1. (30 points) Find the limit

(a) $\lim_{v\to 2} \dfrac{v^3-8}{v^4-16}$ (b) $\lim_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$ (c) $\lim_{\theta\to 0} \dfrac{\tan\theta}{\theta^2\cot 3\theta}$

Solution:

(a) Do factorization for the numerator of the function. $(a^4-b^4) = (a^2-b^2)(a^2+b^2) = (a-b)(a+b)(a^2+b^2)$

$$\lim_{v\to 2}\frac{v^3-8}{v^4-16} = \lim_{v\to 2}\frac{v^3-2^3}{v^4-2^4} = \lim_{v\to 2}\frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim_{v\to 2}\frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{4\cdot 8} = \frac{12}{32} = \frac{3}{8}$$

(b) Rationalize the denominator first.

$$\lim_{x\to 4}\frac{4-x}{5-\sqrt{x^2+9}} = \lim_{x\to 4}\left[\frac{4-x}{5-\sqrt{x^2+9}}\cdot\frac{5+\sqrt{x^2+9}}{5+\sqrt{x^2+9}}\right] = \lim_{x\to 4}\frac{(4-x)(5+\sqrt{x^2+9})}{25-(x^2+9)} = \lim_{x\to 4}\frac{(4-x)(5+\sqrt{x^2+9})}{16-x^2}$$
$$= \lim_{x\to 4}\frac{(4-x)(5+\sqrt{x^2+9})}{(4-x)(4+x)} = \lim_{x\to 4}\frac{5+\sqrt{x^2+9}}{4+x} = \frac{5}{4}$$

(c) Rewrite the function first.

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta^2\cot 3\theta} = \lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}}{\theta^2\cdot\frac{\cos 3\theta}{\sin 3\theta}} = \lim_{\theta\to 0}\left[\left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin 3\theta}{3\theta}\right)\left(\frac{3}{\cos\theta\cos 3\theta}\right)\right] = 1\cdot 1\cdot\frac{3}{1\cdot 1} = 3$$

2. (20 points) For what values of a and b is

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x<2 \\ ax^2-bx+3, & 2\le x<3 \\ 2x-a+b, & x>3 \end{cases}$$

continuous at every x?

Solution:

- Find one-sided limits at $x=2$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\frac{x^2-4}{x-2} = \lim_{x\to 2^-}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-}(x+2) = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(ax^2-bx+3\right) = 4a-2b+3$$

If $f(x)$ is continuous at $x=2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) \Rightarrow 4 = 4a-2b+3 \Rightarrow 4a-2b = 1$.

- Find one-sided limits at $x=3$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\left(ax^2-bx+3\right) = 9a-3b+3$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(2x-a+b) = 6-a+b$$

If $f(x)$ is continuous at $x=3$, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \Rightarrow 9a-3b+3 = 6-a+b \Rightarrow 10a-4b = 3$.

- From above, one can find that $a = \frac{1}{2}$, $b = \frac{1}{2}$.

3. (20 points) A function is defined as

$$f(x) = \frac{1}{x}.$$

If $L = -1$, $c = -1$, and $\epsilon = 0.1$, find an open interval about **c** on which the inequality $\left|f(x) - L\right| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $\left|f(x) - L\right| < \epsilon$ holds.

Solution:

- $|f(x) - L| < \epsilon \Rightarrow |\frac{1}{x} - (-1)| < 0.1 \Rightarrow \left|\frac{1}{x} + 1\right| < 0.1, \quad 0 < |x - c| < \delta \Rightarrow 0 < |x + 1| < \delta.$

$$\left|\frac{1}{x} + 1\right| < 0.1 \Rightarrow -0.1 < \frac{1}{x} + 1 < 0.1 \Rightarrow -\frac{11}{10} < \frac{1}{x} < -\frac{9}{10} \Rightarrow -\frac{10}{11} > x > -\frac{10}{9} \Rightarrow -\frac{10}{9} < x < -\frac{10}{11}$$
$$0 < |x + 1| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$$

If $-\delta - 1 = -\frac{10}{9} \Rightarrow \delta = \frac{1}{9}$. If $\delta - 1 = -\frac{10}{11} \Rightarrow \delta = \frac{1}{11}$. Select the smaller one. Therefore, $\delta = \frac{1}{11}$.

4. (15 points) Show that the equation $-x^3 + 4x + 1 = 0$ has a solution in the interval $(-1, 0)$.

Solution:

- Let $f(x) = -x^3 + 4x + 1$, which is continuous on $(-1, 0)$. Then $f(-1) = -2, f(0) = 1$. By the **intermediate value theorem**, $f(x) = 0$ for some x in the interval: $-1 < x < 0$. That is, $x^3 - 15x + 1 = 0$ has a solution in $(-1, 0)$.

5. (15 points) Find equations for all horizontal and vertical asymptotes for the graph of

$$y = \frac{\sqrt{3x^2 + 4}}{x - 7}$$

Solution:

- Vertical asymptote:

$$\lim_{x \to 7^-} \frac{\sqrt{3x^2 + 4}}{x - 7} = -\infty, \quad \lim_{x \to 7^+} \frac{\sqrt{3x^2 + 4}}{x - 7} = \infty$$

Therefore, its vertical asymptote is $x = 7$.

- Horizontal asymptote:

$$\lim_{x \to \infty} \frac{\sqrt{3x^2 + 4}}{x - 7} = \lim_{x \to \infty} \frac{\sqrt{x^2}\sqrt{3 + \frac{4}{x^2}}}{x - 7} = \lim_{x \to \infty} \frac{|x|\sqrt{3 + \frac{4}{x^2}}}{x - 7} = \lim_{x \to \infty} \frac{x\sqrt{3 + \frac{4}{x^2}}}{x - 7} = \lim_{x \to \infty} \frac{\sqrt{3 + \frac{4}{x^2}}}{1 - \frac{7}{x}} = \frac{\sqrt{3 + 0}}{1 - 0} = \sqrt{3}$$
$$\lim_{x \to -\infty} \frac{\sqrt{3x^2 + 4}}{x - 7} = \lim_{x \to -\infty} \frac{\sqrt{x^2}\sqrt{3 + \frac{4}{x^2}}}{x - 7} = \lim_{x \to -\infty} \frac{|x|\sqrt{3 + \frac{4}{x^2}}}{x - 7} = \lim_{x \to \infty} \frac{-x\sqrt{3 + \frac{4}{x^2}}}{x - 7} = \lim_{x \to -\infty} \frac{-\sqrt{3 + \frac{4}{x^2}}}{1 - \frac{7}{x}} = \frac{-\sqrt{3 + 0}}{1 - 0} = -\sqrt{3}$$

Therefore, its horizontal asymptotes are $y = \sqrt{3}$ and $y = -\sqrt{3}$.