

# Magnet Precalculus CD Matrices

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# Chapter 1

## Matrices

### 1.1 Introduction to Matrices

#### Definition 1.1.1: Matrix

A **matrix** is a rectangular array of variables or constants in rows or columns, usually enclosed in brackets. These constants or variables are known as **elements**.

#### Definition 1.1.2: Element

An element is an individual value within a matrix. Given a matrix  $A$ , a given element inside of  $A$  is notated as  $A_{xy}$ , where  $x$  is the row and  $y$  is the column in which the element is located.

#### Note:-

If either the width or height of a matrix is more than one digit,  $x$  and  $y$  in element notation are generally separated by a dash (e.g.  $A_{10-4}$ )

#### Example 1.1.1 (Find an Element of a Matrix)

$$A = \begin{bmatrix} -8 & 40 & 0 & -1 & 21 \\ 27 & 32 & -29 & 6 & -2 \\ 5 & -7 & 14 & 52 & -35 \end{bmatrix}$$

$$A_{12} = 40$$

$$A_{34} = 52$$

A matrix with  $m$  rows and  $n$  columns is known as an " $m$  by  $n$ " matrix, written as  $m \times n$ . These are its **dimensions**.

#### Example 1.1.2 (Dimensions of a matrix)

Let matrix  $A = \begin{bmatrix} 1 & -8 \\ -4 & 13 \\ -6 & -2 \\ 28 & 0 \end{bmatrix}$ .  $A$  has 4 rows and 2 columns, so its dimensions are  $4 \times 2$ .

### 1.2 Summation of Matrices

Matrices can be summed **only if their dimensions are the same**. The process is as simple as summing all corresponding elements.

**Example 1.2.1** (Sum of Two Matrices)

$$W = \begin{bmatrix} -1 & 9 \\ -11 & 15 \\ 8 & -20 \end{bmatrix}$$

$$Z = \begin{bmatrix} -3 & -2 \\ -16 & 0 \\ 12 & 9 \end{bmatrix}$$

$$W + Z = \begin{bmatrix} -4 & 7 \\ -27 & 15 \\ 20 & -11 \end{bmatrix}$$

$$W - Z = \begin{bmatrix} 2 & 11 \\ 5 & 15 \\ -4 & -29 \end{bmatrix}$$

### 1.3 Multiplication of Matrices

Before we get to the method of matrix multiplication, there is a very important condition that must be met.

Consider two matrices,  $A$  and  $B$ . **They can only be multiplied if  $A$  has the same number of columns as  $B$  has rows.** In other words, if  $A$  had dimensions  $m_1 \times n_1$  and  $B$  had  $m_2 \times n_2$ , they could only be multiplied if  $n_1 = m_2$ . The dimensions of the product matrix are  $m_1 \times n_2$ .

Element  $AB_{hk} = A_{h1}B_{1k} + A_{h2}B_{2k} + A_{h3}B_{3k} + \dots + A_{hn_2}B_{n_2k}$ . So, if  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ , then

$$AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + ch \end{bmatrix}$$
**Example 1.3.1** (Multiplication of Matrices)

$$A = \begin{bmatrix} 9 & -5 \\ -2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 13 & -5 \\ -1 & -7 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (9 \cdot -3) + (-5 \cdot -1) & (9 \cdot 13) + (-5 \cdot -7) & (9 \cdot -5) + (-5 \cdot 2) \\ (-2 \cdot -3) + (4 \cdot -1) & (-2 \cdot 13) + (4 \cdot -7) & (-2 \cdot -5) + (4 \cdot 2) \end{bmatrix} = \begin{bmatrix} -27 + 5 & 117 + 35 & -45 - 10 \\ 6 - 4 & -26 - 28 & -10 + 8 \end{bmatrix} = \begin{bmatrix} -22 & 152 & -55 \\ 2 & -54 & -2 \end{bmatrix}$$

### 1.4 Determinant of a Matrix

Every square matrix has a real number that is its **determinant**. The determinant of matrix  $A$  is denoted as  $\det(A)$  or  $|A|$ .

The determinant of a 2x2 matrix is called a **second-order determinant**. To find a second-order determinant, use the following formula:  $|A| = A_{11}A_{22} - A_{21}A_{12}$

**Example 1.4.1** (Determinant of a 2x2 Matrix)

$$A = \begin{bmatrix} -4 & 3 \\ 5 & -10 \end{bmatrix}$$

$$|A| = (-4 \cdot -10) - (5 \cdot 3) = 40 - 15 = 25$$

The determinant of a 3x3 matrix is called a **third-order determinant**. To find a third-order determinant, use the steps below:

1. Rewrite the first two columns to the right of the matrix
2. Find the sum of the products of each downward diagonal
3. Find the sum of the products of each upward diagonal
4. Subtract the upward diagonal sum from the downward diagonal sum

This can be represented mathematically as  $|A| = (A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}) - (A_{31}A_{22}A_{13} + A_{32}A_{23}A_{11} + A_{33}A_{21}A_{12})$

#### Example 1.4.2 (Determinant of a 3x3 Matrix)

$$A = \begin{bmatrix} 3 & -7 & 2 \\ 5 & 4 & -5 \\ 1 & 5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -7 & 2 \\ 5 & 4 & -5 \\ 1 & 5 & -1 \end{bmatrix} \begin{array}{l} 3 \quad -7 \\ 5 \quad 4 \\ 1 \quad 5 \end{array}$$

Downward diagonal one:  $[3, 4, -1]$

Downward diagonal two:  $[-7, -5, 1]$

Downward diagonal three:  $[2, 5, 5]$

Upward diagonal one:  $[1, 4, 2]$

Upward diagonal two:  $[5, -5, 3]$

Upward diagonal three:  $[-1, 5, -7]$

$$(3 \cdot 4 \cdot -1) + (-7 \cdot 5 \cdot 1) + (2 \cdot 5 \cdot 5) = -12 - 35 + 50 = 3$$

$$(1 \cdot 4 \cdot 2) + (5 \cdot -5 \cdot 3) + (-1 \cdot 5 \cdot -7) = 8 - 75 + 35 = -32$$

$$3 + 32 = \boxed{35}$$

## 1.5 Identity Matrices

The **identity matrix**, denoted  $I$ , is a square matrix that, when multiplied by another matrix, equals that same matrix. An identity matrix contains 1s along the main diagonal and 0s for the remaining elements.

2x2 identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3x3 identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 1.6 Inverse Matrices

Two  $n \times n$  matrices are inverses of each other if and only if the product in both directions equals  $I$ . If matrix  $A$  has an inverse,  $B$ , then  $AB = I$  and  $BA = I$ .

**Question 1: Determine whether the pair of matrices are inverses**

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (-1 \cdot 5) + (2 \cdot 3) & (-1 \cdot 2) + (2 \cdot 1) \\ (3 \cdot 5) + (-5 \cdot 3) & (3 \cdot 2) + (-5 \cdot 1) \end{bmatrix} = \begin{bmatrix} -5 + 6 & -2 + 2 \\ 15 - 15 & 6 - 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} (5 \cdot -1) + (2 \cdot 3) & (5 \cdot 2) + (2 \cdot -5) \\ (3 \cdot -1) + (1 \cdot 3) & (3 \cdot 2) + (1 \cdot -5) \end{bmatrix} = \begin{bmatrix} -5 + 6 & 10 - 10 \\ -3 + 3 & 6 - 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A$  and  $B$  are inverses.

Not all matrices have an inverse. A matrix has no inverse if its determinant is 0.

There is a formula to find the inverse of a matrix. Consider matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ ,

where  $|A| \neq 0$ .

**Example 1.6.1** (Finding the Inverse of a Matrix)

$$A = \begin{bmatrix} 4 & -1 \\ -6 & 3 \end{bmatrix}$$

$$\det(A) = (4 \cdot 3) - (-6 \cdot -1) = 12 - 6 = 6 \therefore A \text{ has an inverse.}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 1 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 1 & \frac{2}{3} \end{bmatrix}$$

## Chapter 2

# Applications of Matrices

### 2.1 Area of a Triangle

Given a triangle with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ , the area of the triangle is  $\frac{1}{2}|\det(x)|$  where  $x = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

**Example 2.1.1** (Finding Triangle Area Using a Matrix)

Triangle  $PQR$  has vertices  $P(-5, -2)$ ,  $Q(3, 9)$ , and  $R(7, -4)$ .

$$x = \begin{bmatrix} -5 & -2 & 1 \\ 3 & 9 & 1 \\ 7 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -2 & 1 \\ 3 & 9 & 1 \\ 7 & -4 & 1 \end{bmatrix} \begin{matrix} -5 & -2 \\ 3 & 9 \\ 7 & -4 \end{matrix}$$

Downward diagonal one:  $[-5, 9, 1]$

Downward diagonal two:  $[-2, 1, 7]$

Downward diagonal three:  $[1, 3, -4]$

Upward diagonal one:  $[7, 9, 1]$

Upward diagonal two:  $[-4, 1, -5]$

Upward diagonal three:  $[1, 3, -2]$

$$(-5 \cdot 9 \cdot 1) + (-2 \cdot 1 \cdot 7) + (1 \cdot 3 \cdot -4) = -45 - 14 - 12 = -71$$

$$(7 \cdot 9 \cdot 1) + (-4 \cdot 1 \cdot -5) + (1 \cdot 3 \cdot -2) = 63 + 20 - 6 = 77$$

$$\det(x) = -71 - 77 = -148$$

$$|-148| = 148$$

$$\frac{148}{2} = \boxed{74}$$