

Advanced Algorithms and Parallel Programming

Elia Ravella

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Part I

Advanced Algorithms

1 Algorithms Recall

An algorithm is a well defined computational procedure that takes some value as input and produces some value as output after a finite number of steps.

Algorithm Performance - Complexity A straightforward performance index for an algorithm is his running time. Obviously, the running time depends on the size of the input, so making the performance index parametric wrt the input size is the best way to compare algorithms. Generally, we look for upper bounds of running time, as a guarantee; but three main analysis can be carried out:

- worst case scenario (big O notation): maximum time of the algorithm for an input of fixed size
- average case scenario (Θ notation): expected time for a fixed size input. To complete an analysis like this one, we need assumption on the distribution of the input
- best case scenario (Ω notation): analyze the *minimum time required to terminate* with the best possible combination of input. It's useful because provides a lower bound of performance for an algorithm (we "can't do better" with that algorithm)

When we compare complexity curves, we must not forget the fact that those are dependent from the input instance size. For sufficiently large inputs, then a n^2 complexity always beats an n^3 , but it's not so when instances are small enough. Use complexity to help structure the design phase, not as the one and only performance index to use.

The Master Method To solve recurrences (analysis of complexity for algorithms expressed as mere time functions of the input) of the form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \quad (1)$$

where $a \geq 1$, $b > 1$ and f is asymptotically positive, we can use the *master method*, that gives use the complexity for three common cases, basing itself on the $f(n)$ function:

- $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$. The total algorithm complexity then is

$$T(n) = \Theta(n^{\log_b a}) \quad (2)$$

- $f(n) = \Theta(n^{\log_b a} \log_e^k n)$ for some $k \geq 0$. The total algorithm complexity then is

$$T(n) = \Theta(n^{\log_b a} \log_e^{k+1} n) \quad (3)$$

- $f(n) = \Theta(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$. The total algorithm complexity then is

$$T(n) = \Theta(f(n)) \quad (4)$$

2 Design Paradigms for Algorithms

2.1 Divide and Conquer

Super classic algorithmic approach to problem solving:

1. Divide the problem into simple subproblems
2. Solve the subproblems, if needed recursively dividing them further
3. Combine the results

Merge Sort DeC applied to sorting results in the famous merge sort algorithm:

1. Split the array
2. Sort the two subarrays
3. Linear time merge

resulting in a total complexity of $T(n) = 2T(\frac{n}{2}) + \Theta(n)$ that can be easily worked out with the master theorem: it falls in the second case with $k = 0$. This leaves us with a total complexity of $\Theta(n \log(n))$.

Binary Search DeC applied to searching an element in a sorted array

1. Check middle element
2. Recursively search a single subarray
3. Find element

total complexity: $T(n) = T(\frac{n}{2}) + \Theta(1)$. Again, through the Master Method, we obtain a complexity of $T(n) = \Theta(\log(n))$

Number Power Computing a^n in underlinear complexity: with divide and conquer is possible. Just apply

$$a^n = \begin{cases} a^{\frac{n}{2}} \cdot a^{\frac{n}{2}} & \text{when the exponent is even} \\ a^{\frac{n-1}{2}} \cdot a^{\frac{n-1}{2}} \cdot a & \text{when the exponent is odd} \end{cases} \quad (5)$$

This algorithm has the same complexity of the binary search one: $T(n) = \Theta(n)$

Matrix Multiplication Row per coloumn multiplication of two matrixes, standard algorithm executes in $\Theta(n^3)$, it mimics the "by hand" procedure. DeC approach: consider each matrix as a 2×2 matrix of submatrixes that are "half the lenght", and then recursively go down to the single elements. in the end, add all.

Resulting complexity:

$$T(n) = \underbrace{8T(\frac{n}{2})}_{\text{the 8 resulting submatrixes}} + \underbrace{\Theta(n^2)}_{\text{adding the matrixes together}} \quad (6)$$

With the master method, we still obtain a complexity of $\Theta(n^3)$ (case 1) so no improvement. The winning idea here is to reduce the number of multiplications needed to find the result of the reduced problem, from 8 to 7, using the Strassen's algorithm. This then changes the first term in the complexity description to $T(n) = 7T(\frac{n}{2}) + \Theta(n^2)$. We still fall in the first case of the Master Method, but this time the difference is in the exponent, reduced from 3 to 2,81. The improvement here is noticeable in not-so-small instances of the problem, of course, and become way more significant when the instances dimension grows.

3 Randomized Algorithms

Randomized algorithms exploits some probabilistic features of the inputs in order to run "at best on the average case". We call a randomized algorithm an algorithm that depends not only on the input it's given, but also on a random number generated inside the algorithm.

To actually evaluate the performances of the algorithm we make use of *indicator random variables* that are simple binary variables that indicates wheter something is happened or not.

3.1 Monte Carlo Algorithms

Monte Carlo algorithms are randomized algorithms that *can return a wrong result*, but with little probability. In the case this probability is non null, the problem is said to have a *two sided error*.

3.1.1 Las Vegas Algorithms

Las Vegas algorithms are randomized algorithms that (differently from Monte Carlo ones) *always return correct results*, but they directly depends on some random choices; this implies that the running time can vary even with identical input, thus rendering the *average running time* very difficult to compute.

3.1.2 Karger's min-cut algorithm

Minimum cut problem: to find the cut (on a connected undirected weighted graph) with the *minimum possible capacity*. The general algorithm computes $n - 1$ times the so called "st-cut problem", that only considers the "weight flow" between two fixed nodes. Karger's approach is based on randomization and contractions¹: by contracting together nodes we will eventually find a cut of the graph, that will be represented by the arcs that go through the "central most contracted node".

So the Karger algorithm translates to:

1. pick a random edge
2. contract it (doing so, remove self-loops)
3. repeat 1 and 2 until the graph has only two nodes
4. check the obtained cut
5. repeat in case

The average complexity is far less than the linear programming formulation of the problem. Moreover, statistically speaking and given enough runs, the Karger's algorithm finds the solution far before the "traditional" linear programming algorithm, that runs in $O(n \cdot m \cdot \log(\frac{n^2}{m}))$ where n and m represent nodes and edges number, respectively.

An improved version of this algorithm was provided by Karger himself, after the observation that is very unlikely to pick a min cut edge *at the beginning* of the algorithm wrt the end of the procedure. The thing that changes here is the last step of the algorithm: instead of restarting from the beginning (so from the complete graph) it restarts from an *intermediate graph* with a reduced number of nodes and then compares the two obtained results. This improvement is huge: it reduces the running time to $n^2 \cdot \log(n)$, with an error probability that's polynomial.

3.1.3 Randomized Quicksort

Classic quicksort is an $O(n \log(n))$ in place sorting algorithm, using the divide and conquer approach. The algorithm can be seen as an "in place merge sort": it divides the algorithm around a **pivot** element and recursively sort the two portions of array.

The random formulation introduces randomness in the pivot selection, in particular in the selection of the indexes that produces the pivot. This ensures that the running time is *independent* of the input order. We end up with an average complexity of $n \log(n)$.

```
# randomized adjustment to quicksort algorithm, where the pivot is
# randomly chosen each time
def rand_partition(arr, bottom, top)
    pivot = Random.new.rand(bottom..top)

    swap = arr[pivot]
    arr[pivot] = arr[top]
    arr[top] = swap
    return partition(arr, bottom, top)
end

# partitioning and swapping subroutine
# this just moves the lower values to the left
```

¹make two nodes coincidental in a graph, "collapse"

```

# and the higher to the rights
def partition(arr, bottom, top)
    last = arr[top]
    first = bottom - 1

    for current in bottom..(top - 1)
        if arr[current] <= last
            first = first + 1

            # swap elements
            swap = arr[current]
            arr[current] = arr[first]
            arr[first] = swap
        end
    end

    # swap the highest left value with the right value
    # from the pivot point of view
    swap2 = arr[first + 1]
    arr[first + 1] = arr[top]
    arr[top] = swap2

    return (first + 1)
end

# simply divide
def quicksort(arr, bottom, top)
    if bottom < top
        pivot = rand_partition(arr, bottom, top)
        quicksort(arr, bottom, pivot - 1)
        quicksort(arr, pivot + 1, top)
    end
end

```

Complexity analysis of quicksort and randomized quicksort: we can see how the PARTITION subprocedure, when it's not randomized, can yield to an optimal case, where the array is split in half, and a worst case, where the array is split in a $n - 1$ long subarray and another of length zero. So the randomized approach ensures that the recursion tree of this procedure is always $\Theta(\log(n))$ and having $O(n)$ work done at each level we end up with $O(n \cdot \log(n))$.

We should focus on the PARTITION subprocedure because it's the core of the complexity of the entire algorithm, and because it has two interesting properties:

1. the time spent in there dominates the complexity of the algorithm
2. it's called *at most* n times

the call itself is a $O(1)$ operation plus the complexity induced by the **for** loop that swaps the elements in the array. So, the total complexity of the algorithm *heavily depends* on the number that the instruction **if** `arr[current] <= last` is executed. Our aim is this, to compute X the number of comparisons performed across all calls to PARTITION. So the total complexity of the algorithm is $O(n + X)$. X_{ij} is so the indicator binary variable for "element i is compared with element j ". Obviously, $X = \sum_{i=1}^{n-1} \sum_{j=1}^n X_{ij}$. If we apply the expected value operator to this formula we are left with the probability that element i is compared with element j .

This is:

$$\begin{aligned}
 Pr\{z_i \text{ is compared to } z_j\} &= Pr\{z_i \text{ or } z_j \text{ are chosen as pivot}\} = \\
 &= \frac{1}{\text{number of elements in subarray}} \times 2 = \\
 &= \frac{2}{i - j + 1}
 \end{aligned}$$

Substituting this equation in $E[X] = E[\sum_{i=1}^{n-1} \sum_{j=1}^n X_{ij}]$ we obtain

$$\begin{aligned} E[X] &= \sum_{i=1}^n \sum_{j=1}^{n-1} \frac{2}{i-j+1} \\ &< \sum_{i=1}^n \sum_{j=1}^{n-1} \frac{2}{i-j} \\ &= \sum_{i=1}^{n-1} O(\log(n)) = O(n \cdot \log(n)) \text{ armonic series} \end{aligned}$$

3.1.4 Randomized Selection - Order Statistics

The algorithm can be described as "find the i -th smallest/greatest number of a collection", so the element with rank i . The naive idea is: sort the collection ($\Theta(n \cdot \log(n))$) then take the i -th element ($\Theta(1)$). Total complexity: $\Theta(n \cdot \log(n))$.

There's an algorithm, derived from quicksort², with expected running time in $\Theta(n)$ to find the i -th element, which has $\Theta(n^2)$ as worst case complexity.

The algorithm is quite simple: it uses the RANDOMIZED-PARTITION function of the quicksort algorithm in order to partition the array around a pivot element (chosen at random): after that, all the smaller elements will be on the left and the greater on the right. Then, it checks if the *order statistic* is equal to the number of elements in the "lesser" subarray, adding 1 to this value to keep track of the pivot element itself. If they're equal, we've found our element. If not, we resort to check a single subarray of the two:

- the greater one if the order statistic is greater than the number of elements in the first subarray
- the smaller one if not

```
def select_ith(arr, head, tail, order_statistic)
    return arr[head] if head == tail

    # so now we've partitioned the array
    pivot_index = rand_partition(arr, head, tail)

    # tentative represents the k-th index of the array
    tentative = pivot_index - head + 1

    if tentative == order_statistic
        return arr[pivot_index]
    elsif order_statistic < tentative
        return select_ith(arr, head, pivot_index - 1, order_statistic)
    else
        return select_ith(
            arr,
            pivot_index + 1,
            tail,
            order_statistic - tentative
        )
    end
end
```

The complexity is given by the recurrence

$$T(n) \leq \sum_{k=1}^n (X_k \cdot (T(\max(k-1, n-k))) + O(n)) \quad (7)$$

Where

²it even uses the same RANDOMIZED-PARTITION function

- X_k is a binary variable that is 1 for the k th partition when it produces a k elements split. This variable is here to help with the expected value analysis, because it has $E[X_k] = \frac{1}{n}$.
- $T(\max(k-1, n-k))$ is the *pessimistic assumption* of this recursion: in order to calculate an upper bound, we hypotize that the element we're looking for is always in the bigger subarray

$$\begin{aligned} \sum_{k=1}^n (X_k \cdot (T(\max(k-1, n-k))) + O(n)) &= \sum_{k=1}^n (X_k \cdot (T(\max(k-1, n-k)))) + O(n) \\ \Rightarrow E[T(n)] &= \sum_{k=1}^n \left(\frac{1}{n} \cdot E[(T(\max(k-1, n-k)))] \right) + O(n) \end{aligned}$$

The problem here is that max function in the center of the recurrence. But, we can analyze the two subproblems to get rid of it. We know that that function will return

$$\max(k-1, n-k) = \begin{cases} k-1 & \text{when } k > \lceil \frac{n}{2} \rceil \\ n-k & \text{when } k \leq \lceil \frac{n}{2} \rceil \end{cases} \quad (8)$$

if n is even, being k spanning from 1 to n , we have that the first $\frac{n}{2}$ terms appears exactly twice, so $\max(k-1, n-k) = 2k$. If not, we simply have an "extra term" that is $\lfloor \frac{n}{2} \rfloor$.

We end up with

$$E[T(n)] = \frac{2}{n} \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} (T(k)) + O(n) \quad (9)$$

where $T(k) = O(n)$ can be proven by substitution.

3.1.5 Primality Test

For "primality test" is intended the boolean procedure to verify if a number is actually a prime number (so is divisible only for itself and 1). The super naive approach is to check if the number is 2, check then parity, then cycle through 3 and $\frac{\sqrt{n}}{2}$ to find a evenly divisor of the considered number.

```
def is_prime(n)
  return false unless n.is_a? Integer
  if n%2 == 0
    return false
  else
    (3..(Integer.sqrt(n)/2).floor()).each {
      |d|
      return false if n%(2*d + 1) == 0
    }
  end
  true
end
```

This algorithm has a $\Theta(\sqrt{n})$ complexity.

The randomized version is a one-sided error, false biased Monte Carlo algorithm: the result is always correct if negative, and can be wrong (with a little probability) if it's positive. This exploits the Fermat's Theorem: "each odd prime numbers divides $2^{p-1} - 1$ ". So a polynomial approach can be: calculate $z = 2^{n-1} \bmod n$ and check if it's 1: then the number it's possibly prime, otherwise surely isn't (it's composite). This test is also called pseudoprimalty test, because numbers that satisfies such property are called *base-a pseudoprimes*, where a is the base of the power(2, in our case). The probability to find a pseudoprime that's not prime in a β -bit encoded number tends to zero when $\beta \rightarrow \infty$. In a 1024-bit encoded number, the probability of random picking a base-2 pseudoprime that is not prime are less than 1 in 10^{41} : this makes this version of the algorithm useful (and insanely fast) for most of the usual applications.


```
def is_pseudoprime(n)
  return false unless n.is_a? Integer
  return (2**(n - 1) % n == 1)
end
```

We actually cannot get rid of the small error margin of this algorithm, due to Carmichael numbers³.

Miller Rabin Las Vegas Version Actually, a Las Vegas version of the primality test does exist, and it's a modification of the previous algorithm that introduces randomization. The two main changes are:

- base a is not fixed, but chosen at random. Also, in an iteration several values are tried
- It performs an additional "nontrivial square root" check, that if succeeds signals a composite number

The nontrivial square root check is fairly simple, as the definition of a nontrivial square root is: a is a nontrivial square root of $1 \bmod n$ if $a^2 \bmod n = 1 \wedge a \neq 1 \wedge a \neq n - 1$. Do not be scared! That only means that a is the square root of a number that's **modulo equivalent** to $1 \bmod n$. We will use this to prove that a number is not composite. Here's some Ruby code that implements that:

```
is_probably_prime = false

def power(base, iter, number)
  # recursive computation of the power of the number
  # in order to find a nontrivial square root

  return 1 if iter == 0

  # computation of the factors
  x = power(base, iter/2, number)
  result = x**2 % number

  # nontrivial square root check
  if (result == 1 and x != 1 and x != number - 1)
    is_probably_prime = false

  # result computation: notice how variable "result" is recomputed
  result = (base * result) % number if iter % 2 == 1

  return result
end

def MR_primality_test(n)
  base = Random.new.rand(2..(n - 1))

  res = power(base, n - 1, n)

  is_probably_prime = true

  return false if res != 1 or !is_probably_prime
  true
end
```

³I don't want to go down this algebraic-arithmetic rabbit hole, ok?

4 Sorting Lower Bounds and Linear Sorting Algorithms

4.1 Comparison Sorts and Lower Bounds

Comparison sort algorithms are algorithms that swaps elements of an ordered collection comparing them one another, to eventually order it. We could see this as a continuous exploration of the "permutations space" of the original collection, until we find the right one. Being a systematic procedure, sorting can be represented as a **decision tree traverse** procedure. This procedure is useful to calculate the theoretical lower bound in complexity for the comparison sort algorithms. If we build a decision tree where every node corresponds to a comparison of our sorting algorithm, and every leaf a permutation of the initial collection, we end up with a representation of all the decisions/comparison that the algorithm should face until it can reach the solution, and the height of that tree would be the worst case running time (each path from the root to a leaf represents an algorithm run). Since the tree must contain more than $n!$ leaves, we have that our decision tree⁴ has $2^h \geq n!$ leaves, where h is the height of the tree. Concisely, a lower bound in the height of the tree corresponds to a lower bound in the number of comparisons needed to sort an array. This leads to the equality

$$h = \Omega(n \cdot \log(n)) \quad (10)$$

Coming from

$$n! \leq 2^{\text{height}} \Rightarrow \text{height} \geq \log(n!) \Rightarrow \underbrace{\text{height} \geq \log\left(\left(\frac{n}{e}\right)^n\right)}_{\text{Stirling's formula}} \Rightarrow \Omega(n \cdot \log(n)) \quad (11)$$

4.2 Linear Time Sorting

The counting sort and the radix sort are two algorithms that sort collections in linear time. How is this possible? They do not rely on *comparisons* and they make strong assumptions on input types.

Counting Sort Counting sort is an *array*-sorting algorithm that deals with *small positive integer numbers*. "Small" in the sense that there should be an upper bound to them. The algorithm uses an auxiliary array where it accumulate the occurrences of every element of the original one (populating the cells of the aux. array by using the original integers as indexes). When the first traversal is completed, we start from the bottom of the sorting array and we confront the value of the element with the value in the aux element associated to it (so that uses that value as index). We put that value at the index pointed, and we decrease the counter (in the second array). At the end of the second traversal, we've sorted the array. $O(n + k)$ complexity, where k is the highest integer number in the array.

Possible ruby implementation (pay attention to the 4-loops construction of the algorithm):

```
def counting_sort(arr)
  auxiliary = Array.new (arr.max + 1)
  output = Array.new arr.length

  # set structures to zero
  auxiliary.map! { |x| x = 0 }
  output.map! { |x| x = 0 }

  # record positions
  arr.each do |i|
    auxiliary[i] += 1
  end

  # accumulate
  i = 0
  while i < auxiliary.length - 1
```

⁴that's a BINARY tree

```

        auxiliary[i + 1] += auxiliary[i]
        i += 1
    end

    # reconstruct array
    i = arr.length - 1
    while i >= 0
        output[
            auxiliary[
                arr[i]
            ]
        ] = arr[i]
        auxiliary[arr[i]] -= 1
        i -= 1
    end

    return output
end

```

Radix Sort Radix sort exploits digit-by-digit sort, using counting sort to sort the digits. The original implementation sorts starting from the most significant digit, but the latter on (the right one) sorts starting from the least significant digit. The application is quite straightforward: counting sort is applied *to the whole number* but considering *only the least significant digit not yet analyzed* for the sortings. The stability of the counting sort (property to preserve relative order between equal numbers) ensures that the final array is sorted.

5 Advanced Data Structures

5.1 Disjoint Sets

Disjoints set are a *collection of collections* where all the subcollections are *disjoint* (no element is in more than one set) and each subset is represented from an element of that set. Usually the choice of the representative element is not crucial to the functioning of the structure.

5.1.1 Operations on a Disjoint Set

Disjoint sets must be capable of performing these operations:

1. **MAKE-SET(x)** the basic wrapper, that puts "x" into a set. Naturally, "x" should not be in another set.
2. **UNION(x, y)** performs the set union of the two sets that contains x and y , S_x and S_y . This operation removes S_x and S_y from the structure (their elements would be duplicated) and merges the two sets into a new one. A representative is either chosen a priori or recomputed.
3. **FIND-SET(x)** returns a reference to S_x .

5.1.2 Implementation: Linked List

A super easy way to implement disjoint sets is through linked lists, where each list represents a subset. The structure for each set holds a pointer to the head and one to the tail of the list, to facilitate operations (as **UNION**).

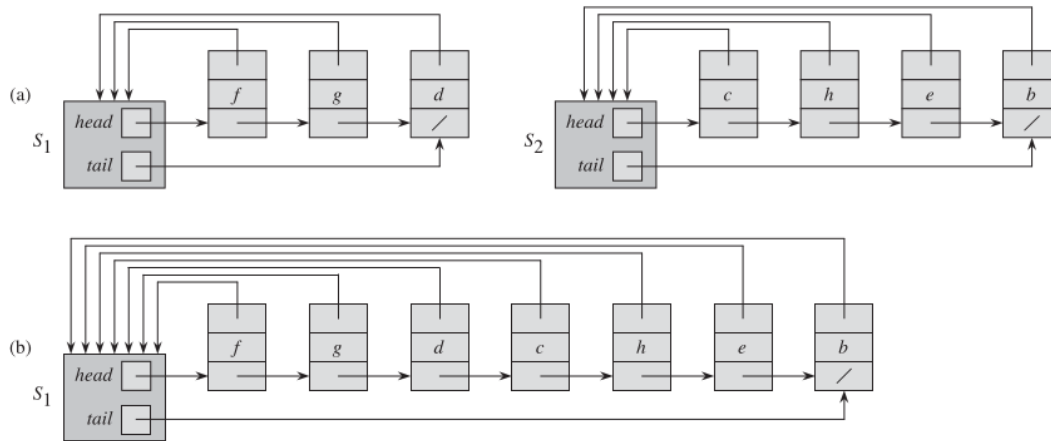
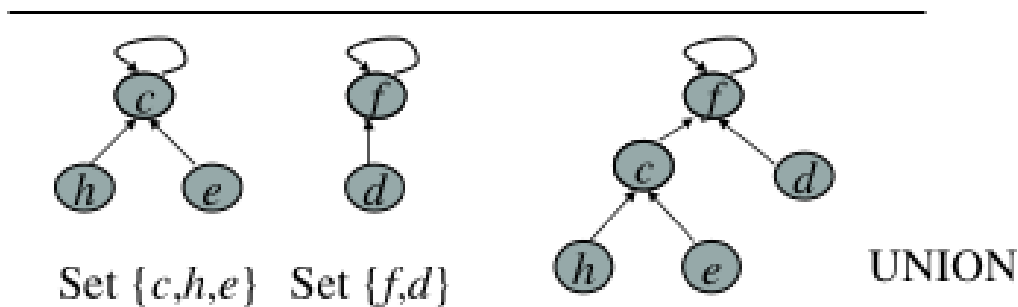


Figure 1: A union performed on a disjoint set implemented through linked lists

The union operation takes $\Theta(n^2)$. Can we do better?

5.1.3 Implementation: Forest

Instead of using linked lists to represent the sets, use instead a tree (also not binary) that has the set-representative in the root element. A disjoint set will then be a forest, where each tree is a set. This time the FIND-SET operation takes $O(\log(\text{element of the subset}))$ and the UNION is straightforward: just "append" a tree to the other's root, $O(1)$



I've spent a week on this DisjointSet class in Ruby, using no optimizations. So, here it is:

```
class DisjointSet
  attr_accessor :sets # forest for sets

  def initialize
    @sets = Array.new
  end

  def make_set(element)
    if (sets.filter { |i| i.has(element) }).length == 0
      @sets.push(Tree.new(element))
    end
  end

  def find_set(element)
```

```

    i = 0
    while i < sets.length
      return sets[i] if sets[i].has(element)
      i = i + 1
    end
    return nil
  end

  def set_insert(representative, element)
    i = 0
    while i < sets.length
      if sets[i].root.value == representative
        @sets[i].insert(element)
        break
      end
      i = i + 1
    end
  end

  # unite the two sets corresponding to the elements
  def union(element1, element2)
    set1 = find_set(element1)
    set2 = find_set(element2)

    return nil if set1.nil? or set2.nil?

    set1.insert_node(set2)

    sets.delete(set2)
  end

  def show
    for i in sets
      print "->"
      i.show
    end
  end
end

```

This implementation can be furtherly optimized, using ranking and path compression:

- Union by ranking: when uniting trees, always look at the rank of the tree itself (that's the height of the tree) and attach the "smaller" to the "greater"
- Path compression: contextually a FIND-SET operation, make all the subtrees' root point directly to the representative (reducing total tree height)

applying both these optimizations (that are merely *heuristics*) we can achieve a worst case running time of $O(m \cdot \alpha(n))$, where $\alpha(n)$ is the inverse Ackermann function (approximable to 4 for most practical application): in most of use cases, it's a linear time running time in the worst case⁵.

5.2 Random Data Structures

Dictionaries A dictionary is a close parent of an hash map, where the values are stored associated to a unique key. Classic example, th DB table with a unique key for each entry. We're introducing fastly dictionaries (and the interface to them: SEARCH(**x**), INSERT(**x**), DELETE(**x**), pretty self explicative. This interface expands can be expanded with union-like and split-like operations.) in order to analyze *random data structures*, strcutres that randomly manage their internal structure. Actually, Skip Lists and Random Treaps are *implementations of dictionaries*.

⁵and that's cool

5.2.1 Random Treaps

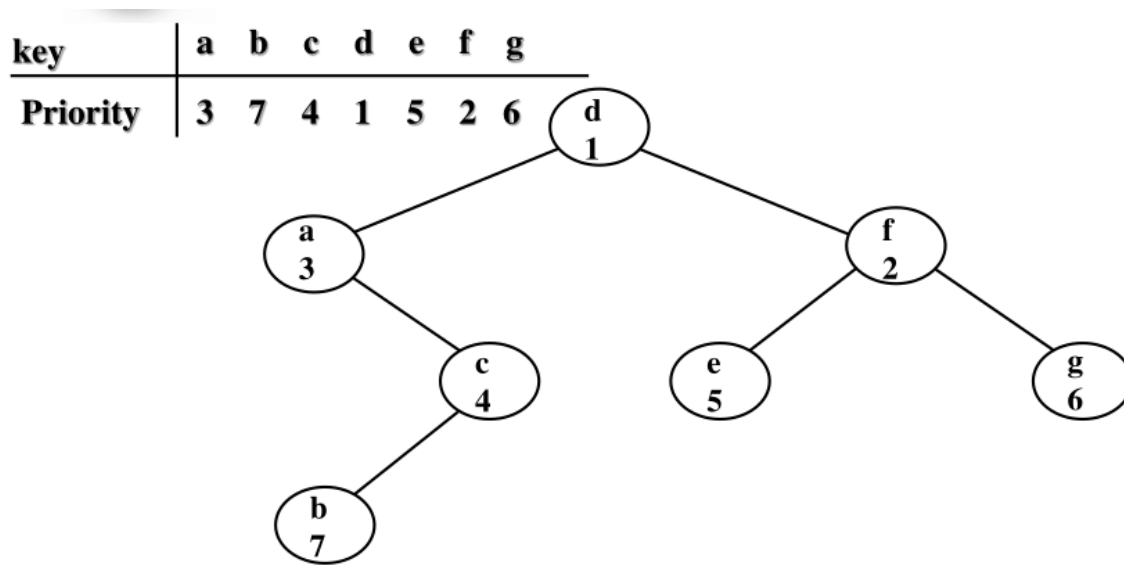
Treaps are binary random trees, *that do not use additional data*, simple to implement and with a limited number of rotations needed per operation (on average).

In a treap, each node contains a element x with its random priority $prio(x)$. They're organized as a binary search tree using the keys, but they also follow a heap-like property: all childs have a priority that's less than their parent priority. An interesting property of treaps is their uniqueness: give elements x_i with $i \in 1..n$ and their priorities, they're represented by a unique treap representing this data⁶.

Treaps: Heaps + Trees Formally, a treap is a binary search tree where both the tree property and the heap property hold:

- for each node, right children have a value higher than the node
- for each node, the priority of the node is always higher than the children's one

As it's clear, each node contains two values: an actual "value" that is stored inside the structure, and a "priority score" that ensures that the heap property is maintained.



Here's a little implementation of Treaps written in Ruby

```
class Treap
  attr_accessor :root

  def initialize(root_node)
    if root_node.is_a? TreapNode
      @root = root_node
    else
      throw "Not a valid node"
    end
  end

  # insert a value in the treap:
  # tree-insert + heap-property-restore
  def insert(value, priority)
    newNode = tree_insert(value, priority, root)
    heap_restore(newNode)
  end
end
```

⁶Proven by induction.

```

private # -----

# binary search tree insert operation
# right values higher
def tree_insert(value, priority, node)
  if value > node.value
    if node.right.nil?
      node.right = TreapNode.new(value, priority)
      node.right.father = node
      return node.right
    else
      tree_insert(value, priority, node.right)
    end
  else
    if node.left.nil?
      node.left = TreapNode.new(value, priority)
      node.left.father = node
      return node.left
    else
      tree_insert(value, priority, node.left)
    end
  end
end

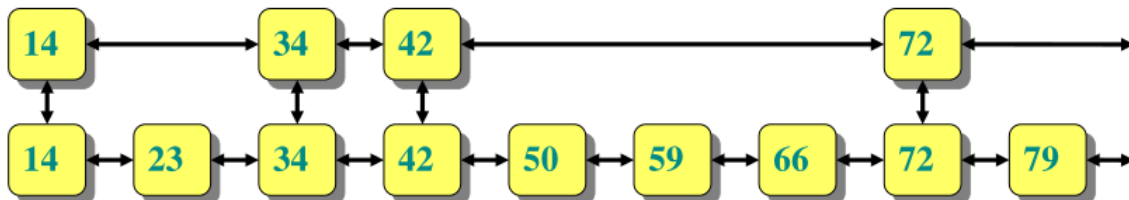
# heap property restoring subroutine
def heap_restore(leaf)
  if leaf != root and leaf.priority > leaf.father.priority
    # need to rotate the tree
    if leaf == leaf.father.left
      rotate_right(leaf)
    else
      rotate_left(leaf)
    end

    heap_restore(leaf.father)
  end
end

```

5.2.2 Skip Lists

Skip lists are sequential data structures (like an ordered array) that can on average maintain $O(\log(n))$ complexity for most operations. The idea is pretty simple: we add an *additional pointer* to some nodes that makes you "skip" portions of the structure when followed.



This creates an "overlay list" where some passages make you skip entire portions of the list, saving some accesses. The **SEARCH(x)** operation starts in the fast line, then "descends" in the lower line when we're close to the x element.

The ideal skip list has half the elements per line wrt the lower line, alla evenly spaced: obviously

managing such a structure during a `INSERT` operation is a mess, so we can randomize something in order to statistically improve the performance of this structure:

Random Skip List On an insert in the bottom level, we *upgrade* the element also to the upper level with a 50% probability, and again for upper level with 50% probability and so on; this makes 25% the probability to be raised to the second level, 12,5% to the third etc. This implementation of skip lists gives us a $O(\log(n))$ search complexity for all search with high probability.

Part II

Parallel Programming

6 Machines and Models

We need a model for our machine in order to design our algorithm in a complete and coherent way, wrt the architecture they will be run on. The most used one is the RAM, or Random Access Machine

Random Access Machine A RAM is characterized by:

1. unbounded number of local cells
2. unbounded cell dimension
3. an instructions set
4. all operation are executed in the same time

These properties of the RAM define the time complexity as the number of intructions executed and the space complexity as the number of memory cells used (for an algorithm, of course).

6.1 Parallel Random Access Machine

The pretty straightforward extension to embed parallelism into a RAM is to multiply the total number of processors in it. To be more general possible, we identify each processor as a single RAM, and we add a number of input and output cells, alongside a *shared memory bank of cells*. Formally, a PRAM is an unbounded collection of RAM processors, with shared memory cells. Given that each operation (read from input, write on output read or write in memory) is done simultaneously by all processors, we have to account for shared memory conflicts. The models of PRAM to avoid conflicts are:

- **Exclusive read:** all processors can read simoultaneously from distinct memory location
- **Exclusive write:** all processors can write simoultaneously to distinct memory location
- **Cuncurrent read:** all processors can read simoultaneously from all memory location
- **Cuncurrent write:** all processors can write simoultaneously to all memory location; which value "wins the race"? Priority methods can be adopted, or allowing cuncurrent writes only if the value is the same from all processors...

The PRAM is to parallel algorithms what the RAM is to simple algorithms: a natural abstraction of a machine that allows equal performance evaluation in a simple manner.

6.2 Extension of Complexity to PRAM

We'll extend the basic definitions to the parallel world:

- $T^*(n)$ complexity for a sequential algorithm on a single processor
- $T_p(n)$ complexity on p processors
- $SU_p(n) = \frac{T^*(n)}{T_p(n)}$ speedup
- $E_p(n) = \frac{T_1(n)}{pT_p(n)}$ effecency: work on 1 over work that can be carried out on p
- T_∞ shortest run time
- Cost: $P(n) \times T(n)$

6.2.1 Matrix Vector Multiplication

Classic $Ax = b$ problem of multiplying matrix A with vector x . The problem is parallel *per se*: every element in the result vector is calculated independently from the others. We split the computations following this concept: every processor will have to read a line of matrix A and all vector b and compute the polynomial value for the result.

Performance evaluation:

- $T_1(N^2) = O(N^2)$
- $T_p(N^2) = O(\frac{N^2}{p})$ linear speedup
- $E_p = \frac{n^2}{p \times \frac{n^2}{p}} = 1$ perfect efficiency

6.2.2 Matrix Matrix Multiplication

Conceptually, just an extension of the previous problem to support matrix per matrix. If n is the length of the matrix (suppose squared) we need n^3 processors⁷. The idea is that each processor calculates the single element of the resulting matrix, so reading a row and a column of the two factors.

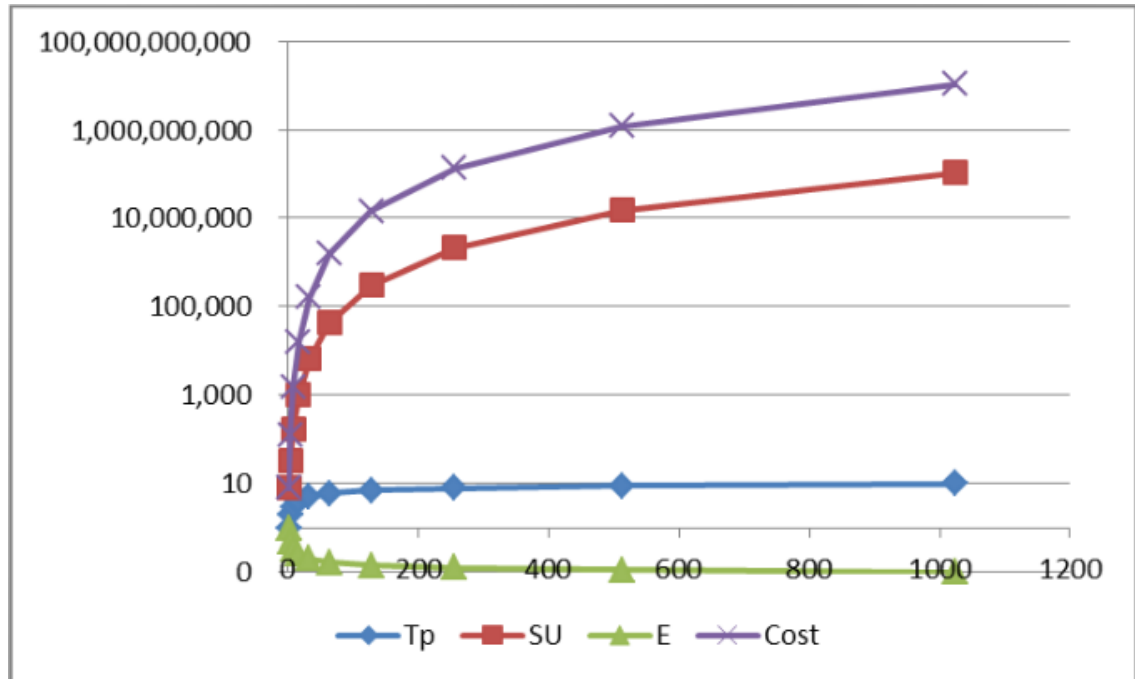
The algorithm:

1. Read row of A and column of $B \rightarrow$ concurrent read
2. multiply elements and sum all together (using n processors for n elements \rightarrow logarithmic cost for the sum)
3. store values \rightarrow exclusive write

Performance:

- $T_1 = n^3$
- $T_{p=n^3} = \log n$ the reduced complexity to $\log n$ is due to the possibility to use n processors to reduce the complexity of a sum to the logarithm of its size
- $SU = \frac{n^3}{\log n}$
- $E_{n^3} = \frac{1}{\log n}$

graph of the performance indexes wrt the number of processors:



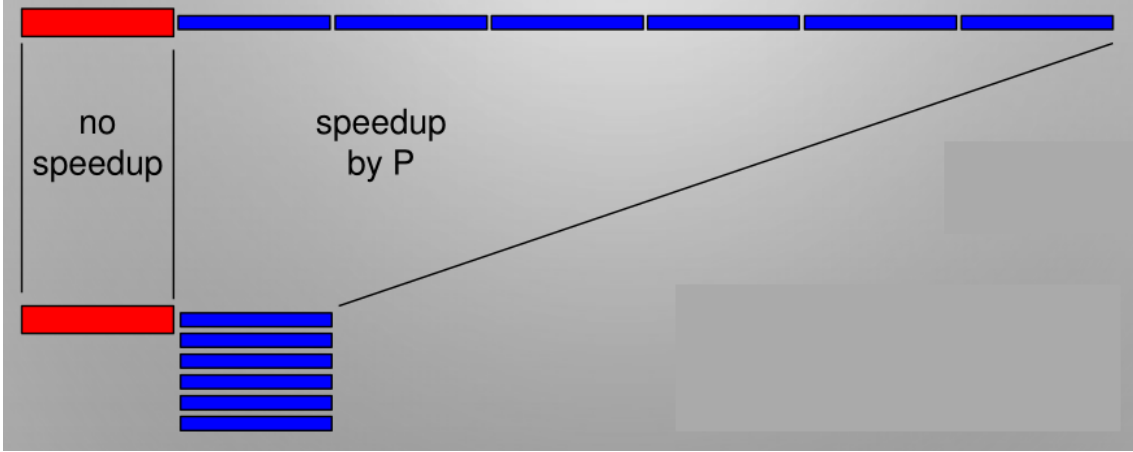
⁷ n processors for the sum, n^2 total sums

7 Amdahl's Law and Gustafson's Law

Amdahl's law is a mathematical formulation for calculating the speedup induced by the introduction of parallelism (multiple processors). The code is divided in *serial* and *parallelizable* segments and the speedup is calculated as

$$\begin{cases} f \text{ the length of the parallelizable part} \\ 1 - f \text{ length of the serial part} \\ P \text{ number of processors} \\ SU(P, f) = \frac{T_1}{T_1 * (1-f) + \frac{T_1 * f}{P}} = \frac{1}{1-f + \frac{f}{P}} \end{cases} \quad (12)$$

The formula is easily visualizable from



where the length of the blue part is f .

This formulation can be used to calculate the speedup upper limit wrt to the number of processors:

$$\lim_{P \rightarrow \infty} SU(P, f) = \frac{1}{1-f} \quad (13)$$

This limit is "pessimistic": for 90% parallelizable problems, Amdahl's law suggests to use no more than 10 processors. Gustafson's formulation "inverted" the reasoning, claiming that the parallelizable part could (more likely) not be of "fixed" length, while this is more likely to happen for the serial one. So he proposed the formula

$$SU = \frac{s + P \cdot (1-s)}{s + (1-s)} = s + P \cdot (1-s) \quad (14)$$

where s represents the length of the serial part.

8 Parallel Algorithms Design and Guidelines

Most problems have a parallel solution. Given this, designing a parallel algorithm that solves a problem is not easy task. We differentiate between

1. Parallel algorithms: the actual algorithm that solves the problem
2. Parallel programs: the target specific implementation of a parallel algorithm

8.1 PCAM Design Methodology

PCAM is the acronym for

1. **Partitioning**: task and data decomposition. The extremization of the "divide" part of the "divide and conquer" approach: the goal of this stage is to identify a large number of subtasks that will be executed in parallel. We take into consideration both the data and the functions:

- (a) Domain decomposition: the data associated to a problem is decomposed (an image into singular color layers, for example) and then the tasks working on separate data are parallelized
- (b) Functional decomposition: the tasks are separated discriminating on the function they carry out, rather than the data they use. The perfect example is the model of an ecosystem: it's easier to model it by separating entities into "plants, carnivores, herbivores" etc rather than splitting the data domain (that's also very difficult, in this case). Pipelining is a kind of functional decomposition approach.

When partitioning, it's important to define "a lot"⁸ of evenly sized tasks⁹.

2. **Communication:** task execution coordination. Tasks need to coordinate in order to achieve the computation target. The communication can be categorized with various metrics: local or global, structured or unstructured (regarding the *internal organization among peers*), static or dynamic (if the peers pool is *already defined or can be discovered*), explicit or implicit (message passing or shared memory), synchronous or asynchronous, point to point or collective.
3. **Agglomeration:** evaluation of the structure. Clustering together tasks and combining them into larger ones is an intermediate "reduce" step. This reduces overhead because it reduces *communication* and avoids a lot of *data replication*. The agglomeration step is performed when we want (for example) to better exploit the locality of computation or data.
4. **Mapping:** resource assignment. At this point, we have decomposed all the tasks, designed the communication between them, clustered some tasks. We end up with a some sort of task graph and we can start to allocate tasks to resources. Distributing tasks on a system of resource is an NP-complete problem, due to the dependency to the communication systems, the single node computational power and the single task complexity.

PCAM is not a sequential process: we may have agglomeration based on possible mappings or communication designed before partitioning.

Parallel Problems are Complex The main steps when writing a parallel program are

1. Analyze the data dependencies of the problem to identify *possible* parallelizations
2. Identify hotspots, so points where most of the work is actually done. Not parallelizing hotspots can lead to bottlenecks
3. Estimate performance against sequential solutions
4. Repeat from 1)

most of the parallelism is identified by hand, a few tools can design a tasks graph or a node network automatically.

⁸a order of magnitude more than the processor count.

⁹in order to fully exploit the parallelization