Geometric Invariant Theory: constructing moduli spaces

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Some classification problems

Question: how will you classify the following objects up to equivalence?

Simple Lie algebras

Dynkin diagrams.

Finitely generated abelian groups

Direct sums of cyclic groups.

Semisimple algebras

Artin-Wedderburn theorem.

Conformal mappings

Liouville's theorem (for dimension > 2).

Observation: only finitely or countably many equivalence classes are involved.

Conventions I

Definition

Given a scheme S, an S-**scheme** X (or a scheme over S), written as X/S, is a scheme equipped with a **structure morphism** $X \to S$. In case S is the affine scheme $\operatorname{Spec} R$, we write X/R instead and call it an R-scheme.

Definition

A **group** S-**scheme** is simply an S-scheme G/S with morphisms $m : G \times_S G \to G$, $i : G \to G$ and $e : S \to G$ such that

(i) $m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$.

(ii) $m \circ (\mathrm{id} \times i) \circ \Delta = m \circ (i \times \mathrm{id}) \circ \Delta = e \circ \pi$.

(iii) $m \circ (e \times \mathrm{id}) = m \circ (\mathrm{id} \times e) = \mathrm{id}$ where $G = S \times_S G = G \times_S S$.

Conventions II

Definition

An **algebraic group over** k is a group k-scheme which is also an integral algebraic scheme over k (an algebraic variety). An algebraic group G/k is called a **linear algebraic group** if it is affine.

Definition

A group G is said to be reductive if its unipotent radical $R_u(G_{\bar{k}}) = \{e\}$ where $G_{\bar{k}} = \operatorname{Spec} \bar{k} \times_k G$.

Definition

A **locally free sheaf of rank** n on a scheme X is a sheaf $\mathcal{F}$ such that there is an open cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$. A locally free sheaf of rank 1 is called an **invertible sheaf**.

What if I know nothing about schemes?

Although most statements in this talk will be given in terms of schemes, there is no need to understand this notion. Indeed, every time a scheme is involved, one could replace the term by “classical affine/projective varieties”, i.e., the solution set of some polynomial equations in the affine space or the projective space. All examples we will give will be varieties.

However, the idea of sheaves is essential in this talk. Sheaves are roughly attachments of algebras to every open set in a topological space.

A problem to keep in mind

Suppose S is a variety. Consider pairs of the form (E, T) where E is a vector bundle of rank n over S and T an endomorphism on E. We call this pair a *family of endomorphisms parametrized by* S. An isomorphism between two families (E_1, T_1) and (E_2, T_2) is an isomorphism $\varphi : E_1 \to E_2$ of vector bundles such that $\varphi \circ T_1 = T_2 \circ \varphi$.

A (coarse) moduli problem

For any variety S we have a set of equivalence classes — families of endomorphisms over S modulo isomorphisms. Find a variety M such that for every variety S, there exists a morphism $S \to M$ whose image corresponds to families parametrized by S.

Moduli spaces I

Let's formalize problem in the last slide.

Definition

A **moduli problem** consists of a collection A of families over schemes such that

(i) There is a notion of equivalence for families over each scheme S.

(ii) For any morphism $\varphi : S \to S'$ and a family X over S, there is a pullback $\varphi^* X$ over S' that is functorial and if $X \sim X'$ then $\varphi^* X \sim \varphi^* X'$.

Denote by $\mathcal{F}(S)$ the set of equivalence classes over S. Then $\mathcal{F}$ is naturally a functor and we denote the moduli problem by $\mathcal{F}$. The goal here is to find the following space.

Definition

A scheme M such that $\mathcal{F}$ is represented by M. We call M a **fine moduli space**.

Moduli spaces II

The definition of fine moduli space is too strong, and quite often such a space does not exist. Thus we want something that might preserve the nice properties of fine moduli spaces.

Definition

A scheme M is a **coarse moduli space** of the moduli problem $\mathcal{F}$ if there is a natural transformation Φ such that

(i) For all algebraically closed field k, $\Phi(\operatorname{Spec} k) : \mathcal{F}(\operatorname{Spec} k) \to M(k)$ is an isomorphism.

(ii) If M' is another scheme with natural transformation $\Phi' : \mathcal{F} \to \operatorname{Hom}(-, M')$, there is a unique morphism $\Pi : \operatorname{Hom}(-, M) \to \operatorname{Hom}(-, M')$ such that $\Pi \circ \Phi = \Phi'$.

Remark: a coarse moduli space, if exists, is unique up to isomorphism. They are also irrelevant to the equivalence relations given by the moduli problem.

Actions of algebraic groups I

Definition

An **action** of an algebraic group G/k on a scheme X/k is a morphism of k-schemes $\sigma : G \times_k X \to X$ such that $\sigma \circ (\mathrm{id}_G \times \sigma) = \sigma \circ (m \times \mathrm{id}_X)$ where m is the multiplication map of G, and given the identity e of G,

$$\sigma \circ (e \times \mathrm{id}_X) : X = \operatorname{Spec} k \times_k X \to X$$

is the identity map.

Definition

If G is linear, let $S = \Gamma(G, \mathcal{O}_G)$. Let $\mu : S \to S \otimes S$ and $\varepsilon : S \to k$ be the induced multiplication and identity. Let R be a k-algebra. A **dual action** of G on R is a homomorphism of k-algebras $\hat{\sigma} : R \to R \otimes_k R$ such that

$$(\mu \otimes \mathrm{id}_R) \circ \hat{\sigma} = (\mathrm{id}_S \otimes \hat{\sigma}) \circ \hat{\sigma} \text{ and } (\varepsilon \otimes \mathrm{id}_R) \circ \hat{\sigma} = \mathrm{id}_R \,.$$

Actions of algebraic group II

Recall the previous definition of reductivity of algebraic groups. We now introduce two different notions of reductivity.

Definition

A linear algebraic group G is **geometrically reductive** if every finite-dimensional linear representation $G \to GL(V)$ and any invariant vector $w \in V^G$, there exists a G-invariant homogeneous polynomial function $f : V \to k$ such that $f(w) = 1$.

Definition

A linear algebraic group G is **linearly reductive** if for any surjective morphism of representations $\varphi : V \to W$, the restricted map $\varphi^G : V^G \to W^G$ is also surjective.

Historical notes

The three notions of reductivity, though defined in very different manners, are closely related. In fact we have

$$\text{Linearly reductive} \subseteq \text{Geometrically reductive} = \text{Reductive}$$

The second equality, known as Mumford's conjecture, is due to Haboush (1975). The first $\subseteq$ turns out to be an equality when the characteristic of the base field of G is zero. However, when the characteristic is positive, the $\subseteq$ can be replaced by $\subsetneq$.

Counterexample

SL_n is geometrically reductive for all $n \geqslant 2$, but the only connected linearly reductive groups in this case are algebraic tori.

Nagata's theorem

Definition

An element x of some k-algebra is **invariant** under the dual action of some group G if $\hat{\sigma}(x) = 1 \otimes x$.

Theorem (Nagata's theorem)

If G is a reductive algebraic group acting dually on a finitely generated *k-algebra R, then R^G is also* finitely generated*.*

Historical notes: some people might call it Hilbert's theorem, but Hilbert only proved this for linearly reductive groups. Nagata provided a counterexample (1958) for Hilbert's 14th problem

Hilbert's 14th problem (original version)

If G is a linear algebraic group acting dually on some finitely generated k-algebra R, is R^G finitely generated?

when he studied linearly reductive groups. Several years later he proved the theorem above (1964).

Geometric quotients

Definition

Given an action σ of G on X, a pair $(Y, \varphi : X \to Y)$ is a **categorical quotient** of X by G if (i) $\varphi \circ \sigma = \varphi \circ p_2$ and (ii) it has the universal property.

Definition

A **geometric quotient** of X by G is a pair (Y, φ) satisfying

(i) As above

(ii) The morphism φ is surjective and the fibers of φ over geometric points are the orbits of geometric points of X

(iii) The morphism φ is submersive: $U \subseteq Y$ is open if and only if $\varphi^{-1}(U)$ is open

(iv) For all open U, $\Gamma(\varphi^{-1}(U), \mathcal{O}_X)$ is isomorphic to $\Gamma(U, \varphi^* \mathcal{O}_X)^G$.

Note: geometric quotients are categorical quotients.

Relation between quotients and moduli spaces

The construction of quotients immediately gives us moduli spaces via the following result.

Definition

A family X parametrized by S is said to have the **local universal property** if for an arbitrary family X' parametrized by S', there is a point $s \in S'$ and a nhds $U \ni s$ such that $X'|_U$ (the family induced from X' by the inclusion morphism $U \hookrightarrow S'$) is equivalent to $\varphi^* X$ for some morphism $U \to S$.

Lemma

Suppose we have a moduli problem $\mathcal{F}$ with X a family over S with the local universal property. If G acts on S and $G \cdot s = G \cdot t$ if and only if the pullbacks of X by the inclusions of s and t are equivalent, then a categorical quotient of S by G is a coarse moduli space of $\mathcal{F}$ if and only if it's an orbit space. In particular, a geometric quotient is a coarse moduli space under the assumption.

Construction of geometric quotients

The goal remains is to construct geometric quotients for groups actions. This is easy for the affine case:

Theorem

If X is an affine scheme over X and G a reductive group acting on X. Then a uniform categorical quotient (Y, φ) exists, φ is universally submersive, and Y is an affine scheme.

Proof.

Take $Y = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)^G$. Then one could deduce the result using Nagata's theorem. $\square$

Corollary

If the action of G on X is closed (i.e., $\sigma : G \times X \to X$ is closed), then (Y, φ) is a geometric quotient.

Separation of orbits

Lemma

If W_1*,* W_2 *are two closed disjoint subsets of* X*, then there is some* $f \in \Gamma(X, \mathcal{O}_x)$ *such that* $f|_{W_1} = 0$ *and* $f|_{W_2} = 1$*.*

Proof.

Let $I_i = \Gamma(W_i, \mathcal{O}_X)$. Then since $R = \Gamma(X, \mathcal{O}_X)^G$ is finitely generated, $(I_1 + I_2) \cap R = I_1 \cap R + I_2 \cap R$ where the LHS contains 1. Thus, there exist $f \in I_1 \cap R$ and $g \in I_2 \cap R$ such that $1 = f + g$. Then f is the desired section. □

Then we can prove the Corollary

Proof.

If $x_1, x_2 \in X$ are two points with $\varphi(x_1) = \varphi(x_2)$ but $O(x_1) \cap O(X_2) = \emptyset$. Then since σ is closed, we can take $W_i = O(x_i)$ in the lemma above, suggesting there is an invariant f s.t. $f(x_1) \neq f(x_2)$. But then $\varphi^* f(\varphi(x_1)) \neq \varphi^* f(\varphi(x_1))$, a contradiction. □

Linearization I

Definition

Let G be an algebraic group, X an algebraic scheme, σ the action of G on X and L an invertible sheaf. Then a G-**linearization** of L is an isomorphism

$$\varphi : \sigma^* L \to p_2^* L$$

where $p_2 : G \times X \to X$ is the projection map such that given m and e the multiplication and identity of G, the following diagram commutes:

$$\begin{tikzcd}
{[\sigma \circ (\mathrm{id}_G \times \sigma)]^* L} \arrow[r, "(\mathrm{id}_G \times \sigma)^* \varphi"] \arrow[d, equal] & {[p_2 \circ (\mathrm{id}_G \times \sigma)]^* L} \arrow[r, equal] & (\sigma \circ p_{23})^* L \arrow[d, "p_{23}^* \varphi"] \\
{[\sigma \circ (m \times \mathrm{id}_X)]^* L} \arrow[rr, "(m \times \mathrm{id}_X)^* \varphi"] & & (p_2 \circ p_{23})^* L
\end{tikzcd}$$

We denote by $\mathrm{Pic}^G(X)$ the group of G-linearized invertible sheaves with multiplication given by tensor products.

Linearization II

Of course we could replace the invertible sheaf L with its corresponding line bundle $\mathcal{L}$, in which case we have an isomorphism $\Phi : (G \times X) \times_X \mathcal{L} \to (G \times X) \times_X \mathcal{L}$ where the fiber products are defined by p_2 and σ resp. Define $\Sigma = p_2 \circ \Phi$, an isomorphism of bundles, and $\pi \circ \Sigma = \sigma \circ (\mathrm{id}_G \times \pi)$, and the following commutes

$$\begin{tikzcd}
G \times G \times \mathcal{L} \arrow[rr, "m \times \mathrm{id}_{\mathcal{L}}"] \arrow[dr, "\mathrm{id}_G \times \Sigma"] \arrow[dd] & & G \times \mathcal{L} \arrow[dr, "\Sigma"] \arrow[dd] & \\
& G \times \mathcal{L} \arrow[rr, "\Sigma"] \arrow[dd] & & \mathcal{L} \arrow[dd] \\
G \times G \times X \arrow[rr, "m \times \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_G \times \sigma"'] & & G \times X \arrow[dr, "\sigma"] & \\
& G \times X \arrow[rr, "\sigma"] & & X
\end{tikzcd}$$

In this case a linearization is just an extension of the action of G to a line bundle on X (which is Σ).

Stability I

With the help of linearizations, one would then be able to define the stability of points.

Definition

We say a geometric point $x \in X$ is **semi-stable** with respect to an invertible sheaf with a G-linearization φ if there exists a global section s of $L^{\otimes r}$ for some r such that $s(x) \neq 0$, s is invariant in the sense that if $\varphi_r : \sigma^* L^{\otimes r} \to p_2^* L^{\otimes r}$ defined by φ (recall how the G-linearizations of invertible sheaves are preserved) then $\varphi_r(\sigma^*(s)) = p_2^*(s)$, and X_s is affine. The set of semi-stable points is denoted by $X^{SS}(L)$.

Definition

A geometric point x is **stable** if there is a global section of $L^{\otimes r}$ such that $s(x) \neq 0$, X_s is affine, s is invariant and G is closed on X_s. The set of such points is $X^S(L)$. If the dimension of the stabilizer of a stable point is zero, we call that point **properly stable**, whose set is denoted by $X_0^S(L)$.

Stability II

In terms of line bundles, the global sections are then invariant in the sense that G acts on $\mathcal{L}$.

Definition

We say a geometric point $x \in X$ is **semi-stable** with respect to a G-linearized line bundle $\mathcal{L}$ if there is an invariant section f of $\mathcal{L}^{\otimes r}$ for some r such that $f(x) \neq 0$ and X_f is affine.

Definition

A semi-stable point x is **stable** if furthermore the action of G is closed on X_s. If the dimension of the stabilizer of a stable point is zero, we call that point **properly stable**.

Projective geometric quotient

Theorem (Existence of geometric quotients)

Suppose X is an algebraic scheme over k and G a reductive algebraic group acting on X. Suppose L is a G-linearized invertible sheaf on X. Then a uniform categorical quotient (Y, φ) of $X^{SS}(L)$ by G exists. Furthermore, (i) *φ is affine,* (ii) *Y is an quasiprojective algebraic scheme, and* (iii) *there is an open subset $\tilde{Y} \subseteq Y$ such that $X^S(L) = \varphi^{-1}(\tilde{Y})$ and $(\tilde{Y}, \varphi)$ is a uniform geometric quotient of $X^S(L)$.*

Proof.

(Point to start:) Take $Y = \operatorname{Proj} \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G$ and glue the morphisms φ_i of the quotients of affine opens of X to get φ. Let $\tilde{Y}$ be the image of X^S in Y. □

We denote by $X /\!\!/_L G$ the quotient Y and $X/_L G$ the quotient $\tilde{Y}$.

Hilbert-Mumford criterion I

Definition

A 1-**parameter subgroup** of G is defined by a homomorphism $\lambda\ \mathbb{G}_m \to G$.

For an algebraic group G that acts by σ on an proper algebraic scheme X/k and some closed point $x \in X$, we define the morphism $\psi_x : G \to X$ by $g \mapsto g \cdot x$. Say λ is a 1-parameter subgroup of G. We embed $\mathbb{G}_m$ into the affine line so that the map $\lambda_x : \psi_x \circ \lambda$ extends uniquely to a morphism $\widehat{\lambda_x}$:

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "\lambda_x"] \arrow[d, hook] & X \arrow[d] \\
\mathbb{A}^1 \arrow[r] \arrow[ur, dashed, "\widehat{\lambda_x}"] & \operatorname{Spec} k
\end{tikzcd}$$

by the valuative criterion for properness. Take the specialization $\widehat{\lambda_x}(0)$ denoted by $\lim_{t\to 0} \lambda(t) \cdot x = x_0$. This point is $\lambda(\mathbb{G}_m)$-invariant since $\widehat{\lambda_x}$ is G-invariant on $\mathbb{G}_m \times \mathbb{G}_m$ which is dense in $\mathbb{G}_m \times \mathbb{A}^1$ and X is separated.

Hilbert-Mumford criterion

On the fiber $\mathcal{L}_{x_0}$ of the line bundle corresponding to our invertible sheaf, $\mathbb{G}_m$ acts via λ, which means it's simply a character of $\mathbb{G}_m$, and thus of the form $\chi : \alpha \mapsto \alpha^r$ for $\alpha \in \mathbb{G}_m$ such that $\kappa(\alpha) = k$. We then define

Definition

Suppose G acts on an algebraic scheme X proper over k, x is a closed point of X, λ a 1-parameter subgroup and $L \in \mathrm{Pic}^G(X)$. Then we define the **Hilbert-Mumford weight** to be $\mu^L(x, \lambda) = -r$.

Clearly (without proof, but one should follow intuitively),

(i) We have $\mu^L(\alpha \cdot x, \lambda) = \mu^L(x, \alpha^{-1} \cdot \lambda \cdot \alpha)$ by definition;

(ii) The map $L \mapsto \mu^L(x, \lambda)$ is a homomorphism from $\mathrm{Pic}^G(X)$ to $\mathbb{Z}$.

(iii) Moving the zero point around, if $f : X \to Y$ is a G-linear morphism of schemes and $L \in \mathrm{Pic}^G(Y)$, then $\mu^{f^*L}(x, \lambda) = \mu^L(f(x), \lambda)$;

(iv) If $\sigma(\alpha) \cdot x \to y$ as $\alpha \to 0$ then $\mu^L(x, \lambda) = \mu^L(y, \lambda)$ (the specialization at zero would be the same).

Computing the HM weight I

From now we assume $X = \mathbb{P}^{n-1}$ for some n. The simplification can be made by the following lemma

Lemma

Suppose $f : X \to Y$ is a finite G-linear morphism, X is proper over k and M an ample invertible sheaf on Y. Then

$$X^{SS}(f^*M) = f^{-1}\left[Y^{SS}(M)\right], \quad X_0^S(f^*M) = f^{-1}\left[Y_0^S(M)\right]$$

in which case we take f is an immersion of X into $\mathbb{P}^{n-1}$ which exists by the assumption of an ample sheaf on X.

In this case the action of $\mathbb{G}_m$ (which is a torus) on an affine space can be diagonalized. Since the action of $\mathbb{G}_m$ on X and a $\mathbb{G}_m$-linearization of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ gives us an action of $\mathbb{G}_m$ on the cone $\mathbb{A}^n$. Thus, choosing a suitable coordinate, for any $\alpha \in (\mathbb{G}_m)_k$ is given by a diagonal matrix $\mathrm{diag}(\alpha^{r_1}, \ldots, \alpha^{r_n})$ with fixed $r_i \in \mathbb{Z}$.

Computing the HM weight II

Lemma

Let x be a closed point of X, let λ be a 1-parameter subgroup and let λ_x specializes to x_0 (extend the morphism by properness). Let x^* be a homogeneous point lying over x. Diagonalize $\mathbb{G}_m$ via λ. Then

$$\mu^{\mathcal{O}_{\mathbb{P}^{n-1}}(1)}(x, \lambda) = \max\{-r_i : x_i^* \neq 0\}$$

if one writes $x^* = (x_1^*, \ldots, x_n^*)$. Furthermore, when $\alpha \to 0$, the morphism $(\sigma^* \circ \lambda_x)(\alpha)$

(i) *has no specialization if and only if $\mu(x, \lambda) > 0$;*

(ii) *has some nonzero specialization if and only if $\mu(x, \lambda) = 0$;*

(iii) *has specialization 0 if and only if $\mu(x, \lambda) < 0$.*

Hilbert-Mumford criterion

Now we could proceed to our ultimate criterion.

Theorem (Hilbert-Mumford criterion)

Let G be a reductive group acting on a separated scheme X which is proper over k. Suppose $L \in \mathrm{Pic}^G(X)$ and assume L is an ample sheaf. Then if $x \in X_k$,

1. *$x \in X^{SS}(L)$ if and only if $\mu^L(x, \lambda) \geqslant 0$ for all 1-parameter subgroup λ;*
2. *$x \in X_0^S(L)$ if and only if $\mu^L(x, \lambda) > 0$ for all 1-parameter subgroup λ.*

Remark

The theorem is called *Hilbert*-Mumford criterion because Hilbert proved it for the case $G = SL_m$ and $X = \mathbb{P}^m$ (as a projective variety). Mumford then generalized this theorem using a similar method (treating the formal power series) for all reductive G and proper separated scheme X.

Examples

1. Binary forms
2. Smooth curves of genus one
3. Ordered points on a line (time?)

Applications of GIT

1. Symplectic quotients (e.g., Hamiltonian reduction)
2. Moduli spaces for vector bundles over curves
3. Analytic quotients
4. Nonreductive geometric invariant theory
5. Moduli spaces of unstable objects