1. (**15** points) Evaluate the following integrals. (5 points for each)

(a) $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$ (b) $\int 3x^5 \sqrt{x^3+1}\, dx$ (c) $\int \frac{\sin(\ln x)}{x}\, dx$

Solution:

(a) Because

$$\begin{aligned}(\csc x - \sec x)(\sin x + \cos x) &= \csc x \sin x + \csc x \cos x - \sec x \sin x - \sec x \cos x \\ &= 1 + \cot x - \tan x - 1 = \cot x - \tan x\end{aligned}$$

therefore,

$$\int (\csc x - \sec x)(\sin x + \cos x)\, dx = \int (\cot x - \tan x)\, dx = \ln|\sin x| - \ln|\sec x| + C$$

(b) Let $u = x^3 + 1 \Rightarrow x^3 = u - 1$, $du = 3x^2\, dx$. Therefore,

$$\begin{aligned}\int 3x^5 \sqrt{x^3+1}\, dx &= \int 3x^2 \cdot x^3 \sqrt{x^3+1}\, dx = \int (u-1)\sqrt{u}\, du = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{5}\left(x^3+1\right)^{\frac{5}{2}} - \frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}} + C\end{aligned}$$

(c) Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$. Therefore,

$$\int \frac{\sin(\ln x)}{x}\, dx = \int \sin u\, du = -\cos u + C = -\cos(\ln x) + C$$

2. (**10** points) Use logarithmic differentiation to find $\frac{dy}{dx}$ if

$$y = \sqrt{\frac{(x+8)^{10}}{(2x+7)^5}}$$

Solution:

- Use logarithmic differentiation.

$$y = \sqrt{\frac{(x+8)^{10}}{(2x+7)^5}} \Rightarrow \ln y = \frac{1}{2}\left[10\ln(x+8) - 5\ln(2x+7)\right] \Rightarrow \frac{y'}{y} = \frac{5}{x+8} - \frac{5}{2x+7}$$

Therefore,

$$y' = y \cdot \left(\frac{5}{x+8} - \frac{5}{2x+7}\right) = \sqrt{\frac{(x+8)^{10}}{(2x+7)^5}}\left(\frac{5}{x+8} - \frac{5}{2x+7}\right)$$

3. (**15** points) Suppose that $f(x)$ is a continuous function, and

$$\int_0^x f(t)\,dt = 1 + 3\sin x + k\cos x.$$

(a) Find $k = ?$ (b) Find $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx$. (c) Find $f'\left(\frac{\pi}{4}\right)$.

Solution:

(a) Based on the properties of the definite integrals, when $x = 0$,

$$\int_0^0 f(t)\,dt = 1 + 3\sin 0 + k\cos 0 \Rightarrow 0 = 1 + k \Rightarrow k = -1$$

(b) Based on the properties of the definite integrals,

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx &= \int_{-\frac{\pi}{2}}^{0} f(x)\,dx + \int_0^{\frac{\pi}{2}} f(x)\,dx = -\int_0^{-\frac{\pi}{2}} f(x)\,dx + \int_0^{\frac{\pi}{2}} f(x)\,dx \\ &= -\left[1 + 3\sin\left(-\frac{\pi}{2}\right) - \cos\left(-\frac{\pi}{2}\right)\right] + \left[1 + 3\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right)\right] = -(-2) + 4 = 6\end{aligned}$$

(c) Based on Fundamental Theorem of Calculus - Part 1,

$$\begin{aligned}&\frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}\left[1 + 3\sin x - \cos x\right] \Rightarrow f(x) = 3\cos x + \sin x \Rightarrow f'(x) = -3\sin x + \cos x \\ &\Rightarrow f'\left(\frac{\pi}{4}\right) = -\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}\end{aligned}$$

4. (**10** points) Find the length of the curve:

$$y = \frac{x^2}{8} - \ln x, \quad 4 \le x \le 8$$

Solution:

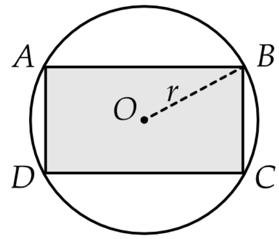
- Find $1 + (y')^2$ first.

$$y = \frac{x^2}{8} - \ln x \Rightarrow y' = \frac{x}{4} - \frac{1}{x} \Rightarrow 1 + (y')^2 = 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2 = \frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2} = \left(\frac{x}{4} + \frac{1}{x}\right)^2$$

The length is,

$$L = \int_4^8 \sqrt{1 + (y')^2}\,dx = \int_4^8 \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2}\,dx = \int_4^8 \left(\frac{x}{4} + \frac{1}{x}\right)dx = \left[\frac{x^2}{8} + \ln|x|\right]\Bigg|_4^8 = 6 + \ln 2$$

5. (**10** points) A rectangle $ABCD$ is inscribed in a circle O with a radius of r. Please find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.



Solution:

- Assume $\overline{AB} = x$ and $\overline{BC} = y$, the area of $ABCD$ is $S = xy$ and the relationship between x, y, and r is

$$x^2 + y^2 = (2r)^2 \Rightarrow y^2 = 4r^2 - x^2 \Rightarrow y = \sqrt{4r^2 - x^2}$$

Therefore,

$$S = xy = x\sqrt{4r^2 - x^2} \Rightarrow \frac{dS}{dx} = \sqrt{4r^2 - x^2} - \frac{x^2}{\sqrt{4r^2 - x^2}} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

When $4r^2 = 2x^2$, $\dfrac{dS}{dx} = 0 \Rightarrow x = \sqrt{2}r$, $y = \sqrt{4r^2 - x^2} = \sqrt{2}r$. Therefore, the dimensions are $\overline{AB} = \overline{BC} = \sqrt{2}r$.

6. (**10** points) Use the shell method to find the volume of the solid obtained by rotating the region bounded by $x + y = 4$ and $x = y^2 - 4y + 4$ about the x-axis.

Solution:

- The curves intersect when $4 - y = y^2 - 4y + 4 \Rightarrow y = 0$ or $y = 3$. The volume is

$$V = \int_0^3 2\pi y\left[(4 - y) - (y^2 - 4y + 4)\right] dy = 2\pi \int_0^3 \left(3y^2 - y^3\right) dy = 2\pi \left[y^3 - \frac{1}{4}y^4\right]\Bigg|_0^3 = \frac{27}{2}\pi$$

7. Let $f(x) = \dfrac{x}{x^2+1}$.
   (a) (**6** points) Find the intervals of increase and decrease.
   (b) (**8** points) Find the intervals of concavity.
   (c) (**4** points) Find the local maximum and minimum values.
   (d) (**6** points) Find the inflection points.

Solution:

(a) The first derivative of this function is

$$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

One can find that $f'(x) > 0$ when $-1 < x < 1$, and $f'(x) < 0$ when $x < -1$ and $1 < x$. Therefore, $f(x)$ is increasing on $(-1, 1)$, decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) The second derivative of this function is

$$f''(x) = \frac{(-2x)(x^2+1)^2 - (1-x^2)(x^2+1)(4x)}{(x^2+1)^4} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

When $-\sqrt{3} < x < 0$ and $x > \sqrt{3}, f''(x) > 0$, therefore, the function is concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. When $x < -\sqrt{3}$ and $0 < x < \sqrt{3}, f''(x) < 0$, therefore, the function is concave downward on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

(c) Based on the result of (a), one can find that $f'(x)$ changes its sign at $x = -1$ and $x = 1$.

Therefore, $f(-1) = -\dfrac{1}{2}$ is its local minimum. $f(1) = \dfrac{1}{2}$ is its local maximum.

(d) Based on the result of (b), one can find that $f''(x)$ changes its sign only at $x = -\sqrt{3}, 0, \sqrt{3}$. The inflection points are

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0,0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right).$$

8. (**6** points) If $f(x) = \sqrt{x-2}$, find $(f^{-1})'(2) =$?

Solution:

- $f'(x) = \dfrac{df}{dx} = \dfrac{1}{2\sqrt{x-2}}$. When $f(a) = 2 \Rightarrow 2 = \sqrt{a-2} \Rightarrow a = 6$. Thus, $f(6) = 2$, $f^{-1}(2) = 6$, and

$$(f^{-1})'(2) = \left.\frac{df^{-1}}{dx}\right|_{x=2} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(2)}} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=6}} = \frac{1}{\left(\dfrac{1}{2\sqrt{6-2}}\right)} = 4$$

9. (**10** points) Find the area of the plane region bounded by the curves $y = \dfrac{1}{x}$ and $2x + 2y = 5$.

Solution:

- The curves intersect when $\dfrac{1}{x} = \dfrac{5-2x}{2} \Rightarrow x = \dfrac{1}{2}$ and $x = 2$. The area is

$$A = \int_{\frac{1}{2}}^{2} \left(\frac{5-2x}{2} - \frac{1}{x} \right) dx = \left[\frac{5}{2}x - \frac{1}{2}x^2 - \ln x \right] \Bigg|_{\frac{1}{2}}^{2} = \frac{15}{8} - 2\ln 2.$$

10. (**10** points) The region bounded by the curve $y = 6 - x^2$ and $y = 2$ is revolved about the x-axis to generate a solid. Please find the volume of the solid.

Solution:

- The curves intersect when $6 - x^2 = 2 \Rightarrow x = -2$ and $x = 2$. Use the washer method. The volume is

$$V = \int_{-2}^{2} \pi \left[\left(6 - x^2\right)^2 - 2^2 \right] dx = \pi \int_{-2}^{2} \left(x^4 - 12x^2 + 32\right) dx = 2\pi \int_{0}^{2} \left(x^4 - 12x^2 + 32\right) dx$$

$$= 2\pi \left[\frac{1}{5}x^5 - 4x^3 + 32x \right] \Bigg|_{0}^{2} = 2\pi \cdot \frac{192}{5} = \frac{384}{5}\pi.$$