Total: **110** points
Note: To get full points, you should write down the procedure **in detail**.

1. Let $f(x, y) = x^2 \ln y$.
   (a) (**5** points) Find the gradient of f.
   (b) (**5** points) Find the directional derivative of f at the point $P(3, 1)$ in the direction toward the point $Q(-5, 7)$.
   (c) (**5** points) Find the maximum **decreasing** rate of change of f at the point $P(3, 1)$. Which is the direction of the maximum **decreasing** rate of change?
   (d) (**5** points) What are the directions of zero change in f at $P(3, 1)$?

Solution:

(a) The gradient vector of f is

$$\nabla f = (2x \ln y)\,\hat{\mathbf{i}} + \left(\frac{x^2}{y}\right)\hat{\mathbf{j}}.$$

(b) At the point $P(3, 1)$, the gradient vector is

$$\nabla f|_P = 0\hat{\mathbf{i}} + \frac{3^2}{1}\hat{\mathbf{j}} = 9\hat{\mathbf{j}}$$

Because $\overrightarrow{PQ} = -8\hat{\mathbf{i}} + 6\hat{\mathbf{j}}$, $\left|\overrightarrow{PQ}\right| = 10$, the direction is

$$\hat{\mathbf{u}} = \frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = -\frac{4}{5}\hat{\mathbf{i}} + \frac{3}{5}\hat{\mathbf{j}}$$

Therefore, the directional derivative is

$$\nabla f|_P \cdot \hat{\mathbf{u}} = 0 \cdot \left(-\frac{4}{5}\right) + 9 \cdot \frac{3}{5} = \frac{27}{5}.$$

(c) The maximum **decreasing** rate of change of f at the point $P(3, 1)$ is $\left|\nabla f|_P\right| = 9$. The direction is $\hat{\mathbf{u}} = -\hat{\mathbf{j}}$

(d) The directions of zero change are $\hat{\mathbf{u}} = \pm\hat{\mathbf{i}}$.

2. (**15** points) Find the maximum and minimum value of the function $f(x,y,z) = x - y + z$ over the unit sphere $x^2 + y^2 + z^2 = 1$

Solution:

- $f(x,y,z) = x - y + z$, $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. Use Lagrange multiplier method. First, find the gradient:

  $$\nabla f = \langle 1, -1, 1 \rangle, \quad \nabla g = \langle 2x, 2y, 2z \rangle.$$

  Then

  $$\nabla f = \lambda \nabla g \Rightarrow 1 = 2x\lambda,\ -1 = 2y\lambda,\ 1 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}, y = -\frac{1}{2\lambda}, z = \frac{1}{2\lambda}$$

  Since

  $$g(x,y,z) = 0 \Rightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 - 1 = 0 \Rightarrow \frac{3}{4\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{3}{4} \Rightarrow \lambda = \pm\frac{\sqrt{3}}{2}$$

  For these two cases,

  $$\lambda = \frac{\sqrt{3}}{2} \Rightarrow (x,y,z) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \Rightarrow f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \sqrt{3}$$
  $$\lambda = -\frac{\sqrt{3}}{2} \Rightarrow (x,y,z) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \Rightarrow f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\sqrt{3}$$

  On the sphere, when $(x,y,z) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, f has a maximum value of $\sqrt{3}$.

  When $(x,y,z) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, f has a minimum value of $-\sqrt{3}$.

3. Let $z = f(x, y) = \sin x \cos y$.

(a) (**5** points) Find the tangent plane of $z = f(x, y)$ at the point $P\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{1}{2}\right)$.

(b) (**5** points) Find the normal line of $z = f(x, y)$ at the point $P\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{1}{2}\right)$.

(c) (**5** points) Use the linearization of $f(x, y)$ at the point $P\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{1}{2}\right)$ to approximate the value of $f\left(\frac{26\pi}{100}, \frac{24\pi}{100}\right)$.

Solution:

- First derivatives of f:

$$\begin{aligned} f_x(x, y) &= \frac{\partial f}{\partial x} = \cos x \cos y \Rightarrow f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2} \\ f_y(x, y) &= \frac{\partial f}{\partial y} = -\sin x \cos y \Rightarrow f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = -\frac{1}{2} \end{aligned}$$

(a) Tangent plane is

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{1}{2}\right) = 0 \Rightarrow \frac{1}{2}x - \frac{1}{2}y - z + \frac{1}{2} = 0$$

(b) Normal line is

$$x = \frac{\pi}{4} + \frac{1}{2}t, \quad y = \frac{\pi}{4} - \frac{1}{2}t, \quad z = \frac{1}{2} - t$$

where t is the parameter.

(c) The linearization of f at $P\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{1}{2}\right)$ is

$$\begin{aligned} L(x, y) &= f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right) \end{aligned}$$

Therefore,

$$f\left(\frac{26\pi}{100}, \frac{24\pi}{100}\right) \approx L\left(\frac{26\pi}{100}, \frac{24\pi}{100}\right) = \frac{1}{2} + \frac{1}{2} \cdot \frac{\pi}{100} - \frac{1}{2} \cdot \left(-\frac{\pi}{100}\right) = \frac{1}{2} + \frac{\pi}{100}$$

4. Consider a polar curve $r = 2\cos\theta, \quad -\pi/2 \le \theta \le \pi/2.$
   (a) (**5** points) Find the exact length of the polar curve.
   (b) (**5** points) Find all points on the curve where the tangent line is horizontal.

Solution:

(a) The arc length is

$$L = \int_{-\pi/2}^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta = \int_{-\pi/2}^{\pi/2} 2\, d\theta = 2\pi$$

(b) Slope of the curve $r = 2\cos\theta$ is

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(2\cos\theta \cdot \sin\theta)}{\frac{d}{d\theta}(2\cos\theta \cdot \cos\theta)} = \frac{2\cos 2\theta}{-2\sin 2\theta} = -\cot 2\theta$$

Therefore

$$\frac{dy}{dx} = 0 \Rightarrow \cot 2\theta = 0 \Rightarrow 2\theta = \pm\frac{\pi}{2} \Rightarrow \theta = \pm\frac{\pi}{4}$$

The points are $\left(\sqrt{2}, \frac{\pi}{4}\right), \left(\sqrt{2}, -\frac{\pi}{4}\right)$

5. (**10** points) Evaluate the following integrals. (5 points for each)
   Hint: You can change the order of integration if necessary.

   (a) $\displaystyle\int_0^1 \int_{y/2}^{1/2} e^{-x^2}\, dx\, dy$ (b) $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\, dy\, dx$

Solution:

(a) Change the order of integration.

$$\int_0^1 \int_{y/2}^{1/2} e^{-x^2}\, dx\, dy = \int_0^{1/2} \int_0^{2x} e^{-x^2}\, dy\, dx = \int_0^{1/2} 2xe^{-x^2}\, dx = \left[-e^{-x^2}\right]\Big|_0^{1/2} = 1 - e^{-1/4}.$$

(b) Change the order of integration.

$$\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\, dy\, dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\, dx\, dy = \int_0^2 \frac{y^3}{y^4+1}\, dy = \left[\frac{1}{4}\ln\left(1+y^4\right)\right]\Big|_0^2 = \frac{1}{4}\ln 17$$

6. (**15** points) Find all the local maxima, local minima, and saddle point(s) of the function

$$f(x,y) = x^3 + y^3 - 3xy + 15$$

Solution:

- $f_x = 3x^2 - 3y, f_y = 3y^2 - 3x \Rightarrow f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3.$

$$f_x = 3x^2 - 3y = 0 \Rightarrow y = x^2$$
$$f_y = 3y^2 - 3x = 0 \Rightarrow x = y^2 \Rightarrow x = x^4 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0,\ x = 1.$$

Thus, the critical points are $(0,0)$, $(1,1)$.

Point $(0,0)$: $f_{xx}f_{yy} - (f_{xy})^2 = -9 < 0 \Rightarrow$ saddle point.

Point $(1,1)$: $f_{xx}f_{yy} - (f_{xy})^2 = 27 > 0, f_{xx} = 6 > 0 \Rightarrow$ local minima.

7. (**10** points) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $e^z - xyz = 0$.

Solution:

- Use implicit differentiation. $F(x,y,z) = e^z - xyz = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-yz}{e^z - xy} = \frac{yz}{e^z - xy}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}$$

8. The following problem will guide you to evaluate the integral

$$\iint_R (3x+2y)(2y-x)^{3/2}\,dx\,dy$$

with the use of substitution method $u = 3x+2y$, $v = 2y-x$. The region R is bounded by the parallelogram with

$$3x+2y=0,\ 3x+2y=16,\ 2y-x=0,\ 2y-x=8$$

Answer the following questions.

(a) (**5** points) Express x, y in terms of u, v.

(b) (**5** points) Find Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

(c) (**5** points) Evaluate the integral.

Solution:

(a) The transformation from (u,v) to (x,y) is

$$x = \frac{1}{4}(u-v)$$
$$y = \frac{1}{8}(u+3v)$$

(b) The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \end{vmatrix} = \frac{1}{8}$$

(c) The integral is

$$\iint_R (3x+2y)(2y-x)^{3/2}\,dx\,dy = \int_0^8 \int_0^{16} uv^{\frac{3}{2}}\frac{1}{8}\,du\,dv = \int_0^8 16v^{\frac{3}{2}}\,dv = \frac{4096}{5}\sqrt{2}$$