A Unified Micromechanical Model for the Mechanical Properties of Two Constituent Composite Materials. Part I: Elastic Behavior

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ABSTRACT: This series of papers reports a new, general, and unified micromechanical model for estimating the three-dimensional mechanical properties of a composite made from two isotropic constituent materials, i.e., continuous fiber and matrix. The present paper focuses on model development and its application to the prediction of the composite elastic property. The most important feature of this model is that the stresses generated in the constituents in a representative volume element of the composite are correlated with a bridging matrix. Based on this bridging matrix, those required quantities for the composite follow easily. A general routine to determine the bridging matrix elements is presented, and a set of explicit expressions of them for simulating a transversely isotropic composite is given. The bridging matrix depends on the physical as well as the geometrical properties of the fiber and matrix materials. For a fixed geometry, the bridging matrix can depend only on the physical properties of the constituents. This feature makes it easy to extend the present bridging matrix to include any inelastic deformation effect from the constituents and to establish a unified model to simulate, in addition, the plastic, strength, rubber-elastic, and laminate failure behaviors of fibrous composites, which will be addressed subsequently. Only linear elastic properties are considered in the present paper. The model has been applied to estimate the elastic properties of two unidirectional composites and a knitted-fabric-reinforced composite. Good agreement has been found between the predicted and available experimental data.

KEY WORDS: unidirectional composite, knitted fabric composite, mechanical property, micromechanical model, unified model, elastic behavior, stiffness, compliance.

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1. INTRODUCTION

IT HAS BEEN recognized that technological development depends on advances in the field of materials. Whatever the field may be, the final limitation on achievement depends on the available materials. In this regard, composites can be considered as an ultimate solution to this problem. Continuous-fiber-reinforced composites constitute a large group of such materials.

A prerequisite to exploiting the properties of any material is a thorough understanding of those properties. Mechanical properties include elastic stiffness/compliance, yield strength (knee stress), tensile strength, ultimate strain, etc. The conventional approach to these properties is based on a judgement that demands an experimental determination of material response characteristics. This approach certainly offers the best advantage for isotropic materials. However, the introduction of the directional dependence of the mechanical properties of anisotropic composite materials along with the variation of these properties with composition, orientation, and packing geometry of the components greatly magnifies the labor and expense involved in experimental determination of the overall material response characteristics. These complications provide a strong practical motivation for the development of a theoretical approach, called a micromechanics theory, to the mechanical properties of the composites. Thus, the purpose of this theory is to predict the mechanical properties of a composite in terms of the known properties of its constituent materials as well as the internal microstructures as reflected by the relative orientation, size, and packing geometry of each of the constituents.

A great variety of micromechanical models exists in the literature when both the fiber and the matrix deformations are confined in the linear elastic range. Surveys about such models as well as their developments have been given, among others, by Chamis and Sendeckyj [1], Hashin [2,3], Halpin [4], and McCullough [5]. Almost all these models were developed based on unidirectional fiber-reinforced composites, and many can be applied to other kinds of fibrous composites giving, in general, good predictions for their linear elastic properties. However, it is difficult to generalize these models to the inelastic deformation situations. Most of these models, without significant modifications, cannot be applied to predict the yield strength (knee stress), the tensile strength, or the nonlinear stress-strain relationship of the composites, since the existing models for estimating composite elastic behavior seldom have any inherent connections with the other mechanical properties of the composites.

On the other hand, the stress or strain shared by the matrix in a representative volume should be determined only by the material properties of the constituents and the geometry of the fiber embedded in the matrix. It should therefore have a fixed and explicit relationship with the stress or strain shared by the fiber, as long as the deformations of the constituents are within the same (elastic or inelastic) range. When the constituent materials deform continuously from one state (e.g., linear elastic) to another (e.g., plastic), the coefficients correlating the two shares should be only influenced by the varied material properties provided that the effect of the constituent deformations is insignificant. These considerations draw the author's attention to look for an explicit coefficient matrix, called a bridging matrix, to correlate the stresses generated in the matrix with those in the fiber. Once this bridging matrix is determined, the Hill's stress concentration matrices [6] are easily obtainable from knowledge of the constituent material properties and the fiber volume fraction, as are the mechanical properties of the composite. A three-dimensional unified micromechanics model is thus developed based on this bridging matrix and is presented in this series of papers. In the present paper, the model development and its use in predicting the linear elastic behavior of the composites are elaborated. Applications of the model to predict the elasto-plastic, strength, rubber-elastic, and multidirectional laminate failure behaviors of fibrous composites will be described in the subsequent papers.

In developing the new unified model, the bridging matrix plays a fundamental role. This paper presents a general routine for the determination of the bridging matrix. A set of explicit expressions for the bridging matrix elements is obtained, based on some well-known elasticity solutions and a semi-empirical approach. The model is then verified with unidirectional composites as well as a weft-knitted-fiber fabric-reinforced composite. High correlation between the predicted results and available experimental data was found.

2. THEORY

Let x_1, x_2, and x_3 be rectangular co ordinates on the three-dimensional geometry of a composite. The fibers in the composite are assumed to be unidirectionally arranged, with their axes parallel to the x_1 direction. Suppose that V_f and V_m are, respectively, the volume fractions of the fibers and the matrix in a representative volume element (RVE) of the composite. The volume averaged stress tensors of the fibers, the matrix, and the composite in the RVE are denoted by $[\sigma_{ij}^f]$, $[\sigma_{ij}^m]$, and $[\sigma_{ij}]$, respectively. The averaged strain tensors are $[\varepsilon_{ij}^f]$, $[\varepsilon_{ij}^m]$, and $[\varepsilon_{ij}]$. By using contracted notation (vector form), these two sets of stresses and strains satisfy the following fundamental relationships [7]:

$$\{\sigma_i\} = V_f\{\sigma_i^f\} + V_m\{\sigma_i^m\} \tag{1}$$

and

$$\{\varepsilon_i\} = V_f\{\varepsilon_i^f\} + V_m\{\varepsilon_i^m\} \tag{2}$$

where $\{d\varepsilon_i\} = \{d\varepsilon_1, d\varepsilon_2, d\varepsilon_3, d\varepsilon_4, d\varepsilon_5, d\varepsilon_6\}^T = \{d\varepsilon_{11}, d\varepsilon_{22}, d\varepsilon_{33}, 2d\varepsilon_{23}, 2d\varepsilon_{13}, 2d\varepsilon_{12}\}^T$ and $\{d\sigma_i\} = \{d\sigma_{11}, d\sigma_2, d\sigma_3, d\sigma_4, d\sigma_5, d\sigma_6\}^T = \{d\sigma_{11}, d\sigma_{22}, d\sigma_{33}, d\sigma_{23}, d\sigma_{13}, d\sigma_{12}\}^T$.

Let us use $[S_{ij}^f]$, $[S_{ij}^m]$, and $[S_{ij}]$ to denote the compliance matrices of the fiber, the matrix, and the composite, respectively, where

$$[S_{ij}^f] = \begin{bmatrix} [S_{ij}^f]_\sigma & 0 \\ 0 & [S_{ij}^f]_\tau \end{bmatrix} \tag{3.1}$$

$$[S_{ij}^m] = \begin{bmatrix} [S_{ij}^m]_\sigma & 0 \\ 0 & [S_{ij}^m]_\tau \end{bmatrix} \tag{3.2}$$

$$[S_{ij}] = \begin{bmatrix} [S_{ij}]_\sigma & 0 \\ 0 & [S_{ij}]_\tau \end{bmatrix} \tag{3.3}$$

In Equations (3.1)–(3.3), $[S_{ij}^f]_\sigma$, $[S_{ij}^m]_\sigma$, and $[S_{ij}]_\sigma$ are the sub-matrices of the corresponding compliances correlating normal stresses with elongation strains, while $[S_{ij}^f]_\tau$, $[S_{ij}^m]_\tau$, and $[S_{ij}]_\tau$ are the sub-matrices correlating shear stresses with shear strains. In isotropic cases, they take the following forms:

$$[S_{ij}^I]_\sigma = \begin{bmatrix} \dfrac{1}{E_I} & -\dfrac{\nu_I}{E_t} & -\dfrac{\nu_I}{E_I} \\ & \dfrac{1}{E_I} & -\dfrac{\nu_I}{E_I} \\ & symmetric & \dfrac{1}{E_I} \end{bmatrix}, \quad I = f, m \tag{4.1}$$

$$[S_{ij}^I]_\tau = \begin{bmatrix} \dfrac{1}{G_I} & 0 & 0 \\ & \dfrac{1}{G_I} & 0 \\ & symmetric & \dfrac{1}{G_I} \end{bmatrix}, \quad I = f, m \tag{4.2}$$

$$[S_{ij}]_\sigma = \begin{bmatrix} \dfrac{1}{E_{11}} & -\dfrac{\nu_{12}}{E_{11}} & -\dfrac{\nu_{13}}{E_{11}} \\ & \dfrac{1}{E_{22}} & -\dfrac{\nu_{23}}{E_{22}} \\ & symmetric & \dfrac{1}{E_{33}} \end{bmatrix} \tag{4.3}$$

$$[S_{ij}]_\tau = \begin{bmatrix} \frac{1}{G_{23}} & 0 & 0 \\ & \frac{1}{G_{13}} & 0 \\ & symmetric & \frac{1}{G_{12}} \end{bmatrix} \tag{4.4}$$

With $[S_{ij}^f]$, $[S_{ij}^m]$, and $[S_{ij}]$, the stresses and strains are connected through

$$\{\varepsilon_i^f\} = [S_{ij}^f]\{\sigma_j^f\} \tag{5.1}$$

$$\{\varepsilon_i^m\} = [S_{ij}^m]\{\sigma_j^m\} \tag{5.2}$$

and

$$\{\varepsilon_i\} = [S_{ij}]\{\sigma_j\} \tag{5.3}$$

Our main goal is to determine $[S_{ij}]$ from the given $[S_{ij}^f]$ and $[S_{ij}^m]$. Suppose there exists a bridging matrix $[A_{ij}]$ such that

$$\{\sigma_i^m\} = [A_{ij}]\{\sigma_j^f\} \tag{6}$$

Substituting Equation (6) into Equation (1) and inverting the resulting matrix equation, we get

$$\{\sigma_i^f\} = (V_f[I] + V_m[A_{ij}])^{-1}\{\sigma_j\} \tag{7.1}$$

where $[I]$ is a unit matrix. Introducing Equation (7.1) into Equation (6) gives

$$\{\sigma_i^m\} = [A_{ij}](V_f[I] + V_m[A_{ij}])^{-1}\{\sigma_j\} \tag{7.2}$$

It is seen that the Hill's stress concentration matrices [6] are obtained as

$$[B_{ij}^f] = (V_f[I] + V_m[A_{ij}])^{-1} \text{ and } [B_{ij}^m] = [A_{ij}](V_f[I] + V_m[A_{ij}])^{-1}$$

which satisfy $V_f[B_{ij}^f] + V_m[B_{ij}^m] = [I]$.

On the other hand, substituting Equations (5.1) and (5.2) into Equation (2) and making use of Equation (6), one obtains

$$\{\varepsilon_i\} = (V_f[S_{ij}^f] + V_m[S_{ij}^m][A_{ij}])\{\sigma_j^f\}$$

Incorporating Equation (7.1) with the last equation and comparing the resulting equation with Equation (5.3), the compliance matrix $[S_{ij}]$ of the composite is seen to be

$$[S_{ij}] = (V_f[S^f_{ij}] + V_m[S^m_{ij}][A_{ij}])(V_f[I] + V_m[A_{ij}])^{-1} \tag{8}$$

Furthermore, the stiffness matrix $[C_{ij}]$ of the composite is simply given by

$$[C_{ij}] = [S_{ij}]^{-1} \tag{9}$$

Therefore, once the bridging matrix, $[A_{ij}]$, is determined, the overall compliance and stiffness matrices of the composite can be calculated from Equations (8) and (9). In addition, as Equations (7.1) and (7.2) explicitly give the stresses shared by the fiber and the matrix in the RVE respectively, an optimal design of the composite is able to achieve upon the knowledge of the constituent material properties as well as of the packing geometry of the fibers. The only assumption inherently implied is the existence of a perfect bond between the fiber and matrix interface in the composite.

It is clear from the above expressions that the most important aspect of the present model is to properly define the bridging matrix, $[A_{ij}]$. The only prerequisite for such a definition is that the overall compliance matrix, $[S_{ij}]$, given by Equation (8) must be symmetric. However, as can be expected, the material properties of the constituents, the relative position of the fiber arrangement, the fiber volume fraction in the composite, and the fiber cross-sectional shape may also affect the bridging matrix. Details will be discussed in the next section.

3. DETERMINATION OF BRIDGING MATRIX

When an isotropic fiber material is unidirectionally arranged in another isotropic matrix, the resulting composite is generally considered transversely isotropic. In such a case, the bridging matrix $[A_{ij}]$ can be sub-divided into

$$[A_{ij}] = \begin{bmatrix} [a_{ij}] & [0] \\ [0] & [b_{ij}] \end{bmatrix}$$

in which $[a_{ij}]$ and $[b_{ij}]$ are 3×3 sub-matrices such that

$$[S_{ij}]_\sigma = (V_f[S^f_{ij}]_\sigma + V_m[S^m_{ij}]_\sigma[a_{ij}])(V_f[I] + V_m[a_{ij}])^{-1} \tag{10}$$

$$[S_{ij}]_\tau = (V_f[S^f_{ij}]_\tau + V_m[S^m_{ij}]_\tau[b_{ij}])(V_f[I] + V_m[b_{ij}])^{-1} \tag{11}$$

Clearly, $[b_{ij}]$ should be diagonal, since $[S_{ij}]_\tau$, $[S^f_{ij}]_\tau$ and $[S^m_{ij}]_\tau$ are all diagonal. Let us examine the sub-matrix $[a_{ij}]$ first. In the case that $[S^f_{ij}]$ and $[S^m_{ij}]$ take the forms of Equations (3.1), (3.2), (4.1), and (4.2), $[S_{ij}]$ represents the compliance matrix of the transversely isotropic material. There are only five independent elastic constants involved in $[S_{ij}]$, with $E_{33} = E_{22}$, $\nu_{13} = \nu_{12}$, and $G_{13} = G_{12}$ in Equations (4.3) and (4.4). Consequently, only five elements of $[A_{ij}]$ can be independent. One of them is $b_{22} = b_{33}$ and the other four are among $[a_{ij}]$. We may thus set $a_{21} = a_{31} = 0$, and take a_{11}, a_{22}, a_{33}, and a_{32} to be independent.

To get the relations relating the remaining elements of the bridging matrix, a_{12}, a_{13}, and a_{23}, with the independent elements, a_{11}, a_{22}, a_{33}, and a_{32}, the symmetric condition on $[S_{ij}]_\sigma$ must be applied. Substituting the chosen $[a_{ij}]$ into Equation (10) and imposing that $S_{ji} = S_{ij}$ for all $i, j = 1, 2, 3$, three algebraic equations are obtained as follows:

$$\alpha_{11}a_{12} + \alpha_{12}a_{13} + \alpha_{13}a_{23} = p_1 \tag{12.1}$$

$$\alpha_{21}a_{12}a_{23} + \alpha_{22}a_{13} + \alpha_{2}\ a_{23} = p_2 \tag{12.2}$$

$$\alpha_{31}a_{12}a_{23} + \alpha_{32}a_{12} + \alpha_{33}a_{13} + \alpha_{34}a_{23} = p_3 \tag{12.3}$$

In Equations (12.1)–(12.3), the parameters α_{ij} and p_i are given by the following:

$$\alpha_{11} = (V_f + V_m a_{33})(S_{11}^f - S_{11}^m)$$

$$\alpha_{12} = -V_m(S_{11}^f - S_{11}^m)a_{32}$$

$$\alpha_{13} = -V_m(S_{12}^f - S_{12}^m)a_{32}$$

$$\alpha_{21} = V_m(S_{11}^f - S_{11}^m)$$

$$\alpha_{22} = -V_f + V_m a_{22})(S_{11}^f - S_{11}^m)$$

$$\alpha_{23} = -[(V_f + V_m a_{11})(S_{12}^f - S_{12}^m) - V_m(S_{13}^f - S_{13}^m)a_{32}]$$

$$\alpha_{31} = V_m(S_{12}^f - S_{12}^m)$$

$$\alpha_{32} = (V_f + V_m a_{33})(S_{31}^f - S_{31}^m)$$

$$\alpha_{33} = [(V_f + V_m a_{22})(S_{12}^f - S_{12}^m) + V_m(S_{31}^f - S_{31}^m)a_{32}]$$

$$\alpha_{34} = -(V_f + V_m a_{11})(S_{22}^f - S_{22}^m)$$

$$p_1 = (V_f + V_m a_{33})(S_{12}^f - S_{12}^m)(a_{11} - a_{22}) - (V_f + V_m a_{11})(S_{13}^f - S_{13}^m)a_{32}$$

$$p_2 = (V_f + V_m a_{22})(S_{13}^f - S_{13}^m)(a_{33} - a_{11})$$

$$p_3 = (V_f + V_m a_{11})(S_{23}^f - S_{23}^m)(a_{33} - a_{22}) - (S_{33}^f - S_{33}^m)a_{32}\,]$$

As Equations (12.1)–(12.3) are nonlinear, there might exist two sets of solutions to a_{12}, a_{13}, and a_{23}.

Choosing a_{12} as the primary variable, two solutions of it are given, respectively, by

$$a_{12}^{I} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad a_{12}^{II} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{13.1}$$

where $a = \alpha_{21}\gamma_2$, $b = \alpha_{21}\beta_2 + \alpha_{22}\gamma_1 + \alpha_{23}\gamma_2$, $c = \alpha_{22}\beta_1 + \alpha_{23}\beta_2 - p_2$,

$$\beta_1 = \frac{[\alpha_{13}(p_3/\alpha_{31} - p_2/\alpha_{21}) - p_1(\alpha_{34}/\alpha_{31} - \alpha_{23}/\alpha_{21})]}{\alpha_{13}(\alpha_{33}/\alpha_{31} - \alpha_{22}/\alpha_{21}) - \alpha_{12}(\alpha_{34}/\alpha_{31} - \alpha_{23}/\alpha_{21})} \tag{13.2}$$

$$\beta_2 = \frac{[p_1(\alpha_{33}/\alpha_{31} - \alpha_{22}/\alpha_{21}) - \alpha_{12}(p_3/\alpha_{31} - p_2/\alpha_{21})]}{\alpha_{13}(\alpha_{33}/\alpha_{31} - \alpha_{22}/\alpha_{21}) - \alpha_{12}(\alpha_{34}/\alpha_{31} - \alpha_{23}/\alpha_{21})} \tag{13.3}$$

$$\gamma_1 = \frac{[-\alpha_{13}(\alpha_{32}/\alpha_{31}) + \alpha_{11}(\alpha_{34}/\alpha_{31} - \alpha_{23}/\alpha_{21})]}{\alpha_{13}(\alpha_{33}/\alpha_{31} - \alpha_{22}/\alpha_{21}) - \alpha_{12}(\alpha_{34}/\alpha_{31} - \alpha_{23}/\alpha_{21})} \tag{13.4}$$

$$\gamma_2 = \frac{[\alpha_{12}(\alpha_{32}/\alpha_{31}) - \alpha_{11}(\alpha_{33}/\alpha_{31} - \alpha_{22}/\alpha_{21})]}{\alpha_{13}(\alpha_{33}/\alpha_{31} - \alpha_{22}/\alpha_{21}) - \alpha_{12}(\alpha_{34}/\alpha_{31} - \alpha_{23}/\alpha_{21})} \tag{13.5}$$

With the formulae (13.1)–(13.5), the other two variables are obtained as

$$a_{13} = \beta_1 + \gamma_1 a_{12}, \qquad a_{23} = \beta_2 + \gamma_2 a_{12} \tag{13.6}$$

The independent variables, a_{11}, a_{22}, a_{33}, and a_{32}, are expected to depend on the elastic constants (Young's modulus and Poisson's ratio) of the matrix and the fibers and possibly on the relative position of the fibers embedded in the matrix as well as fiber cross-sectional shapes. The fiber volume fraction may also have an influence. When the properties of the two materials become the same, the bridging matrix,

$[A_{ij}]$, must be identical to the unit matrix. Hence, the general forms of the independent variables are always able to be expressed as the power series of the material properties, i.e.,

$$a_{11} = 1 + \alpha_{11}(1 - E_m/E_f) + \ldots, \tag{14.1}$$

$$a_{22} = 1 + \alpha_{21}(1 - E_m/E_f) + \ldots, \tag{14.2}$$

$$a_{32} = \alpha_{31}(1 - E_m/E_f) + \alpha_{32}(1 - \nu_m/\nu_f) + \ldots, \tag{14.3}$$

$$a_{33} = 1 + \alpha_{41}(1 - E_m/E_f) + \ldots \tag{14.4}$$

where α's [different from those in Equations (12.1)–(12.3)] are constants which may depend on the geometric properties of the components but are independent of their physical properties. Similarly, a general expression for b_{22} and b_{33} is given as

$$b_{22} = b_{33} = 1 + \alpha_{51}(1 - G_m/G_f) + \ldots \tag{14.5}$$

The expression for the last diagonal element, b_{11}, is rather complicated. However, it is not an independent variable. For a transversely isotropic material, we know that S_{44} is determined by

$$S_{44} = 1/G_{23} = 2(1 + \nu_{23})/E_{22} = 2(S_{22} - S_{23}) \tag{15}$$

Hence, b_{11} is actually not necessary.

The parameters α in Equations (14.1)–(14.5) can be determined through experiments or numerical simulations. The most important, and also the most attractive, feature of the present approach is that when the composite undergoes inelastic deformations, the formal relation (6) will remain unchanged but only with varied constituent material constants in place of the corresponding elastic constants as given in Equations (14.1)–(14.5). This is because when any constituent material deforms from the elastic region to an inelastic one, the fiber packing geometry (the fiber volume fraction, fiber arrangement in the matrix, the fiber cross-sectional shape, etc.) does not change or varies by only a negligibly small amount. Therefore, only the material parameters involved in the independent bridging elements, given by Equations (14.1)–(14.5), will be different. From this point of view, one may need to carry out only linear elastic experiments or simulations on a composite to determine the coefficients α in Equations (14.1)–(14.5). This kind of unified approach to the mechanical behavior of the composites is believed to be unique but very useful in engineering applications.

4. EXPLICIT FORMULAE

Evidently, the most rigorous analysis will depend on the exact parameters α involved in Equations (14.1) to (14.5), which should be obtained experimentally. They can be determined by measuring the overall engineering moduli of the composite because, from Equation (8), these moduli are functions of the considered parameters α. In reality, however, not every experiment can be easily carried out. For some materials such as very thin and flexible composites, accurate measurements of all the engineering moduli are difficult or even impossible to obtain with currently available techniques. Extensive cost may also be a factor in prohibiting some experiments. Therefore, it is highly preferable in practice to give some fixed magnitudes for these parameters, with which the predicted results for the composites will be accurate enough based on engineering judgment and on considerations of the complexities and uncertainties involved. For this purpose, a set of formulae, based on some well-known elasticity solutions and a semi-empirical approach, is determined as follows. Let us imagine that a representative volume element is composed of a concentric cylinder. Based on this, some rigorous analyses have been done and it was found that [7–9]:

$$\sigma_{12}^m = \eta_s \sigma_{12}^f \tag{16.1}$$

where

$$\eta_s = \frac{1}{2}\left(1 + \frac{G_m}{G_f}\right) \tag{16.2}$$

Comparing Equation (16.2) with (14.5), we see that

$$\alpha_{51} = -0.5 \tag{17}$$

Also, experiments have shown that the overall longitudinal stress of a unidirectional fiber-reinforced composite is comparable to the stress of the fiber in the same direction, whereas the overall transverse stress of the composite is comparable to the matrix stress in that direction. It is thus reasonable to assume that the averaged normal stresses between the fiber and the matrix are correlated by

$$\sigma_1^m = a_{11}\sigma_1^f + a_{12}\sigma_2^f + a_{13}\sigma_3^f \tag{18.1}$$

$$\sigma_2^m = a_{22}\sigma_2^f \tag{18.2}$$

$$\sigma_3^m = a_{33}\sigma_3^f \tag{18.3}$$

Because of the axisymmetry of the concentric cylinder geometry, we may further assume that $a_{33} = a_{22}$. Substituting so-defined a_{ij}'s, i.e.

$$a_{21} = a_{31} = a_{23} = a_{32} = 0 \text{ and } a_{33} = a_{22} \tag{19}$$

into Equation (13), it is found that

$$a_{13} = a_{12} = (S_{12}^f - S_{12}^m)(a_{11} - a_{22})/(S_{11}^f - S_{11}^m) \tag{20}$$

Hence, there are only two independent elements, a_{11} and a_{22}, to be defined. Again, substituting Equation (19) into (10) and making some manipulation, the longitudinal Young's modulus is obtained as

$$E_{11} = \frac{(V_f + V_m a_{11})}{(V_f S_{11}^f + V_m a_{11} S_{11}^m)} \tag{21}$$

On the other hand, Hill [6] obtained rigorous and relatively tight bounds on the longitudinal Young's modulus, E_{11}, of a unidirectional fiber-reinforced composite regardless of the fiber form and packing geometry. Hill's results are as follows:

$$\frac{4V_f V_m(\nu_f - \nu_m)^2}{\frac{V_f}{K_m} + \frac{V_m}{K_f} + \frac{1}{G_m}} \le E_{11} - V_f E_f - V_m E_m \le \frac{4V_f V_m(\nu_f - \nu_m)^2}{\frac{V_f}{K_m} + \frac{V_m}{K_f} + \frac{1}{G_f}} \tag{22}$$

in which $K = E/[2(1 - 2\nu)(1 + \nu)]$. Equation (22) indicates that the rule-of-mixtures approach gives a quite accurate prediction for the longitudinal Young's modulus. Supposing that the longitudinal modulus defined by Equation (21) is equal to that given by the rule-of-mixtures formula, we get

$$a_{11} = E_m/E_f \tag{23}$$

or

$$\alpha_{11} = -1 \tag{24}$$

It is noted that with this set of a_{ij}'s, the resulting longitudinal Poisson's ratio ν_{12}, is exactly the same as that given by the rule-of-mixture formula, i.e., $\nu_{12} = V_f \nu_f + V_m \nu_m$.

There remains a_{22}, which defines the transverse modulus E_{22}, to be determined. Many different micromechanical formulae have been proposed for the transverse modulus E_{22}. In fact, one of the main motivations of different micromechanics approaches is to give a distinguished expression for the transverse modulus [5]. However, there is not such a simple formula for which the same tight bounds as for E_{11} given by Equation (22) exist. Tsai and Hahn [7] chose $a_{22} = 0.5$ in a modified rule-of-mixtures formula, and much better estimations were obtained. In light of the fact that $a_{22} = 1$ must be valid when the fiber and the matrix become the same, a formula similar to Equation (16.2) is chosen for a_{22}, i.e.,

$$a_{22} = \frac{1}{2}\left(1 + \frac{E_m}{E_f}\right) \tag{25}$$

which corresponds to $\alpha_{21} = -0.5$ in Equation (14.2).

It should be pointed out that as long as conditions (19) are valid, not only the transverse tensile modulus E_{22} but also the transverse shear modulus G_{23} will depend on the chosen parameter a_{22}. This is different from the situation in Tsai and Hahn's approach [7] where a_{22} had an effect only on E_{22}. In the present and subsequent papers, various calculations have been performed based on the chosen a_{22}, and comparisons with many different experiments have confirmed that the formula (25) is correct and accurate.

By means of the bridging matrix defined above, a set of new formulae for the engineering moduli of the transversely isotropic composite is derived as follows:

$$E_{11} = V_f E_f + V_m E_m \tag{26.1}$$

$$\nu_{12} = \nu_{13} = V_f \nu_f + V_m \nu_m \tag{26.2}$$

$$E_{22} = (V_f + V_m a_{11})(V_f + V_m a_{22})^2 d_3 \tag{26.3}$$

$$\nu_{23} = d_4 / d_3 \tag{26.4}$$

$$G_{12} = G_{13} = (V_f + V_m b_{22}) G_f G_m / (V_f G_m + b_{22} V_m G_f) \tag{26.5}$$

and

$$G_{23} = 0.5 E_{22} / (1 + \nu_{23}) \tag{26.6}$$

where

$$d_3 = S_{11}^f [V_f^3 + V_m V_f^2 (a_{11} + a_{22}) + V_f V_m^2 a_{11} a_{22}]$$
$$+ S_{11}^m [V_f V_m (V_f + V_m a_{33} + V_m a_{11}) a_{22} + V_m^3 a_{11} a_{22} a_{33}]$$
$$- V_f V_m (V_f + V_m a_{33})(S_{12}^f - S_{12}^m) a_{12}$$

$$d_4 = V_f V_m (V_f + V_m a_{33})(S_{12}^f - S_{12}^m) a_{12} - S_{12}^f [V_f^3 + V_f V_m (V_f + V_m a_{11}) a_{22}]$$
$$- S_{12}^m [V_f V_m (V_f + V_m a_{22} + V_m a_{11}) a_{22} + V_m (V_f^2 + V_m^2 a_{22} a_{33}) a_{11}]$$

5. VERIFICATION

Verification for the proposed model is carried out through calculations for the elastic moduli of (a) unidirectional composites and (b) a knitted glass-fiber fabric-reinforced epoxy-matrix composite. Comparisons between predicted results and available experimental data are also made.

5.1 Unidirectional Composite

From Equations (26.1) and (26.2) we see that the longitudinal elastic modulus, E_{11}, and the Poisson's ratio, ν_{12}, given by the present formulae are exactly the same as those given by the rule-of-mixtures formulae. Predictions for E_{11} and ν_{12} from these formulae are sufficiently accurate [7]. Thus, only the other moduli need to be verified. Let us compare the calculated results for the transverse modulus E_{22} and the shear modulus G_{12} with some experimental data. A glass/epoxy composite with $E_f = 73.1$ GPa, $E_m = 3.45$ GPa, $\nu_f = 0.22$, and $\nu_m = 0.35$ is used for both the analytical prediction and the experiment to obtain the transverse modulus E_{22} of the composite. The results are shown in Figure 1, in which the experimental data are taken from Reference [7]. To obtain comparisons for the shear modulus G_{12}, another glass/epoxy composite with $G_f = 30.2$ GPa and $G_m = 1.8$ GPa is used. The experimental data for G_{12} are also taken from Reference [7]. The shear modulus versus fiber volume fraction is shown in Figure 2. From these two figures, we can see that the predictions for both E_{22} and G_{12} from the present formulae, Equations (18.3) and

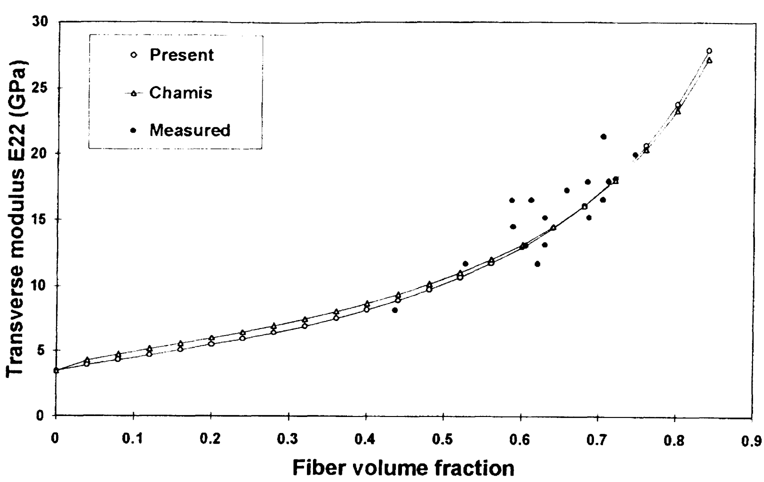

Figure 1. Transverse modulus of unidirectional composite versus fiber volume fraction. Present and Chamis stand for the present and Chamis formulae. The material parameters used are E_f *= 73.1 GPa,* ν_f *= 0.22,* E_m *= 3.45 GPa, and* ν_m *= 0.35.*

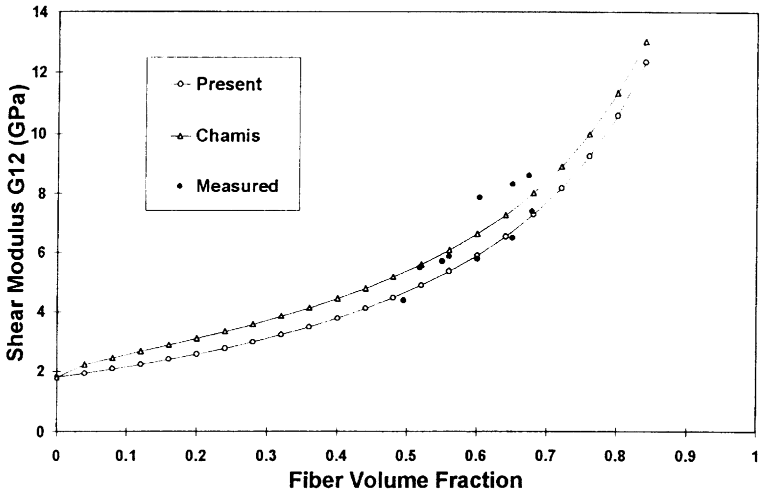

Figure 2. *Shear modulus of unidirectional composite versus fiber volume fraction. The used moduli are* G_f = *30.2 GPa and* G_m = *1.8 GPa.*

(18.5), are quite accurate. Predicted results by using Chamis model formulae [10] are also shown in the figures for comparison. The reason for the author to choose only the Chamis model is that extensive comparisons between the Chamis model and other micromechanical models exist in the literature [11]. It can be seen that the Chamis formulae also give accurate predictions for E_{22} and G_{12}. Therefore, both the present formulae and the Chamis formulae can provide an accurate estimation of the transverse modulus and in-plane shear modulus of a unidirectional fiber-reinforced composite. However, the Chamis formulae can estimate only four independent elastic constants for a transversely isotropic composite if the constituent materials, the fiber and the matrix, are both isotropic [10]. This shortcoming might affect their simulation accuracy in some cases, as shown in the following example.

5.2 Knitted Fabric Reinforced Composite

A more complicated example is chosen to predict the elastic moduli of a plain weft-knitted glass-fiber fabric-reinforced epoxy-matrix composite. Experimental data for the elastic properties of this composite are available in the literature [12], which, together with constituent properties, are summarized in Table 1. Here, "Chamis" refers to the Chamis formulae and "present" to the present model formulae. The volume fraction of the fiber in the composite was 0.095 (V_f = 0.095). Schematic diagrams of the fabric and a unit cell for it are shown in Figures 3(a) and 3(b). As sindicated in Figure 3(b), the fabric unit cell can be further divided into four sub-cells with identical or symmetric geometry. One such sub-cell, when impregnated with the epoxy, can be taken as a composite RVE [12].

Table 1. Elastic properties of plain knitted fiber fabric composites.

Model	E_{xx} (GPa)	E_{yy} (GPa)	E_{zz} (GPa)	G_{xy} (GPa)	G_{xz} (GPa)	G_{yz} (GPa)	ν_{xy}	ν_{xz}	ν_{yz}
Exper.	5.38 (0.33)[1]	4.37 (0.07)[1]					0.48 (0.13)[1]		
Chamis	6.56 (21.9)[2]	5.18 (18.5)[2]	5.01	2.21	1.96	1.90	0.356 (25.8)[2]	0.312	0.308
Present	5.87 (9.1)[2]	4.59 (5.0)[2]	4.50	1.93	1.67	1.60	0.381 (20.1)[2]	0.340	0.375

The parameters used are: $E_f = 74$ GPa, $E_m = 3.6$ GPa, $\nu_f = 0.23$, $\nu_m = 0.35$, $V_f = 0.095$, $d = 0.0445$ cm, $C =$ 2.5 cycle/cm, and $W = 2$ cycle/cm.
[1]Standard scatter deviation of the experiment.
[2]Relative error (%).

Three steps are generally observed in the simulation of a knitted fabric composite. The first step is to subdivide the RVE of the composite (or yarns in the RVE) into sub-volumes (or segments), each of which is considered as a unidirectional (UD) composite. In the second step, a micromechanics model is applied to simulate the properties of all the UD composites in their local coordinate systems. Both the unified model, presented in the previous sections, and the Chamis model will be employed in the present simulation. The third step is to assemble the contributions from all the UD composites to get the overall property of the original RVE. As such, knowledge of yarn orientations, namely, relative positions of yarn segments, in the RVE is essential. This is achieved for the present fabric through the Leaf and

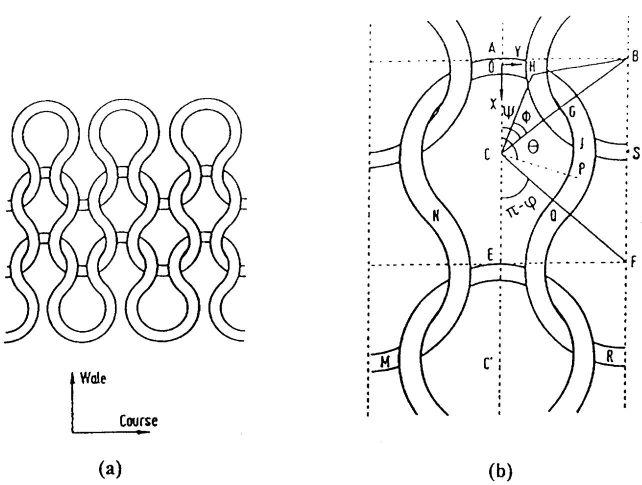

Figure 3. Schematic diagrams of: (a) idealized plain weft knitted fabric and (b) unit cell.

Glaskin model [12,13]. Only three fabric parameters, namely the yarn diameter, d, the fabric wale number, W, and the fabric course number, C, are necessary. With these parameters, the coordinates of a point on the first yarn in the sub-cell [(Figure 3(b)] are given by [12,13]

$$x = ad(1 - \cos\theta),$$

$$y = ad\sin\theta, \qquad 0 \le \theta \le \varphi$$

$$z = \frac{hd}{2}\left(1 - \cos\left(\pi\frac{\theta}{\varphi}\right)\right)$$

where

$$a = \frac{1}{4Wd\sin\varphi}$$

$$\varphi = \pi + \sin^{-1}\left(\frac{C^2 d}{[C^2 + W^2(1 - C^2 d^2)^2]^{1/2}}\right) - \tan^{-1}\left(\frac{C}{W(1 - C^2 d^2)}\right)$$

$$h = \left[\sin\left(\pi\frac{\psi}{\varphi}\right)\sin\left(\pi\frac{\phi}{\varphi}\right)\right]^{-1}$$

$$\psi = \sin^{-1}\left(\frac{2a}{2a - 1}\sin\varphi\right)$$

$$\phi = \cos^{-1}\left(\frac{2a - 1}{2a}\right)$$

The coordinates of a point on the second yarn are obtained just using symmetric conditions, which are given in discretized form as

$$x_1^{2nd} = 2ad - \frac{1}{2W\tan(\psi)}$$

$$y_1^{2nd} = \frac{1}{2W}$$

$$z_1^{2nd} = z_1^{1st}$$

$$x_n^{2nd} = x_1^{2nd} - x_n^{1st}$$

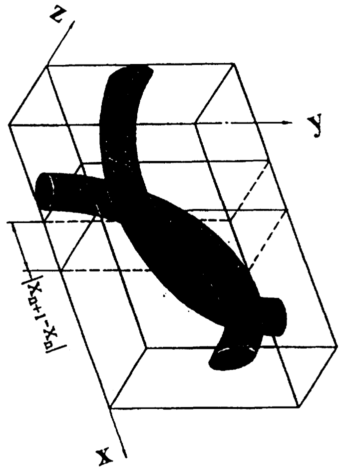

Figure 4. *A typical subdivision of an RVE of the plain weft knitted fabric composite.*

$$y_n^{2nd} = y_1^{2nd} - y_n^{1st}$$

$$z_n^{2nd} = z_n^{1st}, \qquad n \geq 2,3,\ldots$$

where the superscripts 1st and 2nd stand for the first and the second yarns, respectively. As only the relative positions of the yarn segments are important, another sub-cell, or RVE, is chosen for convenience of interpretation, as indicated in Figure 4.

The RVE in Figure 4 is subdivided into sub-volumes along the wale (x-) direction. A UD composite is considered to consist of one yarn segment and extra epoxy matrix, with its local x_1-coordinate along the segment direction. Thus, a sub-volume will be assumed to have contained two UD composites if two yarn segments are involved in this sub-volume. The fiber volume fraction of each UD composite, V_f, is taken as 0.095, equal to the overall fiber volume fraction of the RVE. Having determined the compliance matrices of all the UD composites in their local systems by using a micromechanics model (the present or the Chamis model), the overall compliance matrix of the RVE (namely, the knitted fabric composite) can be obtained through the use of either an iso-stress or iso-strain assumption. By using an iso-stress assumption—namely, the assumption that the stresses on each sub-volume, or, on each UD composite, in the global coordinate system are equal to the overall stresses on the RVE—one has [14]

$$[\overline{S_{ij}}] = \sum_{n=1}^{N-1} \frac{|x_{n+1}^{1st} - x_n^{1st}|}{(2L)} [\overline{S_{ij}}]_n^{1st} + \sum_{n=1}^{N-1} \frac{|x_{n+1}^{2nd} - x_n^{2nd}|}{(2L)} [\overline{S_{ij}}]_n^{2nd}$$

where $(N-1)$ is the total number of discretized segments of one yarn in the RVE, L is the projected length of one yarn on the x axis (wale direction) in the RVE, i.e.

$$L = \left|x_n^{1st} - x_1^{1st}\right| = \left|x_N^{2nd} - x_1^{2nd}\right| \tag{27}$$

and $[\overline{S_{ij}}]_n^{1st}$ or $[\overline{S_{ij}}]_n^{2nd}$ denotes the global compliance matrix of a UD composite. Referring to Figure 5, the transformation formula for the compliance matrix of a UD composite, designated as a yarn segment in Figure 5, is given by (Y = 1st or 2nd)

$$[\overline{S_{ij}}]_{n-1}^{Y} = [T_{ij}]_s [S_{ij}][T_{ij}]_s^T \tag{28}$$

where

$$[T_{ij}]_s =$$

$$\begin{bmatrix} l_1^2 & l_2^2 & l_3^2 & l_2 l_3 & l_3 l_1 & l_1 l_2 \\ m_1^2 & m_2^2 & m_3^2 & m_2 m_3 & m_3 m_1 & m_1 m_2 \\ n_1^2 & n_2^2 & n_3^2 & n_2 n_3 & n_3 n_1 & n_1 n_2 \\ 2m_1 n_1 & 2m_2 n_2 & 2m_3 n_3 & m_2 n_3 + m_3 n_2 & n_3 m_1 + n_1 m_3 & m_1 n_2 + m_2 n_1 \\ 2n_1 l_1 & 2n_2 l_2 & 2n_3 l_3 & l_2 n_3 + l_3 n_2 & n_3 l_1 + n_1 l_3 & l_1 n_2 + l_2 n_1 \\ 2l_1 m_1 & 2l_2 m_2 & 2l_3 m_3 & l_2 m_3 + l_3 m_2 & l_1 m_3 + l_3 m_1 & l_1 m_2 + l_2 m_1 \end{bmatrix} \tag{29.1}$$

$$l_1 = \cos(\theta_x)\sin(\theta_z),\ m_1 = \sin(\theta_x)\sin(\theta_z),\ n_1 = \cos(\theta_z) \tag{29.2}$$

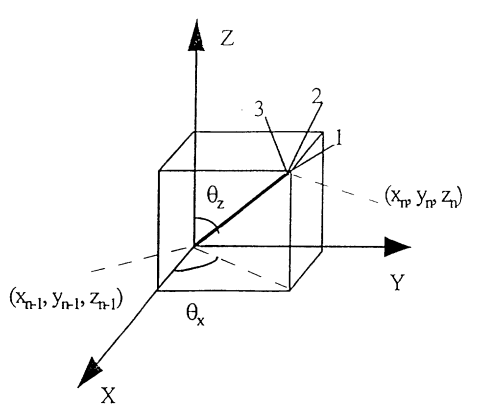

Figure 5. *Orientation of a yarn segment in the global coordinate system.*

$$l_2 = -\sin(\theta_x),\ m_2 = \cos(\theta_x),\ n_2 = 0 \tag{29.3}$$

$$l_3 = -\cos(\theta_x)\cos(\theta_z),\ m_3 = -\sin(\theta_x)\cos(\theta_z)\ n_3 = \sin(\theta_z) \tag{29.4}$$

The results in Table 1 indicate that the present model formulae predict the elastic properties of this complicated fiber architecture reinforced composite slightly better than the Chamis formulae do.

6. CONCLUSION

A new and general micromechanical model for predicting the mechanical properties of composites made from two isotropic constituent materials is presented in this paper. In contrast to most other models, which focus on presenting explicit formulae for the engineering elastic moduli of a composite, this model concentrates on determination of a bridging matrix that correlates the internal stresses generated in the two constituent materials. By means of this bridging matrix, not only can the elastic constants of the composite be estimated, as described in the present paper, but also the strength and inelastic behaviors of the composite can be easily obtained and will be addressed in subsequent papers. The explicit expressions for the bridging matrix given in this paper have been confirmed to be correct and accurate.

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