_{K_{si}})^{\alpha_s}}{A_{si}(1-\tau_{Y_{si}})}$.

1.0.4 Labor allocation

Now, let's find for allocation of labor, capital and output. First, inserting equation 7 to equation 12, we get the following:

$$(1-\tau_{Y_{si}})P_s Y_s^{\frac{1}{\sigma}}\frac{\sigma-1}{\sigma}(A_{si}K_{si}^{\alpha_s}L_{si}^{1-\alpha_s})^{-\frac{1}{\sigma}}A_{si}(1-\alpha_s)\Big(\frac{K_{si}}{L_{si}}\Big)^{\alpha_s} = w$$

Before finding for L_{si}, we need to note that Y_s also depends on Y_{si} and thereby on L_{si} (and P_s does not). But HK probably assumed that there are many firms so that a firm's output will have incremental impact on the industry output. For now, lets follow HK's method, and in the next section, we try out the alternative. So, after a few steps of rearranging the above equation, we get the following:

$$(1-\tau_{Y_{si}})P_s Y_s^{\frac{1}{\sigma}}\frac{\sigma-1}{\sigma}A_{si}^{\frac{\sigma-1}{\sigma}}L_{si}^{-\frac{1}{\sigma}}\Big(\frac{K_{si}}{L_{si}}\Big)^{\frac{\alpha_s(\sigma-1)}{\sigma}}(1-\alpha_s) = w$$

Using equation 14, we get:

$$(1-\tau_{Y_{si}})P_s Y_s^{\frac{1}{\sigma}}\frac{\sigma-1}{\sigma}A_{si}^{\frac{\sigma-1}{\sigma}}L_{si}^{-\frac{1}{\sigma}}\Big(\frac{\alpha_s}{1-\alpha_s}\frac{w}{(1+\tau_{K_{si}})R}\Big)^{\frac{\alpha_s(\sigma-1)}{\sigma}}(1-\alpha_s) = w$$

Finding L_{si} from this, and simplifying, we get:

$$L_{si} = (1-\tau_{Y_{si}})^{\sigma}P_s^{\sigma}Y_s\Big(\frac{\sigma-1}{\sigma}\Big)^{\sigma}A_{si}^{\sigma-1}\frac{\alpha_s^{\alpha_s(\sigma-1)}}{(1-\alpha_s)^{\alpha_s(\sigma-1)-\sigma}}w^{\alpha_s(\sigma-1)-\sigma}(1+\tau_{K_{si}})^{\alpha_s(1-\sigma)}R^{\alpha_s(1-\sigma)} \tag{16}$$

which is why HK says that optimal allocation of L_{si} will be proportional to firm specific TFP levels and distortions on output and capital, that is:

$$L_{si} \propto \frac{A_{si}^{\sigma-1}(1-\tau_{Y_{si}})^{\sigma}}{(1+\tau_{K_{si}})^{\alpha_s(\sigma-1)}} \tag{17}$$

However, when we consider the fact that Y_s also depends on L_{si}. Labor allocation will only depend on the level of firm's capital and firm-specific distortions, but not on their tfp levels. We explore this in the next section.

1.0.5 Capital allocation

As we did find the labor allocation decision for a firm, we follow the same steps to find capital allocation. First, inserting equation 7 to equation 9, we get the following:

$$(1-\tau_{Y_{si}})P_s Y_s^{\frac{1}{\sigma}} \frac{\sigma-1}{\sigma}(A_{si}K_{si}^{\alpha_s}L_{si}^{1-\alpha_s})^{-\frac{1}{\sigma}} A_{si}\alpha_s \Big(\frac{K_{si}}{L_{si}}\Big)^{\alpha_s-1} = (1+\tau_{K_{si}})R$$

Before finding for K_{si}, here we also need to note that Y_s also depends on Y_{si} and therefore on K_{si} (and P_s does not). For now, we ignore this again. Rearranging the above equation, we get the following:

$$\alpha_s(1-\tau_{Y_{si}})P_s Y_s^{\frac{1}{\sigma}} \frac{\sigma-1}{\sigma} A_{si}^{\frac{\sigma-1}{\sigma}} K_{si}^{-\frac{1}{\sigma}} \Big(\frac{K_{si}}{L_{si}}\Big)^{\frac{(1-\alpha_s)(1-\sigma)}{\sigma}} (1+\tau_{K_{si}})^{-1}R^{-1} = 1$$

Using equation 14, we get:

$$\alpha_s(1-\tau_{Y_{si}})P_s Y_s^{\frac{1}{\sigma}} \frac{\sigma-1}{\sigma} A_{si}^{\frac{\sigma-1}{\sigma}} K_{si}^{-\frac{1}{\sigma}} \Big(\frac{\alpha_s}{1-\alpha_s}\frac{w}{(1+\tau_{K_{si}})R}\Big)^{\frac{(1-\alpha_s)(1-\sigma)}{\sigma}} (1+\tau_{K_{si}})^{-1}R^{-1} = 1$$

Finding K_{si} from this, we get:

$$K_{si} = \frac{\alpha_s}{[\alpha_s^{\alpha_s}(1-\alpha_s)^{1-\alpha_s}]^{1-\sigma}}(1-\tau_{Y_{si}})^{\sigma}P_s^{\sigma}Y_s\Big(\frac{\sigma-1}{\sigma}\Big)^{\sigma}A_{si}^{\sigma-1}w^{(1-\alpha_s)(1-\sigma)}(1+\tau_{K_{si}})^{\alpha_s(1-\sigma)-1}R^{\alpha_s(1-\sigma)-1} \tag{18}$$

The optimal allocation of K_{si} will be proportional to firm specific TFP levels and distortions on output and capital, that is:

$$K_{si} \propto \frac{A_{si}^{\sigma-1}(1-\tau_{Y_{si}})^{\sigma}}{(1+\tau_{K_{si}})^{\alpha_s(\sigma-1)-1}} \tag{19}$$

1.0.6 Output allocation

To get the output allocation, we need to use firm production function (7), labor allocation equation (16) and capital allocation (18) equations. We insert both capital and labor allocation equations we derived into the production function and simplify the resulting equation. Since, the equations will be too long, we will provide the final result.

$$Y_{si} = A_{si}^{\sigma}(1-\tau_{Y_s i})^{\sigma}P_s^{\sigma}Y_s\Big(\frac{\sigma-1}{\sigma}\Big)^{\sigma}\frac{\alpha_s^{\alpha_s\sigma}(1-\alpha_s)^{(1-\alpha_s)\sigma}}{w^{(1-\alpha_s)\sigma}(1+\tau_{K_{si}})^{\alpha_s\sigma}R^{\alpha_s\sigma}} \tag{20}$$

which means:

$$Y_{si} \propto \frac{A_{si}^{\sigma}(1-\tau_{Y_{si}})^{\sigma}}{(1+\tau_{K_{si}})^{\alpha_s\sigma}} \tag{21}$$

1.0.7 Marginal Revenue Products

Marginal revenue product of capital. Using equation 11, we get:

$$MRPK_{si} = \alpha_s\frac{\sigma-1}{\sigma}\frac{P_{si}Y_{si}}{K_{si}} = R\frac{1+\tau_{K_{si}}}{1-\tau_{Y_{si}}} \tag{22}$$

Marginal revenue product of labor. Using equation 13, we get:

$$MRPL_{si} = (1 - \alpha_s)\frac{\sigma - 1}{\sigma}\frac{P_{si}Y_{si}}{L_{si}} = \frac{w}{1 - \tau_{Y_{si}}} \tag{23}$$

From here, the next steps HK took was wrong. When estimating weighted marginal products across sectors, HK gave a weight of $\frac{P_s Y_s}{P_{si} Y_{si}}$ instead of $\frac{P_{si} Y_{si}}{P_s Y_s}$. These two weights give very different results. HK's chosen weight favors firms with a larger share of an industry, and inflates marginal revenue products for smaller firms.