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D-modules and Representation Theory

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0 Introduction

The theory of D-modules, where D stands for the sheaf of differential operators, began with the extensive study of algebraic analysis in the Kyoto school led by Sato and Kashiwara. Most notably, in Kashiwara's thesis [23], the theory of D-modules was developed in-depth in an analytic setting. Around the same time, the French mathematician Bernstein proposed an algebraic theory of D-modules, focusing on methods used in algebras and representation theory. A part of the work is rooted in the Borel-Weil theorem, where representations of semisimple algebraic groups were realized as global sections of invertible sheaves on flag varieties. Later becoming the vast geometric representation theory, this part of Bernstein's work culminated in his joint paper with Beilinson [5], which established an equivalence between the representation theory of semisimple algebraic groups and the theory of D-modules on flag variety. To be more precise, given a semisimple Lie algebra $\mathfrak{g}$, a fixed Borel subalgebra $\mathfrak{b}$, and the corresponding flag variety $X = G/B$, there is a correspondence between modules over the sheaf of differential operators twisted by a one-dimensional representation λ of $\mathfrak{b}$ and $U(\mathfrak{g})$-modules with actions of the center $\mathfrak{Z}$ of $U(\mathfrak{g})$ determined by λ. The theorem is outstanding because it classifies not only finite-dimensional representations but all infinite-dimensional ones which are generally difficult to understand in the algebraic setting. One major application of Beilinson-Bernstein is half of the Kazhdan-Lusztig conjecture, while the other half is given by the Riemann-Hilbert correspondence, another crown jewel of the theory of D-modules which will not be discussed in this paper.

In the first section of this essay, we focus on the sheaf of differential operators. Starting with an algebraic introduction to differential operators on algebras over a field of characteristic zero, we study their filtrations and filtered modules over these rings. Later the algebraic data will be globalized and turned into a sheaf of differential operator $\mathcal{D}_X$ on smooth varieties X. We will study the sheaf of differential operators between locally free sheaves, and study the filtration of them. The core of the first section is the structure of $\mathcal{D}_X$, i.e., being generated by the tangent sheaf and structure sheaf, and the structure of its associated graded ring.

The second section is a juicy part of the essay. We start by studying left and right $\mathcal{D}_X$-modules and demonstrating their relations. Then I will delve into the big question of pushforwards and pullbacks of $\mathcal{D}_X$-modules. After realizing the naive definitions of pushforwards and pullbacks do not actually work except for the special examples of immersions, we will investigate a derived version of the constructions. The crucial theorem of Kashiwara regarding the pushforward of D-modules by closed immersions will be the heart of this section. With it proved, we will develop the notion of twisted sheaves of differential operators and the affinity of smooth varieties over twisted sheaves of differential operators. To understand the difference between $\mathcal{O}_X$-coherence and $\mathcal{D}_X$-coherence, we employ the characteristic variety of D-modules and prove the celebrated inequality of Bernstein. At the end, we introduce special D-modules known as holonomic D-modules which are limiting examples in Bernstein's inequality. We will state but not prove several theorems relevant to the properties of holonomic D-modules as they deviate from the purpose of this paper. However, I will illustrate them by producing a great number of examples and demonstrating computational results that draw a connection between holonomic D-modules and systems of partial differential equations with finite-dimensional solution spaces.

The theory of representations finally comes into the picture in the last section. We will first study equivariant sheaves, and understand the Borel-Weil-Bott theorem through explicit computations. Then we will collect the necessary tools such as the Harish-Chandra isomorphism, Kostant's theorem, and a map that computes the global section of twisted sheaves of differential operators of integral weights on flag varieties. Then we will state and prove the Beilinson-Bernstein localization theorem for integral weights. To see what objects would replace our constructions for general weights, I will introduce Lie-Rinehart algebras, which are the algebraic data of Lie algebroids. The Lie algebroids are the tools to construct the correct sheaves in general.

At the very end, I will briefly discuss two different analogous versions of Beilinson-Bernstein for quantum groups $U_q(\mathfrak{g})$ due to Lunts-Rosenberg-Tanisaki, and Backelin-Kremnizer. Even though the representation theory of $U_q(\mathfrak{g})$ at a generic q is almost parallel to the classical $U(\mathfrak{g})$, we encounter several difficulties when trying to replicate the classical constructions. We will work with noncommutative and non-cocommutative bialgebras, representations that are not locally finite, and many interesting oddities that never happen for $U(\mathfrak{g})$. A good thing is that the guiding philosophy of the approach by Backelin and Kremnizer is in fact discussed extensively in previous sections.

In this article, we use normal M, N to denote abstract modules and algebraic objects, and curly $\mathcal{M}$ and $\mathcal{N}$ to denote sheaves of modules. We will be dealing with schemes over a field k of characteristic zero. By algebraic varieties over k I mean separated schemes over k of finite type that are geometrically integral. Given a smooth variety X, I denote by Θ_X, $\Omega_{X/k}$ and $\omega_{X/k} = \det \Omega_{X/k}$ the tangent, cotangent and canonical sheaves on X. I will write TX and T^*X for the tangent and cotangent bundles on X. To prevent confusion, I shall specify all notations before using them and recall them regularly.

The overall reference for the essay is [18]. I will take many interesting arguments and constructions in [16, 32] to enrich the first two sections. In the classical part of the third section, I will mostly rely on [18, 24]. For the quantum analogy, I will stick to the original papers [3, 28, 35].

1 Differential Operators

1.1 Weyl algebras, filtrations and differential operators Let k be a field of characteristic zero. We start with the polynomial ring $R = k[x_1, \ldots, x_n]$. Let the n**th Weyl algebra** be the associative k-algebra generated by elements in R and derivations of R. Write $\partial_i \in \mathrm{Der}_k(R)$ to be the derivative with respect to x_i. Then the generators of R have the following relations: $[\partial_i, \partial_j] = [x_i, x_j] = 0$ (as they commute) and $[\partial_i, x_j] = \delta_{ij}$. When $n = 1$, we omit the index. Since A_n is noncommutative, left and right A-modules are not equivalent in general.

Example 1.1.1. The algebra R is naturally a left A_n-module on which A_n acts as regular differential operators.

Let $P \in A_n$ and consider the partial differential equation $Pu = 0$. Then naturally the quotient $M = A_n/A_nP$ is an A_n-module. Let N be another A_n-module. Then the space of A_n-linear homomorphisms from M to N is

$$\mathrm{Hom}_{A_n}(M, N) = \{\varphi \in \mathrm{Hom}_{A_n}(A_n, N) : \varphi(P) = 0\}$$

Yet $\mathrm{Hom}_{A_n}(A_n, N)$ is isomorphic to N via $\varphi \mapsto \varphi(1)$ with inverse $n \mapsto (Q \mapsto Qn)$. Therefore,

$$\mathrm{Hom}_{A_n}(M, N) = \{u \in N : Pu = P\varphi(1) = \varphi(P) = 0\}$$

meaning $\mathrm{Hom}_{A_n}(M, N)$ is the space of solutions to $Pu = 0$ in N. In general, consider the system of linear equations

$$\sum_{j=1}^{q} P_{ij}u_j = 0, \quad i = 1, 2, \ldots, p$$

To describe the space of solutions, we consider the module M defined by the exact sequence

$$A_n^p \xrightarrow{\psi} A_n^q \to M \to 0$$

where ψ sends

$$(Q_1, \ldots, Q_p) \mapsto \left(\sum_i Q_iP_{i1}, \ldots, \sum_i Q_iP_{iq}\right)$$

That is, M is the cokernel of ψ. Using the same argument, we see that

$$\mathrm{Hom}_{A_n}(M, N) = \{\varphi \in \mathrm{Hom}_{A_n}(A_n^q, N) : \varphi \circ \psi = 0\}$$

Under the map $\varphi \mapsto (\varphi(e_i))_{i=1}^q$, the space $\mathrm{Hom}_{A_n}(A_n^q, N)$ can be identified with N^q. Thus,

$$\mathrm{Hom}_{A_n}(M, N) = \left\{u \in N^q : \forall i \leqslant p, \sum_j P_{ij}u_j = \sum_j P_{ij}\varphi(e_j) = (\varphi \circ \psi)(e_j) = 0\right\}$$

which is the space of solutions to the system of differential equations.

Example 1.1.2. The A_1-orbit of 1 in $k[x]$ is the module $A_1/A_1\partial$, which is simply $k[x]$. Any A_1-homomorphisms from $k[x] \to k[x]$ is a scalar multiplication. Thus, the Hom space corresponds to the space of solutions to $\partial u = 0$, i.e., constants. In general, given any A_n-module N and an element $u \in N$. Define the A_n-module $M(u) = A_n \cdot u$. Then we have a surjection $s : A_n \to M(u)$ given by $P \mapsto Pu$, meaning $M(u) \cong A_n/\ker s$ where $\ker s$ is the space of differential equations satisfied by u. For instance, let N be the module of meromorphic functions on k. Then $M(1/x) = A_1/A_1(\partial x)$; $M(x^r) = A_1/A_1(x\partial - r)$; $M(\log x) = A_1/A_1(\partial x \partial)$; $M(e^x) = A_1/A_1(\partial - 1)$.

Example 1.1.3. Consider the ordinary differential equation $Pu = 0$ where $P = a_n(x)\partial^n + \cdots + a_1(x)\partial + a_0(x)$ for $a_0, \ldots, a_n \in k[x]$. We can reduce the order of the equation as usual: let $u_{i+1} = \partial u_i$ for $i = 0, \ldots, n-2$. Then the equation is equivalent to the system of equations

$\partial u_i - u_{i+1} = 0$ for $i = 0, \dots, n-2$, and $a_n(x)\partial u_{n-1} + a_{n-1}(x)u_{n-1} + \cdots + a_1(x)u_1 + a_0(x)u_0 = 0$. We compute the linear map $\psi : A_1^n \to A_1^n$ describing the linear system to be the n by n matrix

$$\begin{bmatrix} \partial & -1 & \cdots & 0 & 0 \\ 0 & \partial & -1 & \cdots & 0 \\ & & \cdots & & \\ 0 & \cdots & 0 & \partial & -1 \\ a_0(x) & a_1(x) & \cdots & a_{n-2}(x) & a_n(x)\partial + a_{n-1}(x) \end{bmatrix}$$

And the associated A_1-module is $M = \operatorname{coker} \psi$.

Example 1.1.4. In the settings of Example 1.1.3, we can also construct the obvious A_1-module $M = A_1/A_1P$. By the virtue of Example 1.1.2, given a solution u to P, we have $M = A_1 \cdot u$. But then on the set $U = \{x : a_n(x) \neq 0\}$, $M = \bigoplus_{i=0}^n k[x]u^{(i)}$ where $u^{(i)} = \partial^i u$. We therefore identified M with the one in Example 1.1.3.

Definition 1.1.5. Given a k-algebra A, a **filtration** of A is a family $F_\bullet A$ of subspaces

$$0 = F_{-1}A \subseteq F_0 A \subseteq F_1 A \subseteq \cdots$$

such that $F_i F_j A \subseteq F_{i+j} A$, $1 \in F_0$ and $A = \cup_{l=0}^\infty F_l$. Sometimes we write $F_\bullet$ when the base algebra is clear. The **graded algebra associated to** F is the direct sum

$$\operatorname{gr}^F A = \bigoplus_{l=0}^\infty \operatorname{gr}_l^F A = \bigoplus_{l=0}^\infty F_l/F_{l-1}$$

Notice that $\operatorname{gr}^F A$ is naturally a graded k-algebra with well-defined multiplications given by $[x] \cdot [y] = [xy]$. Indeed, for $x, x' \in F_l$ and $y, y' \in F_m$, if $[x'] = [x]$ and $[y'] = [y]$, we have $x'y' = (x+p)(y+q) = xy + py + qx + pq$ for some $p \in F_{l-1}$, $q \in F_{m-1}$, so $py + qx \in F_{l+m-1}$ and $pq \in F_{l+m-2} \subseteq F_{l+m-1}$. On the nth Weyl algebra A_n, we have an **order filtration** given by

$$F_l = \sum_{|\alpha| \leqslant l} R \cdot \partial^\alpha$$

where α is an n-tuple, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. It is easy to deduce the following

Proposition 1.1.6. *Let $F_\bullet$ be the order filtration of A_n. Then*

$$\operatorname{gr}^F A_n \cong k[x_1, \dots, x_n, \xi_1 \dots, \xi_n]$$

which is a commutative k-algebra graded by orders of ξ_i.

Proof. Let $x_1, \dots, x_n$ and $\xi_1, \dots, \xi_n$ be the images of x_i and ∂_i in F_1/F_0. Then by construction F_l/F_{l-1} consists of monomials in terms of x_i and ξ_i such that the sum of degrees of ξ_i is l. It remains to show commutativity. But this is easy, since for any i, j, we have $[\partial_i, x_j] = \delta_{ij} \in F_0$ so $[\xi_i, x_j] = 0$ in F_1/F_0. □

Definition 1.1.7. We say an algebra A admitting a filtration F is **almost commutative** if $\operatorname{gr}^F A$ is commutative.

The associated graded algebra is actually a simple but powerful tool to determine whether two filtered algebras are isomorphic.

Lemma 1.1.8. *Let A and B be two filtered k-algebras with filtrations $F_\bullet A$ and $F_\bullet B$. Given a map $f : A \to B$, if the induced map $\tilde{f} : \operatorname{gr}^F A \to \operatorname{gr}^F B$ is injective (resp. surjective), then f is injective (resp. surjective).*

Proof. The core of this lemma is the exhaustive property of filtrations (assumed in our definition). Suppose $\tilde{f}$ is injective. For any nonzero $a \in A$, there is a minimal k such that $a \in F_k A$. Then a has a nonzero image in $\mathrm{gr}^F A$, i.e., $\tilde{f}(a) \neq 0$. But this is the class of $f(a)$, so $f(a)$ must be nonzero. Suppose $\tilde{f}$ is surjective. Take any $b \in B$. Let k be the minimal number such that $b \in F_k B$. Denote still by b its preimage in $\mathrm{gr}^F_k B$ and a the preimage in $\mathrm{gr}^F_k A$ under $\tilde{f}$. Let $a \in F_k A$ be the element with class a in $\mathrm{gr}^F_k A$. Then $b - f(a)$ vanish in $\mathrm{gr}^F B$, i.e., it's in $F_{k-1}B$. Then we can find another $a' \in F_{k-1}A$ with image $b - f(a)$, in which case $b - f(a) - f(a')$ is zero in $\mathrm{gr}^F B$. Repeat this process, we end with an element in $F_k A$ with image b under f. □

Example 1.1.9. The universal enveloping algebra $U(\mathfrak{g})$ of some Lie algebra $\mathfrak{g}$ has a filtration $PBW_\bullet U(\mathfrak{g})$ by degrees of monomials given by Poincaré-Birkhoff-Witt. In this case $\mathrm{gr}^{PBW} U(\mathfrak{g})$ is commutative because for any $x, y \in \mathfrak{g} \subseteq PBW_1 U(\mathfrak{g})$, we have $xy - yx = [x, y] \in \mathfrak{g} \in PBW_1 U(\mathfrak{g})$ so commutators vanishes in $\mathrm{gr}^{PBW} U(\mathfrak{g})$.

Recall that a **Poisson structure** on an algebra A is a Lie bracket that satisfies the Leibniz relation.

Lemma 1.1.10. *If A is an almost commutative algebra with filtration $F_\bullet A$, then $\mathrm{gr}^F A$ has a canonical Poisson structure.*

Proof. Let $x_i \in \mathrm{gr}^F_i A$ and $x_j \in \mathrm{gr}^F_j A$, and pick representatives $a_i \in F_i A$ and $a_j \in F_j A$. Define the Poisson bracket $\{x_i, x_j\}$ to be the class of $a_i a_j - a_j a_i$ in $\mathrm{gr}^F_{i+j-1} A$ ($[a_i, a_j]$ is an element in $F_{i+j-1}A$ because $\mathrm{gr}^F A$ is commutative). It is easy to check if $b_i \in F_{i-1}A$ then $b_i a_j - a_j b_i \in F_{i+j-2}A$ so the bracket is well-defined in $\mathrm{gr}^F A$. By construction the bracket is a Lie algebra and satisfies the Leibniz relation. □

Example 1.1.11. The Poisson bracket on $\mathrm{gr}^F U(\mathfrak{g})$ is given by $\{x^i, y^j\} = ijx^{i-1}y^{j-1}[x, y]$ for any $x, y \in \mathfrak{g}$. This can be easily seen using the construction of $\{\cdot, \cdot\}$ and Leibniz.

It is also tempting to study filtrations of A-modules when A admits a filtration.

Definition 1.1.12. Let M be an A-module for a filtered k-algebra A with filtration $F_\bullet A$. It has a **filtration** if there is an increasing family of additive subgroups $F_\bullet M$

$$0 = F_{-1}M \subseteq F_0 M \subseteq F_1 M \subseteq \cdots \subseteq M$$

with $F_l A \cdot F_s M \subseteq F_{l+s} M$ and $M = \cup_s F_s M$. We say a filtration on M is **good** if each $F_s M$ is finitely generated over $F_0 A$, and for large enough s, $F_{l+s} M = F_l A \cdot F_s M$ for any l.

Remark 1.1.13. There are various ways to define a good filtration on M, depending on the context. All examples we encounter will be filtrations such that each $F_s M$ is finitely generated as an $F_0 A$-module. We therefore include this condition in the definition of good filtrations.

To determine whether a filtration on an A-module is good, we have the following convenient criterion

Proposition 1.1.14. *If the filtration on A is good (considered as a module over itself), then $F_\bullet M$ is good if and only if $\mathrm{gr}^F M$ is finitely generated over $\mathrm{gr}^F A$.*

Proof. If there exists an s_0 such that $F_{l+s}M = F_l A \cdot F_s M$ for all $s \geqslant s_0$, then $\mathrm{gr}^F M$ is generated by all components $\mathrm{gr}^F_s M = F_s M / F_{s-1} M$ for $s \leqslant s_0$. As each $F_s M$ is finitely generated over $F_0 A$, $\mathrm{gr}^F_s M$ is finitely generated over $\mathrm{gr}^F_0 A$. But then we get a finite number of generators of $\mathrm{gr}^F M$ over $\mathrm{gr}^F A$.

Conversely, let $\{u_i\}$ be a finite set of generators of $\operatorname{gr}^F M$ over $\operatorname{gr}^F A$. We can assume they are homogeneous. Let s_0 be the maximum of their degrees. Then we have $\operatorname{gr}_s^F M = \sum_{t=0}^{s_0} \operatorname{gr}_{s-t}^F A \cdot \operatorname{gr}_t^F M$ for any $s \geqslant s_0$. This suggests

$$F_s M = F_{s-1}M + \sum_{t=0}^{s_0} F_{s-t}A \cdot F_t M = F_{s-1}M + F_{s-s_0}A \cdot \sum_{t=0}^{s_0} F_{s_0-t}A \cdot F_t M$$

Here we used the fact that A has a good filtration. After absorptions, the RHS becomes $F_{s-s_0}A \cdot F_{s_0}M$. Since $F_s M = F_{s-s_0}A \cdot F_{s_0}M$ for all $s \geqslant s_0$ and $F_\bullet A$ is good, we are done. □

Lemma 1.1.15. *If A has a good filtration, then so does M if and only if M is finitely generated over A.*

Proof. Suppose M is finitely generated over A by $m_1, \ldots, m_r$. Define $F_s M = \sum_i F_s A \cdot m_i$. Observe that $F_s A$ is finitely generated over $F_0 A$ from $F_\bullet A$ being good. Every $F_s M$ is therefore finitely generated over $F_0 A$, and the above sum becomes $F_s M = F_s A \cdot F_0 M$, that is, $F_\bullet M$ is good.

On the other hand, if M has a good filtration $F_\bullet M$, then there is some s_0 such that $F_s M = F_{s-s_0}A \cdot F_{s_0}M$ for all $s \geqslant s_0$. Therefore M is generated by elements in $F_{s_0}M$ over A, and by definition $F_{s_0}M$ is finitely generated over $F_0 A \subseteq A$ so we are done. □

Next, we shall follow Grothendieck's approach to define differential operators. Let k be a field of characteristic zero, R a commutative k-algebra, and M an R-bimodule such that elements in k commute with all elements in M. We define a family of subspaces M'_l by $M'_{-1} = 0$ and $M'_l = \{x \in M : \forall r \in R, [r, x] \in M'_{l-1}\}$. Define the submodule $M' = \cup_{l=-1}^{\infty} M'_l$. We call M' the **differential part** of M, and if $M = M'$ then we say M is a **differential bimodule**. Now for any two left R-modules, X, Y, we can define

Definition 1.1.16. The module of **k-linear differential operators from X to Y** is the R-bimodule $\operatorname{Diff}(X, Y) = \operatorname{Hom}_k(X, Y)'$. We denote by $D(R)$ the ring $\operatorname{Diff}(R, R)$ and $D_l(R) = \operatorname{Diff}(R, R)_l$.

Remark 1.1.17. By construction $\operatorname{Diff}(X, Y) = \cup_{l=-1}^{\infty} \operatorname{Diff}(X, Y)_l$ where $f \in \operatorname{Diff}(X, Y)_l$ if and only if

$$[r_{l+1}, [r_l, \ldots, [r_1, f]]] = 0, \quad \forall r_i \in R$$

Moreover we always have $D_l(R)D_m(R) \subseteq D_{l+m}(R)$ (easy to show if one writes $rx = xr + a$ for $x \in D_\bullet R$ and $a \in D_{\bullet-1}R$).

One extremely important property of the constructed filtration is perhaps its compatibility with localizations. Namely, we have the following:

Proposition 1.1.18. *Let M be an R-bimodule. Given any non-nilpotent f in R, take M_f to be the (two-sided) localization of M at f, that is, $M_f = R_f \otimes_R M \otimes_R R_f$. Then the filtered differential part M' localizes to*

$$(M_f)'_l = (M'_l)_f \cong R_f \otimes_R M'_l \cong M'_l \otimes_R R_f$$

for any $l \geqslant 0$. In particular, we have $D_l(R_f) = R_f \otimes_R D_l(R)$ and thus $D(R_f) \cong R_f \otimes_R D(R)$.

Proof. View M as an $R \otimes R$-module. Let J be the submodule in $R \otimes R$ generated by elements of the form $r \otimes 1 - 1 \otimes r$. Then via an inductive argument, we obtain $M'_l = \{x \in M : J^{l+1}x = 0\}$. In this case, we see that $J^{l+1}x = 0$ if and only if for any m we have $J_f^{l+1}(x/f^m) = 0$ where J_f is the kernel of the multiplication $R_f \otimes R_f \to R_f$. Indeed, f is not nilpotent so we can freely remove the powers of f. Thus, $(M'_l)_f = (M_f)'_l$, and the latter two congruences are evident from the universal property of localizations. □

In the case of Weyl algebras, we observed that nice filtrations could carry a lot of information. For any commutative k-algebra R, consider the graded k-algebra $D(R)$. By definition, $D_0(R) = \{f \in \operatorname{End}_k(R) : \forall r \in R, [r,f] = 0\}$. Via the identification $f \mapsto f(1)$, $D_0(R)$ is isomorphic to R. Elements in $D_1(R)$ are then those $P \in \operatorname{End}_k(R)$ such that for any $r \in R$, $[r,P]$ is an element of R, say f_r. Then for any $r_1, r_2 \in R$, $f_{r_1}r_2 = r_1 P(r_2) - P(r_1 r_2)$. But if we consider $\xi_P = P(1) - P \in \operatorname{End}_k(R)$, it is immediate that

$$\xi_P(r_1r_2) = P(1)r_1r_2 - r_1P(r_2) + f_{r_1}r_2 = 2P(1)r_1r_2 - r_1P(r_2) - r_2P(r_1) = r_1\xi_P(r_2) + r_2\xi_P(r_1)$$

where we use $r_2P(r_1) = P(1)r_1r_2 - f_{r_1}r_2$ in the second equality. Hence, the map $P \mapsto (P(1), \xi_P)$ is an isomorphism from $D_1(R)$ to $D_0(R) \oplus \operatorname{Der}_k(R)$ with inverse given by $(f, \xi) \mapsto f + \xi$. To extend the discussion, we have the proposition below

Proposition 1.1.19. *Let R be a commutative k-algebra, if R has rank n and $\operatorname{Der}_k(R)$ is free of rank n over k, then $F_\bullet D(R) = D_\bullet(R)$ is a filtration of R such that*

(i) *We have $\operatorname{gr}_l^F D(R) = D_l(R)/D_{l-1}(R) \cong \operatorname{Sym}^l \operatorname{Der}_k(R)$. That is, as graded R-algebras,*

$$\operatorname{gr}^F D(R) \cong \operatorname{Sym} \operatorname{Der}_k(R)$$

(ii) *As an R-subalgebra of $\operatorname{End}_k(R)$, $D(R)$ is generated by elements $f \in R$, $\xi \in \operatorname{Der}_k(R)$ such that $[\xi, f] = \xi(f)$.*

Proof. (i) The isomorphism $D_0(R) \oplus \operatorname{Der}_k(R) \to D_1(R)$ induces an R-linear map $\operatorname{Der}_k(R) \to \operatorname{gr}_1^F D(R)$. Now if $\operatorname{gr}^F D(R)$ is commutative, the above extends to a map $\operatorname{Sym}\operatorname{Der}_k(R) \to \operatorname{gr}^F D(R)$ respecting the grading since $D(R)$ is graded. Indeed, given $P \in D_l(R)$, $Q \in D_m(R)$, then for any $r \in R$, $[r,[P,Q]] = r[P,Q] - [P,Q]r = rPQ - rQP - PQr + QPr$. Write $rP = Pr + a$ and $rQ = Qr + b$ for some $a \in D_{l-1}(R)$, $b \in D_{m-1}(R)$. In this case,

$$\begin{aligned}
[r,[P,Q]] &= PrQ + aQ - QrP - bP - PQr + QPr \\
&= PQr - QPr - PQr + QPr - Qa + Pb + aQ - bP \\
&= Pb - bP + aQ - Qa \in D_{l+m-1}(R)
\end{aligned}$$

so $[P,Q]$ vanishes in $\operatorname{gr}_{l+m}^F D(R)$. We want to show this is an isomorphism that respects grading, so it suffices to construct an inverse on each graded component $\operatorname{gr}_l^F D(R) \to \operatorname{Sym}^i \operatorname{Der}_k(R)$. For $l = 0$, we have the obvious map $D_0(R) \xrightarrow{\sim} R$. Now given any $P \in D_l(R)$ consider the map $\varphi_P : R \to D_{l-1}(R) \twoheadrightarrow \operatorname{gr}_{l-1}^F D(R)$ given by $f \mapsto [P,f]$. Clearly, φ_P is a derivation in $\operatorname{Der}_k(R, \operatorname{gr}_{l-1}^F D(R))$. Observe that φ_P vanishes for $P \in D_{l-1}(R)$, so we have a map $\operatorname{gr}_l^F D(R) \to \operatorname{Der}_k(R, \operatorname{gr}_{l-1}^F D(R))$ defined by $P \mapsto \varphi_P$ and

$$\operatorname{Der}_k(R, \operatorname{gr}_{l-1}^F D(R)) = \operatorname{Der}_k(R) \otimes \operatorname{gr}_{l-1}^F D(R) \cong \operatorname{Der}_k(R) \otimes \operatorname{Sym}^{l-1} \operatorname{Der}_k(R) \twoheadrightarrow \operatorname{Sym}^l \operatorname{Der}_k(R)$$

where the isomorphism is given by induction. Explicitly, each $P \in D_l(R)$ is sent to ξQ for some $\xi \in \operatorname{Der}_k(R)$ and $Q \in \operatorname{gr}_{l-1}^F D(R)$ such that for each $f \in R$, $Q\xi(f) = [P,f]$. But then $(P - P(1))f = Pf - P(1)f = [P,f](1) = Q\xi(f)(1)$ which in the commutative algebra $\operatorname{gr}^F D(R)$ equals to $\xi(f)Q$. In conclusion, $P \mapsto \varphi_P$ is an inverse of the map $\operatorname{Sym}\operatorname{Der}_k(R) \to \operatorname{gr}^F D(R)$.

(ii) For the proof of the second statement, I'd like to extend a bit by quoting the following lemma

Lemma 1.1.20. *Let A be an almost commutative filtered k-algebra such that $F_0 A = k$ and A is generated by $F_1 A$, then $A \cong U(\mathfrak{g})/I$ for some Lie algebra $\mathfrak{g}$ and an ideal I of $U(\mathfrak{g})$.*

Proof. Since A is almost commutative, the commutator $[a,b] \in F_0 A \subseteq F_1 A$ for any $a, b \in F_1 A$. Thus, $[\cdot,\cdot]$ defines a Lie bracket on $\mathfrak{g} = F_1 A$. Therefore, there exists a map $U(\mathfrak{g}) \to A$ by the universal property of $U(\mathfrak{g})$ applied to the map $\mathfrak{g} \hookrightarrow A$. The map is surjective because A is generated by $\mathfrak{g}$. $\square$

Back to our case of $D(R)$. The vector space $\mathfrak{g} = D_1(R) = R \oplus \mathrm{Der}_k(R)$ clearly has a Lie bracket given by $[f, g] = 0$, $[\xi, f] = \xi(f)$ and $[\xi, \zeta] = \xi\zeta - \zeta\xi$ for any $f, g \in R$ and derivations ξ, ζ. Let I be the ideal identifying $1 \in U(\mathfrak{g})$ with $1 \in R$, multiplications in $U(\mathfrak{g})$ with multiplications in $\mathfrak{g}$. Then we have a map $U(\mathfrak{g})/I \to D(R)$ that respects grading, so we get $\mathrm{gr}^{PBW} U(\mathfrak{g})/I \to \mathrm{gr}^F D(R)$. With the relations defined by I, we get a surjection $\mathrm{Sym}\,\mathrm{Der}_k(R) \to \mathrm{gr}^{PBW} U(\mathfrak{g})/I$. Their composition is precisely the map constructed in (i), so $\mathrm{gr}^{PBW} U(\mathfrak{g})/I \cong \mathrm{gr}^F D(R)$ and by Lemma 1.1.8, $D(R) \cong U(\mathfrak{g})/I$ completing the proof. □

Remark 1.1.21. It is immediately apparent to the reader that differential operators are somehow associated to Lie algebras. We will investigate a much deeper connection later.

1.2 Sheaf of differential operators In this section we investigate differential operators on smooth algebraic varieties. We have so far studied affine/local data of differential operators. Indeed, we should be able to define, globally, differential operators on smooth algebraic varieties over some field k by pasting the data. From now on, fix a field k of characteristic zero. Again, by algebraic varieties over k I mean separated schemes over k of finite type that are geometrically integral. Of course, when k is algebraically closed, algebraic varieties are integral. We make the following definitions

Definition 1.2.1. Let X be a scheme over k. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{O}_X$-modules on X. Let U be an open of X. We define differential operators inductively. We define elements $P \in \mathrm{Diff}(\mathcal{M}|_U, \mathcal{N}|_U)_l$ to be k-morphisms

$$P : \mathcal{M}|_U \to \mathcal{N}|_U$$

such that for all local sections $f \in \mathcal{O}_X$, the morphism $s \mapsto fP(s) - P(fs) \in \mathrm{Diff}(\mathcal{M}|_U, \mathcal{N}|_U)_{l-1}$ (whenever they are defined). By convention for all $l < 0$, set $\mathrm{Diff}(\mathcal{M}|_U, \mathcal{N}|_U)_l = 0$. Let $\mathrm{Diff}(\mathcal{M}|_U, \mathcal{N}|_U)$ be the union of all such modules. Define the sheaf of differential operators of order $\leqslant l$, $F_l\mathcal{D}_X(\mathcal{M}, \mathcal{N})$, to be the sheaf $U \mapsto \mathrm{Diff}(\mathcal{M}|_U, \mathcal{N}|_U)_l$ and the **sheaf of differential operators** to be $\mathcal{D}_X(\mathcal{M}, \mathcal{N}) = \cup_{l=-1}^{\infty} F_l\mathcal{D}_X(\mathcal{M}, \mathcal{N})$.

But to make $\mathcal{D}_X(\mathcal{M}, \mathcal{N})$ quasicoherent, we need extra structure on $\mathcal{M}$ and $\mathcal{N}$.

Lemma 1.2.2. *If $X \to \mathrm{Spec}\, k$ is locally of finite presentation, $\mathcal{M}$ is coherent and $\mathcal{N}$ is quasicoherent, then each $F_l\mathcal{D}_X(\mathcal{M}, \mathcal{N})$ is quasicoherent.*

Proof. We will not prove this, as the general definition is rarely used in this paper. But we *will* prove the quasicoherence of $\mathcal{D}_X$ later (cf. Corollary 1.2.3) as well as a special case of the statement (cf. Corollary 1.2.5). The reader might consult IV.16.8.6 in [17] for a proof of this general statement. □

From now on, we also assume X to be a smooth algebraic variety over k. We have defined a (both left and right) $\mathcal{O}_X$-module $\mathcal{D}_X$ such that on each affine U it's just the algebra of differential operators (apply Lemma 1.2.2 to the coherent $\mathcal{O}_X$-modules $\mathcal{M} = \mathcal{N} = \mathcal{O}_X$). In Proposition 1.1.18, we computed sections of $\mathcal{D}_X$ at distinguished opens in an affine open. It is almost immediate that

Corollary 1.2.3. *Each (both left and right) $\mathcal{O}_X$-module $F_l\mathcal{D}_X$ is quasicoherent over $\mathcal{O}_X$. In particular, $\mathcal{D}_X$ is quasicoherent.*

Proof. On each affine $U = \mathrm{Spec}\, R \subseteq X$ and any non-nilpotent $f \in R$, by Proposition 1.1.18 we have

$$\Gamma(D(f), F_l\mathcal{D}_X) = D_l(R_f) = R_f \otimes_R D_l(R) = D_l(R) \otimes_R R_f$$

which suggests $F_l\mathcal{D}_X$ is quasicoherent as a left and right $\mathcal{O}_X$-module. □

Remark 1.2.4. Although we did not show the quasicoherence of general $\mathcal{D}_X(\mathcal{M},\mathcal{N})$, for $\mathcal{M}$ and $\mathcal{N}$ locally free, we can prove a similar result.

Corollary 1.2.5. *Given locally free $\mathcal{M}$ and $\mathcal{N}$, as $\mathcal{O}_X$-modules, we have*

$$\mathcal{D}_X(\mathcal{M},\mathcal{N}) \cong \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M}^\vee$$

where $\mathcal{M}^\vee$ denotes the locally free dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X)$. In particular, $\mathcal{D}_X(\mathcal{M},\mathcal{N})$ is quasicoherent when $\mathcal{M}$ and $\mathcal{N}$ are locally free.

Proof. First consider the isomorphism

$$\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{E}nd_k(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M}^\vee \xrightarrow{\sim} \mathcal{H}om_k(\mathcal{M},\mathcal{N})$$

given by $n \otimes F \otimes \mu \mapsto (x \mapsto F(\mu(x))n) \in \mathcal{H}om_k(\mathcal{M},\mathcal{N})$. Since $\mathcal{M}$ and $\mathcal{N}$ are locally free, they have local A-bases given by $\{e_i\}_{i\leqslant r}$ and $\{\varepsilon_j\}_{j\leqslant s}$ where r and s are the rank of $\mathcal{M}$ and $\mathcal{N}$. Denote by $\{e_i^\vee\}$ and $\{\varepsilon_j^\vee\}$ the dual bases (exist by local freeness) over A. Then the map above has an inverse

$$(\Phi : \mathcal{M} \to \mathcal{N}) \mapsto \sum_{i,j} \varepsilon_j \otimes \Phi_{ij} \otimes e_i^\vee \in \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{E}nd_k(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M}^\vee$$

where $\Phi_{ij} : \mathcal{O}_X \to \mathcal{O}_X$ is the k-linear morphism defined by

$$\Phi_{ij}(x) = \varepsilon_j^\vee(\Phi(x \cdot e_i)), \quad x \in \mathcal{O}_X.$$

The differential part of $\mathcal{H}om_k(\mathcal{M},\mathcal{N})$ is precisely $\mathcal{D}_X(\mathcal{M},\mathcal{N})$, so it remains show the differential part of the tensor product is $\mathcal{N} \otimes \mathcal{D}_X \otimes \mathcal{M}^\vee$. But this is easy, since the tensor product is taken over $\mathcal{O}_X$ so the action of $\mathcal{O}_X$ can be passed to $\mathcal{E}nd_k(\mathcal{O}_X)$, giving us $\mathcal{D}_X$ in the middle factor. $\square$

Suppose $\dim X = n$. Notice that since X is smooth, on each affine open $U \subseteq X$, $R = \Gamma(U, \mathcal{O}_X)$ is generated by some $x_1, \ldots, x_n$ and $\Gamma(U, \Theta_X)$ is generated by $\partial_1, \ldots, \partial_n$ with Θ_X being the tangent sheaf of X, meaning $\{x_i, \partial_i\}_{i\leqslant n}$ is an étale coordinate system. By Proposition 1.1.19, $\Gamma(U, \mathcal{D}_X)$ is the R-subalgebra generated by elements $f \in R$, $\xi \in \Gamma(U, \Theta_X)$ such that $[\xi, f] = \xi(f)$. Since all $F_l\mathcal{D}_X$ are quasicoherent, both statements in Proposition 1.1.19 can be globalized, obtaining the following result.

Proposition 1.2.6. *Let X be a smooth algebraic variety over k. Then $F_l\mathcal{D}_X$ is a quasicoherent filtration of $\mathcal{D}_X$ such that*

(i) *We have $\operatorname{gr}_l^F \mathcal{D}_X = F_l\mathcal{D}_X/F_{l-1}\mathcal{D}_X \cong \operatorname{Sym}^l \Theta_X$. That is, as graded $\mathcal{O}_X$-algebras,*

$$\operatorname{gr}^F \mathcal{D}_X = \bigoplus_{l=0}^{\infty} \operatorname{gr}_l^F \mathcal{D}_X \cong \operatorname{Sym} \Theta_X$$

(ii) *An a subsheaf of $\mathcal{E}nd_k(\mathcal{O}_X)$, $\mathcal{D}_X$ is generated by elements (i.e., local sections whenever they are defined) $f \in \mathcal{O}_X$ and $\xi \in \Theta_X$ such that $\xi\zeta - \zeta\xi = [\xi, \zeta]$ with the commutator in Θ_X, and $[\xi, f] = \xi(f)$.*

Therefore, on each affine open U we can write

$$\mathcal{D}_U = \bigoplus_{\alpha\in\mathbb{N}^n} \mathcal{O}_U \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \operatorname{gr}^F \mathcal{D}_X = \mathcal{O}_U[\xi_1, \xi_2, \ldots, \xi_n]$$

where ξ_i is the image of ∂_i in $\operatorname{gr}_l^F \mathcal{D}_X$.

2 Algebraic D-Modules

2.1 General theory We start with a general theory of D-modules.

Definition 2.1.1. A **left D-module** on X is an $\mathcal{O}_X$-module endowed with a left $\mathcal{D}_X$-module structure (i.e., on each open subset of X compatible with restrictions). Similarly a **right D-module** is an $\mathcal{O}_X$ module with a right $\mathcal{D}_X$-module structure.

Unless otherwise stated for the purpose of generality, we assume all D-modules in this paper are *quasicoherent*. By (ii) in Proposition 1.2.6, we can check easily if a left action of Θ_X on a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ determines a left D-module structure. The only concern is the action of $\xi \in \Theta_X$ on $\mathcal{M}$. We need for any such $f \in \mathcal{O}_X$ and $\xi \in \Theta_X$,

$$f \cdot (\xi \cdot m) = (f\xi) \cdot m, \quad \xi(f) \cdot m = \xi \cdot (f \cdot m) - f \cdot (\xi \cdot m), \quad [\xi, \zeta] \cdot m = \xi \cdot (\zeta \cdot m) - \zeta \cdot (\xi \cdot m)$$

If all relations above are satisfied, $\mathcal{M}$ is a left D-module. Similarly one can write down the criterion for right D-modules:

$$(m \cdot \xi) \cdot f = m \cdot (\xi f), \quad m \cdot \xi(f) = (m \cdot \xi) \cdot f - (m \cdot f) \cdot \xi, \quad [\xi, \zeta] \cdot m = (m \cdot \xi) \cdot \zeta - (m \cdot \zeta) \cdot \xi$$

I must remind the reader that the associativity axiom for right modules is given by $(x \cdot r) \cdot s = x \cdot (rs)$, which is why the two sides of the minus sign are somehow interchanged. Since $\mathcal{D}_X$ is not commutative, the left and right D-modules cannot be thought as the same for the moment. We will discuss some techniques that will eventually connect the two categories together. Before that, here are some examples:

Example 2.1.2. The $\mathcal{O}_X$-modules $\mathcal{D}_X$ and $\mathcal{O}_X$ are examples of D-modules on X. On $\mathcal{D}_X$, $\mathcal{D}_X$ acts by multiplications (both left and right). On $\mathcal{O}_X$, $\mathcal{D}_X$ acts on the *left* by differentiations.

Example 2.1.3. All A_n-modules in Section 1.1 are (global sections of) left D-modules on the affine space $\mathbb{A}^n_k$.

Example 2.1.4. The canonical bundle $\omega_{X/k} = \det(\Omega_{X/k}) = \wedge^n_{i=1} \Omega_{X/k}$ where $\Omega_{X/k}$ is the cotangent sheaf on X is a *right* D-module. For any $f \in \mathcal{O}_X$, $\xi \in \Theta_X$, the actions on $\omega \in \omega_{X/k}$ are given by

$$\omega \cdot f = f\omega, \quad w \cdot \xi = -\operatorname{Lie}_\xi \omega$$

where the Lie derivative of the top form ω is defined as usual: for any algebraic vector fields $\xi_1, \dots, \xi_n$,

$$\operatorname{Lie}_\xi \omega(\xi_1, \dots, \xi_n) = \xi(\omega(\xi_1, \dots, \xi_n)) - \sum_{i=1}^{n} \omega(\xi_1, \dots, [\xi, \xi_i], \dots, \xi_n)$$

Using a combinatorial argument (or simply assuming properties of Lie derivatives), the reader may check that

$$\operatorname{Lie}_\xi(f\omega) = f \operatorname{Lie}_\xi \omega + \xi(f)\omega = \operatorname{Lie}_{f\xi} \omega, \quad \operatorname{Lie}_{[\xi,\zeta]} \omega = \operatorname{Lie}_\xi \operatorname{Lie}_\zeta \omega - \operatorname{Lie}_\zeta \operatorname{Lie}_\xi \omega$$

and thus

$$\omega \cdot \xi(f) = (\omega \cdot \xi) \cdot f - (\omega \cdot f) \cdot \xi, \quad \omega \cdot [\xi, \zeta] = (\omega \cdot \xi) \cdot \zeta - (\omega \cdot \zeta) \cdot \xi$$

meaning $\omega_{X/k}$ is a right D-module. On each local affine open U with an étale coordinate system $\{x_i, \partial_i\}_{i \leqslant n}$, the action is given by

$$((f dx_1 \wedge \cdots \wedge dx_n) \cdot P)(x, \partial) = ({}^tP(x, \partial) f) dx_1 \wedge \cdots \wedge dx_n$$

where, given that $P(x, \partial) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha \in \mathcal{D}_U$, we define the adjoint of P as

$${}^tP(x, \partial) = \sum_{\alpha \in \mathbb{N}^n} (-\partial)^\alpha a_\alpha(x) \in \mathcal{D}_U$$

We denote by $\mathcal{D}_X^{\mathsf{op}}$ the opposite sheaf of algebras of $\mathcal{D}_X$. The *right* $\mathcal{D}_X$-modules can thus be identified with *left* $\mathcal{D}_X^{\mathsf{op}}$-modules. Indeed, as $\omega_{X/k}$ is a right $\mathcal{D}_X$-module, $\mathcal{D}_X^{\mathsf{op}}$ acts on $\omega_{X/k}$ on the left. Moreover, from the relation between P and ${}^tP$ in Example 2.1.4 and apply the isomorphism in Corollary 1.2.5, we see that

Lemma 2.1.5. *There is an isomorphism*

$$\mathcal{D}_X^{\mathsf{op}} \xrightarrow{\sim} \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_{X/k}^{\vee} \cong \mathcal{D}_X(\omega_{X/k}, \omega_{X/k})$$

of k-algebras.

To continue our discussion of right D-modules, we must investigate how tensor product and Hom spaces interact with left and right D-modules, as they constitute the most essential tools one uses to pass from left to right D-modules. We denote by $\mathrm{Mod}(\mathcal{D}_X)$ and $\mathrm{Mod}(\mathcal{D}_X^{\mathsf{op}})$ the category of left and right D-modules (assumed to be quasicoherent).

Proposition 2.1.6. *For $\mathcal{M}$, $\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_X)$ and $\mathcal{M}', \mathcal{N}' \in \mathrm{Mod}(\mathcal{D}_X^{\mathsf{op}})$, and any $\xi \in \Theta_X$,*

(i) $\mathcal{M} \otimes \mathcal{N} \in \mathrm{Mod}(\mathcal{D}_X)$*:* $\xi(x \otimes y) = \xi(x) \otimes y + x \otimes \xi(y)$.

(ii) $\mathcal{M}' \otimes \mathcal{N} \in \mathrm{Mod}(\mathcal{D}_X^{\mathsf{op}})$*:* $(x' \otimes y)\xi = (x')\xi \otimes y - x' \otimes \xi(y)$.

(iii) $\mathcal{H}om(\mathcal{M}, \mathcal{N}) \in \mathrm{Mod}(\mathcal{D}_X)$*:* $(\xi\varphi)(x) = \xi(\varphi(x)) - \varphi(\xi(x))$.

(iv) $\mathcal{H}om(\mathcal{M}', \mathcal{N}') \in \mathrm{Mod}(\mathcal{D}_X)$*:* $(\xi\varphi)(x) = -(\varphi(x))\xi + \varphi((x)\xi)$.

(v) $\mathcal{H}om(\mathcal{M}, \mathcal{N}') \in \mathrm{Mod}(\mathcal{D}_X^{\mathsf{op}})$*:* $(\varphi\xi)(x) = (\varphi(x))\xi + \varphi(\xi(x))$.

Proof. I think the only case worth checking is probably (ii). Clearly it's compatible with the tensor product over $\mathcal{O}_X$ since $(x'f)\xi \otimes y = (x')(\xi \cdot f) \otimes y - x'\xi(f) \otimes y$. Passing the $\mathcal{O}_X$ coefficients to the second term and apply $[\xi, f] = \xi(f)$ again, we see that $(x'f \otimes y)\xi = (x' \otimes fy)\xi$. It remains to verify the relation $[\xi, f] = \xi(f)$ and $[\xi, \zeta] = \xi\zeta - \zeta\xi$ as actions on the tensor product. This is easy:

$$\begin{aligned}(x' \otimes y)(\xi \cdot f) &= ((x')\xi \otimes y - x' \otimes \xi(y))f \\ &= (x'f)\xi \otimes y + (x' \otimes y) \cdot \xi(f) - x' \otimes (f \cdot \xi)(y) \\ &= (x' \otimes y)(f \cdot \xi) + (x' \otimes y)\xi(f)\end{aligned}$$

and similarly for the commutator $[\xi, \zeta]$. ☐

Now we know that $\omega_{X/k}$ is a right D-module from Example 2.1.4. So tensoring a left D-module on the left by $\omega_{X/k}$ gives us a right D-module. Similarly, for any right D-module $\mathcal{M}$, $\omega_{X/k}^{\vee} \otimes \mathcal{M} = \mathcal{H}om(\omega_{X/k}, \mathcal{M})$ is a left D-module! One thus obtains an equivalence of categories

Proposition 2.1.7. *The functor $\omega_{X/k} \otimes - : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_X^{\mathsf{op}})$ is an equivalence of categories with a quasi-inverse given by $\omega_{X/k}^{\vee} \otimes -$.*

2.2 Pushforwards and pullbacks Like $\mathcal{O}_X$-modules, the pushforwards and pullbacks of D-modules do not automatically gain D-module structures. Thus it is an essential task to define the correct objects that carry the desired structures. Unlike $\mathcal{O}_X$-modules, where pullbacks require a bit more effort, pullbacks of D-modules can be easily defined.

Let $f : X \to Y$ be a morphism of smooth algebraic varieties over k. Suppose $\mathcal{M}$ is a left D-module on Y. We consider the pullback of $\mathcal{M}$ as an $\mathcal{O}_X$-module: $f^*\mathcal{M} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}$. Recall that we have a homomorphism of $\mathcal{O}_X$-modules $df : \Theta_X \mapsto f^*\Theta_Y$: compose $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ with $f_*d_{Y/k} : f_*\mathcal{O}_Y \to f_*\Omega_{Y/k}$ to get a derivation $\mathcal{O}_X \to f_*\Omega_{Y/k}$. By universal property of the cotangent sheaf, we get a map $\Omega_{X/k} \to f_*\Omega_{Y/k}$. By adjointness, there is a map $f^*\Omega_{X/k} \to \Omega_{Y/k}$ and taking its dual we get $df : \Theta_X \mapsto f^*\Theta_Y$. Now we may define a left D-module structure on $f^*\mathcal{M}$. For any $\xi \in \Theta_X$, write $df(\xi) = \sum g_j \otimes \eta_j$ for $g_j \in \mathcal{O}_X$ and $\eta_j \in \Theta_Y$. Then define, for each $x \in \mathcal{O}_X$ and $m \in \mathcal{M}$,

$$\xi(x \otimes m) = \xi(x) \otimes m + x \sum_j g_j \otimes \eta_j(m)$$

Now since $\mathcal{D}_Y$ is a left D-module on Y, the previous procedure gives us a left D-module on X.

Definition 2.2.1. The **transfer bimodule** X **to** Y is $\mathcal{D}_{X\to Y} = f^*\mathcal{D}_Y$.

An essential property of this module is that, as its name suggested, it's a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule, where the left $\mathcal{D}_X$-module structure is defined in the previous paragraph and the right $f^{-1}\mathcal{D}_Y$-module structure is the obvious one. By the absorptions of tensor products, we get $f^*\mathcal{M} \cong \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M}$ as left $\mathcal{D}_X$-modules.

Remark 2.2.2. I want to clarify that $\mathcal{D}_{X\to Y}$ is a *right* $f^{-1}\mathcal{D}_Y$ but $f^{-1}\mathcal{M}$ is a *left* $f^{-1}\mathcal{D}_Y$-module. The difference here made the tensor product possible — since $\mathcal{D}_Y$ is not commutative, tensor products of two left modules make no sense, but tensoring one right module and one left module works.

Definition 2.2.3. Denote the pullback functor by $f^+ = \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}(\cdot) : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_Y)$ as the restriction of f^* to $\mathrm{Mod}(\mathcal{D}_X)$.

Clearly, f^+ is right exact, so it's acceptable compared to standard pushforwards, which we will try to construct now. Given a right D-module M on X, there is an obvious right $f^{-1}\mathcal{D}_Y$-module to define: $\mathcal{M}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ (still, a *right* $\mathcal{D}_X$-module tensoring a *left* $\mathcal{D}_X$-module). Pushing it to Y, we get a right $\mathcal{D}_Y$-module $f_*(\mathcal{M}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y})$. But the functor $f_*(-\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y})$ is not cool. In general, f_* is left exact and $-\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is right exact so a problem occurs whenever we want to do some homological algebra on D-modules.

Given a left D-module $\mathcal{M}$ on X, we can first turn it into a right D-module on X using the functor $\omega_{X/k}\otimes -$. But wait: the right D-module on X can be pushed to Y via $f_*(-\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y})$ defined above. So we get a right D-module on Y. But finally apply $\omega_{Y/k}^\vee \otimes -$ to the right D-module, we get a left D-module on Y. Summarizing, we associated a left $\mathcal{D}_Y$-module

$$f_+\mathcal{M} = \omega_{Y/k}^\vee \otimes_{\mathcal{O}_Y} f_*(\omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$$

To simplify the notation

$$\begin{aligned}
&\omega_{Y/k}^\vee \otimes_{\mathcal{O}_Y} f_*((\omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \\
&\quad = \omega_{Y/k}^\vee \otimes_{\mathcal{O}_Y} f_*((\omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} \mathcal{M}) \\
&\quad = f_*((\omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_{Y/k}^\vee) \otimes_{\mathcal{D}_X} \mathcal{M})
\end{aligned}$$

where the last equality follows from the projection formula. There is also a subtlety in the first equality where we used the fact that

$$(\omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \cong (\omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} \mathcal{M}$$

as right $f^{-1}\mathcal{D}_Y$-modules. The right $\mathcal{D}_X$-module structure of the bracket is given by the right module $\omega_{X/k}$. Note here the action of $f^{-1}\mathcal{D}_Y$ on the right hand side is $((\omega \otimes P) \otimes m)Q = (\omega \otimes PQ) \otimes m$ for any $Q \in \mathcal{D}_Y$. So the map $(\omega \otimes m) \otimes P \mapsto (\omega \otimes P) \otimes m$ is an $f^{-1}\mathcal{D}_Y$-module isomorphism, as the $f^{-1}\mathcal{D}_Y$-module structures are given component-wise.

Definition 2.2.4. The **transfer bimodule** Y **from** X is the $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule $\mathcal{D}_{Y\leftarrow X} = \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_{Y/k}^\vee$.

Definition 2.2.5. For a left D-module $\mathcal{M}$ on X, we define its pushforward $f_+\mathcal{M}$ to be the left $\mathcal{D}_Y$-module $f_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M})$.

Remark 2.2.6. As we mentioned before, f^+ is cool, yet f_+ is not for being neither left nor right exact. But if $f : X \hookrightarrow Y$ is a *closed immersion*, it is affine, which means $R^i f_* = 0$ for all $i > 0$ (since the higher direct images are precisely cohomology groups on preimages of affines, which are affine by our assumption). Thus, f_* is exact so f_+ is exact in this case, as $\mathcal{D}_{Y\leftarrow X}$ is locally free over $\mathcal{D}_X$.

2.3 Derived pushforwards and pullbacks To resolve the problem of nonexactness, we introduce pullbacks and pushforwards of morphisms in derived categories so that no information is lost.

Lemma 2.3.1. *Let X be a topological space and $\mathcal{R}$ be a sheaf of rings on X. Then the category of left $\mathcal{R}$-modules has enough injectives, and for any right $\mathcal{R}$-module $\mathcal{M}$ enough projectives with respect to $\mathcal{M} \otimes_{\mathcal{R}} -$ given by flat $\mathcal{R}$-modules.*

Proof. For enough injectives, see 2.4.3 in [25]. For enough $(\mathcal{M} \otimes_{\mathcal{R}} -)$-projectives, combine 2.4.12 and 2.4.13 in [25]. □

The lemma above allows us to define derived functors for certain functors in the category of D-modules.

Definition 2.3.2. For a sheaf of rings on a locally ringed space X, denote by $D^b(\mathcal{R})$, $D^-(\mathcal{R})$ and $D^+(\mathcal{R})$ the bounded, right bounded and left bounded derived category of the category of *not necessarily quasicoherent* $\mathcal{R}$-modules. On a smooth variety X, denote by $D^{\#}_{qc}(\mathcal{D}_X)$ the full subcategory of $D^{\#}(\mathcal{D}_X)$ consisting of complexes with *quasicoherent cohomology sheaves over* $\mathcal{O}_X$.

Since we have enough $f^+ = (\mathcal{D}_{X\to Y} \otimes -)$-projectives in the category of left $f^{-1}\mathcal{D}_Y$-modules, we can define the left derived functor of f^+ to be $\mathbf{L}f^+ : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ given by

$$\mathbf{L}f^+\mathcal{M}^\bullet = \mathcal{D}_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} \mathcal{M}^\bullet$$

Just to clarify the notations in what follows: we may consider an object in some abelian category as the complex with the object concentrated in degree 0, which is an object in the derived category.

Remark 2.3.3. I also need to address some subtlety here. Our definition of $D^b_{qc}(\mathcal{D}_X)$ is from [18], in which $\mathcal{D}_X$-modules are not forced to be quasicoherent over $\mathcal{O}_X$, although we did require D-modules to be quasicoherent in their definition. If we denote by $\mathrm{Mod}(\mathcal{D}_X)$ the category of (quasicoherent) left D-modules on X, $\mathrm{Mod}^*(\mathcal{D}_X)$ the category of all $\mathcal{D}_X$-modules, then there is an obvious inclusion $\mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}^*(\mathcal{D}_X)$. A complex $\mathcal{M}^\bullet$ of quasicoherent $\mathcal{D}_X$-modules clearly has quasicoherent cohomology sheaves. Therefore, the inclusion induces a map $D^b(\mathrm{Mod}(\mathcal{D}_X)) \to D^b_{qc}(\mathcal{D}_X)$. Indeed, Bernstein showed it's an equivalence of category:

Theorem 2.3.4 (Bernstein)**.** *Let $\mathcal{R}$ be a sheaf of algebras on a quasicompact separated scheme X. Denote by $\mathrm{Mod}(\mathcal{R})$ the category of quasicoherent $\mathcal{A}$-modules. Then there is an equivalence of categories*

$$D^b(\mathrm{Mod}(\mathcal{R})) \xrightarrow{\sim} D^b_{qc}(\mathcal{R})$$

Proof. For a proof see IV.2.10 in [9]. □

Furthermore, the equivalence holds for smaller categories of D-modules: coherent D-modules and holonomic D-modules (to be introduced). We can of course formulate the derived category with complexes of quasicoherent modules (c.f. [16] or [7]). There is no essential difference between the two: we will always construct a quasicoherent resolution. I follow the latter because I find it more interesting. Besides, such a property of cohomologies could be very important, especially in the case of holonomic D-modules.

Definition 2.3.5. We define a shifted derived pullback functor $f^!$ by

$$f^\dagger \mathcal{M}^\bullet = \mathbf{L}f^+\mathcal{M}^\bullet[\dim X - \dim Y]$$

This will be of interest in the next section.

Lemma 2.3.6. *Given $f : X \to Y$ and $g : Y \to Z$, then*

$$\mathbf{L}(g \circ f)^+ \simeq \mathbf{L}f^+ \circ \mathbf{L}g^+, \quad (g \circ f)^\dagger \simeq f^\dagger \circ g^\dagger$$

Proof. It suffices to show that $\mathcal{D}_{X\to Z}$ is isomorphic to $\mathcal{D}_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\to Z}$, in which case we have

$$\begin{aligned}
\mathbf{L}(g \circ f)^+\mathcal{M}^\bullet &= \mathcal{D}_{X\to Z} \otimes^{\mathbf{L}}_{(g\circ f)^{-1}\mathcal{D}_Y} (g \circ f)^{-1}\mathcal{M}^\bullet \\
&\cong \mathcal{D}_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\to Z} \otimes^{\mathbf{L}}_{(g\circ f)^{-1}\mathcal{D}_Y} (g \circ f)^{-1}\mathcal{M}^\bullet \\
&= \mathcal{D}_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} f^{-1}(\mathcal{D}_{Y\to Z} \otimes^{\mathbf{L}}_{g^{-1}\mathcal{D}_Y} g^{-1}\mathcal{M}^\bullet) \\
&= (\mathbf{L}f^+ \circ \mathbf{L}g^+)(\mathcal{M}^\bullet)
\end{aligned}$$

and the second equality follows from the fact $[\dim Y - \dim Z] \circ [\dim X - \dim Y] = [\dim X - \dim Z]$. The claim at the beginning is merely a derived version of the simple fact $\mathcal{D}_{X\to Z} \cong \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\to Z}$ by functoriality of $\mathcal{O}_X$-module pullbacks. Since $\mathcal{D}_Z$ and $\mathcal{D}_Y$ are locally free $\mathcal{O}_Z$- and $\mathcal{O}_Y$-modules resp. (though of infinite rank), the tensor product is exact so the derived version is trivially true. □

Lemma 2.3.7. *The functor $\mathbf{L}f^+$ sends $D^b_{qc}(\mathcal{D}_Y)$ to $D^b_{qc}(\mathcal{D}_X)$.*

Proof. An affine local argument works since $\mathbf{L}f^+$ commutes with pullbacks by inclusions of affines. But the proof would require a special case of Theorem 2.3.4, which we did not prove. So I will refer to (1) in 08DW of [36]. □

Example 2.3.8. When a morphism $f : X \to Y$ is smooth, its pullback f^* is exact for quasicoherent $\mathcal{O}_Y$-modules since f is flat (IV.2.1.3 in [17]). Therefore we have $H^i\mathbf{L}f^+\mathcal{M}^\bullet = 0$ for all $i > 0$ where $\mathcal{M}$ is a D-module on Y. We can drop the derived functor notation in this case.

Example 2.3.9. Given an open immersion $j : X \hookrightarrow Y$, j^* is exact. By definition $\mathcal{O}_X = j^{-1}\mathcal{O}_Y$ and thus $j^{-1}\mathcal{D}_Y = \mathcal{D}_X$. Therefore,

$$\mathcal{D}_{X\to Y} \cong \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X = \mathcal{D}_X$$

in which case $j^\dagger = \mathbf{L}j^+ = j^{-1}$ as $\dim X = \dim Y$.

Next we deal with the task of defining pushforwards. We do this in two ways. Given a general morphism $f : X \to Y$ of smooth algebraic varieties over k, we define

Definition 2.3.10. The derived pushforward f_+ of f is $\mathbf{R}f_*(\mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} -) : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ where we first use a flat resolution to derive the tensor product and then an injective resolution for the pushforward.

Example 2.3.11. Given an open immersion $j : X \hookrightarrow Y$, we've seen $\mathcal{D}_{Y\leftarrow X} = \mathcal{D}_X$ in a previous example, so $j_+ = \mathbf{R}j_*(\mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} -) = \mathbf{R}j_*(\mathcal{D}_X \otimes^{\mathbf{L}}_{\mathcal{D}_X} -) = \mathbf{R}j_*$.

Lemma 2.3.12. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth varieties. Then we have*

$$(g \circ f)_+ \simeq g_+ \circ f_+ : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Z)$$

Proof. We have

$$g_+(f_+\mathcal{M}^\bullet) = \mathbf{R}g_*(\mathcal{D}_{Z\leftarrow Y} \otimes^{\mathbf{L}}_{\mathcal{D}_Y} \mathbf{R}f_*(\mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}^\bullet))$$

and

$$(g \circ f)_+\mathcal{M}^\bullet = \mathbf{R}(g \circ f)_*(\mathcal{D}_{Z\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}^\bullet) = \mathbf{R}g_*(\mathbf{R}f_*(\mathcal{D}_{Z\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}^\bullet))$$

Similar to Lemma 2.3.6, it is easy to verify

$$\mathcal{D}_{Z\leftarrow X} \cong f^{-1}\mathcal{D}_{Z\leftarrow Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X}$$

Therefore, it remains to show

$$\mathcal{D}_{Z\leftarrow Y} \otimes^{\mathbf{L}}_{\mathcal{D}_Y} \mathbf{R}f_*(\mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}^\bullet) \cong \mathbf{R}f_*(f^{-1}\mathcal{D}_{Z\leftarrow Y} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}_Y} \mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}^\bullet)$$

which follows from the projection formula as $\mathcal{D}_{Z\leftarrow Y}$ is a locally free $\mathcal{D}_Y$-module. $\square$

For any morphism $f : X \to Y$, we should expect the image of $D^b_{qc}(\mathcal{D}_X)$ under f_+ to sit inside $D^b_{qc}(\mathcal{D}_Y)$. But this is no longer easy to show. Instead we will have to reformulate the definition of f_+. We can factor it using a projection and a closed immersion, i.e., we take $\iota : X \hookrightarrow Z = X \times Y$ to be the map $\mathrm{id} \times f$ and $p : Z \to Y$ the projection onto Y, meaning $f = p \circ \iota$. We know the *standard* pushforward via ι can be defined easily as $\iota_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M})$, which is quasicoherent. So it remains to compute and study the pushforward via p.

Definition 2.3.13. For any smooth algebraic variety X of dimension n, the **Spencer complex** $\mathrm{Sp}(\mathcal{O}_X)$ of $\mathcal{O}_X$ is the complex

$$0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X \to \mathcal{D}_X \to \mathcal{O}_X \to 0$$

The differentials $d : \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \Theta_X \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{k-1} \Theta_X$ are given by

$$\begin{aligned} d(P \otimes \xi_1 \wedge \cdots \wedge \xi_k) = &\sum_i (-1)^{i+1} P\xi_i \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_k \\ &+ \sum_{i<j} (-1)^{i+j} P \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_k \end{aligned}$$

and the last map $\mathcal{D}_X \to \mathcal{O}_X$ is given by $P \mapsto P(1)$.

Lemma 2.3.14. *The Spencer complex of $\mathcal{O}_X$ is a locally free resolution of the left $\mathcal{D}_X$-module $\mathcal{O}_X$.*

Proof. The Spencer complex $\mathrm{Sp}(\mathcal{O}_X)$ is filtered by subcomplexes

$$F_p \,\mathrm{Sp}(\mathcal{O}_X) : F_{p-n}\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X \to \cdots \to F_{p-1}\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X \to F_p\mathcal{D}_X \to \mathcal{O}_X$$

As the filtration is bounded below and exhaustive, the bounded homology spectral sequence

$$E^1_{p,q} = H_{p+q}(F_p \,\mathrm{Sp}(\mathcal{O}_X)/F_{p-1}\,\mathrm{Sp}(\mathcal{O}_X)) \implies H_{p+q}(\mathrm{Sp}(\mathcal{O}_X))$$

where the convergence is given by the classical convergence theorem. Thus, to show that $\mathrm{Sp}(\mathcal{O}_X)$ is a resolution which in our case is equivalent to being acyclic, it suffices to prove that each

$F_p \operatorname{Sp}(\mathcal{O}_X)/F_{p-1} \operatorname{Sp}(\mathcal{O}_X)$ is acyclic. But observe we have $\operatorname{gr}^F \operatorname{Sp}(\mathcal{O}_X)$ being the direct sum of these components, we can instead try to prove that $\operatorname{gr}^F \operatorname{Sp}(\mathcal{O}_X)$ is acyclic since direct sums are exact. Since Θ_X is locally free, we have

$$\operatorname{gr}^F \operatorname{Sp}(\mathcal{O}_X) : \operatorname{gr}^F \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X \to \cdots \to \operatorname{gr}^F D_X \otimes_{\mathcal{O}_X} \Theta_X \to \operatorname{gr}^F \mathcal{D}_X \to \mathcal{O}_X$$

This is a Koszul resolution of the regular sequence $\partial_1, \ldots, \partial_n$. By the vanishing theorem, the complex is acyclic and $\operatorname{Sp}(\mathcal{O}_X)$ is a resolution of $\mathcal{O}_X$ with each term being locally free over $\mathcal{D}_X$. □

Tensoring $\operatorname{Sp}(\mathcal{O}_X)$ by the locally free canonical sheaf $\omega_{X/k}$, we get a locally free resolution

$$0 \to \mathcal{D}_X \to \cdots \to \Omega^{n-1}_{X/k} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \Omega^n_{X/k} \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_{X/k} \to 0$$

of the right $\mathcal{D}_X$-module $\omega_{X/k}$. Back to our projections $p : Z \to Y$ and $q : Z \to X$. One sees that

$$\mathcal{D}_{Y \leftarrow Z} = \mathcal{D}_Y \boxtimes \omega_{X/k} = p^* \mathcal{D}_Y \otimes_{\mathcal{O}_Z} q^* \omega_{X/k}$$

Indeed, algebraically we have $\Omega_{B \otimes_A C/A} = (\Omega_{B/A} \otimes_A C) \oplus (\Omega_{C/A} \otimes_A B)$. Therefore there is an exact sequence of locally free sheaves of finite rank on Z

$$0 \to p^* \Omega_{Y/k} \to \Omega_{Z/k} \to q^* \Omega_{X/k} \to 0$$

Then taking the determinants, we get $\omega_{Z/k} = p^* \omega_{Y/k} \otimes_{\mathcal{O}_Z} q^* \omega_{X/k}$. The decomposition of $\mathcal{D}_{Y \leftarrow Z}$ then follows from its definition.

Definition 2.3.15. Let $n = \dim X = \dim Z - \dim Y$. For any left D-module $\mathcal{M}$ on Z, the **relative de Rham complex** $\mathcal{A}^\bullet_{Z/Y}$ associated to $\mathcal{M}$ is given by

$$\mathcal{A}^k_{Z/Y}(\mathcal{M}) = \Omega^{n+k}_{Z/Y} \otimes_{\mathcal{O}_Z} \mathcal{M}$$

for $k = -n, \ldots, 0$ where $\Omega^{n+k}_{Z/Y} = \mathcal{O}_Y \boxtimes \Omega^{n+k}_X$. The differentials are given by $d(\omega \otimes s) = d\omega \otimes s + \sum_{j=1}^n (dx_j \wedge \omega) \otimes \partial_j s$ for a coordinate system $\{x_j, \partial_j\}$ of X pulled back to Z.

Remark 2.3.16. Observe that $\mathcal{A}^k_{Z/Y}(q^* \mathcal{D}_X)$ is simply the pullback of the Spencer resolution of $\omega_{X/k}$ to Z, i.e., a locally free resolution of $q^* \omega_{X/k}$. One can therefore see that the relative de Rham complex is induced by the Spencer resolution, i.e., a free resolution of $\mathcal{D}_{Y \leftarrow Z} \otimes_{\mathcal{D}_X} \mathcal{M}$, meaning $\mathcal{D}_{Y \leftarrow Z} \otimes^{\mathbf{L}}_{\mathcal{D}_Z} \mathcal{M} \cong \mathcal{A}^\bullet_{Z/Y}(\mathcal{M})$ as complexes of $p^{-1}\mathcal{D}_Y$-modules, where the right hand side is a locally free complex.

Summarizing our discussion:

Proposition 2.3.17. *Let X, Y be smooth varieties, $Z = X \times Y$, q, p projections to the first, second factor resp. Then for any left $\mathcal{D}_Z$-module $\mathcal{M}$, we have $p_+ \mathcal{M} \cong \mathbf{R} p_* (\mathcal{A}^\bullet_{Z/Y}(\mathcal{M}))$. Moreover, the image of $D^b_{qc}(\mathcal{D}_Z)$ under p_+ sits inside $D^b_{qc}(\mathcal{D}_Y)$.*

Proof. We know $\mathcal{A}^\bullet_{Z/Y}(\mathcal{M})$ is quasicoherent, so the claim follows from the general fact (1) in 08D5 of [36]. □

Given $f_+ = p_+ \circ \iota_+$ where $\iota = \operatorname{id} \times f : X \hookrightarrow Z = X \times Y$ and $p : Z = X \times Y \to Y$, the final task is to check if ι_+ preserves quasicoherent cohomologies. But this is always the case, even for more interesting D-modules, due to the following lemma

Lemma 2.3.18. *If $\iota : X \hookrightarrow Y$ is a closed immersion, we have*

$$H^k(\iota_+ \mathcal{M}^\bullet) \cong \iota_+ (H^k \mathcal{M}^\bullet)$$

for all k and complexes $\mathcal{M}^\bullet$ of $\mathcal{D}_X$-modules on X.

Proof. Recall in Remark 2.2.6, we observed the standard pushforward via ι is exact (both ι_* and $(\mathcal{D}_{Y\leftarrow X}\otimes -)$ are exact). Therefore, $\iota_+\mathcal{M}^\bullet = \iota_*(\mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{D}_X}\mathcal{M}^\bullet)$. The claim is then apparent from exactness. □

The image of $D^b_{qc}(\mathcal{D}_X)$ under ι_+ therefore lies inside $D^b_{qc}(\mathcal{D}_Z)$. Since $f_+ = p_+\circ\iota_+$, combining Proposition 2.3.17, we finally obtain

Proposition 2.3.19. *Let $f : X \to Y$ be an arbitrary morphism of smooth algebraic varieties. Then f_+ sends $D^b_{qc}(\mathcal{D}_X)$ to $D^b_{qc}(\mathcal{D}_Y)$.*

2.4 Closed immersions and Kashiwara's equivalence Suppose $\iota : X \hookrightarrow Y$ is a closed immersion of smooth algebraic varieties over k. As we mentioned, their pullbacks and pushforwards have nice properties. In this section, I will state and prove the first interesting result in this essay: Kashiwara's equivalence on closed immersions. We have defined the (standard) pullback of D-modules on Y in a canonical way, but it is also possible to define another pullback $\iota^! : \mathrm{Mod}(\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_X)$ given by

$$\iota^!\mathcal{M} = \mathcal{H}om_{\iota^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \iota^{-1}\mathcal{M})$$

We use the left $\iota^{-1}\mathcal{D}_Y$ structure of the transfer bimodule and $\iota^{-1}\mathcal{M}$ to get the sheaf hom, and the left $\mathcal{D}_X$-module structure of the sheaf hom is given by the right $\mathcal{D}_X$-module structure of the transfer bimodule via precomposing morphisms with actions of $\mathcal{D}_X$ on the right. This pullback is not completely unreasonable. In fact, it is closely related to the shifted derived pullback $\iota^\dagger\mathcal{M}^\bullet = \mathbf{L}\iota^+\mathcal{M}^\bullet[\dim X - \dim Y]$:

Lemma 2.4.1. *The derived pullback $\iota^\dagger$ is the derived functor of $\iota^!$, that is, $\iota^\dagger\mathcal{M}^\bullet \cong \mathbf{R}\iota^!\mathcal{M}^\bullet$.*

Proof. This lemma is never used in this essay. For a proof see 1.5.16 in [18]. □

To understand the nature of ι_+ and ι^+, we need to write down their local descriptions. Since $\iota : X \hookrightarrow Y$ is a closed immersion of smooth varieties, for any $x \in X$, there is an affine open U of Y with local coordinates $\{y_i, \partial_{y_i}\}_{i=1,\dots,n}$ such that $x \in U$ and $\iota(X)\cap U$ is defined by equations $y_{r+1} = \cdots = y_n = 0$ for some r. Write $x_i = y_i \circ \iota$ for $i = 1,\dots,r$. The preimage $V = \iota^{-1}(U)$ has a local coordinate $\{x_i, \partial_i\}$ (see 0H1G in [36]). The canonical morphism $\Theta_X \to \iota^*\Theta_Y = \mathcal{O}_X\otimes_{\iota^{-1}\mathcal{O}_Y}\Theta_Y$ is given by $\partial_i \mapsto 1\otimes\partial_{y_i}$ in the exact sequence

$$0 \to \Theta_X \to \iota^*\Theta_Y \to \mathcal{N}_{X/Y} = (\mathcal{I}_X/\mathcal{I}_X^2)^\vee \to 0$$

where $\mathcal{I}_X$ is the ideal sheaf of ι.

Example 2.4.2. Consider the subring $\mathcal{D}_Y^X$ of $\mathcal{D}_Y$ given by $\bigoplus_\alpha \mathcal{O}_Y\partial_{y_1}^{\alpha_1}\cdots\partial_{y_r}^{\alpha_r}$. It is clear that $\mathcal{D}_Y^X\otimes_k k[\partial_{y_{r+1}},\dots,\partial_{y_n}] \cong \mathcal{D}_Y$ via $Q\otimes P\mapsto QP$ from their local descriptions. But then $\mathcal{D}_{X\to Y} = \iota^*\mathcal{D}_Y$ which is just $\iota^*\mathcal{D}_Y^X\otimes_k k[\partial_{y_{r+1}},\dots,\partial_{y_n}]$. Here $\iota^*\mathcal{D}_Y^X$ is precisely the image of $\mathcal{D}_X$ under the injective map $\Theta_X \to \iota^*\Theta_Y$ in $\mathcal{D}_{X\to Y}$, so $\mathcal{D}_X \cong \iota^*\mathcal{D}_Y^X$. Hence, affine locally, $\mathcal{D}_{X\to Y} \cong \mathcal{D}_X\otimes_k k[\partial_{y_{r+1}},\dots,\partial_{y_n}]$, a locally free $\mathcal{D}_X$-module of infinite rank. In other words, $\mathcal{D}_{X\to Y} = \mathcal{D}_Y/\mathcal{I}_X\mathcal{D}_Y$.

Example 2.4.3. Similarly we compute $\mathcal{D}_{Y\leftarrow X}$ affine locally. By definition,

$$\mathcal{D}_{Y\leftarrow X} = \iota^{-1}\mathcal{D}_Y\otimes_{\iota^{-1}\mathcal{O}_Y}\mathcal{O}_X\otimes_{\mathcal{O}_X}(\iota^{-1}\omega_{Y/k}^\vee\otimes_{\iota^{-1}\mathcal{O}_Y}\omega_{X/k})$$

We've shown that $\mathcal{D}_Y^X\otimes_k k[\partial_{y_{r+1}},\dots,\partial_{y_n}] \cong \mathcal{D}_Y$, so

$$\mathcal{D}_{Y\leftarrow X} \cong k[\partial_{y_{r+1}},\dots,\partial_{y_n}]\otimes_k(\iota^{-1}\mathcal{D}_Y^X\otimes_{\iota^{-1}\mathcal{O}_Y}\mathcal{O}_X) \cong k[\partial_{y_{r+1}},\dots,\partial_{y_n}]\otimes_k\mathcal{D}_X$$

where we identified $\iota^{-1}\omega_{Y/k}^\vee\otimes_{\iota^{-1}\mathcal{O}_Y}\omega_{X/k}$ with $\mathcal{O}_X$ in the first congruence (we are working closed immersions on affine opens so this is true). The $\iota^{-1}\mathcal{D}_Y$-module structure on this new expression is not obvious. Observe that $\iota^{-1}\mathcal{D}_Y = k[\{\partial_{y_i}\}] \otimes_k \iota^{-1}\mathcal{D}_Y^X$. It suffices to determine the action of each component on the module. The first term $k[\{\partial_{y_i}\}]$ clearly acts on $\mathcal{D}_{Y\leftarrow X}$ by multiplications on the first component. Now, for any $Q \in \iota^{-1}\mathcal{D}_Y^X$, write $QS = \sum S_k Q_k$ for $Q_k \in \iota^{-1}\mathcal{D}_Y^X$ and $S, S_k \in k[\{\partial_{y_i}\}]$. Then for any P of $\mathcal{D}_X$, we have $Q(S \otimes P) = \sum S_k \otimes Q_k P$, where each $Q_k \in \iota^{-1}\mathcal{D}_Y^X$ acts on $\mathcal{D}_X \cong \iota^{-1}\mathcal{D}_Y^X \otimes_{\iota^{-1}\mathcal{O}_Y} \mathcal{O}_X$ in the obvious way. We see that since $\iota^{-1}\mathcal{D}_Y^X$ is the image of $\mathcal{D}_X$ in $\iota^{-1}\mathcal{D}_Y$, the transfer bimodule $\mathcal{D}_{Y\leftarrow X}$ is generated as an $\iota^{-1}\mathcal{D}_Y$-module by $1 \otimes 1$.

Example 2.4.4. It is immediate that $\iota_+\mathcal{M}$ is locally isomorphic to $k[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_k \iota_*\mathcal{M}$ for any left $\mathcal{D}_X$-module $\mathcal{M}$. The pushforward and inverse image preserve the polynomial ring via an easy argument on sections. Intuitively, ι_+ provides actions of $\mathcal{D}_Y$ on $\iota_*\mathcal{M}$ by adding copies of $\partial_{y_{r+1}}, \ldots, \partial_{y_n}$ besides the already existing ∂_i on X.

More importantly, we can directly relate $\iota^!$ to ι_+.

Proposition 2.4.5. *The functors ι_+ and $\iota^!$ are adjoints between the category of all $\mathcal{D}_X$-modules and the category of $\mathcal{D}_Y$-modules, i.e., for any left $\mathcal{D}_X$-module $\mathcal{M}$ and left $\mathcal{D}_Y$-module $\mathcal{N}$,*

$$\mathrm{Hom}_{\mathcal{D}_Y}(\iota_+\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \iota^!\mathcal{N})$$

Proof. The idea is the adjunction between $\mathcal{H}om$ and tensor products, i.e., we have an isomorphism of sheaves of abelian groups

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{H}om_{\iota^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \iota^{-1}N)) \xrightarrow{\sim} \mathcal{H}om_{\iota^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}, \iota^{-1}\mathcal{N}) \tag{1}$$

given by $\varphi \mapsto (P \otimes s \mapsto \varphi(s)(P))$ with inverse $\psi \mapsto (s \mapsto (P \mapsto \psi(P \otimes s)))$. Since ι is an immersion, the canonical map $\iota^{-1}\iota_*$ is an isomorphism of sheaves. Taking the global sections of both sides of Eq. (1), one arrives at

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \iota^!\mathcal{N}) &\cong \mathrm{Hom}_{\iota^{-1}\mathcal{D}_Y}(\iota^{-1}\iota_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}), \iota^{-1}\mathcal{N}) &(2)\\ &\cong \mathrm{Hom}_{\mathcal{D}_Y}(\iota_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}), \iota_*\iota^{-1}\mathcal{N}) &(3)\end{aligned}$$

where the second congruence follows from general adjunction results such as 008Y in [36].

It remains to show the last Hom space is isomorphic to the desired result. But this is not immediate, since in general $\iota_*\iota^{-1}\mathcal{N}$ is not isomorphic to $\mathcal{N}$. But a stalk-wise argument shows that if $\mathcal{N}$ is supported on X, then $\iota_*\iota^{-1}\mathcal{N} \cong \mathcal{N}$. So our aim is to replace $\mathcal{N}$ with a sheaf supported on X. Let $\mathcal{N}^X$ be the subsheaf $V \mapsto \mathcal{N}^X(V)$ with the latter being sections in $\Gamma(\mathcal{N}, V)$ supported on $X \subseteq Y$. To achieve the goal, we claim that for any $\mathcal{D}_X$-module $\mathcal{M}'$ and $\mathcal{D}_Y$-module $\mathcal{N}'$

$$\mathcal{H}om_{\mathcal{D}_Y}(\iota_*\mathcal{M}', \mathcal{N}') \cong \mathcal{H}om_{\mathcal{D}_Y}(\iota_*\mathcal{M}', \mathcal{N}'^X)$$
$$\mathcal{H}om_{\iota^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \iota^{-1}\mathcal{N}') \cong \mathcal{H}om_{\iota^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \iota^{-1}(\mathcal{N}')^X)$$

so that every $\mathcal{N}$ in Eq. (1) and Eq. (2) can be replaced by $\mathcal{N}^X$, in which case we are done. The first congruence is immediate: all sections of $\iota_*\mathcal{M}'$ vanish on small enough opens around points outside X. To wit the second, it suffices to show $\psi(s) \in \iota^{-1}(\mathcal{N}')^X$ for any $\psi \in \mathcal{H}om_{\iota^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \iota^{-1}\mathcal{N}')$ and $s \in \mathcal{D}_{Y\leftarrow X}$. The question is of a local nature, so we can write $\mathcal{D}_{Y\leftarrow X} \cong k[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_k \mathcal{D}_X$ by Example 2.4.3. As an $\iota^{-1}\mathcal{D}_Y$-module, the right hand side is generated by $1 \otimes 1$. Let $\mathcal{I}_X \subseteq \mathcal{O}_Y$ be the ideal sheaf of X. Then a section of $\iota^{-1}\mathcal{N}'$ killed by $\iota^{-1}\mathcal{I}_X$ is a section of $\iota^{-1}(N')^X$ since a section of $\mathcal{N}$ killed by $\mathcal{I}_X$ is supported on X. Now $\iota^{-1}\mathcal{I}_X(1 \otimes 1)$ is zero by construction (use the expression in Example 2.4.3 and the fact that $\mathcal{I}_X$ annihilates $\mathcal{D}_X$), and since ψ is $\iota^{-1}\mathcal{D}_Y$ linear, $\psi(s)$ sits inside $\iota^{-1}(\mathcal{N}')^X$.

In conclusion, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \iota^! \mathcal{N}) &\cong \mathrm{Hom}_{\mathcal{D}_Y}(\iota_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}), \iota_* \iota^{-1} \mathcal{N}^X) \\ &\cong \mathrm{Hom}_{\mathcal{D}_Y}(\iota_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}), \mathcal{N}^X) \\ &= \mathrm{Hom}_{\mathcal{D}_Y}(\iota_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}), \mathcal{N}) = \mathrm{Hom}_{\mathcal{D}_Y}(\iota_+ \mathcal{M}, \mathcal{N})\end{aligned}$$

completing the proof. □

Remark 2.4.6. I have to make the annihilator argument clear here because I personally struggled with the proof above for a while. First, define $\mathcal{M}_X$ to be the submodules of section m such that for any $f \in \mathcal{I}_X$, there is some power of f that kills m. We claim that $\mathcal{M}$ is supported on X if and only if $\mathcal{M}_X = \mathcal{M}$. Note that this claim is local, and as X is Noetherian $\mathcal{I}_X$ is coherent. Suppose $y \notin X$, it can be checked affine locally that there is some $f \in \mathcal{I}_X$ invertible in $\mathcal{O}_{Y,y}$, meaning all sections of $\mathcal{M}_X$ vanish in the stalk at y. Thus, $\mathcal{M}_X$ is supported on X. Conversely, we show that every $\mathcal{D}_Y$-submodule of $\mathcal{M}$ supported on X is contained in $\mathcal{M}_X$, i.e., $\mathcal{M}_X$ is the maximal submodule of $\mathcal{M}$ supported on X. Say m is a section of one such submodule. Let $\mathcal{N}$ be the $\mathcal{O}_X$-submodule generated by m. Then since $\mathcal{N}$ is finite, by the annihilator-support theorem, the support of $\mathcal{N}$ is cut out by its annihilator $\mathcal{J} \subseteq \mathcal{O}_X$. Then Hilbert's Nullstellensatz gives us $\mathcal{I}_X \subseteq \sqrt{\mathcal{J}}$, completing the proof. In the proof of the previous proposition, we used the fact that if $\mathcal{I}_X$ annihilates a section then the section has support in X.

Remark 2.4.7. In the proof of Proposition 2.4.5, we actually showed more. We proved that for any $\psi \in \mathcal{H}om_{\iota^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \iota^{-1}\mathcal{N}')$ and $s \in \mathcal{D}_{Y\leftarrow X}$, $\varphi(s)$ is annihilated by $\iota^{-1}\mathcal{I}_X$. Thus it's easy to identify $\iota^! \mathcal{N}$ with the subsheaf of $\iota^{-1}\mathcal{N}$ annihilated by $\iota^{-1}\mathcal{I}_X$. Recall in Example 2.4.4 we commented that ι_+ "added copies of ∂_{y_i}" to $\iota_*\mathcal{M}$. Note that $\iota_*\mathcal{M}$ as usual is annihilated by $\mathcal{I}_X$. Therefore intuitively, the pullback $\iota^!$ recovers the part annihilated by $\mathcal{I}_X$ in a $\mathcal{D}_Y$-module.

As usual, the category of quasicoherent $\mathcal{D}_X$-modules I denote by $\mathrm{Mod}(\mathcal{D}_X)$. In addition, denote by $\mathrm{Mod}^X(\mathcal{D}_Y)$ the full subcategory of quasicoherent $\mathcal{D}_Y$-modules supported on $X \subseteq Y$ in $\mathrm{Mod}(\mathcal{D}_Y)$. Then

Theorem 2.4.8 (Kashiwara's equivalence)**.** *The standard pushforward functor ι_+ induces an equivalence of categories $\mathrm{Mod}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}^X(\mathcal{D}_Y)$ with a quasi-inverse given by $\iota^!$.*

Proof. Since ι_+ and $\iota^!$ are adjoints by Proposition 2.4.5, we have to show the canonical unit $\mathrm{id} \to \iota^! \iota_+$ and counit $\iota_+ \iota^! \to \mathrm{id}$ are isomorphisms. The problem can be checked affine locally so we may assume $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} R/I$ where R is a k-algebra with generators $y_1, \dots, y_n$ and $I = (y_{r+1}, \dots, y_n)$ for some r. Let X_i be the closed subscheme cut out by the last i elements in the ideal. By an induction on i, it suffices to prove the theorem for $i = 1$, i.e., $X = \operatorname{Spec} R/(y)$ for some y. The local coordinates in Example 2.4.3 is then $\{y, \partial\}$.

The isomorphism $\mathcal{M} \to \iota^! \iota_+ \mathcal{M}$ is not hard to see. Note that $\iota_*\mathcal{M}$ is killed by y. We first claim that $\iota_+\mathcal{M} = k[\partial] \otimes_k \iota_*\mathcal{M}$ is supported on X. Indeed, for any $m \in \iota_*\mathcal{M}$, we have

$$y(\partial \otimes m) = -1 \otimes m + \partial \otimes ym = -1 \otimes m$$

using the expression given in Example 2.4.3 and $[\partial, y] = 1$. Thus, $y^{p+1}(\partial^p \otimes m) = 0$ for any $m \in \iota_*\mathcal{M}$. Therefore, by Remark 2.4.6, $\iota_+\mathcal{M}$ is supported on X. Notice that $\iota^! \iota_+ \mathcal{M}$ is isomorphic to the subsheaf of $\iota^{-1}\iota_+\mathcal{M}$ annihilated by y. The reader has in fact seen this in the proof of Proposition 2.4.5 and Remark 2.4.7! Therefore it remains to compute the subsheaf of $\iota^{-1}\iota_+\mathcal{M}$ killed by $\mathcal{I}_X$. Still we write $\iota^{-1}\iota_+\mathcal{M} = k[\partial] \otimes_k \iota^{-1}\iota_*\mathcal{M} \cong k[\partial] \otimes_k \mathcal{M}$. Then by induction

$$y(\partial^p \otimes m) = -p\partial^{p-1} \otimes m$$

which is zero if and only if $p = 0$ or $m = 0$. Thus, the subsheaf of $\iota^{-1}\iota_+\mathcal{M}$ killed by $\iota^{-1}\mathcal{I}_X$ is precisely isomorphic to $\mathcal{M}$, i.e., $\iota^! \iota_+ \mathcal{M} \cong \mathcal{M}$.

Conversely, fix some $\mathcal{N} \in \mathrm{Mod}^X(\mathcal{D}_Y)$. Then $\iota^!\mathcal{N}$ is (isomorphic to) the subsheaf of $\iota^{-1}\mathcal{N}$ killed by y, i.e., it is the kernel of the map $y : \iota^{-1}\mathcal{N} \to \iota^{-1}\mathcal{N}$. Denote by θ the operator $y\partial : \mathcal{N} \to \mathcal{N}$. For any $j \in \mathbb{Z}$, let $\mathcal{N}^j$ be the eigenspace of θ with an eigenvalue j. We claim that

$$\mathcal{N} = \bigoplus_{j=1}^{\infty} \mathcal{N}^{-j}$$

Since $\mathcal{N}$ is supported on X, as before we have $\mathcal{N} = \cup_{k=1}^{\infty} \ker y^k$. So it suffices to show

$$\ker y^k \subseteq \bigoplus_{j=1}^{k} \mathcal{N}^{-j}$$

For $k = 1$ this is obvious: $yn = 0$ means

$$\theta n = y\partial n = (\partial y - 1)n = -n$$

Now assuming the assertion holds for k. Then for any section n killed by y^{k+1} we have

$$yn \in \ker y^k \subseteq \bigoplus_{j=1}^{k} \mathcal{N}^{-j}$$

Compute that if $n' \in \mathcal{N}^j$,

$$\theta(yn') = y(\theta + 1)n' = (j + 1)yn', \quad \theta(\partial n') = \partial\theta n' - \partial n' = (j - 1)\partial n'$$

From this we see that $\partial y = \theta + 1$ is an isomorphism $\mathcal{N}^j \to \mathcal{N}^j$ for $j \neq -1$, and for $j < -1$, the maps $y : \mathcal{N}^j \to \mathcal{N}^{j+1}$ and $\partial : \mathcal{N}^{j+1} \to \mathcal{N}^j$ are isomorphisms. Applying ∂ to $yn \in \oplus_{j=1}^{k}\mathcal{N}^{-j}$, the direct sum becomes $\oplus_{j=2}^{k+1}\mathcal{N}^{-j}$. Similarly, using the relation $[\partial, y] = 1$ we deduce

$$y^k(\theta n + (k + 1)n) = y^{k+1}\partial n + (k + 1)y^k n = \partial y^{k+1} n = 0$$

meaning $(\theta + k + 1)n \in \ker y^k \subseteq \oplus_{j=1}^{k}\mathcal{N}^{-j}$. The final touch is the difference

$$kn = (\theta + k + 1)n - (\theta + 1)n = (\theta + k + 1)n - \partial yn \in \bigoplus_{j=1}^{k} \mathcal{N}^{-j} + \bigoplus_{j=2}^{k+1} \mathcal{N}^{-j} \subseteq \bigoplus_{j=1}^{k+1} \mathcal{N}^{-j}$$

and as the base field has characteristic zero, n sits inside the direct sum. Since $\partial\mathcal{N}^j \cong \mathcal{N}^{j-1}$, we have $\mathcal{N} = k[\partial] \otimes_k \mathcal{N}^{-1}$. Yet

$$\theta n = -n \iff \partial yn - n = -n \iff yn = 0$$

as ∂ is an isomorphism, meaning $\iota^!\mathcal{N} = \ker y = \iota^{-1}\mathcal{N}^{-1}$. Again, as $\mathcal{N}^{-1}$ is supported on X, $\iota_*\iota^{-1}\mathcal{N}^{-1} \cong \mathcal{N}^{-1}$, i.e.,

$$\mathcal{N} \cong k[\partial] \otimes_k \iota_*\iota^{-1}\mathcal{N}^{-1} \cong k[\partial] \otimes_k \iota_*\iota^!\mathcal{N} = \iota_+\iota^!\mathcal{N}$$

completing the proof. $\square$

Kashiwara's equivalence is not trivial and demonstrates that $\mathcal{D}_X$-module structures are very rigid. An immediate contrast arises from the annihilators of $\iota_+\mathcal{M}$ and $\iota_*\mathcal{F}$ (here $\mathcal{M}$ any D-module and $\mathcal{F}$ is an arbitrary $\mathcal{O}_X$-module). We see that $\mathcal{I}_X$ annihilates $\iota_*\mathcal{F}$ but not $\iota_+\mathcal{M}$, even though sections in the latter are killed by powers of $\mathcal{I}_X$.

Example 2.4.9. Given a closed immersion $\iota : X \to Y$, we can construct an obvious D-module on Y by $\mathcal{B}_{X|Y} = \iota_+ \mathcal{O}_X$ (here I'm using the notation from [16, 18], although I'm not sure what $\mathcal{B}$ stands for). Using the same coordinate system in Example 2.4.3, the module can locally be written as $k[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_k \iota_* \mathcal{O}_X \cong k[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_k \mathcal{O}_Y / \mathcal{I}_X$. Let $\mathcal{J}$ be the $\mathcal{D}_Y$-submodule generated by $\{\partial_{y_i}\}_{i=1,\ldots,r}$ and $\{y_i\}_{i=r+1,\ldots,n}$. Then $\mathcal{B}_{X|Y} = \mathcal{D}_Y / \mathcal{J}$.

Suppose X is a point $y \in Y$, cut out by a maximal ideal $\mathfrak{m}_y = (y_1, \ldots, y_n)$. In this case $r = 0$ so $\mathcal{B}_{y|Y} = \mathcal{D}_Y / \mathcal{D}_Y \mathfrak{m}_y$. Take some $\delta_y = 1 \bmod \mathfrak{m}_y$, we can rewrite $\mathcal{B}_{y|Y}$ as $\mathcal{D}_Y \cdot \delta_y$. In the sense of Example 1.1.2, $\mathcal{B}_{y|Y}$ corresponds to the system of differential equations given by $y_i u = 0$ for all i, which has the Dirac delta function at y as a solution. Since D-modules on y are simply k-vector spaces, $\mathrm{Mod}^y(\mathcal{D}_Y)$ is equivalent to the category of k-vector spaces, where $\mathcal{B}_{y|Y}$ corresponds to k. Thus, D-modules on Y supported by y are direct sums of $\mathcal{B}_{y|Y}$. On the other hand, $\mathcal{O}_X$-modules supported at a point are much more flexible. For instance, take $Y = \operatorname{Spec} k[t]$, y the origin and $\mathcal{N} = \mathcal{O}_Y / (t^2)$. Then clearly $\mathcal{N}$ is supported at y and t acts nontrivially on $\mathcal{N}$. Thus, $\mathcal{N}$ is not the pushforward of any $\mathcal{O}_{\{y\}}$-modules. In general, D-modules cannot have nontrivial nilpotent sections, contrary to $\mathcal{O}_X$-modules.

To discuss other important properties of D-modules and results of Kashiwara's equivalence, we will introduce more types of D-modules that fit into the theory.

2.5 D-affinity Suppose a scheme X is Noetherian. A well-known result due to Serre states that X is affine if and only if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and quasicoherent $\mathcal{O}_X$-modules $\mathcal{F}$, and $\mathcal{F}$ is zero if it has no global sections. In other words, the global section functor $\Gamma(X, -)$ is exact. We will study a similar property of varieties X by considering quasicoherent $\mathcal{D}_X$-modules.

Definition 2.5.1. A smooth algebraic variety X is **$\mathcal{D}_X$-affine** if the global section functor $\Gamma(X, -) : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\Gamma(X, \mathcal{D}_X))$ is exact, and if a D-module $\mathcal{M}$ has no global sections, $\mathcal{M} = 0$.

Lemma 2.5.2. *Smooth affine varieties are $\mathcal{D}_X$-affine.*

Proof. The global section functor of a smooth affine variety X is exact in the category of $\mathcal{O}_X$-modules, and so is its restriction to $\mathrm{Mod}(\mathcal{D}_X)$. The second condition is clear for quasicoherent modules on affine schemes. □

Remark 2.5.3. There are very few known examples of $\mathcal{D}_X$-affine spaces. It is conjectured that the only $\mathcal{D}_X$-affine varieties are affine varieties and partial flag varieties (to be introduced).

One particularly important (and so far the first concrete) consequence of Kashiwara's equivalence is the $\mathcal{D}_X$-affinity of projective spaces:

Theorem 2.5.4. *The fiber product of a projective space and a smooth affine variety is $\mathcal{D}_X$-affine.*

Proof. Let $X = \mathbb{P}^n_k \times_k Y$ where $Y = \operatorname{Spec} A$ is smooth affine. Consider the open subscheme $\widetilde{V} = \mathbb{A}^{n+1}_k \backslash \{0\}$ in $V = \mathbb{A}^{n+1}_k$ and the variety $\widetilde{X} = \widetilde{V} \times Y$. Denote by $\pi : \widetilde{X} \to X$ the quotient map and $j : \widetilde{V} \hookrightarrow V$ the open immersion. The multiplicative group $\mathbb{G}_m$ acts on $\widetilde{V}$ by scaling and thus we have an action $\sigma : \mathbb{G}_m \times \widetilde{V} \times Y \to \widetilde{V} \times Y$. The coaction $\sigma^\#$ of σ defines a map $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to k[t, t^{-1}] \otimes_k \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ given by $x_i \otimes a \mapsto t \otimes x_i \otimes a$ (here $a \in \Gamma(Y, \mathcal{O}_Y)$). By the properties of group scheme actions, we obtain a $\mathbb{Z}$-grading of $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ given by eigenspaces of t with eigenvalues t^n, consisting of $f \otimes a$ where f is a homogeneous polynomial in $k[x_1, \ldots, x_{n+1}]$ of degree n. Now for any quasicoherent left $\mathcal{D}_X$-module $\mathcal{M}$, we consider the module $\pi^* \mathcal{M} = \mathcal{O}_{\widetilde{X}} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1} \mathcal{M}$. From the grading of $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, the global section of this module has a decomposition

$$\Gamma(\widetilde{X}, \pi^* \mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\widetilde{X}, \pi^* \mathcal{M})^n$$

where $\Gamma(\widetilde{X}, \pi^*\mathcal{M})^n$ is eigenspace of t with eigenvalue t^n. Multiplying by x_i (resp. ∂_i) increases (resp. decreases) the homogenous degree by 1. Now, the **Euler vector field** $\theta = \sum_i x_i\partial_i$ acts on $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ and it computes the degree of homogeneous polynomials in $\Gamma(Y, \mathcal{O}_Y)[x_1, \dots, x_{n+1}]$. If we let θ act on $\pi^*\mathcal{M}$ by $\theta \otimes \mathrm{id}$, then it is immediate that

$$\Gamma(\widetilde{X}, \pi^*\mathcal{M})^n = \{u \in \Gamma(\widetilde{X}, \pi^*\mathcal{M}) : \theta \cdot u = nu\}$$

Moreover, homogeneous sections of degree 0 correspond precisely to the global sections of $\mathcal{M}$ as π is a surjective open map, meaning $\Gamma(\widetilde{X}, \pi^*\mathcal{M})^0 = \Gamma(X, \mathcal{M})$.

Let $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ be an exact sequence of D-modules on X. The quotient map is smooth so π^* is exact. The pushforward j_* is left exact. Thus we need to understand $R^1 j_*\pi^*\mathcal{M}_1$. Note that Example 2.3.11 says $R^1 j_* = H^1(\mathbf{R}j_*) = H^1(j_+)$, so by Proposition 2.3.19, $R^1 j_*\pi^*\mathcal{M}_1$ is a quasicoherent $\mathcal{D}_{V\times Y}$-module. For an open affine $U \subseteq V \times Y$ not intersecting $\{0\} \times Y$, $j : j^{-1}(U) \to U$ is an isomorphism so $R^1 j_*\pi^*\mathcal{M}_1$ is supported on the closed subscheme $\{0\} \times Y$. By Kashiwara's equivalence, we see that there exists some D-module $\mathcal{N}$ on $\{0\} \times Y$ such that

$$R^1 j_*\pi^*\mathcal{M}_1 = \iota_+\mathcal{N} \cong k[\partial_1, \dots, \partial_{n+1}] \otimes_k \iota_* N$$

where $\iota : \{0\} \times Y \hookrightarrow V \times Y$ with a kernel generated by all $\{x_i\}$. Since

$$x_i(\partial_j^k \otimes s) = -k\delta_{ij}(\partial_j^{k-1} \otimes s)$$

the action of θ on global sections of $R^1 j_*\pi^*\mathcal{M}_1$ has negative eigenvalues. Now, $\Gamma(V \times Y, -)$ is exact so we get a long exact sequence

$$0 \to \Gamma(V\times Y, j_*\pi^*\mathcal{M}_1) \to \Gamma(V\times Y, j_*\pi^*\mathcal{M}_2) \to \Gamma(V\times Y, j_*\pi^*\mathcal{M}_3) \to \Gamma(V\times Y, R^1 j_*\pi^*\mathcal{M}_1) \to \cdots$$

Notice that $\Gamma(V \times Y, j_*\pi^*\mathcal{M}_i) = \Gamma(\tilde{X}, \pi^*\mathcal{M}_i)$, and maps in the exact sequences are $A_{n+1} \otimes_k \Gamma(Y, \mathcal{O}_Y)$-module homomorphisms, meaning they commute with θ. Thus, taking the eigenspace with eigenvalue 0 of θ preserves exactness. Together with the fact $\Gamma(\tilde{X}, \pi^*\mathcal{M}_i)^0 = \Gamma(X, \mathcal{M})$, one obtains the exact sequence

$$0 \to \Gamma(X, \mathcal{M}_1) \to \Gamma(X, \mathcal{M}_2) \to \Gamma(X, \mathcal{M}_3) \to 0$$

So $\Gamma(X, -)$ is exact.

It remains to show that if $\Gamma(X, \mathcal{M}) = 0$ then $\mathcal{M} = 0$. Assume otherwise, then as π is surjective and flat, $\pi^*\mathcal{M}$ is nonzero. Take some nonzero $s \in \Gamma(\widetilde{X}, \pi^*\mathcal{M})$ with eigenvalue n. If $n = 0$, we are done. If $n > 0$, then $\theta s = ns \neq 0$, meaning some $\partial_i s \neq 0 \in \Gamma(\widetilde{X}, \pi^*\mathcal{M})^{n-1}$. Repeating the same thing, we get a nonzero element in $\Gamma(\widetilde{X}, \pi^*\mathcal{M})^0 = \Gamma(X, \mathcal{M})$, a contradiction. If $n < 0$, then there must some $x_i s \neq 0$ because otherwise, $s \in \Gamma(\widetilde{X}, \pi^*\mathcal{M}) = \Gamma(V \times Y, j_*\pi^*\mathcal{M})$ is annihilated by the ideal sheaf of $\{0\} \times Y$, meaning it's on $\{0\} \times Y$ and as $\tilde{X}$ is open, s must be zero. Then we get a nonzero element $x_i s \in \Gamma(\widetilde{X}, \pi^*\mathcal{M})^{n+1}$. Again this eventually results in a nonzero global section of $\mathcal{M}$, a contradiction, completing the proof. □

Example 2.5.5. The flag variety of SL_2 is $X = \mathbb{P}^1_k$. The above actually shows that there is an equivalence of categories

$$\mathrm{Mod}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}(\Gamma(X, \mathcal{D}_X))$$

Once we connect $\mathrm{Mod}(\Gamma(X, \mathcal{D}_X))$ with representations of $\mathfrak{sl}_2$, the above would become the celebrated Beilinson-Bernstein localization.

In the near future, the sheaf of algebras of interest would no longer be $\mathcal{D}_X$. It will be replaced by twisted versions of $\mathcal{D}_X$. Recall that the cotangent bundle T^*X of X is naturally a symplectic algebraic variety, with a symplectic form ω. The closed nondegenerate 2-form induces a Poisson bracket on $\mathcal{O}_{T^*X}$ given by $\{f, g\} = \omega(H_f, H_g)$ where H_f and H_g are the Hamiltonian vector fields. As $\pi_*\mathcal{O}_{T^*X} \cong \mathrm{Sym}\,\Theta_X$, we have a Poisson structure on $\mathrm{Sym}\,\Theta_X$. Indeed, the Poisson structure given by $\mathrm{gr}^F \mathcal{D}_X \cong \mathrm{Sym}\,\Theta_X$ is the same as the symplectic one.

Definition 2.5.6. A **sheaf of twisted differential operators** (abbreviation t.d.o.) $\mathcal{D}$ on X is a quasicoherent sheaf of almost commutative algebras (cf. Definition 1.1.7) with a positive filtration $F_\bullet\mathcal{D}$ such that Poisson algebras $\operatorname{gr}^F\mathcal{D}$ and $\operatorname{Sym}\Theta_X$ are isomorphic.

Remark 2.5.7. By definition, if $\mathcal{D}$ is a t.d.o., there is an isomorphism $F_0\mathcal{D} \cong \mathcal{O}_X$. The commutator in $\mathcal{D}$ gives us a derivation $[\xi, \cdot]$ on $F_0\mathcal{D}$ for any $\xi \in F_1\mathcal{D}$, i.e., every section in $F_1\mathcal{D}$ induces a section of Θ_X. In fact, we have $\Theta_X \cong F_1\mathcal{D}/F_0\mathcal{D}$. Thus, an isomorphism $\operatorname{Sym}\Theta_X \cong \operatorname{gr}^F\mathcal{D}$ of $\mathcal{O}_X$-modules automatically respects the Poisson structures. We obtained an equivalent definition of t.d.o.: (i) $F_0\mathcal{D} \cong \mathcal{O}_X$, (ii) $F_1\mathcal{D}/F_0\mathcal{D} \cong \Theta_X$ and (iii) $\operatorname{gr}^F\mathcal{D} \cong \operatorname{Sym}\Theta_X$.

Example 2.5.8. Clearly $\mathcal{D}_X$ is a t.d.o. Suppose $\mathcal{L}$ is an invertible sheaf on X. Then we've seen the sheaf of differential operators $\operatorname{Diff}(\mathcal{L},\mathcal{L})$ which is isomorphic to $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X\otimes_{\mathcal{O}_X}\mathcal{L}^\vee$ by Corollary 1.2.5. From now we denote this sheaf by $\mathcal{D}_X^{\mathcal{L}}$. Note that $F_0\mathcal{D}_X^{\mathcal{L}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L}) = \mathcal{O}_X$. Indeed, it's not difficult to check (using the same argument as $\mathcal{D}_X$ affine locally) that $\mathcal{D}_X^{\mathcal{L}}$ is a t.d.o. Note that $\mathcal{D}_X^{\mathsf{op}} = \mathcal{D}_X^{\omega_{X/k}}$ we previously studied is an example.

Example 2.5.9. If $\mathcal{E}$ is a locally free sheaf of rank > 1, $\operatorname{Diff}(\mathcal{E},\mathcal{E})$ is not a t.d.o. per our definition since $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ is not $\mathcal{O}_X$.

Definition 2.5.10. Given a t.d.o $\mathcal{D}$ on X, we say X is **$\mathcal{D}$-affine** if $\Gamma(X,-)$ is an exact functor on $\operatorname{Mod}(\mathcal{D})$ and every quasicoherent $\mathcal{D}$-module $\mathcal{M}$ is generated by global sections (equivalently $\Gamma(X,\mathcal{M}) = 0$ implies $\mathcal{M} = 0$).

Example 2.5.11. We are not defining something redundant/unnecessary here. The projective line $X = \mathbb{P}^1_k$ is $\mathcal{D}_X$-affine, but $\Gamma(X,\omega_{X/k}) = 0$ for the left $\mathcal{D}_X^{\mathsf{op}}$-module $\omega_{X/k} = \mathcal{O}_X(-2)$. Therefore, $\mathbb{P}^1_k$ is not $\mathcal{D}_X^{\mathsf{op}}$-affine.

Quasicoherent $\mathcal{D}$-modules on $\mathcal{D}$-affine varieties play the role of quasicoherent $\mathcal{O}_X$-modules on affine schemes:

Proposition 2.5.12. *Given a $\mathcal{D}$-affine variety X, the global section functor gives an equivalence of categories*

$$\operatorname{Mod}(\mathcal{D}) \to \operatorname{Mod}(\Gamma(X,\mathcal{D}))$$

with quasi-inverse $\mathcal{D}\otimes_{\Gamma(X,\mathcal{D})} -$.

Remark 2.5.13. I need to clarify the notation here. The tensor product $\mathcal{D}\otimes_{\Gamma(X,\mathcal{D})} M$ is the sheaf associated to the presheaf $U \mapsto \Gamma(U,\mathcal{D})\otimes_{\Gamma(X,\mathcal{D})} M$ where $\Gamma(X,\mathcal{D})$ acts on the first component by restricting to U and multiplying on the left. Given an R-module M, under the same notation $\widetilde{\mathcal{M}} = \mathcal{O}_X\otimes_R M$ on $X = \operatorname{Spec} R$. On each distinguished open $D(f)$, we have $\Gamma(D(f),\widetilde{\mathcal{M}}) = M_f$ which is a localization. Therefore, the functor $\mathcal{D}\otimes_{\Gamma(X,\mathcal{D})} -$, being a D-module analogy of $\mathcal{O}_X\otimes_{\Gamma(X,\mathcal{O}_X)} -$, gives us a sense of localization.

Proof. First notice the two functors are adjoint by the adjunction of Hom and tensor products, and the fact that $\operatorname{Hom}_{\mathcal{D}}(\mathcal{D},-) = \Gamma(X,-)$. Write $D = \Gamma(X,\mathcal{D})$ for convenience. We need to show for any $\mathcal{D}$-module $\mathcal{M}$, $\mathcal{D}\otimes_D\Gamma(X,\mathcal{M}) \cong \mathcal{M}$, and for any D-module N, $N \cong \Gamma(X,\mathcal{D}\otimes_D N)$. Let N be presented by

$$D^{\oplus I} \to D^{\oplus J} \to N \to 0$$

for suitable I and J. Applying first the right exact functor $\mathcal{D}\otimes_D -$ and then the exact functor $\Gamma(X,-)$, we get a commutative diagram with exact rows

$$\begin{tikzcd}
D^I \arrow[r] \arrow[d] & D^J \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\eta"] & 0 \arrow[d] \\
D^I \arrow[r] & D^J \arrow[r] & \Gamma(X,\mathcal{D}\otimes_D N) \arrow[r] & 0
\end{tikzcd}$$

By the four lemma, η is an isomorphism. Since $\mathcal{M}$ is generated by global sections, the map $\mathcal{D} \otimes_D \Gamma(X, \mathcal{M}) \to \mathcal{M}$ is surjective. Let $\mathcal{K}$ be its kernel. Then since $\Gamma(X, -)$ is exact, we get an exact sequence

$$0 \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{D} \otimes_D \Gamma(X, \mathcal{M})) \to \Gamma(X, \mathcal{M}) \to 0$$

We've shown that $\Gamma(X, \mathcal{D} \otimes_D \Gamma(X, \mathcal{M})) \cong \Gamma(X, \mathcal{M})$ so $\Gamma(X, \mathcal{K}) = 0$. Again by $\mathcal{D}$-affinity, the kernel $\mathcal{K}$ is zero, completing the proof. □

2.6 Coherent D-modules A property of D-modules we will treat is coherence. Recall we say a quasicoherent $\mathcal{D}$-module is **coherent** if it is locally finitely generated over $\mathcal{D}$ (we are working with locally Noetherian schemes). We denote by $\mathrm{Mod}_c(\mathcal{D})$ the category of coherent $\mathcal{D}$-modules on X. Coherence of $\mathcal{D}_X$-modules is different from coherence over $\mathcal{O}_X$.

Proposition 2.6.1. *A $\mathcal{D}_X$-module is coherent over $\mathcal{O}_X$ if and only if it's locally free of finite rank over $\mathcal{O}_X$.*

Proof. I will only sketch the proof here since it's nonessential. We want to show that $\mathcal{M}_x$ is locally free of finite rank over $\mathcal{O}_{X,x}$ for each $x \in X$. Let $\{x_i, \partial_i\}$ be a local coordinate system of X such that the maximal ideal $\mathfrak{m}_x$ is generated by the x_i. Then by Nakayama's lemma, there exists a $\kappa(x)$-basis of $\mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x$ which lifts to a spanning set $s_1, \ldots, s_n$ of $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$. Suppose there is an equality

$$\sum_{i=1}^{m} f_i s_i = 0$$

Define the order of $f \in \mathcal{O}_{X,x}$ by $\mathrm{ord}(f) = \max\{p : f \in \mathfrak{m}_x^p\}$. Applying ∂_j to the equality we get

$$0 = \sum_{i=1}^{m} [(\partial_j f_i) s_i + f_i(\partial_j s_i)] = \sum_{i=1}^{m} \left(\partial_j f_i + \sum_{k=1}^{m} a_{ikj} f_k \right) s_i$$

where we write $\partial_j s_i = \sum_k a_{ijk} s_k$. Let f_i be the term with minimal order l. Let ∂_j be the operator that reduces the order of f_i. Then $\partial_j f_i + \sum_{k=1}^{m} a_{ikj} f_k$ sits inside $\mathfrak{m}_x^{l-1}$, i.e., the minimal order of coefficients is reduced by 1. Repeating the reduction, we arrive at a sum with coefficients of minimal order 0. Yet since the s_i are linearly independent over $\kappa(x)$ in $\mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x$, the minimal order must be larger than zero, or otherwise a nontrivial relation exists. □

Remark 2.6.2. A proof using Kashiwara's equivalence can be found in 2.1 in [7]. I do not know a lot about curves so I did not include this proof.

Example 2.6.3. The result in Kashiwara's equivalence can be strengthened for coherent D-modules. Indeed, if $\iota : X \hookrightarrow Y$ is a closed immersion where X is a hypersurface in Y, then $\iota_+\mathcal{M} = k[\partial] \otimes_k \iota_*\mathcal{M}$ is coherent over $\mathcal{D}_Y$ if $\mathcal{M}$ is $\mathcal{D}_X$-coherent. Conversely, $\iota^!\mathcal{N} = \iota^{-1}\mathcal{N}^{-1}$ where $\mathcal{N} = k[\partial] \otimes_k \mathcal{N}^{-1}$ is a coherent D-module on Y supported on X. It is clear that $\mathcal{N}^{-1}$ is locally finitely generated and thus so is $\iota^!\mathcal{N}$. Therefore we have a coherent Kashiwara's equivalence: there is an equivalence of categories

$$\iota_+ : \mathrm{Mod}_c(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}_c^X(\mathcal{D}_Y)$$

with its quasi-inverse $\iota^!$.

The equivalence of categories provided by $\mathcal{D}$-affinity can also be refined to coherent $\mathcal{D}$-modules.

Lemma 2.6.4. *If X is $\mathcal{D}$-affine, the global section functor and localization functor give an equivalence of categories*

$$\mathrm{Mod}_c(\mathcal{D}) \simeq \mathrm{Mod}_f(\Gamma(X, \mathcal{D}))$$

where Mod_f *is the category of finite modules.*

Proof. Clearly $\mathcal{D}_X \otimes M$ for a finite $\Gamma(X, \mathcal{D}_X)$-module M is coherent. Conversely, any coherent $\mathcal{D}_X$-module $\mathcal{M}$ is generated by finitely many global sections as X is quasicompact. $\square$

Recall how we defined good filtrations on modules over filtered rings in Definition 1.1.12. We can say the same thing about $\mathcal{D}_X$-modules.

Definition 2.6.5. A **good filtration** on a $\mathcal{D}_X$-module $\mathcal{M}$ is a filtration $F_\bullet\mathcal{M}$ such that each $F_s\mathcal{M}$ is coherent over $\mathcal{O}_X$ and for large enough s, $F_{l+s}\mathcal{M} = F_l\mathcal{D}_X \cdot F_s\mathcal{M}$ for any l.

Example 2.6.6. The filtration of $\mathcal{D}_X$ is good.

Proposition 1.1.14 and Lemma 1.1.15 can thus be translated into global statements:

Proposition 2.6.7. *A filtration $F_\bullet\mathcal{M}$ on a $\mathcal{D}_X$-module $\mathcal{M}$ is good if and only if $\operatorname{gr}^F \mathcal{M}$ is coherent over $\operatorname{gr}^F \mathcal{D}_X \cong \operatorname{Sym}\Theta_X \cong \pi_*\mathcal{O}_{T^*X}$.*

Lemma 2.6.8. *A $\mathcal{D}_X$-module $\mathcal{M}$ has a good filtration if and only if it's a coherent $\mathcal{D}_X$-module.*

Proof. Lemma 1.1.15 only says a D-module $\mathcal{M}$ is coherent if and only if it admits a locally good filtration. If $\mathcal{M}$ has a globally good filtration, then it's locally good and thus coherent. We need to show the converse, i.e., if $\mathcal{M}$ is coherent then it has a globally good filtration.

Our algebraic variety X can be covered by finitely many affine opens $\{U_i\}$. Then each $\Gamma(U_i, \mathcal{M})$ is finitely generated over $\Gamma(U_i, \mathcal{D}_X)$, say with a set of generators $\{s_i\}$. Let $\mathcal{N}_i$ be the $\mathcal{O}_X$-submodule generated by this set on U_i. We obtain coherent $\mathcal{O}_X$-modules $\mathcal{M}_i$ such that $\mathcal{M}_i|_{U_i} = \mathcal{N}_i$. Then their direct sum maps to a coherent $\mathcal{O}_X$-submodule $\mathcal{M}'$ of $\mathcal{M}$ that generates $\mathcal{M}$ over $\mathcal{D}_X$. Hence we can define a filtration $F_s\mathcal{M} = F_s\mathcal{D}_X \cdot \mathcal{M}'$, each coherent over $\mathcal{O}_X$ and the multiplication relation holds trivially, completing the proof. $\square$

We would like to understand to what extent a coherent D-module fails to be $\mathcal{O}_X$-coherent. This is one of the various purposes of the following geometric object.

Definition 2.6.9. Let $\mathcal{M}$ be a D-module on X with a good filtration $F_\bullet\mathcal{M}$. The **characteristic variety** (or singular support) of $\mathcal{M}$ is

$$\operatorname{ch}(\mathcal{M}) = \operatorname{supp} \pi^* \operatorname{gr}^F \mathcal{M}$$

Remark 2.6.10. $F_\bullet\mathcal{M}$ is good so $\operatorname{gr}^F \mathcal{M}$ is coherent over $\pi_*\mathcal{O}_{T^*X}$. Varieties X and T^*X are locally Noetherian so $\pi^* \operatorname{gr}^F \mathcal{M}$ is coherent over $\mathcal{O}_{T^*X}$, meaning $\operatorname{ch}(\mathcal{M})$ can be given a reduced closed subscheme structure in T^*X. Note that $\operatorname{ch}(\mathcal{M})$ is independent of the good filtration. This is a nontrivial but uninspiring fact. For a proof see [7].

Example 2.6.11. I will compute some characteristic varieties here. Let $X = \operatorname{Spec} k[x]$. As we've shown, $\operatorname{gr}^F \mathcal{D}_X$ is the sheaf associated to $k[x, \xi]$ where ξ is the image of ∂_x, and $T^*X = \mathbb{A}^2_k$. Then $\Gamma(T^*X, \pi^* \operatorname{gr}^F \mathcal{O}_X) = k[x, \xi]/(\xi)$ so the sheaf's support is the curve $\xi = 0$. Similarly, the module $\mathcal{D}_X/\mathcal{D}_X(\partial x)$ has characteristic variety $\{x\xi = 0\}$, the same as $\operatorname{ch}(\mathcal{D}_X/\mathcal{D}_X(x\partial - \lambda))$.

Results in algebraic geometry suggest the characteristic variety $\operatorname{ch}(\mathcal{M})$ is the subvariety given by the ideal

$$\mathcal{I}_{\mathcal{M}} = \sqrt{\operatorname{Ann}_{\operatorname{gr}^F \mathcal{D}_X}(\operatorname{gr}^F \mathcal{M})}$$

since $T^*X = \underline{\operatorname{Spec}} \operatorname{gr}^F \mathcal{D}_X$. Relevant results can be found in [36]. The following result demonstrates the interpretation of characteristic varieties mentioned before, that is, a measurement of lack of $\mathcal{O}_X$-coherence.

Proposition 2.6.12. *A nonzero coherent $\mathcal{D}_X$-module $\mathcal{M}$ is coherent over $\mathcal{O}_X$ if and only if $\operatorname{ch}(\mathcal{M})$ is the zero section of T^*X, isomorphic to X.*

Proof. Suppose $\mathcal{M}$ is coherent over $\mathcal{O}_X$. Then by Proposition 2.6.1, it is locally free of rank $n > 0$ over $\mathcal{O}_X$. Then a good filtration on $\mathcal{M}$ can be obtained by setting $F_i\mathcal{M} = \mathcal{M}$ for all $i \geqslant 0$ and $\operatorname{gr}^F \mathcal{M} = \mathcal{M}$ which is locally $\mathcal{O}_X^n$. Then clearly Θ_X annihilates $\operatorname{gr}^F \mathcal{M}$ and thus $\operatorname{ch}(\mathcal{M})$ sits inside the base space $X \subseteq T^*X$. Since $\operatorname{gr}^F \mathcal{M}$ has support equal to X, $\operatorname{ch}(\mathcal{M}) \cong X$. Conversely, we can assume X is affine of dimension n so $T^*X = X \times \mathbb{A}_k^n$. If $\operatorname{ch}(\mathcal{M})$ is the zero section of T^*X, then

$$\mathcal{I} = \sqrt{\operatorname{Ann}_{\operatorname{gr}^F \mathcal{D}_X}(\operatorname{gr}^F \mathcal{M})} = \sum_{i=1}^n \mathcal{O}_X[\xi_1, \ldots, \xi_n]\xi_i$$

where $F_\bullet\mathcal{M}$ is a good filtration. So for a large enough r, $\mathcal{I}^r$ annihilates $\operatorname{gr}^F \mathcal{M}$ since $\mathcal{I}$ is finitely generated. But then there is some large enough s such that

$$F_{r+s}\mathcal{M} = F_r\mathcal{D}_X \cdot F_s\mathcal{M} = \sum_{|\alpha| \leqslant r} \mathcal{O}_X\partial^\alpha \cdot F_s\mathcal{M} \subseteq F_{r+s-1}\mathcal{M}$$

where the inclusion follows from the fact that ξ^α with $|\alpha| = r$ generates I^r which annihilates $\operatorname{gr}^F_{r+s} \mathcal{M} = F_{r+s}\mathcal{M}/F_{r+s-1}\mathcal{M}$. Therefore, we must have $F_{r+s}\mathcal{M} = \mathcal{M}$ and by goodness the LHS is coherent over $\mathcal{O}_X$. □

Theorem 2.6.13 (Bernstein's inequality)**.** *Given a coherent $\mathcal{D}_X$-module $\mathcal{M}$, then* $\operatorname{ch}(\mathcal{M})$ *has dimension* $\geqslant \dim X$*.*

Remark 2.6.14. The theorem actually holds for every irreducible component of $\operatorname{ch}(\mathcal{M})$. But we will not go into details of that here.

Proof. Let $\iota : S \hookrightarrow X$ be a closed immersion where S is a smooth closed subvariety. Let $\mathcal{M}$ be any coherent $\mathcal{D}_S$-module.

Lemma 2.6.15. *We have* $\dim \operatorname{ch}(\iota_+\mathcal{M}) = \dim \operatorname{ch}(\mathcal{M}) - (\dim S - \dim X)$*.*

Proof. As in the proof of Kashiwara's equivalence, we can reduce to the case of S being a hypersurface of X. Let $\{x_i, \partial_i\}$ be a local coordinate of X such that x_1 defines S. Then $\iota_+\mathcal{M} = k[\partial_1] \otimes_k \iota_*\mathcal{M}$. Since $\mathcal{M}$ is $\mathcal{D}_S$-coherent, by Lemma 2.6.8, we can choose a good filtration $F_\bullet\mathcal{M}$. Then we can define

$$F_j\iota_+\mathcal{M} = \sum_{l=0}^j \sum_{s \leqslant l} k \cdot \partial_1^s \otimes \iota_* F_{j-l}\mathcal{M}$$

which is a good filtration on $\iota_+ M$ since each term is coherent and $F_\bullet\mathcal{M}$ is good. An easy computation shows that

$$F_j\iota_+\mathcal{M}/F_{j-1}\iota_+\mathcal{M} = \bigoplus_{l=0}^j k \cdot \partial_1^l \otimes \iota_*(F_{j-l}\mathcal{M}/F_{j-l-1}\mathcal{M})$$

and thus

$$\operatorname{gr}^F \iota_+\mathcal{M} \cong k[\xi] \otimes_k \operatorname{gr}^F \mathcal{M} \cong k[x, \xi]/(x) \otimes_k \operatorname{gr}^F \mathcal{M}$$

The annihilator of $\operatorname{gr}^F \iota_+\mathcal{M}$ in $\operatorname{gr}^F \mathcal{D}_X$ is thus generated by x and the annihilator of $\operatorname{gr}^F \mathcal{M}$, and the same relation holds for their radicals. Then an exercise on Krull dimensions shows that the characteristic varieties, defined by the radical of annihilators, have dimensions given by the statement of this lemma. □

Now if X has dimension 0 the theorem is clearly true. We can induct on $\dim X$. If $\operatorname{supp}\mathcal{M}$ is the entire X, $\operatorname{ch}(\mathcal{M})$ contains the zero section of T^*X and thus has dimension $\geqslant \dim X$. Otherwise, say $\operatorname{supp}\mathcal{M}$ is a closed proper subset of X, i.e., of dimension strictly less than X. Then we can choose a closed hypersurface S of X containing $\operatorname{supp}\mathcal{M}$. By Kashiwara's equivalence, there is a $\mathcal{D}_S$-module $\mathcal{N}$ such that $\iota_+\mathcal{N} = \mathcal{M}$. Therefore, $\dim\operatorname{ch}(\mathcal{M}) = \dim\operatorname{ch}(\mathcal{N})+1$ by the previous lemma. The induction hypothesis says $\dim\operatorname{ch}(\mathcal{N}) \geqslant \dim S = \dim X - 1$, so we are done. □

Remark 2.6.16. Indeed, the theorem is a consequence of a much stronger fact: the characteristic variety of $\mathcal{M}$ is **involutive** with respect to the symplectic structure of T^*X, meaning the tangent spaces of $\operatorname{ch}(\mathcal{M})$ at $(x,\xi) \in T^*X$ has dimension $\geqslant \frac{1}{2}\dim T_{(x,\xi)}(T^*X) = \dim X$. The fact is equivalent to $\{\mathcal{I}_\mathcal{M}, \mathcal{I}_\mathcal{M}\} \subseteq \mathcal{I}_\mathcal{M}$. The theorem of involutivity is due to Gabber, and a proof of Bernstein's inequality via this route can be found in [16].

2.7 Holonomic D-modules The characteristic variety of a coherent $\mathcal{D}_X$-module reveals much more than a lack of $\mathcal{O}_X$-coherence.

Definition 2.7.1. We say a left coherent $\mathcal{D}_X$-module $\mathcal{M}$ is **holonomic** if $\dim\operatorname{ch}(\mathcal{M}) = \dim X$. Denote by $\operatorname{Mod}_h(\mathcal{D}_X)$ the full subcategory of $\operatorname{Mod}_c(\mathcal{D}_X)$ of holonomic D-modules. Let $D^b_h(\mathcal{D}_X)$ be the derived categories of complexes of $\mathcal{D}_X$-modules with holonomic cohomologies.

Indeed, $\operatorname{Mod}_h(\mathcal{D}_X)$ is an abelian category. If $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ is exact in $\operatorname{Mod}_c(\mathcal{D}_X)$, then as gr and localizations preserve exactness, $\operatorname{ch}(\mathcal{M}_2) = \operatorname{ch}(\mathcal{M}_1)\cup\operatorname{ch}(\mathcal{M}_3)$. Therefore, $\mathcal{M}_2$ is holonomic if and only if $\mathcal{M}_1$ and $\mathcal{M}_3$ are. Although the characteristic variety is not isomorphic to X, their equal dimensions still imply a strong sense of $\mathcal{O}_X$-coherence.

Proposition 2.7.2. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then there exists an open dense subset U of X such that $\mathcal{M}|_U$ is coherent over $\mathcal{O}_U$.*

Proof. Regard X as the zero section of T^*X. Let S be the complement of X in $\operatorname{ch}(\mathcal{M})$. If S is empty, then we are done by Proposition 2.6.12. Suppose $S \neq \emptyset$. The group $\mathbb{G}_m$ acts on fibers of S, so the dimension of each fiber of $\pi|_S$ is positive, meaning $\dim\pi(S) < \dim S \leqslant \dim\operatorname{ch}(\mathcal{M}) = \dim X$. Therefore, there is some nonempty open U in X containing the complement of $\pi(S)$. But then $\operatorname{ch}(\mathcal{M}|_U)$ is completely contained in the zero section of T^*U, so again by Proposition 2.6.12, $\mathcal{M}|_U$ is coherent over $\mathcal{O}_X$, completing the proof. □

Now I will state some theorems without proof.† For proofs, see 3.2.3, 2.6.5 and 2.6.8 in [18]. Like the results on pullbacks and pushforwards of $\mathcal{D}_X$-modules, the same holds for holonomic $\mathcal{D}_X$-modules.

Proposition 2.7.3. *Given a morphism $f : X \to Y$, then f_+ sends $D^b_h(\mathcal{D}_X)$ to $D^b_h(\mathcal{D}_Y)$, and $f^!$ sends $D^b_h(\mathcal{D}_Y)$ to $D^b_h(\mathcal{D}_X)$*

We also want to have a dual notion for $\mathcal{D}_X$-modules. The obvious way would be to define $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega^\vee_{X/k}$. But the functor $\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega^\vee_{X/k}$ is not exact, so we consider

Definition 2.7.4. The **Grothendieck-Serre duality functor** $\mathbb{D} : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_X)$ of $\mathcal{D}_X$-modules is given by

$$\mathbb{D}\mathcal{M}^\bullet = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega^\vee_{X/k}[\dim X])$$

We shift the complex by $\dim X$ because we want the index in the theorem below to be zero.

†Holonomic D-modules form an essential part of the theory. They are related to important objects such as perverse sheaves. However, proofs of results stated here deviate significantly from the purpose of this essay. So to make some room for later in-depth discussions, I will omit their proofs. But to compensate, we will go over some examples and illustrate the main ideas.

Proposition 2.7.5. *The duality functor sends $D^b_c(\mathcal{D}_X)$ to $D^b_c(\mathcal{D}_X)^{\mathsf{op}}$. the double dual is isomorphic to identity.*

Theorem 2.7.6. *A coherent D-module $\mathcal{M}$ is holonomic if and only if $H^i(\mathbb{D}\mathcal{M}) = 0$ for all $i \neq 0$. Moreover, if $\mathcal{M}$ is holonomic then $H^0(\mathbb{D}\mathcal{M})$ is holonomic. In particular, $H^0(\mathbb{D}(-))$ gives a duality on $\mathrm{Mod}_h(\mathcal{D}_X)$.*

Example 2.7.7. On the affine line $X = \operatorname{Spec} k[x]$, we've seen that $\operatorname{ch}(\mathcal{O}_X) = \{\xi = 0\}$ which has dimension 1 so $\mathcal{O}_X$ is holonomic. Similarly, modules $\mathcal{D}_X/\mathcal{D}_X(\partial x)$ and $\mathcal{D}_X/\mathcal{D}_X(x\partial - \lambda)$ have characteristic variety $\{x\xi = 0\}$ with two irreducible components of dimension 1, so still holonomic.

Example 2.7.8. Let $\iota : X = \{0\} \hookrightarrow Y = \mathbb{A}^1_k$. In Example 2.4.9, we showed that $\mathcal{B}_{X|Y} = \iota_+\mathcal{O}_X \cong \mathcal{D}_Y/\mathcal{D}_Y \cdot y$. We have an exact sequence

$$0 \to \mathcal{D}_Y \to \mathcal{D}_Y \to \mathcal{B}_{X|Y} \to 0$$

Applying $\mathcal{H}om_{\mathcal{D}_Y}(-, \mathcal{D}_Y)$ to obtain an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{B}_{X|Y}, \mathcal{D}_Y) \to \mathcal{D}_Y \to \mathcal{D}_Y$$

so $\mathcal{E}xt^0_{\mathcal{D}_Y}(\mathcal{B}_{X|Y}, \mathcal{D}_Y) = \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{B}_{X|Y}, \mathcal{D}_Y) = \ker y = 0$ and $\mathcal{E}xt^1_{\mathcal{D}_Y}(\mathcal{B}_{X|Y}, \mathcal{D}_Y) = \mathcal{D}_Y/\mathcal{D}_Y \cdot y$ as Y has dimension one so all other cohomologies vanish. Then $H^0(\mathbb{D}\mathcal{B}_{X|Y}) = \mathcal{E}xt^1_{\mathcal{D}_Y}(\mathcal{B}_{X|Y}, \mathcal{D}_Y) \otimes \omega^\vee_{Y/k} \cong \mathcal{B}_{X|Y}$, and all other $H^i(\mathbb{D}\mathcal{B}_{X|Y})$ vanish. Therefore $\mathcal{B}_{X|Y}$ is holonomic by Theorem 2.7.6.

Example 2.7.9. A trivial counter-example: on $\mathbb{A}^2_k$, the D-module $\mathcal{D}_X$ is not holonomic because very easily we can compute, by definition, that $H^{-2}(\mathbb{D}\mathcal{D}_X) = \mathcal{D}_X \otimes \omega^\vee_{X/k}$, which is nonzero.

Example 2.7.10. Still take $X = \mathbb{A}^2_k$. For $P \in \mathcal{D}_X$, write $\sigma(P)$ to be the symbol of P in $\operatorname{gr}^F \mathcal{D}_X$ (replacing ∂_i with ξ_i). Bernstein and Lunts showed that if $\sigma(P)$ satisfies certain conditions, then $\mathcal{D}_X/\mathcal{D}_X \cdot P$ is a simple, non-holonomic $\mathcal{D}_X$-module.

Example 2.7.11. To extend the above example, suppose $X = \mathbb{A}^n_k$ and fix a left A_n-ideal I. By Proposition 2 in [30], the characteristic variety of the $\mathcal{D}_X$-module A_n/I is the zero locus of the ideal $\operatorname{int}(I)$ in $\mathbb{A}^{2n}_k$, where $\operatorname{int}(I)$ is the vector space (which turns out to be an ideal) spanned by $\{\operatorname{int}(P) : P \in I\}$ where $\operatorname{int}(P)$ is obtained by taking the sum of monomials of maximal order in ξ of $\sigma(P)$ (e.g., $\operatorname{int}(\partial_1^2 + \partial_2\partial_1 + x_1^{10}\partial_2) = \xi_2^2 + \xi_1\xi_2$). Therefore, A_n/I is holonomic if and only if the Krull dimension of $\operatorname{int}(I)$ is n.

Remark 2.7.12. In Example 2.7.9, $\mathcal{D}_X = \mathcal{D}_X/\mathcal{D}_X \cdot 0$ is the D-module corresponding to the system of equations $0 \cdot u = 0$, whose solution space is infinite-dimensional. Indeed, holonomic D-modules are those corresponding to systems of partial equations with a finite-dimensional solution space. In general, given an ideal of A_n, this notion is captured by the **holonomic rank** of I, defined as $\dim_{k(\underline{x})} k(\underline{x})[\underline{\xi}]/k(\underline{x})[\underline{\xi}] \cdot \operatorname{int}(I)$. In some cases, this is the dimension of the solution space (c.f. 1.4.19 in [31] which is a special case of 2.2.1 in one of the fundamental pieces in the theory of D-modules: Kashiwara's Master's thesis [23]). By 1.4.9 in [31], if A_n/I is holonomic then the holonomic rank of I is finite, though the converse is false.

Example 2.7.13. Consider the heat equation $\partial_1 u = \alpha\partial_2^2 u$ where $\alpha \neq 0$ in $\mathbb{A}^2_k$. The D-module associated to it is A_2/I where I is $(\partial_1 - \alpha\partial_2^2)$ and $\operatorname{int}(I) = (\alpha\xi_2^2)$. Therefore, $\operatorname{ch}(A_2/I)$ is a hypersurface in $\mathbb{A}^4_k$, which has dimension 3, meaning A_2/I is not holonomic. Indeed, using methods `inw(I, {0, 0, 1, 1})` and `holonomicRank` in the `Dmodules` package of `macaulay2`, we can compute $\operatorname{int}(I)$ and the holonomic rank of I which is infinite, agreeing with the standard results in analysis. Similarly, let $J = (x_1\partial_1, x_2\partial_2 + x_2^n)$ for any $n \neq 0$. Then $\operatorname{int}(J) = (x_1\xi_1, x_2\xi_2)$ and thus A_2/J is holonomic. Again we can compute its holonomic rank which is 1, the dimension of its solution space $\{c\exp(-x^{n-1}/(n-1))\}$.

3 D-Modules on Flag Varieties

3.1 Flag varieties and equivariant sheaves We assume a basic understanding of algebraic groups and flag varieties. In this section let k be an algebraically closed field of characteristic zero. Let G be a connected semisimple linear algebraic group over k. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G. Denote by Δ and Δ^+ the root system and the set of positive roots of G, by Π the set of simple roots, by Q and Q^+ the root and nonnegative root lattices, and by P and P^+ the weight and nonnegative weight lattices. For a root $\alpha \in \Delta$, denote by $\alpha^\vee$ its coroot. Let H be a fixed maximal torus in G and B a Borel subgroup containing H. The Lie algebras of H and B I denote by $\mathfrak{h}$ and $\mathfrak{b}$, which are Cartan and Borel subalgebras of $\mathfrak{g}$. As a consequence of Chevalley's theorem, given a closed subgroup variety P of G, the homogeneous space G/P exists and is a smooth algebraic variety since G is smooth. Moreover, G/P is complete if and only if P is parabolic (i.e., containing a Borel subgroup). Since G is linear, G/P is projective. In particular, G/B is a smooth projective variety. For details see [29].

Definition 3.1.1. The **flag variety** of G is the smooth projective variety $X = G/B$.

Of course, standard results on algebraic groups show that X can be identified with the set $\mathcal{B}$ of all Borel subgroups of G, on which G acts by conjugation. Sheaves on X are closely related to the representation theory of G.

Definition 3.1.2. Let G be any group scheme and X a scheme over X. Suppose $\sigma : G \times X \to X$ is an action of G on X. A G**-equivariant sheaf** on X is an $\mathcal{O}_X$-module $\mathcal{M}$ with an isomorphism

$$\varphi : \sigma^*\mathcal{M} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{M}$$

such that $\mathrm{pr}_{23}^*\varphi \circ (\mathrm{id} \times \sigma)^*\varphi = (m \times \mathrm{id})^*\varphi$ as morphisms of quasicoherent modules on $G \times G \times X$ where $m : G \times G \to G$ the multiplication.

If X is the flag variety of the algebraic group G, we have an obvious action of B on X so this allows us to talk about B-equivariant sheaves on X. Let G act on itself in the obvious way. The quotient $\pi : G \to X$ is a B**-torsor** in the category of schemes with the fppf topology. Indeed, given an open affine cover $\{U_i\}$ of X, we obtain an fppf covering $\{U_i \to X\}$. Now by 9.19 in [29], $G \times B \cong G \times_X G$ via $(g, b) \mapsto (g, gb)$ and by 4.43 in [14], $\pi : G \to X$ is a B-torsor. Again, apply 4.46 in [14], we see that

Proposition 3.1.3. *The category of quasicoherent sheaves on X is equivalent to the category of B-equivariant quasicoherent sheaves on G.*

Denote by $\mathrm{Mod}_{\mathcal{O}_B}(\mathcal{O}_G)$ the category of B-equivariant quasicoherent $\mathcal{O}_G$-modules. Notice that B and G are both affine, so we probably can write out a version of Proposition 3.1.3 in terms of plain modules. I will follow some excellent introductions in [15] to illustrate the results. Let's first translate the group scheme action into algebras. Write $G = \mathrm{Spec}\, R$. Then the group scheme structure on G is equivalent to a **Hopf algebra** structure on R, which we recall is a k-algebra with a comultiplication map $\Delta : R \to R \otimes R$, a counit map $\varepsilon : R \to k$ and an antipode $S : R \to R$ compatible with the k-algebra structure on R.

Definition 3.1.4. A **right** R**-comodule** is a vector space M over k with a linear map $\rho_M : M \to M \otimes R$ such that $(\mathrm{id} \otimes \Delta) \circ \rho_M = (\rho_M \otimes \mathrm{id}) \circ \rho_M$ and $(\mathrm{id} \otimes \varepsilon) \circ \rho_M = \mathrm{id}$. Write $\mathrm{coMod}(R)$ as the category of R-comodules.

It's immediate that the category is monoidal. Given a k-algebra A that is also an R-comodule such that its algebra structures are comodule maps, denote by $\mathrm{Mod}_R(A)$ the full subcategory of A-modules in $\mathrm{coMod}(R)$ such that the module map $A \otimes_k M \to M$ is a map of R-comodules. Denote by $\rho_A : A \to A \otimes_k R$ the comodule structure on A. Let $p : A \to A \otimes_k R$ be $a \mapsto a \otimes 1$.

We have functors $\rho_A^* : \mathrm{Mod}(A) \to \mathrm{Mod}(A \otimes R)$ and $p^* : \mathrm{Mod}(A) \to \mathrm{Mod}(A \otimes B)$ both sending an A-module M to the vector space $M \otimes R$, but the action of $a \otimes r \in A \otimes R$ on $\rho_A^* M$ is given by Sweedler notations

$$(a \otimes r)(m \otimes r') = (\rho_A(a)_{(1)} \cdot m) \otimes (\rho_A(a)_{(2)} r r')$$

while the action on p^*M is simply component-wise. These functors have left adjoints $\rho_{A,*}$ and p_* resp., given by $N \mapsto N$ such that a acts on $\rho_{A,*}N$ by $a \cdot n = \rho_A(a)n$, and on p_*N as $a \cdot n = (a \otimes 1)n$.

Proposition 3.1.5. *An A-module M sits inside* $\mathrm{coMod}(R)$*, i.e., in* $\mathrm{Mod}_R(A)$ *if and only if we have an isomorphism $\psi : \rho_A^* M \xrightarrow{\sim} p^* M$ satisfying cocycle conditions similar to equivariant sheaves.*

Proof. Let $M \in \mathrm{Mod}_R(A)$. Then by definition, $A \otimes M \to M$ is a map of comodules, i.e., $\rho_M(a \cdot m) = \rho_A(a) \cdot \rho_M(m)$ for all $a \in A$ and $m \in M$. Notice that $M \otimes R = \rho_{A,*}p^*M$, so under this A-module structure ρ_M is an A-module map $M \to \rho_{A,*}p^*M$. By adjunction, we get an $A \otimes R$-linear map $\psi : \rho_A^* M \to p^* M$ which is the identity on vector space levels. Thus, ψ is an isomorphism. It's easy to check its cocycle conditions. $\square$

Write $B = \mathrm{Spec}\, L$. Then the action of B on G is the same as a right L-comodule structure $R \to R \otimes L$ on R such that the k-algebra map $R \otimes R \to R$ is an L-comodule homomorphism. We therefore have the following

Proposition 3.1.6. *There is an equivalence of categories*

$$\mathrm{Mod}_{\mathcal{O}_B}(\mathcal{O}_G) \xrightarrow{\sim} \mathrm{Mod}_L(R)$$

Example 3.1.7. Given a G-equivariant $\mathcal{O}_X$-module $\mathcal{M}$, there is a canonical G-module structure on $\Gamma(X, \mathcal{M})$. Consider the coaction $\varphi^\# : \Gamma(X, \mathcal{M}) \to \Gamma(G, \mathcal{O}_G) \otimes \Gamma(X, \mathcal{M})$ given by the G-equivariant structure φ. If we write $\varphi^\#(s) = \sum f_i \otimes s_i$ where $f_i \in \Gamma(G, \mathcal{O}_G)$ then the G-action on s is given by $g \cdot s = \sum f_i(g) s_i$.

Example 3.1.8. At last, given two locally free G-equivariant sheaves $\mathcal{M}$ and $\mathcal{N}$ on X, we can define a G-action on $\mathrm{Hom}_k(\mathcal{M}, \mathcal{N})$ as follows. Let $\sigma(g)$ be the composition of $g \times X \hookrightarrow G \times X$ and σ. Then there are isomorphism $\varphi_{\mathcal{M}}(g) : \mathcal{M} \xrightarrow{\sim} \sigma(g)^*\mathcal{M}$ and $\varphi_{\mathcal{N}}(g) : \mathcal{N} \xrightarrow{\sim} \sigma(g)^*\mathcal{N}$ induced by the G-equivariant structure. For a morphism $\psi : \mathcal{M} \to \mathcal{N}$, define

$$g \cdot \psi = \varphi_{\mathcal{N}}(g)^{-1} \circ \sigma(g)^*\psi \circ \varphi_{\mathcal{M}}(g)$$

which is clearly a G-action since the $\varphi(g)$ are compatible with the group law.

3.2 Equivariant vector bundles I will quickly establish an equivariant version of the correspondence between locally free sheaves of finite rank on $X = G/B$ and equivariant vector bundles on X.

Definition 3.2.1. We say a vector bundle V on X is G**-equivariant** if it has a G-action such that for all $g \in G$, $x \in X$, the action $g : V_x \to V_{gx}$ is a linear isomorphism where $V_x = V \times_X \mathrm{Spec}\, \kappa(x)$ is the fiber at x.

Let $\mathcal{V}$ be the sheaf of sections of V. Then $\mathcal{V}$ is G-equivariant if and only if V is G-equivariant. To see this, the fiber of $\mathrm{pr}_2^* \mathcal{V}$ at $(g, x) \in G \times X$, i.e. its stalk at (g, x) tensoring the residue field, is precisely V_x, and the fiber of $\sigma^*\mathcal{V}$ is V_{gx} (simply evaluate sections of V at (g, x)). Then the equivalence is evident. Note that we can construct induced representations of B-modules on G. The following theorem identifies G-equivariant vector bundles on flag varieties with B-modules via induced representations of the latter.

Proposition 3.2.2. *Given a B-module M, there is a G-equivariant vector bundle V on X such that $V_x = M$ where x is the point in $X = \mathcal{B}$ corresponding to the Borel subgroup B.*

Proof. Let $G \times M$ be the trivial vector bundle on G. Then B acts on $G \times M$ by $b \cdot (g, m) = (gb^{-1}, b \cdot m)$. Let V_M be the quotient $(G \times M)/B$ (everything is affine in our case so the quotient exists), which is a vector bundle $\pi_M : V_M \to X$ sending $[(g, m)]$ to $gB \in X$. The canonical action of G on $G \times M$ induces a G-action on V_M given by

$$g \cdot [(g', m)] = [(gg', m)]$$

which is clearly well-defined since the action of b is on the right in the first coordinate. Locally, we see that $g : V_{M,g'B} \to V_{M,gg'B}$ is simply the identity map on M so a linear isomorphism. Thus V is G-equivariant on G/B and clearly $V_{M,x} = M$ for $x = 1B$. $\square$

Remark 3.2.3. Let $\mathcal{V}_M$ be the sheaf of sections on V_M. Indeed, given two B-modules M_1, M_2 and any open $U \subseteq X$, we have a map $\Gamma(U, \mathcal{V}_{M_1}) \otimes \Gamma(U, \mathcal{V}_{M_2}) \to \Gamma(U, \mathcal{V}_{M_1 \otimes M_2})$ given by $f_1 \otimes f_2 \mapsto (g \mapsto f_1(g) \otimes f_2(g))$. This gives us an isomorphism $\mathcal{V}_{M_1} \otimes \mathcal{V}_{M_2} \cong \mathcal{V}_{M_1 \otimes M_2}$. Similar relations hold for symmetric and exterior products.

From now on, we assume G is simply connected and write $B = HN$ where N is the unipotent radical of B and H the maximal torus fixed at the beginning. Every integral weight λ of $\mathfrak{g}$ gives rise to a character e^λ of H. A character of B maps N to the unipotent radical of $\mathbb{G}_m$, which is trivial. Therefore, one-dimensional B-modules are equivalent to characters of H. For any integral weight λ defining a character of H, by the previous constructions we obtain a vector bundle V_λ and a G-equivariant sheaf $\mathcal{V}_\lambda$. Write $\mathcal{L}(\lambda) = \mathcal{V}_\lambda$.

Example 3.2.4. Let us compute all constructions we mentioned for $G = SL_2$. Let B be the Borel subgroup of all upper triangular matrices and H the maximal torus of diagonal matrices. Then it's easy to see that G/B is $\mathbb{P}^1$. Let ρ be the half sum of all positive roots, so the character e^ρ sends $\operatorname{diag}(a, a^{-1})$ to a. It is immediate from the definition that the character group $\widehat{H} = P = \mathbb{Z}\rho$. Note that $\pi : G \to X$ is sends $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to $[a : c]$. Cover $\mathbb{P}^1$ by open affines U_0 and U_∞ containing $[0 : 1]$ and $[1 : 0]$ respectively. Let $z = a/c$ and $w = c/a$ be the local coordinates on U_0 and U_∞. Then points in U_0 and U_∞ are classes with representatives

$$\begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix}$$

The preimage of U_0 and U_∞ in the vector bundle $V_{n\rho}$ are trivial bundles $U_0 \times k$ and $U_1 \times k$, and they glue by

$$[(z,t)] = \left[\left(\begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix}, t\right)\right] = \left[\left(\begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} w & 1 \\ 0 & z \end{bmatrix}, \begin{bmatrix} w & 1 \\ 0 & z \end{bmatrix}^{-1} t\right)\right] = [(w, z^n t)]$$

Thus, we see that $\mathcal{L}(n\rho) = \mathcal{O}_X(-n)$. Now the G-action on $V_{n\rho}$ is given by

$$\gamma \cdot [(z,t)] = \gamma \cdot \left[\left(\begin{bmatrix} z & -1 \\ 1 & 0 \end{bmatrix}, t\right)\right] = \left[\left(\begin{bmatrix} \gamma \cdot z & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} j(\gamma, z) & -c \\ 1 & j(\gamma, z)^{-1} \end{bmatrix} t\right)\right] = [(\gamma \cdot z, j(\gamma, z)^n t)]$$

where $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $\gamma \cdot z = (az + b)/(cz + d)$ and $j(\gamma, z) = cz + d$.

Theorem 3.2.5. (i) *If $-\lambda \in P^+$, then $H^i(X, \mathcal{L}(\lambda)) = 0$ for all $i > 0$.*

(ii) *If for all $\alpha \in \Delta^+$, we have $\alpha^\vee(\lambda) \leqslant 0$, then $\mathcal{L}(\lambda)$ is generated by global sections.*

(iii) *The invertible sheaf $\mathcal{L}(\lambda)$ is ample if and only if for all $\alpha \in \Delta^+$, we have $\alpha^\vee(\lambda) < 0$.*

Proof. See 4.5 and 4.4 in [20] for proofs. The first statement is called the Kempf vanishing theorem, which is true even if k has a positive characteristic. $\square$

And without proof we state the Borel-Weil-Bott theorem, the origin of geometric representations. Let ρ be the half sum of all positive roots and $W = N_G(H)/H$ be the Weyl group of G. Recall for any $w \in W$, it has a length $l(w)$ equal to the minimal number of reflections $\sigma_\alpha : \beta \mapsto 2(\alpha, \beta)/(\alpha, \alpha)\alpha$ required to express w. We say λ is **singular** if there is some root α such that $\alpha^\vee(\lambda - \rho) = 0$.

Definition 3.2.6. We define $w \star \lambda = w(\lambda - \rho) + \rho$.

Theorem 3.2.7 (Borel-Weil-Bott)**.** *If λ is singular, all cohomology groups of $\mathcal{L}(\lambda)$ vanish. Otherwise, there is a* unique *$w \in W$ such that $-w \star \lambda \in P^+$. In this case,*

$$H^i(X, \mathcal{L}(\lambda)) = \begin{cases} L^-(w \star \lambda), & i = l(w) \\ 0, & \textit{otherwise} \end{cases}$$

where $L^-(\mu)$ denotes the lowest weight module of G with lowest weight μ.

Remark 3.2.8. Some authors choose to define $\mathcal{L}(\lambda)$ using $-\lambda$, and statements here must be adjusted accordingly.

Example 3.2.9. We can verify Borel-Weil-Bott for $G = SL_2$, $X = \mathbb{P}^1$. In Example 3.2.4, we showed that $\mathcal{L}(n\rho) = \mathcal{O}_X(-n)$. If $n \leqslant 0$, $\mathcal{O}_X(-n)$ has global sections spanned by homogeneous polynomials $x_0^k x_1^{-n-k}$. Restricted to U_0 they become $1, z, \ldots, z^{-n}$. Then as $V_{n\rho}|_{U_0}$ is trivial, sections are given by $s : x \mapsto [(x, f)]$ where f is a polynomial of degree $\leqslant -n$. The action $(g \cdot s)(x) = g \cdot s(g^{-1}x)$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = (dz - b)^k(-cz + a)^{-n-k} = (dx_0 - bx_1)^k(-cx_0 + ax_1)^{-n-k}$$

Plugging E, F and H into the matrix, one deduces that $\Gamma(X, \mathcal{O}_X(-n))$ is the highest weight module of weight $-n\rho$ and the lowest weight module of weight $n\rho$. On the other hand, $n \leqslant 0$ and $1 \star n\rho \in -P^+$, which gives the same result by Borel-Weil-Bott. When $n = 1$, that $n\rho$ is singular so $H^i(X, \mathcal{O}_X(-1)) = 0$ for all i which coincides with the result in algebraic geometry. If $n > 1$, we have

$$-1 \star n\rho = \rho - n\rho + \rho = (2 - n)\rho \in -P^+$$

Then $H^1(X, \mathcal{L}(n\rho)) = H^1(X, \mathcal{O}_X(-n)) = L^-((2 - n)\rho)$ which also agrees with standard results in algebraic geometry.

3.3 Centers and invariants Still let $\mathfrak{g}$ be the Lie algebra of G. We consider the universal enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{Z}$ be its center. Since it commutes with $\mathfrak{h}$ and $\mathfrak{n}$, $\mathfrak{Z}$ acts on any highest weight module M with highest weight λ by a character f_λ. Therefore we have a map $\varphi_z : \mathfrak{h}^* \to k$ sending a weight λ to $f_\lambda(z)$. Recall the algebra $U(\mathfrak{g})$ has a decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$$

where $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and the previously fixed Borel subalgebra is given by $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Let p be the projection of $U(\mathfrak{g})$ to $U(\mathfrak{h})$. Since $\mathfrak{h}$ is abelian, $U(\mathfrak{h}) = \operatorname{Sym} \mathfrak{h} = k[\mathfrak{h}^*]$. Indeed, if $z \in \mathfrak{Z}$, $z \in U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n})$. Notice that every highest weight module is a quotient of the **Verma module** M_λ given by

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$$

where k_λ is the vector space spanned by a vector v such that $\mathfrak{h}$ acts by λ and $\mathfrak{n}$ acts trivially. As $\mathfrak{n}$ kills v, we see that $\varphi_z(\lambda) = p(z)(\lambda)$ where $p(z) \in k[\mathfrak{h}^*]$. Thus, we get an algebra homomorphism $\varphi : \mathfrak{Z} \to U(\mathfrak{h})$ sending z to φ_z.

Definition 3.3.1. The **Harish-Chandra homomorphism** $\gamma : \mathfrak{Z} \to U(\mathfrak{h})$ is given by $\gamma = \tau \circ \varphi$ where $\tau : h \mapsto h - \rho(h) \cdot 1$ is an automorphism of $U(\mathfrak{h})$.

Theorem 3.3.2 (Harish-Chandra)**.** *The Harish-Chandra homomorphism satisfies:*

(i) *The map γ is independent of different Δ^+ and is injective.*

(ii) *For every $w \in W$, we have $\gamma(z)(w\lambda) = \gamma(z)(\lambda)$, i.e., $\gamma(z) \in U(\mathfrak{h})^W$.*

(iii) *The map $\gamma : \mathfrak{Z} \to U(\mathfrak{h})^W$ is an isomorphism.*

Proof. See 23.2 and 23.3 in [19]. □

Definition 3.3.3. Define the central character χ_λ associated to λ to be $\chi_\lambda(z) = \gamma(z)(\lambda)$.

Proposition 3.3.4. *Any character $\mathfrak{Z} \to k$ is χ_λ for some $\lambda \in \mathfrak{h}^*$. Two central characters χ_λ and χ_μ are the same if and only if there is some $w \in W$ such that $w \star \lambda = \mu$.*

Proof. The claims follow from the observation that since $\mathfrak{Z} = \mathrm{Sym}(\mathfrak{h})^W$, the GIT quotient of $\mathfrak{h}^* = \operatorname{Spec} \mathrm{Sym}(\mathfrak{h})$ by W is precisely $\operatorname{Spec} \mathfrak{Z}$; and the quotient map $\pi : \mathfrak{h}^* \to \operatorname{Spec} \mathfrak{Z}$ is induced by the injective homomorphism $\gamma : \mathfrak{Z} \to \mathrm{Sym}(\mathfrak{h})$, so π is surjective. By a theorem of Nagata, $\mathrm{Sym}(\mathfrak{h})^W$ is finitely generated. Therefore, $\operatorname{Spec} \mathfrak{Z}$ is locally of finite type over k, meaning the induced map given by composition with π on k-points $\mathfrak{h}^*(k) \to (\operatorname{Spec} \mathfrak{Z})(k)$ is surjective. But on global sections level, the k-points of $\operatorname{Spec} \mathfrak{Z}$ are precisely central characters, and k-points of $\mathfrak{h}^*$ are just weights of $\mathfrak{g}$. Thus, all central characters are of the form $\gamma(z)(\lambda)$. The second claim is immediate from our construction. □

It remains to study the algebra $U(\mathfrak{h})^W = \mathrm{Sym}(\mathfrak{h})^W$. Now, for the algebraic group G and any G-module V, we can define the action of G on $k[V]$ by $g \cdot f = f \circ g^{-1}$. Again, by the theorem of Nagata, if G is reductive (in our case semisimple), $k[V]^G$ is finitely generated over k. Let G act on $\mathfrak{g}$ by the adjoint representation. Then

Theorem 3.3.5 (Chevalley's restriction theorem)**.** *The map $k[\mathfrak{g}] \to k[\mathfrak{h}]$ restricted to $k[\mathfrak{g}]^G$ is an isomorphism $k[\mathfrak{g}]^G \xrightarrow{\sim} k[\mathfrak{h}]^W$.*

Proof. See 10.1.1 in [18]. □

Let $\mathscr{N}$ be the **nilpotent cone** of $\mathfrak{g}$ which we recall to be the set of nilpotent elements in $\mathfrak{g}$. The set is an irreducible closed subvariety of $\mathfrak{g}$. We may write $\det(t - \mathrm{ad}(x))$ as $t^n + \sum_{i=0}^{n-1} f_i(x)t^i$. By nilpotency, $\mathscr{N}$ is the closed subvariety defined by $f_1, \ldots, f_{n-1}$. But then $\mathscr{N} = \mathrm{Ad}(G)(\mathfrak{b} \cap \mathscr{N}) = \mathrm{Ad}(G)\mathfrak{n}$ which is irreducible. Since $\mathfrak{b}$ is a B-module under the adjoint representation, we can construct a vector bundle $V_\mathfrak{b}$ as in Proposition 3.2.2. We can then define morphisms $\rho_0 : V_\mathfrak{b} \to \mathfrak{g}$ and $\theta : V_\mathfrak{b} \to \mathfrak{h}$ by

$$\rho_0([(g, x)]) = \mathrm{Ad}(g)x, \quad \theta([(g, x)]) = p(x)$$

where p is the projection from $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ to $\mathfrak{h}$. Note that $\theta^{-1}(0)$ is the cotangent bundle on the flag variety X since the space orthogonal to $\mathfrak{b}$ is $\mathfrak{n}$ in $\mathfrak{g}$ with respect to the Killing form. Therefore, $\mathfrak{n}$ can be identified with $(\mathfrak{g}/\mathfrak{b})^*$ via the Killing form. But the latter is T^*X, and the former is $\theta^{-1}(0)$. We quote the following theorems without proof (see 8.2 in [11])

Theorem 3.3.6. *Let Y_+ be the subspace of $Y(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})^G$ consisting of sums of homogeneous elements of positive degrees. Then $\mathscr{N}$ is defined by the ideal $\mathrm{Sym}(\mathfrak{g}) \cdot Y_+$ which is prime. Moreover, $\mathscr{N}$ is a normal variety containing an open dense G-orbit.*

Theorem 3.3.7 (Kostant)**.** *The morphism $\rho_0 : \theta^{-1}(0) = T^*X \to \mathscr{N}$ is a resolution of singularity.*

3.4 Beilinson-Bernstein In this section we will state the Beilinson-Bernstein theorem. The first goal is to understand the representation theoretic nature of the sheaf of differential operators $\mathcal{D}_X$ on flag varieties. Let X be any smooth variety with an action of a linear algebraic group G on X and $\mathfrak{g}$ be the Lie algebra of G. Given a locally free sheaf $\mathcal{V}$ of finite rank on X with a G-equivariant structure given by $\varphi : \sigma^*\mathcal{V} \to \mathrm{pr}_2^*\mathcal{V}$. For any $x \in \mathfrak{g}$, define $\partial_x \in \mathrm{End}_k(\mathcal{V})$ by

$$\partial_x s = (i^* \circ \varphi^{-1})((x \boxtimes \mathrm{id}) \cdot \varphi(\sigma^* s))$$

where $i : X \hookrightarrow G \times X$ is given by $x \mapsto (1, x)$, and the action of x in the tensor product is regarded as its image under the canonical map $\mathfrak{g} \to \Gamma(G, \mathcal{D}_G)$ which is a invariant G-invariant vector field. Recall that $U(\mathfrak{g})$ can be identified with the space of left invariant vector fields on G. Define $\tau_{\mathcal{V}}$ to be the map $x \mapsto \partial_x$.

Lemma 3.4.1. *Suppose $\mathcal{V}$ is a locally free G-equivariant sheaf on X. Then for any $x \in \mathfrak{g}$, $f \in \mathcal{O}_X$ and $s \in \mathcal{V}$, we have*

$$\tau_{\mathcal{V}}(x)(f \cdot s) = f \cdot \tau_{\mathcal{V}}(x)(s) + \tau_{\mathcal{O}_X}(f) \cdot s$$

Proof. This is immediate since the action of x is a left invariant derivation and φ is $\mathcal{O}_{G\times X}$-linear. $\square$

We can therefore extend $\tau_{\mathcal{V}}$ to a ring homomorphism $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^{\mathcal{V}})$. To see this, compute that for any $f, g \in \mathcal{O}_X$, we have $[f, \tau_{\mathcal{V}}(x)](s) = f \cdot \tau_{\mathcal{V}}(x)(s) - f \cdot \tau_{\mathcal{V}}(x)(s) + \tau_{\mathcal{O}_X}(f) \cdot s = \tau_{\mathcal{O}_X}(f) \cdot s \in \Gamma(X, F_0\mathcal{D}_X^{\mathcal{V}})$. Therefore, $\Gamma(X, \mathcal{D}_X^{\mathcal{V}})$ becomes a $U(\mathfrak{g})$-module.

Still let G be a simply connected, connected semisimple algebraic group over k an algebraically closed field of characteristic zero. Denote by Δ the root system, Δ^+ the set of positive roots and P be the weight lattice of G. By ρ we mean the half sum of positive roots. For any $\lambda \in P$, let $\mathcal{L}(\lambda)$ be the G-equivariant invertible sheaf on $X = G/B$. Let $\mathcal{D}_\lambda$ be the t.d.o. $\mathcal{D}_X^{\mathcal{L}(\lambda+\rho)}$. Let τ_λ denote the map we defined above associated to $\mathcal{L}(\lambda + \rho)$, and χ_λ the central character associated to λ. Let $\mathfrak{Z}$ be the center of $U(\mathfrak{g})$. We have adjoint functors $\mathcal{D}_\lambda \otimes_{U(\mathfrak{g})} -$ and $\Gamma(X, -)$ between $\mathrm{Mod}(\mathcal{D}_\lambda)$ and $\mathrm{Mod}(\mathfrak{g})$ by Hom-tensor product adjunction. The first key theorem in [5] describes τ_λ explicitly.

Theorem 3.4.2. *Given $\lambda \in P$, for any $z \in \mathfrak{Z}$, $\tau_\lambda(z)$ is the multiplication by $\chi_\lambda(z)$. Moreover, the homomorphism τ_λ is surjective with kernel $U(\mathfrak{g}) \cdot \ker \chi_\lambda$.*

The second theorem shows the $\mathcal{D}_\lambda$-affinity of X for special λ.

Theorem 3.4.3. *For $\lambda \in P$,*

(i) *If $\alpha^\vee(\lambda) \leqslant 0$ for all $\alpha \in \Delta^+$,[†] then $\Gamma(X, -)$ is exact.*

(ii) *If $\alpha^\vee(\lambda) < 0$ for all roots $\alpha \in \Delta^+$, then any $\mathcal{D}_\lambda$-module $\mathcal{M}$ is generated by global sections.*

In particular, if $\alpha^\vee(\lambda) < 0$ for all $\alpha \in \Delta^+$, then X is $\mathcal{D}_\lambda$-affine.

Remark 3.4.4. Please note that we are proving a weaker version of the original theorem in [5]. Our construction of $\mathcal{D}_\lambda$ relies on the invertible sheaf $\mathcal{L}(\lambda)$, which can be constructed only if $\lambda \in P$. In Section 3.6, we will discuss the general construction in [4, 5] for all $\lambda \in \mathfrak{h}^*$.

Denote by $\mathrm{Mod}(\mathfrak{g}_\lambda)$ the category of $U(\mathfrak{g})$-modules with central character χ_λ.

[†]Usually people state this theorem using the term "dominant weight". But the definition of $\mathcal{L}(\lambda)$ requires us to use anti-dominant weights, while other authors might define $\mathcal{L}(\lambda)$ using $-\lambda$. So to avoid confusions, I choose not to use these terms.

Corollary 3.4.5. *For $\lambda \in P$, if $\alpha^\vee(\lambda) < 0$ for all roots $\alpha \in \Delta^+$, then the functor $\Gamma(X, -)$ induces equivalences*

$$\mathrm{Mod}(\mathcal{D}_\lambda) \xrightarrow{\sim} \mathrm{Mod}(\mathfrak{g}_\lambda), \quad \mathrm{Mod}_c(\mathcal{D}_\lambda) \xrightarrow{\sim} \mathrm{Mod}_f(\mathfrak{g}_\lambda)$$

with the quasi-inverse $\mathcal{D}_\lambda \otimes_{U(\mathfrak{g})} -$.

Proof. By Theorem 3.4.2 $\Gamma(X, \mathcal{D}_\lambda)$ is a quotient of $U(\mathfrak{g})$, so the functors $\mathcal{D}_\lambda \otimes_{U(\mathfrak{g})} -$ and $\mathcal{D}_\lambda \otimes_{\Gamma(X, \mathcal{D}_\lambda)} -$ are isomorphic, and we are done by Proposition 2.5.12 and Lemma 2.6.4. □

3.5 Proof of the theorems The general idea in the proof of Theorem 3.4.2 is to first compute the global sections of $\mathrm{gr}^F \mathcal{D}_\lambda$ using Kostant's theorem, compute the action of the center of $U(\mathfrak{g})$ on $\Gamma(X, \mathcal{D}_\lambda)$ and then show that the filtration $\Gamma(X, F_\bullet \mathcal{D}_\lambda)$ is the same as the PBW filtration of $U(\mathfrak{g})/U(\mathfrak{g}) \cdot \ker \chi_\lambda$ after passing to their associated graded rings. To prove Theorem 3.4.3, we wish to understand what $\mathcal{M}$ is inside $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu) \otimes_k L^+(-\mu)$ for $\mu \in -P^+$, so that we could study maps on cohomologies of $\mathcal{D}_\lambda$-modules by the induced maps on cohomologies of these tensor products, which are better understood due to the ampleness of $\mathcal{L}(\mu)$. We start by proving Theorem 3.4.2. Indeed, by Lemma 1.1.8, we can prove it on the level of associated graded algebras. This is very convenient since we know that $\mathrm{gr}^F \mathcal{D}_\lambda \cong \mathrm{Sym}\, \Theta_X$ by definition. On the other hand, in $U(\mathfrak{g})$ we have a PBW filtration, and we showed that $\tau_\lambda(\mathfrak{g}) \subseteq \Gamma(X, F_1 \mathcal{D}_\lambda)$. The map $\tau_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_\lambda)$ respects the filtration. Since $\Gamma(X, -)$ is left exact, $\Gamma(X, F_l \mathcal{D}_\lambda)/\Gamma(X, F_{l-1} \mathcal{D}_\lambda)$ sits inside $\Gamma(X, \mathrm{gr}_l^{PBW} \mathcal{D}_\lambda)$. Therefore, τ_λ induces

$$\mathrm{gr}\, \tau_\lambda : \mathrm{gr}^{PBW} U(\mathfrak{g}) \to \Gamma(X, \mathrm{gr}^F \mathcal{D}_\lambda)$$

Note that $\mathrm{gr}^{PBW} U(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g})$. Identifying $\mathrm{Sym}(\mathfrak{g})$ with $\Gamma(\mathfrak{g}^*, \mathcal{O}_{\mathfrak{g}^*})$, and $\Gamma(X, \mathrm{gr}^F \mathcal{D}_\lambda) \cong \Gamma(X, \pi_* \mathcal{O}_{T^*X}) = \Gamma(T^*X, \mathcal{O}_{T^*X})$. The map $\mathrm{gr}\, \tau_\lambda : \Gamma(\mathfrak{g}^*, \mathcal{O}_{\mathfrak{g}^*}) \to \Gamma(T^*X, \mathcal{O}_{T^*X})$ is in fact the pullback of the moment map $T^*X \xrightarrow{\rho_0} \mathscr{N} \hookrightarrow \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ where ρ_0 is constructed before Theorem 3.3.7. This is not hard to verify since τ_λ is essentially defined by the action of G on itself.

Proposition 3.5.1. *Let Y_+ be the subspace of $Y(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})^G$ consisting of sums of homogeneous elements of positive degrees. The homomorphism $\mathrm{gr}\, \tau_\lambda$ is surjective with kernel $\mathrm{Sym}(\mathfrak{g}) \cdot Y_+$.*

Proof. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form as usual. The map γ surjects into the nilpotent cone $\mathscr{N}$ of $\mathfrak{g}$. Write $\gamma = \gamma_1 \circ \gamma_2$ where $\gamma_1 : T^*X \to \mathscr{N}$ and $\gamma_2 : \mathscr{N} \hookrightarrow \mathfrak{g}^*$. The morphism $\gamma_1 : T^*X \to \mathscr{N}$ is then a resolution of singularity by Theorem 3.3.7. Therefore, since $\mathscr{N}$ is normal, and γ_1 is birational and finite, $\gamma_{1,*} \mathcal{O}_{T^*X} = \mathcal{O}_{\mathscr{N}}$. Now as γ_1 is surjective, we have an isomorphism $\Gamma(\mathscr{N}, \mathcal{O}_{\mathscr{N}}) \cong \Gamma(T^*X, \mathcal{O}_{T^*X})$. The kernel of the closed immersion γ_2 is the ideal sheaf $\mathcal{I}_{\mathscr{N}}$ of $\mathscr{N}$, and the induced map $\Gamma(\mathfrak{g}^*, \mathcal{O}_{\mathfrak{g}^*}) \to \Gamma(\mathscr{N}, \mathcal{O}_{\mathscr{N}})$ is surjective. The kernel of this map is the global section of $\mathcal{I}_{\mathscr{N}}$ which by Theorem 3.3.6 is the ideal generated by S_+. □

Proposition 3.5.2. *For any $z \in \mathfrak{Z}$, $\tau_\lambda(z) = \chi_\lambda(z)\, \mathrm{id}$.*

Proof. Since $\mathfrak{Z} = U(\mathfrak{g})^G$, $\tau_\lambda(z)$ is also G-invariant. There is a dense G-orbit in $\mathscr{N}$, so the G-invariant part of $\Gamma(\mathscr{N}, \mathcal{O}_{\mathscr{N}})$ are the constants. In this case, $\Gamma(X, \mathrm{gr}^F \mathcal{D}_\lambda)^G = k$. The representations $\Gamma(X, F_{l-1} \mathcal{D}_\lambda)$, $\Gamma(X, F_l \mathcal{D}_\lambda)$ and $\Gamma(X, \mathrm{gr}_l^F \mathcal{D}_\lambda)$ of G decomposes into direct sum of G-invariant parts and their complements. Therefore, taking the G-invariant parts we get an exact sequence

$$0 \to \Gamma(X, F_{l-1} \mathcal{D}_\lambda)^G \to \Gamma(X, F_l \mathcal{D}_\lambda)^G \to \Gamma(X, \mathrm{gr}_l^F \mathcal{D}_\lambda)^G$$

Since X is projective, the G-invariant global sections of $F_0 \mathcal{D}_\lambda = \mathcal{O}_X$ is k. By our previous argument, $\Gamma(X, \mathrm{gr}_0^F \mathcal{D}_\lambda)^G = k$ and $\Gamma(X, \mathrm{gr}_l^F \mathcal{D}_\lambda)^G = 0$ for all $l > 0$ by directness. Via an inductive argument, one sees that $\Gamma(X, F_l \mathcal{D}_\lambda)^G = k$ for all l, and therefore $\Gamma(X, \mathcal{D}_\lambda)^G = k$.

Now we evaluate $\tau_\lambda(z)$ on some nonzero section $s \in \mathcal{L}(\lambda+\rho)$ (here we see $\mathcal{D}_X^{\mathcal{L}}$ as $\mathcal{D}_X(\mathcal{L},\mathcal{L})$). Let v be a nonzero element in the fiber at $1B \in X$ of the vector bundle associated to $\mathcal{L}(\lambda+\rho)$. Let s be the section on the open N^-B/B such that $s(uB) = uv$ for all $u \in N^-$. Then by construction, $\tau_\lambda(h)s = (\lambda+\rho)(h)s$ and $\tau_\lambda(a)s = 0$ for all $a \in \mathfrak{n}^-$. Decompose $z \in \mathfrak{Z}$ as $z = u_1 + u_2$ where $u_1 \in U(\mathfrak{h})$ and $u_2 \in U(\mathfrak{g})\mathfrak{n}^-$. Then we have

$$\tau_\lambda(z)s = \tau_\lambda(u_1)s = (\lambda+\rho)(u_1)s$$

But $(\lambda+\rho)(u_1)$ is precisely $\chi_\lambda(z)$, completing the proof. $\square$

Proof of Theorem 3.4.2. We will show that

$$0 \to J_l = U(\mathfrak{g}) \cdot \ker \chi_\lambda \cap PBW_l U(\mathfrak{g}) \to PBW_l U(\mathfrak{g}) \to \Gamma(X, F_l\mathcal{D}_\lambda) \to 0$$

is an exact sequence for all l through a diagram chasing. To simplify the notations, I will drop PBW and write instead $F_\bullet U(\mathfrak{g})$ for the filtration on $U(\mathfrak{g})$. I will follow [18] to write $I_l = \ker \chi_\lambda \cap F_l U(\mathfrak{g})$,

$$J_l = \sum_{m+n=l} F_m U(\mathfrak{g}) I_n$$

and K_l the lth homogeneous component of $\mathrm{Sym}(\mathfrak{g}) \cdot Y_+$. The following diagram is commutative because inclusions and τ_λ respect gradings.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J_{l-1} \arrow[r] \arrow[d] & F_{l-1}U(\mathfrak{g}) \arrow[r, "\tau_\lambda"] \arrow[d] & \Gamma(X, F_{l-1}\mathcal{D}_\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_l \arrow[r] \arrow[d] & F_l U(\mathfrak{g}) \arrow[r, "\tau_\lambda"] \arrow[d] & \Gamma(X, F_l\mathcal{D}_\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_l \arrow[r] \arrow[d] & \mathrm{Sym}(\mathfrak{g})_l \arrow[r, "\mathrm{gr}\,\tau_\lambda"] \arrow[d] & \Gamma(X, \mathrm{gr}_l^F \mathcal{D}_\lambda) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Suppose the first row is exact. The last row is always exact by Proposition 3.5.1. The second column is trivially exact and the third column is exact by $\Gamma(X,-)$. Now if the first column is exact, then by an easy diagram chasing we see that the second row is exact. For $l > 0$, degree $\leqslant l-1$ elements of I_l die in $\mathrm{Sym}(\mathfrak{g})_l$ and $I_l \subseteq \ker \chi_\lambda \subseteq \mathfrak{Z} = U(\mathfrak{g})^G$. Therefore, J_l does map to K_l, and the kernel of this map is precisely J_{l-1}. To see surjectivity, taking the G-invariant part of the exact sequence $F_l U(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{g})_l \to 0$. Thus for any $x \in \mathrm{Sym}(\mathfrak{g})_l^G$ there is a preimage $y_1 \in \mathfrak{Z} \cap F_l U(\mathfrak{g})$. Since $F_l U(\mathfrak{g})^G \to \mathrm{Sym}(\mathfrak{g})_l^G$ kills constants, x also has the preimage $y = y_1 - \chi_\lambda(y_1) \in \ker \chi_\lambda$. Thus, $y \in I_l \subseteq J_l$ maps to $x \in K_l$.

Now for $l = 0$, we have $F_0 U(\mathfrak{g}) = k$, $\Gamma(X, F_0\mathcal{D}_\lambda) = k$ and $I_0 = 0$. Therefore, we have an exact sequence

$$0 \to J_0 \to F_0 U(\mathfrak{g}) \to \Gamma(X, F_0\mathcal{D}_\lambda) \to 0$$

which completes the proof by the inductive step above. $\square$

Example 3.5.3. We compute the map τ_λ for $G = SL_2$. As before, $X = G/B = \mathbb{P}^1$ and let U_0 and U_∞ be the standard affine open charts. Then $\mathcal{D}_{n\rho}|_{U_0} = \mathcal{D}_{U_0}$ and similarly for $\mathcal{D}_{n\rho}|_{U_\infty}$ since $\mathcal{O}(-n-1)$ is trivial on affine opens. The sheaf $\mathcal{D}_{n\rho}|_{U_0}$ is generated by z and ∂_z with $[\partial_z, z] = 1$ and similarly $\mathcal{D}_{n\rho}|_{U_\infty}$ is generated by w and ∂_w with $[\partial_w, w] = 1$. For convenience let $U = U_0 \cap U_\infty$. On $\Gamma(U, \mathcal{D}_X)$, it is easy to compute that $\partial_w = -z^2\partial_w$. Since $\mathcal{L}((n+1)\rho) =$

$\mathcal{O}_X(-n-1)$ is trivial on affine opens, we obtain an isomorphism $\Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ sending 1 to z^{-n-1} given by transition maps from U_∞ to U_0. In the t.d.o., ∂_w is thus identified with $-z^2\partial_z - (n+1)z$. Moreover, we see that

$$\varphi((e \otimes 1) \cdot \varphi^{-1}(\sigma^* p)) = \sigma^*(-\partial_z p)$$

for any section p of $\mathcal{O}_X(-n-1)$ on U_0. Thus, $\tau_{n\rho}$ sends $e \in U(\mathfrak{sl}_2)$ to $-\partial_z$, and one could verify that $\tau_{n\rho}(f) = z^2\partial_z$ and $\tau_{n\rho}(h) = -2z\partial_z$.

To prove Theorem 3.4.3, we will need some extra maps. For $\mu \in -P^+$, let $p_\mu : \mathcal{O}_X \otimes_k L^-(\mu) \to \mathcal{L}(\mu)$ be the surjective morphism of $\mathcal{O}_X$-modules as $\mathcal{L}(\mu)$ is generated by its global sections which are $L^-(\mu)$ by Theorem 3.2.7. Since $\mathcal{L}(\lambda_1 + \lambda_2) = \mathcal{L}(\lambda_1) \otimes \mathcal{L}(\lambda_2)$ by Remark 3.2.3, we see that $\mathcal{L}(\mu)^\vee = \mathcal{L}(-\mu)$ and $L^-(\mu)^* \cong L^+(-\mu)$. We thus obtained an injection $i_\mu : \mathcal{O}_X \to \mathcal{L}(\mu) \otimes_k L^+(-\mu)$. Tensoring by $\mathcal{D}_\lambda$-module $\mathcal{M}$ on the left, we get morphisms $\overline{p_\mu} : \mathcal{M} \otimes_k L^-(\mu) \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu)$ and $\overline{i_\mu} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu) \otimes_k L^+(-\mu)$ which are also surjective and injective respectively.

Lemma 3.5.4. *The lowest weight module $L^-(\mu)$ has a filtration*

$$0 = L^r \subseteq L^{r-1} \subseteq \cdots \subseteq L^2 \subseteq L^1 = L^-(\mu)$$

of B-submodules such that L^i/L^{i+1} is the one-dimensional B-module with character associated to some $\mu_i \in P$, and $\{\mu_1, \ldots, \mu_{r-1}\}$ are weights of $L^-(\mu)$ such that if $\mu_i < \mu_j$ then $i < j$ and $\mu_i = \mu$ if and only if $i = 1$.

Proof. This is classical representation theory. We can define the filtration using modules spanned by weight vectors, and $\mathfrak{h}$-modules in our case give rise to B-modules. □

Lemma 3.5.5. *The $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_k L^-(\mu)$ has a filtration of G-equivariant modules*

$$0 = \mathcal{V}^r \subseteq \cdots \subseteq \mathcal{V}^2 \subseteq \mathcal{V}^1 = \mathcal{O}_X \otimes_k L^-(\mu)$$

such that $\mathcal{V}^i/\mathcal{V}^{i+1} \cong \mathcal{L}(\mu_i)$ for μ_i satisfying the same conditions in the above lemma.

Proof. The trivial bundle $X \times L^-(\mu)$ has a filtration

$$U^r \subseteq U^{r-1} \subseteq \cdots \subseteq U^2 \subseteq U^1 = X \times L^-(\mu)$$

if we let $U^i = \{(gB, x) : x \in g(L^i)\}$ for the L^i in the lemma above. Define $\mathcal{V}^i$ to be the sheaf of sections of U^i which is an $\mathcal{O}_X$-submodule of $\mathcal{O}_X \otimes_k L^-(\mu)$, completing the proof. □

Proposition 3.5.6. *For all $\lambda \in P$,*

(i) *If $\alpha^\vee(\lambda) < 0$ for all positive roots α, $\ker \overline{p_\mu}$ is a direct summand of $\mathcal{M} \otimes_k L^-(\mu)$ as a sheaf of abelian groups.*

(ii) *If $\alpha^\vee(\lambda) \leqslant 0$ for all positive roots α, $\operatorname{im} \overline{i_\mu}$ is a direct summand of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu) \otimes_k L^-(\mu)$ as a sheaf of abelian groups.*

Proof. By the previous lemma, $\mathcal{M} \otimes_k L^-(\mu)$ has a filtration $\{\overline{\mathcal{V}}^i\}$ such that $\overline{\mathcal{V}}^i/\overline{\mathcal{V}}^{i+1} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu_i)$ obtained by tensoring with $\mathcal{M}$. Using a similar fact on the filtration of $L^+(-\mu)$, we may obtain a filtration

$$0 = \overline{\mathcal{W}}^1 \subseteq \cdots \subseteq \overline{\mathcal{W}}^{r-1} \subseteq \overline{\mathcal{W}}^r = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu) \otimes_k L^+(-\mu)$$

of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu) \otimes_k L^+(-\mu)$ such that $\overline{\mathcal{W}}^{i+1}/\overline{\mathcal{W}}^i \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu - \mu_i)$. For any $z \in \mathfrak{Z}$ and any $\mu \in -P^+$, $(z - \chi_{\lambda+\mu}(z))\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu)$ vanishes since $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}(\mu)$ being a $\mathcal{D}_{\lambda+\mu}$-module is killed by the kernel of $\chi_{\lambda+\mu}$ by Theorem 3.4.2. Therefore, the filtrations of $\mathcal{M}\otimes_k L^-(\mu)$ and $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu)\otimes_k L^+(-\mu)$ suggest that they are killed by

$$\prod_{i=1}^{r-1}(z-\chi_{\lambda+\mu_i}(z)) \text{ and } \prod_{i=1}^{r-1}(z-\chi_{\lambda+\mu-\mu_i}(z))$$

respectively. By a linear algebra argument, since $\mathfrak{Z}$ is commutative, from the above results we can decompose the modules into

$$\mathcal{M}\otimes_k L^-(\mu) = \bigoplus_{\chi \text{ central}} (\mathcal{M}\otimes_k L^-(\mu))^\chi$$
$$\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu)\otimes_k L^+(-\mu) = \bigoplus_{\chi \text{ central}} (\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu)\otimes_k L^+(-\mu))^\chi$$

where $(-)^\chi$ is the generalized eigenspace of the action $\mathfrak{Z}$ with central character χ. Again by the above annihilator results, the central characters of $\mathfrak{Z}$ on $\mathcal{M}\otimes_k L^-(\mu)$ are precisely $\chi_{\lambda+\mu_i}$, and on $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu)\otimes_k L^+(-\mu)$ are $\chi_{\lambda+\mu-\mu_i}$.

Now if $\alpha^\vee(\lambda)<0$ for all positive roots α and $\chi_{\lambda+\mu_i}=\chi_{\lambda+\mu}$, then by Proposition 3.3.4, there is some $w\in W$ such that $w(\lambda)-\lambda+w(\mu_i)-\mu=0$. Yet $w(\lambda)\geqslant\lambda$ since $\alpha^\vee(\lambda)<0$ for all positive roots. Also, $w(\mu_i)$ is a weight of $L^-(\mu)$ by properties of the Weyl group. By minimality, we must have $w(\mu_i)\geqslant\mu$. In conclusion, $\chi(\lambda)-\lambda=w(\mu_i)-\mu=0$ and thus $w=1$ and $\mu_i=\mu$. If $\alpha^\vee(\lambda)\leqslant 0$ and $\chi_{\lambda+\mu-\mu_i}=\chi_\lambda$, then there is some w such that $w(\lambda)-\lambda+\mu_i-\mu=0$. But then $\mu_i=\mu$ by a similar but weaker argument.

On the other hand, $\overline{p_\mu}$ is the morphism $\overline{\mathcal{V}_1}\to\overline{\mathcal{V}_1}/\overline{\mathcal{V}_2}$, so its kernel is eliminated by a large enough power of $z-\chi_{\lambda+\mu}$. Similarly, $\overline{i_\mu}$ is the map $\overline{\mathcal{W}_2}\to\overline{\mathcal{W}_r}$ so its image is killed by a large enough power of $\chi_{\lambda+\mu-\mu}=\chi_\lambda$, completing the proof. □

Proof of Theorem 3.4.3. For the exactness of $\Gamma(X,-)$, assume $\alpha^\vee(\lambda)\leqslant 0$ for all positive roots α. Write $\mathcal{M}=\varinjlim\mathcal{N}$ where $\mathcal{N}$ varies among the coherent $\mathcal{O}_X$-submodules of $\mathcal{M}$. Then $H^k(X,\mathcal{M})=\varinjlim H^k(X,\mathcal{N})$. We will show the map $H^k(X,\mathcal{N})\to H^k(X,\mathcal{M})$ is zero. By Theorem 3.2.5, there is some $\mu\in -P^+$ such that $H^k(X,\mathcal{N}\otimes\mathcal{L}(\mu))=0$ (by ampleness and taking large enough tensor product). Then

$$H^k(X,\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu)\otimes_k L^+(-\mu)) = H^k(X,\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu))\otimes_k L^+(-\mu)=0$$

But (ii) in Proposition 3.5.6 suggests that $\overline{i_\mu}$ induces an injective map

$$H^k(X,\mathcal{M})\to H^k(X,\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu)\otimes_k L^+(-\mu))$$

By naturality of H^k, we see that $H^k(X,\mathcal{N})\to H^k(X,\mathcal{M})$ must be zero, so $\Gamma(X,-)$ is exact.

To show that all left $\mathcal{D}_\lambda$-modules on X are generated by global sections, we define $\mathcal{M}'$ of $\mathcal{M}$ to be the submodule generated by global sections. Let $\mathcal{M}''=\mathcal{M}/\mathcal{M}'$. Assume $\mathcal{M}''\neq 0$. Let $\mathcal{N}$ be a nonzero coherent $\mathcal{O}_X$-submodule of $\mathcal{M}''$, so there is some $\mu\in -P^+$ such that $\Gamma(X,\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{L}(\mu))\neq 0$. Therefore, $\Gamma(X,\mathcal{M}''\otimes_{\mathcal{O}_X}\mathcal{L}(\mu))$ does not vanish. Yet $\Gamma(X,\mathcal{M}''\otimes_k L^-(\mu))\to\Gamma(X,\mathcal{M}''\otimes_{\mathcal{O}_X}\mathcal{L}(\mu))$ is surjective by Proposition 3.5.6 so $\Gamma(X,\mathcal{M}'')\neq 0$. But this is impossible since $\mathcal{M}'$ is generated by global sections of $\mathcal{M}$ and $\Gamma(X,-)$ is exact. □

3.6 Lie algebroids: a more general approach Recall I mentioned that there is a general construction of t.d.o. that doesn't rely on λ being integral. In this section I will briefly introduce the necessary language used in the general construction of t.d.o. for all $\lambda\in\mathfrak{h}^*$. However, some statements in this section will be taken for granted, because we don't have the necessary language such as pullbacks of modules over t.d.o.s and strong G-equivariance. The first objects we will study are Lie-Rinehart algebras over fields. They are the algebraic data of Lie algebroids which will be discussed and used later. I will refer to [1, 6] as the main sources of this section.

Definition 3.6.1. Let R be a commutative k-algebra. A **Lie-Rinehart algebra** over R is a pair $(\mathfrak{l}, \eta)$ where $\mathfrak{l}$ is a Lie algebra over k and also an R-module, and $\eta : \mathfrak{l} \to \mathrm{Der}_k(R)$ a map of both Lie algebras and R-modules called the **anchor map** such that

$$[x, ry] = r[x, y] + \eta(x)(r)y$$

for all $x, y \in \mathfrak{l}$ and $r \in R$.

Example 3.6.2. The R-module $\mathrm{Der}_k(R)$ with the identity anchor map is a Lie-Rinehart algebra. Given a Lie algebra $\mathfrak{l}$ over k, we may consider the free R-module generated by $\mathfrak{l}$, i.e., $R \otimes_k \mathfrak{l}$ on which we have a Lie bracket $[\cdot, \cdot]$ that is an R-bilinear extension of the Lie bracket on $\mathfrak{l}$, and a trivial anchor map $\eta = 0$.

Example 3.6.3. We may also twist the above construction by a Lie algebra map $\tau : \mathfrak{l} \to \mathrm{Der}_k(R)$. We still take the vector space $R \otimes_k \mathfrak{l}$, but with a Lie bracket defined by

$$[f \otimes x, g \otimes y] = fg \otimes [x, y] + f\tau(x)(g) \otimes y - g\tau(y)(f) \otimes x$$

and the anchor map $\eta_\tau : f \otimes x \mapsto f \cdot \tau(x)$, we get a new Lie-Rinehart algebra $(R \otimes_k \mathfrak{l}, \eta_\tau)$ called a **transformation Lie-Rinehart algebra**. The reader may verify it's a Lie-Rinehart algebra.

An **enveloping algebra of** $(\mathfrak{l}, \eta)$ is an associative k-algebra U with a map $i_R : R \to U$ of k-algebras and a map $i_\mathfrak{l} : \mathfrak{l} \to U$ of Lie algebras such that for all $r \in R$ and $x \in \mathfrak{l}$,

$$i_\mathfrak{l}(rx) = i_R(r)i_\mathfrak{l}(x), \quad [i_\mathfrak{l}(x), i_R(r)] = i_R(\eta(x)(r))$$

Take the enveloping algebra with i_R and $i_\mathfrak{l}$ universal among all and denote it by $U_R(\mathfrak{l})$. We can construct it explicitly. The sum $R \oplus \mathfrak{l}$ has a Lie algebra structure given by

$$[f + x, g + y] = (\eta(x)(g) - \eta(y)(f)) + [x, y]$$

that is, it's the semi-direct product of these Lie algebras. There is an inclusion $\iota : R \oplus \mathfrak{l} \to U(R \oplus \mathfrak{l})$ given by PBW of Lie algebras. Let U be the k-subalgebra generated by its images, and I be the two-sided ideal generated by $\iota(f + 0) \cdot \iota(g + x) - \iota(fg + fx)$.

Proposition 3.6.4. *The algebra* $U_R(\mathfrak{l}) = U/I$ *is the universal enveloping algebra of* $\mathfrak{l}$ *with* $i_R : r \mapsto \iota(r + 0)$ *and* $i_\mathfrak{l} : x \mapsto \iota(0 + x)$*.*

Proof. The maps i_R and $i_\mathfrak{l}$ clearly satisfy the desired relations. The universal property is not hard to check. □

Example 3.6.5. The universal enveloping algebra of $R \otimes_k \mathfrak{l}$ with a given map $\tau : \mathfrak{l} \to \mathrm{Der}_k(R)$ in Example 3.6.3 is isomorphic to $R \otimes_k U(\mathfrak{l})$ as k-vector spaces.

Theorem 3.6.6 (Poincaré-Birkhoff-Witt, rinehart-lie-rinehart)**.** *If* $\mathfrak{l}$ *is finite dimensional, then* $\mathrm{Sym}(\mathfrak{l})$ *is isomorphic to* $\mathrm{gr}\, U_R(\mathfrak{l})$*.*

Example 3.6.7. *Crucial:* consider the linear algebraic group $G = \mathrm{Spec}\, R$ with Lie algebra $\mathfrak{g}$. We know $R \otimes_k \mathfrak{g} \cong \mathrm{Der}_k(R)$ as k-vector spaces via the map $r \otimes x \mapsto r\tau(x)$ where $\tau : \mathfrak{g} \to \mathrm{Der}_k(R)$ sends x to the left invariant vector field defined by x from the theory of group schemes. Recall $\mathrm{Der}_k(R)$ is Lie-Rinehart with the identity anchor map, and $(\mathfrak{g}, \tau)$ naturally becomes Lie-Rinehart. It is immediate that $r \otimes x \mapsto r\tau(x)$ is in fact an isomorphism of Lie-Rinehart algebras over R, where the anchor map of $R \otimes_k \mathfrak{g}$ is $\eta_\tau : r \otimes x \mapsto r\tau(x)$ as in Example 3.6.3. But on the other hand, $\mathrm{gr}\, U_R(\mathrm{Der}_k(R)) \cong \mathrm{Sym}\, \mathrm{Der}_k(R)$ by Theorem 3.6.6, where the right hand side is precisely $\mathrm{gr}^F D(R)$ by Proposition 1.1.19. Then by Lemma 1.1.8, the algebras $U_R(\mathrm{Der}_k(R))$

and $D(R)$ are isomorphic. Thus by 1.23 in [6], the algebra $D(R)$ is isomorphic to $R \otimes_k U(\mathfrak{g})$ where the algebra structure of the tensor product is given by, in Sweedler notations,

$$(r \otimes x)(r' \otimes x') = \sum r(x_{(1)} \triangleright r') \otimes x_{(2)} x'$$

where $\triangleright$ is the Hopf action of $U(\mathfrak{g})$ on R which in our case is the action of differential operators. We denote by $R\#U(\mathfrak{g})$ this algebra and call it the **smash product** of R and $U(\mathfrak{g})$.

Definition 3.6.8. On a smooth algebraic variety X, a **Lie algebroid** is a quasicoherent $\mathcal{O}_X$-module $\mathcal{L}$ and a morphism of $\mathcal{O}_X$-modules $\eta : \mathcal{L} \to \Theta_X$ such that on each affine open $U \subset X$, $(\Gamma(U, \mathcal{L}), \eta_U)$ is a Lie-Rinehart algebras over $\Gamma(U, \mathcal{O}_X)$.

Remark 3.6.9. In global languages, we have a k-linear Lie bracket $[\cdot, \cdot] : \mathcal{L} \otimes_k \mathcal{L} \to \mathcal{L}$ commuting with η and for any local sections $x, y \in \mathcal{L}$, $f \in \mathcal{O}_X$, we have

$$[x, fy] = r[x, y] + \eta(x)(f)y$$

By Serre-Swan, we see that the category of locally free Lie algebroid of finite rank on an affine X is equivalent to the category of Lie-Rinehart algebras projective over $\Gamma(X, \mathcal{O}_X)$.

Example 3.6.10. As in Example 3.6.3, given a Lie algebra $\mathfrak{l}$ and a map $\tau : \mathfrak{l} \to \Gamma(X, \Theta_X)$, we can define a Lie algebroid $\mathcal{O}_X \otimes_k \mathfrak{l}$ (as usual this denotes the sheaf $U \mapsto \Gamma(U, \mathcal{O}_X) \otimes_k \mathfrak{l}$ on affine opens) with a Lie bracket

$$[f \otimes x, g \otimes y] = fg \otimes [x, y] + f\tau(x)(g) \otimes y - g\tau(y)(f) \otimes x$$

after restrictions if necessary for local sections $f, g \in \mathcal{O}_X$ and $x, y \in \mathfrak{l}$. The anchor map of this Lie algebroid is given by $\eta_\tau : f \otimes x \mapsto f \cdot \tau(x)$.

Definition 3.6.11. We define the universal enveloping algebra $U_{\mathcal{O}_X}(\mathcal{L})$ of a Lie algebroid $(\mathcal{L}, \eta)$ on X to be the sheaf given by $U \mapsto U_{\Gamma(U, \mathcal{O}_X)}(\Gamma(U, \mathcal{L}))$ on affine opens.

Recall we've defined a map $\tau_{\mathcal{V}} : \mathfrak{g} \to \Theta_X$ for a given G-equivariant locally free sheaf $\mathcal{V}$ on X. Let $\mathcal{G}$ be the Lie algebroid defined by $\tau_{\mathcal{O}_X} : \mathfrak{g} \to \Gamma(X, \Theta_X)$. We can then consider the universal enveloping algebra $U_{\mathcal{O}_X}(\mathcal{G})$ which in the spirit of Example 3.6.7 is the sheaf of differential operators. By Example 3.6.5, the underlying $\mathcal{O}_X$-module is $\mathcal{O}_X \otimes_k U(\mathfrak{g})$. On this module, the Lie bracket in Example 3.6.10 is determined by, for any $\xi \in \mathfrak{g}$ and $f \in \mathcal{O}_X$,

$$[\xi, f] = f[\xi, 1] + \tau_{\mathcal{O}_X}(\xi)(f)$$

where I suppressed the tensor symbol. The map $\tau_{\mathcal{O}_X}$ naturally extends to an $\mathcal{O}_X$-algebra morphism $\tau_{\mathcal{O}_X} : U_{\mathcal{O}_X}(\mathcal{G}) \to \mathcal{D}_X$. Define $\tilde{\mathfrak{g}}$ to be the kernel of $\tau_{\mathcal{O}_X}|_{\mathcal{G}}$. Then we can consider the subspace

$$\sum_{a \in \tilde{\mathfrak{g}}} U_{\mathcal{O}_X}(\mathcal{G})(a - \lambda(a))$$

which turns out to be a two-sided ideal. Set

$$\mathcal{D}_X(\lambda) = U_{\mathcal{O}_X}(\mathcal{G}) / \sum_{a \in \tilde{\mathfrak{g}}} U_{\mathcal{O}_X}(\mathcal{G})(a - \lambda(a))$$

We quote the following theorem

Theorem 3.6.12. *The above $\mathcal{D}_X(\lambda)$ are the only strong G-equivariant t.d.o.s on X.*

Proof. See 4.9.2 in [24]. I will leave the strong G-equivariance here undefined. □

If $\lambda \in P$, given maps $\tau_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_\lambda)$ and $\mathcal{O}_X \hookrightarrow \mathcal{D}_\lambda$, we have a map $U_{\mathcal{O}_X}(\mathcal{G}) \to \mathcal{D}_\lambda$. It can be shown that this map vanishes on $\sum_{a\in\tilde{\mathfrak{g}}} U_{\mathcal{O}_X}(\mathcal{G})(a - (\lambda+\rho)(a))$, so we get a map $\mathcal{D}_X(\lambda+\rho) \to \mathcal{D}_\lambda$. Then as the associated graded algebras are both isomorphic to $\operatorname{Sym}\Theta_X$, we have $\mathcal{D}_X(\lambda+\rho) \cong \mathcal{D}_\lambda$. Therefore, most arguments we employed in the proof of Theorem 3.4.2 and Theorem 3.4.3 can be applied to $\mathcal{D}_X(\lambda+\rho)$ after adjustments. The approach by Lie algebroid is in fact widely used in various generalizations and analogs of Beilinson-Bernstein, say the p-adic analytic version in [1] (which is a very general and powerful construction), the quantum version in [3, 35], and the characteristic p version in [8].

3.7 Quantum Beilinson-Bernstein at generic q To further demonstrate the power of Lie algebroids, I will briefly introduce a version of Beilinson-Bernstein for quantum groups. The history of quantum groups started with the integrality of a quantum system, which could be determined by solving for R-matrices in the quantum Yang-Baxter equation. Drinfeld, Jimbo, Manin and others discovered and studied comprehensively a special class of Hopf algebras relevant to Yang-Baxter equations in the 1980s. One of the most crucial quantum groups is the following Hopf algebra. Fix an algebraically closed field k of characteristic zero. The directing references for this section are the original papers [3, 28, 35] of Lunts, Rosenberg, Backelin, Kremnizer and Tanisaki, and the Ph.D. thesis of Nicolas Dupré [12].

Definition 3.7.1. Suppose $\mathfrak{g}$ is a semisimple Lie algebra (or a Kac-Moody algebra in general) over k with a fixed Cartan subalgebra $\mathfrak{h}$, and $q \neq 0, \pm 1 \in k$. Let $\{e_i, f_i, h_i\}_{i\in I}$ be a Cartan-Weyl basis of $\mathfrak{g}$. The Drinfeld-Jimbo **quantum universal enveloping algebra** $U_q(\mathfrak{g})$ for the Cartan datum $(\mathfrak{h}, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I}, P, P^\vee)$ of $\mathfrak{g}$ is the associative algebra over k generated by $\{e_i, f_i, q^h : i \in I, h \in P^\vee\}$ such that for $d_i = (\alpha_i, \alpha_i)/2$, $a_{ij} = \alpha_i^\vee(\alpha_j)$:

(i) For all $h, h' \in P^\vee$, $q^0 = 1, q^h q^{h'} = q^{h+h'}$, $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$, and $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$,

(ii) For all $i, j \in I$, $e_i f_i - f_i e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}$.

(iii) And summing over k,

$$(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0, \quad (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$$

where $\left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]_q = \frac{[a]_q!}{[a-b]_q![b]_q!}$ with $[a]_q = (q^a - q^{-a})/(q - q^{-1})$.

There is a Hopf algebra structure on $U_q(\mathfrak{g})$ given by

$$\begin{array}{ccc} \Delta(e_i) = e_i \otimes 1 + q^{d_i h_i} \otimes e_i & S(e_i) = -q^{-d_i h_i} e_i & \varepsilon(e_i) = 0 \\ \Delta(f_i) = f_i \otimes q^{-d_i h_i} + 1 \otimes f_i & S(f_i) = -f_i q^{d_i h_i} & \varepsilon(f_i) = 0 \\ \Delta(q^{\pm h_i}) = q^{\pm h_i} \otimes q^{\pm h_i} & S(q^{h_i}) = q^{-h_i} & \varepsilon(q^{h_i}) = 1 \end{array}$$

When $q \to 1$, the classical limit of $U_q(\mathfrak{g})$ is precisely the Hopf algebra $U(\mathfrak{g})$, although strictly speaking for some authors $U_q(\mathfrak{g})$ is not a deformation algebra of $U(\mathfrak{g})$. This quantum group is particularly interesting because for a specific $q \neq 0, \pm 1$, $U_q(\mathfrak{g})$ actually makes sense, and its representations are rich.

In this section, we will study $U_q(\mathfrak{g})$ at q *not a root of unity*. From now on, write $U_q = U_q(\mathfrak{g})$ and $K_i = q^{d_i h_i}$ for convenience. Let $U_q(\mathfrak{n}^+)$, $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{h})$ be the subalgebras of U_q generated by $\{e_i\}$, $\{f_i\}$ and $\{h_i\}$ respectively. Let $U_q(\mathfrak{b}^\pm)$ be the subalgebra generated generated by $U_q(\mathfrak{n}^\pm)$ and $U_q(\mathfrak{h})$. It is not hard to show that $U_q(\mathfrak{b}^\pm)$ are Hopf subalgebras using the structures on U_q. Similar to the classical case, there is a triangular decomposition $U_q \cong U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h}) \otimes U_q(n^-)$ and a Poincaré-Birkhoff-Witt theorem for U_q, although we shall not spell out the details. Now, for a root $\alpha \in Q$ (recall we defined pretty early that Q was the root lattice), if $\alpha = \sum_i n_i \alpha_i$ then we write $K_\alpha = \prod_i K_i^{n_i}$. Given a representation (V, ρ_V) of the algebra U_q, an additive function $\Lambda : Q \to k$ is a **weight** of U_q wrt ρ_V if the weight space

$$V_\Lambda = \{v \in V : \forall \alpha \in Q, \rho_V(K_\alpha)v = \Lambda(\alpha)v\}$$

is nonzero. If V decomposes into weight spaces, we say it's a **highest weight module** with highest weight Λ if there is a nonzero e_Λ in V_Λ such that $U_q \cdot e_\Lambda = V$ and $T(e_i)e_\Lambda = 0$ for all i. We say ρ_V is a **representation of type 1** if it is a highest weight module $L_q^+(\Lambda)$ with a highest weight Λ of the form $\Lambda(\alpha) = q^{\lambda(\alpha)}$ for some $\lambda \in \mathfrak{h}^*$.

Theorem 3.7.2. *If q is not a root of unity, then every finite-dimensional representation of U_q is the tensor product of a representation of type 1 and a one-dimensional representation of U_q.*

Proof. See 7.1.1.5 in [26]. □

Theorem 3.7.3. *Every finite-dimensional representation of U_q is completely reducible.*

Proof. See 5.12 in [21]. □

It is therefore tempting to ask if there is an analog of Beilinson-Bernstein for U_q at q not a root of unity, as the representation theory of U_q is almost parallel to the classical case. One would soon realize that the main issues we have are: the absence of a nice quantized geometric object corresponding to the classical flag varieties, and the notion of differential operators in noncommutative settings, so we can't build sheaves or D-modules in the classical way. One essential reason why Beilinson-Bernstein is true is the duality between $U(\mathfrak{g})$ and $\mathcal{O}(G) = \Gamma(G, \mathcal{O}_G)$. There is a nondegenerate Hopf pairing $\langle \cdot, \cdot \rangle : U(\mathfrak{g}) \times \mathcal{O}(G) \to k$ in the sense that, in Sweedler notations, $\langle uv, x \rangle = \langle u, x_{(1)} \rangle \langle v, x_{(2)} \rangle$, $\langle u, xy \rangle = \langle u_{(1)}, x \rangle \langle u_{(2)}, y \rangle$, $\langle 1, x \rangle = \varepsilon(x)$, $\langle u, 1 \rangle = \varepsilon(u)$ and $\langle S(u), x \rangle = \langle u, S^{-1}(x) \rangle$. Here the comultiplication of $U(\mathfrak{g})$ is given by $\Delta(u) = 1 \otimes u + u \otimes 1$ and of $\mathcal{O}(G)$ is given by $\Delta(x) = x \otimes x$. Of course, the pairing induces/is given by the action of $\mathfrak{g}$ we used to define τ_λ. In the remarkable work [13] of Faddeev, Reshetikhin and Takhtajan, a quantization of $\mathcal{O}(G)$ is constructed via the so-called FRT construction wrt the standard R-matrix in $U_q(\mathfrak{g})$. We denote the resulting Hopf algebra by $\mathcal{O}_q(G)$. Indeed, we still have Hopf pairings $\langle \cdot, \cdot \rangle_q$ between $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$. So it is reasonable to start with $\mathcal{O}_q(G)$ which is a subalgebra of the Hopf dual of $U_q(\mathfrak{g})$.

There are two major approaches to resolving this difficulty, and we will briefly describe them. The first one is formulated and proposed in a series of papers [27, 28] by Lunts and Rosenberg, and later completed by Tanisaki in [35]. The guiding principle of their work is the combination of the Gabriel-Rosenberg reconstruction theorem and a theorem of Serre, stating that given a $\mathbb{Z}$-graded ring A generated by its degree one elements, the category of quasicoherent sheaves on $\operatorname{Proj} A$ is equivalent to the localization of the category of graded A-modules by the subcategory of torsion graded A-modules. Artin and Zhang developed the theory of noncommutative projective schemes in [2]. Given a $\mathbb{Z}$-graded ring A possibly noncommutative, they defined the **general projective scheme** $\operatorname{Proj}_{\mathbb{Z}} A$ to be triple $(\mathrm{Gr}_{\mathbb{Z}} A/\mathrm{Tor}_{\mathbb{Z}}, \mathscr{A}, s)$ where (i) $\mathrm{Gr}_{\mathbb{Z}} A/\mathrm{Tor}_{\mathbb{Z}} A$ is the localization of left $\mathbb{Z}$-graded A-modules by the full subcategory of torsion modules, (ii) $\mathscr{A}$ together with a module whose endomorphisms recover A and an autoequivalence induced by the shift operator. The flag varieties $X = G/B$ are GIT quotients of the base affine space $\tilde{X} = G/N$ by the action of the maximal torus H on the right. The base affine space $\tilde{X} = G/N$ has global sections being the P-graded ring $A = \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \oplus_{\lambda \in P^+} L^+(\lambda)$ (this is a consequence of the fact that $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathcal{O}(G)^N$ which decomposes into the desired direct sum by analyzing the Peter-Weyl decomposition of $\mathcal{O}(G)$). In fact, the category of quasicoherent sheaves on X is then equivalent to the localization $\operatorname{Proj}_P A = \mathrm{Gr}_P A/\mathrm{Tor}_P A$ of P-graded left A-modules by the subcategory of torsion modules. The key observation in the quantum setting is the decomposition

Proposition 3.7.4. *We have*

$$A_q = \{\varphi \in \mathcal{O}_q(G) : \forall x \in U_q(\mathfrak{n}^-), \varphi \cdot x = \varepsilon(x)\varphi\} \cong \bigoplus_{\lambda \in P^+} L_q^+(q^\lambda)$$

where $\mathcal{O}_q(G)$ is regarded as a U_q-bimodule via $x \cdot \varphi = \varphi_{(1)}\langle x, \varphi_{(2)}\rangle_q$ and $\varphi \cdot x = \langle x, \varphi_{(1)}\rangle_q \varphi_{(2)}$. In particular, A_q is a P-graded k-algebra with zero degree elements being k.

Proof. The standard argument still relies on the Peter-Weyl decomposition of $\mathcal{O}_q(G)$, which can be found in most texts on representations of U_q. □

We define the **category of quasicoherent sheaves** $\mathrm{Mod}(G_q/B_q)$ on the quantized flag variety as $\mathrm{Proj}_P\, A_q = \mathrm{Gr}_P A_q/\mathrm{Tor}_P A_q$ (B_q, G_q are just notations). To define differential operators in $\mathrm{Mod}(G_q/B_q)$, Lunts and Rosenberg developed a nice theory of differential operators on noncommutative rings in [27]. But Tanisaki realized that their construction resulted in something too large for Beilinson-Bernstein. Instead, in [35], he defined the **ring of differential operators on the quantized flag variety** to be the subalgebra D_q in $\mathrm{End}_k(A_q)$ generated by l_φ, r_φ, ∂_x and σ_λ where $\varphi \in A_q$, $x \in U_q$ and $\lambda \in P$. Here l_φ and r_φ are left and right multiplications by φ, ∂_x is the left action of x on A_q and σ_λ is the multiplication by $q^{(\lambda,\mu)}$ for homogeneous elements of degree μ in A_q. It is not hard to define a grading of D_q. Let $\mathrm{Gr}_{P,\lambda} D_q$ be the full subcategory of $\mathrm{Gr}_P D_q$ of modules M with σ_μ acting on the ξth homogeneous component of M by $q^{(\mu,\lambda+\xi)}$. We call the localization $\mathrm{Mod}_\lambda(D_q) = \mathrm{Gr}_{P,\lambda} D_q/(\mathrm{Tor}_P D_q \cap \mathrm{Gr}_{P,\lambda} D_q)$ the **category of D-modules on the quantized flag variety**. Tanisaki proved that

Theorem 3.7.5 ([35])**.** *Given $\lambda \in P^+$, $\mathrm{Mod}_\lambda(D_q)$ is equivalent to the category of U_q-modules with central character ζ_λ sending $x = f \otimes K_\alpha \otimes e$ first to $\varepsilon(e)\varepsilon(f)\alpha$ in $k[P]$ and then applying the morphism $k[P] \to k$ defined by $\mu \mapsto q^{(\lambda,\mu)}$.*

Before we discuss the second approach by Backelin and Kremnizer, let's recall the results of Proposition 3.1.3 and Proposition 3.1.6. We showed that quasicoherent sheaves on G/B are equivalent to $\mathcal{O}(G)$-modules that are $\mathcal{O}(B)$-comodules. We know $\mathcal{O}(G)$ and $\mathcal{O}(B)$ have deformations $\mathcal{O}_q(G)$ and $\mathcal{O}_q(B)$, so we can try to work with a quantum version of $\mathrm{Mod}_{\mathcal{O}(B)}(\mathcal{O}(G))$ in the notations of Proposition 3.1.6. We have a map $\mathcal{O}_q(G) \to \mathcal{O}_q(B)$ which makes $\mathcal{O}_q(G)$ into a $\mathcal{O}_q(B)$-comodule. Define a **B_q-equivariant sheaf** to be a triple (M, α, β) where M is a vector space, $\alpha : \mathcal{O}_q(G) \otimes M \to M$ is a left $\mathcal{O}_q(G)$-module structure and $\beta : M \to M \otimes \mathcal{O}_q(B)$ is a right $\mathcal{O}_q(B)$-comodule structure such that α is a map of right $\mathcal{O}_q(B)$-comodules. Denote by $\mathrm{Mod}_{B_q}(G_q)$ the category of such sheaves. Backelin and Kremnizer showed that this category is very nice and is in fact equivalent to $\mathrm{Mod}(G_q/B_q)$ constructed by Lunts and Rosenberg. This construction gives us the benefit of working with deformation of algebras on G, which are very easy to formulate and construct and since G is affine. The global section functor in this setting is easily defined by taking the B_q-invariants: for $M \in \mathrm{Mod}_{B_q}(G_q)$, $\Gamma(M) = \{m \in M : \beta_M(m) = m \otimes 1\}$ where $\beta(m)$ is the $\mathcal{O}_q(B)$-comodule structure of M.

To define the ring of differential operators, we recall the crucial example Example 3.6.7. We showed that $D(\mathcal{O}(G)) = \mathcal{O}(G)\#U(\mathfrak{g})$. So why not define **the ring of differential operators on the quantized flag variety** as $D_q = \mathcal{O}_q(G)\#U_q(\mathfrak{g})$? Let's do this! Well, a problem immediately appears: D_q is not in $\mathrm{Mod}_{B_q}(G_q)$. The Hopf algebra U_q action on D_q are induced by: (i) on $\mathcal{O}_q(G)$ by $x \cdot r = r_{(1)}\langle x, r_{(2)}\rangle_q$ and (ii) on U_q by the standard adjoint representation $\mathrm{ad}(x)(y) = \sum x_{(1)} y S(x_{(2)})$ (in the classical limit $q = 1$ we get $x_{(1)} = x_{(2)} = x$ for all $x \in \mathfrak{g}$ so $\mathrm{ad}(x)(y)$ is the classical adjoint representation). Therefore the action is $x \cdot (r \otimes y) = (x_{(1)} \cdot r) \otimes \mathrm{ad}(x_{(2)})(y)$. But a $\mathcal{O}_q(B)$-coaction on D_q relies on a crucial property of the representation: local finiteness (c.f. I.9.16 in [10], or the notion of integrable modules in the classical setting). In classical settings, $U(\mathfrak{g})$ is a locally finite representation of itself via the adjoint action, i.e., $\dim_k \mathrm{ad}(U(\mathfrak{g}))x < \infty$ for all $x \in U(\mathfrak{g})$. But this is not true for $U_q(\mathfrak{g})$! However, by [21], the finite part $U_q^f = \{x \in U_q : \dim_k \mathrm{ad}(U_q)x < \infty\}$ is a subalgebra of U_q and the finite part

$D_q^f = \mathcal{O}_q(G)\#U_q^f$ sits inside $\mathrm{Mod}_{B_q}(G_q)$. We then define **a (B_q, λ)-equivariant D_q-module** to be a triple (M, α, β) where M is a vector space with a left D_q-action α and a right $\mathcal{O}_q(B)$-action β which induces an $U_q(\mathfrak{b})$-action such that (i) the $U_q(\mathfrak{b})$-action of $\alpha \otimes \mathrm{id}$ on $M \otimes k_\lambda$ is $\beta \otimes \lambda$ and (ii) the map α is $U_q(\mathfrak{b})$-linear. Denote by $\mathrm{Mod}(D_q, B_q, \lambda)$ the category of all (B_q, λ)-equivariant D_q-modules. Further define $U_{q,\lambda} = U_q^f / J_\lambda$ where J_λ is the annihilator of the Verma module M_λ in U_q^f. Backelin and Kremnizer proved that

Theorem 3.7.6 ([3])**.** *The categories* $\mathrm{Mod}(D_q, B_q, \lambda)$ *and* $\mathrm{Mod}(U_{q,\lambda}^f)$ *are equivalent.*

How are the two approaches related? Strictly speaking they are not equivalent. But if we take D_q^f to be the subalgebra of $\mathrm{End}_k(A_q)$ generated by l_φ, ∂_x and σ_λ where $\varphi \in A_q$, $x \in U_q$ and $\lambda \in P$ in Tanisaki's approach (no more right translations), then by 9.1 in the preprint [33],

Proposition 3.7.7. *The categories* $\mathrm{Mod}(D_q, B_q, \lambda)$ *and* $\mathrm{Mod}_\lambda(D_q^f)$ *are equivalent.*

Remark 3.7.8. I will end this paper by a final remark. The quantum analogs of both Theorem 3.4.2 and Theorem 3.4.3 were proved in the two different approaches. The proof of quantum Theorem 3.4.2 in [3] is highly similar to the one we did in the classical setting, contrary to an extremely different one in [35]. However, observed by both Tanisaki in [34] and Dupré in his thesis, the computation of global sections in [3] has gaps. Their argument relies on results in [22], in which a filtration of U_q^f is defined and a quantum version of Kostant's theorem is proved. Tanisaki claims that the filtration does not induce the PBW filtration on $U(\mathfrak{g})$ when $q \to 1$. Yet I was unable to follow the paper [22] and I couldn't even locate the crucial results due to its complexity. So unfortunately I cannot comment on the issue of Backelin and Kremnizer's proof. But luckily, we have Proposition 3.7.7 and the fact that Tanisaki produced a successful proof of the global section theorem in [33].

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