Foundations of Operations Research

Elia Ravella, Juri Sacchetta

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Contents

Chapter 1

Graphs

1.1 Definitions

Definition 1 (Graph) *A graph is a pair $G = (N, E)$, with N a set of nodes (vertices) and $E \subseteq NxN$ a set of pairs of nodes.*

The intuitive representation is the classic web of interconnected nodes.
There are two type of graphs, depending on how is possibile to traverse the edges:

1. **Undirected** graph $G = (N, E)$ with E set of *edges*, i.e. unordered pairs of nodes, denoted by $\{i, j\}$ for some $i, j \in N$.
2. **Direct** graph $G = (N, A)$ with A set of *arcs*, i.e. ordered pairs of nodes, denoted by (i, j) for some $i, j \in N$.

Note the different name of the "edges/arcs" (and syntax) based on in which way is it possible to traverse them.

Definition 2 (adjacent nodes) *Two nodes are said to be **adjacent** if there's an edge connecting them.*

Definition 3 *An edge e is incident in a node v if v is an endpoint of e .*

Definition 4

***Undirected graphs**: The degree of a node is the number of incident edges*

***Directed graphs**: The in-degree (out-degree) of a node is the number of arcs that have it as head (tail).*

Definition 5 (Path)

***A path** from $v_1 \in N$ to $v_k \in N$ is a sequence of consecutive edges $p = \{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{k-1}, v_k\}$
with $\{v_l, v_{l+1}\} \in E$ for $l = 1, ..., k-1$.*

***A directed path** from $v_1 \in N$ to $v_k \in N$ is a sequence of consecutive arcs $(v_l, v_{l+1}) \in E$, for $l = 1, ..., k-1$.*

A **path** is a subset of the E set composed of consecutive edges (the endpoint of an edge is the "start"point of the next one[1]) that connects some edges in the graph.

Definition 6 *Two nodes are **connected** if exists a path that links them.*

Definition 7 *A graph con be **connected** to, if and only if there exists a path the connects **all its nodes**.*

Definition 8 *A graph is **directed** if some of the edges can be traversed in only one way. These edges are called **arcs**.*

Definition 9 *A **cycle** or **circuit** is a directed path that end where it starts.*

Definition 10 *A graph is **complete** if for all pair of nodes there's an edge connecting them.*

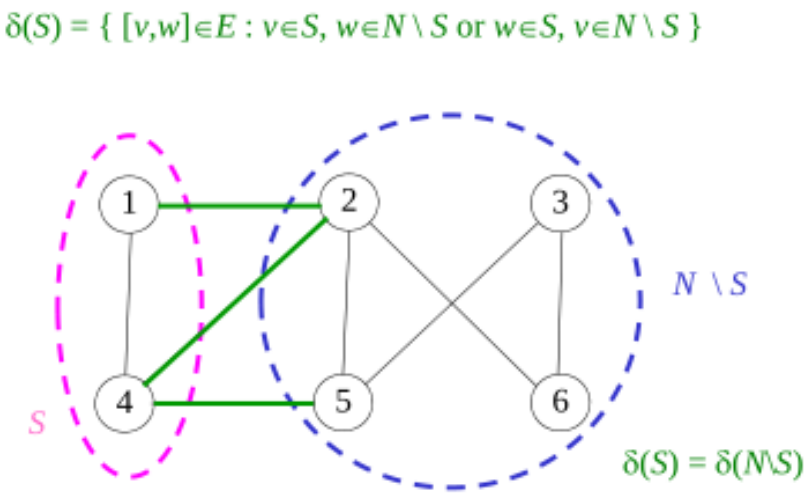
Reasoning on the definition 10, we can calculate the upper bounds for the number of edges given the nodes:

$$\begin{cases} e \leq \frac{n \times (n-1)}{2} \text{ if undirected} \\ e \leq n \times (n-1) \text{ if directed} \end{cases} \tag{1.1}$$

where intuitively "e" is the number of edges and "n" the number of nodes.
Given this two inequalities, we can redefine completeness:

Definition 11 *a graph is complete if and only if $e = \frac{n \times (n-1)}{2}$ or $e = n \times (n-1)$ depending on the graph being directed.*

Definition 12 *Given an undirected graph $G = (N, E)$ we can take $S \subset G$ subsets of nodes; we then observe $\delta(S)$ the cut induced by S is the subsets of **edges** that connects S to $N \setminus S$.*



[1] not in a directional way! we are talking about non directed graphs, the two consecutive edges just in fact share an endpoint

Same definition holds for directed graphs, but this time we can distinguish among *outgoing* and *incoming* cuts, wheter the edges are all exiting S or ending in it.

Definition 13 *A graph is said to be* ***bipartite*** *if we find a partition* $N = (N_1, N_2)$ *such as* $\delta(N_i) = \emptyset$. *So* ***the two subsets are disconnected****.*

A bit of foreshadowing: if a cut from a partial tree to another portion of the graph holds a minimum cost arc, then *there exists a spanning tree with that arc.* This is called the **cut property**.

Data Representations for Graphs

Depending one the edges / nodes ratio, two different data structures can hold a valid representation for a graph. When $e \approx n^2$ (a dense graph) then an adjacency matrix is the best choice. Otherwise (sparse graphs) linked lists of successors for each node.

1.2 Graph reachability problem

The reachability problem can be formulated as "given a graph, determine all the nodes that are reachable from a starting one, decided a priori. A node is reachable is there's a path from the starting one and itself".

To solve this problem, we can use the **breadth-first search algorithm**.

Algorithm 1: Graph reachability problem

```
1 Q ← {s}; M ← ∅
2 while Q ≠ ∅ do
3 |   u ← node in Q; Q ← Q\{u}
4 |   M ← M ∪ {u}/* label u                                  */
5 |   for v ∈ S(u) do
6 |   |   if v ∉ M and v ∉ Q then  Q ← Q ∪ {v}
```

Figure 1.1: Breadth-first search algorithm pseudo code

Note: Here Q is managed as FIFO queue.

1.2.1 Complexity of the reachability algorithm

Refer to 1.1 image.

At each iteration of the while loop:

1. Select one node u in Q, extract it from Q and insert it in M.
2. For all nodes v directly reachable from u and not already in M or Q, insert v in Q.
3. Repeat form 1.

Since each node u is inserted in Q at most once and each arc (u, v) is considered at most once, the overall complexity is $O(n+m)$, where $n = |N|$, $m = |A|$. **Note**: For dense graphs, $m = O(n^2)$.

1.3 Subgraphs, Trees, Spanning Trees

We call $G' = (N', E')$ a subgraph of $G = (N, E)$ if and only if

$$\begin{cases} N' \subseteq N \\ E' \subseteq E \\ \text{all edges inside } E' \text{ connects nodes in } N' \end{cases} \tag{1.2}$$

Definition 14 (Tree) *A tree $G_t = (N', E')$ is a subgraph of a network that's both **connected** and **acyclic**.*

Note: A tree is *spanning* if it contains all the nodes of the original graph. The *leaves* of a tree are the nodes with unitary degree.

Properties of trees:

- Every tree with a number of nodes greater than 2 has at least 2 leaves
- All trees have number of arcs equal to number of nodes minus 1
- Every pair of nodes in a tree is connected by a unique path (acyclic connected graph)
- By adding an edge to a tree, we create a *unique cycle*. This also mean that if we now remove *any* of the edges of this new unique cycle we go back to a spanning tree
- A complete Graph with n nodes has exactly n^{n-2} spanning trees

1.4 Minimum Cost Spanning Tree

Problem: find in a graph $G = (N, E)$ with *weighted arcs*[2] a subgraph that is a tree and has minimum total cost of edges.

The *minimum total cost* condition is verified when the sum of the costs along edges cannot be decreased by swapping edges in and out from the tree.

Theorem 1 (Cayley, 1889) *A complete graph with n nodes ($n \geq 1$) has n^{n-2} spanning trees.*

1.4.1 Prim's Algorithm

The Prim's algorithm is a simple iterative algorithm to build a spanning tree starting from an initial node.
The algorithm adopts an incremental *greedy*[3] strategy: it adds to the (initially empty) tree the nearest node every time (nearest = connected with the minimum cost outgoing edge) until the tree has the same number of nodes as the original graph.
The algorithm

- Input: Connected $G = (N, E)$ with edge costs.

[2] numerated arcs with a "cost" associated to them

[3] A greedy algorithm constructs a feasible solution iteratively by making at each step a *locally optimal* choice, without reconsidering previous choices.

- Output: Subset $T \subseteq N$ of edges of G such that $G_T = (N, T)$ is a minimum cost spanning tree of G.

The Method:

1. initialize the output: $S = (N', T)$ where N' is composed only of the initial chosen node and $T = \emptyset$ is the set of edges.
2. add to S's nodes the "nearest" node from the ones connected to it, and add to T the lowest-cost edge. Careful: we are watching at all the nodes reachable from the outgoing cut of S, not only the ones from a single node.
3. repeat step 2 until the graph is covered.

Prim's algorithm is *exact*[4].
This algorithm exploits the **cut property**.

1.5 Shortest Path and Shortest Path Trees

Finding the shortest path on a graph from node A to node B is another classical problem. The most used algorithm is the famous Dijkstra's algorithm.

Dijkstra Algorithm Dijkstra algorithm is similar to the Prim's one, but applied in a different direction. Dijkstra starts from a subset of the nodes (called the "unvisited nodes") and considers themone at a time. At each iteration, a node is selected from the unvisited set (the first is chosen a priori, will be the *initial* node) and for all his neighbours is updated the distance. When all neighbours have been inspected, the node is marked as "visited"[5]. Next node to be selected is the *closest* and the algorithm restarts. It finishes when the "unvisited" set is empty. More schematically:

1. Select the closest node (first one a priori)
2. Check his neighbours and update their distance
3. When all neighbours have been inspected, go to point one. If no more unvisited nodes exists, end.

At the end of the algorithm we have a *shortest path tree*[6] that highlights all the shortest paths from the initial node to all other nodes in the graph. Taking the *closest* node at each iteration checking the outgoing cut from the "visited nodes" set ensures the exactness of Dijkstra algorithm.

Floyd Warshall Algorithm Negative-cost edges, if present, do not allow to apply Dijkstra algorithm. But they are a further threat to shortest path problems: if a *negative cost cycle* exists, there's no minimum cost path that's also finite between two nodes that traverse that cycle. This is a ill-defined problem, and the Floyd Warshall algorithm detects it. This algorithm uses two matrixes:

[4]Formally proven to provide an optimal solution for each instance of a problem
[5]and never checked again
[6]that's **different** from a minimum cost spanning tree

a distance and a predecessor one. At each iteration of the algorithm, a *triangular check* is performed: for all his neighbours, an undirect path is searched (a multi step path, generally with a single intermediate node) to shorten the distance wrt the direct path.

1. initialize a counter h at 1, it will be our index for the nodes
2. for each value of the counter from 1 to the number of nodes, perform the check $d_{ij} > d_{ih} + d_{hj}$ to evaluate if there is an alternative two-steps path that reduces the distance between nodes i and j.
3. if a loop with negative cost is detected (so a node can go from itself to itself with less than 0 distance passing through another node) that the problem is flagged as ill and the algorithm terminates

At the end, combining the two matrixes, we have a shortest path tree of the original graph. This time, it can be calculated with negative cost arches.

Direct Acyclic Graphs and Dynamic Programming

If we add the hypothesis that the graph we're working on (to find the shortes path or the minimum cost spanning tree) does not contain cycles, we can study some more interesting solutions to such a problem. Moreover, DAGs[7] are well suited to model a wide range of problems.

Topological Ordering DAGs can be *topologically ordered*, which means that we can label the nodes as if the graph represents an order relation between them (in the algebraic sense). The formal definition is that

$$\forall (i,j) \in A \,|\, i < j \tag{1.3}$$

where A is the set of edges. This looks like a trivial additional property to a graph, but having a *strictly monotone ordering of nodes* is super helpful.
To order nodes in such a way, however, the algorithm is quite simple:

1. find a node with no incoming edges[8] and assign the smallest index available
2. remove the node from the graph and return to point 1

Dynamic Programming The "dynamic programming" approach has a bit of a confusing name. The underlying idea is that we can exploit a recursive relation between costs along a shortest path, using the "shortest subpath" property: given π_{ij} shortest path $i \to j$ we can divide it in $\pi_{it} + c_{tj}$ where π_{it} is the *shortest subpath from i to t* and the second term is the cost of the last step. If we define $L(i)$ as the cost of the shortest path from the initial node to node i we have that

$$L(t) = min\{L(i)_{i \text{ predecessor of } t} + c_{it}\} \tag{1.4}$$

This is a recursive relation that can be expanded to all nodes in the path. I'll put here the example by the professor:

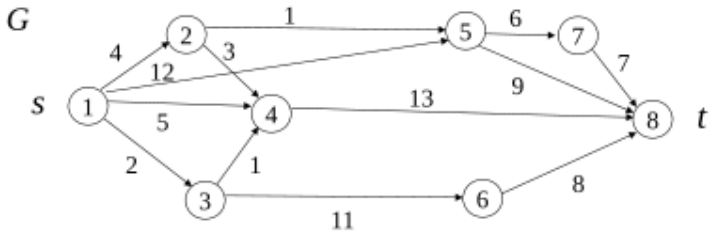
[7] directed acyclic graphs
[8] the existence of such a node is proved by the acyclicity condition

$L(1) = 0$ pred(1)=1

$L(2) = L(1) + c_{12} = 0 + 4 = 4$ pred(2)=1

$L(3) = L(1) + c_{13} = 0 + 2 = 2$ pred(3)=1

$L(4) = \min_{i=1,2,3}\{L(i) + c_{i4}\} = \min\{0 + 5, 4 + 3, 2 + 1\} = 3$ pred(4)=3



[.] indicates the predecessor of node i in a shortest path from 1 to i

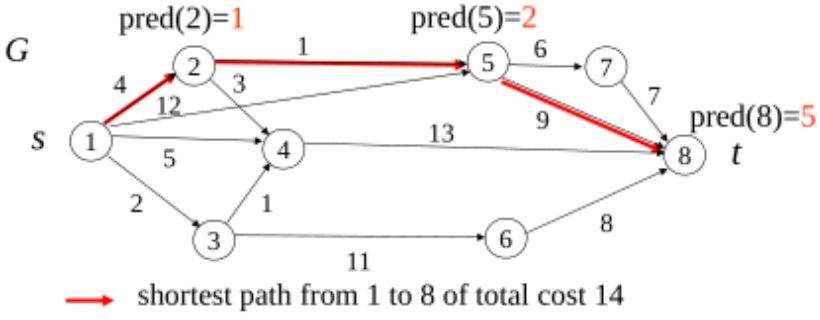
$L(5) = \min_{i=1,2}\{L(i) + c_{i5}\} = \min\{0 + 12, 4 + 1\} = 5$ pred(5)=2

$L(6) = \min_{i=3}\{L(i) + c_{i6}\} = \min\{2 + 11\} = 13$ pred(6)=3

$L(7) = L(5) + c_{57} = 5 + 6 = 11$ pred(7)=5

$L(8) = \min_{i=4,5,6,7}\{L(i) + c_{i8}\} = \min\{3+13, 5+9, 13+8, 11+7\} = 14$

pred(8)=5



Project Planning

As shown in the Examples, we can use graphs to express dependency relationships. A direct application of this model is used when organizing various activitie inside a project: each phase will depend from some previous phases, there will be an "initial" phase with no predecessors and a final phase with no successors. The model adopted here is the one that represents

- *activities* as arcs
- *activity duration* with arcs weight
- *checkpoints* as nodes

This formulation leads to a weighted directed acyclic graph. It's acyclic *by nature*: it would be meaningless (and also logically incorrect) if activities in a project had circular relationships.
This model helps in finding the *minimum overall project duration*: being directed and acyclic, this will be the duration of the *longest path* from the initial node to the final (we cannot compress the project time beyond that).

Critical Path Method The CPM aims to provide an optimal schedule for a project, with allotted time for each activity and the possible slack of each one, given the DAG of a project activities precedences. The algorithm is:

1. Construct the DAG of activities and precedences
2. Topologically sorts the nodes
3. For each node, calculate
    - The earliest start time starting from inital activity. This would be (at the final node) the minimum overall project duration
    - The latest time an activity can be started to NOT increase the minimum project duration (starting from the end, this time)
4. For each activity now we can calculate the *slack*: $T_{max} - T_{min} - d$ where d is the duration of the activity itself

The slack of each activity indicates if it could be *deliberately delayed* without affecting the overall project duration. There exist activities with null slack: these are called critical activities and cannot be delayed in any case. They compose the critical path from the beginning to the end of the project.

Network Flows

Network flows problems are the ones that involves "distribution of a given resource over a network". Needless to say, they boils down to find the optimal throughput of a network given its *Directed Weighted Graph* representation. In this kind of problems, we can remove the acyclicity hypothesis. The graph model we're dealing with is like:

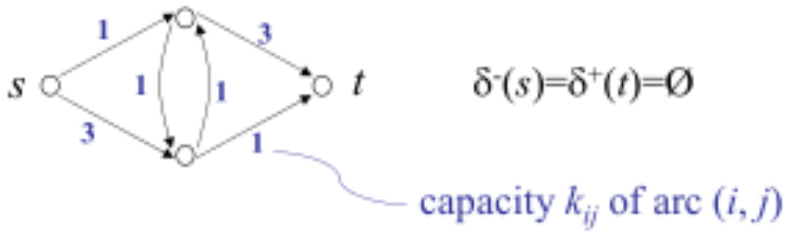


Figure 1.2: Arcs weights assume a different meaning here: they are not more a cost or a lenght, but instead a *width* or *capacity* as shown in the blue note

We introduce the *feasible flow vector*: it's a tuple of values representing the **occupated fraction** of each edge of the network; it should respects two major constraints:

1. Capacity constraints: $\forall(i,j) \in E \,|\, (0 \leq x_{ij} \leq k_{ij})$ where x_{ij} is the value of the flow in the feasible flow vector, k_{ij} the maximum capacity for that arc and E the set of edges;

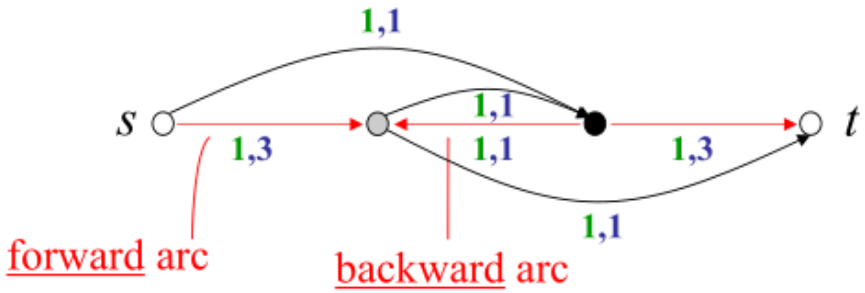
2. Flow balance constraints: $\forall h \in N \,|\, (\sum(x_{ih}) = \sum(x_{hj}))$ that simply states that for each node, the amount that enters the node is the same amount that exits it

We can then calculate the *value of flow* $\phi = \sum(x_{sj})$, where s is the initial node and j all the nodes reached by the *outgoing cut* of s; this represents the actual maximum flow that can go through the network.[9]

Ford Fulkerson's Algorithm Another algorithm, this time to resolve a network flow problem. Again, this is just the algorithmic version of the intuitive solution for this problem: keep sending stuff on non-saturated channels until the network is maxed out.
Said so, we need some additional definition to handle easily this algorithm:

- a *backward arc* is an edge of the network that "sends units backward" with regard to a simil-topological ordering of the nodes. We can "flat out" the graph of the network to visually understand what a backward arc is:

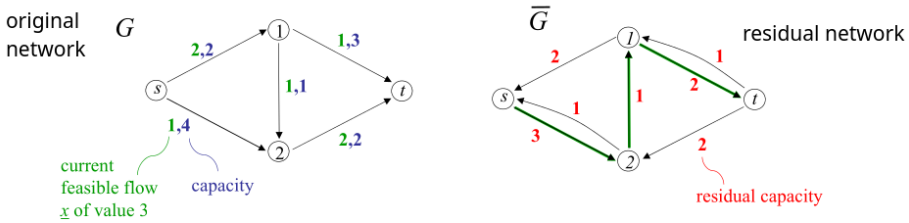
[9]The maximum flow is directly associated with the minimum capacity of a cut. Will see later how these, in fact, are mathematically correlated: *they're dual problems*, or two representations of the same problem.



- an *augmenting path* (wrt the current ϕ) is a path through all the network (so start to end) where

$$\begin{cases} x_{ij} < k_{ij} \text{ for all forward arcs} \\ x_{ij} > 0 \text{ for all backward arcs} \end{cases} \tag{1.5}$$

- the *residual network* of a graph is a graph itself buil keeping track of the possible changes that can be applied to the flows on the arcs. This means, if we have a saturated channel with capacity 3 we will have in the residual network the same channel but *reversed*: this signals that we have a residual capacity of 3 units *in the opposite direction*.



So, the algorithm is structured in 2 main phases:

1. Search for an augmenting path on the actual graph, then rebuild the residual graph. Repeat this point until no more trivial augmenting path can be found

2. Look for an augmenting path *on the residual graph*. Using the second graph is useful when some edges could be desaturated in order to maximize the flow. The algorithm stops when on the residual graph there are no more paths from s to t

Chapter 2

Linear Programming

2.1 Linear Programming Problems

A linear programming (LP) problem is an *optimization* problem in the form

$$\begin{cases} \min f(\underline{x}) \\ \text{s.t. } \underline{x} \in X \subseteq \mathbb{R}^n \end{cases} \tag{2.1}$$

where:

- $f : X \to \mathbb{R}$ is a *linear function*
- the *feasible region* $X \subseteq \mathbb{R}^n$ is a combination of linear functions.

Definition 15 $\underline{x}^* \in \mathbb{R}^n$ *is an* optimal solution *of the LP* (2.1) *if* $f(\underline{x}^*) \leq f(\underline{x}), \forall \underline{x} \in X$

LP problems can also be formulated with matrixes:

$$\begin{cases} \min z = \underline{c}^t \, \underline{x} \\ A\underline{x} \geq \underline{b} \\ \underline{x} \geq \underline{0} \end{cases} \tag{2.2}$$

Assumptions of LP Models LP models and the algorithms operating on them work under several assumptions:

1. Linearity of the objective function and constraints
2. Divisibility: The variables can take fractional (rational) values.
3. Parameters are considered as constants which can be estimated with a sufficient degree of accuracy.

Note: LP "sensitivity analysis" allows to evaluate how sensitive an optimal solution is with respect to small changes in the parameter values (see end of Chapter 2).

2.2 LP Problems Forms

The most general for for a linear programming model we can define is:

$$\begin{cases} \min/\max\ z = \underline{c}^t\,\underline{x} & \\ A_1\underline{x} \geq \underline{b_1} & \\ A_2\underline{x} \leq \underline{b_2} & \\ A_3\underline{x} = \underline{b_3} & \\ x_j \geq 0 & \text{for some } j \\ x_j \text{ free} & \text{for others j} \end{cases} \tag{2.3}$$

This form is usually referred to as *canonical form.*

Standard Form The standard form is the "most restrictive" one, and also one of the easiest to work with

Definition 16 (Standard form)

$$\begin{cases} min\ z = \underline{c}^t\,\underline{x} & \\ A\underline{x} = \underline{b} & \textit{only} \text{ equality constraints} \\ \underline{x} \geq \underline{0} & \textit{all} \text{ nonnegative } \textit{variables} \end{cases} \tag{2.4}$$

In this form, all constraints must be equalities, and all variables must be non negative.
Pay attention: in the original general model, we have the non-negativity constraints only on some variables.

Transforming to Standard Form We can easily transform a general formulation of a LP problem in standard one by applying these simple transformations:

- $\max(\underline{c}^t\underline{x}) = -\min(-\underline{c}^t\underline{x})$ to change objective function
- all inequalities are transformed adding a *slack* variable:

$$\underline{a}^t\underline{x} \leq \underline{b} \Rightarrow \begin{cases} \underline{a}^t\underline{x} + s = \underline{b} \\ s \geq 0 \end{cases} \tag{2.5}$$

  or a *surplus* variable in case it's a $\geq$ inequality

$$\underline{a}^t\underline{x} \geq \underline{b} \Rightarrow \begin{cases} \underline{a}^t\underline{x} - s = \underline{b} \\ s \geq 0 \end{cases} \tag{2.6}$$

- when a variable is unrestricted in sign it's "splitted" in its positive and negative part:

$$x_j \in \mathbb{R} \Rightarrow \begin{cases} x_j = x'_j - x''_j \\ x'_j \geq 0 \\ x''_j \geq 0 \end{cases} \tag{2.7}$$

2.3 Geometry of a LP Problem

2.3.1 Feasible Region as an Hyperplane

Definition 17 (Level curve) *A level curve of value z of a function f is the set of points in $\mathbb{R}^n$ where f is constant and takes value z.*

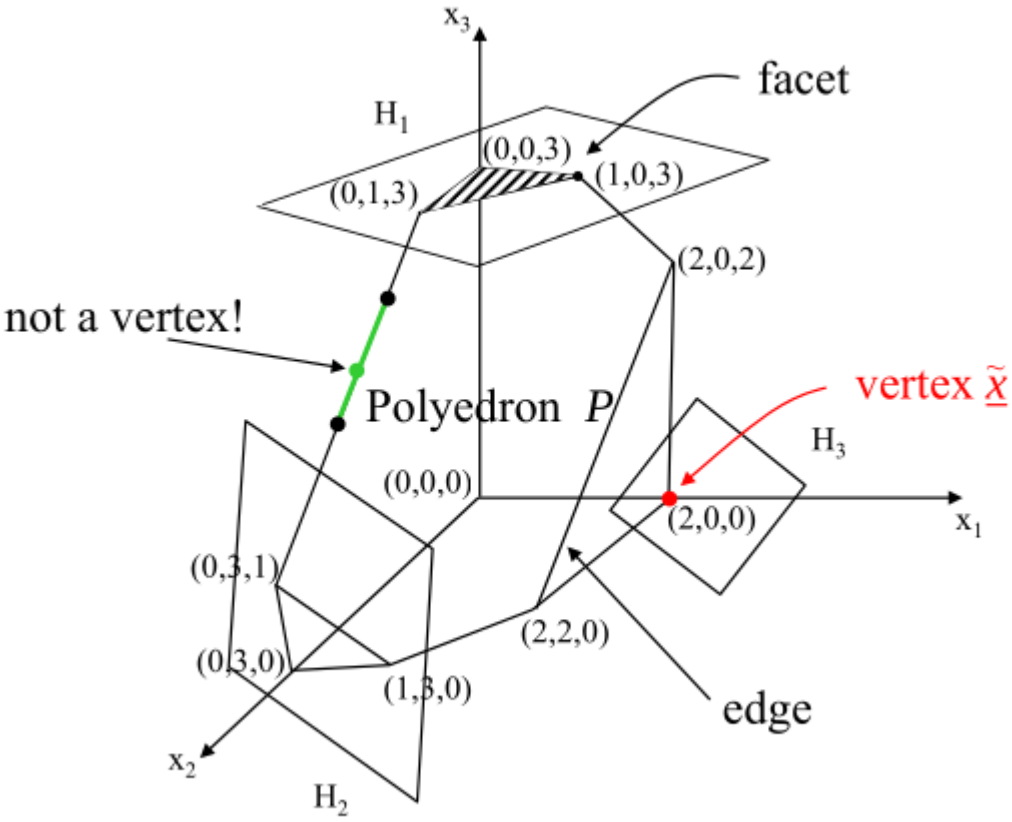
Definition 18

$H = \{x \in \mathbb{R}^n \,|\, \underline{a}^t x = \underline{b}\}$ *is an* hyperplane*;*

$H^- = \{x \in \mathbb{R}^n \,|\, \underline{a}^t x \leq \underline{b}\}$ *is an* affine half-space*.*

Note: Each inequality constraint ($a^T x \leq b$) defines an affine half-space in the variable space.

So, the *feasible region* of an LP problem, defined as *a region delimited by hyperplanes*, is called a *polyhedron*. A polyhedron (so a polygon in two dimension, a 3D figure in three dimension and so on) is a convex set of $\mathbb{R}$ as generated by convex sets.



Definition 19 (Convex combination) *We call a convex combination a linear combination where all coefficients are non negative and sum to 1.*

Definition 20 (Vertex) *A* vertex *of P is a point of P which cannot be expressed as a convex combination of two other distinct points of P.*

Don't worry if you're confused: all "polyhedra theory" has a lot of dark spots[1].

Property 1 *A non-empty polyhedron $P = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = b, x \geq 0\}$ (in standard form) or $P = \{\underline{x} \in R^n : A\underline{x} \geq b, x \geq 0\}$ (in canonical form) has a finite number (≥ 1) of vertices.*

Example: a polyhedron with only one vertex is a quarter of plane, for example.

Theorem 2 (Representation of polyhedra - Weyl-Minkowski) *Every point x of a polyhedron P can be expressed as a convex combination of its vertices $x^1, ..., x^k$ plus (if needed) an unbounded feasible direction d of P:*

$$\underline{x} = \lambda_1 \underline{x}^1 + ... + \lambda_k \underline{x}^k + d$$

where the multipliers $\lambda_i \geq 0$ satisfy $\lambda_1 + ... + \lambda_k = 1$.

So, we can represent *every point of a polyhedron* as a convex combination of its *vertices* plus (if needed) a vector that never ends inside the polyhedron (for unbounded polyhedron).

x = x1 + x2 +d

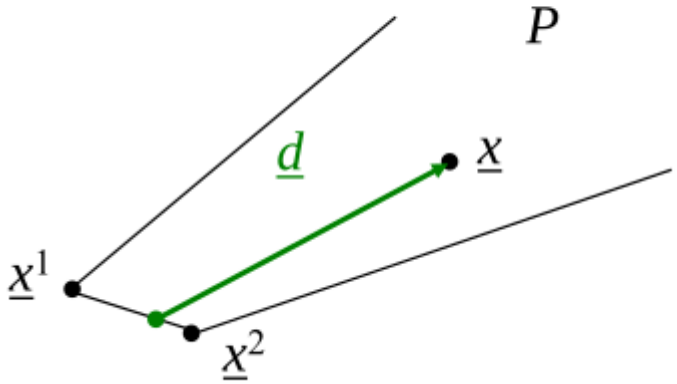


Figure 2.1: Every point of this unbounded polyhedron can be represented as a combination of x_1 and x_2 and $\underline{d}$, where $\underline{d}$ is called *unbounded feasible direction* of the polyhedron.

Only for completeness, we can define bounded polyhedron as:

Definition 21 (Polytope) *A polytope is a bounded polyhedron, that is, it has the only unbounded feasible direction $d = 0$.*

[1] check out they wikipedia page

2.3.2 Fundamental Theorem of Linear Programming

Theorem 3 (Fundamental theorem of Linear Programming) *Consider a LP $min\{c^T x : x \in P\}$, where $P \subseteq R^n$ is a non-empty polyhedron of the feasible solutions (in standard or canonical form).*
Then either:

- *there exists (at least) one optimal vertex or*
- *the value of the objective function is unbounded below on P.*

This foggy theorem has a simple intuitive interpretation, in my mind: the feasible region is represent as a subset of $\mathbb{R}$ and reprents *all the solutions of the system of constraints*. To minimize the objective function, we must "move around" this subspace in a well-defined direction, that is the opposite of the gradient of the objective function itself. Being this a *straight line*, we will eventually hit a boundary of the polyhedron. We could then "slide" onto it until we are "stuck" and every move we make is not anymore bettering the solution. This can happen *only* on the intersection of boundaries (so the vertices), because it's the only point where a "direction" to better the solution finds his boundary[2].

This theorem allows us to do determine a priori which are our solutions, just looking at the way the feasible region is defined. Also, being *finite*, we can cycle through the solutions to "easily" find the right one.

2.4 Algebraic Characterization of the LP Problem

We've seen we can exploit the geometry of the feasible region to find the solutions. We need an algebraic formulation to do that algorithmically.

The Feasible Vector Space

Classic LP problem: $P = \{x \in \mathbb{R} \mid Ax = b, x \geq \underline{0}\}$ in standard form. We assume from now on that A is full rank. This leads to two possible scenarios:

- A is a square matrix $\Rightarrow$ there exists only a *unique* solution to the system: $x = A^{-1}b$.
- A is a rectangular matrix with more columns than rows: this leads to infinite solutions of the system.

Obviously, having the exact number of constraints equal to the variables one is not common. We focus on the second case.
If we partition the matrix $A = [B \mid N]$, where B is a basis[3] and N the "remaining" vectors, we can redefine the solutions' vector as:

$$x = (x_B, x_N) \tag{2.8}$$

[2] I find this mumbo jumbo more clear than the slides definition, ok? In my head it's clear this way.
[3] set of linearly indipendent vectors

where the two components of the vector will be long as the rank of the matrix and the lenght of the remaining matrix. We then substitute into the feasible region definition

$$Bx_B + Nx_N = \underline{b} \tag{2.9}$$

and so

$$x_B = B^{-1}\underline{b} - B^{-1}Nx_N \tag{2.10}$$

from this last equation, we can define

- a basic solution is a solution setting all non base variables x_N to zero, so $x_B = B^{-1}\underline{b}$
- a basic *feasible* solution is a basic solution that has all components greater or equal to zero
- the variable in x_B are called basic variables and the ones in x_N non basic variables

Now we link the algebraic formulation with the geometrical interpretation:

Theorem 4 *$x \in \mathbb{R}^n$ is a basic feasible solution $\Leftrightarrow$ x is a vertex of $P = \{x \in \mathbb{R}^n \mid Ax = \underline{b}, x \geq \underline{0}\}$*

2.5 The Simplex Method

Dantzig's simplex method is an algorithm to find the best basic feasible solution of a LP problem: it examines a sequence of BFSs[4] with *non increasing objective function values* until an optimal solution is reached, or the problem is found to be *unbounded*.

The Algorithm

1. check for infeasibility of the problem
2. find an initial vertex
3. move from a current vertex to a *better adjacent vertex* (or establish that the problem is unbounded)
4. determine if the current vertex is *optimal*

Infeasibility Check To check if a problem is infeasible, we create an auxiliary problem with artificial variables (from the original problem) as:

$$P = \begin{cases} \min z = \underline{c}^t x \\ Ax = \underline{b} x \geq \underline{0} \end{cases} \tag{2.11}$$

$$P_{aux} = \begin{cases} \min v = \sum_{i=1}^{m} y_i \\ Ax + Iy = \underline{b} \\ x \geq \underline{0}, y \geq \underline{0} \end{cases} \tag{2.12}$$

obviously, there exists a BFS composed by all elements in y. We face two cases now:

[4] Basic feasible solutions

1. $v > 0$ the problem is ***infeasible***
2. if $v = 0$, the problem is feasible and we must manipulate the matrix in order to obtain a BFS that only depends on the original x_i variables

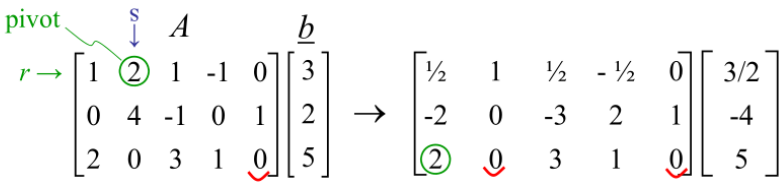
Don't forget we have to turn to the original problem and recompute the objective function.

Initial Vertex Selection The selection of a starting initial BFS is a problem per se. If the problem is easy enough we'll have the initial matrix already in a $A = [N \mid I]$ form, where I represents the identity matrix $\Rightarrow$ a basic feasible solution. In that case, we have already a perfect indication of which variables are in the basis and at which cost, so we can directly go to the "movement" phase (that's the one that really optimizes the objective function value).
In the other case (that's obviously more realistic) we have to resort again at the "auxiliary problem" with artificial variables: the initial basis we find is the one determined by the y_i vectors.

Movement Across Vertices To "move" from a vertex to another means to swap some of the basic vertices out of the solution and "bring in" some of the vectors that were in the N matrix. *Algebrically*, this means to *expressing the basic variables in terms of the non basic ones*. **Remember**: non basic variables are set to **zero**. Algorithmically, this is done through a simple pivoting operation on the combined $[B|N]b$ matrix.
Given $Ax = \underline{b}$

1. select a coefficient $a_{rs} \neq 0$ of table A where r is the row index and s is the column index
2. divide row r for a_{rs} (also in order to have $a_{rs} = 1$)
3. for alla other rows, subtract row r multiplied by the coefficient on column s: that is, a_{is}

This will render the s column all zeroes except the 1 on row r. Don't forget we're applying this procedure also the the $\underline{b}$ vector.



The choice of r and s are crucial: in fact, they select which variable enters the basis (through the pivot column) and which leaves it (through the pivot row). This choice can be taken

- randomly
- using heuristics
- usign **Bland's rule**, that also avoid cycles in the algorithm

Bland's Rule Bland's rule dictate a choice for rows and columns to enter/leave the basis. It states:

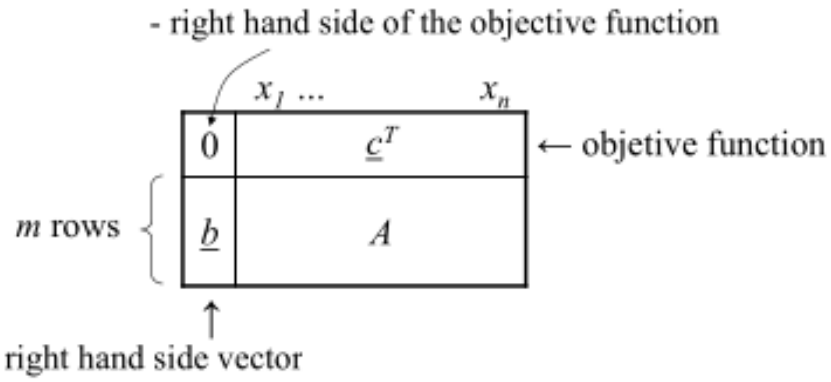
1. select s the *first column with negative reduced cost*
2. then select r the row with the *smallest* $\frac{b_r}{a_{rs}}$ ratio **among the positive ones**

This simple criterion avoids cycles in the Simplex algorithm, that can be stuck in a vertex for some degenerate LP problems. In practice, due to the additional complexity added and the rarity of such problems, Bland's rule is never used.

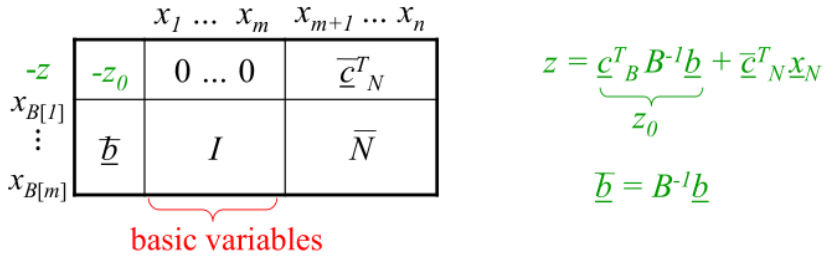
Tableau Representation As spoiled when talking about movement across vertices, the Simplex method has a peculiar way to represent the matrixes and vectors it uses to ease the computing problem.
Given

$$P = \begin{cases} z = \underline{c}^t x \\ Ax = \underline{b} \end{cases} \tag{2.13}$$

We initialize the tableau of values as



Then we proceed (through the pivoting operation) to put the tableau in canonical form:



Optimality Check To check if a solution is optimal we cannot only watch the feasible region, we must take into account also the objective function. We repeat the reasoning made for finding the expressions of BFSs: given $x_B = B^{-1}\underline{b} - B^{-1}N\underline{x_N}$ basic solution and posing x_N equal to zero to find the basic feasible solution, we can express the $\{\min(\underline{c}^t x)\}$ objective function as

$$\underline{c}^t x = (c_B, c_N) * \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}\underline{b} - B^{-1}N\underline{x_N} \\ x_N \end{pmatrix} \tag{2.14}$$

we can now separate

$$\underline{c}^t x = \underbrace{c_B^t B^{-1} b}_{z_0} + \underbrace{(c_N^t - c_B^t B^{-1} N) x_N}_{\overline{c_N}} \tag{2.15}$$

where

- z_0 is a *constant value* and is called cost of the basic feasible solution
- the second term $\overline{c_N}$ is a function *only of the non basic variables* and is the *reduced cost* o the non basic variables

We can also define de vector of reduced costs wrt the basis itself: $\overline{c} = \underline{c}^t - \underline{c}_B^t B^{-1} A$ that is

$$\begin{pmatrix} c_B^t - c_B^t B^{-1} B \\ c_N^t - c_B^t B^{-1} N \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{c_N} \end{pmatrix} \tag{2.16}$$

The reduced costs gives a measure on how much the objective function changes wrt a change in the value of the variable. Also, they provide a sufficent (but not usually necessary) condition for **optimality**: if all reduced costs of non basic variables are non negative, then the basic feasible solution associated to that non negative variables is optimal. So, more formally:
Given $P = \{\underline{c}^t x : [B|N]x = \underline{b}, x \geq \underline{0}\}$

$$\overline{c_N} \geq \underline{0} \Rightarrow (x_B, x_N) \text{ s.t. } x_B = B^{-1}\underline{b} \geq \underline{0} \wedge x_N = \underline{0}, x_B \text{ is optimal} \tag{2.17}$$

2.6 Duality

The **duality** concept for linear programming can be explained as: to any minimization (maximization) LP problem we can associate a closely related maximization (minimization) LP problem *based on the same parameters*. For example, the famous maximum network flow problem is the dual of the minimum capacity cut problem.

Building the Dual Problem

Given a maximization problem $P = \{\max \underline{c}^t x, Ax \leq \underline{b}\}$ any feasible solution provides a *lower bound* of the objective function value. "Which one is the best lower bound"? Can we "flip" the problem to a minimization one?
The basic idea is to find a *linear combination of the constraints* in such a way that the obtained value for the constraints *dominates*[5] the objective function.

[5] has a higher value

So, a new set of variables should be introduced (the coefficents of the linear combination) and should be linked to the *coefficents of the objective value*. These new constraints will create the new feasible region. The objective function the must be changed also: it has to tune each new constraint with the original value of such equation. An example will do the trick.

Example Given

$$\begin{cases} \max z = 4x_1 + x_2 + 5x_3 + 3x_4 \\ \begin{pmatrix} 1 & -1 & -1 & 3 \\ 5 & 1 & 3 & 8 \\ -1 & 2 & 3 & -5 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 55 \\ 3 \end{pmatrix} \\ x \geq \underline{0} \end{cases} \tag{2.18}$$

If we just multiply the second row of matrix A for $\frac{5}{3}$, we obtain a disequality that dominates the objective function. We also obtain such a constraint if we add the second and the third row of the matrix. This two operations generate a *valid constraint* that also *brings information about the objective function*: it gives the obj an upper bound.
Generalizing this reasoning, we can define a linear combination of the constraints, and also linarly combine them with the right hand side vector b, to have actually a whole new set of constraints that is also consistent

$$y_1*(\text{ first line of A })+y_2*(\text{ second line of A })+y_3*(\text{ third line of A }) \leq y_1+55y_2+3y_3 \tag{2.19}$$

Now we can factorize the above equation in order to isolate the x_i variables. Then, we turn back to imposing that the *coefficient for x_i must be greater or equal to the one of the objective function* in order to dominate it.

$$\begin{pmatrix} 1 & 5 & -1 \\ -1 & 1 & 2 \\ -1 & 3 & 3 \\ 3 & 8 & -5 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 4 \\ 1 \\ 5 \\ 3 \end{pmatrix} \tag{2.20}$$

we can see as the last vector is just the $\underline{c}$ vector of the original problem. Given that we're now looking for an upper bound of z, and also the *lowest* upper bound, the dual problem will be a minimization problem; also, every "constraint coefficient" variable will contribute to the objective function proportionally to their right hand side coefficient of the original problem. We can now formulate the full dual problem:

$$\begin{cases} \min v = y_1 + 55y_2 + 3y_3 \\ \begin{pmatrix} 1 & 5 & -1 \\ -1 & 1 & 2 \\ -1 & 3 & 3 \\ 3 & 8 & -5 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 4 \\ 1 \\ 5 \\ 3 \end{pmatrix} \\ y \geq \underline{0} \end{cases} \tag{2.21}$$

General Form of the Dual Problem

As could be derived from the example, the relation between th **primal** and **dual** problem is

$$(P) = \begin{cases} \max z = \underline{c}^t x \\ Ax \leq \underline{b} \\ x \geq \underline{0} \end{cases} \tag{2.22}$$

$$(D) = \begin{cases} \min v = \underline{b}^t y \\ A^t y \geq \underline{c} \\ y \geq \underline{0} \end{cases} \tag{2.23}$$

The duality relation is symmetric: if D is the dual of P, then the dual of D is P.
When dealing with problems in standard form, the non negativity constraint on the vector y is not more needed.
Another formulation for this problem that keeps into account the basic and nonbasic solutions is:

$$(P) = \begin{cases} \min z = c_B x_B + c_N x_N \\ Bx_b + Nx_N = b \\ x_B,\, x_N \geq \underline{0} \end{cases} \tag{2.24}$$

$$(D) = \begin{cases} \max v = yb \\ yB \leq c_B \\ yN \leq c_N \end{cases} \tag{2.25}$$

Weak Duality Theorem

Given the classical formulation for a linear programming problem and its dual:

$$(P) = \begin{cases} \max z = \underline{c}^t x \\ Ax \leq \underline{b} \\ x \geq \underline{0} \end{cases} \tag{2.26}$$

$$(D) = \begin{cases} \min v = \underline{b}^t y \\ A^t y \geq \underline{c} \\ y \geq \underline{0} \end{cases} \tag{2.27}$$

the *weak duality theorem* states that for every feasible solution $\overline{x} \in X$ of (P) and every feasible solution $\overline{y} \in Y$ of (D) we have

$$\underline{b}^t \overline{y} \leq \underline{c}^t \overline{x} \tag{2.28}$$

This is easy to visualize if we imagine the two problems (and in particular, the two feasible regions) as opposite but with the same objective: one is trying to reach the optimal solution by increasing the objective function value, the other doing the opposite reducing it. This leads us to the next important property of dual problems:

Strong Duality Theorem

Given primal and dual problem as in (31) and (32), and given that they are both *bounded* and *feasible*, then if x^* is an optimal solution for (P) and y^* is an optimal solution for (D) we have that

$$\underline{c}^t x^* = \underline{b}^t y^* \tag{2.29}$$

Again, this is just an extension of what we've said for the weak duality theorem: the visualization is also very simple, as shown in the figure

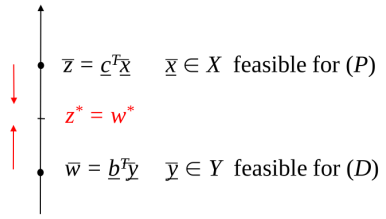


Figure 2.2: Relation between feasible regions and optimal solutions of dual problems.

in the end, we have 4 possible cases out of 9 possible combination of problems, that are explained in the table below:

	∃ an optimal solution	unbounded problem	infeasible problem
∃ an optimal solution	Granted by the strong duality theorem	Not possible (SDT)	Not possible (SDT)
unbounded problem		not possible (WDT)	consequnce of weak du ity theorem
infeasible problem			only possible combinati for empty feasible regio

where obviously columns and rows represent the possible condition of the primal and dual problem. As said, the relation is symmetric (so only half of the matrix is shown) and SDT and WDT means Strong Duality Theorem and Weak Duality Theorem.

Optimality Conditions

The strong duality theorem gives us a powerful tool to prove the optimality of a solution of the linear problem: if we find x^* optimal solution for the primal problem and y^* optimal solution for the dual, it must hold that $\underline{c}^t x^* = \underline{b}^t y^*$. This can be easily proven by substituting the definition of the vectors inside the equation given:

$$\begin{cases} Ax^* = \underline{b} \\ A^t y^* = \underline{c} \\ \Rightarrow \; y^{*t}\underline{b} = y^{*t}Ax^* = \underline{c}^t x^* \end{cases} \tag{2.30}$$

Complementary Slackness

Following along the reasoning that gaves us the new Optimality Conditions, we can see a lso a relation between *slack variables* of optimal solution. Let's say $x^* \in X$ and $y^* \in Y$ optimal solutions for the primal and the dual problem respectively. We can notice that

$$\begin{cases} \forall i \mid y_i^*(a_i^t x^* - \underline{b}_i) = 0 \\ \forall j \mid (\underline{c}_i^t - y^{*t} a_j) x_j^* = 0 \end{cases} \tag{2.31}$$

Where i and j represents respectively the row index and the column index of the A table. We can see that at *optimality*, the product of each variable with the corresponding slack variable of the constraint of the relative dual is null. This *again* gives us a way to easily detect optimality in a solution and to find the dual one: it's enough to check the complementary slackness equations once found a solution.

2.7 Sensitivity Analisys

"How much does an optimal solution change wrt the variations of some of its parameters?"
This is sensitivity analysis, so calculate and quantify the effect produced on the optimal objective function value by a variation in the parameter value. This is useful when adjustments must be made, and we have to choose which parameter ensures the maximum or minimum variation overall. Remember: in sensitivity analysis *everything* is a parameter, from the optimal solution values to the right hand side vector of the feasible region definition.

Optimality Intervals

A direct application of sensitivity analysis is to find the interval in which a basis B remains optimal[6]. In this case, we can vary the cost coefficents and the right hand side termns in order to verify the intervl of validity of a base.

Tweaking the $\underline{b}$ Vector We have the optimal solution $\underline{x}^* = \begin{pmatrix} B^{-1}(\underline{b}) \\ \underline{0} \end{pmatrix}$, we modify *slightly* the solutions vector, we obtain:

$$x^* = \begin{pmatrix} B^{-1}(\underline{b} + \delta \underline{e}) \\ \underline{0} \end{pmatrix} \tag{2.32}$$

where as usual the nullifiedpart of the vector is the one associated to the non basic variables, and e represents a column of the identity matrix. In systesys, we're modifyin just a single entry of the solutions vector, by δ. We just apply the definition of optimality for a basis and have that B *remains optimal as long as*

$$B^{-1}(\underline{b} + \delta \underline{e}) \geq \underline{0} \Rightarrow B^{-1}\underline{b} \geq -\delta B^{-1}\underline{e} \tag{2.33}$$

Obviously, changing the value of the basic vectors impacts the value of the objective function, which goes from $\{\underline{c}^t B^{-1}\underline{b}\}$ to $\{\underline{c}^t B^{-1}(\underline{b} + \delta \underline{e})\}$.

[6] so that $B^{-1}\underline{b}$ remains non negative with x_N set to zero, and th reduced costs of the variables in the basis remain non negative too.

Tweaking the $\underline{c}$ Vector We increment just as before the cost coefficient vector as $\underline{c}' = \underline{c} + \delta\underline{e}$. The basis remains optimal as long as

$$\underline{c}_N^{'t} - \underline{c}_B^{'t} B^{-1} N \geq \underline{0} \tag{2.34}$$

In this case, we're changing the objective function value, but *differently from the previous case* the x^* vector remain unchanged.