Bayesian Filtering and Smoothing: Exercise Set 1

In this exercise set, symbols for random variables and random vectors are underlined.

1. Consider the bivariate random vector $\underline{\mathbf{x}}$ whose probability density function is
$$p_{\underline{\mathbf{x}}}(\mathbf{x}) = \begin{cases} 1/\pi & \text{if } x_1^2 + x_2^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$
Find the marginal distributions $p_{\underline{x}_1}(x_1)$ and $p_{\underline{x}_2}(x_2)$. Are the random variables $\underline{x}_1$ and $\underline{x}_2$ independent?

2. (continued) Find $\mathsf{E}[\underline{\mathbf{x}}]$ and $\mathsf{var}[\underline{\mathbf{x}}]$. Are the random variables $\underline{x}_1$ and $\underline{x}_2$ uncorrelated?

3. (continued) Find the mean and covariance matrix of the random vector
$$\underline{\mathbf{y}} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \underline{\mathbf{x}} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{1}$$

4. (continued) Use the Chebyshev inequality to find the radius of a $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$-centred circle in $\mathbb{R}^2$ that contains at least 90% of the probability of $\underline{\mathbf{y}}$. Plot the circle.

   Hint: in MATLAB, use `axis equal` to ensure that the data units are the same in every direction.

5. (continued) Generate and plot 200 independent samples of $\underline{\mathbf{y}}$ on the same figure.

6. (continued) Find the estimate $\hat{\mathbf{y}}$ of the random vector $\underline{\mathbf{y}}$ that minimises the *mean square error*
$$\begin{aligned} \mathsf{E}[\|\underline{\mathbf{y}} - \hat{\mathbf{y}}\|^2] &= \mathsf{E}[(\underline{y}_1 - \hat{y}_1)^2 + (\underline{y}_2 - \hat{y}_2)^2] \\ &= \iint \left((y_1 - \hat{y}_1)^2 + (y_2 - \hat{y}_2)^2\right) p_{\underline{\mathbf{y}}}(y_1, y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2 \end{aligned}$$

Answers

1. $p_{\underline{x}_1}(x_1) = \frac{2}{\pi}\sqrt{1 - x_1^2}$ for $|x_1| < 1$. 2. $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, $\frac{1}{4}\mathbf{I}$ 3. $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$, $\frac{1}{2}\left[\begin{smallmatrix} 1 & -1 \\ -1 & 2 \end{smallmatrix}\right]$ 4. $\sqrt{15}$
6. $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$