Bayesian Filtering and Smoothing: Exercise Set 6

1. The *leapfrog* method[1] for solving the pendulum SDE $\ddot{u} + g\sin(u) = w$ is
$$\begin{aligned}\hat{u}_k &= u_k + \frac{\Delta}{2} v_k \\ v_{k+1} &= v_k - g\Delta\sin(\hat{u}_k) + w_k\sqrt{q^c\Delta}, \qquad w_k \sim \mathrm{N}(0,1) \\ u_{k+1} &= \hat{u}_k + \frac{\Delta}{2} v_{k+1}\end{aligned}$$
Use this method to simulate a track in $0 \le t \le 5$, with $\Delta = 0.01$, $q^c = 1$, $u(0) = 1.5$, and $\dot{u}(0) = 0$. Also, generate noisy measurements $y_{5k} = \sin(u_{5k}) + r_{5k}$, $r_{5k} \sim \mathrm{N}(0, 0.1)$. (Hint: modify `pendulum_demo.m`)

2. (continued) Find $\mathbf{Q}$, $\mathbf{f}$, $\mathbf{F_x}$, $\mathbf{F_{xx}^{(1)}}$, and $\mathbf{F_{xx}^{(2)}}$ for the leapfrog-discretised pendulum SDE.

3. (continued) Estimate the track from the measurements using EKF and EKF2 and initial distribution $\mathbf{x}_0 \sim \mathrm{N}\left(\left[\begin{smallmatrix}1.6\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0.1 & 0\\0 & 0.1\end{smallmatrix}\right]\right)$.

4. A sensor located at $(0.5, 0.1)$ is tracking a target that moves on the x axis. The target's position is $(u, 0)$ and the target's state is $\mathbf{x} = \left[\begin{smallmatrix}u\\\dot{u}\end{smallmatrix}\right]$. The sensor measures the range:
$$y_k = h(\mathbf{x}_k) + r_k, \qquad h(\mathbf{x}_k) = \left\| \left[\begin{smallmatrix}0.5 - \mathbf{x}_k^{[1]}\\0.1\end{smallmatrix}\right] \right\|, \; r_k \sim \mathrm{N}(0, R)$$
Find $\mathbf{H_x}$ and $\mathbf{H_{xx}}$.

5. (continued) Use the following code to generate an integrated Wiener process track and a sequence of noisy range measurements. Estimate the track using EKF and EKF2 and initial distribution $\mathbf{x}_0 \sim \mathrm{N}\left(\left[\begin{smallmatrix}4\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0.01 & 0\\0 & 0.01\end{smallmatrix}\right]\right)$.

```
rng('default'); rng(0);
DT=0.1; nk=100; Qc=1; r=0.1; x0=[4;0];
T=(1:nk)*DT; X=zeros(1,nk); Y=zeros(1,nk);
Ak=[1,DT;0,1]; Qk=Qc*[DT^3/3,DT^2/2;DT^2/2,DT];
x=x0;
for k=1:nk
  x=mvnrnd(Ak*x,Qk)'; X(k)=x(1);
  Y(k)=norm([0.5-x(1);0.1])+sqrt(r)*randn;
end
```

6. (continued) Repeat, replacing `rng(0)` by: (a) `rng(12)`, (b) `rng(23)`. Explain the results.

Answers

2. $\mathbf{Q} = q^c\left[\begin{smallmatrix}\Delta^3/4 & \Delta^2/2\\ \Delta^2/2 & \Delta\end{smallmatrix}\right]$, $\mathbf{f}(\mathbf{x}) = \left[\begin{smallmatrix}x_1 + x_2\Delta\\ x_2\end{smallmatrix}\right] - \left[\begin{smallmatrix}\Delta/2\\1\end{smallmatrix}\right] g\Delta\sin(x_1 + x_2\Delta/2)$,
   $\mathbf{F_x}(\mathbf{x}) = \left[\begin{smallmatrix}1 & \Delta\\0 & 1\end{smallmatrix}\right] - \left[\begin{smallmatrix}\Delta/2 & \Delta^2/4\\ 1 & \Delta/2\end{smallmatrix}\right] g\Delta\cos(x_1 + x_2\Delta/2)$,
   $\mathbf{F_{xx}^{(2)}}(\mathbf{x}) = \left[\begin{smallmatrix}1 & \Delta/2\\ \Delta/2 & \Delta^2/4\end{smallmatrix}\right] g\Delta\sin(x_1 + x_2\Delta/2)$, $\mathbf{F_{xx}^{(1)}}(\mathbf{x}) = \mathbf{F_{xx}^{(2)}}(\mathbf{x})\Delta/2$

4. $\mathbf{H_x} = [-(0.5 - \mathbf{x}^{[1]})/h(\mathbf{x}),\ 0]$, $\mathbf{H_{xx}} = 0.01(h(\mathbf{x}))^{-3}\left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right]$

5. RMS error is 0.1505 for both filters   6. (a) EKF diverges; (b) both diverge

[1] K. Burrage, I. Lenane, G. Lythe, Numerical methods for second-order stochastic differential equations, *SIAM J. on Scientific Computing* **29**(1) 2007, 245–264.