1. Find the following limits. (10 pts)

$a. \lim_{t\to 0}\frac{cos2t-4cost+3}{t^4}$ (5pts) $b. \lim_{x\to\infty}\left(x\sqrt{\frac{x-1}{x+1}}-x\right)$ (5pts)

(a). $\lim_{t\to 0}\frac{\cos 2t-4\cos t+3}{t^4} = \lim_{t\to 0}\frac{(2\cos^2 t-1)-4\cos t+3}{t^4}$

$= \lim_{t\to 0}\frac{2(1-2\cos t+\cos^2 t)}{t^4} = 2\cdot\lim_{t\to 0}\frac{(\cos t-1)^2}{t^4}$

$= 2\cdot\lim_{t\to 0}\frac{(\cos t-1)}{t^2}\cdot\frac{(\cos t-1)}{t^2}\cdot\frac{(\cos t+1)}{(\cos t+1)}\cdot\frac{(\cos t+1)}{(\cos t+1)}$

$= 2\cdot\lim_{t\to 0}\frac{-\sin^2 t}{t^2}\cdot\frac{-\sin^2 t}{t^2}\cdot\frac{1}{\cos t+1}\cdot\frac{1}{\cos t+1}$

$= (2)\cdot(1)\cdot(1)\cdot(\frac{1}{2})\cdot(\frac{1}{2}) = \boxed{\frac{1}{2}}$ ※

(b). $\lim_{x\to\infty}\left(x\cdot\sqrt{\frac{x-1}{x+1}}-x\right) = \lim_{x\to\infty}x\left(\sqrt{\frac{x-1}{x+1}}-1\right)$

$=\lim_{x\to\infty}x\cdot\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}-\frac{\sqrt{x+1}}{\sqrt{x+1}}\right) = \lim_{x\to\infty}x\cdot\left(\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x+1}}\right)\cdot\left(\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}\right)$

$= \lim_{x\to\infty}\left[\frac{-2x}{\sqrt{x^2-1}+(x+1)}\right] = \lim_{x\to\infty}\frac{x(-2)}{x\cdot(\sqrt{1-\frac{1}{x^2}}+1+\frac{1}{x})} = \lim_{x\to\infty}\frac{-2}{\sqrt{1-\frac{1}{x^2}}+1+\frac{1}{x}} = \boxed{-1}$ ※

2. $f(x)=\begin{cases} cosx\,, x<0 \\ \alpha+x^2\,, 0\le x<1 \\ \beta x\,,\ x\ge 1\end{cases}$ and is continuous at every x, find $\alpha+\beta$. (10pts)

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1 \Rightarrow \lim_{x\to 0} f(x) = 1$$

$\because$ $f(x)$ continuous at every x

$$\therefore f(0) = \alpha = \lim_{x\to 0} f(x) = 1 \Rightarrow \boxed{\alpha = 1}$$

$\because$ $f(x)$ continuous at every x

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \alpha + 1 = 2 \Rightarrow \lim_{x\to 1} f(x) = 2$$

$$\therefore f(1) = \beta = \lim_{x\to 1} f(x) = 2 \Rightarrow \boxed{\beta = 2}$$

$$\boxed{\therefore \alpha + \beta = 3}$$

※

3. Find the tangent and normal line of $F(x, y) = 2x^2 - y^3 + 4xy - 2x = 0$ at Point $P(x, y) = (1, -2)$. (10 pts)

$$4x - 3y^2\left(\frac{dy}{dx}\right) + 4y + \left(\frac{dy}{dx}\right)(4x) - 2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x+4y-2}{3y^2-4x}$$

$$\text{slope}\Big|_{(x,y)=(1,-2)} = \frac{dy}{dx}\Big|_{(x,y)=(1,-2)} = \frac{4-8-2}{12-4} = \frac{-6}{8} = \frac{3}{-4}$$

$$\text{normal line slope}\Big|_{(x,y)=(1,-2)} = \frac{4}{3}$$

① tangent line :

$$y-(-2) = \frac{3}{-4}\cdot(x-1) \Rightarrow \boxed{3x+4y+5=0}$$

※

② normal line:

$$y-(-2) = \frac{4}{3}(x-1) \Rightarrow \boxed{4x-3y-10=0}$$

※

4. Given that $f'(0) = 2$, $\lim_{x\to 0}\frac{f(6x)-f(\sin x)}{x} = ?$

$$\lim_{x\to 0}\frac{f(6x)-f(\sin x)}{x}$$

$$\Rightarrow \lim_{x\to 0}\frac{[f(6x)-f(0)]-[f(\sin x)-f(0)]}{x-0}$$

$$\Rightarrow \lim_{x\to 0}\left\{6\cdot\left[\frac{f(6x)-f(0)}{6x}\right]-\frac{\sin x}{x}\left[\frac{f(\sin x)-f(0)}{\sin x}\right]\right\}$$

$\because x\to 0 \Rightarrow 6x\to 0$, $x\to 0\Rightarrow \sin x\to 0$

$$\therefore 6\cdot\left[\lim_{6x\to 0}\frac{f(6x)-f(0)}{6x-0}\right]-\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{\sin x\to 0}\frac{f(\sin x)-f(0)}{\sin x-0}\right.$$

$$\Rightarrow 6\cdot f'(0)-1\cdot f'(0)=5\cdot f'(0)=\boxed{10}$$

※

5. If $f\left(\frac{x-1}{x+1}\right)=2\cos\left(\frac{\pi}{2}x\right)$, find f'(0) = ?

$$f\left(\frac{x-1}{x+1}\right) = 2\cos\left(\frac{\pi}{2}x\right)$$

$$\Rightarrow f'\left(\frac{x-1}{x+1}\right) \cdot \frac{d}{dx}\left(\frac{x-1}{x+1}\right) = -2 \cdot \sin\frac{\pi}{2}x \cdot \left(\frac{\pi}{2}\right)$$

$$\Rightarrow f'\left(\frac{x-1}{x+1}\right) \cdot \left[\frac{(x+1)-(x-1)}{(x+1)^2}\right] = -\pi \sin\frac{\pi}{2}x$$

$$\Rightarrow f'\left(\frac{x-1}{x+1}\right) = -\frac{\pi}{2}(x+1)^2 \cdot \sin\frac{\pi}{2}x$$

let $x=1 \Rightarrow \frac{x-1}{x+1} = 0$

$$\Rightarrow f'(0) = -\frac{\pi}{2} \cdot (2)^2 \cdot \sin\frac{\pi}{2}$$

$$= \boxed{-2\pi}$$

6. Find the derivatives of the following functions. (20pts)

a. $k(x) = x^2 \sec\left(\frac{1}{x}\right)$ (5pts) b. $g(z) = \frac{(z-1)(z^2+z+1)}{z^3}$ (5pts)

c. $r(\theta) = \sin(\theta^2)\cos(2\theta)$ (5pts) d. $y(t) = (1 + tan^4(\frac{t}{12}))^3$ (5pts)

(a)

. $k(x) = x^2 \sec\left(\frac{1}{x}\right) \Rightarrow k'(x) = x^2 \frac{d}{dx}\left(\sec\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right) \cdot \frac{d}{dx}(x^2) = x^2 \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + 2x\sec\left(\frac{1}{x}\right)$

$= x^2 \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + 2x\sec\left(\frac{1}{x}\right) = 2x\sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)$

(b). $g(z) = \frac{(z-1)(z^2+z+1)}{z^3} = \frac{z^3+\cancel{z^2}+\cancel{z}-\cancel{z^2}-\cancel{z}-1}{z^3} = 1-\frac{1}{z^3}$

$g'(z) = 3\cdot\frac{1}{z^4} = \boxed{\frac{3}{z^4}}$

$^{*}\begin{cases} a^3-b^3 = (a-b)(a^2+b^2+ab) \\ a+b = (a+b)(a^2+b^2-ab) \end{cases}$

(c). $r(\theta) = \sin(\theta^2)\cos(2\theta)$

$r'(\theta) = \cos(\theta^2)(2\theta)\cos(2\theta) + (-\sin(2\theta))\cdot(2)\cdot(\sin(\theta^2))$

$= 2\theta\cos(\theta^2)\cos(2\theta) - 2\sin(\theta^2)\sin(2\theta)$

$= \boxed{2\left[\theta\cos(\theta^2)\cos(2\theta) - \sin(\theta^2)\sin(2\theta)\right]}$

(d). $y(t) = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^3$

$y'(t) = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\cdot\left[4\tan^3\left(\frac{t}{12}\right)\right]\left[\sec^2\left(\frac{t}{12}\right)\right]\left(\frac{1}{12}\right)$

$= \boxed{\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\right]\left[\sec^2\left(\frac{t}{12}\right)\right]}$

7. find the slope of the curve at the given points

$$y^2 + x^2 = y^4 - 2x \quad \text{at} \quad (-2, 1) \text{ and } (-2, -1)$$

$y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$ $\Rightarrow$ $2y\frac{dy}{dx} + 2x = 4y^3\frac{dy}{dx} - 2$ $\Rightarrow$ $2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} = -2 - 2x$
$\Rightarrow$ $\frac{dy}{dx}(2y - 4y^3) = -2 - 2x$ $\Rightarrow$ $\frac{dy}{dx} = \frac{x+1}{2y^3-y}$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{(-2,1)} = -1$ and $\left.\frac{dy}{dx}\right|_{(-2,-1)} = 1$

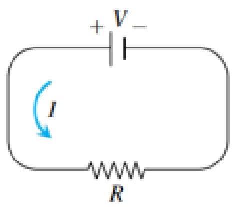
8. The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation Suppose that V is increasing at the rate of 1 volt sec while I is decreasing at the rate of 1/3 amp sec. Let t denote time in seconds. (15pts)

a. What is the value of dV/dt?

b. What is the value of dI/dt?

c. What equation relates dR/dt to dV/dt and dI/dt?

d. Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing?

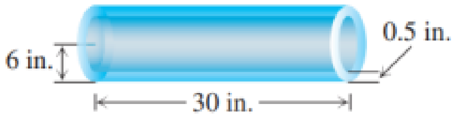


(a) $\frac{dV}{dt} = 1$ volt/sec (b) $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec

(c) $\frac{dV}{dt} = R\left(\frac{dI}{dt}\right) + I\left(\frac{dR}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - R\,\frac{dI}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - \frac{V}{I}\,\frac{dI}{dt}\right)$

(d) $\frac{dR}{dt} = \frac{1}{2}\left[1 - \frac{12}{2}\left(-\frac{1}{3}\right)\right] = \left(\frac{1}{2}\right)(3) = \frac{3}{2}$ ohms/sec, R is increasing

9. Estimate the volume of material in a cylindrical shell with length 30 in., radius 6 in., and shell thickness 0.5 in. (10pts)



The volume of a cylinder is $V = \pi r^2 h$. When h is held fixed, we have $\frac{dV}{dr} = 2\pi rh$, and so $dV = 2\pi rh\,dr$. For $h = 30$ in., $r = 6$ in., and $dr = 0.5$ in., the volume of the material in the shell is approximately $dV = 2\pi rh\,dr = 2\pi(6)(30)(0.5)$ $= 180\pi \approx 565.5\,\text{in}^3$.