# The Faddeev-Reshetikhin-Takhtajan Construction

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# Review I

## Definition

Recall that we say a Hopf algebra $A$ is **quasitriangular** if there is a **universal** $R$**-matrix** $R \in A \otimes A$ that is invertible and satisfies

- For all $a \in A$, $(\sigma \circ \Delta)(a) = R\Delta(a)R^{-1}$;
- $(\Delta \otimes \mathrm{id})(R) = R^{13}R^{23}$ and $(\mathrm{id} \otimes \Delta)(R) = R^{13}R^{12}$.

Note that here if $R = \sum a_i \otimes b_i$ then $R^{13}$ means $\sum a_i \otimes 1 \otimes b_i$, etc. Note that given two $A$-modules $V$ and $W$, we may define an isomorphism $c^R_{V,W} : V \otimes W \to W \otimes V$ $c^R_{V,W}(v \otimes w) = \sum a_i w \otimes b_i v$ In fact, for any three $A$-modules $U, V, W$,

$$(c^R_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c^R_{U,W})(c^R_{U,V} \otimes \mathrm{id}_W)$$
$$= (\mathrm{id}_W \otimes c^R_{U,V})(c^R_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c^R_{V,W})$$

which means $c^R_{V,V}$ is a solution to the QYBE over any $A$-module $V$ (the *universal* $R$-matrix can be used to generate $R$-matrices).

# Review II

In Bobby's talk, we also defined *dual quasitriangular Hopf algebras*:

**Definition**

A **dual quasitriangular Hopf algebras** $A$ is a Hopf algebra over $k$ with a **universal** $r$**-form** $r : A \otimes A \to k$ and another linear form $r^{-1}$ such that

- $r * r^{-1} = r^{-1} * r = \varepsilon \otimes \varepsilon$;
- $\mu \circ \sigma = r * \mu * r^{-1}$;
- $r \circ (\mu \otimes \mathrm{id}) = r_{13} * r_{23}$, $r \circ (\mathrm{id} \otimes \mu) = r_{13} * r_{12}$.

where the involution $*$ means $f * g = (f \otimes g)(\Delta(x))$. If you write $c^R_{V,W}$ in the last slide in terms of maps and dualize the diagram, you will get an isomorphism $c^r_{V,W} : V \otimes W \to W \otimes V$ satisfying a similar relation. In particular, $c^r_{V,V}$ is a solution to the QYBE.

*Exercise:* Define the map $c^r_{V,W}$.

# The FRT construction I

The FRT construction, first proposed in Faddeev, N.Reshetikhin and Takhtajan 1988, provides a way to reverse the process we reviewed. That is, given an $R$-matrix, we construct a quasitriangular Hopf algebra with a universal $R$-matrix that induces the orginal $R$-matrix.

## Theorem (FRT bialgebra version)

*Suppose $V$ is a vector space of dimension $N$ and $c \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ is a solution to the QYBE. Then there exists a dual quasitriangular bialgebra $A(c)$ generated by intermediates $T_i^j$ (for $i, j = 1, \ldots, N$) with a unique universal $r$-form on $A(c) \otimes A(c)$ such that $c^r_{V,V} = c$ and*

$$r(T_i^m \otimes T_j^n) = c_{ji}^{mn}$$

*where $c(v_i \otimes v_j) = \sum c_{ij}^{mn} v_m \otimes v_n$.*

Q: Why do we want dual quasitriangular bialgebras? *Exercise:* Show that if finite-dimensional (Hopf or bialgebra) $H$ is quasitriangular, its dual $H^*$ is dual quasitriangular.

# The FRT construction I

The bialgebra $A(c)$ here is constructed as the algebra generated by $T_i^j$ modulo the ideal generated by

$$C_{ij}^{mn} = \sum_{k,l \leqslant N} c_{ij}^{kl} T_k^m T_l^n - \sum_{k,l \leqslant N} T_i^k T_j^l c_{kl}^{mn}$$

for $i, j, m, n = 1, \ldots, N$. We also need $\varepsilon(T_i^j) = \delta_{ij}$. The $r$-form and its inverse can be defined by

$$r(T_i^m \otimes T_j^n) = c_{ji}^{mn}, \quad r^{-1}(T_i^m \otimes T_j^n) = (c^{-1})_{ij}^{mn}$$

and extended to the whole algebra. It is *nontrivial* to check it's well-defined and a universal $r$-form, but it's tedious so we won't go over the proof. Please refer to Kassel for a very detailed proof.

Now we explore the quantization of the algebra of regular functions $\Gamma(GL_2) = k[x, y, z, w, t]/(t \cdot \det -1)$ and $\Gamma(SL_2) = k[x, y, z, w]/(\det -1)$.

# Example I

In the commutative case, the algebra of regular functions on the algebraic group $GL_2$ is given by $k[x, y, z, w, t]/(t \cdot \det -1)$ where $\det = xw - yz$. To quantize it, we no longer work with the commutative ring $k[x, y, z, w]$. Instead, let

$$M_q(2) = k\{a, b, c, d\}/J_q$$

where $k\{a, b, c, d\}$ is the free algebra generated by $a, b, c, d$ (noncommutative) and $J_q$ the two-sided ideal generated by relations

$$ba = qab, db = qbd, ca = qac, dc = qcd, bc = cb, ad - da = (q^{-1} - q)bc$$

*Exercise:* Show that $\det_q = ad - q^{-1}bc$ is central in $M_q(2)$ (i.e., it commutes with all elements). Before we proceed, let's record the "multiplication table" in $M_q(2)$. Let $\alpha = q - q^{-1}$.

$$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \end{bmatrix} \begin{bmatrix} a^2 & b^2 & ab & ba \\ c^2 & d^2 & cd & dc \\ ac & bd & ad & bc \\ ca & db & cb & da \end{bmatrix} = \begin{bmatrix} a^2 & b^2 & ab & ba \\ c^2 & d^2 & cd & dc \\ ac & bd & ad & bc \\ ca & db & cb & da \end{bmatrix} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \end{bmatrix}$$

# Example I

We may similarly define $GL_q(2) = M_q(2)[t]/(t \cdot \det_q - 1)$ and $SL_q(2) = M_q(2)/(\det_q - 1)$.

## Lemma

*Define*

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I$$

*Then $M_q(2), GL_q(2)$ and $SL_q(2)$ are bialgebras. Furthermore, the latter two are Hopf algebras with the antipode $S(t) = t^{-1}$ and*

$$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det_q^{-1} \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}.$$

## Proof.

This can be left as an exercise. Careful: for $GL_q(2)$ and $SL_q(2)$ we need to extend $\Delta(t) = t \otimes t$ and $\varepsilon(t) = 1$. Moreover, we need to check that they are well-defined on the quotients (easy). $\square$

# Example I

Let $V$ be a two dimensional vector space and $c$ an automorphism of $V \otimes V$ described, with respect to some basis, by the matrix

$$c = q^{-1/2} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \end{bmatrix}$$

Let $T_1^1 = a, T_1^2 = b, T_2^1 = c$ and $T_2^2 = d$. The relations defining $A(c)$ are given by (convince yourself this is true!)

$$\begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \end{bmatrix} \begin{bmatrix} a^2 & b^2 & ab & ba \\ c^2 & d^2 & cd & dc \\ ac & bd & ad & bc \\ ca & db & cb & da \end{bmatrix} = \begin{bmatrix} a^2 & b^2 & ab & ba \\ c^2 & d^2 & cd & dc \\ ac & bd & ad & bc \\ ca & db & cb & da \end{bmatrix} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \end{bmatrix}$$

## Corollary

*The bialgebra $M_q(2)$ is isomorphic to $A(c)$ which is dual quasitriangular.*

## Proof.

The FRT construction. □

# Example I

## Corollary

*The universal $r$-form on $A(c)$ expressed in the matrix form is $c$ with its last two rows switched.*

## Proof.

Super easy exercise. But you need to make sure your basis looks like $a \otimes a, b \otimes b, a \otimes b, b \otimes a, c \otimes c, \dots, d \otimes b, c \otimes b, d \otimes a$. □

## Proposition

*As a result, the Hopf algebras $GL_q(2)$ and $SL_q(2)$ are dual quasitriangular with the universal $r$-form $r$.*

## Proof.

It suffices to show that $r$ is well-defined on the quotients. We need to use the fact that $r(xy \otimes \det_q) = r(x \otimes \det_q) r(y \otimes \det_q)$. See the next slide. □

# Example I

## Proof.

First, to construct $r$ in *the proof of the FRT construction*, one must show that in order to make $r$ a universal $r$-form we must have (convince yourself according to the axioms)

$$r(x \otimes yz) = \sum r(x_{(1)} \otimes y) r(x_{(2)} \otimes z), r(xy \otimes z) = \sum r(x \otimes z_{(1)}) r(y \otimes z_{(2)})$$

and $r(1 \otimes T_i^j) = r(T_i^j \otimes 1) = \varepsilon(T_i^j) = \delta_{ij}$ which means for any $x \in A(c)$ we have

$$r(1 \otimes x) = r(x \otimes 1) = \varepsilon(x)$$

The goal is to show that $r((1 - \det_q) \otimes x) = r(x \otimes (1 - \det_q)) = 0$ so that $r$ is trivial on the ideal generated by $1 - \det_q$.
*Step 1:* Verify that $\Delta(\det_q) = \det_q \otimes \det_q$ with a direct computation.
*Step 2:* Use Step 1 and the previous discussion to show that $r(x \otimes \det_q) = r(\det_q \otimes x) = \varepsilon(x)$ for $x = a, b, c, d$ and thus any $x$.
*Step 3:* Complete the proof with Step 2 and the properties of $r$. $\square$