Part III — Local Fields

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Michaelmas 2016

A brief summary of important ideas and results in the course

0 Introduction

Lecture 1

If we look at $f(x_1, \cdots, x_n) \in \mathbb{Z}[x_1, \cdots, x_n]$, what are the ways we can look for solutions $\mathbf{a} \in \mathbb{Z}^n$?

One way would be to look over $\mathbb{R}$, but the point of this course is to package all of the information modulo $p^n \forall n \geq 0$ together.

1 Basic Theory

1.1 Some Generalities

Definition 1 (Absolute value)**.** Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ s.t.

(i) $|x| = 0 \iff x = 0$

(ii) $|xy| = |x| \cdot |y| \quad \forall x, y \in K$

(iii) $|x+y| \leq |x| + |y| \quad \forall x, y \in K$

Example. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ with $|z| = \sqrt{z\bar{z}}$

Note that $||x| - |y|| \leq |x-y|$ $\forall x, y$. Also, an absolute value defines a metric $d(x,y) = |x-y|$ on K

Definition 2 (Valued Field)**.** A **valued field** is a field with an absolute value.

Definition 3 (Equivalent)**.** If K is a field, then two absolute values $|\cdot|, |\cdot|'$ are **equivalent** if they induce the same topology.

Exercise 4. Using notation as in Definition 3, prove that TFAE

(i) $|\cdot|$ and $|\cdot|'$ are equivalent

(ii) $\forall x \in \quad |x| < 1 \Rightarrow |x|' < 1$

(iii) $\exists s \in \mathbb{R}_{>0}$ s.t. $|x|^s = |x|' \quad x \in K$

Exercise 5. Let K be a valued field. Then the completion $\hat{K}$ of K is independent of $|\cdot|$ up to equivalence, and it is a valued field with an absolute value extending $|\cdot|$.

Definition 6 (Archimedean)**.** An absolute value $|\cdot|$ on a field K is called **non-Archimedean** if it satisfies the strong triangle inequality, i.e.

$$|x+y| \leq \max(|x|, |y|)$$

Otherwise, the absolute value is **Archimedean**.

Unless otherwise mentioned, **all absolute values will be non-Archimedean**. Also, all **absolute values are assumed to be non-trivial**.

Definition. If K is a valued field, then the **valuation ring** of K is $\mathcal{O} = \{x \ : \ |x| \leq 1\}$.

Proposition 7. (i) $\mathcal{O}$ is an open subring of K

(ii) $\forall r \in (0,1]$, $\{x \ : \ |x| < r\}$ and $\{x \ : \ |x| \leq r\}$ are open ideals of $\mathcal{O}$

(iii) $\mathcal{O}^{\times} = \{x \ : \ |x| = 1\}$

Proof. Fairly trivial - obvious proof for each section works. □

Proposition 8. Let K be a valued field. For parts (ii) and (iii), assume that K is complete.

(i) Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$, then (x_n) is Cauchy.

(ii) Let (x_n) be a sequence in K. If $x_n - x_{n+1} \to 0$, then (x_n) converges.

(iii) Let $\sum_{n=0}^{\infty} y_n$ be a series in K. If $y_n \to 0$, then $\sum_{n=0}^{\infty} y_n$ converges.

Proof. The first follows from the Archimedean assumption - use epsilons and that:

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - \cdots - x_n| \leq \max(|x_m - x_{m-1}|, \cdots, |x_{n+1} - x_n|)$$

The other two follow easily from the first. □

Definition 9 (Integral Over a Ring)**.** Let $R \subseteq S$ be rings, then $s \in S$ is **integral over** R if there exists a monic $f(x) \in R[x]$ s.t. $f(s) = 0$.

Proposition 10. Let $R \subseteq S$ be rings

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