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# Assignment 2
## Linear Optimisation

→ **LP Formulation**

(1) PRIMAL PROBLEM : Max - Flow.

Let $G = (V, E)$ be the graph given to us. We have an adjacency matrix, $A$ for $G$, where $A_{ij}$ tells us the maximum possible flow from vertex $i$ to $j$.

~~Also~~ Let $f_{ij}$ be the flow from vertex $i$ to vertex $j$. Obviously, we will have,

$$f_{ij} \geq 0 \quad \forall (i,j) \in E.$$

Also, the conditions for max-flow problem dictate that the amount of inflow into a node is equal to outflow for all non-source, non-sink nodes

$$\text{Total inflow to node } i = \sum_{h:(h,i) \in E} f_{hi}$$

$$\text{Total outflow of node } i = \sum_{j:(i,j) \in E} f_{ij}$$

Thus,

$$\sum_{h:(h,i) \in E} f_{hi} = \sum_{j:(i,j) \in E} f_{ij}$$

Or

$$\sum_{h:(h,i) \in E} f_{hi} - \sum_{j:(i,j) \in E} f_{ij} = 0 \quad \forall i \in V \setminus \{S, T\}$$

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(NOTE: I have assumed that ~~any of it~~ there is only one source and one sink in the graph, like in the given flow networks in the question.

A ~~general~~ more general ~~form~~ formulation would be:

$$\sum_{k:(k,i)\in E} f_{ki} - \sum_{j:(i,j)\in E} f_{ij} = b_i$$

where $b_i$ is the external supply at node $i$. But I ignore $b_i$ here to keep things simple)

Lastly, the flow in each edge must not exceed its capacity, given by $A_{ij}$.

Thus, $f_{ij} \leq A_{ij} \quad \forall (i,j)\in E$

Lastly, what we need to maximise is the total flow in the network. This can be given by the total outflow out of the source node, $S$.

$\therefore$ our objective is,

$$\text{maximise} \sum_{p:(S,p)\in E} f_{Sp}$$

Hence, our final primal problem is,

$$\text{maximise} \sum_{p:(S,p)\in E} f_{Sp}$$

st, ① $\sum_{k:(k,i)\in E} f_{ki} - \sum_{j:(i,j)\in E} f_{ij} = 0 \quad \forall i\in V\setminus\{S,T\}$

② $f_{ij} \leq A_{ij} \quad \forall (i,j)\in E$

③ $f_{ij} \geq 0 \quad \forall (i,j)\in E$

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~~Answer~~

(2) DUAL PROBLEM: Min-Cut

We will formulate the Lagrangian of this problem to get the dual.

Let $f = [f_{ij}]^T$ be the vector containing flows of all edges. $f \in \mathbb{R}^{|E| \times 1}$. Let $\lambda^1 \in \mathbb{R}^{|E| \times 1}$, & $\lambda^2 \in \mathbb{R}^{|E| \times 1}$, $\nu \in \mathbb{R}^{|V - \{S,T\}| \times 1}$ be 3 vectors s.t,

$$\mathcal{L}(f, \lambda^1, \lambda^2, \nu) = -\sum_{p:(s,p)\in E} f_{sp} + \underbrace{\sum_{(i,j)\in E} \lambda^1_{ij}(f_{ij} - C_{ij})}_{\mathrm{I}}$$

$$+ \sum_{(i,j)\in E} \lambda^2_{ij}(-f_{ij}) + \underbrace{\sum_{i\in V\setminus\{S,T\}} \nu_i \left(\sum_{h:(h,i)\in E} f_{hi} - \sum_{j:(i,j)\in E} f_{ij}\right)}_{\mathrm{III}}$$

This is the Lagrangian function of the PRIMAL PROBLEM that we just formulated. Note that $\lambda^1 \geq 0$ and $\lambda^2 \geq 0$

~~Now,~~

$$\mathcal{L}(f, \lambda^1, \lambda^2, \nu) = \underbrace{-\sum_{i:(s,i)\in E} f_{si} + \sum_{i:(s,i)\in E} \lambda^1_{ij} f_{ij} + \sum_{i:(i,t)\in E} f_{it} + \sum_{\substack{(i,j)\in E \,\wedge\\ i,j\in V\setminus\{S,T\}}} \lambda^1_{ij} f_{ij} - \sum_{(i,j)\in E} \lambda^1_{ij} C_{ij}}_{\mathrm{I}}$$

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\left.
\begin{aligned}
&-\sum_{(i,j)\in E} \lambda^2_{ij} f_{ij} + \sum_{i\in V\setminus\{s,t\}} \nu_i f_{si} - \sum_{i\in V\setminus\{s,t\}} \nu_i f_{it} \\
&+ \sum_{\substack{(i,j)\in E \,\wedge \\ i,j\in V\setminus\{s,t\}}} (\nu_j - \nu_i) f_{ij}
\end{aligned}
\right\} \text{IV}
$$

Remark:
I was merely broken into 3 separate summation terms. For III, we just separated the edges with terminal and non-terminal nodes to arrive at this arrangement

$$
= \sum_{i\in V\setminus\{s,t\}} (-1 + \lambda^1_{si} + \nu_i - \lambda^2_{si}) f_{si}
$$

$$
+ \sum_{\substack{(i,j)\in E \,\wedge \\ i,j\in V\setminus\{s,t\}}} (-\lambda^2_{ij} + \nu_j - \nu_i + \lambda^1_{ij}) f_{ij}
$$

$$
+ \sum_{i\in V\setminus\{s,t\}} (\lambda^1_{it} - \lambda^2_{it} - \nu_i) f_{it} - \sum_{(i,j)\in E} \lambda^1_{ij} c_{ij}
$$

Now, consider the dual function, i.e.,

$$
g(\lambda^1, \lambda^2, \nu) = \inf_f \mathcal{L}(f, \lambda^1, \lambda^2, \nu)
$$

$$
= \begin{cases}
-\sum_{(i,j)\in E} \lambda^1_{ij} c_{ij}, & \text{if } (\lambda^1_{it} - \lambda^2_{it} - \nu_i) = 0 \quad \forall i \in V\setminus\{s,t\} \\
& (-1 + \lambda^1_{si} + \nu_i - \lambda^2_{si}) = 0 \quad \forall i \in V\setminus\{s,t\} \\
& (-\lambda^2_{ij} + \nu_j - \nu_i + \lambda^1_{ij}) = 0 \quad \forall (i,j)\in E \wedge i,j \in V\setminus\{s,t\} \\
-\infty, & \text{o/w}
\end{cases}
$$

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now, we try to maximise the dual ~~objective~~ func$^n$ as it gives a lower bound to the primal objective.

$$\text{maximise} \left( -\sum_{(i,j)\in E} \lambda'_{ij} c_{ij} \right)$$

st, ① $\cancel{\sum_{i\in V\setminus\{s,t\}}} (\lambda'_{it} - \lambda^2_{it} - \nu_i) = 0 \;\forall\; i \in V\setminus\{s,t\}$

② $\cancel{\sum_{i\in V\setminus\{s,t\}}} (-1 + \lambda'_{si} + \nu_i - \lambda^2_{si}) = 0 \;\forall\; i\in V\setminus\{s,t\}$

③ $\cancel{\sum_{(i,j)\in E,\, i,j\in V\setminus\{s,t\}}} (-\lambda^2_{ij} + \nu_j - \nu_i + \lambda'_{ij}) = 0$
$\forall\; (i,j)\in E \wedge i,j \in V\setminus\{s,t\}$

$\because \lambda_2 \geq 0$ and is not involved in the objective and we are maximising -ve of a +ve valued func$^n$ we can re-write this as;

$$\text{minimise} \sum_{(i,j)\in E} \lambda'_{ij} c_{ij}$$

st, ① $\cancel{\sum_{i\in V\setminus\{s,t\}}} (\lambda'_{it} \cancel{\lambda^2_{it}} - \nu_i) \geq 0 \;\forall\; i\in V\setminus\{s,t\}$

② $\lambda'_{si} + \nu_i \geq 1 \;\forall\; i\in V\setminus\{s,t\}$

③ $\lambda'_{ij} - \nu_i + \nu_j \geq 0 \;\forall\; (i,j)\in E \wedge i,j\in V\setminus\{s,t\}$

④ $\lambda'_{ij} \geq 0 \;\forall\; (i,j)\in E$

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Now, we use the capacity networks given in the question to formulate the LPs.

ⓐ FLOW 1 ( non-zero, non-self-looping edges are numbered left-to-right, row by row. eg. $f_{SA} = f_1$ ; $f_{SB} = f_2$, $f_{AB} = f_3$ and so on .... if $f_{ii}$ appears then it is ignored )

PRIMAL maximise $f_1 + f_2$

st,

$$f_1 + f_5 - f_3 - f_4 = 0 \quad \text{(for node A)}$$

$$f_2 + f_3 - f_6 = 0 \quad \text{(for node B)}$$

$$f_4 + f_9 - f_7 - f_8 = 0 \quad \text{(for node C)}$$

$$f_6 - f_9 - f_{10} = 0 \quad \text{(for node D)}$$

$f_1 \leq 16$ $\quad f_4 \leq 12$ $\quad f_7 \leq 9$

$f_2 \leq 13$ $\quad f_5 \leq 4$ $\quad f_8 \leq 20$

$f_3 \leq 10$ $\quad f_6 \leq 14$ $\quad f_9 \leq 7$

$f_i \geq 0 \ \forall\, 1 \leq i \leq 10$ $\quad f_{10} \leq 7$

(This LP will converted to standard form before being solved in a dual-phase simplex in the code).

DUAL minimise $\sum_{i=1}^{10} c_i \lambda'_i$ ( $c_i$ = capacity of edge i )

st,

$\lambda'_1 + v_A \geq 1$ $\qquad \lambda'_7 - v_C + v_B$ $\qquad \lambda'_i \geq 0$

$\lambda'_2 + v_B \geq 1$ $\qquad \lambda'_8 - v_C \geq 0$ $\qquad \forall 1 \leq i \leq 10$

$\lambda'_3 - v_A + v_B \geq 0$ $\qquad \lambda'_9 - v_D + v_C \geq 0$

$\lambda'_4 - v_A + v_C \geq 0$ $\qquad \lambda'_{10} - v_D \geq 0$

$\lambda'_5 - v_B + v_A \geq 0$ $\qquad$ ( $\lambda'_i$ corresponds to $f_i$, a variable for each edge )

$\lambda'_6 - v_B + v_D \geq 0$

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(b) FLOW 2 (Assuming similar numbering for edges as in previous flow.)

PRIMAL : maximise $f_1 + f_2 + f_3$

~~st, $f_i \geq C_i$ $\forall i \in \{1, 2, \ldots, 30\}$ ($C_i$ = capacity of edge $i$)~~

for node A: $f_1 + f_6 + f_{16} - f_4 - f_5 = 0$

for node B: $f_2 - f_6 \cancel{+f} - f_7 = 0$

for node C: $f_3 + f_7 + f_{11} - f_8 - f_9 - f_{10} = 0$

for node D: $f_8 + f_{14} + f_{17} - f_{11} - f_{12} - f_{13} = 0$

for node E: $f_4 - f_{14} - f_{15} = 0$

for node F: $f_5 - f_{16} - f_{17} - f_{18} = 0$

for node G: $f_9 + f_{12} + f_{23} - f_{19} - f_{20} - f_{21} - f_{22} = 0$

for node H: $f_{10} + f_{13} + f_{19} + f_{28} - f_{23} - f_{24} - f_{25} = 0$

for node I: $f_{15} + f_{20} - f_{26} - f_{27} = 0$

for node J: $f_{21} + f_{24} + f_{26} - f_{28} - f_{29} = 0$

$f_i \leq C_i$ $(\forall i \ 1 \leq i \leq 29)$ $C_i$ = capacity of edge $i$

$f_i \geq 0$ $\forall \ 1 \leq i \leq 29$

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**DUAL:** minimise $\sum_{i=1}^{29} c_i \lambda'_i$ ($c_i \rightarrow$ capacity of edge i)

st, $\lambda'_i \geq 0 \quad \forall \; 1 \leq i \leq 29$

$\lambda'_1 + \nu_A \geq 1$

$\lambda'_2 + \nu_B \geq 1$

$\lambda'_3 + \nu_C \geq 1$

$\lambda'_4 - \nu_A + \nu_E \geq 0$

$\lambda'_5 - \nu_A + \nu_F \geq 0$

$\lambda'_6 - \nu_B + \nu_A \geq 0$

$\lambda'_7 - \nu_B + \nu_C \geq 0$

$\lambda'_8 - \nu_C + \nu_D \geq 0$

$\lambda'_9 - \nu_C + \nu_G \geq 0$

$\lambda'_{10} - \nu_C + \nu_H \geq 0$

$\lambda'_{11} - \nu_D + \nu_C \geq 0$

$\lambda'_{12} - \nu_D + \nu_G \geq 0$

$\lambda'_{13} - \nu_D + \nu_H \geq 0$

$\lambda'_{14} - \nu_E + \nu_D \geq 0$

$\lambda'_{15} - \nu_E + \nu_I \geq 0$

$\lambda'_{16} - \nu_F + \nu_A \geq 0$

$\lambda'_{17} - \nu_F + \nu_D \geq 0$

$\lambda'_{18} - \nu_F \geq 0$

$\lambda'_{19} - \nu_G + \nu_H \geq 0$

$\lambda'_{20} - \nu_G + \nu_I \geq 0$

$\lambda'_{21} - \nu_G + \nu_J \geq 0$

$\lambda'_{22} - \nu_G \geq 0$

$\lambda'_{23} - \nu_H + \nu_G \geq 0$

$\lambda'_{24} - \nu_H + \nu_J \geq 0$

$\lambda'_{25} - \nu_H \geq 0$

$\lambda'_{26} - \nu_I + \nu_J \geq 0$

$\lambda'_{27} - \nu_I \geq 0$

$\lambda'_{28} - \nu_J + \nu_H \geq 0$

$\lambda'_{29} - \nu_J \geq 0$