Part III — Combinatorics

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A brief summary of important ideas and results in the course

1 Introduction

1.1 Notation and Terminology

Lecture 1

Notation. X, Y sets, then $\mathbb{P}(X)$ is the power set.
$[n] = \{1, \cdots, n\}, [m, n] = \{m, \cdots, n\}$
Much of the time, $|X| = n$ and $X = [n]$ in this course
Level sets: $X^{(r)} = \{A \in \mathbb{P}(X) : |A| = r\}, X^{(<r)} = \{A \in \mathbb{P}(X) : |A| < r\}$
$X^{(\omega)} = \{A \in \mathbb{P}(X) : A \text{ finite}\}$
If G is a graph, $E(G)$ is its edge set, $e(G)$ the number of edges, $V(G)$ the vertex set and $|G|$ the number of vertices. G may also be used to mean its vertex set

Definition (Set System)**.** $\mathscr{A} \subset \mathbb{P}(X)$ is a **set system** or **family of sets** and can be identified with a bipartite graph $G_{\mathscr{A}}(U, W)$ with $U = \mathscr{A}, W = \bigcup_{A \in \mathscr{A}} A$ or $W = X$.

Note that here the edge $Ax \in E(G_{\mathscr{A}}) \Leftrightarrow x \in A$. Basically, we're forming a bipartite graph where each vertex in one set is identified with the sets in $\mathscr{A}$, then each vertex in the other are just all the elements that appear in X.

Definition (Set of Distinct Representatives)**.** Given $\mathscr{A} \subset \mathbb{P}(X)$, a **set of distinct representatives** (an **SDR**) is an injection $f : \mathscr{A} \to X$ s.t. $f(A) \in A \; \forall A \in \mathscr{A}$

If we look at the bipartite graph of an SDR as above, then we see that it corresponds to a complete matching $U \to W$ (i.e. $\mathscr{A}$ into all elements of X that $\mathscr{A}$ hits)

Theorem 1 (Hall, 1935)**.** A set system $\mathscr{A}$ has an SDR iff $\forall \mathscr{A}' \subset \mathscr{A}$:

$$\left| \bigcup_{A \in \mathscr{A}'} A \right| \geq |\mathscr{A}'|$$

If we look at a bipartite graph in the context of set systems, we see that every bipartite graph corresponds to a multiset of sets, giving us a reformulation of Hall:

Theorem 1 (Alternate Version)**.** A bipartite graph $G(U, W)$ has a complete matching iff, $\forall S \subset U$,

$$|\Gamma(S)| \geq |S| \tag{1}$$

Where $\Gamma(S)$ is the set of neighbours of S.

Proof 1. The necessary direction is trivial, so we look only at sufficiency.
Assume G is edge minimal w.r.t. (1). Here we look for a contradiction - suppose that G is not a matching, so we can find two edges out of one vertex in U - say uw_1 and uw_2. Use minimality to get two sets S_1, S_2 in U with the same number neighbours as vertices and u is the only neighbour of w_i in S_i.
Then, use $|S_1 \cap S_2| \leq |\Gamma(S_1 \cap S_2)|$ and manipulate the RHS using conditions on the S_i and their intersection and minimality to get $|\Gamma(S_1 \cap S_2)| \leq |S_1 \cap S_2|$, so those two expressions are equal.
Then, delete u from $S_1 \cap S_2$ and look at that set's size (and the size of its neighbours) to get a contradiction. □

Proof 2. The necessary direction is trivial, so we look only at sufficiency.
Assume G is edge minimal w.r.t. (1). Here we induct. Look at $\xi = \{E \subset U : |\Gamma(E)| = |E| > 0\} \neq \phi$.
Split into the cases about whether or $\exists E \in \xi$ where $E \neq U$. If it exists, then form the sub-graphs from $H = E \cup \Gamma(E)$ and $G \setminus H$ - both satisfy (1) so done by induction.
Otherwise, pick an edge uw - then $G \setminus \{u, w\}$ satisfies (1) so add back u and w and we're done. □

Corollary 2. $G(U, W)$ bipartite, $d(u) \geq d(w)$ $\forall u \in U, w \in W$. Then $\exists$ a complete matching $U \to W$.

Proof. Pick $d(u) \geq d \geq d(w)$, look at a general $S \subset U$ to get $d|S| \leq e(S, \Gamma(S)) \leq d|\Gamma(S)|$ □

Lecture 2

Definition ((r,s)-regular)**.** A bipartite graph $G(U, W)$ is (r,s)**-regular** if $d(u) = r$ and $d(w) = s \quad \forall u \in U, \ \forall w \in W$

Note that $r|U| = e(G) = s|W|$ and that Corollary 2 implies that $G(U, W)$ being (r,s)-regular gives us a complete matching from U to W if $|U| \leq |W|$

Corollary 3. Let $0 \leq i, j \leq n$, with $\binom{n}{i} \leq \binom{n}{j}$. Then $\exists$ a complete matching $f : [n]^{(i)} \to [n]^{(j)}$ s.t. $f(A) \subset A$ if $j \leq i$ and $f(A) \supset A$ if $i \leq j$.

Theorem 4. Let $G = G(U, W)$ be a connected (r,s)-regular graph. Then, for $\phi \neq A \subset U$,

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

With equality iff $A = U$

Proof. Use $r|A| = e(A, \Gamma(A)) \leq s|\Gamma(A)|$ and divide by $|W|$ to get the result (noting $r|U| = s|W|$). If equality holds, then all edges out of $\Gamma(A)$ go into A, so $A = U$ or the else graph is disconnected. □

Definition (Partially Ordered Set)**.** A **partially ordered set**, or **poset** is a set with a binary relation $\leq$ that satisfies reflexivity ($a \leq a$), antisymmetry ($a \leq b, b \leq a \Rightarrow a = b$) and transitivity ($a \leq b, b \leq c \Rightarrow a \leq c$)

Definition (The Cube)**.** We define **the cube**: $Q^n \cong \mathbb{P}(n) \cong [2]^n \cong$ the set of all $0-1$ sequences.

Remarks.

- Q^n can be considered as a graph - AB is an edge iff $|A \Delta B| = 1$ (look at an n dimensional cube with vertices containing all coordinates with 0 or 1 and put $j \in A$ if $x_j = 1$)
- It is also a poset via $A < B$ if $A \subset B$
- Q^n has a natural orientation: $\overrightarrow{AB}$ if $A = B \cup \{a\}$
- For the order induced on Q^n, see Dilworth's Theorem

2 Sperner Systems

Definition (Sperner)**.** A set system $\mathscr{A} \subset \mathbb{P}(n)$ is **Sperner** if $A, B \in \mathscr{A},\ A \neq B \Rightarrow A \not\subset B$. That is no two non-equal sets in $\mathscr{A}$ contain each other.

The simplest example of Sperner sets are the level sets, $X^{(r)}$

Definition (Chain)**.** In a poset, a **chain** is a linearly ordered set, $c_1 < c_2 < \cdots < c_m$

Definition (Antichain)**.** In a poset, an **antichain** has no two elements that are comparable.

A set-system being Sperner is the same as being an antichain with the operation $\subseteq$.

Theorem 1 (Sperner, 1928)**.** If $\mathscr{A} \subset \mathbb{P}(n)$ is Sperner, then $|\mathscr{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Proof. For $0 \leq r \leq n-1$, we can easily find a complete matching between $X^{(r)}$ and $X^{(r+1)}$ from the smaller set to the larger set (where $AB \in E\left(X^{(r)}, X^{(r+1)}\right)$ iff $A \subset B$ or $B \subset A$). All these matchings put together create a collection of $\binom{n}{\lfloor n/2 \rfloor}$ paths in Q^n (look from the largest layer in the middle outward - i.e. the layer where $r = \lfloor n/2 \rfloor$), each of which is a chain. Hence our poset is covered by $\binom{n}{\lfloor n/2 \rfloor}$, giving $|\mathscr{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$ $\square$

Definition (Weight)**.** The **weight** $w(A)$ of a set $A \in \mathbb{P}(n)$ is $w(A) = \frac{1}{\binom{n}{|A|}}$

Theorem 2 (LYM or YMBL[1])**.** Let $\mathscr{A}$ be a Sperner system on X, with $|X| = n$. Then

$$w(\mathscr{A}) := \sum_{A \in \mathscr{A}} w(A) \leq 1$$

Proof. Look at the maximal chains in $\mathbb{P}(X)$, $A_0 \subset A_1 \subset \cdots \subset A_n$, with $A_i \in X^{(i)}$ (so $|A_i| = i$). By just picking elements one at a time, we can see that the number of maximal chains is $n!$. Each such chain has ≤ 1 element of $\mathscr{A}$, as $\mathscr{A}$ is Sperner. But also, every $A \in \mathscr{A}$ is in $|A|!\,(n-|A|)!$ chains.
Hence, $\sum_{A \in \mathscr{A}} |A|!\,(n-|A|)! \leq n!$ which gives the result. $\square$

Lecture 3

Theorem 2 (Alternate Version)**.** If $\mathscr{A} \subset \mathbb{P}(n)$ is an anti-chain, then $w(\mathscr{A}) \leq 1$

Corollary 3. If $\mathscr{A} \subset \mathbb{P}(n)$ is a Sperner system, then $|\mathscr{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$, with equality iff $\mathscr{A}$ is $X^{(\lfloor n/2 \rfloor)}$ or $X^{(\lceil n/2 \rceil)}$.

Proof. The inequality is immediate.
With equality, we see that $\mathscr{A} \subset X^{(\lfloor n/2 \rfloor)} \cup X^{(\lceil n/2 \rceil)}$ quickly[2], so the claim holds for n even - now we check $n = 2k+1$. Write $\mathscr{A}_k = \mathscr{A} \cup X^{(k)}, \mathscr{A}_{k+1} = \mathscr{A} \cup X^{(k+1)}$, which yields a bipartite graph (if both non-empty, else claim holds).
Then, using Theorem 4 from Section 1, $|\mathscr{A}| = |\mathscr{A}_k| + |\mathscr{A}_{k+1}| < |\mathscr{A}_k| + |\Gamma(\mathscr{A}_{k+1})| \leq \binom{n}{k}$. $\square$

Definition (k-Sperner)**.** We say that $\mathscr{A} \subset \mathbb{P}(n)$ is k-Sperner if it does not contain $A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_{k+1}$ - i.e. it doesn't contain a chain of length $k+1$.

Note that a Sperner system is 1-Sperner.

Corollary 4 (Erdős, 1945)**.** If $\mathscr{A} \in \mathbb{P}(n)$ is k-Sperner, then $|\mathscr{A}|$ is at most the sum of the k largest binomial coefficients.

[1] Yamamoto, 1954; Meshalkin, 1963; Bollobás, 1965; Lubell, 1966
[2] Indeed, if A not in here, then $w(A)$ is not minimal, so we can't have $\binom{n}{\lfloor n/2 \rfloor}$ terms

Theorem (Littlewood and Offord - LO[3])**.** $z_i \in \mathbb{C}$, $|z_i| \geq 1$, then the number of $\sum_1^n \pm z_i$ within k of each other is $\leq ck\frac{2^n \log n}{\sqrt{n}}$ for some constant c.

Theorem 5 (Erdős, 1945)**.** Let $x_1, \cdots, x_n \in \mathbb{R}$ with $x_i \geq 1$. Then the number of sums $\sum_1^n \pm x_i$ in an open interval J of length $2k$ is at most the sum of the k largest binomial coefficients.

Proof. Define $\mathscr{A} = \{A \in \mathbb{P}(n) : \sum_{i \in A} x_i - \sum_{j \notin A} x_j \in J\}$ - this is k-Sperner, as moving x_i from one sum to the other gives a change in value of at least 2, so a chain of $k+1$ sets would need a change of at least $2k$. Hence, we're done. $\square$

Conjecture (Erdős, 1945)**.** If $x_1, \cdots, x_n \in E$, where E is a normed space and $||x_i|| \geq 1$, then the number of sums $\sum x_i$ less than distance 2 from each other is $\binom{n}{\lfloor n/2 \rfloor}$

Definition (Symmetric Chain)**.** A chain $A_0 \subset A_1 \subset \cdots \subset A_k$ is **symmetric** if $|A_{i+1}| = |A_i| + 1 \; \forall i$ and $|A_0| + |A_k| = n$

Theorem 6 (Kleitman and Katona)**.** $\mathbb{P}(n)$ has a decomposition into symmetric chains.

Proof. Induct on n - the base case is $\{\phi, \{1\}\}$.
If $\mathbb{P}(n-1) = \dot{\bigcup}_{i=1}^q \mathscr{C}_i$ with $\mathscr{C}_i = \{A_0 \subset \cdots A_k\}$, then form

$$\mathscr{C}_i' = \{A_0, \cdots, A_{k-1}, A_k \cup \{n\}\}$$

and

$$\mathscr{C}_i'' = \{A_0 \cup \{n\}, \cdots, A_{k-1} \cup \{n\}\}$$

Then $\mathscr{C}_i'$ and $\mathscr{C}_i''$ partition $\mathbb{P}(n)$ into symmetric chains. $\square$

Lecture 4

Remarks. Note that we've formed a partition into $\binom{n}{\lfloor n/2 \rfloor}$ symmetric chains, with $\binom{n}{i} - \binom{n}{i-1}$ chains of length $n+1-2i$ (so 1 chain of length $n+1$, $n-1$ chains of length $n-1$, etc.)

For the next few results, let E be a normed space, $x_1, \cdots, x_n \in E$, $||x_i|| \geq 1 \forall i$ and, for $A \in \mathbb{P}(n)$, $x_A = \sum_{i \in A} x_i$

Definition (Scattered)**.** Call $\mathscr{D} \subset \mathbb{P}(n)$ **scattered** if $||x_A - x_B|| \geq 1 \; \forall A, B \in \mathscr{D}$.

Definition (Symmetric Partition)**.** Call a partition $\mathbb{P}(n) = \dot{\bigcup}_1^s \mathscr{D}_j$ **symmetric** if there are precisely $\binom{n}{i} - \binom{n}{i-1}$ sets $\mathscr{D}_i$ of cardinality $n+1-2i$.

Theorem 7 (Kleitman, 1970)**.** E and $(x_i)_1^n$ as above. Then, $\mathbb{P}(n)$ has a symmetric partition into scattered sets.

Proof. [4] Induct on n - the base case is $\{\phi, \{1\}\}$.
Otherwise, let $\mathbb{P}(n-1) = \dot{\bigcup}_{i=1}^q \mathscr{D}_i$ be a scattered partition into scattered sets and let $\mathscr{D}_j = \{A_1, \cdots, A_m\}$.
In each $\mathscr{D}_j$, pick A_m so that $\left|\left|x_{A_m} + x_n - x_{A_j}\right|\right| \geq ||x_n|| \; \forall \, 1 \leq j < m$. This is possible - indeed, if not, then we can find a function $g : [n] \to [n]$ such that $\left|\left|x_{A_i} + x_n - x_{A_{g(i)}}\right|\right| < ||x_n|| \; \forall i \in [n]$ and $g(i) \neq i$. It's clear then that we must be able to find $k > 1$ and $r \in [n]$ with $g^k(r) = r$. But then $\left|\left|x_{A_{g^i(r)}} + x_n - x_{A_{g^{i+1}(r)}}\right|\right| < ||x_n|| \; \forall \, 0 \leq i < k$. Add all these up and divide the LHS by k to get a contradiction via the triangle inequality. Then, write

$$\mathscr{D}_j' = \{A_1, \cdots, A_m, A_m \cup \{n\}\}$$
$$\mathscr{D}_j'' = \{A_1 \cup \{n\}, \cdots, A_{m-1} \cup \{n\}\}$$

[3] I don't think this is examinable
[4] Different in lectures - this avoids support functionals

That $\mathscr{D}_j'$ and $\mathscr{D}_j''$ form a symmetric partition is clear. To see scattered, note that it's trivially true for $\mathscr{D}_j''$ and most of $\mathscr{D}_j'$ - we only need to check the case $\left|\left|x_{A_m \cup \{n\}} - x_{A_j}\right|\right| \geq 1$. But this is true by our choice of A_m above - indeed $\left|\left|x_{A_m \cup \{n\}} - x_{A_j}\right|\right| = \left|\left|x_{A_m} + x_n - x_{A_j}\right|\right| \geq ||x_n|| \geq 1$ □

Theorem 8 (Kleitman[5], 1970)**.** If $\mathscr{A} \subset \mathbb{P}(n)$ s.t. $\forall A, B \in \mathscr{A}$, $||x_A - x_B|| < 1$, then $|\mathscr{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Proof. Take a symmetric partition $\mathbb{P}(n) = \dot{\bigcup}_{j=1}^{s} \mathscr{D}_j,\ s = \binom{n}{\lfloor n/2 \rfloor}$ into scattered sets. Then $|\mathscr{A} \cup \mathscr{D}_j| \leq 1$ because of the scattered condition, so $|\mathscr{A}| \leq s = \binom{n}{\lfloor n/2 \rfloor}$. □

[5] A conjecture of Erdoős in 1945

3 The Kruskal-Katona Theorem

Definition (Lower Shadow)**.** If $\mathscr{A} \subset X^{(r)}$, then the **lower shadow of** $\mathscr{A}$, is given by $\partial\mathscr{A} = \{B \in X^{(r-1)} : B \subset A \text{ for some } A \subset \mathscr{A}\}$

Fact. $|\partial\mathscr{A}| \geq |\mathscr{A}|\binom{n}{r-1}/\binom{n}{r} = |\mathscr{A}|\frac{r}{n-r+1}$ with equality iff $\mathscr{A}$ is ϕ or $X^{(r)}$.

Aim. We ask an in-between question - given m and r, we'd like to find $\mathscr{B} \subset X^{(r)}$ with $|\mathscr{B}| = m$ s.t. $|\partial\mathscr{B}| \leq |\partial\mathscr{A}|$ for any given $\mathscr{A} \subset X^{(r)}$ with $|\mathscr{A}| = m$.

Definition (Lexicographical Order)**.** In the lexicographical order, $A < B$ if $\min(A\Delta B) \in A$. It can be summarised as "prioritise using small numbers".

Definition (Colex Order)**.** In the colex order, $A < B$ if $\max(A\Delta B) \in B$, or alternatively if $\sum_{i\in A} 2^i < \sum_{j\in B} 2^j$. It can be summarised as "prioritise using small numbers".

Example (Lex and Colex orders)**.** For $n = 6, r = 3$ we get the total orders:
Lex: $123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, \cdots, 456$
Colex: $123, 124, 134, 234, 125, 135, 235, \cdots, 456$

Lecture 5

Notation. We define $\mathscr{A} + D = \{A \cup D \ : \ A \in \mathscr{A}\}$ and:

$$\begin{aligned}
\mathscr{B}^{(r)}(m_r, \cdots, m_s) = [m_r]^{(r)} &\cup \left([m_{r-1}]^{(r-1)} + \{m_r + 1\}\right) \\
&\cup \left([m_{r-2}]^{(r-2)} + \{m_{r-1} + 1, m_r + 1\}\right) \\
&\cup \cdots \\
&\cup \left([m_s]^{(s)} + \{m_{s+1} + 1, m_{s+2} + 1, \cdots, m_r + 1\}\right)
\end{aligned}$$

Furthermore, $b^{(r)}(m_r, \cdots, m_s) = |\mathscr{B}^{(r)}(m_r, \cdots, m_s)| = \sum_{j=s}^{r} \binom{m_j}{j}$

$$\text{Note that } \partial\mathscr{B}^{(r)}(m_r, \cdots, m_s) = \mathscr{B}^{(r-1)}(m_r, \cdots, m_s),$$
$$\text{So } b^{(r-1)}(m_r, \cdots, m_s) = \sum_{j=s}^{r} \binom{m_j}{j-1}$$

Lemma 1. For $\ell, r \in \mathbb{N}$, there exists unique $m_r > \cdots > m_s$ s.t. $\ell = \sum_{j=s}^{r} \binom{n_j}{j}$. Also, the first ℓ sets in $X^{(r)}$ under the colex ordering is $\mathscr{B}^{(r)}(m_r, \cdots, m_s)$.

Proof. The sets in $X^{(r)}$ not larger than $A = \{a_1, \cdots, a_r\}$ with $a_1 < \cdots < a_r$ in colex is $\mathscr{B}^{(r)}(a_r - 1, a_{r-1} - 1, \cdots, a_{s+1} - 1, a_s)$, where $s = \min\{i \ : \ a_i < a_{i+1} - 1\}$ - i.e. the first index from the smallest upwards that doesn't have the numbers in consecutive order[6].
That this works is a simple check. $\square$

Definition. An i, j-compression, for $i, j \in X$, $A \in \mathbb{P}(X)$ is:

tests

[6] e.g. in $\{3, 4, 7, 9\}$, this would be between 4 and 7, so we get $\mathscr{B}^{(4)}(8, 6, 4)$, i.e. $s = 2$

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