Notes on Sheaves and Schemes

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Contents

0 A Few Words

This is a note on the theory of schemes. It mixed up definitions from various sources. I have tried to pick the most succinct ones so the proofs and statements can be significantly simplified. This note is definitely not a well-written one, and is never intended to be one. It's more like a record of my learning process.

While reading/reviewing/entertaining using this note, one may observe an increasing use of category-theoretic languages and proofs. This funny pattern is due to my inability to think abstractly when first started reading the texts. The writer has to rely on explicit constructions and expressions to understand certain proofs. But after reading some relevant materials, I became familiar with certain formal properties of the objects, and no longer relied on explicit constructions.

1 Sheaves

1.1 Presheaves and sheaves

Let X be a topological space and Top_X be the category of open sets in X with inclusion maps as the only morphisms. Then a *presheaf* on X is just a contravariant functor $\mathscr{F} : \mathsf{Top}_X \to \mathcal{C}$ where $\mathcal{C}$ is an arbitrary category. Contravariant functors $\mathscr{F} : \mathcal{B} \to \mathcal{C}$ can be seen as (covariant) functors $\mathscr{F}' : \mathcal{B}^{\mathsf{op}} \to \mathcal{C}$ such that $Ff = F'f^{\mathsf{op}}$ (this is how Lean defines presheaves!). Hartshorne requires the functor to preserve initial objects. But this is considered as the "wrong" definition. Grothendieck only defines the presheaves as contravariant functors (see 3.1.1 in the 1971 Springer's version of EGA I).

We write $\rho_{VU} : \mathscr{F}(V) \to \mathscr{F}(U)$ to be the morphism associated to $\iota_{UV} : U \hookrightarrow V$, giving it the name *the restriction map*. For a fixed open set U of X, the group $\mathscr{F}(U)$ is called the *section* of $\mathscr{F}$ over U, and we denote by $s|_V$ for the element $\rho_{UV}(s)$ for all s in the section (this notation has no ambiguity since we know to which $\mathscr{F}(U)$ s belongs).

The *stalk* $\mathscr{F}_p$ of a presheaf $\mathscr{F}$ at some point $p \in X$ is defined to be the direct limit

$$\mathscr{F}_p = \varinjlim_{U \ni p} \mathscr{F}(U)$$

over all open nhds of p (some may define it over any filtered set of open nhds around p, but the direct limit is isomorphic to this one). A morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is simply a natural transformation of $\mathscr{F}$ to $\mathscr{G}$. That is, for any open sets U, V, φ

carries the morphisms $\varphi(U)$ and $\varphi(V)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \Big\downarrow{\scriptstyle \rho^{\mathscr{F}}_{UV}} & & \Big\downarrow{\scriptstyle \rho^{\mathscr{G}}_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

Elements of the stalk at p are equivalent classes $[(U, s)]$ where $s \in \mathscr{F}(U)$ and $[(U, s)] = [(V, t)]$ if and only if there is an open subset $W \subseteq U \cap V$ such that $s|_W = t|_W$ with additions defined by

$$[(U, s)] + [(V, t)] = [(W = U \cap V, s|_W + t|_W)]$$

For any U, there is a canonical map $\eta_U : \mathscr{F}(U) \to \mathscr{F}_p$ defined by $s \mapsto [(U, s)]$. This map is clearly a homomorphism.

Two easy consequences: for any $s \in \mathscr{F}(U)$ and $W \subseteq V \subseteq U$, $\varphi(U)(s)|_V = \varphi(V)(s|_V)$ and $(s|_V)\,|_W = s|_W$.

Then, for any morphism φ of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ induces a morphism of stalks $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$

$$\varphi_p[(U, s)] = [(U, \varphi(U)(s))]$$

This map is well-defined since if $[(U, s)] = [(V, t)]$, that is, there is a nhds W around p such that $s|_W = t|_W$ ($\rho^{\mathscr{F}}_{UW}(s) = \rho^{\mathscr{F}}_{VW}(t)$). Then

$$\begin{aligned} \varphi(U)(s)|_W &= (\rho^{\mathscr{G}}_{UW} \circ \varphi(U))(s) = (\varphi(W) \circ \rho^{\mathscr{F}}_{UW})(s) \\ &= (\varphi(W) \circ \rho^{\mathscr{F}}_{VW})(t) = (\rho^{\mathscr{G}}_{VW} \circ \varphi(V))(t) = \varphi(V)(s)|_W \end{aligned}$$

meaning $[(U, \varphi(U)(s))] = [(V, \varphi(V)(t))]$.

A presheaf $\mathscr{F}$ is a *sheaf* if and only if (1) for any open set U and open covering $\{U_i\}$ of U, if $s, t \in \mathscr{F}(U)$ are elements s.t. $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$ and (2) for any open set U and open covering $\{U_i\}$ of U, if $s_i \in \mathscr{F}(U_i)$ are elements such that for all i, j, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is an $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for each i. By the first condition, the resulting s is clearly unique. Therefore, a sheaf is essentially a presheaf which can be defined locally whose elements can be glued together. By the gluing condition, $\mathscr{F}(\emptyset)$ must be terminal.

We have the following theorem on sheaves:

Theorem 1.1.1. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on some topological space X, then $\varphi : \mathscr{F} \to \mathscr{G}$ is a natural isomorphism if and only if for each $p \in X$, the induced map φ_p is an isomorphism.*

Proof. Fix a $p \in X$. If φ is a natural isomorphism then each $\varphi(U)$ is an isomorphism. Therefore, for any $[(V,t)]$ of $\mathscr{F}_p$, φ_p admits a preimage $[(V,(\varphi(U))^{-1}(t))]$. Also, if $[(U,\varphi(U)(s))] = [(V,\varphi(V)(t))]$ then there is a nhds W of p such that $(\varphi(W) \circ \rho_{UW})(s) = (\varphi(W) \circ \rho_{VW})(t)$. But since $\varphi(W)$ is an isomorphism, $s|_W = t|_W$. Thus, $[(U,s)] = [(V,t)]$. Therefore φ_p is an isomorphism.

Conversely, suppose each φ_p is a isomorphism. Let $s,t \in \mathscr{F}(U)$ be elements such that $\varphi(U)(s) = \varphi(U)(t)$. Then since each φ_p is injective, the images satisfy $[(U,\varphi(U)(s))] = [(U,\varphi(U)(t))]$. Since φ_p is injective, $[(U,s)] = [(U,t)]$. Then there is a nhds $W_p \subseteq U$ of p such that $s|_{W_p} = t|_{W_p}$. The W_p form an open cover of U and thus by condition (1) of sheaves, we have $s = t$. Therefore, $\varphi(U)$ is a monomorphism.

To show that $\varphi(U)$ is surjective, let $t \in \mathscr{G}(U)$. Since φ_p is surjective, there exists some $[(V_p,s_p)]$ such that $\varphi_p([(V_p,s_p)]) = [(U,t)]$. Then $\varphi(V_p)(s_p)$ and $t|_{V_p}$ are two elements in $\mathscr{G}$ such that

$$[(V_p,\varphi(V_p)(s_p))] = [(V_p,t|_{V_p})]$$

since the LHS is $[(U,t)]$ and the RHS is the the restriction. Then there exists some nhds W_p of p such that

$$\varphi(W_p)(s_p|_{W_p}) = (\varphi(V_p)(s_p))\,|_{W_p} = \left(t|_{V_p}\right)|_{W_p} = t|_{W_p}$$

in $\mathscr{G}(W_p)$. Write $r_p = s_p|_{W_p}$. Now for any $p,q \in X$, the image of $r_p|_{W_p \cap W_q}$ under $\varphi(W_p \cap W_q)$ is

$$\varphi(W_p \cap W_q)(r_p|_{W_p \cap W_q}) = (\varphi(W_p)(r_p))\,|_{W_p \cap W_q} = \left(t|_{W_p}\right)|_{W_p \cap W_q} = t_{W_p \cap W_q}$$

and similarly for $r_q|_{W_p \cap W_q}$. Thus, as we have proved that $\varphi(W_p \cap W_q)$ is injective, $r_p|_{W_p \cap W_q} = r_q|_{W_p \cap W_q}$. By the gluing condition, there exists an element s of $\mathscr{F}(U)$ such that $s|_{W_p} = r_p$. Note for all p $\varphi(U)(s)|_{W_p} = \varphi(W_p)(s|_{W_p}) = \varphi(W_p)(r_p) = t|_{W_p}$, by the locality condition, $\varphi(U)(s) = t$. Therefore, $\varphi(U)$ is surjective, completing the proof. □

1.2 Sheafifications

Given a morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ with values in the category of abelian groups, we define the *presheaf kernel* of φ to be a map: $\ker\varphi : U \mapsto \ker\varphi(U)$. Similarly, the *presheaf cokernel* is $\operatorname{coker}\varphi : U \mapsto \operatorname{coker}\varphi(U)$ and the *presheaf image* of φ is the map $\operatorname{im}\varphi : U \mapsto \operatorname{im}\varphi(U)$. The contravariant functors $\ker\varphi, \operatorname{coker}\varphi$ and $\operatorname{im}\varphi$ send the morphisms $\iota_{UV} : U \to V$ in Top_X to restrictions of $\rho^{\mathscr{F}}_{VU}$, $\rho^{\mathscr{G}}_{VU}$ and $\rho^{\mathscr{G}}_{VU}$

to $\ker\varphi(V)$, $\operatorname{coker}\varphi(V)$ and $\operatorname{im}\varphi(V)$ respectively. Note that the codomain of these morphisms are kernels, cokernels and images. For instance, if $s \in \ker\varphi(U)$, then

$$0 = (\rho^{\mathscr{G}}_{UV} \circ \varphi(U))(s) = (\varphi(V) \circ \rho^{\mathscr{F}}_{UV})(s)$$

This means $s|_V$ is indeed in $\ker\varphi(V)$. Thus, $\ker\varphi$, $\operatorname{coker}\varphi$ and $\operatorname{im}\varphi$ are presheaves.

As a matter of fact, if φ is a morphism of sheaves, $\ker\varphi$ forms a sheaf. If $s, t \in \ker\varphi(U)$ and $\{U_i\}$ is an open cover of U then clearly $s|_{U_i} = t|_{U_i}$ for all i implies $s = t$ since the restrictions of s and t are exactly the same as their restrictions defined by the sheaf $\mathscr{F}$. If $s_i \in \ker\varphi(U_i)$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all i, j, we can produce an $s \in \mathscr{F}(U)$ via the locality of $\mathscr{F}$ such that $s|_{U_i} = s_i$ for each i. We claim that $s \in \ker\varphi(U)$. Since $0 = \varphi(U_i)(s|_{U_i}) = \varphi(U)(s)|_{U_i}$ for all i, we have $\varphi(U)(s) = 0$ by locality. Thus, $s \in \ker\varphi(U)$, completing the proof.

An important before giving the sheafification.

Lemma 1.2.1. *Two morphisms of sheaves $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ are equal if and only the induced morphisms agree on stalks*

$$\varphi_p = \psi_p, \quad \forall p \in X$$

Proof. The only if direction is obvious since the induced morphisms (which are induced by universal properties) are unique. Suppose that for all $\in X$, $\varphi_p = \psi_p$. Then for any open $U \subseteq X$, we want to show that $\varphi(U) = \psi(U)$. By definition of the induced morphism, for any $p \in U$ and $s \in \mathscr{F}(U)$,

$$[(U, \varphi(U)(s))] = \varphi_p[(U, s)] = \psi_p[(U, s)] = [(U, \psi(U)(s))]$$

Thus, there exists a nhds $p \in W_p \subseteq U$ such that $\varphi(U)(s)|_{W_p} = \psi(U)(s)|_{W_p}$. The family $\{W_p\}$ is an open cover of U and the locality of sheaves suggests $\varphi(U)(s) = \psi(U)(s)$. Therefore, $\varphi = \psi$, completing the proof. □

We introduce the *sheafification* of a presheaf in a "less category theory way".

Theorem 1.2.1 (sheafification)**.** *Given a presheaf $\mathscr{F}$, there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ such that for any sheaf $\mathscr{G}$ and morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi : \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$. Furthermore, $\mathscr{F}^+$ is unique up to (natural) isomorphisms.*

Proof. For any open set U, let $\mathscr{F}^+(U)$ be the set of functions $s : U \to \bigsqcup_{p\in U} \mathscr{F}_p$ such that for each p, $s_p \in \mathscr{F}_p$ and there exists a nhds $p \in V \subseteq U$ and $t \in \mathscr{F}(V)$ s.t. for all $q \in V$ we have

$$s_q = [(V, t)]_q$$

where we denote by s_p the value $s(p)$ and $[(\cdots)]_p$ the equivalence class of $(\cdots)$ in $\mathscr{F}_p$.

If $U \subseteq V$, $\mathscr{F}^+$ sends the inclusion to the restriction map: $(s : V \to \bigsqcup_{p\in V} \mathscr{F}_p) \mapsto (s|_U : U \to \bigsqcup_{p\in U} \mathscr{F}_p)$ (the codomain changed due to our condition on s_p). Therefore $\mathscr{F}^+$ is a presheaf. To show that $\mathscr{F}^+$ is a sheaf, let U be any open set and $\{U_i\}$ an open cover of U. Then since $\mathscr{F}^+(U)$ is a set of functions, the locality condition is immediate. If there exists $s_i \in \mathscr{F}^+(U_i)$ such that they coincides on intersections, then the function defined by $s_p = (s_i)_p$ (well-defined) is in $\mathscr{F}^+(U)$ because the nhds V_i produced by the conditions on the s_i can be directly taken to s.

We now construct θ and ψ. Each morphism $\varphi : \mathscr{F} \to \mathscr{G}$ induces morphisms on stalks: $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$. We can compose φ_p with the canonical injections of $\mathscr{G}_p$ into the disjoint union of them. Then by the universal property of disjoint unions, we have a map

$$\bigsqcup_{p\in U} \varphi_p : \bigsqcup_{p\in U} \mathscr{F}_p \to \bigsqcup_{p\in U} \mathscr{G}_p$$

Note for each $r \in \mathscr{F}_p$, $\left(\bigsqcup_{p\in U} \varphi_p\right)(r) = \varphi_p(r) \in \mathscr{G}_p$.

For any $s \in \mathscr{F}^+(U)$, we can compose it with $\sqcup_U \varphi_p$ to get a map $\varphi^*(U)(s)$ from $U \to \sqcup G_p$. Indeed, the resulting composition is in $\mathscr{G}^+(U)$: for any $p \in U$, there exists a nhds $p \in V_p \subseteq U$ and $t_p \in \mathscr{F}(V_p)$ such that for all $q \in V$, $s_q = [(V_p, t_p)]_q$. Now

$$\varphi^*(U)(s)_q = \varphi_q(s_q) = \varphi_q([V_p, t_p]_q) = [(V_p, \varphi(V_p)(t_p))]_q$$

Thus, the $\varphi(V_p)(t_p) \in V_p$ are the desired elements and $\varphi^* : \mathscr{F}^+ \to \mathscr{G}^+$ is a morphism of presheaves. (Clearly $\varphi^*(U)(s)|_V = \varphi^*(V)(s|_V)$ since the restrictions maps are nothing but restrictions of functions).

We define $\theta : \mathscr{F} \to \mathscr{F}^+$ to be the natural transformation $\theta(U) : s \mapsto (p \mapsto [(U, s)]_p)$. The function $p \mapsto [(U, s)]_p$ is in $\mathscr{F}^+(U)$ since $\{U\}$ would be the desired cover with s the desired element. Also, for any $q \in V \subseteq U$, $\left(\theta(U)(s)|_V\right)_q = [(U, s)]_q = [(V, s|_V)]_q = \left(\theta(V)(s|_V)\right)_q$. Thus, θ is a natural transformation. We also define the same θ' on $\mathscr{G}$ to $\mathscr{G}^+$. We claim that

Lemma 1.2.2. *For $\mathscr{F}$ a presheaf, the morphism θ defined above induces an isomorphism $\theta_p : \mathscr{F}_p \to \mathscr{F}_p^+$ on the stalk at each p.*

Proof. We first describe θ_p. It sends $[(U, s)]_p$ to $[(U, \theta(U)(s))]_p = [(U, q \mapsto [(U, s)]_q)]_p$. If $[(U, s)]_p$ and $[(V, t)]_p$ has the same image under θ_p, then there exists a nhds of p, $W \subseteq U \cap V$ such that

$$(q \mapsto [(U, s)]_q)|_W = (q \mapsto [(V, t)]_q)|_W$$

In particular, as $p \in W$, $[(U,s)]_p = [(V,t)]_p$, i.e., θ_p is injective. Now for any $[(U, q \mapsto [(U,s)]_q)]_p \in F_p^+$, there exists a nhds $V \subseteq U$ of p and some $t \in \mathcal{F}(V)$ such that for any $r \in V$,$(q \mapsto [(U,s)]_q)_r = [(U,s)]_r = [(V,t)]_r$. Obviously, $\theta_p([(V,t)]_p) = [(V, q \mapsto [(V,t)]_q)]$. But for all $r \in V$ we have $\left((q \mapsto [(V,t)]_q)|_V\right)(r) = [(V,t)]_r = [(U,s)]_r = \left((q \mapsto [(U,s)]_q)\right)|_V(r)$. Thus, the two functions agree on V, meaning they are in the same germ. Therefore, $\theta_p([(V,t)]_p) = [(U,s)]_p$. Thus, θ_p is an isomorphism. □

Now since $\mathscr{G}$ is a sheaf, we have proved that the bijectivities of all θ'_p gives the bijectivity of θ'. Therefore, $\theta' : \mathscr{G} \to \mathscr{G}^+$ is an isomorphism. Let

$$\psi = \theta'^{-1} \circ \varphi^* : \mathscr{F}^+ \to \mathscr{G}$$

We claim that $\varphi = \psi \circ \theta$. For any open U and $s \in \mathscr{F}(U)$, we have

$$\begin{aligned}(\varphi^* \circ \theta)(U)(s) &= (\varphi^*(U) \circ \theta(U))(s) = \varphi^*(U)(q \mapsto [(U,s)]_q) = q \mapsto \varphi_q([(U,s)]_q) \\ &= q \mapsto [(U, \varphi(U)(s))]_q = (\theta'(U) \circ \varphi(U))(s) = (\theta' \circ \varphi)(U)(s)\end{aligned}$$

Thus, $\varphi = \psi \circ \theta$.

It remains to show that ψ is unique. Let ψ' be another map such that $\varphi = \psi' \circ \theta$. By constructions, all quadrilateral and the leftmost triangular diagrams in the following diagram commute:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[rrr, "\alpha"] \arrow[dr, "\theta(U)"] \arrow[dd, "\varphi(U)"] & & & \mathscr{F}_p \arrow[dl, "\theta_p"] \arrow[dd, "\varphi_p"] \\
& \mathscr{F}^+(U) \arrow[r, "\gamma"] \arrow[dl, "\psi(U)"] & \mathscr{F}_p^+ \arrow[dr, "\psi_p"] & \\
\mathscr{G}(U) \arrow[rrr, "\beta"] & & & \mathscr{G}_p
\end{tikzcd}$$

Then

$$\varphi_p \circ \alpha = \beta \circ \varphi = \beta \circ \psi \circ \theta = \psi_p \circ \gamma \circ \theta = \psi_p \circ \theta_p \circ \alpha$$

for all canonical morphisms α. This means for any $[(W,r)]$, we have $\varphi_p \circ \alpha' = \psi_p \circ \theta_p \circ \alpha'$ where α' is the canonical morphism from $\mathscr{F}(W)$ to $\mathscr{F}_p$. Thus, $\varphi_p = \psi_p \circ \theta_p$. Similarly, $\varphi_p = \psi'_p \circ \theta_p$. Since each θ_p is isomorphic, $\psi_p = \psi'_p$ for all $p \in X$. But by Lemma 1.2.2, $\psi = \psi'$, completing the proof. □

The sheafification of a sheaf is obviously itself: the identity morphism of a sheaf $\mathscr{F}$ factors through $\mathscr{F}^+$: $\mathrm{id} = \varphi \circ \theta$ by Lemma 1.2.1. We have shown that the induced maps θ_x on stalks are isomorphisms in the proof above. But now both $\mathscr{F}$ and $\mathscr{F}^+$ are sheaves, by Theorem 1.1.1, $\theta : \mathscr{F} \to \mathscr{F}^+$ is an isomorphism of sheaves.

1.3 Subsheaves

A *sub(pre)sheaf* on X of an (abelian) (pre)sheaf $\mathscr{F}$ is a (pre)sheaf $\mathscr{F}'$ such that for any open $U \subseteq X$, $\mathscr{F}'(U)$ is a subgroup of $\mathscr{F}(U)$, the restriction maps $\rho_{UV}^{\mathscr{F}'}$ are the restrictions of $\rho_{UV}^{\mathscr{F}}$ and $\rho_{UV}^{\mathscr{F}}(\mathscr{F}'(U)) \subseteq \mathscr{F}'(V)$. Note the stalks $\mathscr{F}'_p$ are subgroups of $\mathscr{F}_p$ if we consider the universal property of limits.

If φ is a morphism of sheaves we call $\ker\varphi : U \mapsto \ker\varphi(U)$ the *kernel* of φ. We have shown it is in fact a sheaf. A morphism of sheaves is injective if $\ker\varphi = 0$, that is, the kernels of all $\varphi(U)$ are zero. The *image* of φ is the sheafification $\operatorname{im}\varphi$ of the presheaf image of φ (the sheafification of $U \mapsto \operatorname{im}\varphi(U)$). We know that φ factors through $\operatorname{im}\varphi$. We say that this morphism is surjective if $\operatorname{im}\varphi = \mathscr{G}$. We call the sheafification of the presheaf cokernel of φ the *cokernel* of φ, denoted by $\operatorname{coker}\varphi$.

If $\mathscr{F}'$ is a subpresheaf of the presheaf $\mathscr{F}$, the *quotient presheaf* $\mathscr{F}/\mathscr{F}'$ is the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$. If $\mathscr{F}'$ is a subsheaf of the sheaf $\mathscr{F}$, the *quotient sheaf* $\mathscr{F}/\mathscr{F}'$ is the sheafification of the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$.

Lemma 1.3.1. *For any $p \in X$, the stalk on the quotient presheaf at p is just*

$$(\mathscr{F}/\mathscr{F}')_p = \mathscr{F}_p/\mathscr{F}'_p$$

Proof. By definition the sequence on the top of the following diagram is exact for all U:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & (\mathscr{F}/\mathscr{F}')(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'_p \arrow[r] & \mathscr{F}_p \arrow[r] & (\mathscr{F}/\mathscr{F}')_p \arrow[r] & 0
\end{tikzcd}$$

Thus, as direct limits in the category of abelian groups preserve exactness, the lower sequence is exact. □

In Lemma 1.2.1 we have proved that the stalks on the sheafification of a presheaf are isomorphic to those on the presheaf, so the previous lemma (Lemma 1.3.1) also holds for stalks on quotient sheaves.

Let $f : X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X. The *direct image (or pushforward)* sheaf $f_*\mathscr{F}$ **on** Y is $V \mapsto \mathscr{F}(f^{-1}(V))$ (the locality and gluing conditions can be easily checked using the continuity of f: if V_i is a cover of Y then $f^{-1}(V_i)$ is a cover of X). For any sheaf $\mathscr{G}$ on Y, we call the sheafification $f^{-1}\mathscr{G}$ of the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$ the *inverse image (or pullback)* sheaf. If Z is a subset of X equipped with the subspace topology, then $\iota^{-1}\mathscr{F}$ is the *restriction* of $\mathscr{F}$ to Z where ι is the (continuous) inclusion map, denoted by $\mathscr{F}|_Z$.

Observe that stalk on the presheaf of $f^{-1}\mathscr{G}$ at any $p \in X$ is

$$\varinjlim_{U \ni p} \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$$

Elements of this stalk has the form $[(W, [(T, s)])]$ where $W \ni p$, $T \supseteq f(W) \ni f(p)$ and $s \in \mathscr{G}(T)$. We map this to $[(T, s)] \in \mathscr{G}_{f(p)}$. This map is well-defined since if the restrictions of two classes $[(T, s)]$ and $[(T', s')]$ agree on an open subset of $W \cap W'$, the restriction of s and s' agree on an open subset of $T \cap T'$ as the restriction maps are induced by direct limits. This map is obviously injective (since we can plug the output into the input) and surjective (the image of $[(f^{-1}(T), [(T, s)])]$ is $[(T, s)]$; $f^{-1}(T)$ is open by f's continuity). Thus, the stalks on the presheaf are isomorphic to $\mathscr{G}_{f(p)}$. By Lemma 1.2.1, the stalk of $f^{-1}\mathscr{G}$ at $p \in X$ is isomorphic $\mathscr{G}_{f(p)}$. In particular, the stalk of $\mathscr{F}|_Z$ at $p \in Z$ is nothing else but $\mathscr{F}_p$.

It is therefore important to remember that, if Y is open in X, then the restriction $\mathscr{F}|_Y$ contains the same data as $\mathscr{F}$: we have $\mathscr{F}|_Y(U) \cong \mathscr{F}(U)$ for all open $U \subseteq Y$. This is because the map $u : \varinjlim_{V \supseteq U} \mathscr{F}(V) \to \mathscr{F}(U)$ sending the class $[(V, t)]$ to $t|_U$ is an isomorphism. Clearly it is well-defined, injective (as restrictions being equal is exactly the definition of the equivalence classes) and surjective ($[(U, s)]$ for any $s \in \mathscr{F}(U)$). Therefore the sheafification of the restriction of the presheaf is just $\mathscr{F}$ itself, as a functor on $\mathsf{Top}_U^{\mathsf{op}}$ (which is still a sheaf since all sheaf conditions are local properties and can be passed to an open subspace).

1.4 Exploring the functoriality of sheafification, pullback, and pushforward

Let X be a topological space. The map $^+ : \mathsf{PSh}(X) \to \mathsf{Sh}(X)$. This map defines a functor which sends a morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ to a natural transformation carrying data $\varphi^+(U) : s \mapsto \sqcup_{p \in U} \varphi_p \circ s$ for some $s \in \mathscr{F}^+(U)$ and U open in X.

Let ι be the inclusion functor $\mathsf{Sh}(X) \to \mathsf{PSh}(X)$ from the category of sheaves on X to the category of presheaves on X. We claim that

Lemma 1.4.1. *The sheafification $^+$ is a left adjoint of the inclusion ι.*

Proof. It suffices to prove that for any presheaf $\mathscr{F}$ and sheaf $\mathscr{G}$, $\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathscr{F}^+, \mathscr{G}) = \mathrm{Hom}_{\mathsf{PSh}(X)}(\mathscr{F}, \iota(\mathscr{G}))$. This is an immediate result of Theorem 1.2.1: every morphism $\mathscr{F} \to \mathscr{G} = \iota(\mathscr{G})$ induces a **unique** morphism $\mathscr{F}^+ \to \mathscr{G}$! $\square$

Here I want to explore/summarize some basic properties of f_* and f^{-1} as functors, given a continuous map $f : X \to Y$. We have a pushforward functor $f_* : \mathsf{PSh}(X) \to$

$\mathsf{PSh}(Y)$. The image of objects under f_* is defined in the previous subsection (in which the restriction maps are $\eta_{UV} = \rho_{f^{-1}(U)f^{-1}V}$ for open sets $U, V \subseteq Y$). The morphisms are also clear: if $\mathscr{F}$ and $\mathscr{G}$ are two presheaves on X and $\varphi : \mathscr{F} \to \mathscr{G}$ morphism of presheaves, then $f_*\varphi$ would be a morphism of presheaves on Y, which carries the data $f_*\varphi(V) = \varphi(f^{-1}(V))$. These results are quite natural considering the definition of $f_*\mathscr{F}$. We have proved in the last subsection that the pushforward of sheaves are themselves sheaves, so f_* also gives a functor $\mathsf{Sh}(X) \to \mathsf{Sh}(Y)$.

Then we proceed to study the pullback functor f^{-1}. We first study the functor on presheaves, defined by $f^{-1}_{\text{pre}}\mathscr{G} : U \mapsto \varinjlim_{V \supseteq f(U)} \mathscr{G}(U)$ (actually now, at the time I'm writing this part, it seems natural to define the pullback in this way, as $f(U)$ is not guaranteed to be open) where $\mathscr{G}$ is a presheaf on Y. To understand the induced restriction maps one can refer to the subsection 1.1 for a similar definition. But we also quote a result in category theory here for clarity:

Lemma 1.4.2. *Consider diagrams $\mathscr{F} : \mathcal{I} \to \mathcal{C}$ and $\mathscr{G} : \mathcal{J} \to \mathcal{C}$ whose colimits exist and $\varphi : \mathcal{I} \to \mathcal{J}$ a functor with $\theta : \mathscr{F} \to \mathscr{G} \circ \rho$ a natural transformation, then there is a unique morphism $\theta_* : \varinjlim_{\mathcal{I}} \mathscr{F} \to \varinjlim_{\mathcal{J}} G$ such that the following diagram*

$$\begin{tikzcd}
\mathscr{F}(i) \arrow[r] \arrow[d, "\theta"'] & \varinjlim_{\mathcal{I}} \mathscr{F} \arrow[d, "\theta_*"] \\
\mathscr{G}(\theta(i)) \arrow[r] & \varinjlim_{\mathcal{J}} \mathscr{G}
\end{tikzcd}$$

commutes.

Proof. The proof is quite clear if we apply colimits' universal properties twice. Consider the natural transformation θ and the cocone $\left(\varinjlim_{\mathcal{J}} \mathscr{G}, t\right)$ of $\mathscr{G}$. For each index i we can define $r(i)$ to be the composition of morphisms $t(\varphi(i))$ and $\theta(i) : \mathscr{F}(i) \to \mathscr{G}(\varphi(i))$, getting a new cocone $\left(\varinjlim_{\mathcal{J}} \mathscr{G}, r\right)$ of $\mathscr{F}$ (the commutativity of the cocone can be checked easily). Then by the universal property of $\varinjlim_{\mathcal{I}} \mathscr{F}$, there is a unique map $\theta_* : \varinjlim_{\mathcal{I}} \mathscr{F} \to \varinjlim_{\mathcal{J}} G$ that commutes with the cocone morphisms if one decomposes each map $r(i)$ into $t(\varphi(i)) \circ \theta(i)$. $\square$

Remark. Lemma 1.4.2 can be used to induce restriction maps on $f^{-1}_{\text{pre}}\mathscr{F}$ with $\mathscr{F}$ on Y. Take open sets $V \subseteq U$ in X and index categories $\mathcal{I} = \{U' \supseteq f(U)\}$ and $\mathcal{J} = \{V' \subseteq f(V)\}$. Then clearly we have an inclusion functor: $\iota : \mathcal{I} \to \mathcal{J}$ since if

$U' \supseteq f(U)$ then it contains $f(V)$. We also have a natural transformation θ simply given by the restriction morphisms of $\mathscr{F}$, from $\mathscr{F}$ restricted to $\mathcal{I}$ to its restriction to $\mathcal{J}$. Then by Lemma 1.4.2 we get a unique morphism $\theta_* : f^{-1}_{\mathrm{pre}}\mathscr{F}(U) \to f^{-1}_{\mathrm{pre}}\mathscr{F}(V)$. Note that the resulting morphisms are associative by their uniqueness.

Lemma 1.4.3. *Given two topological spaces X, Y and a continuous map f, then the pullback of presheaves f^{-1}_{pre} is a left adjoint of f_*.*

Proof. We claim that, for any two presheaves $\mathscr{F}$ on X, $\mathscr{G}$ on Y,

$$\mathrm{Hom}_{\mathsf{PSh}(X)}(f^{-1}_{\mathrm{pre}}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_{\mathsf{PSh}(Y)}(\mathscr{G}, f_*\mathscr{F})$$

Take any open $V \subseteq Y$. Then V is an open nhds of $f(f^{-1}(V))$. Therefore, we have a canonical map $i_{\mathscr{G}}(V) : \mathscr{G}(V) \to f^{-1}_{\mathrm{pre}}\mathscr{G}(f^{-1}(V))$ (the cocone morphism given by the colimit). Therefore we have a canonical morphism: $i_{\mathscr{G}} : \mathscr{G} \to f_* f^{-1}_{\mathrm{pre}}\mathscr{G}$ (the images of restriction maps are also given by the induced maps on colimits).

Similarly, if $U \subseteq X$ is open, then for every open set V containing $f(U)$, there is a map $t(V) : f_*\mathscr{F}(V) \to \mathscr{F}(U)$ given by the restriction map of $\mathscr{F}$ from $f^{-1}(V)$ to U. These maps given by open nhds of $f(U)$ are clearly compatible with the restriction maps of $f_*\mathscr{F}$ (since they are all given by $\mathscr{F}$). Thus, we have a cocone $(\mathscr{F}(U), t)$ of $f_*\mathscr{F}$. And by the universal property of colimits we get a canonical map $j_{\mathscr{F}}(U) : f^{-1}_{\mathrm{pre}} f_*\mathscr{F}(U) \to \mathscr{F}(U)$.

Now for any morphism $\varphi : f^{-1}_{\mathrm{pre}}\mathscr{G} \to \mathscr{F}$, the corresponding map from $\mathscr{G} \to f_*\mathscr{F}$ is given by $f_*\varphi \circ i_{\mathscr{G}}$. For any morphism $\psi : \mathscr{G} \to f_*\mathscr{F}$, the corresponding map from $f^{-1}_{\mathrm{pre}}\mathscr{G} \to \mathscr{F}$ is given by $j_{\mathscr{F}} \circ f^{-1}_{\mathrm{pre}}\psi$.

It remains to check that these two constructions are inverses, i.e.,

$$j_{\mathscr{F}} \circ f^{-1}_{\mathrm{pre}}(f_*\varphi \circ i_{\mathscr{G}}) = \varphi$$

and

$$f_*(j_{\mathscr{F}} \circ f^{-1}_{\mathrm{pre}}\psi) \circ i_{\mathscr{G}} = \psi$$

The verification is quite trivial if one draws the commutative diagrams. It is only important to notice facts like $(j_{\mathscr{F}} \circ \eta)(V) = \mathrm{id}$ if $\eta(V)$ is the cocone map from $f_*\mathscr{F}(V) \to f^{-1}_{\mathrm{pre}} f_*\mathscr{F}(f^{-1}(V))$ (draw the cocones and the answer will be clear!). $\square$

By our definition of f^{-1} as $(f^{-1}_{\mathrm{pre}})^+$ together with Lemma 1.4.1 and Lemma 1.4.3, we have the following corollary:

Corollary 1.4.1. *The pullback functor f^{-1} is a left adjoint of f_* on sheaves.*

2 Spectra and Schemes

2.1 Sheaves of rings

Given a commutative ring with unity A and its $\operatorname{Spec} A$ with the Zariski topology. For any open set U of $\operatorname{Spec} A$, we let

$$\Delta(U) = A \backslash \bigcup_{\mathfrak{p} \in U} \mathfrak{p}$$

This set is multiplicative since if $a, b \in \Delta(U)$, $a, b \notin \mathfrak{p}$ and thus $ab \notin \mathfrak{p}$ for all $\mathfrak{p} \in U$. This means $ab \in \Delta(U)$ and obviously $1 \in \Delta(U)$, giving us a multiplicative set $\Delta(U)$. Let $\mathscr{O}'$, *the structure presheaf of* A, to be the functor defined by

$$\mathscr{O}' : U \mapsto \Delta(U)^{-1} A$$

and if $U \subseteq V$, the corresponding restriction map is

$$\rho_{UV}^{\mathscr{O}'} : \frac{a}{s} \mapsto \frac{a}{s}$$

since $U \subseteq V$ gives us $\Delta(V) \subseteq \Delta(U)$. Clearly $\mathscr{O}'$ is a presheaf.

We define the *structure sheaf of the ring* A, $\mathscr{O}$, to be the sheafification of $\mathscr{O}'$. We make the following claim:

Lemma 2.1.1. *The stalk of $\mathscr{O}'$ at any prime ideal $\mathfrak{p}$ is just the localization $A_{\mathfrak{p}}$.*

Proof. If $\mathfrak{p} \in U$, then $\Delta(U) \subseteq \Delta(\mathfrak{p})$ which means we have a homomorphism

$$\varphi_{U,\mathfrak{p}} : \mathscr{O}'(U) \to A_{\mathfrak{p}}$$

defined by $\frac{a}{s} \mapsto \frac{a}{s}$. By the universal property of direct limits, this map induces a morphism $\mathscr{O}'_{\mathfrak{p}} \to A_{\mathfrak{p}}$. We show that $\varphi_{\mathfrak{p}}$ is an isomorphism. For any $[(U, \frac{b}{s})] \in \mathscr{O}'_{\mathfrak{p}}$ in the kernel of the map, we have

$$\varphi_{\mathfrak{p}}([(U, b/s)]) = 0 = \varphi_{U,\mathfrak{p}}(b/s) = b/s \in A_{\mathfrak{p}}$$

Thus, there exists a $t \in A \backslash \mathfrak{p}$ such that $tb = 0$. But then the restriction of b/s to $U \cap \{\mathfrak{p} : t \notin \mathfrak{p}\}$ is zero. Thus, $[(U, b/s)] = 0$ and $\varphi_{\mathfrak{p}}$ is injective. For any $\frac{a}{s} \in A_{\mathfrak{p}}$ with $s \notin \mathfrak{p} \in U$, we have therefore $\varphi_{U,\mathfrak{p}}(a/s) = a/s$. Therefore the image $[(U, a/s)]$ in $\mathscr{O}'_{\mathfrak{p}}$ is mapped to a/s by $\varphi_{\mathfrak{p}}$. Thus, $\varphi_{\mathfrak{p}}$ is an isomorphism. $\square$

Therefore, elements of the sheafification $\mathscr{O}$ are functions

$$s : U \to \bigsqcup_{\mathfrak{p}\in U} \mathscr{O}'_{\mathfrak{p}} = \bigsqcup_{\mathfrak{p}\in U} A_{\mathfrak{p}}$$

such that for any $\mathfrak{p} \in U$, $s_{\mathfrak{p}} \in A_{\mathfrak{p}}$ and there exists a nhds $\mathfrak{p} \in V \subseteq U$ and some $a, f \in A$ satisfying for all $\mathfrak{q} \in U$, $f \notin \mathfrak{q}$ and $s_{\mathfrak{q}} = a/f \in A_{\mathfrak{p}}$ (Hartshorne's definition). Clearly, such elements form a commutative ring with unity. We define the *spectrum* of A to be the pair $(\operatorname{Spec} A, \mathscr{O})$.

Denote by $D(f)$ the open complement $\operatorname{Spec} A \backslash V(f)$.

Lemma 2.1.2. *The open sets $D(f)$ form a basis of* $\operatorname{Spec} A$*.*

Proof. It suffices to show that (1) for any open set $\mathfrak{p} \subseteq \operatorname{Spec} A$, there exists some $f \in A$ such that $\mathfrak{p} \in D(f)$ and (2) for any $f, g \in A$, there is some $h \in A$ such that $D(h) \subseteq D(f) \cap D(g)$. For (1), let simply take an element $f \in A\backslash\mathfrak{p}$ (which exists as $\mathfrak{p}$ is prime) then $\mathfrak{p} \in D(f)$. For (2), observe that

$$D(f) \cap D(g) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p} \text{ and } g \notin \mathfrak{p}\} = D(fg)$$

so let $h = fg$, completing the proof. $\square$

Although we have proved that the stalk of the structure sheaf at a point $\mathfrak{p}$ is precisely $A_{\mathfrak{p}}$, but to get a clearer proof and some general results, we will do this again.

Lemma 2.1.3. *Given a ring A and its spectrum* $(\operatorname{Spec} A, \mathscr{O})$*, we have:*

(a) *For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathscr{O}_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$.*

(b) *For any element $f \in A$, we have $\mathscr{O}(D(f)) \cong A_f$.*

(c) *In particular $\mathscr{O}(\operatorname{Spec} A) = A$.*

Proof. (a) Let $\varphi : \mathscr{O}_{\mathfrak{p}} \to A_{\mathfrak{p}}$ be the homomorphism defined by $[(U, s)] \mapsto s_{\mathfrak{p}}$. This is a well-defined map since if $[(U, s)] = [(V, t)]$ then there is an open nhds $p \in W \subseteq U \cap V$ such that $s|_W = t|_W$. In particular as $\mathfrak{p} \in W$, $s_{\mathfrak{p}} = t_{\mathfrak{p}}$. Now for any element $\alpha \in A_{\mathfrak{p}}$ represented by a/f where $a, f \in A$ and $f \notin \mathfrak{p}$, we can define a function t that sends all element $\mathfrak{q} \in D(f)$ to the element represented by a/f in $A_{\mathfrak{q}}$ (recall the definition of $D(f)$). Then clearly $t \in \mathscr{O}(D(f))$. Then $[(D(f), t)] \in \mathscr{O}_{\mathfrak{p}}$ is the preimage of $t_{\mathfrak{p}} = a/f \in A_{\mathfrak{p}}$. Thus, φ is surjective. Take elements $[(U, s)]$ and $[(V, t)]$ of $\mathscr{O}_{\mathfrak{p}}$ such that $\varphi(s) = s_{\mathfrak{p}} = t_{\mathfrak{p}} = \varphi(t)$. By construction, there are $a, f \in A$ with a smaller nhds V of $\mathfrak{p}$ such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s_{\mathfrak{q}} = a/f$, and $b, g \in A$ with a smaller nhds V' of $\mathfrak{p}$ such that for all $\mathfrak{q} \in V'$, $g \notin \mathfrak{q}$ and $t_{\mathfrak{q}} = b/g$. So we can shrink to a smaller nhds $W = V \cap V'$ such that $[(U, s)] = [(W, s|_W)]$, $[(W, t)] = [(V, t|_W)]$. Our assumption suggests that $a/f = (s|_W)_{\mathfrak{p}} = (t|_W)_{\mathfrak{p}} = b/g$ in $A_{\mathfrak{p}}$, so there is some $r \notin \mathfrak{p}$ such that $r(ag - bf) = 0$. Therefore if $\mathfrak{q}$ is a prime ideal such that $f, g, r \notin \mathfrak{q}$, the classes represented by a/f and b/g in $A_{\mathfrak{q}}$ are the same. However the set of such points is exactly $D(f) \cap D(g) \cap D(r)$, which contains $\mathfrak{p}$. Therefore $s|_W$ and $t|_W$ agrees on a nhds of $\mathfrak{p}$, meaning $[(U, s)] = [(W, s|_W)] = [(W, t|_W)] = [(V, t)]$. Therefore φ is an isomorphism.

(b) Let $\psi : A_f \to \mathscr{O}(D(f))$ be the map defined by $a/f^n \mapsto (\mathfrak{p} \mapsto a/f^n \in A_{\mathfrak{p}})$. If $\psi(a/f^n) = \psi(b/f^m)$, then for every $\mathfrak{p} \in D(f)$, the image of a/f^n equals to the image of b/f^m in $A_{\mathfrak{p}}$. Therefore there exists an element $h_{\mathfrak{p}} \notin \mathfrak{p}$ such that $h_{\mathfrak{p}}(af^m - bf^n) = 0$. Let $\mathfrak{a} = \mathrm{Ann}(af^m - bf^n)$. Then clearly $\mathfrak{a} \not\subseteq \mathfrak{p}$ (since $h_{\mathfrak{p}} \in \mathfrak{a}$). Therefore, as $\mathfrak{p} \in D(f)$ is arbitrary, the definition of closed sets in the Zariski topology suggests $V(\mathfrak{a}) \cap D(f) = \emptyset$. Since $D(f) = \operatorname{Spec} A \backslash V(f)$, the empty set condition implies $V(\mathfrak{a}) \subseteq V(f)$. But this is true if and only if $\sqrt{(f)} \subseteq \sqrt{\mathfrak{a}}$ since the radical of an ideal is the intersection of all prime ideals containing that ideal. In particular, $f \in \sqrt{\mathfrak{a}}$. Thus, there exists a power of f, f^k such that

$$f^k(af^m - bf^n) = 0$$

But this means $a/f^n = b/f^m$ in A_f. Therefore ψ is injective.

To show that ψ is surjective, take some $s \in \mathscr{O}(D(f))$. We first produce a finite open cover of $D(f)$ of the form $\{D(h_i)\}$ for some $h_i \in A$ such that for all i, there exist $a_i \in A$ such that for each $\mathfrak{q} \in D(h_i)$, $s_{\mathfrak{q}} = a_i/h_i \in A_{\mathfrak{q}}$. By definition of $s \in \mathscr{O}(D(f))$ (that is, for any $\mathfrak{p} \in U$, $s_{\mathfrak{p}} \in A_{\mathfrak{p}}$ and there exists a nhds $\mathfrak{p} \in V \subseteq U$ and some $a, g \in A$ satisfying for all $\mathfrak{q} \in U$, $g \notin \mathfrak{q}$ and $s_{\mathfrak{q}} = a/g \in A_{\mathfrak{p}}$), there is an open cover $\{V_i\}$ of $D(f)$ such that for any i, there exists $a_i, g_i \in A_i$ with for all $\mathfrak{q} \in V_i$, $g_i \notin \mathfrak{q}$ and $s_{\mathfrak{q}} = a_i/g_i$. By Lemma 2.1.2, open sets $D(h)$ form a basis of the topology. In particular, each V_i is a union of some $D(h)$. Therefore we can redefine the open cover $\{V_i\}$ to be open sets of the form $D(h_i)$ for some $h_i \in A$. In this case the local condition does not change: we still have a_i, g_i inherited from the original V_i (satisfying for any i, there exists $a_i, g_i \in A_i$ with for all $\mathfrak{q} \in D(h_i)$, $g_i \notin \mathfrak{q}$ and $s_{\mathfrak{q}} = a_i/g_i$). Since for all $\mathfrak{q} \in D(h_i)$, we have $g_i \notin \mathfrak{q}$, so $D(h_i) \subseteq D(g_i)$; equivalently, $V(g_i) \subseteq V(h_i)$, meaning $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$. Thus, $h_i^n \in (g_i)$ for some n or $h_i^n = cg_i$ for some $c \in A$. In this case, $a_i/g_i = ca_i/h_i$. We redefine again $h_i = h_i^n$ and $a_i = ca_i$. Note $D(h_i) = D(h_i^n)$ so we still have an open cover $\{D(h_i)\}$ but in this case $s|_{D(h_i)} = a_i/h_i$. Observe that $D(f) \subseteq \cup D(h_i)$ iff $V(\sum(h_i)) = \cap V(h_i) \subseteq V(f)$. This is equivalent to say that

$\sqrt{(f)} \subseteq \sqrt{\sum(h_i)}$. In other words, there exist m and finitely many $b_i \in A$ such that

$$f^m = \sum_i b_i h_i$$

Therefore, $\sqrt{(f)}$ is contained in the radical of the sum of finitely many ideals (h_i), and therefore there is a finite open cover $D(h_1) \cup \cdots \cup D(h_r)$ of $D(f)$. On $D(h_i h_j) = D(h_i) \cap D(h_j)$, we have $a_i/h_i, a_j/h_j \in A_{h_i h_j}$ both representing s (on the whole set $D(h_i h_j)$). Therefore since $\psi : A_{h_i h_j} \to \mathscr{O}(D(h_i h_j))$ is injective, $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Hence for some k,

$$(h_i h_j)^k (a_i h_j - a_j h_i) = h_j^{k+1}(a_i h_i^k) - h_i^{k+1}(a_j h_j^k) = 0$$

Replace each h_i by h_i^{k+1} and each a_i by $a_i h_i^k$, we still have $s = a_i/h_i$ on $D(h_i)$, which is still a finite open cover of $D(f)$. Furthermore, we have $a_i h_j = a_j h_i$ for all i, j. Still, with the radical argument and the open cover condition, write the power $f^l = \sum b_i h_i$ for some $b_i \in A$. Let $a = \sum b_i a_i$. Then

$$h_j a = \sum b_i a_i h_j = \sum b_i a_j h_i = a_j f^l$$

Thus, a/f^l is represented by a_j/h_j on $D(h_j)$, meaning $\psi(a/f^l) = s$ (by the locality condition of sheaves on $\mathscr{O}(D(f))$!), completing the proof.

(c) Obviously $\operatorname{Spec} A = D(1)$ so by part (b) $\mathscr{O}(\operatorname{Spec} A) \cong A_1 \cong A$. $\square$

We usually denote by $\Gamma(U, \mathscr{O})$ the ring $\mathscr{O}(U)$.

2.2 Ringed spaces and locally ringed spaces

The pair $(X, \mathscr{O}_X)$ is called a *ringed space* if $\mathscr{O}_X : \mathsf{Top}_X^{\mathsf{op}} \to \mathsf{Comm}$ is a sheaf of rings on the topological space X (it's just a sheaf with values in the category of commutative ring, nothing more! The previous lemmas on the structure sheaf of a ring is a preparation for the definition of schemes). The sheaf $\mathscr{O}_X$ is called the *structure sheaf of* X. A morphism of ringed spaces is a pair $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathscr{O}_Y \to f_* \mathscr{O}_X$ a morphism of sheaves (on Y! It might feel bit strange to define a morphism of ringed spaces on two different topological spaces with a morphism of sheaves on one of them. So see Stacks project's definition of f-maps and the relevant lemma on morphisms to pushforward sheaves). So $f^\#$ is essentially a natural transformation whose components are homomorphisms $f^\#(U) : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$).

A ringed space $(X, \mathscr{O}_X)$ is said to be a *locally ringed space* if each stalk $\mathscr{O}_{X,p}$ is a local ring. A morphism $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed space is a morphism of ringed space such that for each $p \in X$, the induced map $f_p^\# : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$ (the idea why it's a map to the stalk of a sheaf on X is explained later; however, it should be agreed that there is an induced map from $\mathscr{O}_{Y,f(p)}$ to $(f_*\mathscr{O}_X)_{f(p)}$) is a *local homomorphism*, a homomorphism under which the image of the maximal ideal $\mathfrak{m}_Y$ of $\mathscr{O}_{Y,f(p)}$ is contained in the maximal ideal $\mathfrak{m}_X$ of $\mathscr{O}_{X,p}$ (see this lemma in the Stacks project for equivalent definitions of local ring maps; can be seen as an easy exercise).

We need to make it clear what this induced map $f_p^\#$ is. Explicitly, the map sends the class $[(V, t)]$ in $\mathscr{O}_{Y,f(p)}$ to the class $[(f^{-1}(V), f^\#(V)(t))]$ in $\mathscr{O}_{X,p}$. This map is clearly well-defined since the map $f^\#$ is compatible with restrictions maps. Then this map makes the diagram commute

$$\begin{tikzcd}
\mathscr{O}_Y(V) \arrow[r, "f^\#(V)"] \arrow[d] & \mathscr{O}_X(f^{-1}(V)) \arrow[d] \\
\mathscr{O}_{Y,f(p)} \arrow[r, "f_p^\#(V)"] & \mathscr{O}_{X,p}
\end{tikzcd}$$

To construct with categories, observe there exist maps

$$g_p(f^{-1}(V)) : \mathscr{O}_X(f^{-1}(V)) \to \mathscr{O}_{X,p}$$

which sends elements to its equivalent classes. Also note that these maps are compatible with the restriction maps (since stalks on X, i.e., colimits are required to be targets). Therefore, for any V containing $f(p)$, we can compose $f^\#(V)$ with $g_p(f^{-1}(V))$ to get a new map from $\mathscr{O}_Y(V)$ to $\mathscr{O}_{X,p}$, compatible with restriction maps. Hence by the universal property of colimits, there is a unique map $f_p^\# : \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$. Immediately one can deduce that this map aligns with the explicit description above.

It is important to make sure that we define the composition of two morphisms in this case (because Hartshorne did not!). Take ringed space $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ and $(Z, \mathscr{O}_Z)$, and morphisms $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\#) : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$. We define $(g, g^\#) \circ (f, f^\#) = (g \circ f, g_*(f^\#) \circ g^\#)$. In this case

$$g_*(f^\#) \circ g^\# : \mathscr{O}_Z \xrightarrow{g^\#} g_*\mathscr{O}_Z \xrightarrow{g^\#} g_*\mathscr{O}_Y \xrightarrow{g_*(f^\#)} g_*(f_*\mathscr{O}_X) = (g \circ f)_*\mathscr{O}_X$$

Here we are interpreting g_* as the *direct image functor*. We have defined the image of objects. Any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ is taken by the direct image functor f_*

to $f_*\varphi(V) = \varphi(f^{-1}(V))$. It can be easily checked that this is a morphism of the pushforward sheaves. Also, $g_*(f_*\mathscr{O}_X) = (g \circ f)_*\mathscr{O}_X$ since

$$g_*(f_*\mathscr{O}_X)(V) = f_*\mathscr{O}_X(g^{-1}(V)) = \mathscr{O}_X(g^{-1}(f^{-1}(V))) = \mathscr{O}_X((g \circ f)^{-1}(V))$$

To show that the composition is associative, see my comment under this answer. An *isomorphism* of ringed space is a pair $(f, f^\#)$ with a two-sided inverse, meaning it consists of a homeomorphism f and an isomorphism of sheaves $f^\#$ (on any open $V \subseteq Y$, $\mathrm{id}_V = (f_*(g^\#) \circ f^\#)(V) = f_*(g^\#)(V) \circ f^\#(V)$ and $\mathrm{id}_{g(V)} = (g_*(f^\#) \circ g^\#)(g(V)) = f^\#(V) \circ g^\#(g(V))$).

In remains to show that the composition of morphisms of locally ringed space is a morphism of locally ringed spaces. Given $f : X \to Y$ and $g : Y \to Z$, we want to show that their composition $g_*(f^\#) \circ g^\#$ induces local homomorphisms on stalks $(g \circ f)^\#_p : \mathscr{O}_{Z,g(f(p))} \to \mathscr{O}_{X,p}$. This is actually quite easy since the map $[(V, t)] \mapsto [(f^{-1}(g^{-1}(V)), (g_*(f^\#) \circ g^\#)(V)(t))]$ is the composition of $g^\#_{f(p)}$ and $f^\#_p$. We need to push $f^\#$ down considering the image under $g^\#_{f(p)}$ is the class $[(f^{-1}(V), -)]$.

Clearly, for any ring $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ is a locally ringed space: its stalk at $\mathfrak{p}$ is isomorphic to the local ring $A_\mathfrak{p}$. Then it's natural to ask whether the morphisms of these locally ringed spaces are related to the homomorphisms on the rings. Indeed, they are closely related:

Lemma 2.2.1. *Given rings A, B,*

(a) *If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a morphism of locally ringed spaces:*

$$(f, f^\#) : (\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}).$$

(b) *Any morphism of the locally ringed spaces $(f, f^\#)$ from $\operatorname{Spec} A$ to $\operatorname{Spec} B$ is induced by a homomorphism $\varphi : A \to B$.*

Proof. Let $\varphi : A \to B$ be a homomorphism of rings. Then this map induces a continuous map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ defined by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ (Grothendieck calls it the *map associated to* φ, with the notation ${}^a\varphi$ widely used by Holmes, etc). For any prime ideal $\mathfrak{p}$ of B, the map $\iota_B : b \mapsto b/1 \in B_\mathfrak{p}$ composed with φ produces a map $A \to B_\mathfrak{p}$. Then, by the universal property of localization and the fact that all elements in $\varphi^{-1}(\mathfrak{p})$ are mapped to a unit in $B_\mathfrak{p}$, there is a unique map $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$ such that $\iota_B \circ \varphi = \varphi_\mathfrak{p} \circ \iota_A$. In fact, we have $\varphi_\mathfrak{p}(a/g) = \varphi(a)/\varphi(g)$. For arbitrary open set $V \subseteq \operatorname{Spec} A$, the elements in $\mathscr{O}_{\operatorname{Spec} A}(V)$ are functions $V \to \sqcup_{\mathfrak{p} \in V} A_\mathfrak{p}$ satisfying a local property. Then for any $\mathfrak{p} \in f^{-1}(V)$ we have $f(\mathfrak{p}) \in V$. Applying s and its property, we get an element $(s \circ f)(\mathfrak{p}) \in A_{f(\mathfrak{p})}$. Again apply $\varphi_\mathfrak{p}$ one gets $(\varphi_\mathfrak{p} \circ s \circ f)(\mathfrak{p}) \in B_\mathfrak{p}$. Thus, we get a homomorphism of rings $f^\#(V) : s \mapsto (\mathfrak{q} \mapsto \varphi_\mathfrak{q} \circ s \circ f)$. It remains to check that the resulting function is in $\mathscr{O}(f^{-1}(V))$. For any $\mathfrak{p} \in f^{-1}(V)$, i.e., $f(\mathfrak{p}) \in V$, there exists an open nhds $f(\mathfrak{p}) \in W \subseteq V$ and $a, g \in B$ such that for all $\mathfrak{i} \in W$, $g \notin \mathfrak{i}$ and $s_\mathfrak{i} = a/g \in A_\mathfrak{i}$. Let $U = f^{-1}(V), W' = f^{-1}(W)$ which are open. Let $b = \varphi(a), h = \varphi(g)$. Then for any $\mathfrak{q} \in W'$, we have (1) $(s \circ f)_\mathfrak{q} = a/g$ since $f(\mathfrak{q}) \in W$; (2) $h \notin \mathfrak{q}$ by our construction; and (3) $(\varphi_\mathfrak{q} \circ s \circ f)_\mathfrak{q} = \varphi_\mathfrak{q}(a/g) = \varphi(a)/\varphi(g) = b/h \in B_\mathfrak{q}$. With the obvious compatibility with the restriction maps, i.e., restrictions of functions, we get a morphism $f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_{\operatorname{Spec} B}$.

For any $\mathfrak{p} \in \operatorname{Spec} B$, we have shown in Lemma 2.1.3 that there are isomorphisms $\psi_A : \mathscr{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \to A_{f(\mathfrak{p})}$ and $\psi_B : \mathscr{O}_{\operatorname{Spec} B, \mathfrak{p}} \to B_\mathfrak{p}$ defined by $[(U, s)] \mapsto s_{f(\mathfrak{p})}$ and $[(V, t)] \mapsto t_\mathfrak{p}$. Also recall the induced map $f^\#_\mathfrak{p} : O_{\operatorname{Spec} A, f(\mathfrak{p})} \to \mathscr{O}_{\operatorname{Spec} B, \mathfrak{p}}$ defined by $[(U, s)] \mapsto [(f^{-1}(U), f^\#(U)(s))]$. Writing out each map we see the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \arrow[r, "f^\#_\mathfrak{p}"] \arrow[d, "\psi_A"'] & \mathscr{O}_{\operatorname{Spec} B, \mathfrak{p}} \arrow[d, "\psi_B"] \\
A_{f(\mathfrak{p})} \arrow[r, "\varphi_\mathfrak{p}"] & B_\mathfrak{p}
\end{tikzcd}$$

Then since $\varphi_\mathfrak{p}$ is a local ring map (an easy result), $f^\#_\mathfrak{p}$ is a local ring map. Thus, $f^\#$ is a morphism of locally ringed spaces.

(b) Suppose we have a morphism of locally ringed spaces $(f, f^\#)$. Then $f^\#$'s value on $\operatorname{Spec} A$ as a map $\mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to \mathscr{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)$ gives us, by part (c) of Lemma 2.1.3, a homomorphism $\varphi : A \to B$. Then for any $\mathfrak{p} \in \operatorname{Spec} B$, the homomorphism $\varphi : A \to B$ is compatible with the map $f^\#_\mathfrak{p} : A_{f(\mathfrak{p})} \to B_\mathfrak{p}$. Note that an element b in B is sent via an isomorphism (proved) to the function $t \in \mathscr{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)$ which sends each $\mathfrak{p}$ in $D(1) = \operatorname{Spec} B$ to the class represented by $b/1$ in $B_\mathfrak{p}$. Then this function will be sent to the germ $[(D(1), t)]$ in the stalk at $\mathfrak{p}$. But then this germ is sent to $t_\mathfrak{p} = b/1 \in B_\mathfrak{p}$. Therefore, if ι_B, ι_A are the localization homomorphisms from $B \to B_\mathfrak{p}$ and $A \to A_{f(\mathfrak{p})}$ respectively, then $f^\#_\mathfrak{p} \circ \iota_A = \iota_B \circ \varphi$. Then by the universal property of localizations, the map $f^\#_\mathfrak{p}$ sends a/s to $\varphi(a)/\varphi(s)$. But the locality of this map shows the set $\{a/s : \varphi(a)/\varphi(t) \in \mathfrak{p}A_\mathfrak{p}\}$ is the maximal ideal $f(\mathfrak{p})A_{f(\mathfrak{p})}$. The clearly the preimage of $\mathfrak{p}$ under is precisely $f(\mathfrak{p})$, that is, $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$. Then by our argument in part (a), φ induces the map $f^\#$. $\square$

2.3 Affine schemes and schemes

An *affine scheme* is a locally ringed space $(X, \mathscr{O}_X)$ isomorphic to the spectrum of some ring A. A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ where every point $x \in X$ has a nhds U such that the subspace U with the restricted sheaf $\mathscr{O}_X|_U = \iota^{-1}\mathscr{O}_X$ is an affine scheme (note the stalks of restricted sheaves are the same as the stalks of $\mathscr{O}_X$, so they are also local rings, meaning each pair $(U, \mathscr{O}_X|_U)$ is indeed a locally ringed space). The space X is called the *underlying topological space* and $\mathscr{O}_X$ has been named its structure sheaf. A morphism of schemes is just a morphism of locally ringed space. (Grothendieck (in early versions of EGA) called (and Hartshorne used to call) the schemes "preschemes"; the term is then changed in the 1971 edition of EGA).

Instead of producing a topological space that is an (affine) scheme, we investigate the simplest affine schemes: the spectra of different rings.

Take any field k then the affine scheme $(\operatorname{Spec} k, \mathscr{O}_{\operatorname{Spec} k})$ consists of a topological space of only one point (the zero ideal) and the structure sheaf carrying only one piece of data: k (why is this the case? by construction we have $\mathscr{O}_{\operatorname{Spec} k}((0))$ the ring of functions which send (0) to elements in $k_{(0)} = k$; so this ring is isomorphic to k). Still take the field k but the spectrum $(\operatorname{Spec} k[x], \mathscr{O})$. The underlying topological space contain one point called the *generic point* corresponding to the zero ideal and *closed points* corresponding to the maximal ideals in $k[x]$ (which are precisely the nonzero prime ideals since $k[x]$ is a PID). We call this affine scheme the *affine line*, denoted by $\mathbb{A}^1_k$. When k is algebraically closed, the maximal ideals are linear and are thus in one-to-one correspondence with elements in k.

Consider an algebraically closed k. Define the *affine plane* $\mathbb{A}^2_k$ to be the spectrum $(\operatorname{Spec} k[x, y], \mathscr{O})$. The *closed points* (i.e., points correspond to maximal ideals) correspond to ordered pairs of elements in k (since the maximal ideals are of the form $(x - a, y - b)$). Note that the set of closed points $\operatorname{MaxSpec} k[x, y]$ is homeomorphic to the affine variety $\mathbb{A}^2$. (Each Zariski closed set $V(I)$ is mapped to $\{(x - a, y - b) : f(a, b) = 0, \forall f \in I\}$. Clearly $I \subseteq (x - a, y - b)$ so the image is $V(I)$ in $\operatorname{MaxSpec} k[x, y]$. The map is clearly invertible.) For each irreducible polynomial $f(x, y)$ there is a point η such that its closure consists of closed points $(x - a, y - b)$ such that $f(a, b) = 0$. The point η is called the *generic point* of the curve $f(x, t) = 0$.

Here is a counter-example for part (b) of Lemma 2.2.1 where the locality condition is not satisfied. Let R be a discrete valuation ring, i.e., a principal ideal domain with only one nonzero maximal ideal. Then the spectrum $\operatorname{Spec} R$ is an affine scheme with only two points: the zero ideal and the unique maximal ideal (prime ideals in PIDs are maximal). The stalk at the first point, the zero ideal, is the field of fractions

$K(R)$ of R. The stalk at the second point, the maximal ideal, is the ring itself which is local by assumptions (elements in the complement of maximal ideals are units so the localization homomorphism is an isomorphism). There is a morphism of ringed spaces: the map $f : \operatorname{Spec} K(R) \to \operatorname{Spec} R$ sending the zero ideal to the maximal ideal in $\operatorname{Spec} R$ and the morphism of sheaves $f^{\#} : \mathscr{O}_{\operatorname{Spec} K(R)} \to \mathscr{O}_{\operatorname{Spec} R}$ associated to the inclusion map as a morphism on stalks $R_{\mathfrak{m}} = R \hookrightarrow K(R) = K(R)_{(0)}$. This morphism $(f, f^{\#})$ is not induced by a homomorphism from R to $K(R)$ since the map on stalks is not local (we have shown that morphisms induced by homomorphisms must be morphisms of locally ringed space)!

2.4 Gluing sheaves and schemes

Before gluing schemes we first glue sheaves. Given a topological space X, an indexed family of open subsets $\{U_i \subseteq X_i\}_{i\in I}$ and sheaves $\mathscr{F}_i$ on X with isomorphisms $\varphi_{ij} : \mathscr{F}_j|_{U_i\cap U_j} \to \mathscr{F}_i|_{U_i\cap U_j}$ where $\varphi_{ii} = \mathrm{id}$ (to understand these maps see my comment in the end of 1.3). Let $\mathscr{F} : \mathsf{Top}_X^{\mathsf{op}} \to \mathsf{Comm}$ be the functor which maps $V \subseteq X$ to the ring of tuples

$$\mathscr{F}(V) = \left\{(s_i)_{i\in I} : s_i \in \mathscr{F}_i(V \cap U_i), \forall i, j, \varphi_{ij}(V \cap U_i \cap U_j)(s_j|_{V\cap U_i\cap U_j}) = s_i|_{V\cap U_i\cap U_j}\right\}$$

We claim that this is a sheaf on X. First it is clear that it's a presheaf: the set is a ring since each $\varphi_{ij}(V \cap U_i \cap U_j)$ is an isomorphism of rings (so we have additions, additive inverses, multiplications, etc.). Furthermore the restrictions maps are of the form $\rho_{UV} = (\rho_i)_{i\in I}$ where ρ_i is the restriction map of $U \cap U_i$ to $V \cap U_i$ in $\mathscr{F}_i$. If $(s_i)_{i\in I} \in \mathscr{F}(U)$ then we have $\rho_{UV}((s_i)) = (\rho_i(s_i))_{i\in I}$. But then by the commutativity of φ_{ij} with the restriction maps, $\varphi_{ij}(V \cap U_i \cap U_j)(\rho_j(s_j)|_{V\cap U_i\cap U_j})$ is simply

$$\varphi_{ij}(U \cap U_j)(s_j)|_{V\cap U_i\cap U_j} = s_i|_{U\cap U_i\cap U_j}|_{V\cap U_i\cap U_j} = s_i|_{V\cap U_i\cap U_j}$$

meaning $\rho_{UV}((s_i)) \in \mathscr{F}(V)$. So the restriction maps are well-defined. To show that this is a sheaf let $\{W_\alpha\}$ be an open cover of V. Then if $(s_i)_{i\in I}|_{W_\alpha} = 0$ for all α, each s_i is zero on $W_\alpha \cap U_i$ for all α. By the locality condition of each $\mathscr{F}_i$, we have $s_i = 0$ on V. Thus, $s = 0$. Similarly given $\{s_\alpha \in \mathscr{F}(W_\alpha)\}$ that are compatible on the overlaps, we have a family of $s_{\alpha,i} \in F(W_\alpha \cap U_i)$ compatible on all $W_\alpha \cap W_{\alpha'} \cap U_i$. Gluing them using $\mathscr{F}_i$ together gives an element s_i in $V \cap U_i$. Clearly the tuple $(s_i)_{i\in I}$ is in $\mathscr{F}(V)$ since the φ_{ij}-condition is satisfied on each W_α, and by the locality property we've proved the φ_{ij}-condition is satisfied globally. The restriction of $(s_i)_{i\in I}$ to each W_α is by construction s_α. Thus, $\mathscr{F}$ is a sheaf, called the *gluing sheaf of X_i along U_i via φ*.

The construction of this gluing sheaf does not require the cocycle condition $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ for any $i, j, k \in I$. But this property is required to make sense of the idea of "gluing": we want a sheaf which can be "cut" into the original family of data, i.e., isomorphisms $\psi_j : \mathscr{F}|_{U_j} \xrightarrow{\sim} \mathscr{F}_j$. Furthermore these isomorphisms should consist of the canonical projections $\psi_j(V) : (s_i)_{i \in I} \in \mathscr{F}(V) = \mathscr{F}|_{U_j}(V) \mapsto s_j \in \mathscr{F}_j(V)$ where $V \subseteq U_j$. If the cocycle property is satisfied, for any $s_j \in\in \mathscr{F}_j(V)$, the inverse of $\psi_j(V)$ is defined by $s_j \mapsto \left(\varphi_{ji}^{-1}(V \cap U_i)(s_j|_{V \cap U_i})\right)_{i \in I}$. It's easy to check that by construction and the cocycle condition, the image obeys the φ_{ij}-condition. Clearly the latter map is the inverse of the projection and therefore we have an isomorphism $\psi_j : \mathscr{F}|_{U_j} \xrightarrow{\sim} \mathscr{F}_j$. It can be further deduced that since for all i, $\mathscr{F}|_{U_i} \cong \mathscr{F}_i$, for any $x \in U_i$ the stalk $\mathscr{F}_x$ is simply $(\mathscr{F}|_{U_i})_x = \mathscr{F}_{i,x}$.

We proceed to glue two schemes (a generalization is available as an exercise in Hartshorne so I won't do it here). Consider two schemes $(X_1, \mathscr{O}_1), (X_2, \mathscr{O}_2)$ and open subsets $V_i \subseteq X_i$. Let $\varphi_{12} : (V_1, \mathscr{O}_1|_{V_1}) \to (V_2, \mathscr{O}_2|_{V_2})$ be an isomorphism of locally ringed spaces (underlying data: φ_{ii} the identities and φ_{21} the inverse of φ_{12}). Define X to be the quotient space $(X_1 \sqcup X_2)/\sim$ where $x \in X_i$ and $x' \in X_j$ are equivalent if and only if $x \in U_i$, $x' \in U_j$ and $x' = \varphi_{ij}(x)$. Clearly this relation is an equivalence relation. Then we have canonical maps $\iota_i : X_i \to X$ such that by the definition of the topology on the disjoint union and the quotient topology, a set $V \subseteq X$ is open if and only if each $\iota_i^{-1}(V)$ is open in X_i (so it's a continuous map). Denote by U_i the open set $\iota_i(X_i)$ in X ($\iota_i^{-1}(U_i) = X_i$ open, and if $i \neq j$, $\iota_j^{-1}(U_i) = \{x \in X_j : [x] = [y] \in \iota_i(X_i)\}$ which means $x \in U_j$ and there is some $y \in U_i$ such that $x = \varphi_{ij}(y)$; this y always exists since we can take $y = \varphi_{ji}(x)$. Thus $\iota_j^{-1}(U_i) = V_j$ which is open). The map $\iota_i : X_i \to U_i$ is bijective since the equivalence relation on elements of the same X_i is plain equality (so injective and by construction surjective). For any open $W \subseteq X_i$, $\iota_j^{-1}(\iota_i(W)) = \varphi_{ji}^{-1}(W \cap V_i)$ (the case when $i = j$ is simple) is open. Thus, each ι_i is a homeomorphism. Now we glue the sheaves $\mathscr{O}_{X,i} = (\iota_i)_* \mathscr{O}_i$ to get a sheaf $\mathscr{O}_X$ on X. Since $\iota_i|_{U_i} = \iota_j|_{U_j} \circ \varphi_{ij}$, the isomorphism of locally ringed spaces $(\varphi_{ij}, \varphi_{ij}^\#)$ gives an isomorphism of sheaves from $\mathscr{O}_{X,i}|_{U_i \cap U_j}$ to $\mathscr{O}_{X,j}|_{U_i \cap U_j}$ by pushing the maps $\varphi_{ij}^\# : \mathscr{O}_j|_{V_j} \to (\varphi_{ij})_* \mathscr{O}_i|_{V_i}$ via $(\iota_j)_*$ to $(\iota_j)_* \mathscr{O}_j|_{V_j} \to (\iota_j)_*(\varphi_{ij})_* \mathscr{O}_i|_{V_i}$. Note that $(\iota_j)_*(\varphi_{ij})_* = (\iota_j \circ \varphi_{ij})_* = (\iota_i)_*$, we have an isomorphism from $\mathscr{O}_{X,i}|_{U_i \cap U_j}$ to $\mathscr{O}_{X,j}|_{U_i \cap U_j}$. The restrictions are simplified because they don't matter in this case — we are just pushing forward some local data. It can be seen that the result is indeed an isomorphism. Since $(\varphi_{ji})_* \varphi_{ij}^\# \circ \varphi_{ji}^\# = \mathrm{id}$,

$$(\iota_j)_* \varphi_{ij}^\# \circ (\iota_i)_* \varphi_{ji}^\# = (\iota_i \circ \varphi_{ji})_* \varphi_{ij}^\# \circ (\iota_i)_* \varphi_{ji}^\# = (\iota_i)_* \left((\varphi_{ji})_* \varphi_{ij}^\# \circ \varphi_{ji}^\# \right) = \mathrm{id}$$

Clearly the new maps satisfy the cocycle condition — there are only two maps! Therefore, $\mathscr{O}_X$ is a sheaf on X such that, by the previous arguments on sheaves, we have $\mathscr{O}_X|_{U_i} = (\iota_i)_* \mathscr{O}_i$. In particular, since U_i covers X, for any $x \in X$ we can find an i such that $x \in U_i$. Therefore, the stalk at x is the stalk $(\mathscr{O}_X|_{U_i})_x = (\mathscr{O}_{X,i})_x$. By since ι_i is a homeomorphism, the stalk is precise the stalk of $\mathscr{O}_i$ at the (unique) preimage of x. But by assumption the stalks are local rings, so $(X, \mathscr{O}_X)$ is a locally ringed space. Furthermore, if the $(X_i, \mathscr{O}_i)$ are schemes, $(X, \mathscr{O}_X)$ is also a scheme. This is because if $(X_i, \mathscr{O}_i)$ is a scheme, then for each $x \in U_i \subseteq X$ we have an isomorphism of locally ringed spaces (we are abusing the notations here: X_i is the affine scheme around the point corresponding to x and U_i is the image of X_i under the restriction of the homeomorphism ι_i) $(\iota_i, \mathrm{id}) : (U_i, \mathscr{O}_X|_{U_i}) \to (X_i, \mathscr{O}_i)$ where we have identity since for the structure sheaves $\mathscr{O}_X|_{U_i} \cong (\iota_i)_*\mathscr{O}_i$ and its inverse is just $\mathscr{O}_i \to (\iota_i^{-1})_*\mathscr{O}_X|_{U_i} \cong (\iota_i^{-1})_*(\iota_i)_*\mathscr{O}_i = \mathscr{O}_i$ which is still the identity. Thus, as X_i is an affine scheme, so is U_i. This means X is a scheme, completing our construction. The resulting scheme is called *the gluing scheme of X_1, X_2 along U_1, U_2 via the isomorphism φ.*

Here is an example: consider affine schemes $X_1 = X_2 = \mathbb{A}^1_\daleth$, $U_1 = U_2 = \mathbb{A}^1_\daleth \backslash p$ for some point p. Then gluing X_1, X_2 along U_1, U_2 we get a new scheme X. This is an example of a scheme that is NOT an affine scheme, which we will prove later.

2.5 Structure sheaves and schemes via topological bases

To be written.

2.6 The Proj of a ring

Let $S = \bigoplus_{d\in\mathbb{N}} S_d$ be a graded ring. An ideal $\mathfrak{a}$ of S is called a *homogeneous ideal* when $\mathfrak{a} = \bigoplus_{d\in\mathbb{N}}(\mathfrak{a} \cap S_d)$. In this case if f is an element of $\mathfrak{a}$, by directness of the sum, each homogeneous component of f should belong to $\mathfrak{a}$.

I here reproduce some commutative algebra results for convenience.

Lemma 2.6.1. *An ideal $\mathfrak{a}$ of S is homogeneous if and only if it's generated by a set of homogeneous elements.*

Proof. The forward direction is clear: if $\mathfrak{a} = \bigoplus_{d\in\mathbb{N}}(\mathfrak{a} \cap S_d)$ then $\mathfrak{a}$ is generated by $\bigcup(\mathfrak{a} \cap S_d)$.

Conversely, if $\mathfrak{a}$ is generated by a set of homogeneous elements $H = \bigcup H_d$ for $H_d \subseteq S_d$. Then for any $f \in \mathfrak{a}$ we have $f = \sum b_h h$ where $h \in H_d$ for some d and $b_h \in S$ are nonzero for finitely many h. We can further write

$$f = \sum_{d\in\mathbb{N}} \sum_{h\in H_d} b_h h.$$

But since b_h can be expressed in homogeneous components $b_h = \sum b_{h,k}$, $b_{h,k}h \in S_{k+d}$ and $b_{h,k}h \in \mathfrak{a}$ as $\mathfrak{a}$ is an ideal, $b_h h \in \bigoplus_{d\leqslant k}(\mathfrak{a}\cap S_k)$ and thus $f = \sum b_h h \in \bigoplus_{d\in\mathbb{N}}(\mathfrak{a}\cap S_d)$. Yet trivially $\bigoplus_{d\in\mathbb{N}}(\mathfrak{a}\cap S_d) \subseteq \mathfrak{a}$ so we have $\mathfrak{a} = \bigoplus_{d\in\mathbb{N}}(\mathfrak{a}\cap S_d)$, completing the proof. $\square$

Lemma 2.6.2. *A homogeneous ideal $\mathfrak{p}$ is prime iff for all homogeneous elements a, b satisfying $ab \in \mathfrak{p}$, we have $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*

Proof. The only if direction is obvious. Conversely, suppose the assumption on homogeneous elements holds. Let $a, b \in S$ such that $ab \in \mathfrak{p}$. Assume, in contradiction, that neither a nor b is in the ideal. Then if we write $a = \sum a_d$ and $b = \sum b_d$ in homogeneous elements, then there must exist some a_d and $b_{d'}$ such that $a_d \notin \mathfrak{p}$ and $b_{d'} \notin \mathfrak{p}$. We can assume d, d' are maximal among such indexes since the two sums above are finite. But as $\mathfrak{p}$ is homogeneous, each homogeneous component of ab must belong to $\mathfrak{p}$. This suggests the $(d+d')$th component

$$\sum_{i+j=d+d'} a_i b_j \in \mathfrak{p}$$

If $(i,j) \neq (d,d')$, then $i > d$ or $j > d'$, meaning a_i or $b_j \in \mathfrak{p}$ so $a_i b_j \in \mathfrak{p}$. But then $a_d b_{d'} \in \mathfrak{p}$ which gives us a contradiction given $a_d, b_{d'} \notin \mathfrak{p}$. $\square$

Let S_+ denote the irrelevant ideal $\bigoplus_{d>0} S_d$. We define the *homogeneous spectrum* $\operatorname{Proj} S = \{\mathfrak{p} \text{ homogeneous prime ideal} : S_+ \subsetneq \mathfrak{p}\}$. It is therefore easy to construct a Zariski topology on $\operatorname{Proj} S$ via closed sets $V_+(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{a} \subseteq \mathfrak{p}\}$ for an arbitrary homogeneous ideal $\mathfrak{a}$. For any homogeneous element f of positive degree, define the open set $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\}$. It is easy to see that they are just the induced closed and open sets from $\operatorname{Spec} S$, so they form a subspace topology.

Let T be a multiplicative set of **homogeneous** elements. Then $T^{-1}S$ gets a natural $\mathbb{Z}$-grading. If $\mathfrak{p}$ is a homogeneous prime ideal, we consider the set T of all homogeneous elements not in $\mathfrak{p}$ and denote the localization $T^{-1}S$ by $S_\mathfrak{p}$ and by $S_{(\mathfrak{p})}$ the subring $(T^{-1}S)_0$. Then $S_{(\mathfrak{p})}$ is a local ring with the unique maximal ideal $\mathfrak{p}S_p \cap S_0$. The proof of this is pretty much the same as the one that proves the localness of $S_\mathfrak{p}$, with the extra care that if a/b has degree zero then b/a (when it makes sense) also has degree zero.

For any homogeneous element f, the multiplicative set $\{1, f, f^2, \dots\}$ consists of only homogeneous elements. The definition of S_f is therefore the same. Similarly, we denote by $S_{(f)}$ the subring of homogeneous elements of degree zero in S_f.

Lemma 2.6.3. *Suppose S is a $\mathbb{Z}$-graded ring with an invertible element f of positive degree. Then the subspace G of homogeneous prime ideals of S is homeomorphic to* $\operatorname{Spec} S_0$.

Proof. Assume f has degree $d > 0$. Since f is invertible, its inverse has homogeneous degree $-d$ (which is immediate if we write f^{-1} in homogeneous components). We define the map $\varphi : G \to \operatorname{Spec} S_0$ by $\mathfrak{p} \mapsto \mathfrak{p} \cap S_0$. We claim that the inverse of this map is $\psi : \mathfrak{p}_0 \mapsto \sqrt{\mathfrak{p}_0 S}$. Clearly if $\mathfrak{p} \in G$, $\mathfrak{p} \cap S_0$ is prime in S_0. So it suffices to construct an inverse of this map. For any $\mathfrak{p}_0 \in \operatorname{Spec} S_0$, the ideal $\mathfrak{p}_0 S$ (the set of finite sums of products ps for some $p \in \mathfrak{p}_0$ and $s \in S$) is homogeneous (generated by the sets $\mathfrak{p}_0 S_i$ of homogeneous elements).

It is required to show that ψ is well-defined, that is, $\sqrt{\mathfrak{p}_0 S}$ is indeed prime. Take any two homogeneous elements a, b such that $ab \in \sqrt{\mathfrak{p}_0 S}$, we have $a^n b^n \in \mathfrak{p}_0 S$ for some natural number n. Then $a^{dn} b^{dn} / f^{n \deg a + n \deg b} \in \mathfrak{p}_0$. Then either $a^{dn}/f^{n \deg a} \in \mathfrak{p}_0$ or $b^{dn}/f^{n \deg b} \in \mathfrak{p}_0$. This means either $a^{dn} \in \mathfrak{p}_0 S$ or $b^{dn} \in \mathfrak{p}_0 S$, in other words $a \in \sqrt{\mathfrak{p}_0 S}$ or $b \in \sqrt{\mathfrak{p}_0 S}$. By Lemma 2.6.2, $\sqrt{\mathfrak{p}_0 S}$ is prime.

We first check $\psi \circ \varphi = \mathrm{id}$. If $p \in \mathfrak{p}$ is homogeneous, then $s^d / f^{\deg s} \in \mathfrak{p} \cap S_0$. Thus, $s^d = f^{\deg s}(s^d / f^{\deg s}) \in (\mathfrak{p} \cap S_0) S$. But recall that $\mathfrak{p}$ is homogeneous, so $\mathfrak{p} \subseteq \sqrt{(\mathfrak{p} \cap S_0) S}$ as the latter is an ideal. The opposite inclusion is immediate: $\sqrt{(\mathfrak{p} \cap S_0) S} \subseteq \sqrt{\mathfrak{p} S} = \sqrt{\mathfrak{p}} = \mathfrak{p}$.

Now we show that $\varphi \circ \psi = \mathrm{id}$. Obverse if $p \in \sqrt{\mathfrak{p}_0 S} \cap S_0$, then since S is graded, $p^n = \sum p_i s_i \in \mathfrak{p}_0$ where $p_i \in \mathfrak{p}_0$ and $s_i \in S_0$ for some n. Then $p^n \in \mathfrak{p}_0$ implying $p \in \mathfrak{p}_0$. Therefore, $\mathfrak{p}_0 S \cap S_0 = \mathfrak{p}_0$ as the other inclusion is obvious.

It remains to show that φ, ψ are both open. Take any open set $D'(g) = G \cap D(g)$ of the basis. If $g = \sum g_i$ where $g_i \in S_i$, then $\varphi(D'(g)) = \bigcup D(g_i^d / f^i) \subseteq \operatorname{Spec}(S_0)$ (can be proved easily on scratch paper). For an open basis set $D(g_0)$ of $\operatorname{Spec} S_0$ with $g_0 \in S_0$, it is easy to see that $\psi(D(g_0)) = D'(g_0)$ in G, which is open. $\square$

Here is another lemma on the spectrum of localizations. We omit the proof since it's standard.

Lemma 2.6.4. *Given a ring R and a multiplicative set S, the canonical map $\varphi : R \to S^{-1}R$ induces a homeomorphism between $\operatorname{Spec} S^{-1}R$ and $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \cap S = \emptyset\}$ where the latter has a subspace topology induced by the spectrum of R whose inverse is $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.*

In particular, since the localization of a homogeneous ideal is homogeneous, the open set $D_+(f) \subseteq D(f)$ corresponds to the homogeneous prime ideals in $\operatorname{Spec} S_f$ (since $\mathfrak{q} \in \operatorname{Spec} S_f$ automatically ensures that $f \notin \varphi^{-1}(\mathfrak{q})$ and $S_+ \not\subseteq \varphi^{-1}(\mathfrak{q})$!). And therefore by Lemma 2.6.3, given a homogeneous element f of positive degree, we get a homeomorphism from $D_+(f)$ to $\operatorname{Spec}(S_{(f)})$.

Lemma 2.6.5. *Given a graded ring S and an element $\mathfrak{p} \in \operatorname{Proj} S$, take some homogeneous element f of positive degree that is not in $\mathfrak{p}$ (which exists by our construction of Proj). Let $\mathfrak{p}'$ be the element in $\operatorname{Spec}(S_{(f)})$ corresponding to $\mathfrak{p}$, then $S_{(\mathfrak{p})} = (S_{(f)})_{\mathfrak{p}'}$.*

Proof. Given any $g/h \in S_{(\mathfrak{p})}$, let

$$\varphi(g/h) = (gh^{\deg f - 1}/f^{\deg g})/(h^{\deg f}/f^{\deg h})$$

Then clearly the degrees check out, and $h^{\deg f}/f^{\deg h} \notin \mathfrak{p}'$ as $h^{\deg f} \notin \mathfrak{p}$ while $\mathfrak{p}' = \{a/f^n : \deg a - n \deg f = 0, a \in \mathfrak{p}\}$. To show that this is well-defined Why?. It's easy to see that this map is multiplicative and additive. To show that it's actually an isomorphism, we construct the inverse: $\psi : (g/f^n)/(h/f^m) \mapsto (gf^m)/(hf^n)$. Need details! $\square$

2.7 Projective schemes

We define the structure sheaf of $\operatorname{Proj} S$ to be the functor $\mathscr{O}$ which sends an open set $U \subseteq \operatorname{Proj} S$ to the set of functions

$$s : U \to \bigsqcup_{\mathfrak{p} \in U} S_{(\mathfrak{p})}, \quad s_{\mathfrak{p}} \in S_{(\mathfrak{p})}$$

and for each $\mathfrak{p} \in U$, there is an open nhds $p \in V \subset U$ and homogeneous $a, f \in S$ of the same degree such that $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$ and $s_{\mathfrak{q}} = a/f \in S_{(\mathfrak{q})}$ (this is really the same thing as what we did about the spectra of rings, only with the extra degree condition). This presheaf $\mathscr{O}$ is clearly a sheaf.

Similar to Lemma 2.1.3, the following lemma gives us a scheme $(\operatorname{Proj} S, \mathscr{O}_{\operatorname{Proj} S})$ called the *homogeneous spectrum of* S:

Lemma 2.7.1. *Given a graded ring S, we have*

(a) *For any $\mathfrak{p} \in \operatorname{Proj} S$, the stalk $\mathscr{O}_{\mathfrak{p}}$ is the ring $S_{(\mathfrak{p})}$, which is local.*

(b) *Each $D_+(f)$ is open. Furthermore, they cover $\operatorname{Proj} S$ and the following locally ringed spaces are isomorphic:*

$$\left(D_+(f), \mathscr{O}|_{D_+(f)}\right) \cong \operatorname{Spec} S_{(f)}$$

where $S_{(f)}$ is the ring of homogeneous elements of degree zero in S_f.