DATA.STAT.620 Bayesian Filtering and Smoothing

Exam 4.5.2021, professor Robert Piché

This is a take-home exam. You are allowed to use any software and to consult any reference materials including books, notes, and on-line material. You are not allowed to collaborate or discuss the exam with any person during the examination. Submit your solutions, in English or in Finnish, on Moodle by noon on 4.5.2021. Include computer codes.

Document your solution, explaining how the results are obtained. Include commented computer codes. If you use course demo and model solution codes, then specify what codes you use and identify the changes you've made.

1. Data file `RSS.mat` contains measurements of the received signal strength of the radio signal from a Wi-Fi transmitter at various locations, and the distances between the receiver and the transmitter. Assuming the measurement model
$$\mathrm{RSS}_k = 2A + \frac{B}{2}\ln(d_k) + v_k$$
with independent $v_k \sim \mathrm{N}(0, 5^2)$, and prior
$$\begin{bmatrix} A \\ B \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} 10 \\ -20 \end{bmatrix}, 20\,\mathbf{I} \right),$$
find 95% confidence intervals for the posterior distribution of A and B in two ways:

   (a) exact formulas

   (b) Robust Adaptive Metropolis algorithm (hint: confidence intervals can be found with `quantile`)

2. A quadcopter drone navigates using ultra-wideband radio ranging to four transponders located at
$$\mathbf{s}^1 = \begin{bmatrix} -20 \\ 10 \\ 0 \end{bmatrix},\ \mathbf{s}^2 = \begin{bmatrix} 30 \\ 0 \\ -100 \end{bmatrix},\ \mathbf{s}^3 = \begin{bmatrix} 70 \\ -100 \\ -200 \end{bmatrix},\ \mathbf{s}^4 = \begin{bmatrix} 40 \\ -150 \\ -300 \end{bmatrix}$$
The motion of the drone is modelled by
$$\mathbf{u}_k = \mathbf{u}_{k-1} + \mathbf{v}_{k-1} + \mathbf{e}^u_{k-1}, \quad \mathbf{e}^u_{k-1} \sim \mathrm{N}(0, \mathbf{I}_3), \quad \mathbf{u}_0 \sim \mathrm{N}(0, 10^2\,\mathbf{I}_3)$$
$$\mathbf{v}_k = \mathbf{v}_{k-1} + \mathbf{e}^v_{k-1}, \quad \mathbf{e}^v_{k-1} \sim \mathrm{N}(0, 0.3^2\,\mathbf{I}_3), \quad \mathbf{v}_0 \sim \mathrm{N}(0, \mathbf{I}_3)$$
The range measurements are modelled as
$$y^i_k = \|\mathbf{u}_k - \mathbf{s}^i\| + w_k + \varepsilon^i_k, \quad \varepsilon^i_k \sim \mathrm{N}(0, 2^2), \qquad i \in \{1,2,3,4\}$$
where w_k is an unknown bias that is modelled as
$$w_k = w_{k-1} + e^w_{k-1}, \quad e^w_{k-1} \sim \mathrm{N}(0, 0.01^2), \quad w_0 \sim \mathrm{N}(0, 10^2)$$
Data file `Drone.mat` contains measurements $y^{1:4}_{1:T}$ and true positions $\mathbf{u}_{0:T}$.

   Estimate and plot the position sequence of the drone from the measurements, and compute the RMSE of the position estimate, using

   (a) a bootstrap particle filter,
   (b) a Rao-Blackwellized particle filter that samples only positions.

3. (a) The autoregressive (AR) model

$$\begin{aligned} x_k \mid x_{k-1} &\sim \mathrm{N}(Ax_{k-1}, Q) \\ y_k \mid x_k &\sim \mathrm{N}(x_k, R) \end{aligned}$$

   can be obtained by discretisation of the Langevin differential equation. Data file `AR.mat` contains measurements $y_{1:100}$ generated from the AR model with $x_0 = 0$, $Q = 0.2^2$, $R = 0.1^2$. Find the MAP estimate of A, assuming the prior distribution $A \sim \mathrm{uniform}([-1, 1])$.

   (b) Prove that the steady-state Kalman filter's gain for the AR model is

$$K = \frac{\frac{Q}{R} + A^2 - 1 + \sqrt{((A-1)^2 + \frac{Q}{R})((A+1)^2 + \frac{Q}{R})}}{\frac{Q}{R} + A^2 + 1 + \sqrt{((A-1)^2 + \frac{Q}{R})((A+1)^2 + \frac{Q}{R})}}$$

4. Data file `Car.mat` contains noisy measurements of the range and angle, relative to the origin, of a moving target:

$$\begin{aligned} r_k \mid x_k, y_k &\sim \mathrm{N}(\sqrt{x_k^2 + y_k^2}, 0.5^2) \\ \theta_k \mid x_k, y_k &\sim \mathrm{N}(\mathrm{atan2}(y_k, x_k), 0.1^2) \end{aligned}$$

   The range finder was turned off for a short time during the experiment; this is indicated by NANs in the data. The data set also includes the true positions.

   Estimate the path of the target using

   (a) unscented filter and fixed-interval smoother
   (b) particle filter and fixed-interval smoother

   Plot and determine the RMSE of each estimated path. Use the motion model from textbook Example 4.3 with

$$q_1^c = q_2^c = 1, \quad \Delta t = 0.1, \quad \mathbf{m}_0 = 0, \quad \mathbf{P}_0 = 100\,\mathbf{I}$$

   In your solution code, make effective use of the fact that the range and angle measurements are conditionally independent given position.