Bayesian Filtering and Smoothing
Exercise Set 12

1. Estimate Q and R for the data of `example12_1.m` using EM.

2. Consider this generated data:

   ```
   dt=0.02;  t=0:dt:2; nt=length(t); X=sin(pi*t); R=0.01;
   rng('default'); Y=X(2:end)+sqrt(R)*randn(1,nt-1);
   ```

   Using the state space model

   $$\mathbf{x}_{k+1}|\mathbf{x}_k \sim \mathrm{N}\left(\begin{pmatrix} 1 & \Delta \\ 0 & 1 \end{pmatrix} \mathbf{x}_k,\ q \begin{pmatrix} \frac{1}{3}\Delta^3 & \frac{1}{2}\Delta^2 \\ \frac{1}{2}\Delta^2 & \Delta \end{pmatrix}\right),$$
   $$\mathbf{y}_k|\mathbf{x}_k \sim \mathrm{N}([1\ 0]x_k,\ R),$$
   $$x_0 \sim \mathrm{N}\left(\left[\begin{smallmatrix} 0 \\ \pi \end{smallmatrix}\right], \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.1 \end{smallmatrix}\right]\right)$$

   find the MAP estimate of q. Assume $p(q) \propto 1$.

3. (continued) Estimate the curve and its derivative using a Bayesian smoother.

4. (continued) Repeat problems 2–3, estimating both q and R. Assume $p(q,R) \propto 1$.

5. (continued) Estimate the curve and its derivative using `https://se.mathworks.com/matlabcentral/fileexchange/63925-derest`

6. Prove that Q^* given by (12.47) maximises $\mathscr{Q}$ as a function of Q, by showing that
   $$\mathrm{d}(\log\det(2\pi Q) + \mathrm{tr}(Q^{-1}Q^*)) = \mathrm{tr}(Q^{-1}(I - Q^*Q^{-1})\mathrm{d}Q)$$
   This identity uses the *differential*; see `https://en.wikipedia.org/wiki/Matrix_calculus#Identities_in_differential_form`

Answers

2. $\hat{q} = 12.25$, 4. $(\hat{q}, \hat{R}) = (11.95, 0.013)$