Total: 100 points

1. (**15** points) Find the slope of the curve when $\theta = 0$ and $\theta = \pi$.

$$r = -1 + \sin\theta$$

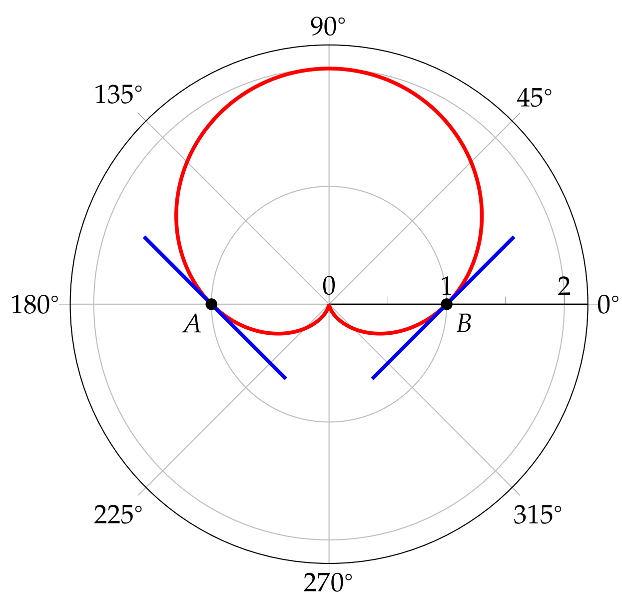
Solution:

- $\theta = 0 \Rightarrow A(-1, 0) \quad \theta = \pi \Rightarrow B(-1, \pi)$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + r\cos\theta}{\cos\theta\cos\theta - r\sin\theta}$$

  - Slope at point A: $\dfrac{\cos(0)\sin(0) + (-1)\cos(0)}{\cos(0)\cos(0) - (-1)\sin(0)} = -1.$
  - Slope at point B: $\dfrac{\cos(\pi)\sin(\pi) + (-1)\cos(\pi)}{\cos(\pi)\cos(\pi) - (-1)\sin(\pi)} = 1.$

The graph of the curve (red) and the tangent lines (blue) at the given points are



2. (**15** points) Find the area shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$

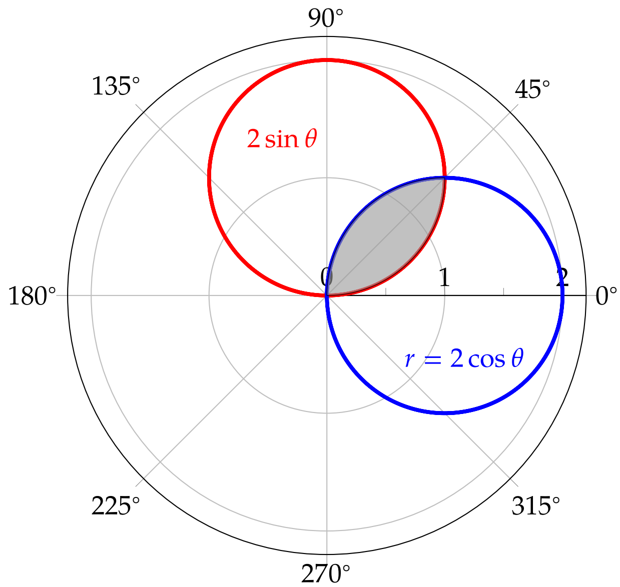
Solution:

- Find intersection:

$$2\cos\theta = 2\sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

Therefore,

$$A = \int_0^{\pi/4} \frac{1}{2}\left(2\sin\theta\right)^2\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\left(2\cos\theta\right)^2\, d\theta = \int_0^{\pi/4} \left(1-\cos 2\theta\right)\, d\theta + \int_{\pi/4}^{\pi/2} \left(1+\cos 2\theta\right)\, d\theta = \frac{\pi}{2} - 1$$



3. (**60** points) Let $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$. (5 points for each)

   (a) Find the gradient of f.

   (b) Find the directional derivative of f at the point $A(1,1)$ in the direction toward the point $B(3,3)$.

   (c) Find the maximum increasing rate of change of f at the point $A(1,1)$. Which is the direction of the maximum increasing rate of change?

   (d) At the point $A(1,1)$, which direction is the direction of zero change?

   (e) Find the tangent plane of $z = f(x,y)$ at the point $(1,1,\frac{1}{\sqrt{2}})$.

   (f) Use linear approximation of $f(x,y)$ at $(1,1)$ to estimate the value of $f(1.01, 0.99)$.

Solution:

(a) $\nabla f = -\dfrac{x}{(x^2+y^2)^{3/2}}\hat{\mathbf{i}} - \dfrac{y}{(x^2+y^2)^{3/2}}\hat{\mathbf{j}}$

(b) At $A(1,1)$, $\nabla f|_A = -\dfrac{\sqrt{2}}{4}\hat{\mathbf{i}} - \dfrac{\sqrt{2}}{4}\hat{\mathbf{j}}$. Direction: $\hat{\mathbf{u}} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} + \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$. Therefore, the directional derivative is $\nabla f|_A \cdot \hat{\mathbf{u}} = -\dfrac{1}{2}$

(c) The maximum **increasing** rate of change of f at the point $A(1,1)$ is $\left|\nabla f|_A\right| = \frac{1}{2}$. The direction is $\hat{\mathbf{u}} = -\dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} - \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$

(d) Because $\nabla f|_{(1,1)} = -\dfrac{\sqrt{2}}{4}\hat{\mathbf{i}} - \dfrac{\sqrt{2}}{4}\hat{\mathbf{j}}$, let the unit vector for the zero change is $\hat{\mathbf{u}} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}}$ where $u_1^2 + u_2^2 = 1$, then

$$\nabla f|_{(1,1)} \cdot \hat{\mathbf{u}} = 0 \Rightarrow u_1 + u_2 = 0 \Rightarrow u_1 = -\frac{1}{\sqrt{2}}, u_2 = \frac{1}{\sqrt{2}} \quad \text{or} \quad u_1 = \frac{1}{\sqrt{2}}, u_2 = -\frac{1}{\sqrt{2}}$$

Thus, the direction is $\hat{\mathbf{u}} = \dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} - \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$ or $\hat{\mathbf{u}} = -\dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} + \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$.

(e) Tangent plane is

$$z - \frac{1}{\sqrt{2}} = f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$\Rightarrow z - \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{4}(x-1) - \frac{\sqrt{2}}{4}(y-1) \Rightarrow \frac{\sqrt{2}}{4}x + \frac{\sqrt{2}}{4}y + z = \sqrt{2}$$

(f) Linear approximation at $(1,1)$ is

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{4}(x-1) - \frac{\sqrt{2}}{4}(y-1).$$

Therefore,

$$f(1.01, 0.99) \approx L(1.01, 0.99) = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{4}(1.01-1) - \frac{\sqrt{2}}{4}(0.99-1) = \frac{1}{\sqrt{2}}.$$

4. (**10** points) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at (π, π, π) if $\sin(x+y) + \sin(y+z) + \sin(z+x) = 0$.

Solution:

Let $F(x, y, z) = \sin(x+y) + \sin(y+z) + \sin(z+x) = 0$

$\Longrightarrow \dfrac{\partial F}{\partial x} = \cos(x+y) + \cos(z+x), \quad \dfrac{\partial F}{\partial y} = \cos(x+y) + \cos(y+z), \quad \dfrac{\partial F}{\partial z} = \cos(y+z) + \cos(z+x)$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{\cos(x+y) + \cos(z+x)}{\cos(y+z) + \cos(z+x)} \Longrightarrow \left.\frac{\partial z}{\partial x}\right|_{(\pi,\pi,\pi)} = -1$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{\cos(x+y) + \cos(y+z)}{\cos(y+z) + \cos(z+x)} \Longrightarrow \left.\frac{\partial z}{\partial y}\right|_{(\pi,\pi,\pi)} = -1$$