Surface Smoothing Based on a Sphere Shape Model: Convergence

Jussi Tohka

Institute of Signal Processing, Tampere University of Technology,
P.O.Box 553, FIN-33101 Tampere, FINLAND, jussi.tohka@tut.fi

A surface smoothing scheme based on the simplex mesh representation of the surface and the sphere prior was presented in [3]. In this document, the convergence of the iterative smoothing scheme is established. This document is not meant to be self-contained although important definitions from [3] are re-given.

A simplex mesh consists of a set $S = \{s_1, \ldots, s_n\}$ of vertex coordinates and a graph $\mathcal{G} = (V, E)$ with $V = \{1, \ldots, n\}$ modeling the neighborhood relations between vertices. Here all vertex coordinates $s_i = (x_i, y_i, z_i)^T \in \mathbb{R}^3$. In a simplex mesh all vertices have exactly three neighbors i.e. the graph $\mathcal{G}$ is 3-regular. The set of neighbors of the vertex i is denoted by N_i.

We define the symmetric $n \times n$ matrix $P = (p_{ij})$ as

$$p_{ij} = \begin{cases} 1 & \text{if} \quad j \in N_i \\ 0 & \text{otherwise} \end{cases} .$$

Further, let the *shape parameter*

$$\alpha = [3 \cos(2 \arctan \frac{2\sqrt{\pi\sqrt{3}}}{3\sqrt{n}})]^{-1}. \tag{1}$$

and

$$\begin{aligned} Q &= I + \lambda[(I - \alpha P)(I - \frac{1}{n}\mathbf{1})]^2 && (2) \\ &= I + \lambda(I - \alpha P + \frac{(3\alpha - 1)}{n}\mathbf{1})^2 && (3) \\ &= I + \lambda[(I - \alpha P)^2 - \frac{(3\alpha - 1)^2}{n}\mathbf{1}], && (4) \end{aligned}$$

where $\mathbf{1}$ denotes the $n \times n$ matrix whose elements are all equal to one. Given vertex coordinates of a simplex mesh, the iterative scheme for x-coordinates is

$$x^{k+1} = x^k - \eta \nabla f(x^k | x^0) = x^k - 2\eta[Qx^k - x^0], \tag{5}$$

where the learning rate η is a positive constant. The same formula is used for updating y and z coordinates.

Proposition 1 *The algorithm in Eq. (5) converges if*

$$0 < \eta < \frac{2}{1 + \lambda[(1 + 3\alpha)^2]}.$$

To prove the proposition, we first recall a standard convergence result concerning the steepest descent algorithm. The proof can be found in [1].

Theorem 1 *Suppose that the square matrix Q is symmetric and positive definite and consider the function $f(x) = x^T Q x + 2b^T x$. Let the largest eigenvalue of Q be L. Let $\{x^k\}$ be a sequence generated by a steepest descent method with a constant step size: $x^{k+1} = x^k - \eta \nabla f(x^k)$. The sequence $\{x^k\}$ converges to $x^* = Q^{-1}b$ for every starting point x_0 if and only if $\eta \in (0, 2/L)$.*

Proof of Proposition 1: We prove that the largest eigenvalue L of

$$Q = I + \lambda[(I - \alpha P)^2 - \frac{(3\alpha - 1)^2}{n}\mathbf{1}]$$

is majorized by $1 + \lambda(1 + 3\alpha)^2$. After this, Proposition 1 follows from Theorem 1. Let L_1 denote the largest eigenvalue of $[(I - \alpha P)^2 - \frac{(3\alpha-1)^2}{n}]$ and L_2 the largest eigenvalue of $(I - \alpha P)^2$.

The eigenvalues of the matrix αP lie within the range $[-3\alpha, 3\alpha]$ by Gerschgorin's theorem. Hence,

$$L_2 \leq (1 + 3\alpha)^2.$$

The interlocking eigenvalues lemma states that the largest eigenvalue of symmetric matrix A is smaller or equal to the largest eigenvalue of the matrix $A + cc^T$, where c is an arbitrary column vector. For a proof, see [2]. Let $c = \frac{(3\alpha-1)}{\sqrt{n}}(1, \ldots, 1)^T$. Then, the largest eigenvalue of

$$(I - \alpha P)^2 = (I - \alpha P)^2 - cc^T + cc^T$$

is greater or equal to the largest eigenvalue of $(I - \alpha P)^2 - cc^T$, or in other words,

$$L_1 \leq L_2 \leq (1 + 3\alpha)^2.$$

The largest eigenvalue of Q is $L = 1 + \lambda L_1$. Hence, $L \leq 1 + \lambda(1 + 3\alpha)^2$.

REFERENCES

[1] O. Axelsson. *Iterative solution methods*. Cambridge University Press, 1994.

[2] D. Bertsekas, *Nonlinear Programming*, 2nd ed. Athena Scientific, 1999.

[3] J. Tohka. Surface smoothing based on a sphere shape model. In J. Tanskanen, editor, *Proc. Nordic Signal Processing Symposium (NORSIG04)*, pages 17 – 20. IEEE 2004.