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Assignment 2
Linear Optimisation

→ LP Formulation

1. PRIMAL PROBLEM : Max - Flow.

Let $G=(V,E)$ be the graph given to us. We have an adjacency matrix, A for G, where A_{ij} tells us the maximum possible flow from vertex i to j.

~~Also~~ ~~to~~ Let f_{ij} be the flow from vertex i to vertex j. Obviously, we will have,

$$f_{ij} \geq 0 \quad \forall (i,j) \in E.$$

Also, the conditions for max-flow problem dictate that the amount of inflow into a node is equal to outflow for all non-source, non-sink nodes

$$\text{Total inflow to node } i = \sum_{h:(h,i)\in E} f_{hi}$$

$$\text{Total outflow of node } i = \sum_{j:(i,j)\in E} f_{ij}$$

Thus,

$$\sum_{h:(h,i)\in E} f_{hi} = \sum_{j:(i,j)\in E} f_{ij}$$

Or

$$\sum_{h:(h,i)\in E} f_{hi} - \sum_{j:(i,j)\in E} f_{ij} = 0 \quad \forall i \in V\setminus\{S,T\}$$

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(NOTE: I have assumed that ~~any of it~~ there is only one source and one sink in the graph, like in the given flow networks in the question.

A ~~general~~ more general ~~fol~~ formulation would be:

$$\sum_{h:(h,i)\in E} f_{hi} - \sum_{j:(i,j)\in E} f_{ij} = b_i$$

where b_i is the external supply at node i. But I ignore b_i here to keep things simple)

Lastly, the flow in each edge must not exceed its capacity, given by A_{ij}.

Thus, $f_{ij} \le A_{ij} \quad \forall (i,j)\in E$

Lastly, what we need to maximise is the total flow in the network. This can be given by the total outflow out of the source node, S.

$\therefore$ our objective is,

$$\text{maximise} \sum_{p:(S,p)\in E} f_{sp}$$

Hence, our final primal problem is,

$$\text{maximise} \sum_{p:(S,p)\in E} f_{sp}$$

st, ① $\displaystyle\sum_{h:(h,i)\in E} f_{hi} - \sum_{j:(i,j)\in E} f_{ij} = 0 \quad \forall\, i\in V\setminus\{S,T\}$

② $f_{ij} \le A_{ij} \quad \forall (i,j)\in E$

③ $f_{ij} \ge 0 \quad \forall (i,j)\in E$

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(2) DUAL PROBLEM: Min-Cut

We will formulate the Legrangian of this problem to get the dual.

Let $f = [f_{ij}]^T$ be the vector containing flows of all edges. $f \in \mathbb{R}^{|E| \times 1}$. Let $\lambda^1 \in \mathbb{R}^{|E| \times 1}$, & $\lambda^2 \in \mathbb{R}^{|E| \times 1}$

$\nu \in \mathbb{R}^{|V - \{S,T\}| \times 1}$ be 3 vectors st,

$$\mathcal{L}(f, \lambda^1, \lambda^2, \nu) = -\sum_{p:(s,p)\in E} f_{sp} + \underbrace{\sum_{(i,j)\in E} \lambda^1_{ij}(f_{ij} - C_{ij})}_{\mathrm{I}}$$

$$+ \sum_{(i,j)\in E} \lambda^2_{ij}(-f_{ij})$$

$$\underbrace{+ \sum_{i\in V\setminus\{S,T\}} \nu_i \left(\sum_{h:(h,i)\in E} f_{hi} - \sum_{j:(i,j)\in E} f_{ij}\right)}_{\mathrm{III}}$$

This is the Legrangian function of the PRIMAL PROBLEM that we just formulated. Note that $\lambda^1 \geq 0$ and $\lambda^2 \geq 0$

~~Now,~~

$$\mathcal{L}(f, \lambda^1, \lambda^2, \nu) = \underbrace{-\sum_{i:(s,i)\in E} f_{si} + \sum_{i:(s,i)\in E} \lambda^1_{ij} f_{ij} + \sum_{i:(i,t)\in E} f_{it} + \sum_{\substack{(i,j)\in E \,\wedge \\ i,j\in V\setminus\{S,T\}}} \lambda^1_{ij} f_{ij} - \sum_{(i,j)\in E} \lambda^1_{ij} C_{ij}}_{\mathrm{I}}$$

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$$\left.\begin{aligned} &- \sum_{(i,j)\in E} \lambda^2_{ij} f_{ij} + \sum_{i\in V\setminus\{s,t\}} \nu_i f_{si} - \sum_{i\in V\setminus\{s,t\}} \nu_i f_{it} \\ &+ \sum_{\substack{(i,j)\in E\,\wedge \\ i,j\in V\setminus\{s,t\}}} (\nu_j - \nu_i) f_{ij} \end{aligned}\right\} \text{IV}$$

Remark:
I was merely broken into 3 separate summation terms. For III, we just separated the edges with terminal and non-terminal nodes to arrive at this arrangement

$$= \sum_{i\in V\setminus\{s,t\}} (-1 + \lambda^1_{si} + \nu_i - \lambda^2_{si}) f_{si}$$

$$+ \sum_{\substack{(i,j)\in E\,\wedge \\ i,j\in V\setminus\{s,t\}}} (-\lambda^2_{ij} + \nu_j - \nu_i + \lambda^1_{ij}) f_{ij}$$

$$+ \sum_{i\in V\setminus\{s,t\}} (\lambda^1_{it} - \lambda^2_{it} - \nu_i) f_{it} - \sum_{(i,j)\in E} \lambda_{ij} c_{ij}$$

Now, consider the dual function, i.e.,

$$\mathcal{Q}(\lambda^1, \lambda^2, \nu) = \inf_f \mathcal{L}(f, \lambda^1, \lambda^2, \nu)$$

$$= \begin{cases} -\sum_{(i,j)\in E} \lambda_{ij} c_{ij}, & \text{if } (\lambda^1_{it} - \lambda^2_{it} - \nu_i) = 0 \;\; \forall i \in V\setminus\{s,t\} \\ & (-1 + \lambda^1_{si} + \nu_i - \lambda^2_{si}) = 0 \;\; \forall i \in V\setminus\{s,t\} \\ & (-\lambda^2_{ij} + \nu_j - \nu_i + \lambda^1_{ij}) = 0 \;\; \forall (i,j)\in E \wedge i,j \in V\setminus\{s,t\} \\ -\infty, & \text{o/w} \end{cases}$$

now, we try to maximise the dual ~~objective~~ func^n as it gives a lower bound to the primal objective.

$$\text{maximise} \left(-\sum_{(i,j)\in E} \lambda'_{ij} c_{ij} \right)$$

st, ① $\cancel{\sum_{i\in V\setminus\{s,t\}}} (\lambda'_{it} - \lambda^2_{it} - \nu_i) = 0 \quad \forall\ i \in V\setminus\{s,t\}$

② $\cancel{\sum_{i\in V\setminus\{s,t\}}} (-1 + \lambda'_{si} + \nu_i - \lambda^2_{si}) = 0 \quad \forall\ i \in V\setminus\{s,t\}$

③ $\cancel{\sum_{(i,j)\in E,\ i,j\in V\setminus\{s,t\}}} (-\lambda^2_{ij} + \nu_j - \nu_i + \lambda'_{ij}) = 0$

$\forall\ (i,j)\in E \wedge i,j \in V\setminus\{s,t\}$

$\because \lambda_2 \geq 0$ and is not involved in the objective and we are maximising -ve of a +ve valued func^n we can re-write this as;

$$\text{minimise} \sum_{(i,j)\in E} \lambda'_{ij} c_{ij}$$

st, ① $\cancel{\sum_{i\in V\setminus\{s,t\}}} (\lambda'_{it} \cancel{-\lambda^2_{it}} - \nu_i) \geq 0 \quad \forall\ i \in V\setminus\{s,t\}$

② $\lambda'_{si} + \nu_i \geq 1 \quad \forall\ i \in V\setminus\{s,t\}$

③ $\lambda'_{ij} - \nu_i + \nu_j \geq 0 \quad \forall\ (i,j)\in E \wedge i,j \in V\setminus\{s,t\}$

④ $\lambda'_{ij} \geq 0 \quad \forall\ (i,j) \in E$

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Now, we use the capacity networks given in the question to formulate the LPs.

Ⓐ **FLOW 1** (non-zero, non-self-looping edges are numbered ~~left-to right~~, row by row. eg. $f_{SA} = f_1$; $f_{SB} = f_2$, $f_{AB} = f_3$ and so on ....
if f_{ii} appears then it is ignored)

PRIMAL maximise $f_1 + f_2$

st,
$$f_1 + f_5 - f_3 - f_4 = 0 \quad \text{(for node A)}$$
$$f_2 + f_3 - f_6 = 0 \quad \text{(for node B)}$$
$$f_4 + f_9 - f_7 - f_8 = 0 \quad \text{(for node C)}$$
$$f_6 - f_9 - f_{10} = 0 \quad \text{(for node D)}$$

$$f_1 \leq 16 \quad f_4 \leq 12 \quad f_7 \leq 9$$
$$f_2 \leq 13 \quad f_5 \leq 4 \quad f_8 \leq 20$$
$$f_3 \leq 10 \quad f_6 \leq 14 \quad f_9 \leq 7$$
$$f_i \geq 0 \ \forall \ 1 \leq i \leq 10 \quad f_{10} \leq 7$$

(This LP will converted to standard form before being solved in a dual-phase simplex in the code).

DUAL minimise $\sum_{i=1}^{10} c_i \lambda_i$ (c_i = capacity of edge i)

st,
$$\lambda'_1 + \nu_A \geq 1$$
$$\lambda'_2 + \nu_B \geq 1$$
$$\lambda'_3 - \nu_A + \nu_B \geq 0$$
$$\lambda'_4 - \nu_A + \nu_C \geq 0$$
$$\lambda'_5 - \nu_B + \nu_A \geq 0$$
$$\lambda'_6 - \nu_B + \nu_D \geq 0$$
$$\lambda'_7 - \nu_C + \nu_B$$
$$\lambda'_8 - \nu_C \geq 0$$
$$\lambda'_9 - \nu_D + \nu_C \geq 0$$
$$\lambda'_{10} - \nu_D \geq 0$$

$\lambda'_i \geq 0$ $\forall 1 \leq i \leq 10$

(λ'_i corresponds to f_i, a variable for each edge)

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(b) FLOW 2 (Assuming similar numbering for edges as in previous flow.)

PRIMAL : maximise $f_1 + f_2 + f_3$

st,

for node A: $f_1 + f_6 + f_{16} - f_4 - f_5 = 0$

for node B: $f_2 - f_6 - f_7 = 0$

for node C: $f_3 + f_7 + f_{11} - f_8 - f_9 - f_{10} = 0$

for node D: $f_8 + f_{14} + f_{17} - f_{11} - f_{12} - f_{13} = 0$

for node E: $f_4 - f_{14} - f_{15} = 0$

for node F: $f_5 - f_{16} - f_{17} - f_{18} = 0$

for node G: $f_9 + f_{12} + f_{23} - f_{19} - f_{20} - f_{21} - f_{22} = 0$

for node H: $f_{10} + f_{13} + f_{19} + f_{28} - f_{23} - f_{24} - f_{25} = 0$

for node I: $f_{15} + f_{20} - f_{26} - f_{27} = 0$

for node J: $f_{21} + f_{24} + f_{26} - f_{28} - f_{29} = 0$

$f_i \leq C_i \quad (\forall\ i\ 1 \leq i \leq 29)$ C_i = capacity of edge i

$f_i \geq 0 \quad \forall\ 1 \leq i \leq 29$

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DUAL: minimise $\sum_{i=1}^{29} c_i \lambda'_i$ ($c_i \to$ capacity of edge i)

st, $\lambda'_i \geq 0 \quad \forall \; 1 \leq i \leq 29$

$\lambda'_1 + \nu_A \geq 1$

$\lambda'_2 + \nu_B \geq 1$

$\lambda'_3 + \nu_C \geq 1$

$\lambda'_4 - \nu_A + \nu_E \geq 0$

$\lambda'_5 - \nu_A + \nu_F \geq 0$

$\lambda'_6 - \nu_B + \nu_A \geq 0$

$\lambda'_7 - \nu_B + \nu_C \geq 0$

$\lambda'_8 - \nu_C + \nu_D \geq 0$

$\lambda'_9 - \nu_C + \nu_G \geq 0$

$\lambda'_{10} - \nu_C + \nu_H \geq 0$

$\lambda'_{11} - \nu_D + \nu_C \geq 0$

$\lambda'_{12} - \nu_D + \nu_G \geq 0$

$\lambda'_{13} - \nu_D + \nu_H \geq 0$

$\lambda'_{14} - \nu_E + \nu_D \geq 0$

$\lambda'_{15} - \nu_E + \nu_I \geq 0$

$\lambda'_{16} - \nu_F + \nu_A \geq 0$

$\lambda'_{17} - \nu_F + \nu_D \geq 0$

$\lambda'_{18} - \nu_F \geq 0$

$\lambda'_{19} - \nu_G + \nu_H \geq 0$

$\lambda'_{20} - \nu_G + \nu_I \geq 0$

$\lambda'_{21} - \nu_G + \nu_J \geq 0$

$\lambda'_{22} - \nu_G \geq 0$

$\lambda'_{23} - \nu_H + \nu_G \geq 0$

$\lambda'_{24} - \nu_H + \nu_J \geq 0$

$\lambda'_{25} - \nu_H \geq 0$

$\lambda'_{26} - \nu_I + \nu_J \geq 0$

$\lambda'_{27} - \nu_I \geq 0$

$\lambda'_{28} - \nu_J + \nu_H \geq 0$

$\lambda'_{29} - \nu_J \geq 0$

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These LPs are properly converted to ~~Du~~ Standard form and solved by the SIMPLEX.

~~Some assumptions in SIMPLEX code:~~

~~(i) Flows are always +ve~~

~~(ii) Cycl In case of cycling, abandon solving~~

~~(iii) In case of degenerate soln~~ ~~a~~ Flow network will not lead to degenerate BFS.

~~It is always the~~

The simplex ~~maximises~~ minimises ~~min~~ -ve of the max-flow objective. Hence result displayed is ~~max~~ actual max-flow.

In case of min-cut, the implemented tableau method returns the -ve of the -ve of the optimal min-cut. Hence ~~max~~, the value is multiplied by -1 and then displayed.