The Belavin-Drinfeld Classification

September 29, 2023

Review of Lukas' talk: the CYBE

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The (unsolved) goal is to classify all solutions of the classical Yang-Baxter equation. Let $\mathfrak{g}$ be an arbitrary semisimple finite-dimensional Lie algebra over $\mathbb{C}$.

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A solution of the classical Yang-Baxter equation is a meromorphic function $r : U \to \mathfrak{g} \otimes \mathfrak{g}$ where $U \subseteq \mathbb{C}^2$ is an open nbhd of the origin satisfying

$$[r^{12}(u_1, u_2), r^{13}(u_1, u_3)] + [r^{12}(u_1, u_2), r^{23}(u_2, u_3)] + [r^{13}(u_1, u_3), r^{23}(u_2, u_3)] = 0$$

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When $r(u_i, u_j)$ only depends on $u_i - u_j$, we may rewrite the equation

CYBE (with conditions)

We look for meromorphic functions $r : U \to \mathfrak{g} \otimes \mathfrak{g}$ where $U \subseteq \mathbb{C}$ is an open nbhd of the origin satisfying

$u = u_1 - u_2 \quad v = u_2 - u_3$

$$[r^{12}(u), r^{13}(u+v)] + [r^{12}(u), r^{23}(v)] + [r^{13}(u+v), r^{23}(v)] = 0$$

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Let κ be the killing form on $\mathfrak{g}$, $\{I_\mu\}$ the orthogonal basis of $\mathfrak{g}$ wrt κ, and $\Omega = \sum_\mu I_\mu \otimes I_\mu$ the Casimir element.

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Definition

A tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$ is **nondegenerate** if $\det(\varphi(t)) \neq 0$. A meromorphic function $r : U \to \mathfrak{g} \otimes \mathfrak{g}$ is **nondegenerate** if there is some $u_0 \in U$ such that $r(u_0)$ is nondegenerate.

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Theorem (Equivalence of nondegeneracy)

The following statements are equivalent for a solution r of the CYBE:

1. *The solution r is nondegenerate,*
2. *The function r has at least one pole in U and there are no proper Lie subalgebras $\mathfrak{a}$ of $\mathfrak{g}$ such that $r(u) \in \mathfrak{a} \otimes \mathfrak{a}$ for any u.*
3. *The function has a pole of order 1 at 0 with residue $\lambda\Omega, \lambda \in \mathbb{C}^\times$.*

Nondegeneracy II

Proof.

Step 1: If r is nondegenerate and holomorphic, then one could show that $r(0)$ would be a constant nondegenerate solution (which does not exist).

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Step 2: If an $\mathfrak{a}$ of such exists then at all u where r is holomorphic, we have $\mathrm{im}(\varphi(r(u))) \subseteq \mathfrak{a}$, which means $r(u)$ is nondegenerate if and only if $\mathfrak{a}$ is $\mathfrak{g}$. But r is nondegenerate so we are done (i $\Longrightarrow$ ii).

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Step 3: The idea is to establish some Lie bracket relations between r and the residue of an arbitrary pole using the CYBE (taking limits). Then show that the image of φ(residue) is an ideal and thus $\mathfrak{g}$ as $\mathfrak{g}$ is simple, which means θ is nondegenerate. The computation of the order of the pole is difficult and omitted. To compute the residue one can define a subalgebra using $\varphi(\theta)$ and show that, via Schur's lemma, $\varphi(\theta)$ must be $\lambda\,\mathrm{id}$ to get $\mathfrak{a} = \mathfrak{g}$. By the definition of φ we see that $\theta = \lambda\Omega$ (ii $\Longrightarrow$ iii).

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Step 4: Take the limit $\lim \overbrace{u\varphi(r(u))}^{\in \mathrm{End}(\mathfrak{g})}$ when $u \to 0$ and use the linearity of φ to get $\lambda\,\mathrm{id}$. Then $\lim \det(u\varphi(r(u)))$ is nonzero, meaning $\det(\varphi(r(u)))$ is not zero around 0.

$\lim u\, r(u) = \lambda\Omega$ $\qquad$ $\varphi(\lambda\Omega) = \lambda\,\mathrm{id}$

$\square$

Nondegeneracy III

Lemma

A nondegenerate solution is unitary $r(u) + r^{21}(-u) = 0$.

$r^{21} = \sigma \circ r$

Nondegeneracy III

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Proof.

Write $CYBE(u_1, u_2, u_3)$ as the sum

$$[r^{12}(u_{12}), r^{13}(u_{13})] + [r^{12}(u_{12}), r^{23}(u_{23})] + [r^{13}(u_{13}), r^{23}(u_{23})]$$

where $u_{ij} = u_i - u_j$.

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where $u_{ij} = u_i - u_j$. The CYBE says $CYBE(u_1, u_2, u_3) = 0$. Now we may switch without changing the equality u_1, u_2 and the first two components of each r^{ij}. This gives us $CYBE(u_2, u_1, u_3)^{213} = 0$. Summing the two equalities up we get
$[r^{12}(u_{12}) + r^{21}(u_{21}), r^{13}(u_{13}) + r^{23}(u_{23})] = 0$

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Constant solutions I

Constant CYBE

Let $\mathfrak{g}$ be a simple Lie algebra. A **constant solution** of the CYBE is some $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that

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Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ so that we may decompose $\mathfrak{g}$ into $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Let Γ be the set of simple roots of $\mathfrak{g}$ and $e_\alpha, f_\alpha, h_\alpha$ be the Chevalley generators.

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Definition

A **Belavin-Drinfeld triple** is a triple (Γ_1, Γ_2, T) where Γ_1, Γ_2 are subsets of Γ and T a bijection preserving the bilinear form on roots such that for all $\alpha \in \Gamma_1$, there exists some $k > 0$ such that $T^k(\alpha) \in \Gamma_2 \backslash \Gamma_1$. We say $\alpha \prec \beta$ if there exists $k > 0$ such that $T^k(\alpha) = \beta$.

Constant solutions II

For any $x \in \mathfrak{g} \otimes \mathfrak{g}$, write x_0 for its components in $\mathfrak{h} \otimes \mathfrak{h}$.

Theorem (Belavin-Drinfeld classification of constant r-matrices)

Let $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfy

$$r_0 + r_0^{21} = \Omega_0$$

(Γ_1, Γ_2, T) : discrete

r_0 : continuous

Then

$$r = r_0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{0 < \alpha \prec \beta} e_{-\alpha} \wedge e_\beta$$

where $x \wedge y = x \otimes y - y \otimes x$ is a solution to the CYBE. Furthermore, any constant solution $\tilde{r}$ is equivalent to an r-matrix of this form in the sense that there exists a $\psi \in \operatorname{Aut} \mathfrak{g}$ such that $\tilde{r} = (\psi \otimes \psi)(r)$.

$\tilde{r} + \tilde{r}^{21} = \Omega$

$r(u) = \dfrac{\Omega}{e^u - 1} + r$

Constant solutions III

Proof.

Recall the map φ and define $f = \varphi(r)$, i.e., $r = (f \otimes \mathrm{id})(\Omega)$.

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Step 1: We first prove that the CYBE and the Ω_0 condition are equivalent to, respectively,

$$[f(x), f(y)] = f\left([x, f(y)] - [f^*(x), y]\right), \; f + f^* = 1$$

where f^* is the adjoint of f wrt to the Killing form.

①

② Its Cayley trans— is 0

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Step 2: We define the Cayley transform of f as the map
$\theta : \mathrm{im}(f-1)/\ker f \to \mathrm{im}\, f/\ker(f-1)$ given by

$$\theta: C_2/C_2^{\perp} \xrightarrow{\ \sim\ } C_1/C_1^{\perp}$$

$$\theta = \frac{f}{f-1}$$

$$(f-1)(w) + \ker f \mapsto f(w) + \ker(f-1)$$

Then show that $\ker f = \mathrm{im}(f-1)^{\perp}$, $\ker(f-1) = \mathrm{im}\, f^{\perp}$ and the map θ is orthogonal. Furthermore, f satisfies the two equalities iff $\mathrm{im}\, f$, $\mathrm{im}(f-1)$ are Lie and θ is an iso. Write $C_1 = \mathrm{im}\, f$ and $C_2 = \mathrm{im}(f-1)$. □

Constant solutions III

Proof.

Step 3: Let L be a subset of simple roots, we define

$$\mathfrak{h}_L = \bigoplus_{\alpha \in L} \mathfrak{h}_\alpha, \mathfrak{g}_L = \bigoplus_{\alpha \in \mathbb{Z}L} \mathfrak{g}_\alpha \oplus \mathfrak{h}_L, \mathfrak{n}_L^\pm = \bigoplus_{\alpha > 0 \notin \mathbb{Z}L} \mathfrak{g}_{\pm\alpha}$$

Write $\mathfrak{p}_L^\pm = \mathfrak{g}_L \oplus \mathfrak{n}_L^\pm$. It can be shown that $(\mathfrak{p}_L^\pm \oplus V)^\perp \subseteq \mathfrak{p}_L^\pm \oplus V$. Now we define $C_1 = \mathfrak{p}_{\Gamma_1}^+ + V_1$ and $C_2 = \mathfrak{p}_{\Gamma_2}^- + V_2$ for any $\Gamma_i \subseteq \Gamma$ and some $V_1, V_2 \subseteq \mathfrak{h}$. The simplest iso $\theta : C_1/C_1^\perp \to C_2/C_2^\perp$ can be induced by a Killing form preserving bijection $T : \Gamma_1 \to \Gamma_2$:

$$\theta(X_\alpha) = X_{T(\alpha)}, X = e, f, h$$

Constant solutions III

$V^{\perp} \cap \mathfrak{h}_L^{\perp} \subseteq V$

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Step 4.1: The final step is complicated. We need to construct a map $f : \mathfrak{g} \to \mathfrak{g}$ whose Cayley transform is θ. This f is defined by parts $f_0 : \mathfrak{h} \to \mathfrak{h}$ and $f_{\pm} : \mathfrak{n}_{\pm} \to \mathfrak{n}_{\pm}$. ∎

Constant solutions IV

Proof.

Step 4.2: In order for f to have the Cayley transform θ, we must have $f_\bullet + f_\bullet^* = 1$, $f_\pm - 1$ both invertible and $f_\pm/(f_\pm - 1) = \psi^\pm$ where $\psi^\pm = \theta$ on $\mathfrak{g}_\alpha$ for $\alpha \in \mathbb{Z}^\pm\Gamma_1$ and zero otherwise. However, ψ^+ is invertible if and only if for any $\alpha \in \Gamma_1$ there exists $k > 0$ such that $T^k(\alpha) \in \Gamma_2 \backslash \Gamma_1$ (Easy! Try this!).

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Step 5: Let (Γ_1, Γ_2, T) be a BD-triple, and choose some $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ in the theorem. This r_0 defines a f_0 via φ. Define $f_+ = \frac{\psi^+}{\psi^+ - 1}$ and $f_- = 1 - f_+^*$. Then $f = \varphi(r)$ where r is given by the theorem, and θ is the Cayley transform of f.

$f = f_0 - \sum_{n\geq 0} (\psi^+)^n + \sum_{m \geq 0} (\psi^+)^{*m}$.

Step 6: To prove the second statement, we start with an r-matrix r and $f = \varphi(r) : \mathfrak{g} \to \mathfrak{g}$. Using kernels of $(f - \lambda)^n$ we will be able to decompose $\mathfrak{g}$ into $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ preserved by f. Define $C_1 = \operatorname{im}(f - 1)$ and $C_2 = \operatorname{im} f$ and θ the Cayley transform of f. Then there exist Γ_1, $\Gamma_2 \subseteq \Gamma$ such that $C_1 = \mathfrak{p}^+_{\Gamma_1} + V_1$ and $C_2 = \mathfrak{p}^-_{\Gamma_2} + V_2$ and a map T defined by $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{T(\alpha)}$. The triple (Γ_1, Γ_2, T) is a BD-triple. □

CYBE with a spectral parameter I

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Let $r : U \to \mathbb{C} \otimes \mathbb{C}$ be a nondegenerate meromorphic solution to the CYBE, then

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- $r(u) = \frac{\lambda\Omega}{u} + O(1)$ *around* 0,
- $r(u) + r^{21}(-u) = 0.$

Furthermore, if $\Gamma = 0$, r *is equivalent to a rational function* f *in* u*; if* $\Gamma \cong \mathbb{Z}$ *(rank 1),* r *is equivalent to a trigonometric function* $f(e^u)$*; if* $\Gamma \cong \mathbb{Z}^2$ *then* r *is equivalent to an elliptic function. The equivalence is given by a meromorphic function* $\varphi : U \to \mathrm{Aut}\,\mathfrak{g}$ *such that* $(\varphi(u) \otimes \varphi(v))r(u-v)$ *is the desired form in* $u - v$.

CYBE with a spectral parameter II

Lemma

For $u, v \in U$, $r(u+v)$ is a rational function of $r(u)$ and $r(v)$.

Theorem (Weierstrass)

Let r be a meromorphic function such that there exists some rational function f with

$$r(u+v) = f(r(u), r(v))$$

Then r is rational, trigonometric or elliptic.

CYBE with a spectral parameter II

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Then r *is rational, trigonometric or elliptic.*

To understand the poles of r we need the following lemma

Lemma

For any pole z *of* r, *there exists some* $A_z \in \mathrm{Aut}\,\mathfrak{g}$ *such that*

$$r(u+z) = (A_z \otimes 1) r(u)$$

CYBE with a spectral parameter III

$\lambda\Omega$

Remark

But then we have $\lim_{u\to 0} ur(u+z_1+z_2) = (A_{z_1}A_{z_2}\otimes 1)\lim_{u\to 0} ur(u)$, which means r has a pole at z_1+z_2 as well: Γ is a discrete subgroup of $\mathbb{C}$.

CYBE with a spectral parameter III

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Now the image of r is multi-dimensional, so we need the following generalization:

Theorem (Myberg)

Let $r : \mathbb{C} \to \mathbb{C}^n$ be a meromorphic function satisfying $r(u + v) = f(r(u), r(v))$ for some rational function f. Then there exists $n > 0$, a qausiabelian function $\tilde{r}$ and a vector $x \in \mathbb{C}^n$ such that $r(u) = \tilde{r}(u \cdot x)$, and $\tilde{r}(x + y) = f(\tilde{r}(x), \tilde{r}(y))$.

CYBE with a spectral parameter III

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By qausiabelian we mean there is a coordinate system $(z_1,\ldots,z_n)$, integers $p+q+r=n$ and vectors $x_1,\ldots,x_{2r}$ such that if $z_{p+q+1},\ldots,z_n$ are fixed, the function is rational in terms of $z_1,\ldots,z_p, e^{z_{p+1}},\ldots,e^{z_{p+q}}$ and the vectors are linearly independent over $\mathbb{R}$ periods of the function.

CYBE with a spectral parameter IV

Proof.

Elliptic case: It is relatively easier when Γ has rank 2. Let z_1, z_2 be two LI generators of the lattice. It suffices to show that r has periods nz_1, mz_2 for some $n, m > 0$, i.e., $A_{z_1}^n = A_{z_2}^m = 1$

CYBE with a spectral parameter IV

Proof.

Elliptic case: It is relatively easier when Γ has rank 2. Let z_1, z_2 be two LI generators of the lattice. It suffices to show that r has periods nz_1, mz_2 for some $n, m > 0$, i.e., $A_{z_1}^n = A_{z_2}^m$ $= 1$. But the group generated by A_{z_1}, A_{z_2} cannot be infinite in $\operatorname{Aut}\mathfrak{g}$, otherwise there is some $a \in \mathfrak{g}$ such that $ha = a$ for all $h \in \langle A_{z_1}, A_{z_2} \rangle$ (fact in algebraic groups). Then the function $(a^\vee \otimes 1)r$ is Γ-periodic with all poles simple in Γ, which is impossible by the residue theorem. □

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An interesting result in Lie algebra is that if φ_1, φ_2 are two automorphisms of $\mathfrak{g}$ of finite order, then $\mathfrak{g} \cong \mathfrak{sl}_2$.

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An interesting result in Lie algebra is that if φ_1, φ_2 are two automorphisms of $\mathfrak{g}$ of finite order, then $\mathfrak{g} \cong \mathfrak{sl}_n$. Thus elliptic solutions only exist when $\mathfrak{g} \cong \mathfrak{sl}_n$.

Proof.

Other cases: The trigonometric and rational cases require the Myberg theorem. The proof is not complicated, but quite technical. □

Some examples

Example

The functions $r(u) = \Omega/u$ (Yang's r-matrix) and $r(u) = \Omega/u + r_0$ for some skew-symmetric constant r-matrix r_0 are clearly rational solutions.

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Let r_0 be a constant solution of the CYBE such that $r_0 + r_0^{21} = \Omega$. Then the Baxterization of r_0, defined by

$$r(u) = \frac{r_0 + e^u r_0^{21}}{e^u - 1}$$

is a trigonometric solution to the CYBE.

Explicit formulas of trigonometric solutions

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