Tail bounds for Matrix De-noising application

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Motivation I

- **Overview:** Matrix de-noising is one of the popular topics in Machine Learning. Many existing papers have focused on the assumption that the data matrix has low rank. In this presentation we want to focus on one specific de-noising algorithm on low rank matrices and analyze it from the probability perspective.

Motivation II

One important application of matrix de-noising is image de-noising

(a) Ground Truth (b) Noisy Input($\sigma = 30$) (c) De-noised Image

Figure: Example of image de-noising

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Problem Formulation I

Oracle: We can sample any matrix with noise where the noise has distribution $\mathcal{N}(0, \sigma^2)$ for some σ.

Problem: Given a low-rank matrix $\boldsymbol{A}^{m \times n}$ with $rk(\boldsymbol{A}) = r \ll \min\{m, n\}$, we infect it with Gaussian noise and we get a noisy observation of the matrix.

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{G}$$

where $[\boldsymbol{G}^{m \times n}]_{i,j} \sim \mathcal{N}(0, \sigma^2)$.

Our goal is to recover our initial matrix $\boldsymbol{A}$ from the noisy counterpart $\boldsymbol{C}$.

Problem Formulation II

We pick linear operator to recover the matrix, and we can get our estimator $\hat{\boldsymbol{A}} = \boldsymbol{C}\boldsymbol{X}$ for some linear operator $\boldsymbol{X}$. Then finding the best estimator involves solving the optimization problem

$$\begin{aligned} \min_{\boldsymbol{X}} \quad & \mathbb{E}\left\|\hat{\boldsymbol{A}} - \boldsymbol{A}\right\|_F^2 \\ \text{s.t.} \quad & \hat{\boldsymbol{A}} = \boldsymbol{C}\boldsymbol{X} \\ & rk(\hat{\boldsymbol{A}}) \leq r \end{aligned}$$

Problem Solution

It can be shown that

$$\boldsymbol{X}^* = \arg\min_{\boldsymbol{X}} \mathbb{E} \left\| \hat{\boldsymbol{A}} - \boldsymbol{A} \right\|_F^2 \tag{1}$$

$$= (\boldsymbol{A}^\top \boldsymbol{A} + \sigma^2 m \boldsymbol{I})^{-\frac{1}{2}} \left[(\boldsymbol{A}^\top \boldsymbol{A} + \sigma^2 m \boldsymbol{I})^{-\frac{1}{2}} \boldsymbol{A}^\top \boldsymbol{A} \right]_r \tag{2}$$

where $[\boldsymbol{M}]_r = \boldsymbol{U}_r \boldsymbol{\Sigma}_r \boldsymbol{V}_r^\top$ is the rank-r truncation of $\boldsymbol{M}$ if $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$ is its SVD.

Notice that the optimal matrix $\boldsymbol{X}^*$, involves $\boldsymbol{A}$ in the formula.

$$\boldsymbol{X}^* = (\boldsymbol{A}^\top \boldsymbol{A} + \sigma^2 m \boldsymbol{I})^{-\frac{1}{2}} \left[(\boldsymbol{A}^\top \boldsymbol{A} + \sigma^2 m \boldsymbol{I})^{-\frac{1}{2}} \boldsymbol{A}^\top \boldsymbol{A} \right]_r$$

WARNING: We are not allowed to use $\boldsymbol{A}$ in our expression!!!

Proposed Solution

Observation:

$$\mathbb{E}\left[\boldsymbol{C}^\top \boldsymbol{C}\right] = \boldsymbol{A}^\top \boldsymbol{A} + \sigma^2 m \boldsymbol{I}$$

Proposed linear operator:

$$\tilde{\boldsymbol{X}} = (\boldsymbol{C}^\top \boldsymbol{C})^{-\frac{1}{2}} \left[(\boldsymbol{C}^\top \boldsymbol{C})^{-\frac{1}{2}}(\boldsymbol{C}^\top \boldsymbol{C} - \sigma^2 m \boldsymbol{I})\right]_r$$

We use $\tilde{\boldsymbol{A}} = \boldsymbol{C}\tilde{\boldsymbol{X}}$ to approximate $\hat{\boldsymbol{A}} = \boldsymbol{C}\boldsymbol{X}^*$.

Challenge: Is this estimation accurate?

- The main challenge is to find probability tail bounds that shows our solution concentrates around and is unlikely to deviate far from the optimal solution.
- Namely, we want to show

$$\mathbb{E}\left\|\boldsymbol{C}\tilde{\boldsymbol{X}} - \boldsymbol{C}\boldsymbol{X}^*\right\|_2 < \varepsilon$$

w.h.p for some small ε.

- Notice

$$\mathbb{E}\left\|\boldsymbol{C}\tilde{\boldsymbol{X}} - \boldsymbol{C}\hat{\boldsymbol{X}}\right\|_2 \leq \mathbb{E}\left[\|\boldsymbol{C}\|_2 \left\|\tilde{\boldsymbol{X}} - \boldsymbol{X}^*\right\|_2\right]$$

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Here some of the existing probability tools will be presented for scalar random variables.

- Sub-Gaussian Bounds:
  - A random variable X is sub-Gaussian if it satisfies: $\forall \lambda \in R$

$$\mathbb{E}\left[e^{\lambda(X-\mu)}\right] \leq e^{\frac{\sigma^2\lambda^2}{2}}$$

and the concentration bound is:

$$\mathbb{P}\left[|X-\mu| \geq t\right] \leq 2e^{-\frac{t^2}{2\sigma^2}}$$

- Sub-exponential Bounds:
  - A random variable X is sub-exponential if it satisfies: $\exists v, \alpha > 0$, such that $\forall |\lambda| < \frac{1}{\alpha}$

$$\mathbb{E}\left[e^{\lambda(X-\mu)}\right] \leq e^{\frac{v^2\lambda^2}{2}}$$

    and it's a generalization of the sub-Gaussian random variables. The sub exponential tail bound can be fomulated as

$$\mathbb{P}[X-\mu \geq t] \leq \begin{cases} e^{-\frac{t^2}{2v^2}} & \text{if } 0 \leq t \leq \frac{v^2}{\alpha} \\ e^{-\frac{-t}{2a}} & \text{for } t \geq \frac{v^2}{a} \end{cases}$$

- Bernstein Bounds:
  - Can be applied to sum of series of centered bounded i.i.d random variables. Let $S_1, ..., S_n$ be random variables with $\mathbb{E}[S_i] = 0$ and $|S_i| \leq L$ and let

$$Z = \sum_{i=1}^{n} S_i, \ \ Var(Z) = \sum_{i=1}^{n} Var(S_i)$$

then we have

$$\mathbb{P}(|Z| \geq t) \leq 2e^{-\frac{t^2/2}{Var(Z)+Lt/3}}$$

- In general, it is hard to extend the bounds derived for scalar random variables to random matrices. In our case if we can represent our random matrices by a sum of bounded i.i.d random matrices, we can apply the matrix Bernstein bound.

Matrix Bernstein bound(J. Tropp, 2011) If $\boldsymbol{S}_1, ..., \boldsymbol{S}_n$ are centered i.i.d random matrices such that $\forall i, \boldsymbol{S}_i \in R^{d_1 x d_2}$ and $||\boldsymbol{S}_i|| \leq L$ and

$$\boldsymbol{Z} = \sum_{i=1}^{n} \boldsymbol{S}_i$$

Let $v(\boldsymbol{Z})$ denote the matrix variance statistic of the sum:

$$v(\boldsymbol{Z}) = \max\{||\mathbb{E}[\boldsymbol{Z}^* \boldsymbol{Z}]||, ||\mathbb{E}[\boldsymbol{Z}\boldsymbol{Z}^*]||\} \ \forall t \geq 0$$

Then the concentration bound is:

$$\mathbb{P}(||\boldsymbol{Z}|| \geq t) \leq (d_1 + d_2) e^{-\frac{t^2/2}{v(\boldsymbol{Z}) + Lt/3}}$$

Also we can get a bound for the expectation of $\boldsymbol{Z}$:

$$\mathbb{E}[||\boldsymbol{Z}||] \leq \sqrt{2v(\boldsymbol{Z}) \log{(d_1 + d_2)}} + \frac{1}{3} L \log{(d_1 + d_2)}$$

Norm Bound for Matrix Gaussian & Rademacher Series (J. Tropp, 2014) Consider a finite sequence $\{\boldsymbol{B}_k\}$ of fixed complex matrices with dimension $d_1 d_2$, and let $\{\gamma_k\}$ be a finite sequence of independent standard normal variables. Introduce the matrix Gaussian series

$$\boldsymbol{Z} = \sum_k \gamma_k \boldsymbol{B}_k$$

Let $v(\boldsymbol{Z})$ denote the matrix variance statistic of the sum:

$$v(\boldsymbol{Z}) = \max\{||\mathbb{E}[\boldsymbol{Z}^*\boldsymbol{Z}]||, ||\mathbb{E}[\boldsymbol{Z}\boldsymbol{Z}^*]||\}$$
$$= \max\left\{\left\|\sum_k \boldsymbol{B}_k \boldsymbol{B}_k^*\right\|, \left\|\sum_k \boldsymbol{B}_k^* \boldsymbol{B}_k\right\|\right\}$$

Then:

$$\mathbb{E}\|\boldsymbol{Z}\| \leq \sqrt{2v(\boldsymbol{Z})\log(d_1 + d_2)}$$

and $\forall t \geq 0$

$$\mathbb{P}(\|\boldsymbol{Z}\| \geq t) \leq (d_1 + d_2)\exp\left(\frac{-t^2}{2v(\boldsymbol{Z})}\right)$$

Matrix Concentration Inequalities V

From Scalar Random Variables to Random Matrices

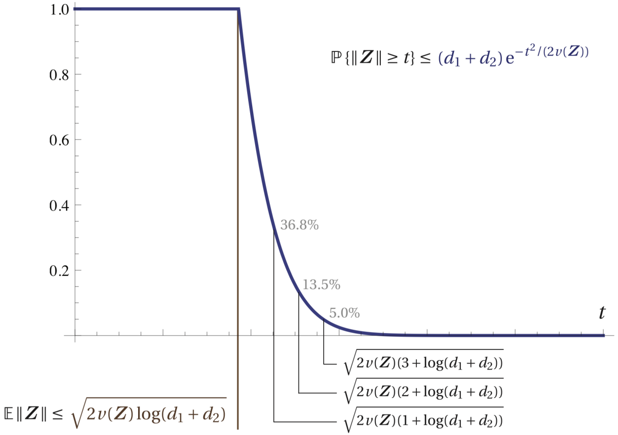

Figure: plot for tail bound behavior for matrix Gaussian series(J. Tropp, 2011)

Using Random Matrix Bounds I

Our Attempt to Bound the Solution

We can simplify the optimal and the proposed solution to

$$\boldsymbol{X}^* = \left[\boldsymbol{I} - \sigma^2 m(\boldsymbol{A}^\top \boldsymbol{A} + \sigma^2 m \boldsymbol{I})^{-1}\right]_r$$
$$\tilde{\boldsymbol{X}} = \left[\boldsymbol{I} - \sigma^2 m(\boldsymbol{C}^\top \boldsymbol{C})^{-1}\right]_r$$

So for $\tilde{\boldsymbol{X}}$ to be concentrating around $\boldsymbol{X}^*$ it has to that $\boldsymbol{C}^\top \boldsymbol{C}$ concentrates around $\boldsymbol{A}^\top \boldsymbol{A} + \sigma^2 m \boldsymbol{I}$

Using Random Matrix Bounds II

Our Attempt to Bound the Solution

We have

$$
\begin{aligned}
&\mathbb{E}\left\|\boldsymbol{C}^\top\boldsymbol{C} - (\boldsymbol{A}^\top\boldsymbol{A} + \boldsymbol{\sigma}^2 m\boldsymbol{I})\right\| \\
=&\mathbb{E}\left\|\boldsymbol{A}^\top\boldsymbol{A} + \boldsymbol{G}^\top\boldsymbol{A} + \boldsymbol{G}\boldsymbol{A}^\top + \boldsymbol{G}^\top\boldsymbol{G} - \boldsymbol{A}^\top\boldsymbol{A} - \sigma^2 m\boldsymbol{I}\right\| \\
\leq& 2\left\|\boldsymbol{A}\right\|\mathbb{E}\left\|\boldsymbol{G}\right\| + \mathbb{E}\left\|\boldsymbol{G}\right\|^2 + \sigma^2 m
\end{aligned}
$$

The question reduces to finding the upper bound for $\mathbb{E}\,\|\boldsymbol{G}\|$, notice $[\frac{1}{\sigma}\boldsymbol{G}^{m\times n}]_{i,j} \sim \mathcal{N}(0,1)$, we can re-scale and express $\boldsymbol{G}$ using a Gaussian series, take $\gamma_{ij} = \frac{g_{ij}}{\sigma}$ and denote:

$$\boldsymbol{G}_r = \frac{1}{\sigma}\boldsymbol{G} = \sum_{i=1}^{m}\sum_{j=1}^{n}\gamma ij\mathbf{E}_{ij}$$

we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}_{ij}\mathbf{E}_{ij}^{*} = \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}_{ii} = n\mathbf{I}_m$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}_{ij}^{*}\mathbf{E}_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{E}_{jj} = m\mathbf{I}_n$$

Then

$$v(\boldsymbol{G}) = \frac{1}{\sigma^2}\max\{\|n\boldsymbol{I}_m\|, \|m\boldsymbol{I}_n\|\} = \frac{1}{\sigma^2}\max\{m, n\}$$

Thus

$$\mathbb{E}\|\boldsymbol{G}\| \leq \frac{1}{\sigma}\sqrt{2\max\{m, n\}\log(m+n)}$$

and

$$\mathbb{P}(\|\boldsymbol{G}\| \geq t) \leq (m+n)\exp\left(\frac{-t^2\sigma^2}{2\max\{m, n\}}\right)$$

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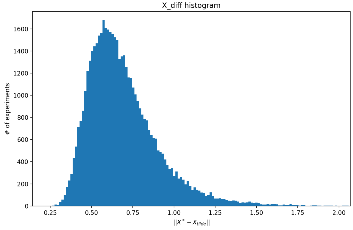
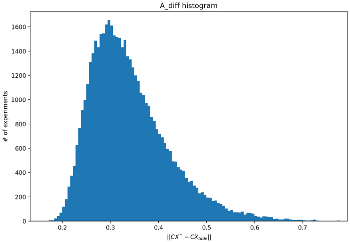
Empirical Results
Experiments

Experiment settings: $\boldsymbol{A} \sim \mathcal{N}(5, 1)$, $\boldsymbol{G} \sim \mathcal{N}(0, 0.01)$. We sample $\boldsymbol{A}$ for 50000 times and plot the histogram for both $\left\|\boldsymbol{X}^* - \tilde{\boldsymbol{X}}\right\|_2$ and $\left\|\boldsymbol{C}\boldsymbol{X}^* - \boldsymbol{C}\tilde{\boldsymbol{X}}\right\|_2$



(a) $\left\|\boldsymbol{X}^* - \tilde{\boldsymbol{X}}\right\|_2$



(b) $\left\|\boldsymbol{C}\boldsymbol{X}^* - \boldsymbol{C}\tilde{\boldsymbol{X}}\right\|_2$