Magnet Precalculus CD
Matrices

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Chapter 1

Introduction to Matrices

Definition 1.0.1: Matrix

A **matrix** is a rectangular array of variables or constants in rows or columns, usually enclosed in brackets. These constants or variables are known as **elements**.

Definition 1.0.2: Element

An element is an individual value within a matrix. Given a matrix A, a given element in side of A is notated as A_{xy}, where x is the row and y is the column in which the element is located.

Note:-

If either the width or height of a matrix is more than one digit, x and y in element notation are generally separated by a dash (e.g. A_{10-4})

Example 1.0.1 (Find an Element of a Matrix)

$$A = \begin{bmatrix} -8 & 40 & 0 & -1 & 21 \\ 27 & 32 & -29 & 6 & -2 \\ 5 & -7 & 14 & 52 & -35 \end{bmatrix}$$

$A_{12} = 40$

$A_{34} = 52$

A matrix with m rows and n columns is known as an "m by n" matrix, written as mxn. These are its **dimensions**.

Example 1.0.2 (Dimensions of a matrix)

Let matrix $A = \begin{bmatrix} 1 & -8 \\ -4 & 13 \\ -6 & -2 \\ 28 & 0 \end{bmatrix}$. A has 4 rows and 2 columns, so its dimensions are 4x2.

1.1 Summation of Matrices

Matrices can be summed **only if their dimensions are the same**. The process is as simple as summing all corresponding elements.

Example 1.1.1 (Sum of Two Matrices)

$$W = \begin{bmatrix} -1 & 9 \\ -11 & 15 \\ 8 & -20 \end{bmatrix}$$

$$Z = \begin{bmatrix} -3 & -2 \\ -16 & 0 \\ 12 & 9 \end{bmatrix}$$

$$W + Z = \begin{bmatrix} -4 & 7 \\ -27 & 15 \\ 20 & -11 \end{bmatrix}$$

$$W - Z = \begin{bmatrix} 2 & 11 \\ 5 & 15 \\ -4 & -29 \end{bmatrix}$$

1.2 Multiplication of Matrices

Before we get to the method of matrix multiplication, there is a very important condition that must be met.

Consider two matrices, A and B. **They can only be multiplied if A has the same number of columns as B has rows**. In other words, if A had dimensions $m_1 x n_1$ and B had $m_2 x n_2$, they could only be multiplied if $n_1 = m_2$. The dimensions of the product matrix are $m_1 x n_2$

Element $AB_{hk} = A_{h1}B_{1k} + A_{h2}B_{2k} + A_{h3}B_{3k} + ... + A_{hn_2}B_{n_2k}$. So, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + ch \end{bmatrix}$

Example 1.2.1 (Multiplication of Matrices)

$$A = \begin{bmatrix} 9 & -5 \\ -2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 13 & -5 \\ -1 & -7 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (9 \cdot -3) + (-5 \cdot -1) & (9 \cdot 13) + (-5 \cdot -7) & (9 \cdot -5) + (-5 \cdot 2) \\ (-2 \cdot -3) + (4 \cdot -1) & (-2 \cdot 13) + (4 \cdot -7) & (-2 \cdot -5) + (4 \cdot 2) \end{bmatrix} = \begin{bmatrix} -27 + 5 & 117 + 35 & -45 - 10 \\ 6 - 4 & -26 - 28 & -10 + 8 \end{bmatrix} = \begin{bmatrix} -22 & 152 & -55 \\ 2 & -54 & -2 \end{bmatrix}$$

1.3 Determinant of a Matrix

Every square matrix has a real number that is its **determinant**. The determinant of matrix A is denoted as $\det(A)$ or $|A|$.

The determinant of a 2x2 matrix is called a **second-order determinant**. To find a second-order determinant, use the following formula: $|A| = A_{11}A_{22} - A_{21}A_{12}$

Example 1.3.1 (Determinant of a 2x2 Matrix)

$$A = \begin{bmatrix} -4 & 3 \\ 5 & -10 \end{bmatrix}$$

$$|A| = (-4 \cdot -10) - (5 \cdot 3) = 40 - 15 = 25$$

The determinant of a 3x3 matrix is called a **third-order** determinant. To find a third-order determinant, use the steps below:

1. Rewrite the first two columns to the right of the matrix

2. Find the sum of the products of each downward diagonal

3. Find the sum of the products of each upward diagonal

4. Subtract the upward diagonal sum from the downward diagonal sum

This can be represented mathematically as $|A| = (A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}) - (A_{31}A_{22}A_{13} + A_{32}A_{23}A_{11} + A_{33}A_{21}A_{12})$

Example 1.3.2 (Determinant of a 3x3 Matrix)

$$A = \begin{bmatrix} 3 & -7 & 2 \\ 5 & 4 & -5 \\ 1 & 5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -7 & 2 \\ 5 & 4 & -5 \\ 1 & 5 & -1 \end{bmatrix} \begin{matrix} 3 & -7 \\ 5 & 4 \\ 1 & 5 \end{matrix}$$

Downward diagonal one: $[3, 4, -1]$
Downward diagonal two: $[-7, -5, 1]$
Downward diagonal three: $[2, 5, 5]$

Upward diagonal one: $[1, 4, 2]$
Upward diagonal two: $[5, -5, 3]$
Upward diagonal three: $[-1, 5, -7]$

$(3 \cdot 4 \cdot -1) + (-7 \cdot 5 \cdot 1) + (2 \cdot 5 \cdot 5) = -12 - 35 + 50 = 3$
$(1 \cdot 4 \cdot 2) + (5 \cdot -5 \cdot 3) + (-1 \cdot 5 \cdot -7) = 8 - 75 + 35 = -32$
$3 + 32 = \boxed{35}$

1.4 Identity Matrices

The **identity matrix**, denoted I, is a square matrix that, when multiplied by another matrix, equals that same matrix. An identity matrix contains 1s along the main diagonal and 0s for the remaining elements.

2x2 identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3x3 identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.5 Inverse Matrices

Two nxn matrices are inverses of each other if and only if the product in both directions equals I. If matrix A has an inverse, B, then $AB = I$ and $BA = I$.

Question 1: Determine whether the pair of matrices are inverses

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (-1 \cdot 5) + (2 \cdot 3) & (-1 \cdot 2) + (2 \cdot 1) \\ (3 \cdot 5) + (-5 \cdot 3) & (3 \cdot 2) + (-5 \cdot 1) \end{bmatrix} = \begin{bmatrix} -5 + 6 & -2 + 2 \\ 15 - 15 & 6 - 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} (5\cdot -1)+(2\cdot 3) & (5\cdot 2)+(2\cdot -5) \\ (3\cdot -1)+(1\cdot 3) & (3\cdot 2)+(1\cdot -5) \end{bmatrix} = \begin{bmatrix} -5+6 & 10-10 \\ -3+3 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

A and B are inverses.
Not all matrices have an inverse. A matrix has no inverse if its determinant is 0.

1.6 Finding the Inverse of a 2x2 Matrix

There is a formula to find the inverse of a 2x2 matrix. If matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $|A| \neq 0$.

Example 1.6.1 (Finding the Inverse of a 2x2 Matrix)

$A = \begin{bmatrix} 4 & -1 \\ -6 & 3 \end{bmatrix}$

$\det(A) = (4\cdot 3) - (-6\cdot -1) = 12 - 6 = 6 \therefore A$ has an inverse.

$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 1 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 1 & \frac{2}{3} \end{bmatrix}$

1.7 Finding the Inverse of a 3x3 Matrix

Consider some 3x3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. The following steps can be taken to find its inverse.

1. Find the determinant of the matrix
2. Transpose the original matrix. All rows become columns
3. Place the determinants of each 2x2 minor matrix in a matrix B. The matrix at B_{nm} is A with row n and column m removed. For instance, $B_{11} = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$.

   In this case, $B = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & \begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix} = \begin{bmatrix} ei-hf & di-gf & dh-ge \\ bi-hc & ai-gc & ah-gb \\ bf-ec & af-dc & ae-db \end{bmatrix}$

4. Create the adjugate matrix, denoted $\mathrm{Adj}(B) = B \cdot \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

   In this case, $\mathrm{Adj}(B) = \begin{bmatrix} ei-hf & gf-di & dh-ge \\ hc-bi & ai-gc & gb-ah \\ bf-ec & dc-af & ae-db \end{bmatrix}$

5. Divide each term of the adjugate matrix by the determinant

1.8 Cramer's Rule

Cramer's Rule, named after the Swiss mathematician Gabriel Cramer, uses the coefficient matrix and determinants to solve a system of linear equations. It is as follows:

Given $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (the coefficient matrix). If $|A| = 0$, then the system has a unique solution given by $x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{|A|}$ and $y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{|A|}$.

Cramer's rule also applies to 3x3 matrices.

Given $\begin{cases} ax + by + cz = j \\ dx + ey + fz = k \\ gx + hy + iz = l \end{cases}$, let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ (the coefficient matrix). If $|A| = 0$, then the system has a unique solution given by $x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{|A|}$, $x = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{|A|}$, and $x = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{|A|}$.

Chapter 2

Applications of Matrices

2.1 Area of a Triangle

Given a triangle with vertices (x_1, y_2), (x_2, y_2), (x_3, y_3), the area of the triangle is $\frac{1}{2}|\det(x)|$ where $x = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

Example 2.1.1 (Finding Triangle Area Using a Matrix)

Triangle PQR has vertices $P(-5, -2)$. $Q(3, 9)$, and $R(7, -4)$.

$x = \begin{bmatrix} -5 & -2 & 1 \\ 3 & 9 & 1 \\ 7 & -4 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc} -5 & -2 & 1 \\ 3 & 9 & 1 \\ 7 & -4 & 1 \end{array}\right] \begin{array}{cc} -5 & -2 \\ 3 & 9 \\ 7 & -4 \end{array}$

Downward diagonal one: $[-5, 9, 1]$
Downward diagonal two: $[-2, 1, 7]$
Downward diagonal three: $[1, 3, -4]$

Upward diagonal one: $[7, 9, 1]$
Upward diagonal two: $[-4, 1, -5]$
Upward diagonal three: $[1, 3, -2]$

$(-5 \cdot 9 \cdot 1) + (-2 \cdot 1 \cdot 7) + (1 \cdot 3 \cdot -4) = -45 - 14 - 12 = -71$
$(7 \cdot 9 \cdot 1) + (-4 \cdot 1 \cdot -5) + (1 \cdot 3 \cdot -2) = 63 + 20 - 6 = 77$
$\det(x) = -71 - 77 = -148$
$|-148| = 148$
$\frac{148}{2} = \boxed{74}$

2.2 Matrix Equation

A system of linear equations can be written as a matrix equation:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$
$$\therefore$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 2.2.1 (Using an Inverse Matrix to Solve a System of Equations)

$$\begin{cases} 6x + 8y = -16 \\ 5x + 3y = -28 \end{cases}$$

$$A = \begin{bmatrix} 6 & 8 \\ 5 & 3 \end{bmatrix}^{-1}$$

$$|A| = (6 \cdot 3) - (5 \cdot 8) = 18 - 40 = -22$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 3 & -8 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{-22} & \frac{8}{22} \\ \frac{5}{22} & \frac{6}{-22} \end{bmatrix} = \begin{bmatrix} \frac{3}{-22} & \frac{4}{11} \\ \frac{5}{22} & \frac{3}{-11} \end{bmatrix}$$

$$B = \begin{bmatrix} -16 \\ -28 \end{bmatrix}$$

$$A^{-1} \cdot B = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3 \cdot -16}{-22} + \frac{4 \cdot -28}{11} \\ \frac{5 \cdot -16}{22} + \frac{3 \cdot -28}{-11} \end{bmatrix} = \begin{bmatrix} \frac{24}{11} + \frac{-112}{11} \\ \frac{-40}{11} + \frac{84}{11} \end{bmatrix} = \begin{bmatrix} \frac{-88}{11} \\ \frac{44}{11} \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$x = -8$
$y = 4$

Example 2.2.2 (3rd-Degree System of Equations Solved With Matrices)

$$\begin{cases} -x + 5y - 3z = -40 \\ 7x - 2y - z = 27 \\ -2x - y + 6z = -5 \end{cases}$$

$$\begin{bmatrix} -1 & 5 & -3 \\ 7 & -2 & -1 \\ -2 & -1 & 6 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -40 \\ 27 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{vmatrix} -1 & 5 & -3 \\ 7 & -2 & -1 \\ -2 & -1 & 6 \end{vmatrix} = (12 + 10 + 21) - (-12 - 1 + 210) = 43 - 197 = -154$$

$$\begin{bmatrix} -1 & 5 & -3 \\ 7 & -2 & -1 \\ -2 & -1 & 6 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} -1 & 7 & -2 \\ 5 & -2 & -1 \\ -3 & -1 & 6 \end{bmatrix}$$

2.3 Augmented Matrix

The **augmented matrix** of a system of linear equations uses *all* the coefficients and constants of the system in one matrix. For instance, given a system of equations $\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \\ ix + jy + kz = l \end{cases}$, its augmented matrix is $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$. Generally, a vertical line is drawn between the coefficients and the constants in the matrix.

2.4 Row-Echelon Form

A system is in **row-echelon** form (or triangular form) when the last equation contains just one variable, each equation above contains the variable(s) below and an additional variable, and the leading coefficient of each equation is one.

Example 2.4.1 (System in Row-Echelon Form)

This is an example of a system in row-echelon form:

$$\begin{cases} x + ay + bz = c \\ y + dz = e \\ z = f \end{cases}$$

Row-echelon form is convenient because the remaining variables can be solved by back-substitution.

2.5 Elementary Row Operations

The following operations, called elementary row operations, produce equivalent systems and can be applied to an augmented matrix to transform it to row-echelon form:

- Interchange any two rows
- Multiply a row by a non-zero constant
- Add one row to another (so $R_1 - R_2 = \begin{bmatrix} A_{11} - A_{21} & A_{12} - A_{22} \end{bmatrix}$)

2.6 Gaussian Elimination

The following process of solving a system of equations is known as Gaussian Elimination, named after the German mathematician Carl Friedrich Gauss:

1. Write the system as an augmented matrix
2. Transform the matrix to row-echelon form using elementary row operations
3. Use the row-echelon form of the augmented matrix to rewrite the system of equations
4. Use back-substitution to solve

Example 2.6.1 (Using Gaussian Elimination to Solve a Lineary System of Equations)

$$\begin{cases} 4x - 4y + 8z = 12 \\ x - 2y + z = 11 \\ -x + y - 3z = -1 \end{cases}$$

$$\begin{bmatrix} 4 & -4 & 8 & 12 \\ 1 & -2 & 1 & 11 \\ -1 & 1 & -3 & 1 \end{bmatrix}$$

$$\frac{1}{4}R_1 \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & -2 & 1 & 11 \\ -1 & 1 & -3 & 1 \end{bmatrix}$$

$$-1R_3 \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & -2 & 1 & 11 \\ 1 & -1 & 3 & -1 \end{bmatrix}$$

$$R_3 - R_1 \rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & -2 & 1 & 11 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R_1 - R_2 \rightarrow \begin{bmatrix} 0 & 1 & 1 & -8 \\ 1 & -2 & 1 & 11 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \rightarrow \begin{bmatrix} 1 & -2 & 1 & 11 \\ 0 & 1 & 1 & -8 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{cases} x - 2y + z = 11 \\ y + z = -8 \\ z = -1 \end{cases}$$

$$\begin{cases} x = -2 \\ y = -7 \\ z = -1 \end{cases}$$

Example 2.6.2 (Using Gaussian Elimination to Solve a Lineary System of Equations)

$$\begin{cases} -3x + 3y + 9z = -30 \\ -x - 2y + 3z = -31 \\ 2x - y - 4z = 19 \end{cases}$$

$$\begin{bmatrix} -3 & 3 & 9 & -30 \\ -1 & -2 & 3 & -31 \\ 2 & -1 & -4 & 19 \end{bmatrix}$$

$$\frac{1}{3}R_1 \rightarrow \begin{bmatrix} -1 & 1 & 3 & -10 \\ -1 & -2 & 3 & -31 \\ 2 & -1 & -4 & 19 \end{bmatrix}$$