## Definitions of the models and expressions for the RGEs

In this analysis, we compute the Renormalization Group Equations (RGEs) for the Standard Model (SM) with added multiplets, according to the paper https://arxiv.org/pdf/2404.07897. We compute only the one-loop RGEs using the SARAH package https://sarah.hepforge.org/ (or https://gitlab.in2p3.fr/-goodsell/sarah). We use following convention of beta function: $\mu \frac{dX}{d\mu} = \frac{1}{16\pi^2} \beta[X]$.

From the SM, we have three gauge couplings $g_1$, $g_2$ and $g_3$ with the assumed GUT normalization for $g_1$ given by $g_1 \to \sqrt{\frac{5}{3}}\, g_1$. In computations, we assume contributions only from the third generation for Yukawa couplings. For simplicity, we consider here the contribution only from the top-quark Yukawa coupling $y_t$.

For the implementation of our model in SARAH, we need to rewrite the model potential in tensor notation.

### 2-plet

For the 2-plet we use following quartic potential:

$$V_4 = \frac{1}{2}\bar{\lambda}_1 \left(H^\dagger H\right)^2 + \frac{1}{2}\bar{\lambda}_2 \left(\chi^\dagger \chi\right)^2 + \bar{\lambda}_3 \left(H^\dagger H \chi^\dagger \chi\right)_3 + \bar{\lambda}_4 \left(H^\dagger H \chi^\dagger \chi\right)_4$$

where

$\left(H^\dagger H \chi^\dagger \chi\right)_3 = H^{*i} H_i \chi^{*a} \chi_a$

$\left(H^\dagger H \chi^\dagger \chi\right)_4 = H^{*i} H_j \chi^{*j} \chi_i$

The tensor notation for the 2-plet is:

$\chi_1$ = c

$\chi_2$ = d

The equivalence between the notations in the potential above and 2-plet in https://arxiv.org/pdf/2404.07897 is established through:

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$\bar{\lambda}_1 \to \lambda_1$

$\bar{\lambda}_2 \to \lambda_2$

$\bar{\lambda}_3 \to \lambda_3 - \frac{\lambda_4}{4}$

$\bar{\lambda}_4 \to \frac{\lambda_4}{2}$

$\beta$ functions for the Yukawa and gauge couplings:

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$\beta[y_t] == -\frac{1}{20} y_t \left(17 g_1^2 + 45 g_2^2 + 160 g_3^2 - 90 y_t^2\right)$

$\beta[g_1] == \frac{1}{10}\left(41 + 4 Y^2\right) g_1^3$

$\beta[g_2] == -3 g_2^3$

$\beta[g_3] == -7 g_3^3$

$\beta$ functions for the quartic couplings:

### 3-plet

For the 3-plet we use following quartic potential:

$$V_4 = \frac{1}{2}\bar{\lambda}_1 \left(H^\dagger H\right)^2 + \frac{1}{2}\bar{\lambda}_2 \left(\chi^\dagger \chi\right)^2 + \bar{\lambda}_3 \left(H^\dagger H \chi^\dagger \chi\right)_3 + \bar{\lambda}_4 \left(H^\dagger H \chi^\dagger \chi\right)_4 + \bar{\lambda}_5 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_5$$

where

$\left(H^\dagger H \chi^\dagger \chi\right)_3 = H^{*i} H_i \chi^{*ab} \chi_{ab}$

$\left(H^\dagger H \chi^\dagger \chi\right)_4 = H^{*i} H_j \chi^{*ja} \chi_{ia}$

$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 = \chi^{*ij} \chi_{ib} \chi^{*ab} \chi_{aj}$

The tensor notation for the 3-plet is:

$\chi_{11}$ = -d/sqrt(2)

$\chi_{12}$ = c

$\chi_{21}$ = -e

$\chi_{22}$ = d/sqrt(2)

The equivalence between the notations in the potential above and 3-plet in https://arxiv.org/pdf/2404.07897 is established through:

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$\bar{\lambda}_1 \to \lambda_1$

$\bar{\lambda}_2 \to \lambda_2 + \frac{4\lambda_5}{3}$

$\bar{\lambda}_3 \to \lambda_3 - \frac{\lambda_4}{2}$

$\bar{\lambda}_4 \to -\lambda_4$

$\bar{\lambda}_5 \to -\frac{2\lambda_5}{3}$

$\beta$ functions for the Yukawa and gauge couplings:

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$\beta[y_t] == -\frac{1}{20} y_t \left(17 g_1^2 + 45 g_2^2 + 160 g_3^2 - 90 y_t^2\right)$
$\beta[g_1] == \frac{1}{10} \left(41 + 6 Y^2\right) g_1^3$
$\beta[g_2] == -\frac{5 g_2^3}{2}$
$\beta[g_3] == -7 g_3^3$

$\beta$ functions for the quartic couplings:

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$\beta[\lambda_1] == \frac{27 g_1^4}{100} + \frac{9 g_2^4}{4} - 12 y_t^4 + \frac{9}{10} g_1^2 \left(g_2^2 - 2 \lambda_1\right) - 9 g_2^2 \lambda_1 + 12 y_t^2 \lambda_1 + 12 \lambda_1^2 + 6 \lambda_3^2 + \lambda_4^2$
$\beta[\lambda_2] == \frac{108}{25} Y^4 g_1^4 + 18 g_2^4 + \frac{24}{5} g_2^2 \left(3 Y^2 g_1^2 - 5 \lambda_2\right) - \frac{36}{5} Y^2 g_1^2 \lambda_2 + 14 \lambda_2^2 + 4 \lambda_3^2 + \lambda_4^2 + \frac{16 \lambda_2 \lambda_5}{3} + \frac{32 \lambda_5^2}{9}$
$\beta[\lambda_3] == \frac{27}{25} Y^2 g_1^4 + 6 g_2^4 - \frac{33}{2} g_2^2 \lambda_3 + 6 y_t^2 \lambda_3 + 6 \lambda_1 \lambda_3 + 8 \lambda_2 \lambda_3 + 4 \lambda_3^2 - \frac{9}{10} g_1^2 \left(\lambda_3 + 4 Y^2 \lambda_3\right) + 2 \lambda_4^2 + \frac{8 \lambda_3 \lambda_5}{3}$
$\beta[\lambda_4] == \frac{9}{10} g_1^2 \left(8 Y g_2^2 - \left(1 + 4 Y^2\right) \lambda_4\right) + \frac{1}{6} \lambda_4 \left(-99 g_2^2 + 4 \left(9 y_t^2 + 3 \lambda_1 + 3 \lambda_2 + 12 \lambda_3 - 4 \lambda_5\right)\right)$
$\beta[\lambda_5] == 9 g_2^4 - \frac{3 \lambda_4^2}{2} + \frac{4}{5} \lambda_5 \left(-9 Y^2 g_1^2 + 15 \lambda_2 + 5 \lambda_5\right) - \frac{12}{5} g_2^2 \left(9 Y^2 g_1^2 + 10 \lambda_5\right)$

## 4-plet

For the 4-plet we use following quartic potential:

$$V_4 = \frac{1}{2} \bar{\lambda}_1 \left(H^\dagger H\right)^2 + \frac{1}{2} \bar{\lambda}_2 \left(\chi^\dagger \chi\right)^2 + \bar{\lambda}_3 \left(H^\dagger H \chi^\dagger \chi\right)_3 + \bar{\lambda}_4 \left(H^\dagger H \chi^\dagger \chi\right)_4 + \bar{\lambda}_5 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_5$$

where
$\left(H^\dagger H \chi^\dagger \chi\right)_3 = H^{*i} H_i \chi^{*abc} \chi_{abc}$
$\left(H^\dagger H \chi^\dagger \chi\right)_4 = H^{*i} H_j \chi^{*jab} \chi_{iab}$
$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 = \chi^{*ijk} \chi_{ijc} \chi^{*abc} \chi_{abk}$

The symmetric tensor notation for the 4-plet is:
$\chi_{111}$ = c
$\chi_{112}$ = d/sqrt(3)
$\chi_{122}$ = e/sqrt(3)
$\chi_{222}$ = f

The equivalence between the notations in the potential above and 4-plet in https://arxiv.org/pdf/2404.07897 is established through:

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$\bar{\lambda}_1 \to \lambda_1$
$\bar{\lambda}_2 \to \lambda_2 + \frac{9 \lambda_5}{5}$
$\bar{\lambda}_3 \to \lambda_3 - \frac{3 \lambda_4}{4}$
$\bar{\lambda}_4 \to \frac{3 \lambda_4}{2}$
$\bar{\lambda}_5 \to -\frac{9 \lambda_5}{10}$

$\beta$ functions for the Yukawa and gauge couplings:

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$\beta[y_t] == -\frac{1}{20} y_t \left(17 g_1^2 + 45 g_2^2 + 160 g_3^2 - 90 y_t^2\right)$
$\beta[g_1] == \frac{1}{10} \left(41 + 8 Y^2\right) g_1^3$
$\beta[g_2] == -\frac{3 g_2^3}{2}$
$\beta[g_3] == -7 g_3^3$

$\beta$ functions for the quartic couplings:

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$\beta[\lambda_1] == \frac{27 g_1^4}{100} + \frac{9 g_2^4}{4} - 12 y_t^4 + \frac{9}{10} g_1^2 \left(g_2^2 - 2 \lambda_1\right) - 9 g_2^2 \lambda_1 + 12 y_t^2 \lambda_1 + 12 \lambda_1^2 + 8 \lambda_3^2 + \frac{5 \lambda_4^2}{2}$
$\beta[\lambda_2] == \frac{108}{25} Y^4 g_1^4 + \frac{297 g_2^4}{4} + \frac{9}{5} g_2^2 \left(18 Y^2 g_1^2 - 25 \lambda_2\right) - \frac{36}{5} Y^2 g_1^2 \lambda_2 + 16 \lambda_2^2 + 4 \lambda_3^2 + \frac{9 \lambda_4^2}{4} + \frac{72 \lambda_2 \lambda_5}{5} + \frac{342 \lambda_5^2}{25}$
$\beta[\lambda_3] == \frac{27}{25} Y^2 g_1^4 + \frac{45 g_2^4}{4} - 27 g_2^2 \lambda_3 + 6 y_t^2 \lambda_3 + 6 \lambda_1 \lambda_3 + 10 \lambda_2 \lambda_3 + 4 \lambda_3^2 - \frac{9}{10} g_1^2 \left(\lambda_3 + 4 Y^2 \lambda_3\right) + \frac{15 \lambda_4^2}{4} + \frac{36 \lambda_3 \lambda_5}{5} + \frac{3 \lambda_4 \lambda_5}{10}$
$\beta[\lambda_4] == -27 g_2^2 \lambda_4 + \frac{9}{10} g_1^2 \left(8 Y g_2^2 - \left(1 + 4 Y^2\right) \lambda_4\right) + \frac{2}{5} \lambda_4 \left(15 y_t^2 + 5 \lambda_1 + 5 \lambda_2 + 20 \lambda_3 - 9 \lambda_5\right)$
$\beta[\lambda_5] == -\frac{5 \lambda_4^2}{2} - 45 g_2^2 \lambda_5 + 12 \lambda_2 \lambda_5 + \frac{8 \lambda_5^2}{5} - \frac{36}{5} Y^2 g_1^2 \left(5 g_2^2 + \lambda_5\right)$

## 5-plet

For the 5-plet we use following quartic potential:

$$V_4 = \frac{1}{2} \bar{\lambda}_1 \left(H^\dagger H\right)^2 + \frac{1}{2} \bar{\lambda}_2 \left(\chi^\dagger \chi\right)^2 + \bar{\lambda}_3 \left(H^\dagger H \chi^\dagger \chi\right)_3 + \bar{\lambda}_4 \left(H^\dagger H \chi^\dagger \chi\right)_4 + \bar{\lambda}_5 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 + \bar{\lambda}_6 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_6$$

where
$\left(H^\dagger H \chi^\dagger \chi\right)_3 = H^{*i} H_i \chi^{*abcd} \chi_{abcd}$

$\left(H^\dagger H \chi^\dagger \chi\right)_4 = H^{*i} H_j \chi^{*jabc} \chi_{iabc}$
$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 = \chi^{*ijkl} \chi_{ijkd} \chi^{*abcd} \chi_{abcl}$
$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_6 = \chi^{*ijkl} \chi_{ijcd} \chi^{*abcd} \chi_{abkl}$

The symmetric tensor notation for the 5-plet is:
$\chi_{1111}$ = c
$\chi_{1112}$ = d/sqrt(4)
$\chi_{1122}$ = e/sqrt(6)
$\chi_{1222}$ = f/sqrt(4)
$\chi_{2222}$ = g

The equivalence between the notations in the potential above and 5-plet in https://arxiv.org/pdf/2404.07897 is established through:

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$\bar{\lambda}_1 \to \lambda_1$
$\bar{\lambda}_2 \to \lambda_2 + \frac{8\,\lambda_5}{7} + \frac{4\,\lambda_6}{5}$
$\bar{\lambda}_3 \to \lambda_3 - \lambda_4$
$\bar{\lambda}_4 \to 2\,\lambda_4$
$\bar{\lambda}_5 \to \frac{8\,\lambda_5}{7} - \frac{8\,\lambda_6}{5}$
$\bar{\lambda}_6 \to -\frac{12\,\lambda_5}{7} + \frac{6\,\lambda_6}{5}$

$\beta$ functions for the Yukawa and gauge couplings:

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$\beta[y_t] == -\frac{1}{20}\, y_t \left(17\, g_1^2 + 45\, g_2^2 + 160\, g_3^2 - 90\, y_t^2\right)$
$\beta[g_1] == \frac{1}{10} \left(41 + 10\, Y^2\right) g_1^3$
$\beta[g_2] == \frac{g_2^3}{6}$
$\beta[g_3] == -7\, g_3^3$

$\beta$ functions for the quartic couplings:

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$\beta[\lambda_1] == \frac{27\, g_1^4}{100} + \frac{9\, g_2^4}{4} - 12\, y_t^4 + \frac{9}{10}\, g_1^2 \left(g_2^2 - 2\,\lambda_1\right) - 9\, g_2^2\,\lambda_1 + 12\, y_t^2\,\lambda_1 + 12\,\lambda_1^2 + 10\,\lambda_3^2 + 5\,\lambda_4^2$
$\beta[\lambda_2] == \frac{108}{25}\, Y^4 g_1^4 + 216\, g_2^4 + \frac{72}{5}\, g_2^2 \left(4\, Y^2 g_1^2 - 5\,\lambda_2\right) - \frac{36}{5}\, Y^2 g_1^2\,\lambda_2 + 18\,\lambda_2^2 + 4\,\lambda_3^2 + 4\,\lambda_4^2 + 16\,\lambda_2\,\lambda_5 + \frac{544\,\lambda_5^2}{49} + \frac{16\,\lambda_2\,\lambda_6}{5} + \frac{128\,\lambda_5\,\lambda_6}{35} + \frac{32\,\lambda_6^2}{25}$
$\beta[\lambda_3] == \frac{27}{25}\, Y^2 g_1^4 + 18\, g_2^4 - \frac{81}{2}\, g_2^2\,\lambda_3 + 6\, y_t^2\,\lambda_3 + 6\,\lambda_1\,\lambda_3 + 12\,\lambda_2\,\lambda_3 + 4\,\lambda_3^2 - \frac{9}{10}\, g_1^2 \left(\lambda_3 + 4\, Y^2\,\lambda_3\right) + 6\,\lambda_4^2 + 8\,\lambda_3\,\lambda_5 + \frac{8\,\lambda_3\,\lambda_6}{5}$
$\beta[\lambda_4] == -\frac{81}{2}\, g_2^2\,\lambda_4 + 6\, y_t^2\,\lambda_4 + 2\,\lambda_1\,\lambda_4 + 2\,\lambda_2\,\lambda_4 + 8\,\lambda_3\,\lambda_4 + \frac{9}{10}\, g_1^2 \left(8\, Y g_2^2 - \left(1 + 4\, Y^2\right)\lambda_4\right) - 4\,\lambda_4\,\lambda_5 - \frac{8\,\lambda_4\,\lambda_6}{5}$
$\beta[\lambda_5] == -63\, g_2^4 - \frac{7\,\lambda_4^2}{2} - \frac{36}{5}\, g_2^2 \left(7\, Y^2 g_1^2 + 10\,\lambda_5\right) + \frac{4}{35}\,\lambda_5 \left(-63\, Y^2 g_1^2 + 105\,\lambda_2 + 45\,\lambda_5 - 28\,\lambda_6\right)$
$\beta[\lambda_6] == 90\, g_2^4 - 5\,\lambda_4^2 - \frac{440\,\lambda_5^2}{49} - \frac{36}{5}\, Y^2 g_1^2\,\lambda_6 + 12\,\lambda_2\,\lambda_6 + \frac{32\,\lambda_5\,\lambda_6}{7} + 4\,\lambda_6^2 - 72\, g_2^2 \left(Y^2 g_1^2 + \lambda_6\right)$

## 6-plet

For the 6-plet we use following quartic potential:

$$V_4 = \frac{1}{2}\,\bar{\lambda}_1 \left(H^\dagger H\right)^2 + \frac{1}{2}\,\bar{\lambda}_2 \left(\chi^\dagger \chi\right)^2 + \bar{\lambda}_3 \left(H^\dagger H \chi^\dagger \chi\right)_3 + \bar{\lambda}_4 \left(H^\dagger H \chi^\dagger \chi\right)_4 + \bar{\lambda}_5 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 + \bar{\lambda}_6 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_6$$

where
$\left(H^\dagger H \chi^\dagger \chi\right)_3 = H^{*i} H_i \chi^{*abcde} \chi_{abcde}$
$\left(H^\dagger H \chi^\dagger \chi\right)_4 = H^{*i} H_j \chi^{*jabcd} \chi_{iabcd}$
$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 = \chi^{*ijklm} \chi_{ijkle} \chi^{*abcde} \chi_{abcdm}$
$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_6 = \chi^{*ijklm} \chi_{ijkde} \chi^{*abcde} \chi_{abclm}$

The symmetric tensor notation for the 6-plet is:
$\chi_{11111}$ = c
$\chi_{11112}$ = d/sqrt(5)
$\chi_{11122}$ = e/sqrt(10)
$\chi_{11222}$ = f/sqrt(10)
$\chi_{12222}$ = g/sqrt(5)
$\chi_{22222}$ = h

The equivalence between the notations in the potential above and 6-plet in https://arxiv.org/pdf/2404.07897 is established through:

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$\bar{\lambda}_1 \to \lambda_1$

$\bar{\lambda}_2 \to \lambda_2 + \frac{5\,\lambda_5}{9} + \frac{10\,\lambda_6}{7}$

$\bar{\lambda}_3 \to \lambda_3 - \frac{5\,\lambda_4}{4}$

$\bar{\lambda}_4 \to \frac{5\,\lambda_4}{2}$

$\bar{\lambda}_5 \to \frac{35\,\lambda_5}{18} - \frac{15\,\lambda_6}{7}$

$\bar{\lambda}_6 \to -\frac{20\,\lambda_5}{9} + \frac{10\,\lambda_6}{7}$

$\beta$ functions for the Yukawa and gauge couplings:

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$\beta[y_t] == -\frac{1}{20}\, y_t \left(17\, g_1^2 + 45\, g_2^2 + 160\, g_3^2 - 90\, y_t^2\right)$

$\beta[g_1] == \frac{1}{10} \left(41 + 12\, Y^2\right) g_1^3$

$\beta[g_2] == \frac{8\, g_2^3}{3}$

$\beta[g_3] == -7\, g_3^3$

$\beta$ functions for the quartic couplings:

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$\beta[\lambda_1] == \frac{27\, g_1^4}{100} + \frac{9\, g_2^4}{4} - 12\, y_t^4 + \frac{9}{10}\, g_1^2 \left(g_2^2 - 2\,\lambda_1\right) - 9\, g_2^2\,\lambda_1 + 12\, y_t^2\,\lambda_1 + 12\,\lambda_1^2 + 12\,\lambda_3^2 + \frac{35\,\lambda_4^2}{4}$

$\beta[\lambda_2] == \frac{108}{25}\, Y^4\, g_1^4 + \frac{2025\, g_2^4}{4} + 15\, g_2^2 \left(6\, Y^2\, g_1^2 - 7\,\lambda_2\right) - \frac{36}{5}\, Y^2\, g_1^2\,\lambda_2 + 20\,\lambda_2^2 + 4\,\lambda_3^2 + \frac{25\,\lambda_4^2}{4} + \frac{56\,\lambda_2\,\lambda_5}{3} + \frac{988\,\lambda_5^2}{81} + 8\,\lambda_2\,\lambda_6 + \frac{56\,\lambda_5\,\lambda_6}{9} + \frac{232\,\lambda_6^2}{49}$

$\beta[\lambda_3] == \frac{27}{25}\, Y^2\, g_1^4 + \frac{105\, g_2^4}{4} - 57\, g_2^2\,\lambda_3 + 6\, y_t^2\,\lambda_3 + 6\,\lambda_1\,\lambda_3 + 14\,\lambda_2\,\lambda_3 + 4\,\lambda_3^2 - \frac{9}{10}\, g_1^2 \left(\lambda_3 + 4\, Y^2\,\lambda_3\right) + \frac{35\,\lambda_4^2}{4} + \frac{28\,\lambda_3\,\lambda_5}{3} + 4\,\lambda_3\,\lambda_6$

$\beta[\lambda_4] == -57\, g_2^2\,\lambda_4 + 6\, y_t^2\,\lambda_4 + 2\,\lambda_1\,\lambda_4 + 2\,\lambda_2\,\lambda_4 + 8\,\lambda_3\,\lambda_4 + \frac{9}{10}\, g_1^2 \left(8\, Y\, g_2^2 - \left(1 + 4\, Y^2\right)\lambda_4\right) - \frac{44\,\lambda_4\,\lambda_5}{15} - \frac{124\,\lambda_4\,\lambda_6}{35}$

$\beta[\lambda_5] == -216\, g_2^4 - \frac{9\,\lambda_4^2}{2} - \frac{36}{5}\, Y^2\, g_1^2\,\lambda_5 + 12\,\lambda_2\,\lambda_5 + \frac{326\,\lambda_5^2}{75} - \frac{3}{5}\, g_2^2 \left(108\, Y^2\, g_1^2 + 175\,\lambda_5\right) - \frac{88\,\lambda_5\,\lambda_6}{175} - \frac{2592\,\lambda_6^2}{1225}$

$\beta[\lambda_6] == 84\, g_2^4 - 7\,\lambda_4^2 - \frac{6188\,\lambda_5^2}{2025} - \frac{36}{5}\, Y^2\, g_1^2\,\lambda_6 + 12\,\lambda_2\,\lambda_6 - \frac{436\,\lambda_5\,\lambda_6}{225} + \frac{764\,\lambda_6^2}{175} - \frac{21}{5}\, g_2^2 \left(24\, Y^2\, g_1^2 + 25\,\lambda_6\right)$

## 7-plet

For the 7-plet we use following quartic potential:

$$V_4 = \frac{1}{2}\,\bar{\lambda}_1 \left(H^\dagger H\right)^2 + \frac{1}{2}\,\bar{\lambda}_2 \left(\chi^\dagger \chi\right)^2 + \bar{\lambda}_3 \left(H^\dagger H \chi^\dagger \chi\right)_3 + \bar{\lambda}_4 \left(H^\dagger H \chi^\dagger \chi\right)_4 + \bar{\lambda}_5 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 + \bar{\lambda}_6 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_6 + \bar{\lambda}_7 \left(\chi^\dagger \chi \chi^\dagger \chi\right)_7$$

where

$\left(H^\dagger H \chi^\dagger \chi\right)_3 = H^{*i} H_i\, \chi^{*abcdef} \chi_{abcdef}$

$\left(H^\dagger H \chi^\dagger \chi\right)_4 = H^{*i} H_j\, \chi^{*jabcde} \chi_{iabcde}$

$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_5 = \chi^{*ijklmn} \chi_{ijklmf}\, \chi^{*abcdef} \chi_{abcden}$

$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_6 = \chi^{*ijklmn} \chi_{ijklef}\, \chi^{*abcdef} \chi_{abcdmn}$

$\left(\chi^\dagger \chi \chi^\dagger \chi\right)_7 = \chi^{*ijklmn} \chi_{ijkdef}\, \chi^{*abcdef} \chi_{abclmn}$

The symmetric tensor notation for the 7-plet is:

$\chi_{111111}$ = c

$\chi_{111112}$ = d/sqrt(6)

$\chi_{111122}$ = e/sqrt(15)

$\chi_{111222}$ = f/sqrt(20)

$\chi_{112222}$ = g/sqrt(15)

$\chi_{122222}$ = h/sqrt(6)

$\chi_{222222}$ = i

The equivalence between the notations in the potential above and 7-plet in https://arxiv.org/pdf/2404.07897 is established through:

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$\bar{\lambda}_1 \to \lambda_1$

$\bar{\lambda}_2 \to \lambda_2 + \frac{18\,\lambda_5}{77} + \frac{25\,\lambda_6}{21} + \frac{4\,\lambda_7}{7}$

$\bar{\lambda}_3 \to \lambda_3 - \frac{3\,\lambda_4}{2}$

$\bar{\lambda}_4 \to 3\,\lambda_4$

$\bar{\lambda}_5 \to \frac{18\,\lambda_5}{11} - \frac{12\,\lambda_7}{7}$

$\bar{\lambda}_6 \to \frac{45\,\lambda_5}{77} - \frac{75\,\lambda_6}{14} + \frac{30\,\lambda_7}{7}$

$\bar{\lambda}_7 \to -\frac{180\,\lambda_5}{77} + \frac{100\,\lambda_6}{21} - \frac{20\,\lambda_7}{7}$

$\beta$ functions for the Yukawa and gauge couplings:

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$$\beta[y_t] == -\frac{1}{20}\, y_t \left(17\, g_1^2 + 45\, g_2^2 + 160\, g_3^2 - 90\, y_t^2\right)$$
$$\beta[g_1] == \frac{1}{10}\left(41 + 14\, Y^2\right) g_1^3$$
$$\beta[g_2] == \frac{37\, g_2^3}{6}$$
$$\beta[g_3] == -7\, g_3^3$$

$\beta$ functions for the quartic couplings:

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$$\beta[\lambda_1] == \frac{27\, g_1^4}{100} + \frac{9\, g_2^4}{4} + \frac{9}{10}\, g_1^2 \left(g_2^2 - 2\,\lambda_1\right) - 9\, g_2^2\, \lambda_1 + 2\left(-6\, y_t^4 + 6\, y_t^2\, \lambda_1 + 6\,\lambda_1^2 + 7\left(\lambda_3^2 + \lambda_4^2\right)\right)$$
$$\beta[\lambda_2] == \frac{108}{25}\, Y^4\, g_1^4 + 1026\, g_2^4 + \frac{72}{5}\, g_2^2 \left(9\, Y^2\, g_1^2 - 10\,\lambda_2\right) - \frac{36}{5}\, Y^2\, g_1^2\, \lambda_2 + 22\,\lambda_2^2 + 4\,\lambda_3^2 + 9\,\lambda_4^2 + \frac{144\,\lambda_2\,\lambda_5}{7} + \frac{77040\,\lambda_5^2}{5929} + \frac{80\,\lambda_2\,\lambda_6}{7} + \frac{4040\,\lambda_5\,\lambda_6}{539} + \frac{300\,\lambda_6^2}{49} + \frac{16\,\lambda_2\,\lambda_7}{7} + \frac{288\,\lambda_5\,\lambda_7}{539} + \frac{400\,\lambda_6\,\lambda_7}{147} + \frac{32\,\lambda_7^2}{49}$$
$$\beta[\lambda_3] == \frac{27}{25}\, Y^2\, g_1^4 + 36\, g_2^4 - \frac{153}{2}\, g_2^2\, \lambda_3 + 6\, y_t^2\, \lambda_3 + 6\,\lambda_1\,\lambda_3 + 16\,\lambda_2\,\lambda_3 + 4\,\lambda_3^2 - \frac{9}{10}\, g_1^2 \left(\lambda_3 + 4\, Y^2\, \lambda_3\right) + 12\,\lambda_4^2 + \frac{72\,\lambda_3\,\lambda_5}{7} + \frac{40\,\lambda_3\,\lambda_6}{7} + \frac{8\,\lambda_3\,\lambda_7}{7}$$
$$\beta[\lambda_4] == -\frac{153}{2}\, g_2^2\, \lambda_4 + \frac{9}{10}\, g_1^2 \left(8\, Y\, g_2^2 - \left(1 + 4\, Y^2\right)\lambda_4\right) + \frac{2}{7}\,\lambda_4 \left(21\, y_t^2 + 7\,\lambda_1 + 7\,\lambda_2 + 28\,\lambda_3 - 6\,\lambda_5 - 15\,\lambda_6 - 4\,\lambda_7\right)$$
$$\beta[\lambda_5] == -495\, g_2^4 - \frac{11\,\lambda_4^2}{2} - \frac{36}{5}\, Y^2\, g_1^2\, \lambda_5 + 12\,\lambda_2\,\lambda_5 + \frac{1996\,\lambda_5^2}{539} - \frac{36}{5}\, g_2^2 \left(11\, Y^2\, g_1^2 + 20\,\lambda_5\right) - \frac{60\,\lambda_5\,\lambda_6}{49} - \frac{220\,\lambda_6^2}{441} + \frac{128\,\lambda_5\,\lambda_7}{49} - \frac{440\,\lambda_6\,\lambda_7}{147}$$
$$\beta[\lambda_6] == -54\, g_2^4 - 9\,\lambda_4^2 - \frac{25056\,\lambda_5^2}{5929} - \frac{36}{5}\, Y^2\, g_1^2\, \lambda_6 + 12\,\lambda_2\,\lambda_6 + \frac{2952\,\lambda_5\,\lambda_6}{539} + \frac{230\,\lambda_6^2}{147} - \frac{72}{5}\, g_2^2 \left(9\, Y^2\, g_1^2 + 10\,\lambda_6\right) - \frac{1728\,\lambda_5\,\lambda_7}{539} - \frac{16\,\lambda_6\,\lambda_7}{49}$$
$$\beta[\lambda_7] == 315\, g_2^4 - \frac{21\,\lambda_4^2}{2} + \frac{5472\,\lambda_5^2}{847} - \frac{1420\,\lambda_5\,\lambda_6}{77} - \frac{10\,\lambda_6^2}{21} - \frac{36}{5}\, Y^2\, g_1^2\, \lambda_7 + 12\,\lambda_2\,\lambda_7 + \frac{432\,\lambda_5\,\lambda_7}{77} + \frac{40\,\lambda_6\,\lambda_7}{21} + 4\,\lambda_7^2 - \frac{36}{5}\, g_2^2 \left(21\, Y^2\, g_1^2 + 20\,\lambda_7\right)$$

## 8-plet

For the 8-plet we use following quartic potential:

$$V_4 = \frac{1}{2}\,\bar\lambda_1 \left(H^\dagger H\right)^2 + \frac{1}{2}\,\bar\lambda_2 \left(\chi^\dagger \chi\right)^2 + \bar\lambda_3 \left(H^\dagger H\, \chi^\dagger \chi\right)_3 + \bar\lambda_4 \left(H^\dagger H\, \chi^\dagger \chi\right)_4 + \bar\lambda_5 \left(\chi^\dagger \chi\, \chi^\dagger \chi\right)_5 + \bar\lambda_6 \left(\chi^\dagger \chi\, \chi^\dagger \chi\right)_6 + \bar\lambda_7 \left(\chi^\dagger \chi\, \chi^\dagger \chi\right)_7$$

where

$\left(H^\dagger H\, \chi^\dagger \chi\right)_3 = H^{*i} H_i\, \chi^{*abcdefg} \chi_{abcdefg}$

$\left(H^\dagger H\, \chi^\dagger \chi\right)_4 = H^{*i} H_j\, \chi^{*jabcdef} \chi_{iabcdef}$

$\left(\chi^\dagger \chi\, \chi^\dagger \chi\right)_5 = \chi^{*ijklmnr} \chi_{ijklmng}\, \chi^{*abcdefg} \chi_{abcdefr}$

$\left(\chi^\dagger \chi\, \chi^\dagger \chi\right)_6 = \chi^{*ijklmnr} \chi_{ijklmfg}\, \chi^{*abcdefg} \chi_{abcdenr}$

$\left(\chi^\dagger \chi\, \chi^\dagger \chi\right)_7 = \chi^{*ijklmnr} \chi_{ijklefg}\, \chi^{*abcdefg} \chi_{abcdmnr}$

The symmetric tensor notation for the 8-plet is:

$\chi_{1111111}$ = c

$\chi_{1111112}$ = d/sqrt(7)

$\chi_{1111122}$ = e/sqrt(21)

$\chi_{1111222}$ = f/sqrt(35)

$\chi_{1112222}$ = g/sqrt(35)

$\chi_{1122222}$ = h/sqrt(21)

$\chi_{1222222}$ = i/sqrt(7)

$\chi_{2222222}$ = j

The equivalence between the notations in the potential above and 8-plet in https://arxiv.org/pdf/2404.07897 is established through:

Out[*]//TableForm=

$$\bar\lambda_1 \to \lambda_1$$
$$\bar\lambda_2 \to \lambda_2 + \frac{7\,\lambda_5}{78} + \frac{49\,\lambda_6}{66} + \frac{7\,\lambda_7}{6}$$
$$\bar\lambda_3 \to \lambda_3 - \frac{7\,\lambda_4}{4}$$
$$\bar\lambda_4 \to \frac{7\,\lambda_4}{2}$$
$$\bar\lambda_5 \to \frac{161\,\lambda_5}{156} + \frac{245\,\lambda_6}{132} - \frac{35\,\lambda_7}{12}$$
$$\bar\lambda_6 \to \frac{119\,\lambda_5}{52} - \frac{343\,\lambda_6}{44} + \frac{21\,\lambda_7}{4}$$
$$\bar\lambda_7 \to -\frac{175\,\lambda_5}{52} + \frac{245\,\lambda_6}{44} - \frac{35\,\lambda_7}{12}$$

$\beta$ functions for the Yukawa and gauge couplings:

Out[*]//TableForm=

$$\beta[y_t] == -\frac{1}{20}\, y_t \left(17\, g_1^2 + 45\, g_2^2 + 160\, g_3^2 - 90\, y_t^2\right)$$
$$\beta[g_1] == \frac{1}{10}\left(41 + 16\, Y^2\right) g_1^3$$
$$\beta[g_2] == \frac{65\, g_2^3}{6}$$
$$\beta[g_3] == -7\, g_3^3$$

$\beta$ functions for the quartic couplings:

Out[∘]//TableForm=

$$\beta[\lambda_1] == \frac{27\,g_1^4}{100} + \frac{9\,g_2^4}{4} - 12\,y_t^4 + \frac{9}{10}\,g_1^2\left(g_2^2 - 2\,\lambda_1\right) - 9\,g_2^2\,\lambda_1 + 12\,y_t^2\,\lambda_1 + 12\,\lambda_1^2 + 16\,\lambda_3^2 + 21\,\lambda_4^2$$

$$\beta[\lambda_2] == \frac{108}{25}\,Y^4\,g_1^4 + \frac{7497\,g_2^4}{4} + \frac{63}{5}\,g_2^2\left(14\,Y^2\,g_1^2 - 15\,\lambda_2\right) - \frac{36}{5}\,Y^2\,g_1^2\,\lambda_2 + 24\,\lambda_2^2 + 4\,\lambda_3^2 + \frac{49\,\lambda_4^2}{4} + 22\,\lambda_2\,\lambda_5 + \frac{13\,775\,\lambda_5^2}{1014} + 14\,\lambda_2\,\lambda_6 + \frac{1211\,\lambda_5\,\lambda_6}{143} + \frac{5047\,\lambda_6^2}{726} + 6\,\lambda_2\,\lambda_7 + \frac{109\,\lambda_5\,\lambda_7}{117} + \frac{511\,\lambda_6\,\lambda_7}{99} + \frac{53\,\lambda_7^2}{18}$$

$$\beta[\lambda_3] == \frac{27}{25}\,Y^2\,g_1^4 + \frac{189\,g_2^4}{4} - 99\,g_2^2\,\lambda_3 + 6\,y_t^2\,\lambda_3 + 6\,\lambda_1\,\lambda_3 + 18\,\lambda_2\,\lambda_3 + 4\,\lambda_3^2 - \frac{9}{10}\,g_1^2\left(\lambda_3 + 4\,Y^2\,\lambda_3\right) + \frac{63\,\lambda_4^2}{4} + 11\,\lambda_3\,\lambda_5 + 7\,\lambda_3\,\lambda_6 + 3\,\lambda_3\,\lambda_7$$

$$\beta[\lambda_4] == \frac{9}{10}\,g_1^2\left(8\,Y\,g_2^2 - \left(1 + 4\,Y^2\right)\lambda_4\right) + \frac{1}{21}\,\lambda_4\left(-2079\,g_2^2 + 126\,y_t^2 + 42\,\lambda_1 + 42\,\lambda_2 + 168\,\lambda_3 - 11\,\lambda_5 - 91\,\lambda_6 - 59\,\lambda_7\right)$$

$$\beta[\lambda_5] == -936\,g_2^4 - \frac{13\,\lambda_4^2}{2} - \frac{36}{5}\,Y^2\,g_1^2\,\lambda_5 + 12\,\lambda_2\,\lambda_5 + \frac{6428\,\lambda_5^2}{1911} - \frac{9}{5}\,g_2^2\left(52\,Y^2\,g_1^2 + 105\,\lambda_5\right) - \frac{82\,\lambda_5\,\lambda_6}{33} - \frac{26\,\lambda_6^2}{121} + \frac{2182\,\lambda_5\,\lambda_7}{441} - \frac{260\,\lambda_6\,\lambda_7}{99} - \frac{650\,\lambda_7^2}{441}$$

$$\beta[\lambda_6] == -396\,g_2^4 - 11\,\lambda_4^2 - \frac{19\,228\,\lambda_5^2}{3549} - \frac{36}{5}\,Y^2\,g_1^2\,\lambda_6 + 12\,\lambda_2\,\lambda_6 + \frac{1642\,\lambda_5\,\lambda_6}{273} + \frac{38\,\lambda_6^2}{11} - \frac{9}{5}\,g_2^2\left(88\,Y^2\,g_1^2 + 105\,\lambda_6\right) - \frac{440\,\lambda_5\,\lambda_7}{819} - \frac{226\,\lambda_6\,\lambda_7}{63} - \frac{44\,\lambda_7^2}{63}$$

$$\beta[\lambda_7] == 324\,g_2^4 - \frac{27\,\lambda_4^2}{2} + \frac{77\,824\,\lambda_5^2}{24\,843} - \frac{1888\,\lambda_5\,\lambda_6}{429} - \frac{1688\,\lambda_6^2}{363} - \frac{36}{5}\,Y^2\,g_1^2\,\lambda_7 + 12\,\lambda_2\,\lambda_7 - \frac{914\,\lambda_5\,\lambda_7}{1911} + \frac{34\,\lambda_6\,\lambda_7}{33} + \frac{214\,\lambda_7^2}{49} - \frac{27}{5}\,g_2^2\left(36\,Y^2\,g_1^2 + 35\,\lambda_7\right)$$

## General case

There is also possible to write general RGEs for $\lambda_{1-4}$ by assuming $\lambda_5$, $\lambda_6$ and $\lambda_7$ are equal zero for arbitrary hypercharge *Y* and isospin *J*.

Out[∘]//TableForm=

$$\beta[\lambda_1] == \frac{27\,g_1^4}{100} + \frac{9}{10}\,g_1^2\,g_2^2 + \frac{9\,g_2^4}{4} - 12\,y_t^4 - \frac{9}{5}\,g_1^2\,\lambda_1 - 9\,g_2^2\,\lambda_1 + 12\,y_t^2\,\lambda_1 + 12\,\lambda_1^2 + \left(2 + 4\,J\right)\lambda_3^2 + \frac{1}{6}\,J\,(1 + J)\left(1 + 2\,J\right)\lambda_4^2$$

$$\beta[\lambda_2] == \frac{108}{25}\,Y^4\,g_1^4 + \frac{72}{5}\,J^2\,Y^2\,g_1^2\,g_2^2 + 6\left(J^2 + 2\,J^4\right)g_2^4 - \frac{36}{5}\,Y^2\,g_1^2\,\lambda_2 - 12\,J\,(1 + J)\,g_2^2\,\lambda_2 + \left(10 + 4\,J\right)\lambda_2^2 + 4\,\lambda_3^2 + J^2\,\lambda_4^2$$

$$\beta[\lambda_3] == \frac{27}{25}\,Y^2\,g_1^4 + 3\,J\,(1 + J)\,g_2^4 - \left(\frac{9}{2} + 6\,J\,(1 + J)\right)g_2^2\,\lambda_3 + 6\,y_t^2\,\lambda_3 + 6\,\lambda_1\,\lambda_3 + 4\,(1 + J)\,\lambda_2\,\lambda_3 + 4\,\lambda_3^2 - \frac{9}{10}\,g_1^2\left(\lambda_3 + 4\,Y^2\,\lambda_3\right) + J\,(1 + J)\,\lambda_4^2$$

$$\beta[\lambda_4] == -\left(\left(\frac{9}{2} + 6\,J\,(1 + J)\right)g_2^2\,\lambda_4\right) + 6\,y_t^2\,\lambda_4 + 2\,\lambda_1\,\lambda_4 + 2\,\lambda_2\,\lambda_4 + 8\,\lambda_3\,\lambda_4 + \frac{9}{10}\,g_1^2\left(8\,Y\,g_2^2 - \left(1 + 4\,Y^2\right)\lambda_4\right)$$

$\beta$ functions for the Yukawa coupling $y_t$ for all muliplets are the same

Out[∘]= $$\beta[y_t] == -\frac{1}{20}\,y_t\left(17\,g_1^2 + 45\,g_2^2 + 160\,g_3^2 - 90\,y_t^2\right)$$

$\beta$ functions for the gauge couplings

Out[∘]//TableForm=

$$\beta[g_1] == \frac{1}{10}\left(41 + 8\,Y^2\right)g_1^3$$

$$\beta[g_2] == \left(-\frac{19}{6} + \frac{1}{9}\,J\,(1 + J)\left(1 + 2\,J\right)\right)g_2^3$$

$$\beta[g_3] == -7\,g_3^3$$

## Computing time

The computing time for RGEs increases exponentially with the size of the multiplet.

**Figure 1:** Computing time *versus* size of the multiplet in logarithmic scale. Computations were performed with a 13th Gen Intel i9-13900K CPU.

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