1. (**30** points) Evaluate the following integrals. (10 points for each)

(a) $\displaystyle\int \sin x\sqrt{1+\cos x}\,dx$ (b) $\displaystyle\int_9^{64} \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\,dx$ (c) $\displaystyle\int x^2\sqrt{2+x}\,dx$

Solution:

(a) Let $u = 1+\cos x \Rightarrow du = -\sin x\,dx$. Therefore,

$$\int \sin x\sqrt{1+\cos x}\,dx = \int \sqrt{u}\,(-du) = -\int u^{\frac{1}{2}}\,du = -\frac{2}{3}u^{\frac{3}{2}} + C = -\frac{2}{3}(1+\cos x)^{\frac{3}{2}} + C$$

(b) Let $u = 1+\sqrt{x} \Rightarrow du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx \quad x: 9 \to 64 \Rightarrow u: 4 \to 9$. Therefore,

$$\int_9^{64} \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\,dx = \int_4^9 \frac{2}{\sqrt{u}}\,du = 2\int_4^9 u^{-\frac{1}{2}}\,du = \left[4u^{\frac{1}{2}}\right]\Big|_4^9 = 4\cdot(3-2) = 4.$$

(c) Let $u = 2+x \Rightarrow du = dx$, $x = u-2$. Therefore,

$$\int x^2\sqrt{2+x}\,dx = \int (u-2)^2\sqrt{u}\,du = \int \left(u^2-4u+4\right)u^{\frac{1}{2}}\,du = \int \left(u^{\frac{5}{2}} - 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right)du$$
$$= \frac{2}{7}u^{\frac{7}{2}} - \frac{8}{5}u^{\frac{5}{2}} + \frac{8}{3}u^{\frac{3}{2}} + C = \frac{2}{7}(2+x)^{\frac{7}{2}} - \frac{8}{5}(2+x)^{\frac{5}{2}} + \frac{8}{3}(2+x)^{\frac{3}{2}} + C$$

2. (**20** points) Find the areas of the regions enclosed by $y = 7-2x^2$ and $y = x^2+4$.

Solution:

- **Step 1:** Find the lower and upper limits of integration:

  $7-2x^2 = x^2+4 \Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x = -1,\ x = 1$

  The intersection is at $x=-1$ and $x=1$. The limits of integration is from -1 to 1.

- **Step 2:** Find the area:

  $$A = \int_{-1}^{1}\left[(7-2x^2)-(x^2+4)\right]dx = \int_{-1}^{1}\left(3-3x^2\right)dx = 4.$$

3. (**20** points) Answer the following question.

   (a) If $g(y) = \int_3^y f(x)\,dx$, please write down the relationship between $g'(y)$ and $f(y)$.

   (b) If $f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt$, find $f'(x)$

   (c) If $g(y) = \int_3^y f(x)\,dx$ and $f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt$, based on the results of (a) and (b), please find $g''\left(\frac{\pi}{6}\right)$.

   (d) Find $g(3)$.

Solution:

(a) Use Fundamental Theorem of Calculus Part 1:

$$g'(y) = \frac{d}{dy}\int_3^y f(x)\,dx = f(y) \Rightarrow g'(y) = f(y)$$

(b) Use Fundamental Theorem of Calculus Part 1 and the Chain Rule:

$$f'(x) = \frac{d}{dx}\int_0^{\sin x} \sqrt{1+t^2}\,dt = \sqrt{1+\sin^2 x}\cdot\cos x$$

(c) Based on the results of (a) and (b),

$$g'(x) = f(x) \Rightarrow g''(x) = f'(x) = \sqrt{1+\sin^2 x}\cdot\cos x \Rightarrow g''\left(\frac{\pi}{6}\right) = \sqrt{1+\sin^2\frac{\pi}{6}}\cdot\cos\frac{\pi}{6} = \sqrt{1+\frac{1}{4}}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}$$

(d) $g(3) = \int_3^3 f(x)\,dx = 0.$

4. (**15** points) Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the y-axis.

Solution:

- Find the volume by using cross-section (washer method). The cross-section is a washer with inner radius $x = y$ and outer radius $x = \sqrt{y}$. Therefore, the volume is

$$V = \int_0^1 \left[\pi\left(\sqrt{y}\right)^2 - \pi y^2\right] dy = \int_0^1 \pi\left(y - y^2\right) dy = \pi\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]\Bigg|_0^1 = \frac{\pi}{6}$$

5. (**15** points) Use the shell method to find the volumes of the solids generated by revolving the regions bounded by $y = x+2$, $y = x^2$ about the line $x = -1$.

Solution:

- Use the shell method. The volume is

$$V = \int_{-1}^2 2\pi\left(x+1\right)\left(x+2-x^2\right) dx = 2\pi\int_{-1}^2 \left(2+3x-x^3\right) dx = 2\pi\left[2x + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]\Bigg|_{-1}^2 = \frac{27}{2}\pi.$$