Total: 100 points

1. (**90** points) Evaluate the following integrals. (15 points for each)

(a) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

(b) $\displaystyle\int \frac{-4x-1}{x^2\left(x^2+1\right)}\,dx$

(c) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,d\theta$

(d) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3\theta\sin 2\theta\,d\theta$

(e) $\displaystyle\int \sin\left(\ln x\right)\,dx$

(f) $\displaystyle\int x^4\ln x\,dx$

Solution:

(a) Use trigonometric substitution. $x = 3\sin\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} \Rightarrow dx = 3\cos\theta\,d\theta \Rightarrow \sqrt{9-x^2} = |3\cos\theta| = 3\cos\theta$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{3\cos\theta}\cdot 3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta = 9\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

Because $x = 3\sin\theta \Rightarrow \cos\theta = \dfrac{\sqrt{9-x^2}}{3}$, $\quad \sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\dfrac{x}{3}\cdot\dfrac{\sqrt{9-x^2}}{3}$. Therefore,

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right] + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

(b) Do partial fraction decomposition first. Let

$$\frac{-4x-1}{x^2\left(x^2+1\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

Thus,

$$-4x-1 = Ax\left(x^2+1\right) + B\left(x^2+1\right) + (Cx+D)\,x^2$$

Then,

$$\begin{aligned}
\text{Let } x = 0 &\Rightarrow B = -1\\
\text{Coefficient of } x^3 &\Rightarrow A + C = 0\\
\text{Coefficient of } x^2 &\Rightarrow B + D = 0 \Rightarrow D = 1\\
\text{Coefficient of } x^1 &\Rightarrow A = -4 \Rightarrow C = 4
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \frac{-4x-1}{x^2\left(x^2+1\right)}\,dx &= \int \frac{-4}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{4x+1}{x^2+1}\,dx = -4\ln|x| + \frac{1}{x} + \int \frac{4x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx\\
&= -4\ln|x| + \frac{1}{x} + 2\ln\left|x^2+1\right| + \tan^{-1}x + K
\end{aligned}$$

where K is the integration constant.

Solution:

(c) Use integration by part. $u = \sec\theta \Rightarrow du = \sec\theta\tan\theta\, d\theta, \quad dv = \sec^2\theta\, d\theta \Rightarrow v = \tan\theta$. Therefore,

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta &= [\sec\theta\tan\theta]\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\tan^2\theta\, d\theta = \left[\sqrt{2} - \left(-\sqrt{2}\right)\right] - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\left(\sec^2\theta - 1\right) d\theta \\
&= 2\sqrt{2} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\, d\theta = 2\sqrt{2} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta + [\ln|\sec\theta + \tan\theta|]\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= 2\sqrt{2} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta + \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right) = 2\sqrt{2} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)
\end{aligned}$$

Therefore,

$$2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta = 2\sqrt{2} + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \Rightarrow \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \sqrt{2} + \frac{1}{2}\ln\left(3 + 2\sqrt{2}\right).$$

(d) $\sin 2\theta = 2\sin\theta\cos\theta$. Thus,

$$\int_0^{\frac{\pi}{2}} \cos^3\theta \sin 2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \left(\cos^3\theta \cdot 2\sin\theta\cos\theta\right) d\theta = 2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\, d\theta$$

Let $u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta, \quad \theta : 0 \to \dfrac{\pi}{2} \Rightarrow u : 1 \to 0$. Therefore,

$$2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\, d\theta = 2\int_1^0 \left(-u^4\right) du = -2\left[\frac{1}{5}u^5\right]\Big|_1^0 = -2\left(0 - \frac{1}{5}\right) = \frac{2}{5}$$

(e) Let $u = \ln x \Rightarrow x = e^u$, $du = \dfrac{1}{x}\, dx$. Therefore,

$$\int \sin(\ln x)\, dx = \int \sin(u)\, x\, du = \int \sin(u)\, e^u\, du = \int e^u \sin(u)\, du$$

Use integration by parts, one can find that

$$\begin{aligned}
&\int e^u \sin(u)\, du = -e^u\cos(u) + \int e^u\cos(u)\, du = -e^u\cos(u) + \left[e^u\sin(u) - \int e^u\sin(u)\, du\right] \\
&\Rightarrow 2\int e^u\sin(u)\, du = e^u(\sin u - \cos u) \Rightarrow \int e^u\sin(u)\, du = \frac{e^u}{2}(\sin u - \cos u) + C
\end{aligned}$$

Therefore,

$$\int \sin(\ln x)\, dx = \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$$

(f) Let $u = \ln x$, $dv = x^4\, dx \Rightarrow du = \dfrac{1}{x}\, dx$, $v = \dfrac{1}{5}x^5$. Then

$$\int x^4\ln x\, dx = \frac{1}{5}x^5\ln x - \int \frac{1}{5}x^5 \cdot \frac{1}{x}\, dx = \frac{1}{5}x^5\ln x - \frac{1}{5}\int x^4\, dx = \frac{1}{5}x^5\ln x - \frac{1}{25}x^5 + C$$

2. (**10** points) Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Solution:

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow y = \pm b\sqrt{1 - \dfrac{x^2}{a^2}}$. Therefore, the area can be expressed as $A = 4\displaystyle\int_0^a b\sqrt{1 - \dfrac{x^2}{a^2}}\, dx$.

  Let $x = a\sin\theta, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} \to dx = a\cos\theta\, d\theta, \quad \sqrt{1 - \dfrac{x^2}{a^2}} = \cos\theta$. Therefore,

$$A = 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\, dx = 4b\int_0^{\frac{\pi}{2}} (\cos\theta \cdot a\cos\theta)\ d\theta = 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = 4ab \cdot \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = 4ab \cdot \frac{\pi}{4} = \pi ab$$