Total: 100 points

1. (**25** points) Find the points on the curve $x^2y = 2$ nearest the origin.

Solution:

- Goal: to minimize $f(x,y) = d^2 = x^2 + y^2$ subject to the constraint $g(x,y) = x^2y - 2 = 0$ where d is the distance between the point (x,y) and the origin. Because $\nabla f = \langle 2x, 2y \rangle, \quad \nabla g = \left\langle 2xy, x^2 \right\rangle, \quad \nabla f = \lambda \nabla g, g(x,y) = 0$,

$$\begin{aligned} &2x = \lambda \cdot 2xy \Rightarrow \lambda = \frac{1}{y} \\ &2y = \lambda \cdot x^2 \Rightarrow 2y = \frac{1}{y} \cdot x^2 \Rightarrow x^2 = 2y^2 \\ &x^2y = 2 \Rightarrow x^2y = 2y^2 \cdot y = 2 \Rightarrow y^3 = 1 \Rightarrow y = 1,\, x^2 = 2 \Rightarrow x = \pm\sqrt{2} \end{aligned}$$

  Note that $x \neq 0$, therefore, we can write $\lambda = 1/y$ only from $2x = \lambda 2xy$ and omit the case $x = 0$. From the result above, $\left(\pm\sqrt{2}, 1\right)$ are the points on the curve $x^2y = 2$ nearest the origin.

2. (**25** points) Find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

Solution:

- Find critical points.

$$\begin{aligned} &f_x(x,y) = 4x + 3y - 5 = 0 \\ &f_y(x,y) = 3x + 8y + 2 = 0 \end{aligned}$$

  After solving this equation, one can find that the critical point is $(x,y) = (2,-1)$.

- Check Hessian at this point $(x,y) = (2,-1)$.

$$f_{xx}(2,-1) = 4 > 0, f_{yy}(2,-1) = 8, f_{xy}(2,-1) = 3 \Rightarrow \textbf{Hessian}: f_{xx}(2,-1) \cdot f_{yy}(2,-1) - \left(f_{xy}(2,-1)\right)^2 = 23 > 0$$

- Based on this result, $f(x,y)$ has a local minimum at $(x,y) = (2,-1)$. The value of the local minimum is $f(2,-1) = -6$.

3. (**30** points) Write an equivalent double integral with the order of integration reversed.

(**a**) $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} f(x,y)\,dx\,dy$ (**b**) $\displaystyle\int_0^{\ln 2}\int_{e^y}^{2} f(x,y)\,dx\,dy$ (**c**) $\displaystyle\int_0^{8}\int_{\sqrt[3]{x}}^{2} f(x,y)\,dy\,dx$

Solution:

(a) $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} f(x,y)\,dx\,dy = \int_{-3}^{3}\int_0^{\frac{1}{2}\sqrt{9-x^2}} f(x,y)\,dy\,dx$

(b) $\displaystyle\int_0^{\ln 2}\int_{e^y}^{2} f(x,y)\,dx\,dy = \int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx$

(c) $\displaystyle\int_0^{8}\int_{\sqrt[3]{x}}^{2} f(x,y)\,dy\,dx = \int_0^2\int_0^{y^3} f(x,y)\,dx\,dy$

4. (**20** points) Change the Cartesian integral into an equivalent polar integral. Then **evaluate the polar integral**.

$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\,dy\,dx$$

Solution:

- The integral is

$$\begin{aligned}\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\,dy\,dx &= \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4} r\,dr\,d\theta = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r^{-3}\,dr\,d\theta = \int_0^{\pi/4}\left[-\frac{1}{2}\frac{1}{r^2}\right]\Bigg|_{\sec\theta}^{2\cos\theta} \\ &= \int_0^{\pi/4}\left[\left(-\frac{1}{2}\cdot\frac{1}{4\cos^2\theta}\right)-\left(-\frac{1}{2}\cdot\frac{1}{\sec^2\theta}\right)\right]d\theta = \int_0^{\pi/4}\left(\frac{1}{2}\cos^2\theta-\frac{1}{8}\sec^2\theta\right)d\theta \\ &= \frac{\pi}{16}.\end{aligned}$$