7 LIMITE IN ZVEZNOST FUNKCIJ

1. Izračunajte limiti:

$$(a)\quad \lim_{x\to 0}\frac{\sin(2x)}{\sqrt{2x+9}-3}; \qquad (b)\quad \lim_{x\to\infty}\left(\frac{x^3}{3x^2+2}-\frac{2x^2}{6x+1}\right)$$

Rešitev: (a) 6; (b) $\frac{1}{18}$

a) $$\lim_{x\to 0}\frac{\sin 2x}{\sqrt{2x+9}-3} = \lim \frac{\sin 2x\,(\sqrt{2x+9}+3)}{(\sqrt{2x+9}-3)(\sqrt{2x+9}+3)} =$$

$$= \lim_{x\to 0}\frac{\sin 2x\,(\sqrt{2x+9}+3)}{2x+9-9} = \lim_{x\to 0}\frac{\sin 2x\,(\sqrt{2x+9}+3)}{2x} =$$

$$= \sqrt{9}+3 = 6$$

b) $$\lim_{x\to\infty}\left(\frac{x^3}{3x^2+2}-\frac{2x^2}{6x+1}\right) = \lim_{x\to\infty}\left(\frac{x^3(6x+1)-2x^2(3x^2+2)}{(3x^2+2)(6x+1)}\right)$$

$$= \lim_{x\to\infty}\frac{6x^4+x^3-6x^4-4x^2}{18x^3+3x^2+12x+2} = \lim_{x\to\infty}\frac{x^3-4x^2}{18x^3+3x^2+12x+2} =$$

$$= \frac{1}{18}$$

2. Določite a tako, da bo funkcija

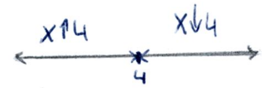
$$f(x) = \begin{cases} \frac{\sqrt{1+6x}-5}{\sqrt{x}-2} & x > 4 \\ a & x \leq 4 \end{cases}$$

zvezna.

Rešitev:

Funkcija bo zvezna, če bo $\lim_{x\searrow 4} f(x) = \lim_{x\nearrow 4} f(x) = f(4)$. Z množenjem števca in imenovalca z izrazom $(\sqrt{1+6x}+5)(\sqrt{x}+2)$ ali z uporabo l'Hospitalovega pravila dobimo $\lim_{x\searrow 4} = \frac{12}{5}$. Ker je $\lim_{x\nearrow 4} f(x) = f(4) = a$, bo funkcija zvezna za $a = \frac{12}{5}$.

- $\lim\limits_{x\uparrow 4} f(x) = \lim\limits_{x\downarrow 4} f(x) = f(4)$



- $\lim\limits_{x\downarrow 4} \frac{\sqrt{1+6x}-5}{\sqrt{x}-2} = \lim\limits_{x\downarrow 4} \frac{(\sqrt{1+6x}-5)(\sqrt{1+6x}+5)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{1+6x}+5)(\sqrt{x}+2)} =$

$= \lim\limits_{x\downarrow 4} \frac{(1+6x-25)(\sqrt{x}+2)}{(x-4)(\sqrt{1+6x}+5)} = \lim\limits_{x\downarrow 4} \frac{(6x-24)(\sqrt{4}+2)}{(x-4)(\sqrt{1+6x}+5)} =$

$= \lim\limits_{x\downarrow 4} \frac{6(x-4)(\sqrt{4}+2)}{(x-4)(\sqrt{1+6x}+5)} = \lim\limits_{x\downarrow 4} \frac{6(\sqrt{4}+2)}{(\sqrt{1+6x}+5)} =$

$= \frac{6(\sqrt{4}+2)}{(\sqrt{1+6\cdot 4}+5)} = \frac{24}{5+5} = \frac{24}{10} = \frac{12}{5}$

- $\lim\limits_{x\uparrow 4} f(x) = \lim\limits_{x\uparrow 4} a = a = \lim\limits_{x\downarrow 4} f(x) = f(4) = a = \frac{12}{5}$

3. Določite parametra a in b tako, da bo funkcija

$$f(x) = \begin{cases} \frac{\sqrt{x+5}-3}{4-x} & x < 4 \\ ax+b & 4 \leq x \leq 6 \\ e^{\frac{1}{6-x}} & x > 6 \end{cases}$$

zvezna.

Rešitev:

Funkcija bo zvezna povsod, če parametra a in b določimo tako, da bo $\lim_{x\nearrow 4} f(x) = \lim_{x\searrow 4} f(x) = f(4)$ in $\lim_{x\nearrow 6} f(x) = \lim_{x\searrow 6} f(x) = f(6)$. Ker je $\lim_{x\nearrow 4} f(x) = -\frac{1}{6}$ in $\lim_{x\searrow 4} f(x) = f(4) = 4a+b$, dobimo pogoj $4a+b = -\frac{1}{6}$. Ker je $\lim_{x\nearrow 6} f(x) = f(6) = 6a+b$ in $\lim_{x\searrow 6} f(x) = 0$, dobimo še pogoj $6a+b = 0$. Torej bo funkcija zvezna, če bo $a = \frac{1}{12}$ in $b = -\frac{1}{2}$.

- $\lim\limits_{x\uparrow 4} f(x) = \lim\limits_{x\downarrow 4} f(x) = f(4)$

- $\lim\limits_{x\uparrow 6} f(x) = \lim\limits_{x\downarrow 6} f(x) = f(6)$

- $\lim\limits_{x\uparrow 4} f(x) = \lim\limits_{x\uparrow 4} \frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(4-x)\ (\sqrt{x+5}+3)} = \lim\limits_{x\uparrow 4} \frac{x+5-9}{(4-x)(\sqrt{x+5}+3)} =$

$= \lim\limits_{x\uparrow 4} \frac{x-4}{(4-x)(\sqrt{x+5}+3)} = \lim\limits_{x\uparrow 4} \frac{-(4-x)}{(4-x)(\sqrt{x+5}+3)} =$

$= \frac{-1}{\sqrt{4+5}+3} = \frac{-1}{6}$

- $\lim\limits_{x\downarrow 4} f(x) = \lim\limits_{x\downarrow 4} (ax+b) = 4a+b \quad \left(= \lim\limits_{x\uparrow 4} f(x) = -\frac{1}{6} \right)$

- $\lim\limits_{x\downarrow 6} f(x) = \lim\limits_{x\downarrow 6} e^{\frac{1}{6-x}} = e^{-\infty} = 0$ ker $x > 6 \Rightarrow e^{\frac{1}{6-6{,}0001}} = e^{\frac{1}{-0{,}0001}}$

- $\lim\limits_{x\uparrow 6} f(x) = \lim\limits_{x\uparrow 6} (ax+b) = 6a+b = \lim\limits_{x\downarrow 6} f(x) = 0$

- $6a+b = 0$
  $b = -6a$

  $4a+b = -\frac{1}{6}$
  $4a-6a = -\frac{1}{6}$
  $-2a = -\frac{1}{6} \quad /:(-2)$
  $a = \frac{1}{12}$

  $b = -6a$
  $b = -6 \cdot \frac{1}{12} = -\frac{1}{2}$

4. Določite a tako, da bo funkcija

$$f(x) = \begin{cases} \frac{\sin 2x}{\sqrt{9-x}-3} & x < 0 \\ a & x \geq 0 \end{cases}$$

zvezna v točki $x = 0$.

Rešitev:

Funkcija bo zvezna, če bo $\lim_{x \nearrow 0} f(x) = \lim_{x \searrow 0} f(x) = f(0)$. Ker je $\lim_{x \searrow 0} f(x) = f(0) = a$ in z uporabo l'Hospitalovega pravila dobimo $\lim_{x \nearrow 0} = -12$, bo funkcija zvezna za $a = -12$.

- $\lim\limits_{x \uparrow 0} f(x) = \lim\limits_{x \downarrow 0} f(x) = \lim(0)$

- $\lim\limits_{x \uparrow 0} f(x) = \lim\limits_{x \uparrow 0} \frac{\sin 2x\,(\sqrt{9-x}+3)}{\sqrt{9-x}-3\,(\sqrt{9-x}+3)} = \lim\limits_{x \uparrow 0} \frac{\sin 2x(\sqrt{9-x}+3)}{9-x-9} =$

$= \lim\limits_{x \uparrow 0} \frac{\sin 2x(\sqrt{9-x}+3)}{-x} \frac{(2)}{(2)} = \lim\limits_{x \uparrow 0} \frac{2\sin 2x(\sqrt{9-x}+3)}{-2x} =$

$= \lim\limits_{x \uparrow 0} \frac{2\,(\sqrt{9-x}+3)}{-1} = -2\,(\sqrt{9}+3) = -2(6) = -12$

- $\lim\limits_{x \downarrow 0} f(x) = \lim\limits_{x \downarrow 0} a = a = \lim\limits_{x \uparrow 0} f(x) = -12 = a$

5. Določi parametra a in b tako, da bo funkcija

$$f(x) = \begin{cases} \arctan \frac{1}{3+x} & x < -3 \\ ax + b & -3 \leq x \leq 0 \\ \frac{\sin(3x)}{\sqrt{x+4}-2} & x > 0 \end{cases}$$

zvezna.

Rešitev:

Funkcija bo zvezna, če parametra a in b določimo tako, da bo $\lim_{x \nearrow -3} f(x) = \lim_{x \searrow -3} f(x) = f(-3)$ in $\lim_{x \nearrow 0} f(x) = \lim_{x \searrow 0} f(x) = f(0)$. Ker je $\lim_{x \nearrow -3} f(x) = -\frac{\pi}{2}$ in $\lim_{x \searrow -3} f(x) = f(-3) = -3a + b$, dobimo pogoj $-3a + b = -\frac{\pi}{2}$. Ker je $\lim_{x \nearrow 0} f(x) = f(0) = b$ in $\lim_{x \searrow 0} f(x) = 12$, dobimo še pogoj $b = 12$. Torej bo funkcija zvezna, če bo $a = \frac{24+\pi}{6}$.

- $\lim\limits_{x\uparrow -3} f(x) = \lim\limits_{x\downarrow -3} f(x) = f(-3)$

  $\lim\limits_{x\uparrow 0} f(x) = \lim\limits_{x\downarrow 0} f(x) = f(0)$

  Da je $f(x)$ zvezna (neprekinjena) more tu držat.

- $\lim\limits_{x\uparrow -3} f(x) = \lim\limits_{x\uparrow -3} \arctan\frac{1}{3+x} = \arctan -\infty = -\frac{\pi}{2}$

  ker $x < -3 \Rightarrow \frac{1}{3+(-3{,}00001)} = \frac{1}{-0{,}00001} = -\infty$

- $\lim\limits_{x\downarrow -3} f(x) = \lim\limits_{x\downarrow -3} (ax+b) = -3a + b \quad \left(= \lim\limits_{x\uparrow -3} f(x) = -\frac{\pi}{2}\right)$

- $\lim\limits_{x\downarrow 0} f(x) = \lim\limits_{x\downarrow 0} \frac{(\sin 3x)\ (\sqrt{x+4}+2)}{(\sqrt{x+4}-2)\ (\sqrt{x+4}+2)} = \lim\limits_{x\downarrow 0} \frac{\sin 3x\ (\sqrt{x+4}+2)}{x+4-4} =$

  $= \lim\limits_{x\downarrow 0} \frac{\sin 3x\ (\sqrt{x+4}+2)\ (\cdot 3)}{x\ (\cdot 3)} = \lim\limits_{x\downarrow 0} \frac{\sin 3x\ \ 3(\sqrt{x+4}+2)}{3x} =$

  $= \lim\limits_{x\downarrow 0} 3(\sqrt{x+4}+2) = 3(\sqrt{4}+2) = 3\cdot 4 = 12$

- $\lim\limits_{x\uparrow 0} f(x) = \lim\limits_{x\uparrow 0} (ax+b) = a\cdot 0 + b = b \quad \left(= \lim\limits_{x\downarrow 0} f(x) = 12\right)$

- $-3a + b = -\frac{\pi}{2}$ $\qquad\qquad b = 12$

  $-3a + 12 = -\frac{\pi}{2}$

  $-3a = -\frac{\pi}{2} - 12 \quad / : (-3)$

  $a = \frac{\pi}{6} + 4$

6. Določite a in b tako, da bo funkcija

$$f(x) = \begin{cases} \frac{1}{x+2} + \frac{4}{x^2-4} & x < -2 \\ ax + b & -2 \leq x \leq 0 \\ \frac{\cos(3x)\sin\sqrt{x}}{\sqrt{x}} & x > 0 \end{cases}$$

zvezna povsod.

Rešitev:

Funkcija bo zvezna povsod, če parametra a in b določimo tako, da bo $\lim_{x\nearrow -2} f(x) = \lim_{x\searrow -2} f(x) = f(-2)$ in $\lim_{x\nearrow 0} f(x) = \lim_{x\searrow 0} f(x) = f(0)$. Ker je $\lim_{x\nearrow -2} f(x) = -\frac{1}{4}$ in $\lim_{x\searrow -2} f(x) = f(-2) = -2a+b$, dobimo pogoj $-2a+b = -\frac{1}{4}$. Ker je $\lim_{x\nearrow 0} f(x) = f(0) = b$ in $\lim_{x\searrow 0} f(x) = 1$, dobimo še pogoj $b = 1$. Torej bo funkcija zvezna, če bo $a = \frac{5}{8}$.

- $\lim_{x\uparrow -2} f(x) = \lim_{x\downarrow -2} f(x) = f(-2)$

  $\lim_{x\uparrow 0} f(x) = \lim_{x\downarrow 0} f(x) = f(0)$

- $\lim_{x\uparrow -2} f(x) = \lim_{x\uparrow -2} \left(\frac{1(x-2)}{x+2\,(x-2)} + \frac{4}{x^2-4}\right) = \lim_{x\uparrow -2} \frac{x-2+4}{(x^2-4)} =$

  $= \lim_{x\uparrow -2} \frac{x+2}{(x-2)(x+2)} = \lim_{x\uparrow -2} \frac{1}{(x-2)} = \frac{1}{-2-2} = -\frac{1}{4}$

- $\lim_{x\downarrow -2} f(x) = \lim_{x\downarrow -2} (ax+b) = -2a+b \quad \left(= \lim_{x\uparrow -2} f(x) = -\frac{1}{4}\right)$

- $\lim_{x\uparrow 0} f(x) = \lim_{x\uparrow 0} (ax+b) = a\cdot 0 + b = b$

- $\lim_{x\downarrow 0} f(x) = \lim_{x\downarrow 0} \frac{\cos 3x \cdot \sin\sqrt{x}}{\sqrt{x}} = \lim_{x\downarrow 0} \cos 3x = \cos(3\cdot 0) =$

  $= \cos 0 = 1 \quad \left(= \lim_{x\uparrow 0} f(x) = b\right)$

- $-2a+b = -\frac{1}{4} \qquad b = 1$

  $-2a+1 = -\frac{1}{4}$

  $-2a = -\frac{1}{4} - 1$

  $-2a = -\frac{5}{4} \quad /:(-2)$

  $a = \frac{5}{8}$