1. (30 points) Evaluate the limit. If the limit does not exist, please write down the reason. (10 points for each)

(a) $\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x}$ (b) $\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$ (c) $\lim_{y\to 0^+} e^{-\frac{1}{y}} \ln y$

Solution:

(a) $\left(\frac{0}{0}\right)$. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

$$\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{1+x^2}} = \lim_{x\to 0} \frac{1+x^2}{\sqrt{1-x^2}} = 1$$

(b) (∞^0) Let $y = \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right)$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x} = \lim_{x\to\infty} \left(\frac{\ln\left(x^2+1\right) - \ln(x+2)}{x}\right)$$
$$= \lim_{x\to\infty} \left(\frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1}\right) = \lim_{x\to\infty} \left(\frac{x^2+4x-1}{x^3+2x^2+x+2}\right) = 0$$

Thus,

$$\lim_{x\to\infty} \ln y = \ln\left(\lim_{x\to\infty} y\right) = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$$

(c) $\left(\frac{\infty}{\infty}\right)$.

$$\lim_{y\to 0^+} e^{-\frac{1}{y}} \ln y = \lim_{y\to 0^+} \frac{\ln y}{e^{\frac{1}{y}}} = \lim_{y\to 0^+} \frac{\frac{1}{y}}{-\frac{1}{y^2} e^{\frac{1}{y}}} = \lim_{y\to 0^+} -\frac{y}{e^{\frac{1}{y}}} = -\frac{\lim_{y\to 0^+} y}{\lim_{y\to 0^+} e^{\frac{1}{y}}} = -\frac{0}{\infty} = 0$$

2. (15 points) Find $\dfrac{dy}{dx}$. (5 points for each)

(a) $y = e^{\sin x}\left(\ln x^2 + 1\right)$ (b) $y = 2\left(\ln x\right)^{x/2}$ (c) $y = e^{\tanh 2x}$

Solution:

(a) $y = e^{\sin x}\left(\ln x^2 + 1\right)$

$$\begin{aligned} y' &= \frac{d}{dx}\left(e^{\sin x}\right)\cdot\left(\ln x^2+1\right) + e^{\sin x}\cdot\frac{d}{dx}\left(\ln x^2+1\right) \\ &= e^{\sin x}\cdot\cos x\cdot\left(\ln x^2+1\right) + e^{\sin x}\cdot\frac{2}{x} \\ &= e^{\sin x}\cdot\left[\cos x\left(\ln x^2+1\right)+\frac{2}{x}\right] \end{aligned}$$

(b) Use logarithmic differentiation. $y = 2\left(\ln x\right)^{x/2} \Rightarrow \ln y = \ln 2 + \frac{x}{2}\ln\ln x$

$$\ln y = \ln 2 + \frac{x}{2}\ln\ln x \Rightarrow \frac{y'}{y} = \frac{1}{2}\ln\ln x + \frac{x}{2}\frac{1}{\ln x}\cdot\frac{1}{x} \Rightarrow y' = y\cdot\left(\frac{1}{2}\ln\ln x + \frac{1}{2}\frac{1}{\ln x}\right) = \left(\ln x\right)^{x/2}\left(\ln\ln x + \frac{1}{\ln x}\right)$$

(c) Use the Chain rule

$$\frac{dy}{dx} = e^{\tanh 2x}\cdot\frac{d}{dx}\left(\tanh 2x\right) = e^{\tanh 2x}\cdot 2\,\mathrm{sech}^2\,2x = 2e^{\tanh 2x}\,\mathrm{sech}^2\,2x$$

3. (10 points) In the following two problems, does $f(x)$ grows faster, slower, or at the same rate as $g(x)$ when $x \to \infty$ (5 points for each)

(a) $f(x) = \tan^{-1}\dfrac{1}{x}$, $g(x) = \dfrac{1}{x}$ (b) $f(x) = \dfrac{x}{100}$, $g(x) = xe^{-x}$

Solution:

(a) Evaluate the limit:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\tan^{-1}\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{-\frac{x^{-2}}{1+x^{-2}}}{-x^{-2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x^2}} = 1$$

Therefore, $f(x)$ and $g(x)$ grow at the same rate when $x \to \infty$.

(b) Evaluate the limit:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\frac{x}{100}}{xe^{-x}} = \lim_{x\to\infty}\frac{e^x}{100} = \infty$$

Therefore, $f(x)$ grows faster than $g(x)$ when $x \to \infty$.

4. (20 points) Evaluate the integrals (10 points for each)

(a) $\displaystyle\int \frac{x2^{x^2}}{1+2^{x^2}}\,dx$ (b) $\displaystyle\int_1^e \frac{8\ln 3\log_3 x}{x}\,dx$

Solution:

(a) Let $u = 1 + 2^{x^2} \Rightarrow du = \ln 2 \cdot 2^{x^2} \cdot (2x)\,dx \Rightarrow \dfrac{1}{2\ln 2}\,du = x2^{x^2}\,dx$

$$\int \frac{x2^{x^2}}{1+2^{x^2}}\,dx = \frac{1}{2\ln 2}\int \frac{1}{u}\,du = \frac{1}{2\ln 2}\ln|u| + C = \frac{1}{2\ln 2}\ln\left(1+2^{x^2}\right) + C$$

(b) Change the base first.

$$\int_1^e \frac{8\ln 3\log_3 x}{x}\,dx = \int_1^e \frac{8\ln 3\frac{\ln x}{\ln 3}}{x}\,dx = \int_1^e \frac{8\ln x}{x}\,dx$$

Let $u = \ln x \Rightarrow du = \dfrac{1}{x}\,dx$, $\quad x : 1 \to e \Rightarrow u : 0 \to 1$. Therefore,

$$\int_1^e \frac{8\ln x}{x}\,dx = \int_0^1 8u\,du = 4u^2\Big|_0^1 = 4$$

5. (25 points) Consider the equation of the **tractrix**

$$y = a\operatorname{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2},\ a > 0$$

(a) (10 points) Prove that $\dfrac{d}{dx}\left(\operatorname{sech}^{-1}x\right) = -\dfrac{1}{x\sqrt{1-x^2}}$. (**Hint**): $\tanh^2 x = 1 - \operatorname{sech}^2 x$

(b) (5 points) Use the result of (a), find $\dfrac{dy}{dx}$.

(c) (5 points) Assume one point $P(x_0, y_0)$ is on the tractrix curve. What is the equation of tangent line through P.

(d) (5 points) Let L be the tangent line to the tractrix at the point P. When L intersects the y-axis at the point Q, show that $\overline{PQ} = a$.

Solution:

(a) Utilize the inverse differentiation.

$$y = \left(\operatorname{sech}^{-1}x\right) \Rightarrow x = \operatorname{sech}y \Rightarrow 1 = -\operatorname{sech}y\tanh y \cdot y' \Rightarrow y' = -\frac{1}{\operatorname{sech}y\tanh y} = -\frac{1}{\operatorname{sech}y \cdot \sqrt{1-\operatorname{sech}^2 y}}$$

$$\Rightarrow y' = -\frac{1}{x \cdot \sqrt{1-x^2}}$$

(b) Use the Chain rule.

$$y' = a \cdot \left(-\frac{1}{\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}}\right) \cdot \frac{1}{a} + \frac{x}{a^2-x^2} = -\frac{a^2}{x\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}} = \frac{x^2-a^2}{x\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{x}$$

(c) The slope of the tangent line at $P(x_0, y_0)$ is $m = -\dfrac{\sqrt{a^2-x_0^2}}{x_0}$. Therefore, the tangent line is

$$y - y_0 = m(x - x_0) \Rightarrow y = -\frac{\sqrt{a^2-x_0^2}}{x_0}(x - x_0) + y_0$$

where $y_0 = a\operatorname{sech}^{-1}\dfrac{x_0}{a} - \sqrt{a^2-x_0^2}$

(d) On the tangent line, when $x = 0$,

$$y = -\frac{\sqrt{a^2-x_0^2}}{x_0}(0 - x_0) + y_0 = \sqrt{a^2-x_0^2} + y_0$$

The coordinate of Q is $(0, y_0)$. Therefore

$$\overline{PQ} = \sqrt{(x_0-0)^2 + \left[y_0 - \left(\sqrt{a^2-x_0^2} + y_0\right)\right]^2} = \sqrt{x_0^2 + a^2 - x_0^2} = \sqrt{a^2} = a$$