Mathematics

Internal Assessment

How many pieces? (HL TYPE I)

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In this work I will investigate the maximum number of pieces obtained when an n-dimensional object is cut.

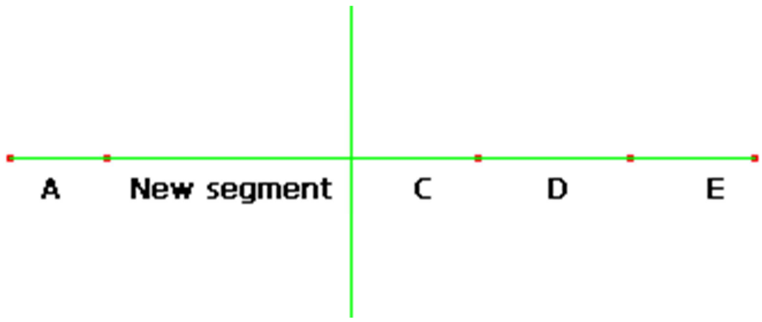
I will do this by going step by step from one dimensional object to more dimensional objects.

One-dimensional object

Firstly, I will investigate, what is the maximum number of pieces obtained, when I cut a finite one-dimensional object – a line segment.

When I cut a line segment, using one cut I can divide only one existing segment. Therefore, by cutting a line segment, the total (and maximum) number of segments of the line segment increases by one.

Example: In the image below an example is described. The maximum number of segments increased by one from 4 to 5 after segment C was divided into segments "C" and "New segment".



Out of this, I can formulate a recursive expression for the maximum number of segments after n cuts.

$S_{(0)} = 1$

$S_{(n)} = S_{(n-1)} + 1$

Initially, there is one whole line segment. For every other cut the number of segments is equal to the previous number of segments + 1.

Out of this we can also see the non-recursive relationship for number of segments:

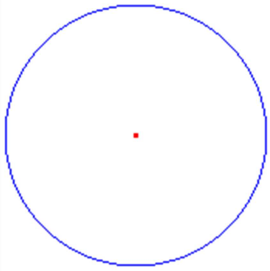
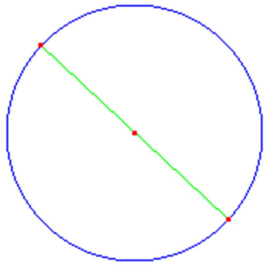
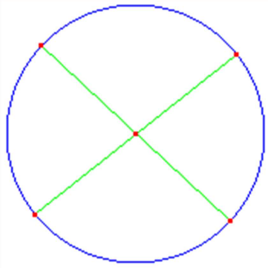
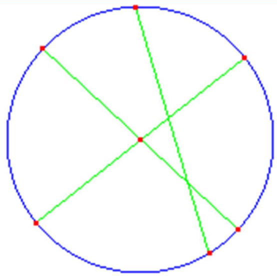
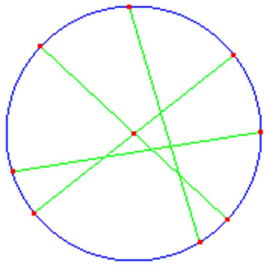
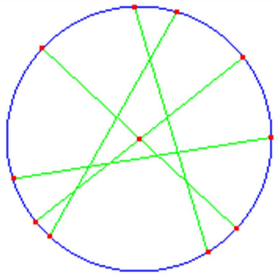
$S_{(n)} = n + 1$

Because of its clarity, I believe the formal proof is not required. This relationship simply states, that the number of segments after n cuts equals 1 (the initial whole) + n (since every cut creates one new segment).

Two-dimensional object

In two dimensions, I am going to investigate how many regions are created when a finite 2-dimensional object – a circle – is divided by n chords.

In order to get the first impression of what happens in two dimensions, I will draw the circle with 1 to 5 chords when the maximum possible value of regions is created.

Number of cuts	Drawing	Number of regions
0		1
1		2
2		4
3		7
4		11
5		16

We can see that if we make a new cut but we do not cross any existing cut, there is one new region created. Moreover, for every crossed cut another region is created.

This is enough data to for a recursive rule for number of maximum regions after *n* cuts.

$R_{(0)} = 1$

$R_{(n)} = R_{(n-1)} + 1 + (n-1) = R_{(n-1)} + n$

Initially, there is 1 region – the whole circle. By every other n^{th} cut – if is different from all previous cuts and does not intersect other two planes in their interaction - we create a maximum of other *n* regions.

If we want to find a non-recursive relationship, it is firstly useful to create a table which shows a rise of number of segments for n cuts. We can easily achieve this using spreadsheet software.

Number of cuts	Maximum number of regions as they increase	Maximum number of regions
0	1	= 1
1	1 + 1	= 2
2	1 + 1 + 2	= 4
3	1 + 1 + 2 + 3	= 7
4	1 + 1 + 2 + 3 + 4	= 11
5	1 + 1 + 2 + 2 + 4 + 5	= 16

Using this table we can formulate our assumption – a conjuncture for the non-recursive relationship between the maximum number of regions after *n* cuts.

Because a sum of arithmetical progression is apparent in the table, our conjuncture for the maximum number of regions after *n* cuts goes as following:

$$R_{(n)} = \frac{(1+n)n}{2} + 1$$

However, in order to be sure whether this non-recursive expression is truly related to the recursive expression defined above, we have to formally **prove it**. We can do this using the **mathematical induction**.

1) If n=1 then

   $R_{(1)} = \frac{(1+1)*1}{2} + 1 = 2$

   This is true – we receive two regions after 1 cut, indeed.

2) If n=k then

   $$R_{(k)} = \frac{(1+k)k}{2} + 1$$

   and

$$R_{(k+1)} = \frac{(1+k)k}{2} + 1 + (k+1) = \frac{(k+1)(k+2)}{2} + 1 = \frac{\big(1+\ (k+1)\big) * (k+1)}{2} + 1$$
$$= R_{(k+1)}$$

3) Since if the relationship is valid for k^{th} term it is valid for $k+1^{th}$ element and it is valid for the first term, it is valid for all terms.

   Q.E.D.

Lastly, it might be interesting to see this relationship concerning the relationship for segments from one-dimension. Therefore, I will express it as R = X + S, where S is the number of segments as we calculated before.

In order to do this, we have to determine the X. We can do this by subtracting S from R.

$$X = R - S = \frac{(1+n)n}{2} + 1 - (1+\ n) = \frac{(n-1)n}{2} = X$$

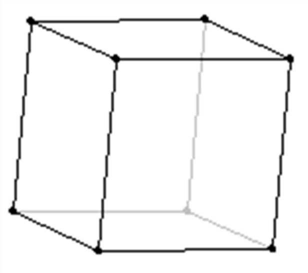
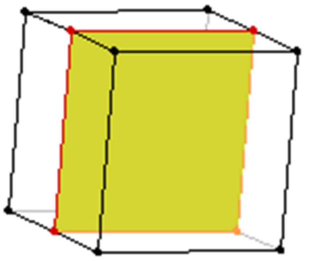
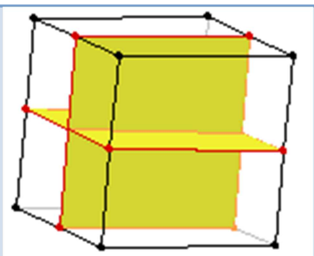
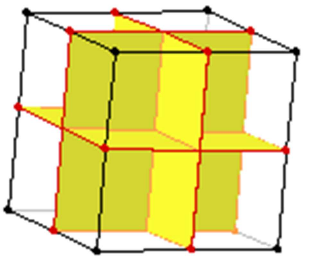
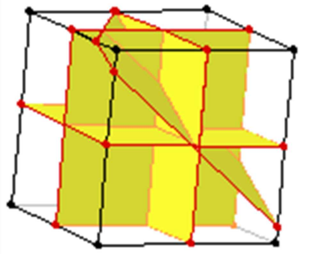
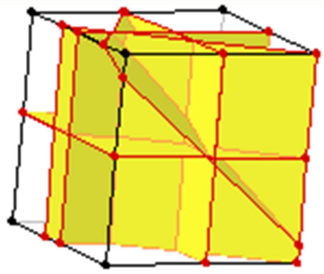
Therefore

$$R = X + S = \frac{(n-1)n}{2} + (1+n)$$

Three-dimensions

Now, I am going to find out how is the maximum number of parts of a finite three-dimensional object – a cuboid – which can be created by *n* cuts.

Firstly, I will do the same thing I have done for two-dimensions. I am going to determine – using computer software – what the maximum number of parts is when cuboid is cut by 1 to 5 cuts.

Number of cuts	Illustration	Number of parts
0		1
1		2
2		4
3		8
4		15
5		26

In order to make things clearer, we can put the number of parts we have determined above in a clear table:

Number of cuts	Maximum number of regions as they increase	Maximum number of parts
0	1	= 1
1	1 + 1	= 2
2	1 + 1 + 2	= 4
3	1 + 1 + 2 + 4	= 8
4	1 + 1 + 2 + 4 + 7	= 15
5	1 + 1 + 2 + 4 + 7 + 11	= 26

In order to maximize the number of newly created parts we want to cross all existing cuts. Therefore, n^{th} cut, if different from all previous and it does not intersect any other two previous cuts in their intersection, creates additional $\frac{(1+(n-1))(n-1)}{2}+1=\frac{n(n-1)}{2}+1$ parts as it is clear out of the previous table.

This also means we are able to formulate a recursive expression for the maximum number of parts after n^{th} cut:

$P_{(0)} = 1$

$P_{(n)} = P_{(n-1)} + \frac{n(n-1)}{2}+1$

Due to the last table we are also able to formulate a conjecture expressing this relationship in a non-recursive way. It goes like this:

$$P_{(n)} = \frac{n^3+5n}{6}+1$$

However, it is still a conjecture as far as it is not proven. So we need a formal proof to show that this non-recursive relationship is really related to the recursive relationship. This proof can be done again using the **mathematical induction**.

1) If n=1 then

$$P_{(1)} = \frac{1^3+5}{6}+1 = 2$$

This is true – we receive two parts after 1 cut, indeed.

2) If n=k then

$$P_{(k)} = \frac{k^3+5k}{6}+1$$

and

$$P_{(k)} = \frac{k^3+5k}{6}+1+\frac{n(n-1)}{2}+1 = \frac{(k+1)^3+5(k+1)}{6}+1$$

3) Since if the relationship is valid for k^{th} term it is valid for $k+1^{th}$ element and it is valid for the first term, it is valid for all terms.

Q.E.D.

Once again, to see the difference among different dimensions, it might be interesting to express P as P = Y + X + S. In order to do this, we have to determine the value of Y. We can do this as Y = P – (X+ S).

Therefore

$$Y = \frac{n^3+5n}{6} + 1 - \left(\frac{(n-1)n}{2} + (1+n)\right) = \frac{(n-2)(n-1)n}{6}$$

thus

$$P = \frac{(n-2)(n-1)n}{6} + \frac{(n-1)n}{2} + n + 1$$

Four-dimensions

The problem with four dimensions is that since we live in 3D world, I can't draw a reasonable sketch which could be used for illustration of cuts. I can, however, use my result from lower dimensions and make an assumption – if the difference between the following dimensions remains unchanged – what should the relationship among parts created and number of cuts should look like.

First, let's look on the differences between following dimensions in terms of the previous dimension. If we look on the relationship for maximum number of parts in 3D, we see:

$$P = \frac{(n-2)(n-1)n}{6} + \frac{(n-1)n}{2} + n + 1$$

Out of this we are able to see that by every dimension a new addition to the function is created and it is equal to the

$$Nth\ dimension\ difference = Previous\ addend * \frac{(n-(n-1))}{n}$$

Therefore, the Z should look like, according to this assumption, as

$$Z = \frac{(n-3)(n-2)(n-1)n}{24}$$

Another – completely unrelated assumption – is following: The differences between following terms in 3D were actually the values from $n-1^{th}$ element from 2D. Therefore, it is clever to anticipate that in 4D the differences will be identical with actual values from 3D. According to this, the recurrence for Q, as the number of parts created by division of four dimensional object by n cuts should look like:

$Q_{(0)} = 1$

$Q_{(n)} = Q_{(n-1)} + \frac{(n-1)^3+5(n-1)}{6} + 1$

The question now is, whether these independent guesses are self-consistent.

So, is my guess correct?

Once again, I can the mathematical induction to show and proof that both the recursive and non-recursive relationship give me the exact same result.

Firstly, I need to get the total relationship for Q.

$$Q = Z + Y + X + S = \frac{(n-3)(n-2)(n-1)n}{24} + \frac{(n-2)(n-1)n}{6} + \frac{(n-1)n}{2} + n + 1$$

Now, let's use the mathematical induction again:

1) If n=1 then

$$Q_{(1)} = \frac{(1-3)(1-2)(1-1)1}{24} + \frac{(1-2)(1-1)1}{6} + \frac{(1-1)1}{2} + 1 + 1 = 2$$

This is true – we receive two parts after 1 cut, indeed.

2) If n=k then

$$Q_{(k)} = \frac{(k-3)(k-2)(k-1)k}{24} + \frac{(k-2)(k-1)k}{6} + \frac{(k-1)k}{2} + k + 1$$

and

$$Q_{(k)} = \frac{(k-3)(k-2)(k-1)k}{24} + \frac{(k-2)(k-1)k}{6} + \frac{(k-1)k}{2} + k + 1$$
$$+ \frac{(n-1)^3 + 5(n-1)}{6} + 1$$
$$= \frac{((k+1)-3)((k+1)-2)((k+1)-1)(k+1)}{24}$$
$$+ \frac{((k+1)-2)((k+1)-1)(k+1)}{6} + \frac{((k+1)-1)(k+1)}{2} + (k+1)$$
$$+ 1$$

3) Since if the relationship is valid for k[th] term it is valid for k+1[th] element and it is valid for the first term, it is valid for all terms.

Q.E.D.

This relationship for Q is already in the form Q = Z + Y + X + S.

$$Q = Z + Y + X + S = \frac{(n-3)(n-2)(n-1)n}{24} + \frac{(n-2)(n-1)n}{6} + \frac{(n-1)n}{2} + n + 1$$

Since my guesses are self-consistent, the odds are that I am correct and this indeed is the right relationship for the maximum number of parts obtained by n cuts of four-dimensional object. Moreover, using the similar method I am able to predict the relationship for other dimensions as well.

$$Nth\ dimension\ difference = Previous\ addend * \frac{(n-(n-1))}{n}$$