# Part III — Additive Combinatorics

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*A brief summary of important ideas and results in the course*

# 0 Introduction

Lecture 1

**Notation.** $[N] = 1, 2, \cdots, N$

**Theorem** (Szemerdi's theorem)**.** $\forall \delta > 0$, $\forall k$, $\exists N$ $\forall A \subset [N]$ s.t. $|A| \geq \delta N$ and $A$ contains an arithmetic progression of length $k$.

This theorem looks like it might be nicely solvable or have a counter-example, but we need hard machinery to attack it. Developing such tools will be the goal of the course.

# 1 Discrete Fourier Analysis and Roth's Theorem

**Notation.** Let $N \in \mathbb{N}$ and $\omega = e^{2\pi\imath/N}$.
Write $\mathbb{Z}_N$ for the cyclic group of integers mod $N$.
Write $\mathbb{E}_x f(x) = N^{-1} \sum_{x \in \mathbb{Z}_N} f(x)$ (same notation also used for other averages).

**Definition** (Discrete Fourier Transform)**.** If $f : \mathbb{Z}_N \to \mathbb{C}$, we define this as

$$\hat{f}(r) = \mathbb{E}_x f(x) \omega^{-rx}$$

Roughly speaking, expectations are taken in "physical space" and sums in "frequency space". This gives rise to two inner products:

$$< f, g > = \mathbb{E}_x f(x) \overline{g(x)}$$
$$< \hat{f}, \hat{g} > = \sum_r \hat{f}(r) \overline{\hat{g}(x)}$$

**Fact** (Parserval's Identity)**.** $< f, g > = < \hat{f}, \hat{g} >$

**Definition** (Convolution)**.** The convolution of $f$ and $g$ is:

$$f * g(x) = \mathbb{E}_{y+z=x} f(y) g(z)$$
$$\hat{f} * \hat{g}(x) = \sum_{s+t=r} \hat{f}(s) \hat{g}(t)$$

**Fact** (Convolution Law)**.** For $f, g : \mathbb{Z}_N \to \mathbb{C}$, $f \hat{*} g(r) = \hat{f}(r) \hat{g}(r)$

**Fact** (Inversion Formula)**.** For $f : \mathbb{Z}_N \to \mathbb{C}$, we get $f(x) = \sum_r \hat{f}(r) \omega^{rx}$

**Definition** (Dilation Notation)**.** If $a$ is invertible mod $N$, then we write $f_a(x) = f(a^{-1}x)$

**Fact** (Dilation Rule)**.** $\hat{f_a}(r) = \hat{f}(ar)$

**Definition** (Indicators)**.** If $A \subset \mathbb{Z}_N$, write

$$A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Note that if $|A| = \alpha N$, then $\hat{A}(0) = \mathbb{E}_x A(x) = \alpha$

**Definition** (Norms)**.** We define the norms $||f||_p = (\mathbb{E}_x |f(x)|^p)^{1/p}$ and $||\hat{f}||_p = \left( \sum_r |\hat{f}(r)|^p \right)^{1/p}$

So, if $A \subset \mathbb{Z}_N$, then $||A||_2^2 = < A, A > = \alpha$
Also, by Parserval, we get $\sum_r \left| \hat{A}(r) \right|^2 = \alpha$

Lecture 2

**Theorem** (Roth)**.** For every $\delta > 0$, $\exists N$ s.t. every $A \subset [N]$ of size at least $\delta N$ contains an arithmetic progression of at least 3.

*Basic Ideas.* We use a density increment argument. If $A$ has density $\alpha$ and has no 3 term A.P., then we can find a reasonably long A.P. $P$ s.t. $|A \cap P|/|P| >> \alpha$. Then, we're done or pass to $P$ and start again with a better density. Iterate, and as $\alpha \leq 1$, we get an A.P..
Consider $A \subset \mathbb{Z}_N$ and define $B = C = A \cap [N/3, 2N/3]$. Observe that $(x, y, z)$ an A.P. in $A \times B \times C$ in $\mathbb{Z}_N$, then it also is in $[N]$.
Let $N$ be odd and $\alpha$ the density of $A$.
If $|B| < \frac{\alpha N}{5}$, then either $|A \cap [1, N/3)|$ or $|A \cap (2N/3, N]|$ is at least $\frac{2\alpha N}{5}$, so we pass to one of these and increment $A$'s density.
Hence assume $|B| = |C| \geq \frac{\alpha N}{5}$.
We define the *3AP Density* of $(A, B, C)$, the probability that a random $(x, y, z)$ with $x + z = 2y$ lies in $A \times B \times C$ as

$$\mathbb{E}_{x+z=2y} A(x)B(y)C(z)$$

Then we get, after some manipulation and calling $\beta = \gamma = ||\hat{B}||_2 = ||\hat{C}||_2$ the density of $B$ (and $C$):

$$\mathbb{E}_{x+z=2y} A(x)B(y)C(z) = \alpha\beta\gamma + \sum_{r \neq 0} \hat{A}(r)\hat{B}(-2r)\hat{C}(r)$$

But

$$\left| \sum_{r \neq 0} \hat{A}(r)\hat{B}(-2r)\hat{C}(r) \right| \leq \max_{r \neq 0} \left| \hat{A}(r) \right| ||\hat{B}||_2 ||\hat{C}||_2$$

So, given $\max_{r \neq 0} \left| \hat{A}(r) \right| \leq \frac{\alpha^2}{10}$ and $\frac{\alpha^3}{50} > \frac{1}{N}$, we get 3APs, so assume we have an $r$ such that $\left| \hat{A}(r) \right| \geq \frac{\alpha^2}{10}$

**Lemma.** Let $\epsilon > 0, r \in \mathbb{Z}_N$. Then $[N]$ can be partitioned into APs of length at least $\frac{\epsilon\sqrt{N}}{8\pi}$ on which the function $x \mapsto \omega^{rx}$ varies by at most $\epsilon$.

*Proof.* Let $m = \left\lfloor \sqrt{N} \right\rfloor$ and look at $1, \omega^r, \cdots, \omega^{mr}$, two of which are sufficiently close, terms $u$ and $v$. Write $t = v - u$ and use this to get $\left| \omega^{btr} - \omega^{atr} \right| \leq (b - a)\frac{2\pi}{m}$, then partition $[N]$ into congruence classes mod $t$ and partition each congruence class. □

Lecture 3

**Definition** (Balanced Function)**. Balanced Function** $f$ of $A$ is $f(x) = A(x) - \alpha$

Then $\mathbb{E}_x f(x) = 0$ and for $r \neq 0$, $\hat{f}(r) = \hat{A}(r)$. From before, we can pick $r$ s.t. $|\hat{f}(r)| \geq \alpha^2/10$. If we let $P_i$ be partitions given by the lemma, then we get:

$$\frac{\alpha^2}{10} \leq N^{-1} \left| \sum_x f(x)\omega^{-rx} \right| \leq \frac{\alpha^2}{20} + N^{-1} \sum_i \left| \sum_{x \in P_i} f(x) \right|$$

So, $N^{-1} \sum_i \left| \sum_{x \in P_i} f(x) \right| \geq \frac{\alpha^2}{20} N$.
Also, $\sum_i \sum x \in P_i f(x) = 0$. Using these results, we can conclude that there is an $i$ such that

$$\sum_{x \in P_i} f(x) \geq \frac{\alpha^2}{40} |P_i|$$

That is, $|A \cap P_i| \geq \left(\alpha + \alpha^2/40\right) |P_i|$ □

**Theorem** (Behrend, 1947)**.** For every $N, \exists$ a subset $A \subset [N]$ of size $N/\exp\left(c\sqrt{\log N}\right)$

*Proof.* Let $m$ and $d$ be positive integers, write in this proof $[N] = \{0, 1, \cdots, N-1\}$ and consider the grid $[m]^d$.
Note that in $\mathbb{R}^d$, no sphere contains points $x, y, z$ with $x + z = 2y$ (i.e. a 3AP). But $\sum x_i^2$ can take at most $m^2 d$ values in $[m]^d$, so there's a sphere that goes through at least $\frac{m^d}{m^2 d}$ points.
Then, write a map $\phi$ that sends points in $[m]^d$ to a base $2m$ number in the natural way, which gives $\phi(x) + \phi(z) = 2\phi(y)$ iff $x + z = 2y$. Hence, we've constructed a set of at least $m^{d-2}/d$ numbers, all less than $(2m)^d$ with no 3AP. So, we let $N = (2m)^d$ and, as $d$ is arbitrary, we choose it to get the most points - $d = \sqrt{\log N}$ is a pretty good choice, which gives us the bound. $\square$

# 2 Bohr Sets and Bogolyubov's Method

**Definition** (Bohr Set)**.** Let $K \subset \mathbb{Z}_N$ and $\epsilon > 0$. Then, the **Bohr set** is defined as:

$$B(K, \epsilon) = \{x \in \mathbb{Z}_N : |1 - \omega^{rx}| \leq \epsilon \, \forall r \in K\}$$

# Index