1.10

(a)

$f(x) = \sqrt{1+x^2}$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$

$f''(x) = \frac{x' \cdot \sqrt{1+x^2} - (\sqrt{1+x^2})' \cdot x}{(\sqrt{1+x^2})^2} = \sqrt{1+x^2}^{-\frac{3}{2}}$

$f'''(x) = -\frac{3x}{(1+x^2)^{\frac{5}{2}}}$

$f(0) = 1 \quad f'(0) = 0 \quad f''(0) = 0 \quad f'''(0) = 0$

By Taylor's Theorem:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots \approx f(0) = 1$$

Hence

$$\lim_{x \to 0} f(x) = \frac{\sqrt{1+x^2} - 1}{x^2} = \frac{(\sqrt{1+x^2} - 1)(\sqrt{1+x^2} + 1)}{x^2(\sqrt{1+x^2} + 1)} = \frac{x^2}{x^2(\sqrt{1+x^2} + 1)} = \frac{1}{\sqrt{1+x^2} + 1} = \frac{1}{2}$$

(b)

Since $f(x)$ is an even function so W.L.O.G we analyze for $x > 0$, as we observe the value of $\sqrt{1+x^2}$ is a little larger than 1 for smaller values of x. In floating-point numbers just to the right of $x = 1$ we have $x = 1 + \epsilon$, $x = 1 + 2\epsilon \cdots$ Then due to round-off rule of the system, for $x^2 < \epsilon/2$, $1 + x^2$ will be round to 1 and that causes

$$f(x) = \frac{\sqrt{1} - 1}{x^2} = 0$$

And for real number $b = \sqrt{\frac{\epsilon}{2}} = 10^{-8}$, if $x < b$, we have $f(x) = 0$.

Additionally the oscillations in the graph is caused by $\sqrt{1+x^2}$-value passes from one floating point interval (such as $(1, 1 + \frac{\epsilon}{2})$) to the next one (such as $(1 + \frac{\epsilon}{2}, 1 + \frac{3\epsilon}{2})$) and when $x = 10^{-8}$, $f(x)$ is jumping in to a value less than 1. Also notice that in every interval the function could be expressed as

$$f(x) = \frac{c \cdot \epsilon}{x^2}, \quad \exists c \in \mathbb{R}_+$$

so the function produces a hyperbolic curve in every interval.

1.8(a) Extra credit

For $k \geq 1$, since $e^{-k} < 1$ we have

$$\frac{e^k}{1+e^k} = \frac{1}{e^{-k}+1} > \frac{1}{1+1} = \frac{1}{2}$$

Also notice that

$$\frac{e^k}{1+e^k} < 1$$

thus we have partial sum inequality

$$\sum_{k=1}^{1000} \frac{1}{2} < \sum_{k=1}^{1000} \frac{e^k}{1+e^k} < \sum_{k=1}^{1000} 1$$

since for $k = 0$

$$\frac{1}{2} \leq \frac{e^k}{1+e^k} < 1$$

then we have

$$\sum_{k=1}^{1000} \frac{1}{2} + \frac{1}{2} < \sum_{k=1}^{1000} \frac{e^k}{1+e^k} + \frac{e^0}{1+e^0} < \sum_{k=1}^{1000} 1 + 1$$

which is

$$\frac{1001}{2} < \sum_{k=0}^{1000} \frac{e^k}{1+e^k} < 1001$$