The Faddeev-Reshetikhin-Takhtajan Construction

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Topological free modules

Let $K = \mathbb{C}[[h]]$ be the algebra of formal power series in h over $\mathbb{C}$. Take some constant $p > 1$ we have the h-**adic norm**

$$|a_n h^n + a_{n+1} h^{n+1} + \cdots| = p^{-n}$$

which induces the h-**adic topology** on K in the usual way. Given a $\mathbb{C}$-vector space V we may define $V[[h]]$ to be topological $\mathbb{C}[[h]]$-module

$$V[[h]] = \left\{ \sum_{i=0}^{\infty} v_i h^i : v_i \in V \right\}$$

with a similar h-adic norm.

Definition

A **topologically free** module is a K-module of the form $V[[h]]$ for some $\mathbb{C}$-vector space V.

Note that all topologically free modules are complete with respect to the h-adic norm.

Topological free modules

Morphisms of topologically free modules are simply $\mathbb{C}[[h]]$-module maps. These maps are naturally continuous:

Lemma

As usual, we have the identification $V[[h]] = \varprojlim V[[h]]/h^n V[[h]]$.

We now have a problem: the usual tensor product of finite-dimensional modules no longer works properly. For instance, if V is infinite dimensional $V[[h]] \ncong V \otimes_{\mathbb{C}} K$; $\mathbb{C}[[x]] \otimes_{\mathbb{C}} \mathbb{C}[[y]] \ncong \mathbb{C}[[x, y]]$ (not the h-adic topology).

Definition

The **topological tensor product** of two K-modules M, N is defined to be the h-adic completion

$$M \hat{\otimes} N = \varprojlim \,(M \otimes_K N)/h^n(M \otimes_K N)$$

Note that $V[[h]] \hat{\otimes} W[[h]] = (V \otimes W)[[h]]$. So the topological tensor product of two topologically free modules is topologically free.

Deformation algebras

Definition

A **deformation (Hopf, bi-) algebra** over K is a topologically free K-module A with the required maps defined on topological tensors products.

Given some $a \in A$, a formal power series

$$f(a) = \sum_{n=1}^{\infty} c_i a^n h^n$$

defines an element in A, regarded as an inverse limit. Therefore it makes sense to talk about e^h and e^{ha} for any $a \in A$.

Observation

The quotient A/hA is a $\mathbb{C}$-vector space which also gains a usual (Hopf, bi-) algebra structure from the maps on the topological algebra A restricted to A/hA. If $A_0 \cong A/hA$, we say A is a **deformation of** A_0.

h-formal groups

Definition

A **formal group** is a topological Hopf algebra isomorphic to $U(\mathfrak{g})^*$ for some Lie algebra $\mathfrak{g}$. An h-**formal group** is a deformation Hopf algebra H such that H/hH is a formal group.

Let $A^\circ = \mathrm{Hom}_K(A, K)$ be the algebra of K-linear maps on A. We define the dual algebra of an h-formal group H to be the space of continuous K-linear maps $H \to K$.

Proposition

Let A be a QUEA. Then A° where $K = \mathbb{C}[[h]]$ is an h-formal group. Conversely, the dual algebra of an h-formal group H is a QUEA.

Lemma

The dual $U_h(\mathfrak{sl}_2)^\circ$ is isomorphic to $SL_h(2)$ as deformation Hopf algebras.

The FRT construction II

Now consider GL_n, SL_n as algebraic groups. We may still define the completion

$$U(\mathfrak{sl}_n)^\circ = \varprojlim \Gamma(SL_n)/J^i, U(\mathfrak{gl}_n)^\circ = \varprojlim \Gamma(GL_n)/J^i$$

where J is the kernel of ε.

Theorem (FRT deformation version)

Let W be a finite dimensional $\mathbb{C}$-vector space, and $R \in (\mathrm{End}(W) \otimes \mathrm{End}(W))[[h]]$ an R-matrix such that $R \equiv \mathrm{id} \pmod{h}$ satisfying the Hecke relation:

$$(\sigma \circ R - p)(\sigma \circ R + q) = 0$$

for $p, q \equiv 1 \pmod{h}$. Define the $\mathbb{C}[[h]]$-algebra $H(R)$ to be completion of the algebra H_0 generated by T_i^j and similar relations defined by R. Then $H(R)$ is an h-formal group and $H(R)/hH(R) \cong U(\mathfrak{gl}_n)^\circ$. Moreover, there is a universal r-form r induces R on the $H(R)$-comodule $W[[h]]$.

Comments and Example I

In the last slide, by completion we mean the h-adic completion of $\varprojlim H_0/(\ker\varepsilon)^i$. The Hecke relation ensures the deformation is flat, or in some sense satisfies the *Ore condition* that governs localizations. Define the R-matrix

$$R = e^h \sum E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (e^h - e^{-h}) \sum_{i<j} E_{ij} \otimes E_{ji}$$

Exercise: Compute the case $n = 2$. Show that it satisfies the Hecke relation.

Example I

Define the q-determinant

$$\det_q = \sum_{\gamma \in S_n} (-e^{-h})^{|\gamma|} T^1_{\gamma(1)} \cdots T^n_{\gamma(n)}$$

where $|\gamma|$ is the length of the permutation. If R satisfies the Hecke-relation, then we can localize $H(R)$ at $\det_q$ to get $H(R)_{\det_q}$.

Proposition

Let I be the two-sided ideal in $H(R)_{\det_q}$ generated by $\det_q^{-1}$. Then

$$H(R)_{\det_q}/I \cong U_h(\mathfrak{sl}_n)^\circ$$

The case $n = 2$ is precisely what we did in the previous slides (although we need to rearrange the R-matrix).

Properties of topological free modules

Proposition

Let V be a $\mathbb{C}$-vector space and $\{e_i\}$ a basis of V.

- *The K-module generated by $\{e_i\}$ is dense in $V[[h]]$.*
- *For any separated, completed K-module M, we have a bijection*

$$\mathrm{Hom}_K(V[[h]], M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, M)$$

Proof.

For the first part, show that if $f = \sum v_n h^n$ we may define

$$f_n = \sum_{k=0}^{n-1} v_k h^k = \sum_{k=0}^{n-1} \sum_{i=0}^{r} a_{ik} e_i h^k = \sum_{i=0}^{r} \left(\sum_{k=0}^{n-1} a_{ik} h^k \right) e_i \in W$$

and that $|f - f_n|_h = p^{-n}$. For the second part, given a map $g : V \to M$ define $g_n = \sum g : V[[h]]/h^n V[[h]] \to M/h^n M$ and take the completion. □

Quantum universal enveloping algebras (again)

Definition

A **quantum universal enveloping algebra** is a deformation algebra A such that $A/hA \cong U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$.

Define $U_h(\mathfrak{sl}_2)$ as the deformation algebra generated by E, F, H (for $q = e^h$) such that $\varepsilon(E) = \varepsilon(F) = \varepsilon(H) = 0$,

$$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}, [H, E] = 2E, [H, F] = -2F$$

with $\Delta(E) = E \otimes q^H + 1 \otimes E$, $\Delta(F) = F \otimes 1 + q^{-H} \otimes F$, $\Delta(H) = H \otimes 1 + 1 \otimes H$; $S(E) = -Eq^H$, $S(F) = q^H F$, $S(H) = -H$.

Lemma

The deformation algebra $U_h(\mathfrak{sl}_2)$ is a QUEA: $U_h(\mathfrak{sl}_2)/hU_h(\mathfrak{sl}_2) = U(\mathfrak{sl}_2)$.

Proof.

The proof is rather easy. The only thing you need is the power series of $\frac{q^H - q^{-H}}{q - q^{-1}}$. □

Quantum universal enveloping algebras

It is also possible to deform $U(\mathfrak{g})$ for a semisimple Lie algebra (or a Kac-Moody algebra in general).

Definition

The **Drinfeld-Jimbo quantum groups** are the h-deformations of $U(\mathfrak{g})$ generated by elements behaving like E_i, F_i, H_i, and some Serre relations.

$$[H_i, H_j] = 0, \qquad [H_j, E_i] = a_{ij} E_i, \qquad [H_j, F_i] = -a_{ij} F_i, \tag{9.2}$$

$$[E_i, F_j] = \delta_{ij} \frac{q^{d_i h H_i} - q^{-d_i h H_i}}{q^{d_i h} - q^{-d_i h}}, \tag{9.3}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \tag{9.4}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} F_i^{1-a_{ij}-k} F_j F_i^k = 0, \tag{9.5}$$

(from Etingof & Schiffmann)

Structures on QUEA

As K-modules, we always have $A \cong U(\mathfrak{g})[[h]]$ and $A^{\hat{\otimes} n} \cong (U(\mathfrak{g})^{\otimes n})[[h]]$. By the previous proposition, the maps $\mu, \iota, \Delta, \varepsilon$ are all determined by their restrictions to $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\mathbb{C}$, etc., which are maps of the form

$$\mu(a \otimes b) = \sum \mu_n(a \otimes b) h^n$$

for any $a, b \in U(\mathfrak{g})$, $\mu_n : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ and μ_0 the multiplication in $U(\mathfrak{g})$. Furthermore, the universal R-matrix on A is given by (consider the first line of this slide)

$$R = \sum R_n h^n, \quad R_n \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

such that $R_0 = 1 \otimes 1$. Now if we write $R = 1 \otimes 1 + rh + O(h^2)$, we may define

Definition

The **quasiclassical limit** of R is the matrix $r = R_1 \in A \otimes A$. And R is a **quantization** of r (from now r no longer means a form).

More about quantization

Lemma

Let $R \in (A \otimes A)[[h]]$ *be a solution to the QYBE for some associative algebra* A *such that* $R_0 = 1 \otimes 1$. *Then the quasiclassical limit of* R *is a solution to the classical YBE:*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

Write $R_{21} = \sigma(R)$. Define $t \in A \otimes A$ to be the quasiclassical limit of RR_{21}. Then easily we see that $t = r + r_{21}$ and thus t is symmetric.

Proposition

Let (H, R) *be a quasitriangular quantum universal enveloping algebra quantizing* $U(\mathfrak{g})$ *and* r *be the quasiclassical limit of* R. *Then* **canonical** 2-**tensor** $t = r + r_{21} \in (S^2(\mathfrak{g}))^{\mathfrak{g}}$ *in the sense that (the commutator!)*

$$[\Delta(x), t] = 0, \quad \forall x \in \mathfrak{g}$$

More about quantization

To understand the theorem, we will need the vague idea of a **Lie bialgebra**, a Lie algebra with a map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta([X, Y]) = (\mathrm{ad}_X \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_X)\delta(Y) - (\mathrm{ad}_Y \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}_Y)\delta(X)$$

If H is a quantization of $U(\mathfrak{g})$ then we may define on $\mathfrak{g}$

$$\delta(a) = \frac{\Delta(\tilde{a}) - \sigma(\Delta(\tilde{a}))}{h} \pmod{h}$$

where $\tilde{a}$ is any lift of a to H. Fact: δ turns $\mathfrak{g}$ into a Lie bialgebra.
Step 0: Use the Hexagon relations to show that $(\Delta_0 \otimes \mathrm{id})(r) = r_{13} + r_{23}$ and $(\mathrm{id} \otimes \Delta_0)(r) = r_{12} + r_{13}$. Deduce that $r \in \mathfrak{g} \otimes \mathfrak{g}$ and $t \in S^2(\mathfrak{g})$.
It remains to see that $t = r + r_{21}$ is invariant.
Step 1: Show that if R is a universal R-matrix of H and r its quasiclassical limit, then $\delta(a) = \partial r := [\Delta(a), r]$.
Step 2: Use Step 1 and the skew-symmetric construction of δ, show that $[\Delta(x), t] = 0$.

Example II

Let $\mathfrak{g}$ be the Heisenberg Lie algebra generated by x, y, z such that $[x, y] = z$, $[x, z] = [y, z] = 0$. Then $t = z \otimes z \in (S^2(\mathfrak{g}))^{\mathfrak{g}}$ since z sits inside the center of $\mathfrak{g}$. Let H be the bialgebra $U(\mathfrak{g})[[h]]$ and $R = e^{ht/2}$. Then

Proposition

The algebra (H, R) is a quantum universal enveloping algebra of $\mathfrak{g}$ whose canonical 2-tensor quantizes t.

Proof:

Example II

Example II