1 Cheng's RGEs

Cheng *et al.* [1] write the quartic part of the scalar potential in the general form

$$V_4 = \frac{1}{24} \sum_{i,j,k,l} f_{ijkl}\, \phi_i \phi_j \phi_k \phi_l, \tag{1}$$

where the ϕ_i are *real* scalar fields. The renormalization-group equation (RGE) for the couplings f_{ijkl} then is [1]

$$16\pi^2 \frac{\mathrm{d} f_{ijkl}}{\mathrm{d}t} = \sum_{m,n} \left(f_{ijmn} f_{mnkl} + f_{ikmn} f_{mnjl} + f_{ilmn} f_{mnjk} \right). \tag{2}$$

2 The 2HDM

In the case $J = 1/2$,

$$H = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \chi = \begin{pmatrix} c \\ d \end{pmatrix}. \tag{3}$$

In the usual notation for the two-Higgs-doublet model [2],

$$\begin{aligned} V_4 &= \frac{\bar\lambda_1}{2} \left(H^\dagger H\right)^2 + \frac{\bar\lambda_2}{2} \left(\chi^\dagger \chi\right)^2 + \bar\lambda_3 \left(H^\dagger H\right)\left(\chi^\dagger \chi\right) + \bar\lambda_4 \left(H^\dagger \chi\right)\left(\chi^\dagger H\right) && \text{(4a)} \\ &= \frac{\bar\lambda_1}{2} (A+B)^2 + \frac{\bar\lambda_2}{2} (C+D)^2 \\ &\quad + \bar\lambda_3 (A+B)(C+D) + \bar\lambda_4 \left[AC + BD + 2\,\mathrm{Re}\left(a^* b c d^*\right)\right]. && \text{(4b)} \end{aligned}$$

Notice that there are no terms with coefficients $\bar\lambda_{5,6,7}$ because the hypercharge of χ is arbitrary. In the notation of [3], on the other hand,

$$\begin{aligned} V_4 &= \frac{\lambda_1}{2} (A+B)^2 + \frac{\lambda_2}{2} (C+D)^2 \\ &\quad + \lambda_3 (A+B)(C+D) + \lambda_4 \left[\frac{A-B}{2}\,\frac{C-D}{2} + \mathrm{Re}\left(a^* b c d^*\right)\right]. && \text{(5a)} \end{aligned}$$

The equivalence between the two notations is made through

$$\bar\lambda_1 = \lambda_1, \quad \bar\lambda_2 = \lambda_2, \quad \bar\lambda_3 = \lambda_3 - \frac{\lambda_4}{4}, \quad \bar\lambda_4 = \frac{\lambda_4}{2}. \tag{6}$$

The RGE for $\bar\lambda_{1,2,3,4}$ are, according to [2],

$$\begin{aligned} 16\pi^2 \frac{\mathrm{d}\bar\lambda_1}{\mathrm{d}t} &= 12\bar\lambda_1^2 + 4\bar\lambda_3^2 + 2\bar\lambda_4^2 + 4\bar\lambda_3\bar\lambda_4, && \text{(7a)} \\ 16\pi^2 \frac{\mathrm{d}\bar\lambda_2}{\mathrm{d}t} &= 12\bar\lambda_2^2 + 4\bar\lambda_3^2 + 2\bar\lambda_4^2 + 4\bar\lambda_3\bar\lambda_4, && \text{(7b)} \\ 16\pi^2 \frac{\mathrm{d}\bar\lambda_3}{\mathrm{d}t} &= 4\bar\lambda_3^2 + 2\bar\lambda_4^2 + \left(\bar\lambda_1 + \bar\lambda_2\right)\left(6\bar\lambda_3 + 2\bar\lambda_4\right), && \text{(7c)} \\ 16\pi^2 \frac{\mathrm{d}\bar\lambda_4}{\mathrm{d}t} &= 4\bar\lambda_4^2 + 2\left(\bar\lambda_1 + \bar\lambda_2\right)\bar\lambda_4 + 8\bar\lambda_3\bar\lambda_4. && \text{(7d)} \end{aligned}$$

It follows from Eqs. (7) and (6) that

$$\begin{aligned}
16\pi^2 \frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= 12\lambda_1^2 + 4\lambda_3^2 + \frac{\lambda_4^2}{4}, &\text{(8a)}\\
16\pi^2 \frac{\mathrm{d}\lambda_2}{\mathrm{d}t} &= 12\lambda_2^2 + 4\lambda_3^2 + \frac{\lambda_4^2}{4}, &\text{(8b)}\\
16\pi^2 \frac{\mathrm{d}\lambda_3}{\mathrm{d}t} &= 6\left(\lambda_1 + \lambda_2\right)\lambda_3 + 4\lambda_3^2 + \frac{3}{4}\,\lambda_4^2, &\text{(8c)}\\
16\pi^2 \frac{\mathrm{d}\lambda_4}{\mathrm{d}t} &= 2\left(\lambda_1 + \lambda_2\right)\lambda_4 + 8\lambda_3\lambda_4. &\text{(8d)}
\end{aligned}$$

Equations (8) are the RGEs for the case $Y = 1/2$. Notice that Eqs. (8) are actually simpler than Eqs. (7).

3 RGEs for complex scalar fields

Let us work instead in terms of *complex* scalar fields Φ_i. Suppose V_4 may be written as

$$V_4 = \sum_{i,k,j,l} \lambda_{ikjl}\, \Phi_i^* \Phi_k^*\, \Phi_j \Phi_l, \tag{9}$$

where

$$\lambda_{ikjl} = \lambda_{kijl} = \lambda_{iklj} = \lambda_{kilj} = \lambda_{jlik}^* = \lambda_{ljik}^* = \lambda_{jlki}^* = \lambda_{ljki}^*. \tag{10}$$

For instance, in the case $J = 1/2$ we write

$$a = \Phi_1, \quad b = \Phi_2, \quad c = \Phi_3, \quad d = \Phi_4, \tag{11}$$

and

$$\begin{aligned}
V_4 = {}& \frac{\lambda_1}{2}\left(\Phi_1^*\Phi_1^*\,\Phi_1\Phi_1 + \Phi_2^*\Phi_2^*\,\Phi_2\Phi_2\right) + \frac{\lambda_2}{2}\left(\Phi_3^*\Phi_3^*\,\Phi_3\Phi_3 + \Phi_4^*\Phi_4^*\,\Phi_4\Phi_4\right) &\text{(12a)}\\
&+\lambda_1\,\Phi_1^*\Phi_2^*\,\Phi_1\Phi_2 + \lambda_2\,\Phi_3^*\Phi_4^*\,\Phi_3\Phi_4 &\text{(12b)}\\
&+\left(\lambda_3 + \frac{\lambda_4}{4}\right)\left(\Phi_1^*\Phi_3^*\,\Phi_1\Phi_3 + \Phi_2^*\Phi_4^*\,\Phi_2\Phi_4\right) &\text{(12c)}\\
&+\left(\lambda_3 - \frac{\lambda_4}{4}\right)\left(\Phi_1^*\Phi_4^*\,\Phi_1\Phi_4 + \Phi_2^*\Phi_3^*\,\Phi_2\Phi_3\right) &\text{(12d)}\\
&+\frac{\lambda_4}{2}\left(\Phi_1^*\Phi_4^*\,\Phi_2\Phi_3 + \Phi_2^*\Phi_3^*\,\Phi_1\Phi_4\right). &\text{(12e)}
\end{aligned}$$

This means that

$$
\begin{aligned}
\lambda_{1111} = \lambda_{2222} &= \frac{\lambda_1}{2}, && \text{(13a)}\\
\lambda_{3333} = \lambda_{4444} &= \frac{\lambda_2}{2}, && \text{(13b)}\\
\lambda_{1212} = \lambda_{1221} = \lambda_{2112} = \lambda_{2121} &= \frac{\lambda_1}{4}, && \text{(13c)}\\
\lambda_{3434} = \lambda_{3443} = \lambda_{4334} = \lambda_{4343} &= \frac{\lambda_2}{4}, && \text{(13d)}\\
\lambda_{1313} = \lambda_{1331} = \lambda_{3113} = \lambda_{3131} = \lambda_{2424} = \lambda_{2442} = \lambda_{4224} = \lambda_{4242} &= \frac{\lambda_3}{4} + \frac{\lambda_4}{16}, && \text{(13e)}\\
\lambda_{1414} = \lambda_{1441} = \lambda_{4114} = \lambda_{4141} = \lambda_{2323} = \lambda_{2332} = \lambda_{3223} = \lambda_{3232} &= \frac{\lambda_3}{4} - \frac{\lambda_4}{16}, && \text{(13f)}\\
\lambda_{1423} = \lambda_{4123} = \lambda_{1432} = \lambda_{4132} = \lambda_{2314} = \lambda_{3214} = \lambda_{2341} = \lambda_{3241} &= \frac{\lambda_4}{8}, && \text{(13g)}
\end{aligned}
$$

and all the other λ_{ikjl} are zero.

In the case $J = 1$ one has

$$a = \Phi_1, \quad b = \Phi_2, \quad c = \Phi_3, \quad d = \Phi_4, \quad e = \Phi_5, \tag{14}$$

where a and b form the doublet and c, d, and e form the triplet; and

$$\begin{aligned}
V_4 \;=\;& \frac{\lambda_1}{2}\left(\Phi_1^*\Phi_1+\Phi_2^*\Phi_2\right)^2+\frac{\lambda_2}{2}\left(\Phi_3^*\Phi_3+\Phi_4^*\Phi_4+\Phi_5^*\Phi_5\right)^2 && \text{(15a)}\\
&+\lambda_3\left(\Phi_1^*\Phi_1+\Phi_2^*\Phi_2\right)\left(\Phi_3^*\Phi_3+\Phi_4^*\Phi_4+\Phi_5^*\Phi_5\right) && \text{(15b)}\\
&+\lambda_4\left[\frac{\left(\Phi_1^*\Phi_1-\Phi_2^*\Phi_2\right)\left(\Phi_3^*\Phi_3-\Phi_5^*\Phi_5\right)}{2}\right.\\
&\left.+\frac{\Phi_1\Phi_2^*}{\sqrt{2}}\left(\Phi_3^*\Phi_4+\Phi_4^*\Phi_5\right)+\frac{\Phi_1^*\Phi_2}{\sqrt{2}}\left(\Phi_3\Phi_4^*+\Phi_4\Phi_5^*\right)\right] && \text{(15c)}\\
&+\frac{\lambda_5}{3}\left(4\Phi_3^*\Phi_3\Phi_5^*\Phi_5+\Phi_4^*\Phi_4\Phi_4^*\Phi_4-2\Phi_3^*\Phi_5^*\Phi_4\Phi_4-2\Phi_4^*\Phi_4^*\Phi_3\Phi_5\right) && \text{(15d)}\\
=\;& \frac{\lambda_1}{2}\left(\Phi_1^*\Phi_1^*\Phi_1\Phi_1+\Phi_2^*\Phi_2^*\Phi_2\Phi_2\right) && \text{(15e)}\\
&+\frac{\lambda_2}{2}\left(\Phi_3^*\Phi_3^*\Phi_3\Phi_3+\Phi_5^*\Phi_5^*\Phi_5\Phi_5\right) && \text{(15f)}\\
&+\left(\frac{\lambda_2}{2}+\frac{\lambda_5}{3}\right)\Phi_4^*\Phi_4^*\Phi_4\Phi_4 && \text{(15g)}\\
&+\lambda_1\,\Phi_1^*\Phi_2^*\Phi_1\Phi_2+\lambda_2\left(\Phi_3^*\Phi_4^*\Phi_3\Phi_4+\Phi_4^*\Phi_5^*\Phi_4\Phi_5\right)+\left(\lambda_2+\frac{4\lambda_5}{3}\right)\Phi_3^*\Phi_5^*\Phi_3\Phi_5 && \text{(15h)}\\
&+\left(\lambda_3+\frac{\lambda_4}{2}\right)\left(\Phi_1^*\Phi_3^*\Phi_1\Phi_3+\Phi_2^*\Phi_5\Phi_2\Phi_5\right) && \text{(15i)}\\
&+\left(\lambda_3-\frac{\lambda_4}{2}\right)\left(\Phi_1^*\Phi_5^*\Phi_1\Phi_5+\Phi_2^*\Phi_3\Phi_2\Phi_3\right) && \text{(15j)}\\
&+\lambda_3\left(\Phi_1^*\Phi_4^*\Phi_1\Phi_4+\Phi_2^*\Phi_4^*\Phi_2\Phi_4\right) && \text{(15k)}\\
&+\frac{\lambda_4}{\sqrt{2}}\left(\Phi_2^*\Phi_3^*\Phi_1\Phi_4+\Phi_2^*\Phi_4^*\Phi_1\Phi_5+\Phi_1^*\Phi_4^*\Phi_2\Phi_3+\Phi_1^*\Phi_5^*\Phi_2\Phi_4\right) && \text{(15l)}\\
&-\frac{2\lambda_5}{3}\left(\Phi_4^*\Phi_4^*\Phi_3\Phi_5+\Phi_3^*\Phi_5^*\Phi_4\Phi_4\right). && \text{(15m)}
\end{aligned}$$

This means that

$$
\begin{aligned}
\lambda_{1111} = \lambda_{2222} &= \frac{\lambda_1}{2}, && \text{(16a)}\\
\lambda_{3333} = \lambda_{5555} &= \frac{\lambda_2}{2}, && \text{(16b)}\\
\lambda_{4444} &= \frac{\lambda_2}{2} + \frac{\lambda_5}{3}, && \text{(16c)}\\
\lambda_{1212} = \lambda_{1221} = \lambda_{2112} = \lambda_{2121} &= \frac{\lambda_1}{4}, && \text{(16d)}\\
\lambda_{3434} = \lambda_{3443} = \lambda_{4334} = \lambda_{4343} = \lambda_{4545} = \lambda_{4554} = \lambda_{5445} = \lambda_{5454} &= \frac{\lambda_2}{4}, && \text{(16e)}\\
\lambda_{3535} = \lambda_{3553} = \lambda_{5335} = \lambda_{5353} &= \frac{\lambda_2}{4} + \frac{\lambda_5}{3}, && \text{(16f)}\\
\lambda_{1313} = \lambda_{1331} = \lambda_{3113} = \lambda_{3131} = \lambda_{2525} = \lambda_{2552} = \lambda_{5225} = \lambda_{5252} &= \frac{\lambda_3}{4} + \frac{\lambda_4}{8}, && \text{(16g)}\\
\lambda_{1515} = \lambda_{1551} = \lambda_{5115} = \lambda_{5151} = \lambda_{2323} = \lambda_{2332} = \lambda_{3223} = \lambda_{3232} &= \frac{\lambda_3}{4} - \frac{\lambda_4}{8}, && \text{(16h)}\\
\lambda_{1414} = \lambda_{1441} = \lambda_{4114} = \lambda_{4141} = \lambda_{2424} = \lambda_{2442} = \lambda_{4224} = \lambda_{4242} &= \frac{\lambda_3}{4}, && \text{(16i)}\\
\lambda_{1423} = \lambda_{4123} = \lambda_{1432} = \lambda_{4132} = \lambda_{2314} = \lambda_{3214} = \lambda_{2341} = \lambda_{3241} & \\
= \lambda_{1524} = \lambda_{5124} = \lambda_{1542} = \lambda_{5142} = \lambda_{2415} = \lambda_{4215} = \lambda_{2451} = \lambda_{4251} &= \frac{\lambda_4}{4\sqrt{2}}, && \text{(16j)}\\
\lambda_{4435} = \lambda_{4453} = \lambda_{3544} = \lambda_{5344} &= -\frac{\lambda_5}{3}, && \text{(16k)}
\end{aligned}
$$

and all the other λ_{ikjl} are zero.

The RGE for the λ_{ikjl} are

$$
16\pi^2 \frac{\mathrm{d}\lambda_{ikjl}}{\mathrm{d}t} = \sum_{m,n} \left[4\, \lambda_{ikmn}\, \lambda_{mnjl} + 8 \left(\lambda_{imjn}\, \lambda_{knlm} + \lambda_{imln}\, \lambda_{knjm} \right) \right]. \tag{17}
$$

4 The case $J = 1/2$

When $J = 1/2$ one has

$$
\begin{aligned}
8\pi^2 \frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{1111}}{\mathrm{d}t} &= \sum_{m,n} \left(4\, \lambda_{11mn}\, \lambda_{mn11} + 16\, \lambda_{1m1n}\, \lambda_{1n1m} \right) && \text{(18a)}\\
&= 4\, \lambda_{1111}\, \lambda_{1111} + 16\, \lambda_{1111}\, \lambda_{1111} + 16\, \lambda_{1212}\, \lambda_{1212} \\
&\quad + 16\, \lambda_{1313}\, \lambda_{1313} + 16\, \lambda_{1414}\, \lambda_{1414} && \text{(18b)}\\
&= 20 \left(\frac{\lambda_1}{2} \right)^2 + 16 \left(\frac{\lambda_1}{4} \right)^2 \\
&\quad + 16 \left(\frac{\lambda_3}{4} + \frac{\lambda_4}{16} \right)^2 + 16 \left(\frac{\lambda_3}{4} - \frac{\lambda_4}{16} \right)^2 && \text{(18c)}\\
&= 6\lambda_1^2 + 2\lambda_3^2 + \frac{\lambda_4^2}{8}; && \text{(18d)}
\end{aligned}
$$

$$
\begin{aligned}
4\pi^2 \frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{1212}}{\mathrm{d}t} &= \sum_{m,n} \left[4\,\lambda_{12mn}\,\lambda_{mn12} + 8\left(\lambda_{1m1n}\,\lambda_{2n2m} + \lambda_{1m2n}\,\lambda_{2n1m}\right)\right] && \text{(19a)}\\
&= 4\,\lambda_{1212}\,\lambda_{1212} + 4\,\lambda_{1221}\,\lambda_{2112} \\
&\quad +8\,\lambda_{1111}\,\lambda_{2121} + 8\,\lambda_{1212}\,\lambda_{2222} + 8\,\lambda_{1313}\,\lambda_{2323} + 8\,\lambda_{1414}\,\lambda_{2424} \\
&\quad +8\,\lambda_{1221}\,\lambda_{2112} + 8\,\lambda_{1423}\,\lambda_{2314} && \text{(19b)}\\
&= 8\left(\frac{\lambda_1}{4}\right)^2 + 8\,\frac{\lambda_1}{2}\,\frac{\lambda_1}{4} + 8\,\frac{\lambda_1}{4}\,\frac{\lambda_1}{2} + 16\left(\frac{\lambda_3^2}{16} - \frac{\lambda_4^2}{256}\right) \\
&\quad +8\left(\frac{\lambda_1}{4}\right)^2 + 8\left(\frac{\lambda_4}{8}\right)^2 && \text{(19c)}\\
&= 3\lambda_1^2 + \lambda_3^2 + \frac{\lambda_4^2}{16}; && \text{(19d)}
\end{aligned}
$$

$$
\begin{aligned}
2\pi^2 \frac{\mathrm{d}\lambda_4}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{1423}}{\mathrm{d}t} &= \sum_{m,n} \left[4\,\lambda_{14mn}\,\lambda_{mn23} + 8\left(\lambda_{1m2n}\,\lambda_{4n3m} + \lambda_{1m3n}\,\lambda_{4n2m}\right)\right] && \text{(20a)}\\
&= 4\,\lambda_{1414}\,\lambda_{1423} + 4\,\lambda_{1441}\,\lambda_{4123} + 4\,\lambda_{1423}\,\lambda_{2323} + 4\,\lambda_{1432}\,\lambda_{3223} \\
&\quad +8\,\lambda_{1221}\,\lambda_{4132} + 8\,\lambda_{1423}\,\lambda_{4334} \\
&\quad +8\,\lambda_{1331}\,\lambda_{4123} + 8\,\lambda_{1432}\,\lambda_{4224} && \text{(20b)}\\
&= \frac{\lambda_4}{8}\left[16\left(\frac{\lambda_3}{4} - \frac{\lambda_4}{16}\right) + 8\,\frac{\lambda_1}{4} + 8\,\frac{\lambda_2}{4} + 16\left(\frac{\lambda_3}{4} + \frac{\lambda_4}{16}\right)\right] && \text{(20c)}\\
&= \frac{\lambda_4}{8}\left(2\lambda_1 + 2\lambda_2 + 8\lambda_3\right); && \text{(20d)}
\end{aligned}
$$

$$
\begin{aligned}
4\pi^2 \frac{\mathrm{d}\lambda_3}{\mathrm{d}t} + \pi^2 \frac{\mathrm{d}\lambda_4}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{1313}}{\mathrm{d}t} &= \sum_{m,n} \left[4\,\lambda_{13mn}\,\lambda_{mn13}\right. \\
&\quad \left. +8\left(\lambda_{1m1n}\,\lambda_{3n3m} + \lambda_{1m3n}\,\lambda_{3n1m}\right)\right] && \text{(21a)}\\
&= 4\,\lambda_{1313}\,\lambda_{1313} + 4\,\lambda_{1331}\,\lambda_{3113} + 8\,\lambda_{1111}\,\lambda_{3131} \\
&\quad +8\,\lambda_{1212}\,\lambda_{3232} + 8\,\lambda_{1313}\,\lambda_{3333} + 8\,\lambda_{1414}\,\lambda_{3434} \\
&\quad +8\,\lambda_{1331}\,\lambda_{3113} + 8\,\lambda_{1432}\,\lambda_{3214} && \text{(21b)}\\
&= (\lambda_1 + \lambda_2)\left[\left(\lambda_3 + \frac{\lambda_4}{4}\right) + \frac{1}{2}\left(\lambda_3 - \frac{\lambda_4}{4}\right)\right] \\
&\quad +16\left(\lambda_3 + \frac{\lambda_4}{4}\right)^2 + 8\left(\frac{\lambda_4}{8}\right)^2; && \text{(21c)}
\end{aligned}
$$

$$
\begin{aligned}
4\pi^2 \frac{\mathrm{d}\lambda_3}{\mathrm{d}t} - \pi^2 \frac{\mathrm{d}\lambda_4}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{1414}}{\mathrm{d}t} &= \sum_{m,n} [4\,\lambda_{14mn}\,\lambda_{mn14} \\
&\quad +8\,(\lambda_{1m1n}\,\lambda_{4n4m} + \lambda_{1m4n}\,\lambda_{4n1m})] && \text{(22a)} \\
&= 4\,\lambda_{1414}\,\lambda_{1414} + 4\,\lambda_{1441}\,\lambda_{4114} + 4\,\lambda_{1423}\,\lambda_{2314} \\
&\quad +4\,\lambda_{1432}\,\lambda_{3214} + 8\,\lambda_{1111}\,\lambda_{4141} + 8\,\lambda_{1212}\,\lambda_{4242} \\
&\quad +8\,\lambda_{1313}\,\lambda_{4343} + 8\,\lambda_{1414}\,\lambda_{4444} + 8\,\lambda_{1441}\,\lambda_{4114} && \text{(22b)} \\
&= (\lambda_1 + \lambda_2)\left[\left(\lambda_3 - \frac{\lambda_4}{4}\right) + \frac{1}{2}\left(\lambda_3 + \frac{\lambda_4}{4}\right)\right] \\
&\quad +16\left(\lambda_3 - \frac{\lambda_4}{4}\right)^2 + 8\left(\frac{\lambda_4}{8}\right)^2. && \text{(22c)}
\end{aligned}
$$

This confirms Eqs. (8).

5 The case $J = 1$

When $J = 1$ one has

$$
\begin{aligned}
8\pi^2 \frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{1111}}{\mathrm{d}t} &= \sum_{m,n} (4\,\lambda_{11mn}\,\lambda_{mn11} + 16\,\lambda_{1m1n}\,\lambda_{1n1m}) && \text{(23a)} \\
&= 4\,\lambda_{1111}\,\lambda_{1111} + 16\,\lambda_{1111}\,\lambda_{1111} + 16\,\lambda_{1212}\,\lambda_{1212} \\
&\quad +16\,\lambda_{1313}\,\lambda_{1313} + 16\,\lambda_{1414}\,\lambda_{1414} + 16\,\lambda_{1515}\,\lambda_{1515} && \text{(23b)} \\
&= 6\lambda_1^2 + 3\lambda_3^2 + \frac{\lambda_4^2}{2}; && \text{(23c)}
\end{aligned}
$$

$$
\begin{aligned}
8\pi^2 \frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{3333}}{\mathrm{d}t} &= \sum_{m,n} (4\,\lambda_{33mn}\,\lambda_{mn33} + 16\,\lambda_{3m3n}\,\lambda_{3n3m}) && \text{(24a)} \\
&= 4\,\lambda_{3333}\,\lambda_{3333} + 16\,\lambda_{3131}\,\lambda_{3131} + 16\,\lambda_{3232}\,\lambda_{3232} \\
&\quad +16\,\lambda_{3333}\,\lambda_{3333} + 16\,\lambda_{3434}\,\lambda_{3434} + 16\,\lambda_{3535}\,\lambda_{3535} && \text{(24b)} \\
&= 7\lambda_2^2 + 2\lambda_3^2 + \frac{\lambda_4^2}{2} + \frac{16\lambda_5^2}{9} + \frac{8\lambda_2\lambda_5}{3}; && \text{(24c)}
\end{aligned}
$$

$$
\begin{aligned}
4\pi^2 \frac{\mathrm{d}\lambda_3}{\mathrm{d}t} = 16\pi^2 \frac{\mathrm{d}\lambda_{1414}}{\mathrm{d}t} &= \sum_{m,n} (4\,\lambda_{14mn}\,\lambda_{mn14} + 8\,\lambda_{1m1n}\,\lambda_{4n4m} + 8\,\lambda_{1m4n}\,\lambda_{4n1m}) && \text{(25a)} \\
&= 4\,\lambda_{1414}\,\lambda_{1414} + 4\,\lambda_{1441}\,\lambda_{4114} + 4\,\lambda_{1423}\,\lambda_{2314} + 4\,\lambda_{1432}\,\lambda_{3214} \\
&\quad +8\,\lambda_{1111}\,\lambda_{4141} + 8\,\lambda_{1212}\,\lambda_{4242} + 8\,\lambda_{1313}\,\lambda_{4343} + 8\,\lambda_{1414}\,\lambda_{4444} \\
&\quad +8\,\lambda_{1515}\,\lambda_{4545} + 8\,\lambda_{1441}\,\lambda_{4114} + 8\,\lambda_{1542}\,\lambda_{4215} && \text{(25b)} \\
&= \lambda_3^2 + \frac{\lambda_4^2}{2} + \left(\frac{3\lambda_1}{2} + 2\lambda_2 + \frac{2\lambda_5}{3}\right)\lambda_3; && \text{(25c)}
\end{aligned}
$$

$$2\sqrt{2}\,\pi^2\,\frac{\mathrm{d}\lambda_4}{\mathrm{d}t} = 16\pi^2\,\frac{\mathrm{d}\lambda_{1423}}{\mathrm{d}t} = \sum_{m,n}\left(4\,\lambda_{14mn}\,\lambda_{mn23} + 8\,\lambda_{1m2n}\,\lambda_{4n3m} + 8\,\lambda_{1m3n}\,\lambda_{4n2m}\right) \tag{26a}$$

$$\begin{aligned} &= 4\,\lambda_{1414}\,\lambda_{1423} + 4\,\lambda_{1441}\,\lambda_{4123} + 4\,\lambda_{1423}\,\lambda_{2323} \\ &\quad +4\,\lambda_{1432}\,\lambda_{3223} + 8\,\lambda_{1221}\,\lambda_{4132} + 8\,\lambda_{1423}\,\lambda_{4334} \\ &\quad +8\,\lambda_{1524}\,\lambda_{4435} + 8\,\lambda_{1331}\,\lambda_{4123} + 8\,\lambda_{1432}\,\lambda_{4224} \end{aligned} \tag{26b}$$

$$= \frac{\lambda_4}{4\sqrt{2}}\left(2\lambda_1 + 2\lambda_2 + 8\lambda_3 - \frac{8\lambda_5}{3}\right); \tag{26c}$$

$$-\frac{16\pi^2}{3}\,\frac{\mathrm{d}\lambda_5}{\mathrm{d}t} = 16\pi^2\,\frac{\mathrm{d}\lambda_{4435}}{\mathrm{d}t} = \sum_{m,n}\left(4\,\lambda_{44mn}\,\lambda_{mn35} + 8\,\lambda_{4m3n}\,\lambda_{4n5m} + 8\,\lambda_{4m5n}\,\lambda_{4n3m}\right) \tag{27a}$$

$$\begin{aligned} &= 4\,\lambda_{4444}\,\lambda_{4435} + 4\,\lambda_{4435}\,\lambda_{3535} + 4\,\lambda_{4453}\,\lambda_{5335} \\ &\quad +8\,\lambda_{4334}\,\lambda_{4453} + 8\,\lambda_{4132}\,\lambda_{4251} + 8\,\lambda_{4435}\,\lambda_{4554} \\ &\quad +8\,\lambda_{4554}\,\lambda_{4435} + 8\,\lambda_{4251}\,\lambda_{4132} + 8\,\lambda_{4453}\,\lambda_{4334} \end{aligned} \tag{27b}$$

$$= \frac{\lambda_4^2}{2} - \frac{4\lambda_5}{3}\left(3\lambda_2 + \lambda_5\right). \tag{27c}$$

6 The case $J = 3/2$

When $J = 3/2$ one has

$$V_4 = \frac{\lambda_1}{2}\left(A+B\right)^2 + \frac{\lambda_2}{2}\left(C+D+E+F\right)^2 + \lambda_3\left(A+B\right)\left(C+D+E+F\right) \tag{28a}$$

$$+\lambda_4\left\{\frac{\left(A-B\right)\left(3C+D-E-3F\right)}{4} + \left[\frac{ab^*}{2}\left(\sqrt{3}\,c^*d + 2d^*e + \sqrt{3}\,e^*f\right) + \text{H.c.}\right]\right\} \tag{28b}$$

$$+\frac{\lambda_5}{5}\left[\left|\sqrt{6}\,ce - \sqrt{2}\,d^2\right|^2 + \left|3cf - de\right|^2 + \left|\sqrt{6}\,df - \sqrt{2}\,e^2\right|^2\right], \tag{28c}$$

where $A = |a|^2$, $B = |b|^2$, and so on. We denote $\Phi_1 = a$, $\Phi_2 = b$, and so on. Then the notation (9) holds, with

$$\lambda_{1111} = \lambda_{2222} = \frac{\lambda_1}{2}, \tag{29}$$

$$\lambda_{1212} = \lambda_{2121} = \lambda_{1221} = \lambda_{2112} = \frac{\lambda_1}{4}, \tag{30}$$

$$\lambda_{3333} = \lambda_{6666} = \frac{\lambda_2}{2}, \tag{31a}$$

$$\lambda_{4444} = \lambda_{5555} = \frac{\lambda_2}{2} + \frac{2\lambda_5}{5}, \tag{31b}$$

$$\lambda_{3434} = \lambda_{4343} = \lambda_{4334} = \lambda_{3443} = \lambda_{5656} = \lambda_{6565} = \lambda_{6556} = \lambda_{5665} = \frac{\lambda_2}{4}, \tag{32a}$$

$$\lambda_{3535} = \lambda_{5353} = \lambda_{5335} = \lambda_{3553} = \lambda_{4646} = \lambda_{6464} = \lambda_{6446} = \lambda_{4664} = \frac{\lambda_2}{4} + \frac{3\lambda_5}{10}, \tag{32b}$$

$$\lambda_{4545} = \lambda_{5454} = \lambda_{5445} = \lambda_{4554} = \frac{\lambda_2}{4} + \frac{\lambda_5}{20}, \tag{32c}$$

$$\lambda_{3636} = \lambda_{6363} = \lambda_{6336} = \lambda_{3663} = \frac{\lambda_2}{4} + \frac{9\lambda_5}{20}, \tag{32d}$$

$$\lambda_{1313} = \lambda_{3131} = \lambda_{1331} = \lambda_{3113} = \lambda_{2626} = \lambda_{6262} = \lambda_{2662} = \lambda_{6226} = \frac{\lambda_3}{4} + \frac{3\lambda_4}{16}, \tag{33a}$$

$$\lambda_{1414} = \lambda_{4141} = \lambda_{1441} = \lambda_{4114} = \lambda_{2525} = \lambda_{5252} = \lambda_{2552} = \lambda_{5225} = \frac{\lambda_3}{4} + \frac{\lambda_4}{16}, \tag{33b}$$

$$\lambda_{1515} = \lambda_{5151} = \lambda_{1551} = \lambda_{5115} = \lambda_{2424} = \lambda_{4242} = \lambda_{2442} = \lambda_{4224} = \frac{\lambda_3}{4} - \frac{\lambda_4}{16}, \tag{33c}$$

$$\lambda_{1616} = \lambda_{6161} = \lambda_{1661} = \lambda_{6116} = \lambda_{2323} = \lambda_{3232} = \lambda_{2332} = \lambda_{3223} = \frac{\lambda_3}{4} - \frac{3\lambda_4}{16}, \tag{33d}$$

$$\lambda_{2314} = \lambda_{1423} = \lambda_{3214} = \lambda_{2341} = \lambda_{2516} = \lambda_{1625} = \lambda_{5216} = \lambda_{2561} = \frac{\sqrt{3}\,\lambda_4}{2}, \tag{34a}$$

$$\lambda_{2415} = \lambda_{1524} = \lambda_{4215} = \lambda_{2451} = \lambda_4, \tag{34b}$$

$$\lambda_{4435} = \lambda_{4453} = \lambda_{5344} = \lambda_{3544} = \lambda_{5546} = \lambda_{5564} = \lambda_{4655} = \lambda_{6455} = -\frac{\sqrt{3}\,\lambda_5}{5}, \tag{35a}$$

$$\lambda_{3645} = \lambda_{6354} = \lambda_{6345} = \lambda_{3654} = \lambda_{4536} = \lambda_{5463} = \lambda_{4563} = \lambda_{5436} = -\frac{3\lambda_5}{20}. \tag{35b}$$

Now,

$$16\pi^2 \, \frac{\mathrm{d}\lambda_{iiii}}{\mathrm{d}t} = 4 \sum_{m,n} \left| \lambda_{iimn} \right|^2 + 16 \sum_m \left| \lambda_{imim} \right|^2 . \tag{36}$$

Hence,

$$8\pi^2 \, \frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = 20 \left| \lambda_{1111} \right|^2 + 16 \left(\left| \lambda_{1212} \right|^2 + \left| \lambda_{1313} \right|^2 + \left| \lambda_{1414} \right|^2 + \left| \lambda_{1515} \right|^2 + \left| \lambda_{1616} \right|^2 \right) \tag{37a}$$

$$= 6\lambda_1^2 + 4\lambda_3^2 + \frac{5}{4}\,\lambda_4^2; \tag{37b}$$

7 The theory without quartic terms in χ

7.1 Half-integer J

7.1.1 $J = 1/2$

In this case there is

$$H = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \Phi_3 \\ \Phi_4 \end{pmatrix}. \tag{38}$$

The quartic part of the potential is

$$
\begin{aligned}
V_4 &= \frac{\lambda_1}{2}\left(|\Phi_1|^2 + |\Phi_2|^2\right)^2 && \text{(39a)}\\
&+\lambda_3\left(|\Phi_1|^2 + |\Phi_2|^2\right)\left(|\Phi_3|^2 + |\Phi_4|^2\right) && \text{(39b)}\\
&+\lambda_4\left[\frac{|\Phi_1|^2 - |\Phi_2|^2}{2}\ \frac{|\Phi_3|^2 - |\Phi_4|^2}{2} + \frac{\Phi_1\Phi_2^*}{2}\ \Phi_3^*\Phi_4 + \frac{\Phi_1^*\Phi_2}{2}\ \Phi_3\Phi_4^*\right]. && \text{(39c)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lambda_{1111} = \lambda_{2222} &= \frac{\lambda_1}{2}, && \text{(40a)}\\
\lambda_{1212} = \lambda_{2112} = \lambda_{1221} = \lambda_{2121} &= \frac{\lambda_1}{4}, && \text{(40b)}\\
\lambda_{1313} = \lambda_{3113} = \lambda_{1331} = \lambda_{3131} = \lambda_{2424} = \lambda_{4224} = \lambda_{2442} = \lambda_{4242} &= \frac{\lambda_3}{4} + \frac{\lambda_4}{16}, && \text{(40c)}\\
\lambda_{1414} = \lambda_{4114} = \lambda_{1441} = \lambda_{4141} = \lambda_{2323} = \lambda_{3223} = \lambda_{2332} = \lambda_{3232} &= \frac{\lambda_3}{4} - \frac{\lambda_4}{16}, && \text{(40d)}\\
\lambda_{2314} = \lambda_{3214} = \lambda_{2341} = \lambda_{3241} = \lambda_{1423} = \lambda_{4123} = \lambda_{1432} = \lambda_{4132} &= \frac{\lambda_4}{8}. && \text{(40e)}
\end{aligned}
$$

7.1.2 $J = 3/2$

In this case there is

$$
H = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \Phi_5 \\ \Phi_3 \\ \Phi_4 \\ \Phi_6 \end{pmatrix}. \tag{41}
$$

The quartic part of the potential is

$$
\begin{aligned}
V_4 &= \frac{\lambda_1}{2}\left(|\Phi_1|^2 + |\Phi_2|^2\right)^2 && \text{(42a)}\\
&+\lambda_3\left(|\Phi_1|^2 + |\Phi_2|^2\right)\left(|\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2 + |\Phi_6|^2\right) && \text{(42b)}\\
&+\lambda_4\left[\frac{|\Phi_1|^2 - |\Phi_2|^2}{2}\ \frac{|\Phi_3|^2 - |\Phi_4|^2 + 3\,|\Phi_5|^2 - 3\,|\Phi_6|^2}{2}\right.\\
&+\frac{\Phi_1\Phi_2^*}{2}\left(\sqrt{3}\,\Phi_5^*\Phi_3 + 2\Phi_3^*\Phi_4 + \sqrt{3}\,\Phi_4^*\Phi_6\right)\\
&\left.+\frac{\Phi_1^*\Phi_2}{2}\left(\sqrt{3}\,\Phi_3^*\Phi_5 + 2\Phi_4^*\Phi_3 + \sqrt{3}\,\Phi_6^*\Phi_4\right)\right]. && \text{(42c)}
\end{aligned}
$$

Therefore,

$$\lambda_{1111} = \lambda_{2222} = \frac{\lambda_1}{2}, \tag{43a}$$

$$\lambda_{1212} = \lambda_{2112} = \lambda_{1221} = \lambda_{2121} = \frac{\lambda_1}{4}, \tag{43b}$$

$$\lambda_{1313} = \lambda_{3113} = \lambda_{1331} = \lambda_{3131} = \lambda_{2424} = \lambda_{4224} = \lambda_{2442} = \lambda_{4242} = \frac{\lambda_3}{4} + \frac{\lambda_4}{16}, \tag{43c}$$

$$\lambda_{1414} = \lambda_{4114} = \lambda_{1441} = \lambda_{4141} = \lambda_{2323} = \lambda_{3223} = \lambda_{2332} = \lambda_{3232} = \frac{\lambda_3}{4} - \frac{\lambda_4}{16}, \tag{43d}$$

$$\lambda_{1515} = \lambda_{5115} = \lambda_{1551} = \lambda_{5151} = \lambda_{2626} = \lambda_{6226} = \lambda_{2662} = \lambda_{6262} = \frac{\lambda_3}{4} + \frac{3\lambda_4}{16}, \tag{43e}$$

$$\lambda_{1616} = \lambda_{6116} = \lambda_{1661} = \lambda_{6161} = \lambda_{2525} = \lambda_{5225} = \lambda_{2552} = \lambda_{5252} = \frac{\lambda_3}{4} - \frac{3\lambda_4}{16}, \tag{43f}$$

$$\begin{aligned} &\lambda_{1325} = \lambda_{3125} = \lambda_{1352} = \lambda_{3152} = \lambda_{2513} = \lambda_{2531} = \lambda_{5213} = \lambda_{5231} \\ &= \lambda_{1624} = \lambda_{6124} = \lambda_{1642} = \lambda_{6142} = \lambda_{2416} = \lambda_{2461} = \lambda_{4216} = \lambda_{4261} = \frac{\sqrt{3}\,\lambda_4}{8}, \end{aligned} \tag{43g}$$

$$\lambda_{2314} = \lambda_{3214} = \lambda_{2341} = \lambda_{3241} = \lambda_{1423} = \lambda_{4123} = \lambda_{1432} = \lambda_{4132} = \frac{\lambda_4}{4}. \tag{43h}$$

7.1.3 $J = 5/2$

In this case there is

$$H = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \Phi_7 \\ \Phi_5 \\ \Phi_3 \\ \Phi_4 \\ \Phi_6 \\ \Phi_8 \end{pmatrix}. \tag{44}$$

The quartic part of the potential is

$$\begin{aligned} V_4 \;=\;& \frac{\lambda_1}{2} \left(|\Phi_1|^2 + |\Phi_2|^2 \right)^2 & (45a) \\ & + \lambda_3 \left(|\Phi_1|^2 + |\Phi_2|^2 \right) \left(|\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2 + |\Phi_6|^2 + |\Phi_7|^2 + |\Phi_8|^2 \right) & (45b) \\ & + \lambda_4 \left[\frac{|\Phi_1|^2 - |\Phi_2|^2}{2}\ \frac{|\Phi_3|^2 - |\Phi_4|^2 + 3\,|\Phi_5|^2 - 3\,|\Phi_6|^2 + 5\,|\Phi_7|^2 - 5\,|\Phi_8|^2}{2} \right. \\ & + \frac{\Phi_1 \Phi_2^*}{2} \left(\sqrt{5}\,\Phi_7^* \Phi_5 + \sqrt{8}\,\Phi_5^* \Phi_3 + \sqrt{9}\,\Phi_3^* \Phi_4 + \sqrt{8}\,\Phi_4^* \Phi_6 + \sqrt{5}\,\Phi_6^* \Phi_8 \right) \\ & \left. + \frac{\Phi_1^* \Phi_2}{2} \left(\sqrt{5}\,\Phi_5^* \Phi_7 + \sqrt{8}\,\Phi_3^* \Phi_5 + \sqrt{9}\,\Phi_4^* \Phi_4 + \sqrt{8}\,\Phi_6^* \Phi_4 + \sqrt{5}\,\Phi_8^* \Phi_6 \right) \right]. & (45c) \end{aligned}$$

Therefore,

$$
\begin{aligned}
\lambda_{1111} = \lambda_{2222} &= \frac{\lambda_1}{2}, && \text{(46a)}\\
\lambda_{1212} = \lambda_{2112} = \lambda_{1221} = \lambda_{2121} &= \frac{\lambda_1}{4}, && \text{(46b)}\\
\lambda_{1313} = \lambda_{3113} = \lambda_{1331} = \lambda_{3131} = \lambda_{2424} = \lambda_{4224} = \lambda_{2442} = \lambda_{4242} &= \frac{\lambda_3}{4} + \frac{\lambda_4}{16}, && \text{(46c)}\\
\lambda_{1414} = \lambda_{4114} = \lambda_{1441} = \lambda_{4141} = \lambda_{2323} = \lambda_{3223} = \lambda_{2332} = \lambda_{3232} &= \frac{\lambda_3}{4} - \frac{\lambda_4}{16}, && \text{(46d)}\\
\lambda_{1515} = \lambda_{5115} = \lambda_{1551} = \lambda_{5151} = \lambda_{2626} = \lambda_{6226} = \lambda_{2662} = \lambda_{6262} &= \frac{\lambda_3}{4} + \frac{3\lambda_4}{16}, && \text{(46e)}\\
\lambda_{1616} = \lambda_{6116} = \lambda_{1661} = \lambda_{6161} = \lambda_{2525} = \lambda_{5225} = \lambda_{2552} = \lambda_{5252} &= \frac{\lambda_3}{4} - \frac{3\lambda_4}{16}, && \text{(46f)}\\
\lambda_{1717} = \lambda_{7117} = \lambda_{1771} = \lambda_{7171} = \lambda_{2828} = \lambda_{8228} = \lambda_{2882} = \lambda_{8282} &= \frac{\lambda_3}{4} + \frac{5\lambda_4}{16}, && \text{(46g)}\\
\lambda_{1818} = \lambda_{8118} = \lambda_{1881} = \lambda_{8181} = \lambda_{2727} = \lambda_{7227} = \lambda_{2772} = \lambda_{7272} &= \frac{\lambda_3}{4} - \frac{5\lambda_4}{16}, && \text{(46h)}\\
\lambda_{2715} = \lambda_{7215} = \lambda_{2751} = \lambda_{7251} = \lambda_{1527} = \lambda_{1572} = \lambda_{5127} = \lambda_{5172} & \\
= \lambda_{2618} = \lambda_{6218} = \lambda_{2681} = \lambda_{6281} = \lambda_{1826} = \lambda_{1862} = \lambda_{8126} = \lambda_{8162} &= \frac{\sqrt{5}\,\lambda_4}{8}, && \text{(46i)}\\
\lambda_{2513} = \lambda_{5213} = \lambda_{2531} = \lambda_{5231} = \lambda_{1325} = \lambda_{1352} = \lambda_{3125} = \lambda_{3152} & \\
= \lambda_{2416} = \lambda_{4216} = \lambda_{2461} = \lambda_{4261} = \lambda_{1624} = \lambda_{1642} = \lambda_{6124} = \lambda_{6142} &= \frac{\sqrt{8}\,\lambda_4}{8}, && \text{(46j)}\\
\lambda_{2314} = \lambda_{3214} = \lambda_{2341} = \lambda_{3241} = \lambda_{1423} = \lambda_{4123} = \lambda_{1432} = \lambda_{4132} &= \frac{3\lambda_4}{8}. && \text{(46k)}
\end{aligned}
$$

7.1.4 The RGE for λ_1

Let us denote

$$t \equiv \ln \mu \tag{47}$$

and

$$\mathcal{D} \equiv 16\pi^2 \, \frac{\mathrm{d}}{\mathrm{d}t}. \tag{48}$$

Then, from $\lambda_{1111} = \lambda_1/2$ and from

$$\mathcal{D}\lambda_{iiii} = \sum_{m,n} \left(4\left|\lambda_{iimn}\right|^2 + 16\left|\lambda_{imin}\right|^2 \right) \tag{49}$$

it follows that

$$
\begin{aligned}
\frac{\mathcal{D}\lambda_1}{2} = \mathcal{D}\lambda_{1111} &= \sum_m \left(4\left|\lambda_{11mm}\right|^2 + 16\left|\lambda_{1m1m}\right|^2 \right) && \text{(50a)}\\
&= 20\left|\lambda_{1111}\right|^2 + 16\left|\lambda_{1212}\right|^2 + 16 \sum_{m=3}^{2J+3} \left|\lambda_{1m1m}\right|^2 && \text{(50b)}\\
&= 6\lambda_1^2 + 16 \sum_{m=3}^{2J+3} \left|\lambda_{1m1m}\right|^2, && \text{(50c)}
\end{aligned}
$$

where we have used $\lambda_{1111} = \lambda_1/2$ and $\lambda_{1212} = \lambda_1/4$. Now,

$$\sum_{m=3}^{2J+3} |\lambda_{1m1m}|^2 = \begin{cases} 2\left(\frac{\lambda_3}{4}\right)^2 + 2\left(\frac{\lambda_4}{16}\right)^2 & \Leftarrow \quad J = \frac{1}{2}, \\ 4\left(\frac{\lambda_3}{4}\right)^2 + 2\left(\frac{\lambda_4}{16}\right)^2 + 2\left(\frac{3\lambda_4}{16}\right)^2 & \Leftarrow \quad J = \frac{3}{2}, \\ 6\left(\frac{\lambda_3}{4}\right)^2 + 2\left(\frac{\lambda_4}{16}\right)^2 + 2\left(\frac{3\lambda_4}{16}\right)^2 + 2\left(\frac{5\lambda_4}{16}\right)^2 & \Leftarrow \quad J = \frac{5}{2}. \end{cases} \tag{51}$$

Therefore,

$$\frac{\mathcal{D}\lambda_1}{2} = 6\lambda_1^2 + (2J+1)\,\lambda_3^2 + \frac{\lambda_4^2}{8}\,A, \tag{52}$$

where $A = 1$ for $J = 1/2$, $A = 1 + 9$ for $J = 3/2$, $A = 1 + 9 + 25$ for $J = 5/2$, and so on. In general,

$$\begin{aligned} A &= \sum_{k=1}^{2J} k^2 - 4 \sum_{k=1}^{J-1/2} k^2 && (53a) \\ &= \frac{2J\,(2J+1)\,(4J+1)}{6} - 4\,\frac{(J-1/2)\,(J+1/2)\,(2J)}{6} && (53b) \\ &= \frac{J\,(8J^2+6J+1)}{3} - \frac{(4J^2-1)\,J}{3} && (53c) \\ &= \frac{J\,(4J^2+6J+2)}{3} && (53d) \\ &= \frac{2J\,(J+1)\,(2J+1)}{3}. && (53e) \end{aligned}$$

Thus, finally,

$$\mathcal{D}\lambda_1 = 12\lambda_1^2 + 2\,(2J+1)\,\lambda_3^2 + \frac{J\,(J+1)\,(2J+1)}{6}\,\lambda_4^2. \tag{54}$$

Equation (54) holds *for half-integer* J.

7.1.5 The RGEs for λ_3 and λ_4

We use $\lambda_{1313} = \lambda_3/4 + \lambda_4/16$, $\lambda_{1414} = \lambda_3/4 - \lambda_4/16$, and

$$\mathcal{D}\lambda_{1k1k} = \sum_{m,n} \left(4\,|\lambda_{1kmn}|^2 + 8\,\lambda_{1m1n}\lambda_{knkm} + 8\,|\lambda_{1mkn}|^2\right). \tag{55}$$

Therefore,

$$\begin{aligned} \frac{\mathcal{D}\lambda_3}{4} + \frac{\mathcal{D}\lambda_4}{16} &= \sum_{m,n} \left(4\,|\lambda_{13mn}|^2 + 8\,\lambda_{1m1n}\lambda_{3n3m} + 8\,|\lambda_{1m3n}|^2\right), && (56a) \\ \frac{\mathcal{D}\lambda_3}{4} - \frac{\mathcal{D}\lambda_4}{16} &= \sum_{m,n} \left(4\,|\lambda_{14mn}|^2 + 8\,\lambda_{1m1n}\lambda_{4n4m} + 8\,|\lambda_{1m4n}|^2\right). && (56b) \end{aligned}$$

7.2 Integer J

7.2.1 $J = 0$

In this case there is

$$H = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \Phi_3 \end{pmatrix}. \tag{57}$$

The quartic part of the potential is

$$V_4 = \frac{\lambda_1}{2} \left(|\Phi_1|^2 + |\Phi_2|^2 \right)^2 + \lambda_3 \left(|\Phi_1|^2 + |\Phi_2|^2 \right) |\Phi_3|^2 . \tag{58}$$

Therefore,

$$\lambda_{1111} = \lambda_{2222} = \frac{\lambda_1}{2}, \tag{59a}$$

$$\lambda_{1212} = \lambda_{2112} = \lambda_{1221} = \lambda_{2121} = \frac{\lambda_1}{4}, \tag{59b}$$

$$\lambda_{1313} = \lambda_{3113} = \lambda_{1331} = \lambda_{3131} = \lambda_{2323} = \lambda_{3223} = \lambda_{2332} = \lambda_{3232} = \frac{\lambda_3}{4}. \tag{59c}$$

7.2.2 $J = 1$

In this case there is

$$H = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \Phi_4 \\ \Phi_3 \\ \Phi_5 \end{pmatrix}. \tag{60}$$

The quartic part of the potential is

$$\begin{aligned}
V_4 \;=\; & \frac{\lambda_1}{2} \left(|\Phi_1|^2 + |\Phi_2|^2 \right)^2 && \text{(61a)} \\
& + \lambda_3 \left(|\Phi_1|^2 + |\Phi_2|^2 \right) \left(|\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2 \right) && \text{(61b)} \\
& + \lambda_4 \left[\frac{\left(|\Phi_1|^2 - |\Phi_2|^2 \right) \left(|\Phi_4|^2 - |\Phi_5|^2 \right)}{2} \right. \\
& \left. + \frac{\Phi_1 \Phi_2^*}{\sqrt{2}} \left(\Phi_4^* \Phi_3 + \Phi_3^* \Phi_5 \right) + \frac{\Phi_1^* \Phi_2}{\sqrt{2}} \left(\Phi_4 \Phi_3^* + \Phi_3 \Phi_5^* \right) \right]. && \text{(61c)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_{1111} = \lambda_{2222} &= \frac{\lambda_1}{2}, && \text{(62a)}\\
\lambda_{1212} = \lambda_{2112} = \lambda_{1221} = \lambda_{2121} &= \frac{\lambda_1}{4}, && \text{(62b)}\\
\lambda_{1313} = \lambda_{3113} = \lambda_{1331} = \lambda_{3131} = \lambda_{2323} = \lambda_{3223} = \lambda_{2332} = \lambda_{3232} &= \frac{\lambda_3}{4}, && \text{(62c)}\\
\lambda_{1414} = \lambda_{4114} = \lambda_{1441} = \lambda_{4141} = \lambda_{2525} = \lambda_{5225} = \lambda_{2552} = \lambda_{5252} &= \frac{\lambda_3}{4} + \frac{\lambda_4}{8}, && \text{(62d)}\\
\lambda_{1515} = \lambda_{5115} = \lambda_{1551} = \lambda_{5151} = \lambda_{2424} = \lambda_{4224} = \lambda_{2442} = \lambda_{4242} &= \frac{\lambda_3}{4} - \frac{\lambda_4}{8}, && \text{(62e)}\\
\lambda_{2413} = \lambda_{4213} = \lambda_{2431} = \lambda_{4231} = \lambda_{1324} = \lambda_{3124} = \lambda_{1342} = \lambda_{3142} & && \\
= \lambda_{2315} = \lambda_{3215} = \lambda_{2351} = \lambda_{3251} = \lambda_{1523} = \lambda_{5123} = \lambda_{1532} = \lambda_{5132} &= \frac{\lambda_4}{4\sqrt{2}}. && \text{(62f)}
\end{aligned}$$

7.2.3 $J = 2$

In this case there is

$$H = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \Phi_6 \\ \Phi_4 \\ \Phi_3 \\ \Phi_5 \\ \Phi_7 \end{pmatrix}. \tag{63}$$

The quartic part of the potential is

$$\begin{aligned}
V_4 &= \frac{\lambda_1}{2} \left(|\Phi_1|^2 + |\Phi_2|^2\right)^2 && \text{(64a)}\\
&+\lambda_3 \left(|\Phi_1|^2 + |\Phi_2|^2\right) \left(|\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2 + |\Phi_6|^2 + |\Phi_7|^2\right) && \text{(64b)}\\
&+\lambda_4 \left[\frac{\left(|\Phi_1|^2 - |\Phi_2|^2\right) \left(2\,|\Phi_6|^2 + |\Phi_4|^2 - |\Phi_5|^2 - 2\,|\Phi_7|^2\right)}{2} \right. && \\
&+\frac{\Phi_1 \Phi_2^*}{2} \left(2\Phi_6^* \Phi_4 + \sqrt{6}\, \Phi_4^* \Phi_3 + \sqrt{6}\, \Phi_3^* \Phi_5 + 2\Phi_5^* \Phi_7\right) && \\
&\left. +\frac{\Phi_1^* \Phi_2}{2} \left(2\Phi_6 \Phi_4^* + \sqrt{6}\, \Phi_4 \Phi_3^* + \sqrt{6}\, \Phi_3 \Phi_5^* + 2\Phi_5 \Phi_7^*\right) \right]. && \text{(64c)}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\lambda_{1111} = \lambda_{2222} &= \frac{\lambda_1}{2}, & \text{(65a)}\\
\lambda_{1212} = \lambda_{2112} = \lambda_{1221} = \lambda_{2121} &= \frac{\lambda_1}{4}, & \text{(65b)}\\
\lambda_{1313} = \lambda_{3113} = \lambda_{1331} = \lambda_{3131} = \lambda_{2323} = \lambda_{3223} = \lambda_{2332} = \lambda_{3232} &= \frac{\lambda_3}{4}, & \text{(65c)}\\
\lambda_{1414} = \lambda_{4114} = \lambda_{1441} = \lambda_{4141} = \lambda_{2525} = \lambda_{5225} = \lambda_{2552} = \lambda_{5252} &= \frac{\lambda_3}{4} + \frac{\lambda_4}{8}, & \text{(65d)}\\
\lambda_{1515} = \lambda_{5115} = \lambda_{1551} = \lambda_{5151} = \lambda_{2424} = \lambda_{4224} = \lambda_{2442} = \lambda_{4242} &= \frac{\lambda_3}{4} - \frac{\lambda_4}{8}, & \text{(65e)}\\
\lambda_{1616} = \lambda_{6116} = \lambda_{1661} = \lambda_{6161} = \lambda_{2727} = \lambda_{7227} = \lambda_{2772} = \lambda_{7272} &= \frac{\lambda_3 + \lambda_4}{4}, & \text{(65f)}\\
\lambda_{1717} = \lambda_{7117} = \lambda_{1771} = \lambda_{7171} = \lambda_{2626} = \lambda_{6226} = \lambda_{2662} = \lambda_{6262} &= \frac{\lambda_3 - \lambda_4}{4}, & \text{(65g)}\\
\lambda_{2413} = \lambda_{4213} = \lambda_{2341} = \lambda_{4231} = \lambda_{1324} = \lambda_{3124} = \lambda_{1342} = \lambda_{3142} & & \\
= \lambda_{2315} = \lambda_{3215} = \lambda_{2351} = \lambda_{3251} = \lambda_{1523} = \lambda_{5123} = \lambda_{1532} = \lambda_{5132} &= \frac{\sqrt{6}\,\lambda_4}{8}, & \text{(65h)}\\
\lambda_{2614} = \lambda_{6214} = \lambda_{2641} = \lambda_{6241} = \lambda_{1426} = \lambda_{4126} = \lambda_{1462} = \lambda_{4162} & & \\
= \lambda_{2517} = \lambda_{5217} = \lambda_{2571} = \lambda_{5271} = \lambda_{1725} = \lambda_{7125} = \lambda_{1752} = \lambda_{7152} &= \frac{\lambda_4}{4}. & \text{(65i)}
\end{aligned}
$$

7.2.4 $J = 3$

In this case there is

$$
H = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \Phi_8 \\ \Phi_6 \\ \Phi_4 \\ \Phi_3 \\ \Phi_5 \\ \Phi_7 \\ \Phi_9 \end{pmatrix}. \tag{66}
$$

The quartic part of the potential is

$$
\begin{aligned}
V_4 \;=\;& \frac{\lambda_1}{2}\left(|\Phi_1|^2 + |\Phi_2|^2\right)^2 && \text{(67a)}\\
&+\lambda_3\left(|\Phi_1|^2 + |\Phi_2|^2\right)\left(|\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2 + |\Phi_6|^2 + |\Phi_7|^2 + |\Phi_8|^2 + |\Phi_9|^2\right) && \text{(67b)}\\
&+\lambda_4\left[\frac{\left(|\Phi_1|^2 - |\Phi_2|^2\right)\left(3\,|\Phi_8|^2 + 2\,|\Phi_6|^2 + |\Phi_4|^2 - |\Phi_5|^2 - 2\,|\Phi_7|^2 - 3\,|\Phi_9|^2\right)}{2}\right.\\
&+\frac{\Phi_1\Phi_2^*}{2}\left(\sqrt{6}\,\Phi_8^*\Phi_6 + \sqrt{10}\,\Phi_6^*\Phi_4 + \sqrt{12}\,\Phi_4^*\Phi_3\right.\\
&\left.+\sqrt{12}\,\Phi_3^*\Phi_5 + \sqrt{10}\,\Phi_5^*\Phi_7 + \sqrt{6}\,\Phi_7^*\Phi_9\right)\\
&+\frac{\Phi_1^*\Phi_2}{2}\left(\sqrt{6}\,\Phi_8\Phi_6^* + \sqrt{10}\,\Phi_6\Phi_4^* + \sqrt{12}\,\Phi_4\Phi_3^*\right.\\
&\left.\left.+\sqrt{12}\,\Phi_3\Phi_5^* + \sqrt{10}\,\Phi_5\Phi_7^* + \sqrt{6}\,\Phi_7\Phi_9^*\right)\right]. && \text{(67c)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lambda_{1111} = \lambda_{2222} &= \frac{\lambda_1}{2}, && \text{(68a)}\\
\lambda_{1212} = \lambda_{2112} = \lambda_{1221} = \lambda_{2121} &= \frac{\lambda_1}{4}, && \text{(68b)}\\
\lambda_{1313} = \lambda_{3113} = \lambda_{1331} = \lambda_{3131} = \lambda_{2323} = \lambda_{3223} = \lambda_{2332} = \lambda_{3232} &= \frac{\lambda_3}{4}, && \text{(68c)}\\
\lambda_{1414} = \lambda_{4114} = \lambda_{1441} = \lambda_{4141} = \lambda_{2525} = \lambda_{5225} = \lambda_{2552} = \lambda_{5252} &= \frac{\lambda_3}{4} + \frac{\lambda_4}{8}, && \text{(68d)}\\
\lambda_{1515} = \lambda_{5115} = \lambda_{1551} = \lambda_{5151} = \lambda_{2424} = \lambda_{4224} = \lambda_{2442} = \lambda_{4242} &= \frac{\lambda_3}{4} - \frac{\lambda_4}{8}, && \text{(68e)}\\
\lambda_{1616} = \lambda_{6116} = \lambda_{1661} = \lambda_{6161} = \lambda_{2727} = \lambda_{7227} = \lambda_{2772} = \lambda_{7272} &= \frac{\lambda_3 + \lambda_4}{4}, && \text{(68f)}\\
\lambda_{1717} = \lambda_{7117} = \lambda_{1771} = \lambda_{7171} = \lambda_{2626} = \lambda_{6226} = \lambda_{2662} = \lambda_{6262} &= \frac{\lambda_3 - \lambda_4}{4}, && \text{(68g)}\\
\lambda_{1818} = \lambda_{8118} = \lambda_{1881} = \lambda_{8181} = \lambda_{2929} = \lambda_{9229} = \lambda_{2992} = \lambda_{9292} &= \frac{\lambda_3}{4} + \frac{3\lambda_4}{8}, && \text{(68h)}\\
\lambda_{1919} = \lambda_{9119} = \lambda_{1991} = \lambda_{9191} = \lambda_{2828} = \lambda_{8228} = \lambda_{2882} = \lambda_{8282} &= \frac{\lambda_3}{4} - \frac{3\lambda_4}{8}, && \text{(68i)}\\
\lambda_{2413} = \lambda_{4213} = \lambda_{2431} = \lambda_{4231} = \lambda_{1324} = \lambda_{3124} = \lambda_{1342} = \lambda_{3142} &\\
= \lambda_{2315} = \lambda_{3215} = \lambda_{2351} = \lambda_{3251} = \lambda_{1523} = \lambda_{5123} = \lambda_{1532} = \lambda_{5132} &= \frac{\sqrt{12}\,\lambda_4}{8}, && \text{(68j)}\\
\lambda_{2614} = \lambda_{6214} = \lambda_{2641} = \lambda_{6241} = \lambda_{1426} = \lambda_{4126} = \lambda_{1462} = \lambda_{4162} &\\
= \lambda_{2517} = \lambda_{5217} = \lambda_{2571} = \lambda_{5271} = \lambda_{1725} = \lambda_{7125} = \lambda_{1752} = \lambda_{7152} &= \frac{\sqrt{10}\,\lambda_4}{8}, && \text{(68k)}\\
\lambda_{2816} = \lambda_{8216} = \lambda_{2861} = \lambda_{8261} = \lambda_{1628} = \lambda_{6128} = \lambda_{1682} = \lambda_{6182} &\\
= \lambda_{2719} = \lambda_{7219} = \lambda_{2791} = \lambda_{7291} = \lambda_{1927} = \lambda_{9127} = \lambda_{1972} = \lambda_{9172} &= \frac{\sqrt{6}\,\lambda_4}{8}. && \text{(68l)}
\end{aligned}
$$

7.2.5 The general RGEs

Let us denote

$$t \equiv \ln \mu \tag{69}$$

and

$$\mathcal{D} \equiv 16\pi^2 \, \frac{\mathrm{d}}{\mathrm{d}t}. \tag{70}$$

One then has the fundamental RGE

$$\mathcal{D}\lambda_{ikjl} = \sum_{m,n} \left[4\, \lambda_{ikmn}\, \lambda_{mnjl} + 8 \left(\lambda_{imjn}\, \lambda_{knlm} + \lambda_{imln}\, \lambda_{knjm} \right) \right]. \tag{71}$$

Hence,

$$\mathcal{D}\lambda_{ikik} = \sum_{m,n} \left(4 \left| \lambda_{ikmn} \right|^2 + 8\, \lambda_{imin}\, \lambda_{knkm} + 8 \left| \lambda_{imkn} \right|^2 \right), \tag{72a}$$

$$\mathcal{D}\lambda_{iiii} = \sum_{m,n} \left(4 \left| \lambda_{iimn} \right|^2 + 16 \left| \lambda_{imin} \right|^2 \right). \tag{72b}$$

7.2.6 The RGE for λ_1

From $\lambda_{1111} = \lambda_1/2$ and from Eq. (72b) it follows that

$$\frac{\mathcal{D}\lambda_1}{2} = \mathcal{D}\lambda_{1111} = \sum_{m} \left(4 \left| \lambda_{11mm} \right|^2 + 16 \left| \lambda_{1m1m} \right|^2 \right) \tag{73a}$$

$$= 20 \left| \lambda_{1111} \right|^2 + 16 \left| \lambda_{1212} \right|^2 + 16 \sum_{m=3}^{2J+3} \left| \lambda_{1m1m} \right|^2 \tag{73b}$$

$$= 6\lambda_1^2 + 16 \sum_{m=3}^{2J+3} \left| \lambda_{1m1m} \right|^2, \tag{73c}$$

where we have used $\lambda_{1111} = \lambda_1/2$ and $\lambda_{1212} = \lambda_1/4$. Now,

$$\sum_{m=3}^{2J+3} \left| \lambda_{1m1m} \right|^2 = \begin{cases} \left(\dfrac{\lambda_3}{4} \right)^2 & \Leftarrow \quad J = 0, \\ 3 \left(\dfrac{\lambda_3}{4} \right)^2 + 2 \left(\dfrac{\lambda_4}{8} \right)^2 & \Leftarrow \quad J = 1, \\ 5 \left(\dfrac{\lambda_3}{4} \right)^2 + 2 \left(\dfrac{\lambda_4}{8} \right)^2 + 2 \left(\dfrac{\lambda_4}{4} \right)^2 & \Leftarrow \quad J = 2, \\ 7 \left(\dfrac{\lambda_3}{4} \right)^2 + 2 \left(\dfrac{\lambda_4}{8} \right)^2 + 2 \left(\dfrac{\lambda_4}{4} \right)^2 + 2 \left(\dfrac{3\lambda_4}{8} \right)^2 & \Leftarrow \quad J = 3. \end{cases} \tag{74}$$

Thus,

$$\frac{\mathcal{D}\lambda_1}{2} = 6\lambda_1^2 + (2J+1)\, \lambda_3^2 + \frac{\lambda_4^2}{2}\, A, \tag{75}$$

where

$$A = \sum_{k=1}^{J} k^2. \tag{76}$$

Thus, finally,

$$\mathcal{D}\lambda_1 = 12\lambda_1^2 + 2\left(2J+1\right)\lambda_3^2 + \frac{J\left(J+1\right)\left(2J+1\right)}{6}\,\lambda_4^2. \tag{77}$$

7.2.7 The RGE for λ_3

From $\lambda_{1313} = \lambda_3/4$ and from Eq. (72a) it follows that

$$\begin{aligned}
\frac{\mathcal{D}\lambda_3}{4} = \mathcal{D}\lambda_{1313} &= \sum_{m,n} \left(4\left|\lambda_{13mn}\right|^2 + 8\,\lambda_{1m1n}\lambda_{3n3m} + 8\left|\lambda_{1m3n}\right|^2\right). && \text{(78a)} \\
&= 4\left|\lambda_{1313}\right|^2 + 4\left|\lambda_{1331}\right|^2 + 4\left|\lambda_{1324}\right|^2 + 4\left|\lambda_{1342}\right|^2 \\
&\quad +8\,\lambda_{1111}\lambda_{3131} + 8\,\lambda_{1212}\lambda_{3232} \\
&\quad +8\left|\lambda_{1331}\right|^2 + 8\left|\lambda_{1532}\right|^2 && \text{(78b)} \\
&= \lambda_3^2 + \frac{3}{2}\,\lambda_1\lambda_3 + 8\left(\left|\lambda_{1324}\right|^2 + \left|\lambda_{1532}\right|^2\right) && \text{(78c)} \\
&= \lambda_3^2 + \frac{3}{2}\,\lambda_1\lambda_3 + \frac{J\left(J+1\right)}{4}\,\lambda_4^2. && \text{(78d)}
\end{aligned}$$

Thus,

$$\mathcal{D}\lambda_3 = 4\lambda_3^2 + 6\lambda_1\lambda_3 + J\left(J+1\right)\lambda_4^2. \tag{79}$$

7.2.8 The RGE for λ_4

$$\begin{aligned}
\mathcal{D}\lambda_{1324} &= \sum_{m,n}\left(4\,\lambda_{13mn}\lambda_{mn24} + 8\,\lambda_{1m2n}\lambda_{3n4m} + 8\,\lambda_{1m4n}\lambda_{3n2m}\right) && \text{(80a)} \\
&= 4\left(\lambda_{1313}\lambda_{3124} + \lambda_{1331}\lambda_{1324} + \lambda_{1324}\lambda_{4224} + \lambda_{1342}\lambda_{2424}\right) \\
&\quad +8\left(\lambda_{1221}\lambda_{3142} + \lambda_{1342}\lambda_{3223} + \lambda_{1441}\lambda_{3124}\right) && \text{(80b)} \\
&= 8\lambda_{1324}\left(\lambda_{1313} + \lambda_{2424} + \lambda_{1221} + \lambda_{2323} + \lambda_{1414}\right). && \text{(80c)}
\end{aligned}$$

Hence,

$$\mathcal{D}\lambda_4 = \lambda_4\left(2\lambda_1 + 8\lambda_3\right). \tag{81}$$

References

[1] T. P. Cheng, E. Eichten, and L.-F. Li, *Phys. Rev. D* **9**, 2259 (1973).

[2] G. C. Branco, P. M. Ferreira, L. Lavoura, M. N. Rebelo, M. Sher, and J. P. Silva, *Phys. Rept.* **516**, 1 (2012).

[3] D. Jurčiukonis and L. Lavoura, E-print `arXiv:2303.07897 [hep-ph]`.