

Part III — Combinatorics

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A brief summary of important ideas and results in the course

1 Introduction

1.1 Notation and Terminology

Lecture 1

Notation. X, Y sets, then $\mathbb{P}(X)$ is the power set.

$[n] = \{1, \dots, n\}, [m, n] = \{m, \dots, n\}$

Much of the time, $|X| = n$ and $X = [n]$ in this course

Level sets: $X^{(r)} = \{A \in \mathbb{P}(X) : |A| = r\}, X^{(<r)} = \{A \in \mathbb{P}(X) : |A| < r\}$

$X^{(\omega)} = \{A \in \mathbb{P}(X) : A \text{ finite}\}$

If G is a graph, $E(G)$ is its edge set, $e(G)$ the number of edges, $V(G)$ the vertex set and $|G|$ the number of vertices. G may also be used to mean its vertex set

Definition (Set System). $\mathcal{A} \subset \mathbb{P}(X)$ is a **set system** or **family of sets** and can be identified with a bipartite graph $G_{\mathcal{A}}(U, W)$ with $U = \mathcal{A}, W = \bigcup_{A \in \mathcal{A}} A$ or $W = X$.

Note that here the edge $Ax \in E(G_{\mathcal{A}}) \Leftrightarrow x \in A$. Basically, we're forming a bipartite graph where each vertex in one set is identified with the sets in \mathcal{A} , then each vertex in the other are just all the elements that appear in X .

Definition (Set of Distinct Representatives). Given $\mathcal{A} \subset \mathbb{P}(X)$, a **set of distinct representatives** (an **SDR**) is an injection $f : \mathcal{A} \rightarrow X$ s.t. $f(A) \in A \forall A \in \mathcal{A}$

If we look at the bipartite graph of an SDR as above, then we see that it corresponds to a complete matching $U \rightarrow W$ (i.e. \mathcal{A} into all elements of X that \mathcal{A} hits)

Theorem 1 (Hall, 1935). A set system \mathcal{A} has an SDR if $\forall \mathcal{A}' \subset \mathcal{A}$:

$$\left| \bigcup_{A \in \mathcal{A}'} A \right| \geq |\mathcal{A}'|$$

If we look at a bipartite graph in the context of set systems, we see that every bipartite graph corresponds to a multiset of sets, giving us a reformulation of Hall:

Theorem 1 (Alternate Version). A bipartite graph $G(U, W)$ has a complete matching if, $\forall S \subset U$,

$$|\Gamma(S)| \geq |S| \tag{1}$$

Where $\Gamma(S)$ is the set of neighbours of S .

Proof 1. The necessary direction is trivial, so we look only at sufficiency.

Assume G is edge minimal w.r.t. (1). Here we look for a contradiction - suppose that G is not a matching, so we can find two edges out of one vertex in U - say uw_1 and uw_2 . Use minimality to get two sets S_1, S_2 in U with the same number neighbours as vertices and u is the only neighbour of w_i in S_i .

Then, use $|S_1 \cap S_2| \leq |\Gamma(S_1 \cap S_2)|$ and manipulate the RHS using conditions on the S_i and their intersection and minimality to get $|\Gamma(S_1 \cap S_2)| \leq |S_1 \cap S_2|$, so those two expressions are equal.

Then, delete u from $S_1 \cap S_2$ and look at that set's size (and the size of its neighbours) to get a contradiction. \square

Proof 2. The necessary direction is trivial, so we look only at sufficiency.

Assume G is edge minimal w.r.t. (1). Here we induct. Look at $\xi = \{E \subset U : |\Gamma(E)| = |E| > 0\} \neq \emptyset$.

Split into the cases about whether or $\exists E \in \xi$ where $E \neq U$. If it exists, then form the sub-graphs from $H = E \cup \Gamma(E)$ and $G \setminus H$ - both satisfy (1) so done by induction. Otherwise, pick an edge uw - then $G \setminus \{u, w\}$ satisfies (1) so add back u and w and we're done. \square

Corollary 2. $G(U, W)$ bipartite, $d(u) \geq d(w) \forall u \in U, w \in W$. Then \exists a complete matching $U \rightarrow W$.

Proof. Pick $d(u) \geq d \geq d(w)$, look at a general $S \subset U$ to get $d|S| \leq e(S, \Gamma(S)) \leq d|\Gamma(S)|$ \square

Lecture 2

Definition $((r, s)$ -regular). A bipartite graph $G(U, W)$ is (r, s) -**regular** if $d(u) = r$ and $d(w) = s \quad \forall u \in U, \forall w \in W$

Note that $r|U| = e(G) = s|W|$ and that Corollary 2 implies that $G(U, W)$ being (r, s) -regular gives us a complete matching from U to W if $|U| \leq |W|$

Corollary 3. Let $0 \leq i, j \leq n$, with $\binom{n}{i} \leq \binom{n}{j}$. Then \exists a complete matching $f : [n]^{(i)} \rightarrow [n]^{(j)}$ s.t. $f(A) \subset A$ if $j \leq i$ and $f(A) \supset A$ if $i \leq j$.

Theorem 4. Let $G = G(U, W)$ be a connected (r, s) -regular graph. Then, for $\phi \neq A \subset U$,

$$\frac{|\Gamma(A)|}{|W|} \geq \frac{|A|}{|U|}$$

With equality iff $A = U$

Proof. Use $r|A| = e(A, \Gamma(A)) \leq s|\Gamma(A)|$ and divide by $|W|$ to get the result (noting $r|U| = s|W|$). If equality holds, then all edges out of $\Gamma(A)$ go into A , so $A = U$ or the else graph is disconnected. \square

Definition (Partially Ordered Set). A **partially ordered set**, or **poset** is a set with a binary relation \leq that satisfies reflexivity ($a \leq a$), antisymmetry ($a \leq b, b \leq a \Rightarrow a = b$) and transitivity ($a \leq b, b \leq c \Rightarrow a \leq c$)

Definition (The Cube). We define **the cube**: $Q^n \cong \mathbb{P}(n) \cong [2]^n \cong$ the set of all 0-1 sequences.

Remarks.

- Q^n can be considered as a graph - AB is an edge iff $|A \Delta B| = 1$ (look at an n dimensional cube with vertices containing all coordinates with 0 or 1 and put $j \in A$ if $x_j = 1$)
- It is also a poset via $A < B$ if $A \subset B$
- Q^n has a natural orientation: \overrightarrow{AB} if $A = B \cup \{a\}$
- For the order induced on Q^n , see Dilworth's Theorem

2 Sperner Systems

Definition (Sperner). A set system $\mathcal{A} \subset \mathbb{P}(n)$ is **Sperner** if $A, B \in \mathcal{A}$, $A \neq B \Rightarrow A \not\subset B$. That is no two non-equal sets in \mathcal{A} contain each other.

The simplest example of Sperner sets are the level sets, $X^{(r)}$

Definition (Chain). In a poset, a **chain** is a linearly ordered set, $c_1 < c_2 < \dots < c_m$

Definition (Antichain). In a poset, an **antichain** has no two elements that are comparable.

Theorem 1 (Sperner, 1928). If $\mathcal{A} \subset \mathbb{P}(n)$ is Sperner, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Proof. For $0 \leq r \leq n-1$, we can easily find a complete matching between $X^{(r)}$ and $X^{(r+1)}$ from the smaller set to the larger set (where $AB \in E(X^{(r)}, X^{(r+1)})$ iff $A \subset B$ or $B \subset A$). All these matchings put together create a collection of $\binom{n}{\lfloor n/2 \rfloor}$ paths in Q^n (look from the largest layer in the middle outward - i.e. the layer where $r = \lfloor n/2 \rfloor$), each of which is a chain. Hence our poset is covered by $\binom{n}{\lfloor n/2 \rfloor}$, giving $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$ \square

Definition (Weight). The **weight** $w(A)$ of a set $A \in \mathbb{P}(n)$ is $w(A) = \frac{1}{\binom{n}{|A|}}$

Theorem 2 (LYM or YMBL¹). Let \mathcal{A} be a Sperner system on X , with $|X| = n$. Then

$$w(\mathcal{A}) := \sum_{A \in \mathcal{A}} w(A) \leq 1$$

Proof. Look at the maximal chains in $\mathbb{P}(X)$, $A_0 \subset A_1 \subset \dots \subset A_n$, with $A_i \in X^{(i)}$ (so $|A_i| = i$). By just picking elements one at a time, we can see that the number of maximal chains is $n!$. Each such chain has ≤ 1 element of \mathcal{A} , as \mathcal{A} is Sperner. But also, every $A \in \mathcal{A}$ is in $|A|!(n - |A|)!$ chains.

Hence, $\sum_{A \in \mathcal{A}} |A|!(n - |A|)! \leq n!$ which gives the result. \square

Theorem 2 (Alternate Version). If $\mathcal{A} \subset \mathbb{P}(n)$ is an anti-chain, then $w(\mathcal{A}) \leq 1$

Lecture 3

Corollary 3. If $\mathcal{A} \subset \mathbb{P}(n)$ is a Sperner system, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$, with equality iff \mathcal{A} is $X^{(\lfloor n/2 \rfloor)}$ or $X^{(\lceil n/2 \rceil)}$.

Proof. The inequality is immediate.

With equality, $\mathcal{A} \subset X^{(\lfloor n/2 \rfloor)} \cup X^{(\lceil n/2 \rceil)}$ so the claim holds for n even - now we check $n = 2k + 1$. Write $\mathcal{A}_k = \mathcal{A} \cap X^{(k)}$, $\mathcal{A}_{k+1} = \mathcal{A} \cap X^{(k+1)}$, which yields a bipartite graph (if both non-empty, else claim holds).

Then, using Theorem 4 from Section 1, $|\mathcal{A}| = |\mathcal{A}_k| + |\mathcal{A}_{k+1}| < |\mathcal{A}_k| + |\Gamma(\mathcal{A}_{k+1})| \leq \binom{n}{k}$. \square

Definition (k -Sperner). We say that $\mathcal{A} \subset \mathbb{P}(n)$ is k -Sperner if it does not contain $A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_{k+1}$ - i.e. it doesn't contain a chain of length $k + 1$.

Note that a Sperner system is 1-Sperner.

Corollary 4 (Erdős, 1945). If $\mathcal{A} \in \mathbb{P}(n)$ is k -Sperner, then $|\mathcal{A}|$ is at most the sum of the k largest binomial coefficients.

¹Yamamoto, 1954; Meshalkin, 1963; Bollobás, 1965; Lubell, 1966

Theorem (Littlewood and Offord - LO²). $z_i \in \mathbb{C}$, $|z_i| \geq 1$, then the number of $\sum_1^n \pm z_i$ within k of each other is $\leq ck \frac{2^n \log n}{\sqrt{n}}$ for some constant c .

Theorem 5 (Erdős, 1945). Let $x_1, \dots, x_n \in \mathbb{R}$ with $x_i \geq 1$. Then the number of sums $\sum_1^n \pm x_i$ in an open interval J of length $2k$ is at most the sum of the k largest binomial coefficients.

Proof. Define $\mathcal{A} = \{A \in \mathbb{P}(n) : \sum_{i \in A} x_i - \sum_{j \notin A} x_j \in J\}$ - this is k -Sperner, as moving x_i from one sum to the other gives a change in value of at least 2, so a chain of $k+1$ sets would need a change of at least $2k$. Hence, we're done. \square

Conjecture (Erdős, 1945). If $x_1, \dots, x_n \in E$, where E is a normed space and $\|x_i\| \geq 1$, then the number of sums $\sum x_i$ less than distance 2 from each other is $\binom{n}{\lfloor n/2 \rfloor}$.

Definition (Symmetric Chain). A chain $A_0 \subset A_1 \subset \dots \subset A_k$ is **symmetric** if $|A_{i+1}| = |A_i| + 1 \forall i$ and $|A_0| + |A_k| = n$

Theorem 6 (Kleitman and Katona). $\mathbb{P}(n)$ has a decomposition into symmetric chains.

Proof. Induct on n - the base case is $\{\phi, \{1\}\}$.

If $\mathbb{P}(n-1) = \dot{\bigcup}_{i=1}^q \mathcal{C}_i$ with $\mathcal{C}_i = \{A_0 \subset \dots \subset A_k\}$, then form

$$\mathcal{C}'_i = \{A_0, \dots, A_{k-1}, A_k \cup \{n\}\}$$

and

$$\mathcal{C}''_i = \{A_0 \cup \{n\}, \dots, A_{k-1} \cup \{n\}\}$$

Then \mathcal{C}'_i and \mathcal{C}''_i partition $\mathbb{P}(n)$ into symmetric chains. \square

Lecture 4

Remarks. Note that we've formed a partition into $\binom{n}{\lfloor n/2 \rfloor}$ symmetric chains, with $\binom{n}{i} - \binom{n}{i-1}$ chains of length $n+1-2i$ (so 1 chain of length $n+1$, $n-1$ chains of length $n-1$, etc.)

For the next few results, let E be a normed space, $x_1, \dots, x_n \in E$, $\|x_i\| \geq 1 \forall i$ and, for $A \in \mathbb{P}(n)$, $x_A = \sum_{i \in A} x_i$

Definition (Scattered). Call $\mathcal{D} \subset \mathbb{P}(n)$ **scattered** if $\|x_A - x_B\| \geq 1 \forall A, B \in \mathcal{D}$.

Definition (Symmetric Partition). Call a partition $\mathbb{P}(n) = \dot{\bigcup}_1^s \mathcal{D}_j$ **symmetric** if there are precisely $\binom{n}{i} - \binom{n}{i-1}$ sets \mathcal{D}_i of cardinality $n+1-2i$.

Theorem 7 (Kleitman, 1970). E and $(x_i)_1^n$ as above. Then, $\mathbb{P}(n)$ has a symmetric partition into scattered sets.

Proof. ³ Induct on n - the base case is $\{\phi, \{1\}\}$.

Otherwise, let $\mathbb{P}(n-1) = \dot{\bigcup}_{i=1}^q \mathcal{D}_i$ be a scattered partition into scattered sets and let $\mathcal{D}_j = \{A_1, \dots, A_m\}$.

In each \mathcal{D}_j , pick A_m so that $\|x_{A_m} + x_n - x_{A_j}\| \geq \|x_n\| \forall 1 \leq j < m$. This is possible - indeed, if not, then we can find a function $g : [n] \rightarrow [n]$

²I don't think this is examinable

³Different in lectures - this avoids support functionals

such that $\|x_{A_i} + x_n - x_{A_{g(i)}}\| < \|x_n\| \ \forall i \in [n]$ and $g(i) \neq i$. It's clear then that we must be able to find $k > 1$ and $r \in [n]$ with $g^k(r) = r$. But then $\|x_{A_{g^i(r)}} + x_n - x_{A_{g^{i+1}(r)}}\| < \|x_n\| \ \forall 0 \leq i < k$. Add all these up and divide the LHS by k to get a contradiction via the triangle inequality.

Then, write

$$\begin{aligned}\mathcal{D}'_j &= \{A_1, \dots, A_m, A_m \cup \{n\}\} \\ \mathcal{D}''_j &= \{A_1 \cup \{n\}, \dots, A_{m-1} \cup \{n\}\}\end{aligned}$$

That \mathcal{D}'_j and \mathcal{D}''_j form a symmetric partition is clear. To see scattered, note that it's trivially true for \mathcal{D}''_j and most of \mathcal{D}'_j - we only need to check the case $\|x_{A_m \cup \{n\}} - x_{A_j}\| \geq 1$. But this is true by our choice of A_m above - indeed $\|x_{A_m \cup \{n\}} - x_{A_j}\| = \|x_{A_m} + x_n - x_{A_j}\| \geq \|x_n\| \geq 1$ \square

Theorem 8 (Kleitman⁴, 1970). If $\mathcal{A} \subset \mathbb{P}(n)$ s.t. $\forall A, B \in \mathcal{A}, \|x_A - x_B\| < 1$, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Proof. Take a symmetric partition $\mathbb{P}(n) = \dot{\bigcup}_{j=1}^s \mathcal{D}_j$, $s = \binom{n}{\lfloor n/2 \rfloor}$ into scattered sets.

Then $|\mathcal{A} \cap \mathcal{D}_j| \leq 1$ because of the scattered condition, so $|\mathcal{A}| \leq s = \binom{n}{\lfloor n/2 \rfloor}$. \square

⁴A conjecture of Erdős in 1945

3 The Kruskal-Katona Theorem

Definition (Lower Shadow). If $\mathcal{A} \subset X^{(r)}$, then the **lower shadow of \mathcal{A}** , is given by $\partial\mathcal{A} = \{B \in X^{(r-1)} : B \subset A \text{ for some } A \in \mathcal{A}\}$

Fact. $|\partial\mathcal{A}| \geq |\mathcal{A}| \binom{n}{r-1} / \binom{n}{r} = |\mathcal{A}| \frac{r}{n-r+1}$ with equality iff \mathcal{A} is ϕ or $X^{(r)}$.

We also ask the in-between question - what is $\mathcal{B} \subset X^{(r)}$ s.t. $|\mathcal{B}| = |\mathcal{A}|$ and $|\partial\mathcal{B}| \leq |\partial\mathcal{A}|$?

It's clear that $\partial[k]^{(r)} = [k]^{(r-1)}$, so $\exists \mathcal{B}_1, \mathcal{B}_2, \dots \subset X^{(r)}$ s.t. $|\mathcal{B}_m| = m$ and $|\partial\mathcal{B}_m| \leq |\partial\mathcal{A}| \quad \forall \mathcal{A} \subset X^{(r)}, |\mathcal{A}| = m$

Definition (Lexicographical Order). In the lexicographical order, $A < B$ if $\min(A \Delta B) \in A$. It can be summarised as “prioritise using small numbers”.

Definition (Colex Order). In the colex order, $A < B$ if $\max(A \Delta B) \in B$, or alternatively if $\sum_{i \in A} 2^i < \sum_{j \in B} 2^j$. It can be summarised as “prioritise using small numbers”.

Example (Lex and Colex orders). For $n = 6, r = 3$ we get the total orders:

Lex: 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, \dots , 456

Colex: 123, 124, 134, 234, 125, 135, 235, \dots , 456

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