Hopf Algebras

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$$\mu \circ (\mathrm{id} \otimes \iota) = \mu \circ (\iota \otimes \mathrm{id}) = \mathrm{id} \qquad \text{(identity)}$$
$$\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}) \qquad \text{(associativity)}$$

$$\begin{tikzcd}
A \otimes R \arrow[r, "\mathrm{id} \otimes \iota"] \arrow[d, equal] & A \otimes A \arrow[d, "\mu"] \\
A \arrow[r, "\mathrm{id}"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
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Coalgebras

In algebraic terms, the unit map sends $r \in R$ to $\iota(r) = r1$ in A. To construct the dual diagrams of algebra structures, we may define a coalgebra to be

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$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} \qquad \text{(coidentity)}$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \qquad \text{(associativity)}$$

$$\begin{tikzcd} A \otimes R & A \otimes A \arrow[l, "\mathrm{id} \otimes \varepsilon"'] \\ A \arrow[u, equal] & A \arrow[l, "\mathrm{id}"'] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\mathrm{id} \otimes \Delta"'] \\ A \otimes A \arrow[u, "\Delta \otimes \mathrm{id}"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

Structures on algebras and coalgebras

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$$\begin{tikzcd} A\otimes A \arrow[r, "\sigma"] \arrow[dr, "\mu"'] & A\otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A\otimes A & A\otimes A \arrow[l, "\sigma"'] \\ & A \arrow[ul, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

An algebra homomorphism is an R-linear map $\varphi : A \to B$ such that $\varphi \circ \mu_A = \mu_B \circ (\varphi \otimes \varphi)$ and $\varphi \circ \iota_A = \iota_B$. Naturally a coalgebra homomorphism is a map $\psi : A \to B$ such that $\Delta_B \circ \psi = (\psi \otimes \psi) \circ \Delta_A$ and $\varepsilon_B \circ \psi = \varepsilon_A$. A **coideal** I of a coalgebra A is a submodule such that $\Delta(I) \subseteq I \otimes A + A \otimes I$ (1) and $\varepsilon(I) = 0$. (2)

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$$(f \otimes f)(\Delta(x)) = 0 \Rightarrow \Delta(x) \in \ker(f \otimes f)$$

But $\ker(f \otimes f) = \ker f \otimes A + A \otimes \ker f$ so we must have

$$\Delta(x) \in \ker f \otimes A + A \otimes \ker f$$

$$\Delta(I) \subseteq \underset{I}{\overset{\shortparallel}{\ker f}} \otimes A + A \otimes \underset{I}{\overset{\shortparallel}{\ker f}}$$

Exercise: using the same argument, show that to achieve our goal, we must have $\varepsilon(I) = 0$.

Hopf Algebras

Definition

An R-module A is a **Hopf algebra** if it is both an algebra and a coalgebra such that

bialgebra

1. The multiplication μ and unit ι maps are coalgebra homomorphisms.
2. The comultiplication Δ and counit ε maps are algebra homomorphisms.
3. There is an bijective R-bilinear **antipode** map $S : A \to A$ such that

$$\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \mu \circ (S \otimes \mathrm{id}) \circ \Delta = \iota \circ \varepsilon$$

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We sometimes call the Hopf algebras **"quantum groups"**, but usually the term refers to a Hopf algebra that is neither commutative nor cocommutative.

Some easy exercises to help you understand the formalism

- Show that given Condition 3, Condition 1 is equivalent to Condition 2 in the previous definition.
- A homomorphism of Hopf algebra is a map that is both an algebra and a coalgebra homomorphism. Show that it must commute with the antipode.
- If a Hopf algebra is commutative or cocommutative then the antipode map satisfies $S^2 = \mathrm{id}$.
- A cocommutative *finite-dimensional* Hopf algebra over an algebraically closed field of char zero is the group algebra of some finite group [*Hint:* use the third exercise and the group algebra example in the next slide.] $\mu^* : A^* \to (A \otimes A)^* \supseteq A^* \otimes A^*$
- Prove that the vector space dual of a finite-dimensional Hopf algebra over some field k is a Hopf algebra. [*Hint:* pullback the structure maps of the original bialgebra; $(V \otimes V)^* = V^* \otimes V^*$.]

$(V^*)^* \cong V$ $\quad\quad$ $(V^\circ)^\circ \not\cong V$

Examples: global sections of group schemes and group algebras of finite groups I

We first set up some group objects. Let k be a nice enough field and R a finitely generated commutative k-algebra.

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The algebraic group is equipped with three morphisms:

$$\mu : G \times_k G \to G, i : G \to G, e : \operatorname{Spec} k \to G$$

corresponding to the group structure on the finite group has

$$\cdot : G \times G \to G, \mathrm{inv} : G \to G, 1_G$$

(the "normal" group operations on finite groups).

Examples: global sections of group schemes and group algebras of finite groups II

Since G is affine, $\Gamma(G \times_k G) = \Gamma(G) \otimes \Gamma(G)$. Thus, the three morphisms induce the following pullbacks on k-modules:

$$\Delta : R \to R \otimes R, \varepsilon : R \to k, S : R \to R$$

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If G is a finite group, then we can define

$(f \otimes f)(x, y) = f(xy)$ $S(f)(g) = f(g^{-1})$

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In any case, $\Gamma(G)$ is a Hopf algebra.

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Exercise: Using relations of the three morphisms on the algebraic group G (or the usual group structure on a finite group), show that $(R, \Delta, \varepsilon, S)$ is a Hopf algebra.

Universal enveloping algebras I

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Definition

An **enveloping algebra** of $\mathfrak{g}$ is a pair (U, φ) where U is a unital associative algebra equipped with the Lie bracket $[a, b] = ab - ba$ and $\varphi : \mathfrak{g} \to U$ a Lie algebra homomorphism (regarding U as a Lie algebra). A **universal enveloping algebra** is an enveloping algebra $(U(\mathfrak{g}), \Phi)$ with the universal property (in the category of enveloping algebras of $\mathfrak{g}$).

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Clearly a universal enveloping algebra is unique up to isomorphism if it exists.

Lemma

There exists a universal enveloping algebra of any Lie algebra $\mathfrak{g}$.

Construction: Let $T = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$. Define $U(\mathfrak{g}) = T/\sim$ where the equivalence relation is given by $a \otimes b - b \otimes a = [a, b]$ for $a, b \in \mathfrak{g}$.

Universal enveloping algebras II

If $\mathfrak{g}$ is finite dimensional, let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. Write $[X_i, X_j] = \sum c_{ijk} X_k$.

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Theorem (Poincaré-Birkhoff-Witt)

Given a Lie algebra $\mathfrak{g}$ and an ordered basis $\{x_1, x_2, \ldots\}$ of $\mathfrak{g}$, the monomials $x_{i_1}^{e_1} \cdots x_{i_r}^{e_r}$ form a basis of $U(\mathfrak{g})$. In particular, the basis of $\mathfrak{g}$ is linearly independent in $U(\mathfrak{g})$, so the map $\Phi : \mathfrak{g} \to U(\mathfrak{g})$ is injective.

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nice Hopf algebra $= \overbrace{U(g)}^{\text{cocommutative}} \ltimes \underbrace{\mathbb{C}G}_{\text{commutative}}$

$\Delta(f) = f \otimes f$ grouplike

Exercise: On the universal enveloping algebra of $\mathfrak{g}$, let $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$ and $\varepsilon(x) = 0$. Show that $U(\mathfrak{g})$ is a cocommutative Hopf algebra.

$(\sigma \circ \Delta)(x) = \sigma(x \otimes 1 + 1 \otimes x) = 1 \otimes x + x \otimes 1 = \Delta(x)$

Quantum deformation I

$\mathbb{R}^2$

Let's consider a particle moving on a line. In classical mechanics, we study the **state space** of the position x and momentum p and the **algebra of observables** $\mathcal{O}(X) = \mathbb{C}[x, p]$ generated by differentiable complex functions on X. The Poisson bracket of the generators is $\{x, p\} = 1$, and the observables commute.

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Not operators in general

$$\lim_{\hbar \to 0} \frac{[f_1, f_2]_{\star_\hbar}}{\hbar} = \{f_1, f_2\}$$

Slogan for quantizaion

To make quantum is to make less commutative than before.

Moyal, Weyl and Groenewold: on the algebra of functions, define a noncommutative product $\star_h$ such that $[f_1, f_2]_{\star_h} = ih\{f_1, f_2\} + O(h^3)$.

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Drinfeld: when X is replaced by a Poisson-Lie group G, we deform the "dual" of $C^\infty(G)$.

Quantum deformation II

Some (not necessarily equivalent) dualities between geometry and algebra:

1. Affine k-schemes $\leftrightarrow$ Finitely generated commutative k-algebras
2. Compact topological space $\leftrightarrow$ C^*-algebras of continuous $\mathbb{C}$-functions
3. Smooth manifolds $\leftrightarrow$ Smooth functions $\hookrightarrow$ Commutative algebras
4. Lie groups $\leftrightarrow$ Lie algebras

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Fact: Given a semisimple, connected and simply-connected linear algebraic group G over an algebraically closed field of char zero, $\Gamma(G, \mathcal{O}_G)$ is isomorphic to the **restricted dual**

$$U(\mathfrak{g})^\circ = \{\alpha \in U(\mathfrak{g})^* : \mu^*(\alpha) \in U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*\}$$

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less commutative

To quantize G, we may deform $U(\mathfrak{g})^*$ containing $C^\infty(G)$, which is equivalent to the quantum deformation of $U(\mathfrak{g})$.

less cocommutative

Quantum deformation of $U(\mathfrak{sl}_2)$

Kulish and Sklyanin: Let $q = e^{\hbar} \in \mathbb{C}$ nonzero, not equal to ± 1. Let $U_q(\mathfrak{sl}_2)$ be the quantum deformation generated by $E, F, K = q^H$ and K^{-1} such that

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$[e,f] = h$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \quad KX = q^2 XK, KY = q^{-2} YK$$

with $\Delta(E) = E \otimes K + 1 \otimes E$, $\Delta(F) = F \otimes 1 + K^{-1}F$ and $\Delta(K) = K \otimes K$; $\varepsilon(E) = \varepsilon(F) = 0$ and $\varepsilon(K) = 1$.

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Note that the generator $K = q^H$ here is just an abstract generator, and we want it to have the properties of q^H:

$$q^H = \sum_{n=0}^{\infty} \frac{H^n \log(q)^n}{n!}, [H,E] = 2E, [H,F] = -2F$$

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$$q^H = \sum_{n=0}^{\infty} \frac{H^n \log(q)^n}{n!}, [H, E] = 2E, [H, F] = -2F$$

Exercise: derive the last two relations using the expansion of q^H and the relations of E, F, H in $\mathfrak{sl}_2$. Show that $\lim_{\hbar \to 0} U_q(\mathfrak{sl}_2) = U(\mathfrak{sl}_2)$.

Philosophy: Tannaka duality

Suppose we are working with a Hopf algebra A over some algebraically closed field k. On the category A-Mod and we have a forgetful functor $\mathcal{F} : A\text{-Mod} \to \mathsf{Vec}_k$. Then $\underbrace{\mathrm{End}(\mathcal{F})}_{=\{\text{natural transformations } \mathcal{F} \to \mathcal{F}\}} \cong A$ as associative unital algebras.

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fiber functor

$$(\dagger)\ \mathcal{F} \text{ is exact faithful } \implies \mathrm{End}(\mathcal{F}) \cong A \text{ as bialgebras}$$

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Assuming $(\dagger)$, we have

(left duals and right duals)

$$A\text{-Mod is rigid } \implies \mathrm{End}(\mathcal{F}) \cong A \text{ as Hopf algebras}$$

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$$A\text{-Mod is rigid} \implies \mathrm{End}(\mathcal{F}) \cong A \text{ as Hopf algebras}$$
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Assuming $(\dagger)$, we have

$$\begin{aligned} A\text{-Mod is rigid} &\implies \mathrm{End}(\mathcal{F}) \cong A \text{ as Hopf algebras} \\ A\text{-Mod is rigid and symmetric} &\implies \mathrm{End}(\mathcal{F}) \text{ is cocommutative} \\ A\text{-Mod is rigid and braided} &\implies \text{What structures on } \mathrm{End}(\mathcal{F}) \cong A? \end{aligned}$$

Philosophy: Tannaka duality

Suppose we are working with a Hopf algebra A over some algebraically closed field k. On the category A-Mod and we have a forgetful functor $\mathcal{F} : A\text{-Mod} \to \mathsf{Vec}_k$. Then $\mathrm{End}(\mathcal{F}) \cong A$ as associative unital algebras. We can require more structures on this monoidal $(\otimes)$ category to get stronger isomorphisms.

$$(\dagger)\ \mathcal{F} \text{ is exact faithful} \implies \mathrm{End}(\mathcal{F}) \cong A \text{ as bialgebras}$$

Assuming $(\dagger)$, we have

$$A\text{-Mod is rigid (duals)} \implies \mathrm{End}(\mathcal{F}) \cong A \text{ as Hopf algebras}$$
$$A\text{-Mod is rigid and symmetric} \implies \mathrm{End}(\mathcal{F}) \text{ is cocommutative}$$
$$A\text{-Mod is rigid and braided} \implies \text{What structures on } \mathrm{End}(\mathcal{F}) \cong A?$$

The algebra $\mathrm{End}(\mathcal{F}) \cong A$ has a **quasi-triangular structure** that satisfies the quantum Yang-Baxter equation! This structure is called the **universal R-matrix** of A and it measures the non-cocommutativity of the algebra.