1. (15 points) If a resistor of $R$ ohms is connected across a battery of $E$ volts with internal resistance $r$ ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R+r)^2}$$

If $E$ and $r$ are fixed but $R$ varies, what is the maximum value of the power?

**Solution:**

- Find the first derivative first.

$$P(R) = \frac{E^2 R}{(R+r)^2} \Rightarrow P'(R) = \frac{dP}{dR} = \frac{E^2(R+r)^2 - E^2 \cdot 2R(R=r)}{(R+r)^4} = \frac{E^2(r-R)}{(R+r)^3}$$

When $P'(R) = 0$, one can find that $R = r$. When $R < r$, $P'(R) > 0$ (increasing). When $R > r$, $P'(R) < 0$ (decreasing). Therefore, $P(R)$ have maximum value at $R = r$. The maximum value of the power is $P(r) = \frac{E^2}{4r}$.

2. (15 points) Find the value or values of $c$ that satisfy the Mean Value Theorem for the function

$$f(x) = \frac{1}{x}$$

in the interval $[1, 3]$.

**Solution:**

- The function $f$ is continuous and differentiable on the whole real line except $x = 0$, so $f$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$.

$$f'(c) = \frac{f(3) - f(1)}{3-1} \Rightarrow -\frac{1}{c^2} = \frac{\frac{1}{3} - 1}{3-1} = -\frac{1}{3} \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}$$

However, $c = -\sqrt{3}$ is not located on the interval $[1, 3]$. Therefore, the correct answer is $c = \sqrt{3}$ **only**.

3. (20 points) Suppose **the derivative of the function** $y = f(x)$ is

$$y' = \frac{dy}{dx} = (x-1)^2(x-2).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection?

**Solution:**

- $y' = (x-1)^2(x-2) \Rightarrow y'' = 2(x-1)(x-2) + (x-1)^2 = (x-1)(3x-5)$. One can find that the curve increase on $(2, \infty)$ and decrease on $(-\infty, 2)$. At $x = 2$, there is a local minimum. There is no local maximum. The curve is concave upward on $(-\infty, 1)$ and $(5/3, \infty)$, and concave downward on $(1, 5/3)$. Therefore, there are two inflection points which are located at $x = 1$ and $x = 5/3$.

4. (30 points) The function

$$f(x) = -2x^3 + 6x^2 - 3$$

(a) Identify the coordinates of all critical points.
(b) Identify the coordinates of all local extreme points.
(c) Identify its inflection points.

**Solution:**

(a) $f(x) = -2x^3 + 6x^2 - 3.$

$$f'(x) = -6x^2 + 12x = -6x(x-2)$$

One can find that $f'(x) = 0$ when $x = 0$, $x = 2$. The coordinates of critical points are $(0, -3)$ and $(2, 5)$.

(b) $f'(x) = -6x^2 + 12x \Rightarrow f''(x) = -12x + 12 = -12(x-1)$. When $x = 0$, $f''(0) = 12 > 0$. Therefore, $(0, 3)$ is a local minimum point. When $x = 2$, $f''(0) = -12 < 0$. Therefore, $(2, 5)$ is a local maximum point.

(c) When $f''(x) = 0$, $x = 1$. When $x < 1$, $f''(x) > 0$, and when $x > 1$, $f''(x) < 0$. Therefore, the point $(1, 1)$ is the point of inflection.

5. (20 points) Find the point on the curve $y = \sqrt{x}$ that is the closest to the point $(3, 0)$.

**Solution:**

- A point on the curve can be expressed as $(a, \sqrt{a})$. The distance between $(a, \sqrt{a})$ and $(3, 0)$ is

$$d = \sqrt{(a-3)^2 + \left(\sqrt{a} - 0\right)^2}$$

The square of the distance can be expressed as a function of $a$

$$S(a) = d^2 = (a-3)^2 + \left(\sqrt{a} - 0\right)^2 = (a-3)^2 + a = a^2 - 5a + 9 \Rightarrow S'(a) = \frac{dS}{da} = 2a - 5$$

When $S'(a) = 0$, $a = \frac{5}{2}$. Because $S''(a) = 2 > 0$, therefore, when $a = \frac{5}{2}$, $S(a)$ has a minimum value of $S\left(\frac{5}{2}\right) = \frac{11}{4}$ which means the point $(a, \sqrt{a}) = \left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$ is the closest to the point $(3, 0)$