Scientific Computing
Assignment 2

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Problem 4(c)

Faulty Matrix

$$A_2 = \begin{bmatrix} 1.e-03 & 1.e-03 & 5.e+00 & ... & 1.e-03 & 1.e-03 & 1.e-03 \\ 1.e-03 & 1.e-03 & 1.e-03 & ... & 1.e-03 & 1.e-03 & 1.e-03 \\ 1.e-03 & 1.e-03 & 1.e-03 & ... & 1.e-03 & 1.e-03 & 1.e-03 \\ ... \\ 1.e-03 & 1.e-03 & 1.e-03 & ... & 1.e-03 & 1.e-03 & 5.e+00 \\ 5.e+00 & 1.e-03 & 1.e-03 & ... & 1.e-03 & 1.e-03 & 1.e-03 \\ 1.e-03 & 5.e+00 & 1.e-03 & ... & 1.e-03 & 1.e-03 & 1.e-03 \end{bmatrix}$$

Error: pivot element becomes 0 at some point

Analysis Table

	Input	A_1, x_1	A_2, x_2	A_3, x_3
Statistic	Method			
Error	ge_nop	9.316228e-12	NaN	6.205457e-08
	ge_pp	4.758281e-14	1.282790e-13	3.262358e-15
	np.solve	6.849568e-14	2.665814e-15	2.677728e-15
Residual	ge_nop	2.122088e-11	NaN	6.168661e-08
	ge_pp	1.370217e-13	6.414621e-13	3.336822e-15
	np.solve	9.173675e-14	1.192074e-14	2.842407e-15
Relative Error	ge_nop	1.639523e-12	NaN	1.031253e-08
	ge_pp	8.373896e-15	2.177661e-14	5.421543e-16
	np.solve	1.205426e-14	4.525477e-16	4.449976e-16
Relative Residual	ge_nop	1.890765e-13	NaN	1.890765e-13
	ge_pp	1.220853e-15	2.135603e-14	1.220853e-15
	np.solve	1.220853e-15	2.135603e-14	1.220853e-15
Condition number		1.663415e+03	1.020004e+00	1.302279e+00

Problem 4(d)

- $A_1 x_1 = b_1$:
  - The condition number in this case, is large which is probably, because the matrix is randomly generated using `numpy`. This also means that this linear system is ill-conditioned.
  - Pivoting reduces the error for the system. Hence, Pivoting in the case of this system is beneficial as it leads to larger pivots and smaller multipliers, and hence, a more accurate solution.
- $A_2 x_2 = b_2$:
  - Condition number for the matrix is small. Hence, the system itself is well conditioned. The relative residual is also low in case of `ge_pp` and `np.solve`. Hence, it results in a lower relative error in these cases.

- Without pivoting, this system is unsolvable as it leads to a zero pivot in when solving by `ge_nop`. Hence, pivoting is instrumental to obtaining a solution here.

- $A_3x_3 = b_3$:
  - Condition number for this system too is small and this implies that the system is well conditioned. Also evident, is the low relative error in this case (for `ge_pp` and `np.solve`), which is a direct consequence of a low relative residual and the well-conditioning of the system
  - Without pivoting, we observe that in spite of a better condition number, the relative error of this system is worse than $A_1x_1 = b_1$. Thus, pivoting is instrumental to a more accurate result here as it leads to larger pivots and smaller multipliers, hence more accurate solutions.

Problem 5

Analysis Table

Statistic	Relative Error	Relative Residual	Condition Number
Part			
(i)	1.744068e-14	6.599244e-16	3.243453e+02
(ii)	1.744068e-14	6.599244e-16	3.243453e+02
(iii)	2.118587e-14	5.279811e-16	2.561083e+03
(iv)	1.579943e-14	5.527979e-16	2.276538e+05
(v)	1.014464e-13	2.014274e-17	1.370235e+31

Observations

- The last instance, `part (v)`, has the poorest accuracy.
- Yes, the relative residual is still small in this case. In fact it is the smallest amongst all the cases.
- We know that,
$$\frac{||\Delta \boldsymbol{x}||}{||\boldsymbol{x}||} \leq \text{cond}(\boldsymbol{A}) \frac{||\boldsymbol{r}||}{||\boldsymbol{A}|| \cdot ||\boldsymbol{x}||}$$
  Because of a very large condition number (the largest in all cases) in `part (v)`, we have a very loose bound on the relative error, even with the smallest relative residual. This allows the relative error to increase. Hence, we have a poor error in this case.
- In conclusion this confirms that a low relative residual guarantees a low relative error iff the system is well-conditioned.

Problem 6(a)

The output of the code is:

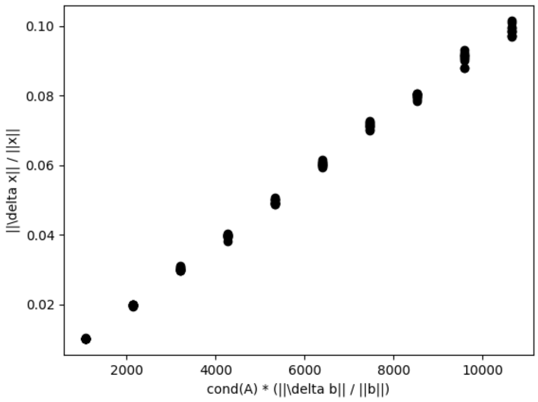
```
A =
 [[ 2 -1  0 ...  0  0  0]
 [-1  2 -1 ...  0  0  0]
 [ 0 -1  2 ...  0  0  0]
 ...
 [ 0  0  0 ...  2 -1  0]
 [ 0  0  0 ... -1  2 -1]
 [ 0  0  0 ...  0 -1  2]]
Condition number of A  =  106657.71164568371
```

Problem 6(b)

Note: I used `numpy.random.rand` for generating random numbers in this question

Problem 6(c)

The plot obtained in this question in displayed below:



Problem 6(d)

No, it is not possible for the vertical length to exceed the horizontal length. The reason for that is that, as we have shown in class,

$$\frac{||\boldsymbol{\Delta x}||}{||\boldsymbol{x}||} \leq \text{cond}(\boldsymbol{A})\left(\frac{||\Delta \boldsymbol{b}||}{||\boldsymbol{b}||} + \frac{||\boldsymbol{E}||}{||\boldsymbol{A}||}\right)$$

, where $\boldsymbol{E}$ and $\boldsymbol{\Delta b}$ are the perturbations in $\boldsymbol{A}$ and $\boldsymbol{b}$ respectively.
Now, in this case, $\boldsymbol{A}$ is fixed, and hence, $||\boldsymbol{E}|| = 0$. Hence, the equation becomes,

$$\frac{||\boldsymbol{\Delta x}||}{||\boldsymbol{x}||} \leq \text{cond}(\boldsymbol{A})\left(\frac{||\Delta \boldsymbol{b}||}{||\boldsymbol{b}||}\right)$$

We can observe that the LHS is the vertical distance of a point and the RHS is the horizontal distance of a point from the axes. Hence, proved.