1. (30 points) Find the derivatives of the following functions

(a) $f(x) = \sin^3\left(\frac{\pi x}{2}\right)$ (b) $f(x) = (1 + \sec x)\sin x$ (c) $f(x) = \frac{x}{\sqrt{x^2+1}}$

Solution:

(a) Use the Chain rule

$$f'(x) = 3\sin^2\left(\frac{\pi x}{2}\right) \cdot \cos\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2} = \frac{3\pi}{2}\cos\left(\frac{\pi x}{2}\right)\sin^2\left(\frac{\pi x}{2}\right)$$

(b) Utilize the product rule. $\sec x \tan x \sin x = \frac{1}{\cos x} \cdot \tan x \cdot \sin x = \frac{\sin x}{\cos x} \cdot \tan x = \tan^2 x$, $1 + \tan^2 x = \sec^2 x$.

$$\begin{aligned} f'(x) &= (1+\sec x)' \cdot \sin x + (1 + \sec x) \cdot (\sin x)' = (\sec x \tan x) \cdot \sin x + (1 + \sec x) \cdot \cos x \\ &= \tan^2 x + (\cos x + 1) = \cos x + \sec^2 x \end{aligned}$$

(c) Use the quotient rule and the Chain rule

$$\begin{aligned} f'(x) &= \frac{1 \cdot \left(x^2+1\right)^{\frac{1}{2}} - x \cdot \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \cdot 2x}{x^2+1} = \frac{\left(x^2+1\right)^{\frac{1}{2}} - x^2\left(x^2+1\right)^{-\frac{1}{2}}}{x^2+1} = \frac{\left(x^2+1\right)^{-\frac{1}{2}} \cdot \left[(x^2+1) - x^2\right]}{x^2+1} \\ &= \frac{1}{\left(x^2+1\right)^{\frac{3}{2}}} \end{aligned}$$

2. (15 points) Use implicit differentiation to find the equation the tangent line to the curve

$$x^2 + y^2 = \left(2x^2 + 2y^2 - x\right)^2$$

at the point $\left(0, \frac{1}{2}\right)$

Solution:

- Do implicit differentiation first.

$$2x + 2y\frac{dy}{dx} = 2\left(2x^2 + 2y^2 - x\right)\left(4x + 4y\frac{dy}{dx} - 1\right)$$

At the point $\left(0, \frac{1}{2}\right)$, replace $x = 0, y = 1/2$ to the equation above.

$$0 + 1 \cdot \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} = 2\left(0 + \frac{1}{2} - 0\right)\left(0 + 2 \cdot \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} - 1\right) \Rightarrow \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} = 2\left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} - 1 \Rightarrow \left.\frac{dy}{dx}\right|_{\left(0,\frac{1}{2}\right)} = 1$$

Therefore, the tangent line is

$$y - \frac{1}{2} = 1 \cdot (x - 0) \Rightarrow y = x + \frac{1}{2}.$$

3. (20 points) Is there a value of b that will make

$$g(x) = \begin{cases} x+b & x<0 \\ \cos x & x \geq 0 \end{cases}$$

continuous at $x=0$? Is there a value of b that will make $g(x)$ differentiable at $x=0$?

Solution:

- To check the continuity:

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x+b = b, \quad \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \cos x = 1$$

If g is continuous at $x=0$,

$$g(0) = \lim_{x\to 0} g(x) = \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) \Rightarrow b = 1$$

- To check the differentiability. For the left-derivative:

$$f'_-(0) = \frac{d}{dx}(x+b)\bigg|_{x=0} = 1$$

For the right-derivative:

$$f'_+(0) = \frac{d}{dx}\cos x\bigg|_{x=0} = 0$$

- Therefore, g is not differentiable at $x=0$ for any value b.

4. (15 points) Find $\dfrac{d^2y}{dx^2}$ if

$$y = \frac{x^2+5x-1}{x^2}$$

Solution:

- $y = \dfrac{x^2+5x-1}{x^2} = 1 + 5x^{-1} - x^{-2}$. Therefore

$$\frac{dy}{dx} = 5\cdot(-1)x^{-2} - (-2)\cdot x^{-3} = -5x^{-2} + 2x^{-3}$$
$$\frac{d^2y}{dx^2} = -5\cdot(-2)x^{-3} + 2\cdot(-3)x^{-4} = 10x^{-3} - 6x^{-4} = \frac{10}{x^3} - \frac{6}{x^4}.$$

5. (20 points) Let $g(x) = \dfrac{f(x)}{x^2}$.

   (a) Find $g'(x)$. Please use $f(x)$ and $f'(x)$ to express $g'(x)$.

   (b) If the equation $y = 3x + 5$ is an equation of the tangent line to the curve $y = f(x)$ at $x = -2$, $f'(-2) =$?

   (c) Based on (b), $f(-2) =$?

   (d) Based on (a), (b), and (c), find $g'(-2) =$?

Solution:

(a) Use the quotient rule.

$$g'(x) = \frac{f'(x) \cdot x^2 - f(x) \cdot 2x}{x^4}$$

(b) The slope of the tangent line to the curve $y = f(x)$ at $x = -2$ is 3. Therefore $f'(-2) = 3$.

(c) For some function $y = f(x)$, the equation of the tangent line to the curve $y = f(x)$ at $x = -2$ can be expressed as

$$y - f(-2) = f'(-2) \cdot [x - (-2)] \Rightarrow y - f(-2) = 3(x + 2) \Rightarrow y = 3x + 6 + f(-2).$$

And we have known that the tangent line is $y = 3x + 5$. Therefore, $6 + f(-2) = 5 \Rightarrow f(-2) = -1$.

(d) Based on (a),

$$g'(-2) = \frac{f'(-2) \cdot (-2)^2 - f(-2) \cdot 2 \cdot (-2)}{(-2)^4} = \frac{3 \cdot 4 - (-1) \cdot (-4)}{16} = \frac{1}{2}.$$