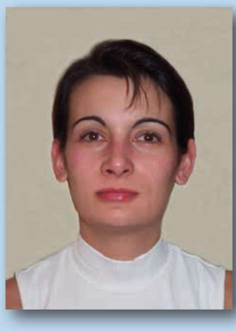
originalni
naučni
rad

UDK 336.781.5 ; 338.51



dr Nataša Kožul
Samostalni ekspert i konsultant za investiciono bankarstvo
nkozul@gmail.com

PREGLED JEDNOFAKTORSKIH MODELA KAMATNIH STOPA SA FOKUSOM NA MODEL HAL I VAJT

Rezime

S obzirom na to da kamatne stope utiču na vrednovanje i utvrđivanje cene svih finansijskih proizvoda, velika pažnja je posvećena prognoziranju kamatnih stopa i konstruisanju krive prinosa. Međutim, derivati kamatnih stopa zavise od razvoja kamatnih stopa u smislu sadašnje i buduće vrednosti novca, kao i kroz njihovu zavisnost od od kamatne stope kao osnovne "aktive". Otuda je razvijeno nekoliko klasa modela za utvrđivanje cene derivata kamatnih stopa. Zasnivajući se na varijabilama koje pokušavaju da obuhvate, oni se u širem smislu klasifikuju kao modeli trenutne stope (kratkoročni) ili tržišni modeli. Mada su kratkoročni modeli zasnovani na nevidljivoj varijabli, dok modeli tržišne stope neposredno odražavaju tržišne stope i cene, ti kratkoročni modeli se široko koriste za utvrđivanje cene derivata, pri čemu se - zavisno od svoje složenosti i imlicitnih pretpostavki i ograničenja - dalje klasifikuju kao jednofaktorski i dvofaktorski modeli. Ovaj rad daje kratak prikaz teorije koja se nalazi u osnovi jednofaktorskog modeliranja kamatnih stopa, sa posebnim fokusom na prednosti i nedostatke Halovog i Vajtovog modela.

Ključne reči: utvrđivanje cene derivata kamatne stope, jednofaktorsko modeliranje, Hal i Vajt

JEL: C52, E43, G12

Rad primljen: 04.02.2013.
Odobren za štampu: 18.02.2013.

original scientific paper

UDC 336.781.5 ; 338.51

OVERVIEW OF ONE-FACTOR INTEREST RATE MODELS WITH THE FOCUS ON HULL AND WHITE

Nataša Kožul MSc, PhD

Independent expert and investment banking consultant
nkozul@gmail.com

Summary

Given that interest rates affect the valuation and pricing of any financial product, much attention has been devoted to interest rate forecasting and yield curve construction. However, interest rate derivatives depend on the evolution of interest rates both in terms of present and future value of money, as well as through their dependence on the interest rate as the underlying 'asset'. Consequently, several classes of models have been developed for pricing interest rate derivatives. Based on the variables they are attempting to capture, they are broadly classified as instantaneous rate or short-rate models, and market models. Although short-rate models are based on an unobservable variable, whilst market rate models directly reflect market rates and prices, the former are widely used in derivatives pricing, whereby - depending on their complexity and implicit assumptions and limitations - they are further classified as one-factor or two-factor models. This paper gives a brief overview of the theory underlying the one-factor interest rate modelling, with the specific focus on the advantages and disadvantages of Hull and White model.

Keywords: interest rate derivatives pricing, one-factor modelling, Hull and White

JEL: C52, E43, G12

Paper received: 04.02.2013

Approved for publishing: 18.02.2013

Jednofaktorsko modeliranje kamatnih stopa

Jednofaktorski modeli kamatnih stopa imaju samo jedan izvor nasumičnosti (tj. neizvesnost ili tržišni rizik). Oni modeliraju kratkoročnu stopu (spot stopu), vodeći do parcijalne diferencijalne jednačine cena različitih klasa derivata kamatne stope. Spot stopa (poznata kao trenutna ili kratkoročna) predstavlja nevidljivu veličinu. Teoretski, slobodno se definiše kao stopa infinitezimalno kratke ročnosti (od jednog momenta do drugog). U praksi, međutim, jednaka je stopi postignutoj na najkraći mogući depozit.

Ovu trenutnu kamatntu stopu r određuje sledeća diferencijalna jednačina:

$$dr = u\,(r,\,t)\,dt + w\,(r,\,t)\,dX \qquad (1)$$

Prvi deo Jednačine 1 predstavlja otklon, tj. devijaciju od teoretskog proseka i deterministički je (jer pretpostavlja da će kamatne stope, nezavisno od toga u kojoj meri fluktuiraju, da teže nekoj srednjoj vrednosti). Nasuprot tome, drugi uslov (volatilnost) je stohastički, jer obuhvata Vinerov process dX kao izvor nasumičnosti (poznat kao standardno Braunijevsko kretanje). Funkcionalne forme $u(r,\ t)$ i $w(r,\ t)$ određuju ponašanje spot stope r. Njihov izbor vodi različitim dobro poznatim modelima, kao što su Ho & Lee, Vasiček, Hal i Vajt, itd.

Da bi gore navedeni model kratkoročne stope imao bilo kakvu praktičnu svrhu, mora biti moguće da se koristi za izvođenje cenovne jednačine proizvoda kojima se trguje na tržištu, kao što su obveznice. S obzirom na to da je obveznica derivat stope, može se posmatrati kao opcija na osnovnu akciju. Međutim, pošto kamata nije utrživa aktiva, ne postoji osnovna aktiva kojom bi se hedžovao derivat. Obveznica može da se hedžuje samo uzimanjem pozicije u drugoj obveznici različite ročnosti.

Na osnovu ovog principa i primenom Itove leme, može se izvesti sledeća jednačina određivanja cene obveznice:

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \qquad (2)$$

gde funkcija $\lambda(r,\ t)$ tek treba da se specifikuje i četiri uslova su respektivno:

- vremensko opadanje
- difuzija
- otklon
- diskont

Tržišna cena rizika

U Jednačini 2 napred, nepoznata funkcija $\lambda(r,\ t)$ može da se specifikuje pretpostavkom da držimo nehedžovanu poziciju ročnosti T. U infinitezimalno malom vremenskom koraku dt vrednost obveznice će se promeniti za:

$$dV = w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r}\right)dt \qquad (3)$$

Korišćenjem jednačine obveznice date pod (2), možemo navedeni izraz napisati kao:

$$dV = w\frac{\partial V}{\partial r}dX + \left(\lambda w\frac{\partial V}{\partial r} + rV\right)dt \qquad (4)$$

Posle preuređenja Jednačine 4, dobijamo:

$$dV - rVdt = w\frac{\partial V}{\partial r}(dX + \lambda dt) \qquad (5)$$

U gornjem izrazu, pošto desna strana sadrži dva uslova-deterministički (u dt) i nasumični (u dX) - portfolio obveznica nije bezrizičan. Deterministički uslov može da se interpretira kao kompenzacija za prihvatanje rizika (nasumični deo). Otuda je funkcija *$\lambda(r,\ t)$ tržišna cena rizika*.

Neutralnost u odnosu na rizik

Rešenje jednačine obveznica date u Jednačini 2 može se tumačiti kao očekivana sadašnja vrednost svih gotovinskih tokova, gde se očekivanje ne odnosi na realnu nasumičnu varijablu, već na *rizično-neutralnu* varijablu. Rizično neutralna stopa neće imati otklon u, već $u - \lambda w$, kako je dato u Jednačini 2. Otuda, kada se određuje cena derivata kamatne stope, treba koristiti sledeći model rizično neutralne kamatne stope:

$$dr = (u - \lambda w)dt + wdX \qquad (6)$$

One-Factor interest rate modelling

One-factor interest rate models have only one source of randomness (i.e. uncertainty or market risk). They are modelling a short-term interest rate (spot rate), leading to a partial differential equation of the prices of various classes of interest rate derivatives. The spot rate (also known as instantaneous or short rate) is an unobservable quantity. Theoretically, it is a loosely defined as a rate of infinitesimally short maturity (from one moment to the next). In practice, however, it is equivalent to the rate achieved on the shortest possible deposit.

This instantaneous interest rate r is governed by the following differential equation:

$$dr = u\,(r,\,t)\,dt + w\,(r,\,t)\,dX \quad (1)$$

The first part of Equation 1 represents drift, i.e. the deviation from the theoretical mean, and is deterministic (as it is assumed that interest rates, irrespective of how much they fluctuate, will converge to some mean value). In contrast, the second term (volatility) is stochastic, as it incorporates Wiener process dX as a source of randomness (also known as standard Brownian motion). The functional forms of $u(r,t)$ and $w(r,t)$ determine the behaviour of the spot rate r. Their choice leads to different well-known models, such as Ho & Lee, Vasicek, Hull & White, etc.

In order for the model of the short rate given above to have any practical purpose, it has to be possible to use it to derive a pricing equation of a market-traded product, such as bond. Given that the bond is a derivative of the rate, it can be viewed as an option on the underlying stock. However, as the rate is not a tradable asset, there is no underlying with which to hedge the derivative. A bond can only be hedged by taking a position in another bond of a different maturity.

Based on this principle, and applying Ito's lemma, the following bond pricing equation can be derived:

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \qquad (2)$$

where the function $\lambda(r,t)$ is yet to be specified and the four terms are respectively:

- time decay
- diffusion
- drift and
- discounting

Market Price of Risk

In Equation 2 above, the unknown function $\lambda(r,t)$ can be specified by assuming that we hold an unhedged bond position of maturity T. In an infinitesimally small time step dt the bond value will change by:

$$dV = w\frac{\partial V}{\partial r}dX + (\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r})dt \qquad (3)$$

Using the bond equation given in (2), we can write the above expression as:

$$dV = w\frac{\partial V}{\partial r}dX + (\lambda w\frac{\partial V}{\partial r} + rV)dt \qquad (4)$$

After rearranging Eq. 4, we obtain:

$$dV - rVdt = w\frac{\partial V}{\partial r}(dX + \lambda dt) \qquad (5)$$

In the above expression, as the right hand side contains two terms-deterministic (in dt) and random (in dX)-the bond portfolio is not riskless. The deterministic term can be interpreted as a reward for accepting the risk (random part). Hence the function $\lambda(r,t)$ is the *market price of risk.*

Risk Neutrality

The solution to the bond equation given by Eq. 2 can be interpreted as the expected present value of all the cashflows, where the expectation does not relate to the real random variable, but rather to the *risk-neutral* variable. This risk neutral rate will not have the drift u, but rather $u - \lambda w$, as given by Eq. 2. Hence, when pricing interest rate derivatives, the following risk-neutral interest rate model should be used:

$$dr = (u - \lambda w)dt + wdX \qquad (6)$$

Tractable Models

The bond pricing equation given above was based on an arbitrary model of interest rates, whereby the risk neutral drift $u - \lambda w$ and the volatility w have not yet been specified. The

Analitički transparentni modeli

Jednačina za određivanje cena obveznica zasnovana je na arbitrarnom modelu kamatnih stopa, gde rizično neutralan otklon $u - \lambda w$ i volatilnost w još nisu specificirani. Izbor tih funkcija mora da vodi do modela za koji rešenje jednačine za obveznice sa zero kuponom može da se pronađe analitički.

Pri tim datim uslovima, možemo uzeti da $u - \lambda w$ i w uzimaju oblik:

$$u(r,t) - \lambda(r,t)w(r,t) = \eta(t) - \gamma(t)r \qquad (7)$$

$$w(r,t) = \sqrt{\alpha(t)r + \beta(t)} \qquad (8)$$

Na odgovarajući način ograničavajući funkcije α, β, γ, η i λ možemo obezbediti da nasumičan hod za r dat jednačinom (1) ima sledeća svojstva:

- **Pozitivne kamatne stope -** izuzev u retkim slučajevima, kamatne stope su tipično pozitivne za sve valute i ročnosti. Sa navedenim modelom, spot stopa može da bude ograničena odozdo pozitivnim brojem ako je $\alpha > 0$ i $\beta \leq 0$. Mada stopa r može još uvek raste u beskonačnost, verovatnoća takvog događaja je nula.
- **Reverzija ka srednjoj vrednosti -** zbog uslova otklona, velika r bi se kretala naniže prema srednjoj vrednosti, dok bi mala stopa u proseku rasla naviše. To je tipično sagledano ponašanje kamatnih stopa na tržištu.

Razlog za izbor funkcionalnih formi datih u (7) i (8) za otklon i volatilnost, respektivno, leži u tome što što oni daju prosto rešenje za jednačinu obveznice (2):

$$Z(r,t;T) = e^{A(t;T) - B(t;T)} \qquad (9)$$

Zamena (9) u jednačini za određivanje cene obveznica (2) omogućava izračunavanje parametara A i B. Dalje, integrisanje tih običnih diferencijalnih jednačina daje vrednosti za parametre α, β, γ, i η koji se koriste za specificiranje skretanja spot stope i volatilnosti. Međutim, ova integracija, generalno, ne može da se izvede eksplicitno.

Specifični jednofaktorski modeli

Stohastička diferencijalna jednačina (l) za rizično neutralnu kamatnu stopu sa rizično neutralnim otklonom i volatilnošću datih u (7) i (8) respektivno predstavljaju generalizovane verzije mnogih poznatih tržišno standardnih modela, od kojih su one najčešće korišćene ukratko date u daljem tekstu.

Vasiček

Vasiček model uvodi restrikcije $\alpha = 0$ i $\beta > 0$, dok su svi ostali parametri vremenski nezavisni. Tako, model dobija formu:

$$dr = (\eta - \gamma r)dt + \beta^{1/2} dX$$

Mada je ovaj model reverzan prema srednjoj vrednosti, kamatne stope mogu lako postati negativne, čineći njegovu primenu nepraktičnom. Spot stopa ima Normalnu distribuciju centriranu na η/γ. Moguće su krive prinosa nagnute nagore, nagnute nadole i konveksne.

Cox, Ingersoll i Ross

Kao i napred, parametri su vremenski nezavisni i $\beta = 0$. Model na taj način postaje:

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r} dX$$

Spot stopa teži srednjoj vrednosti i pozitivna je za $\eta > \alpha/2$. Funkcija gustine verovatnoće je centrirana na η/γ (kao kod Vasičeka), ali je iskrivljena, dajući manje verovatnoće stopama ispod proseka. Kao kod Vasičeka, model dozvoljava krive prinosa sa nagibom nagore, nagibom nadole i konveksne krive. Pošto su kamatne stope dužih ročnosti linearno zavisne od kratkoročne stope r, ročnu strukturu u bilo koje vreme t određuje $r(t)$. Drugim rečima, ročna struktura zavisi od $r(t)$ ali je nezavisna od t.

Ho & Lee

U ovom modelu, $\alpha = \gamma = 0$, $\beta > 0$ su konstante, ali η može biti funkcija vremena. To je dato u:

$$dr = \eta(t)dt + \beta^{1/2} dX$$

Ovaj model je poznat kao "model bez arbitraže" jer pravi izbor η može da proizvede choice of those functions has to lead to a model for which the solution of the bond equation for zero-coupon bonds can be found analytically.

Given those conditions, we can assume that $u - \lambda w$ and w take the form:

$$u(r,t) - \lambda(r,t)w(r,t) = \eta(t) - \gamma(t)r \qquad (7)$$

$$w(r,t) = \sqrt{\alpha(t)r + \beta(t)} \qquad (8)$$

By suitably restricting the functions α, β, γ, η and λ, we can ensure that the random walk for r given by equation (1) has the following properties:

- **Positive interest rates -** except for the rare cases, interest rates are typically positive for all currencies and maturities. With the above model, the spot rate can be bounded from below by a positive number if $\alpha > 0$ and $\beta \leq 0$. Although the rate r can still extend to infinity, the probability of such occurrence is zero.
- **Mean reversion** - due to the drift term, a large r would move down towards the mean, whilst a small rate would on average move up. This is a typical observed behaviour of interest rates in the market.

The reason for choosing the functional forms given in (7) and (8) for drift and volatility, respectively, is that they yield a simple solution to the bond equation (2):

$$Z(r,t;T) = e^{A(t;T) - B\,(t;T)} \qquad (9)$$

Substituting (9) into the bond pricing equation (2) allows for calculating parameters A and B. Furthermore, integration of these ordinary differential equations gives values for the parameters α, β, γ, η used to specify the spot rate drift and volatility. However, this integration, in general, cannot be done explicitly.

Specific one-factor models

The stochastic differential equation (1) for the risk-neutral interest rate with risk neutral drift and volatility given by (7) and (8) respectively are generalised versions of many familiar market-standard models, with the most frequently used ones briefly summarized below.

Vasicek

Vasicek model imposes the restrictions $\alpha = 0$ and $\beta > 0$, whist all other parameters are time-independent. Thus, the model takes the form:

$$dr = (\eta - \gamma r)dt + \beta^{1/2} dX$$

Although this model is mean-reverting, interest rates can easily become negative, making its application impractical. The spot rate is normally distributed with a mean of η/γ. Upward-sloping, downward-sloping and humped yield curves are also possible.

Cox, Ingersoll and Ross

As above, the parameters are independent of time and $\beta = 0$. The model thus becomes:

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}\, dX$$

The spot rate is mean-reverting and is positive for $\eta > \alpha/2$. The probability density function has the mean equal to η/γ (same as Vasicek), but is skewed, assigning lower probabilities to the rates below the mean. As with the Vasicek, the model allows for upward-sloping, downward-sloping and humped yield curves. As the interest rates of longer maturity are linearly dependent on the short rate r, the term structure at any time t is determined by $r(t)$. In other words, the term structure is dependent on $r(t)$ but is independent of t.

Ho & Lee

In this model, $\alpha = \gamma = 0$, $\beta > 0$ and are constant, but η can be a function of time. It is given by:

$$dr = \eta(t)dt + \beta^{1/2} dX$$

This model is known as a 'no-arbitrage model' as the right choice of η can yield the theoretical bond prices that are equal to the observable market prices.

Hull and White

Hull and White extended the above models to incorporate time-dependent parameters. This time-dependence allows the yield curve (and sometimes volatility structure) to be fitted. This model is widely used for pricing interest rate derivatives and is thus described in more detail.

prinos teoretskih cena obveznica koje su jednake viđenim tržišnim cenama.

Hal i Vajt

Hal i Vajt su proširili navedeni model da obuhvati vremenski zavisne parametre. Ova vremenska zavisnost dozvoljava da se kriva prinosa (i ponekad struktura volatilnosti) prilagođava. Ovaj model se široko koristi za određivanje cena kamatnih derivata i zato ga opisujemo sa više detalja.

Model Hal i Vajt

Model Hal i Vajt proširuje Vasičekov model da bi η postalo vremenski zavisno. Alternativno, može se posmatrati i kao Ho i Li model sa parametrom reverzije ka srednjoj vrednosti. Njegova jednačina se daje kao:

$$dr = (\eta(t) - \gamma r)dt + \beta^{1/2} dX$$

U ovom modelu, pod pretpostavkom da su svi drugi parametri procenjeni statistički, možemo da odaberemo η(t) tako da teoretski cene obveznica budu jednake cenama viđenim na tržištu.

Model je sličan Ho i Lee u pogledu analitičke transparentnosti. U proseku, *r* približno prati nagib trenutne krive terminske stope. Ako se udaljava od te krive, vraća se stopi γ. Pošto volatilnu strukturu modela Hal i Vajt određuju β i γ, on može da predstavlja širi spektar volatilnosti u poređenju sa bilo kojim od navedenih modela. Analitički izrazi za volatilnost obveznica, standardnu devijaciju stope zero kupona i standardnu devijaciju trenutne terminske stope daju se analitički respektivno u sledećem:

$$\nu(t, T, \Omega_t) = \frac{w}{\gamma}\left[1 - e^{-\gamma(T-t)}\right] \qquad (10)$$

$$std.dev_Z = \frac{w}{\gamma(T-t)}\left[1 - e^{-\gamma(T-t)}\right] \qquad (11)$$

$$std.dev_{Fwd} = we^{-\gamma(T-t)} \qquad (12)$$

Parametar reverzne stope γ određuje zakrivljenost krive volatilnosti cena obveznica i stope po kojoj standardne devijacije stope zero kupona i trenutne terminske stope opadaju sa ročnošću. Zakrivljenost i pad povećavaju se sa rastom γ. Za γ = 0 model se svodi na Ho i Li.

Analitički izrazi za cene obveznica kao i za kupovne i prodajne opcije na obveznice sa zero kuponom mogu lako da se izvedu.

Pretpostavke i ograničenja jednofaktorinih modela kamatnih stopa

Ova diskusija se odnosi na jednofaktorne modele generalno. Specifične reference na Hal i Vajta daju se gde to odgovara.

Prednosti

- **Analitička transparentnost -** lako je pratiti vezu između modela i jednačine za obveznice i videti efekte promene vrednosti parametara
- **Nedostatak arbitraže** - za razliku od ravnotežnih modela koji imaju početnu ročnu strukturu krive prinosa kao autput, modeli bez arbitraže nju uzimaju kao imput, dajući očekivani budući razvoj krive prinosa kao autput
- **Pozitivne kamatne stope -** izuzev retkih slučajeva (kao što je JPY), kamatne stope su obično pozitivne za sve valute i ročnosti. Kod modela datih u jednačinama (1), (7) i (8), spot stopa može se ograničiti naniže pozitivnim brojem ako je $\alpha > 0$ i $\beta \leq 0$. Stopa *r* i dalje može da raste u beskonačnost, ali sa nulom verovatnoće.
- **Reverzija ka srednjoj vrednosti -** zbog uslova otklona, *r* veće od proseka bi se kretalo naniže, i obrnuto. To je uobičajeno viđeno ponašanje kamatnih stopa na tržištu.

Mane

- **Hedžing** - glavne pretpostavke za jednačine određivanja cena obveznica su (a) sposobnost za delta hedž (stvaranje pozicije u derivatima koja bi neutralisala promene vrednosti osnovne aktive) i (b) odsustvo prilika za arbitražu (gde bi se garantovani profit generisao iz anomalija tržišnih cena). Međutim, da bi ostali delta neutralni, instrumenti moraju da se kupuju i prodaju i to mora da se radi po tržišnim cenama (a ne po teoretskim cenama). Pošto modeli

Hull and White model

Hull and White model extends the Vasicek model to make η time-dependent. Alternatively, it can be seen as the Ho and Lee model with a mean reversion parameter. Its equation is given by:

$$dr = (\eta(t) - \gamma r)dt + \beta^{1/2} dX$$

In this model, assuming that all other parameters are estimated statistically, we can choose η*(t)* so that the theoretical bond prices are equal to the prices observed in the market.

The model has the same amount of tractability as Ho and Lee. On average, *r* approximately follows the slope of the instantaneous forward rate curve. If it moves away from that curve, it reverts back at the rate γ. As the volatility structure of Hull and White model is determined by both β and γ, it can represent a wider range of volatilities than any of the above models. The analytical expressions for bond volatility, standard deviation of zero-coupon rate and standard deviation of instantaneous forward rate are respectively given analytically by:

$$v(t,T,\Omega_t) = \frac{w}{\gamma}\left[1 - e^{-\gamma(T-t)}\right] \quad (10)$$

$$std.dev_Z = \frac{w}{\gamma(T-t)}\left[1 - e^{-\gamma(T-t)}\right] \quad (11)$$

$$std.dev_{Fwd} = we^{-\gamma(T-t)} \quad (12)$$

The reversion rate parameter γ determines the curvature in the bond price volatility curve and the rate at which standard deviations of the zero-coupon rate and instantaneous forward rate decline with maturity. Both the curvature and the decline increase with increasing γ. At γ = 0, the model reduces to Ho and Lee.

Analytical expressions for bond prices as well as call and put options on zero-coupon bonds can be easily derived.

Assumptions and limitations of one-factor interest rate models

This discussion relates to the one-factor models in general. Specific reference to Hull and White is given where appropriate.

Advantages

- **Tractability** - it is easy to follow the link between the model and the bond equation and to see the effect of changing parameter values
- **No-arbitrage -** unlike equilibrium models that have initial yield curve term structure as an output, no-arbitrage models take it as an input, providing the expected future evolution of the yield curve as the output
- **Positive interest rates -** except for the rare cases (such as JPY), interest rates are typically positive for all currencies and maturities. With the model given by equations (1), (7) and (8), the spot rate can be bounded from below by a positive number if $\alpha > 0$ and $\beta \leq 0$. The rate *r* can still extend to infinity, but with zero probability.
- **Mean reversion** - due to the drift term, an *r* greater than the mean would move down, and vice versa. This is a typical observed behaviour of interest rates in the market.

Disadvantages

- **Hedging** - The main assumptions behind the bond pricing equations are (a) ability to delta hedge (create the position in the derivative that would offset the changes in the value of the underlying asset) and (b) absence of arbitrage opportunities (whereby guaranteed profit would be generated due to market price anomalies). However, in order to stay delta neutral, instruments need to be bought and sold and this has to be done at market (rather than theoretical) prices. As the models are actually modelling rates, and bonds are just derivatives of those, in reality, their theoretical prices will be markedly different from the market bond prices. Consequently, unless hedging instruments can be priced correctly (which requires yield curve fitting), the model collapses and cannot be used.
- **Timeliness** - even if the bond market prices are correctly given by the model today, the likelihood of that being true at some point in the future is small. This is because the model assumes that the time-dependent parameters have not changed in time whilst in practice they always do.
- **Curve fitting** - using Taylor expansion, it can

u stvari modeliraju stope, a obveznice su samo izvedenice iz njih, u stvarnosti, njihove teoretske cene biće značajno različite od tržišnih cena obveznica. Otuda, ukoliko se vrednost instrumenata hedžinga ne može korektno proceniti (što zahteva podešavanje krive prinosa), model postaje beskoristan.

- **Blagovremenost** - čak i kada su tržišne cene danas date korektno, mala je verovatnoća da će biti važeće u neko vreme u budućnosti. To je tako zato što model pretpostavlja da se parametri zavisni od vremena nisu promenili vremenom mada se u praksi uvek menjaju.
- **Podešavanje krive -** koristeći Tejlorovu ekspanziju, može se pokazati da je nagib krive prinosa koje daje model jednak 0, 5 od rizično neutralnog otklona. Dalje, zakrivljenost krive prinosa na kratkoročnom kraju zavisi od vremenskog izvoda rizično neutralnog otklona i obrnuto.
  - Posledica toga je da $\eta(t)$ izračunato danas zavisi od zakrivljenosti današnje krive prinosa. Vrlo je uobičajeno da kriva prinosa ima znatan nagib na kratkoročnom kraju, kao i da razlika između kratkoročnih i dugoročnih stopa bude velika, pri čemu kriva postaje ravna prema dužim ročnostima. Zbog toga, kada se danas modelira kriva prinosa, očekujemo da se početan jaki nagib znatno smanjuje u budućnosti. Međutim, prilagođavanje krive prema tržištu na neki budući datum pokazaće da su naša očekivanja pogrešna. Prilagođena kriva izgledaće slična današnjoj (kao da je preneta kroz vreme).
  - Zato **nijedan jednofaktorski model ne može tačno da modelira celokupnu krivu prinosa.** Prihvatljivi rezultati mogu da se očekuju samo ako je kriva prinosa relativno ravna.
- **Nivoi krive naspram oblika krive** - na osnovu izloženog, možemo zaključiti da, korišćenjem jednofaktornih modela, tačno može da se predstavi jedino nivo krive prinosa. Potrebni su dodatni parametri da bi se obuhvatila njena zakrivljenost.

Zaključci

- Model Hal i Vajt (ili bilo koji jednofaktorni model) može tačno da modelira nivo krive, ali ne uspeva da prikaže njenu zakrivljenost. Zato treba da se koristi samo za određivanje cene instrumenata koji zavise od nivoa krive (na pr. digitalne opcije, bermudanske, barijere) a ne od oblika (gde je razlika između dugoročne i kratkoročne stope značajna, na pr. svopovi konstantne ročnosti, opcije na spred, itd).
- Kalibracija na tržišne cene je neophodna ako će se portfolio delta-hedžovati (stvaranje pozicije u derivatima koja bi neutralisala promene vrednosti osnovne aktive); međutim, ovaj proces je obično nepouzdan i treba da se izvodi oprezno
- Ovaj model omogućava reverziju ka srednjoj vrednosti, što je korisno, jer obuhvata generalne trendove na tržištu
- Model Hal i Vajt pretpostavlja pozitivne kamatne stope, što zahteva oprez kada se primenjuje u okruženju niskih kamatnih stopa; međutim, to ne važi za sve jednofaktorske modele
- Model je analitički transparentan i lako ga je razumeti; otuda je jedan od najpopularnijih tržišno standardnih modela. To daje poverenje za njegovo kontinuirano korišćenje.

be shown that the slope of the yield curve given by the model is equal to 0.5 of the risk-neutral drift. Moreover, the curvature of the yield curve at short end depends on the time derivative of the risk-neutral drift and vice versa.

  - The consequence of this is that $\eta(t)$ as calculated today will depend on the curvature of today's yield curve. It is very common for the yield curve to have significant slope at the short end, as well as for the difference between short and long-term rates to be large, whereby the curve flattens out towards longer maturities. As a result, when modelling the yield curve today, we will expect the initial large slope to reduce significantly in the future. However, refitting to the market at some future date will prove our expectations wrong. The refitted curve will look similarly to today's one (as if it were shifted in time).
  - Therefore ***no one-factor model will model the entire yield curve correctly.*** Reasonable results can only be expected if the yield curve is relatively flat.
- **Curve level vs. curve shape -** from the above, we can conclude that, using any of the one-factor models, only the level of the yield curve can be represented correctly. Additional parameters are required in order to capture its curvature.

Conclusions

- Hull and White model (or any one-factor model) can model the curve level correctly, but it fails to capture its curvature. Therefore, it should only be used to price instruments that depend on curve level (e.g. digital options, bermudans, barriers) rather than shape (where the difference between the long and short rate is important, e.g. constant maturity swaps, spread options, etc.).
- Calibration to market prices in necessary if the portfolio is to be delta hedged; however, this process is typically unreliable and should be performed with caution
- This model allows for mean reversion, which is useful, as it captures general trends in the market
- Hull and White model assumes positive interest rates, which requires caution when applying it in a low interest rate environment; however, this is not true for all one-factor models
- The model is tractable and easy to understand; hence, it is one of the most popular market-standard models. This gives confidence in its continued usage.

Literatura / References

1. Hull, J. (1997) *Options, Futures and other Derivative Securities* (3rd ed.), Pretince hall, New Jersey.
2. Neftci, S.N. (2000) *An Introduction to the Mathematics of Financial derivatives* (2nd ed.), Academic Press, San Diego.
3. Stigum, M. and Robinson F.L. (1996) *Money Market & Bond Calculations*, Irwin Times Mirror Higher Education Group, New York.
4. Taylor, J. (1996) *Mastering Derivatives Markets*, Pearson Education, Harlow.
5. Wilmot, P. Howison, S. and Dewyanne, J. (1995) *The Mathematics of Financial Derivatives*, Cambridge University press, Cambridge.
6. Wilmott, P. (2007) *Paul Wilmott Introduces Quantitative Finance* (2nd ed.), John Wiley & Sons, West Sussex.