Problem: Derivation of backpropagation Solutions

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In this problem, you will derive the equations needed to calculate the gradient of the cost function in a feedforward neural network using backpropagation. This gradient is used within learning algorithms (such as gradient descent) that train the neural network.

Recall that the output from the j^{th} neuron in layer ℓ is given by

$$a_j^{(\ell)} = g_\ell\left(z_j^{(\ell)}\right),$$

where g_ℓ is a non-linear activation function and

$$z_j^{(\ell)} = \sum_{i=1}^{n_{\ell-1}} a_i^{(\ell-1)} W_{ij}^{(\ell)} + b_j^{(\ell)},$$

with $W_{ij}^{(\ell)}$ representing the weights and $b_j^{(\ell)}$ the biases of the network. Assume that the cost function C can be expressed as a sum over the dataset $\mathcal{D}$ as

$$C = \frac{1}{|\mathcal{D}|} \sum_{\mathbf{x} \in \mathcal{D}} c(\mathbf{x}),$$

and define the notation

$$\delta_j^{(\ell)} = \frac{\partial c}{\partial z_j^{(\ell)}}.$$

a) Show that the quantity $\delta_j^{(\ell)}$ can be expressed as

$$\delta_j^{(L)} = \frac{\partial c}{\partial a_j^{(L)}} \; g_L'\left(z_j^{(L)}\right),$$

$$\delta_j^{(\ell)} = \sum_{k=1}^{n_{\ell+1}} \delta_k^{(\ell+1)} W_{kj}^{(\ell+1)\,T} g_\ell'\left(z_j^{(\ell)}\right) \qquad (\text{for } \ell < L).$$

b) Show that the partial derivatives of the cost function with respect to the network's weights and biases can be calculated as

$$\frac{\partial c}{\partial W_{ij}^{(\ell)}} = a_i^{(\ell-1)} \delta_j^{(\ell)}, \qquad \frac{\partial c}{\partial b_j^{(\ell)}} = \delta_j^{(\ell)}.$$

Solution: See `Backpropagation_Solution.pdf`